phys481_week10_fourier_part2

November 14, 2017

1 Fourier transform II

1.1 Phys 481 Winter 2017 Week 10

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline
```

1.2 Fourier transform

If the Fourier transform is defined as

$$F(\omega) = \int_{-\infty}^{+\infty} dt \; e^{-i\omega t} \; f(t)$$

and the inverse Fourier transform is defined as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{+i\omega t} \; F(\omega)$$

then we can take the inverse of a forward transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{+i\omega t} \int_{-\infty}^{+\infty} dt' \; e^{-i\omega t'} \; f(t')$$

and rearrange

$$f(t) = \int_{-\infty}^{+\infty} dt' \; f(t') \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{+i\omega t} \; e^{-i\omega t'}$$

and rearrange

$$f(t) = \int_{-\infty}^{+\infty} dt' \; f(t') \; \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; e^{i\omega(t-t')}$$

to get the following result in terms of the Dirac delta function

$$f(t) = \int_{-\infty}^{+\infty} dt' \; f(t') \; \delta(t'-t)$$

1.3 Dirac delta function

The Dirac delta function has the following property

$$\int_{-\infty}^{+\infty} dx\ f(x)\ \delta(x-x') = f(x')$$

so from the definition of Fourier transforms we can also say

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\ e^{i\omega(t-t')}$$

1.4 Examples

1.4.1 Real delta function

$$f(t) = \delta(t-t_0)$$

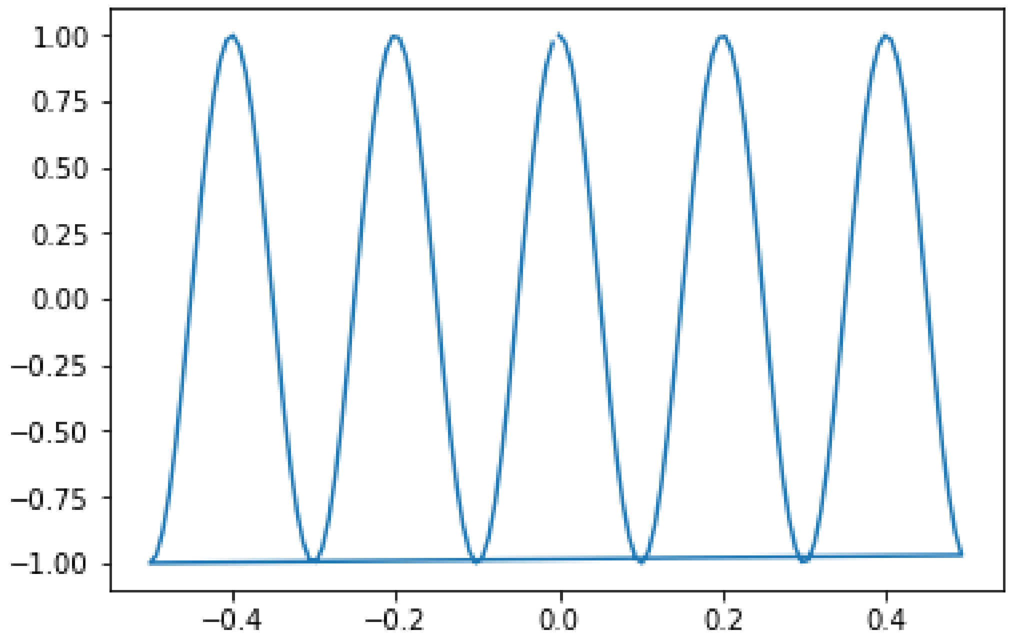
$$F(\omega) = \int_{-\infty}^{+\infty} dt\ e^{-i\omega t}\ \delta(t-t_0)$$

$$F(\omega) = e^{i\omega t_0}$$

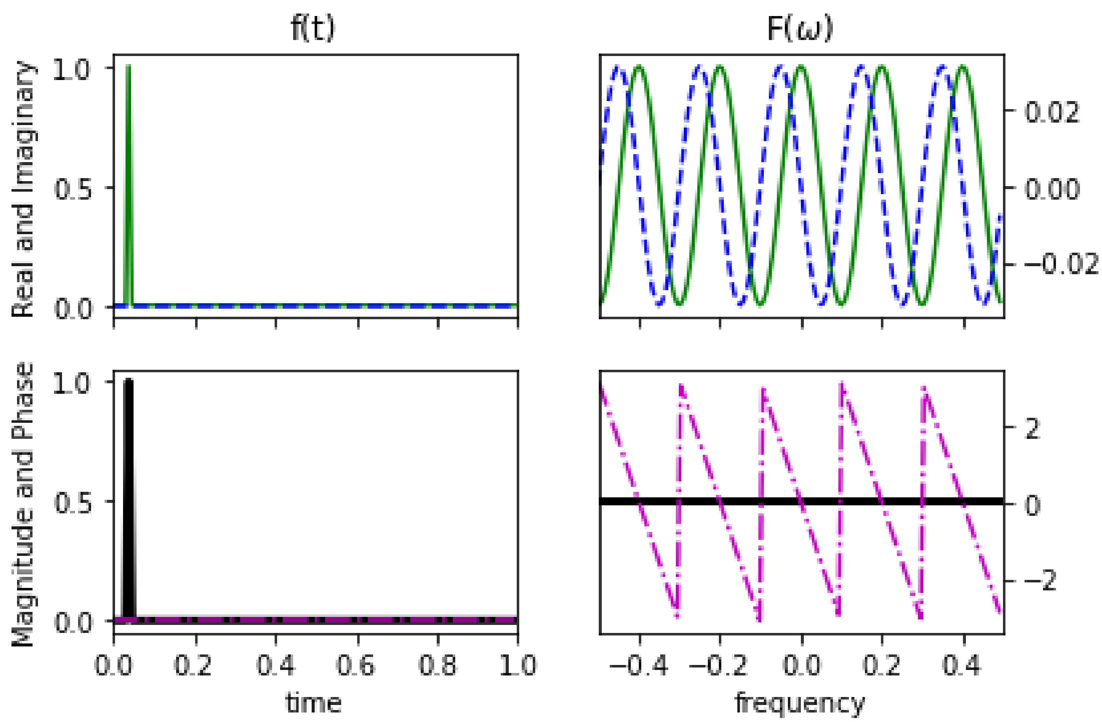
```
In [2]: time = np.linspace(0, 1.0, 129)[0:-1]    # time scale

