Undegraduate Mathematics

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Chapter 1

Preface

This project started as a posible review of all topics revisited in an undergraduate math course.

Chapter 2

Introduction to Mathematics

First, a word about sets. These are the most primitive objects in mathematics. We use the following observation by Cantor

> By an aggregate [set] we are to understand any collection into a whole M of definite and separate objects m of our intution or our thought. These objects we call the elements of M
>
> *Cantor*

We start with an informal definition of Natural numbers, recall that we can put this in stone observations, using the Peano axioms.

Chapter 3

Vector Calculus

3.0.1 Stokes Theorem

Stokes'theorem relates the line integral of a vector field around a simple closed curve C in $\mathbb{R}^3$ to an integral over a surface S for wic C is the boundary.

Stokes Theorem for Graph Consider S that is the graph of a function $f(x,y)$ so that is parametrized by

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$

Stokes' Theorem for Graphs Let S be the oriented surface by a C^2 function $z = f(x,y)$, where $(x,y) \in D$, a region to which Green's theorem applies, and let $\mathbf{F}$ be a C^1 vector field on S. Then if ∂S denotes the oriented boundary curves of S as just defined, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Stokes' Theorem for Parametrized surfaces

Let S be an oriented surfaces defined by a one-to-one parametrization $\Phi : D \subset \mathbb{R}^2 \to S$, where D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary of S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_s (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Chapter 4

Linear Algebra

A vector space V over a filed $\mathbb{F}$ is an abelian group $(V, +)$, for which a binary product, $(a, v) \to av$, of $\mathbb{F} \times V$ into V is defined satisfying the following axioms for all $a, b \in \mathbb{F}$ and $u, v \in V$

- $1v = v$
- $(ab)v = a(bv)$
- $(a + b)v = av + bv$ and $a(v + u) = av + au$

The elements of V are usually referred as vectors; the elements of the underlying field as scalars.

Examples of vector spaces:

- $\mathbb{F}^n$ the space of all F-valued n-tuples with addition and multiplication by scalars defined pointwise. To be able to differentiate between row and column spaces we will denote by the following $\mathbb{F}^n_c$ or $\mathbb{F}^n_r$.
- $M(n, m; \mathbb{F})$ are the default matrices with entries from F.

Definition: A map $\phi : V \to W$ is linear if for all scalars a, b and vectors v_1, v_2, we have

$$\phi(av_1 + bv_2) = a\phi(v_1) + b\phi(v_2)$$

A map ϕ is an isomorphism if it is both bijective and linear.

We observe that the relation of being isomorphic is an equivalence relation.

Definition: A (vector) subspace of a vector space V is a subset that is closed under the operations of addition and multiplication by scalars inherited from V.

An interesting subspace of a vector space is to take W_j be a family of subsets, then we have that the sum of subspaces, is the set

$$\sum W_j = \bigcup_{J_1 \subset J} \left\{ v : v = \sum_{j \in J_1} v_j, v_j \in W_j \right\}$$

4.0.1 Quotient spaces

A subspace W of a vector space V defines an equivalence relation in V

$$x \equiv y \mod W$$

This equivalence relation partitions V into equivalence classes, called the cosets of W in V. For $x \in V$, the coset of W that contains x is the set $\tilde{x} = x + W$ the *translate* of x by W.

Proposition

$$\tilde{x} = x + W$$

Consider an arbitrary element of b of $\tilde{x}$,

$$b \equiv x \mod W$$
$$b - x = w_1$$

For some w_1 in W, so that $b \in x + W$. The other side is similar.

We define the quotient space V/W to be the space whose elements are the cosets of W in V

Definition **Direct Sums** If $V_1, \ldots, V_k$ are vector spaces over $\mathbb{F}$, the (formal) direct sum

$$\oplus_1^k V_j$$

is the set $\{(v_1, \ldots, v_k) : v_j \in V\}$

Chapter 5

Complex Analysis

The central objects are functions from the complex plane to itself

$$f : \mathbb{C} \to \mathbb{C}$$

A more interesting anotation is that f is differentiable in the complex sense. This condition is called holomorphicity, and it shapes all of complex analysis.

A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad h \in \mathbb{C}$$

This encompasses a multiplicity of conditions: so to speak, one for each angle that h can approach zero.

Our main goals is to observe the following properties.

- Contour Integration. If f is holomorphic in ω, then for appropriate closed paths in ω.
$$\int_\gamma f(z)dz = 0$$
- Regularity. If f is holomorphic, then f is indefinitely differentiable
- Analytic continuation. If f and g are holomorphic functions in ω which are equal in an arbitrary small disc in ω, then $f = g$ everywhere in ω

Basic Properties

A complex number takes the form $z = x + iy$ where x, y are real numbers and i is an imaginray number that satisfies $i^2 = -1$, we denote $x =$

$\mathrm{Re}(z), y = \mathrm{Im}(z)$, we observe that the real numbers are precisely those complex numbers for which the imaginary part is zero. An important observation is the multiplication of two complex numbers:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) \end{aligned}$$

We observe that addition correspond naturally to the addition of two vectors, while multiplication consist of a rotation plus a dilatation, multiplication of i is a rotation of $\frac{\pi}{2}$. We define, the absolute value of a complex number z by

$$|z| = (x^2 + y^2)^{\frac{1}{2}}$$

We observe then $|z|$ consists of the distance from the origin to the point (x, y).

We observe that the triangle inequality holds

$$|z + w| <= |z| + |w| \quad \forall z, w \in \mathbb{C}$$

We observe the above since

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) \\ &= |z|^2 + w\overline{z} + z\overline{w} + |w|^2 \\ &= |z|^2 + w\overline{z} + \overline{w\overline{z}} + |w|^2 \end{aligned}$$

Now, we make the observation that

$$Re(a) = \frac{a + \overline{a}}{2}$$

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) \\ &= |z|^2 + 2\,\mathrm{Re}(w\overline{z}) + |w|^2 \\ &\leq |z|^2 + 2|w||\overline{z}| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

The reverse triangle inequality is

$$||z| - |w|| \leq |z - w|$$

and is proven as: with the triangle inequality we have

$$|z| + |w - z| \geq |z + w - z| = |w|$$
$$|w| + |z - w| \geq |w + z - w| = |z|$$

and we observe then

$$|w - z| \geq |w| - |z|$$
$$|z - w| \geq |z| - |w|$$

from absolute values we know that $|w-z| = |z-w|$, and if $t \geq a$ and $t \geq -a$, implies $t \geq |a|$, so that

$$|z - w| \geq ||w| - |z||$$

We already have used $\overline{z}$ to denote the complex conjugate of z. we observe that

$$z\overline{z} = |z|^2 \quad \Longrightarrow \frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Any non-zero complex number z can be written in polar form

$$z = re^{i\theta}$$

From here, we obtain the observation that if z and w are complex numbers, we have that their multiplication is:

$$zw = rse^{i(\theta+\phi)}$$

so multiplication by a complex number corresponds to a homothety in the plane.

5.0.1 Convergence

A sequence $\{z_1, z_2, \ldots\}$ of complex numbers is said to converge to w if

$$\lim_{n\to\infty} |z_n - w| = 0$$

Since absolute values in $\mathbb{C}$ and Euclidean distances in the plane coincide, we see that z_n converges to w if and only if the corresponding sequence of points in ht complex plane converges to the point that correspond to w

In fact the sequence $\{z_n\}$ converges to w if and only if the sequence of real and imaginary parts of z_n converge to the real and imaginary parts of w, respectively