        tfunc = np.zeros( len(time) )    # test function
        tfunc[5] = 1.0                  # only a single frequency is non-zero

        ffunc = np.fft.fft(tfunc)    # Fourier transform
        freq = np.fft.fftfreq(len(tfunc))  # frequency scale
        plt.plot(freq, ffunc)
```

```
/home/bjackel/miniconda3/lib/python3.5/site-packages/numpy/core/numeric.py:531: Com
  return array(a, dtype, copy=False, order=order)
```

```
Out[2]: [<matplotlib.lines.Line2D at 0x7ff4cd0403c8>]
```




```
In [3]: from fft_code import plot_fourier4    # top secret code
        plot_fourier4(tfunc, time)
```



1.4.2 Task: Write a python function called "plot_fourier4(f,t)" to generate sets of 2x2 figures exactly as shown.

1.4.3 Complex harmonic

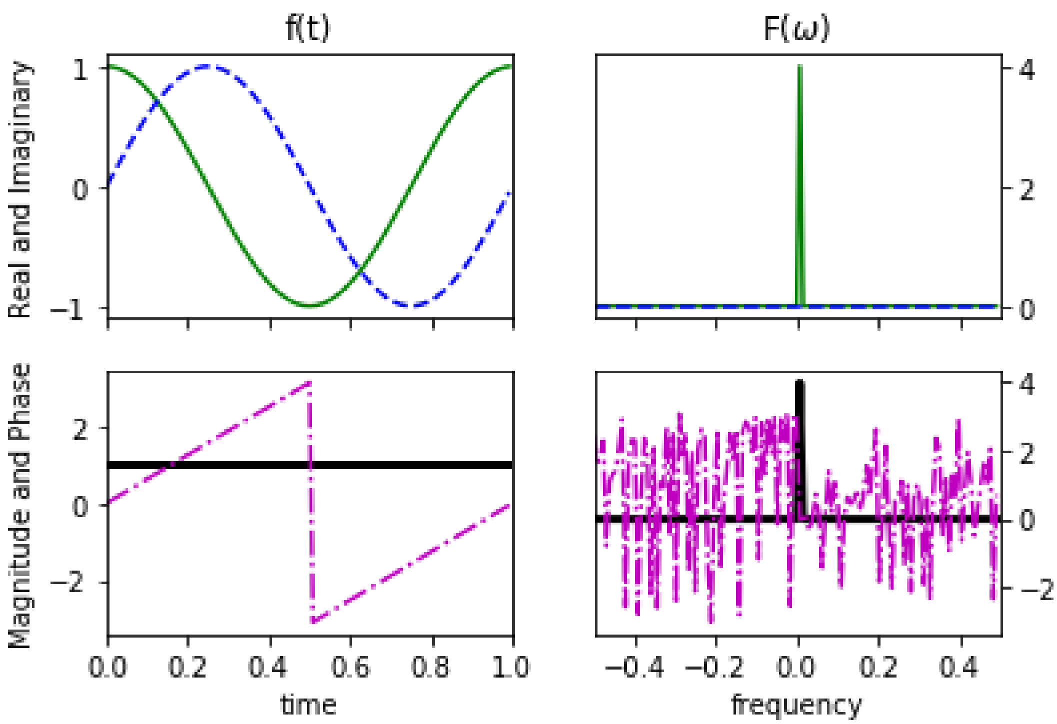
$$f(t) = e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

$$F(\omega) = \int_{-\infty}^{+\infty} dt\ e^{-i\omega t}\ e^{i\omega_0 t}$$

$$F(\omega) = \int_{-\infty}^{+\infty} dt\ e^{i(\omega_0 - \omega)t}$$

$$F(\omega) = \delta(\omega_0 - \omega)$$

```
In [4]: from fft_code import plot_fourier4     # top secret code

        time = np.linspace(0, 1.0, 129)[0:-1]
        omega = 2*np.pi * 1.0
        phi = 0.0
        tfunc = np.exp(0+1j * omega*time + phi)   # $e^{i\omega t + \phi}$

        plot_fourier4(tfunc, time)
```



```
In [5]: # numerical precision makes the phase look noisy
        #
        ffunc = np.fft.fft(tfunc)
        freq = np.fft.fftfreq(len(tfunc))
        print(ffunc)
```

```
[ -4.82703475e-15 +6.49820617e-16j   1.28000000e+02 -1.18691848e-13j
   4.74520722e-15 +3.64521478e-17j   3.58944134e-14 +1.76223867e-15j
   1.98814400e-15 -7.53046752e-16j   1.81657605e-14 +6.18411030e-16j
   1.43113702e-15 +1.25865047e-15j   1.32459412e-14 +1.83840726e-15j
   8.40347080e-17 -5.71172980e-16j   1.19289165e-14 +2.03964288e-15j
   2.26192513e-16 +1.24189090e-15j   8.04760309e-15 +1.30473641e-15j
   7.78995176e-16 +1.45621115e-15j   3.45821096e-15 +1.16688546e-15j
  -6.21079777e-16 -1.20642626e-15j   4.99145753e-15 +5.08222514e-16j
   4.79549948e-16 +3.54681324e-16j   4.51699514e-15 +2.15580828e-15j
   1.39893483e-15 +8.33820953e-16j   3.26558552e-15 +3.88252933e-15j
   1.34064077e-15 +4.23138149e-16j   4.28749840e-15 +3.17863566e-15j
   9.13627410e-18 +1.40562144e-15j   3.52164338e-15 +2.64702820e-15j
  -8.61774660e-16 +4.87973801e-15j  -4.30983194e-15 +2.45036469e-15j
  -1.07018510e-15 -4.04617972e-15j   3.14027296e-15 -2.14330800e-17j
  -8.00398616e-16 +1.26704452e-15j   2.44080523e-15 -2.79823194e-16j
  -6.01861817e-16 -1.24678300e-15j   1.60656515e-15 +9.47816333e-16j
   9.46124981e-16 +7.33087343e-16j   7.07767178e-16 +1.42040518e-15j
   5.59955236e-16 -2.03716451e-15j   1.32362425e-15 +1.99516550e-15j
  -5.51588043e-16 -8.76509069e-16j   1.90912151e-15 +1.31780659e-15j
   6.28676232e-16 +1.03004136e-15j   1.89021830e-15 +1.01888908e-15j
  -1.02136221e-16 -2.78431150e-16j   2.58702008e-15 +1.74965183e-15j
  -1.72208769e-15 -1.83696784e-15j   3.17431029e-15 +2.77196953e-15j
  -1.45568798e-15 +1.42385203e-15j  -4.31300062e-16 +8.03051966e-16j
   1.21732878e-15 -5.92491268e-16j   5.10153277e-16 +1.61402135e-15j
   2.90020261e-15 -4.19757151e-15j   4.09169949e-16 +1.35113589e-15j
   3.76536265e-15 +2.93317170e-15j  -1.87248738e-15 +3.47199898e-16j
   8.51988293e-16 -6.41077482e-16j   2.12125117e-15 +2.36328152e-15j
  -2.46939974e-16 +6.02605541e-16j   2.80901891e-16 +4.99896398e-16j
   1.86609203e-15 -9.93414530e-16j   5.50309225e-15 +1.21831816e-15j
   1.69772436e-15 +2.11911386e-15j  -1.48326892e-16 +3.49404970e-15j
   3.90656693e-16 +6.09826427e-16j  -2.07672347e-15 +3.97134922e-15j
  -7.25925666e-16 -7.33660296e-16j   2.73063035e-15 +2.32324293e-15j
  -1.10778761e-15 +9.27376373e-16j   0.00000000e+00 +2.96083962e-15j
  -1.81219173e-16 +6.60189294e-16j  -3.25003978e-15 +3.70705881e-15j
   1.11382668e-15 -1.15717041e-15j   2.07672347e-15 +3.97134922e-15j
   3.14180199e-16 +1.26120051e-15j   9.18931783e-16 +5.37034610e-15j
  -1.40840263e-15 +2.36330045e-15j  -5.50309225e-15 +1.21831816e-15j
  -1.23225575e-15 -7.79284808e-16j  -2.88744531e-16 +1.38684018e-15j
  -4.50932176e-16 +4.83313209e-16j  -2.12125117e-15 +2.36328152e-15j
  -1.24540537e-15 -4.50757047e-16j   1.24329581e-15 +1.06077887e-15j
  -2.77839190e-15 +2.74907232e-15j  -4.09169949e-16 +1.35113589e-15j
  -2.85048115e-15 -3.91726493e-15j  -6.99210744e-16 +1.24361171e-15j
```

```
-3.61821996e-16 -2.12632576e-16j   4.31300062e-16 +8.03051966e-16j
 8.71159903e-16 +1.18144978e-15j  -3.38324498e-15 +1.59687699e-15j
 1.37995441e-15 -6.33961584e-16j  -2.58702008e-15 +1.74965183e-15j
-1.94848218e-16 -1.40704664e-16j  -1.37850219e-15 +8.80526655e-17j
-3.25216890e-16 +9.88909953e-16j  -1.90912151e-15 +1.31780659e-15j
 9.67897209e-18 -1.05263904e-15j  -5.05851077e-16 +1.09859848e-15j
-1.27432107e-15 -2.04247022e-15j  -7.07767178e-16 +1.42040518e-15j
-1.57193002e-15 +8.81960733e-16j  -1.41123559e-15 +1.35906863e-15j
 1.60356999e-16 -1.40213197e-15j  -2.44080523e-15 -2.79823194e-16j
 1.49810889e-15 +9.35591297e-16j  -2.34416975e-15 +1.31690722e-15j
 9.36645427e-16 -4.79859589e-15j   4.30983194e-15 +2.45036469e-15j
 6.04590537e-16 +4.75243670e-15j  -3.36445404e-15 +2.48503840e-15j
-1.58438270e-16 +1.43730372e-15j  -4.28749840e-15 +3.17863566e-15j
-5.70145549e-16 -1.95182411e-16j  -5.34570477e-15 +3.56386503e-15j
-1.53530858e-15 +4.73453556e-16j  -4.51699514e-15 +2.15580828e-15j
-5.43730326e-16 +1.27453356e-15j  -6.63144268e-15 +1.17005614e-15j
 2.51895590e-16 -6.18101382e-16j  -3.45821096e-15 +1.16688546e-15j
 2.96357852e-16 +1.15501934e-15j  -8.09150580e-15 +1.17857745e-15j
-2.09595243e-16 +5.68165894e-16j  -1.19289165e-14 +2.03964288e-15j
 1.04561791e-16 -1.65684958e-17j  -1.27087268e-14 +1.44552695e-15j
-1.17914142e-15 +1.30686222e-15j  -1.81657605e-14 +6.18411030e-16j
-2.08132793e-15 -1.39280553e-15j  -3.43619013e-14 +1.83823523e-15j]
```

1.4.4 Task: Calculate the Fourier transform $F(\omega)$ for each of the following functions

Constant

$$f(t) = C$$

Single spike at the origin

$$f(t) = \delta(t)$$

Two spikes at $\pm t_0$

$$f(t) = \delta(t - t_0) + \delta(t + t_0)$$

Cosine

$$f(t) = \cos(\omega_0 t)$$

Sine

$$f(t) = \sin(\omega_0 t)$$

Tophat

$$f(t) = 1 \qquad \text{if} \quad -t_0 \leq t \leq +t_0 \qquad \text{else} \quad 0$$

1.4.5 Task: change the time scale from

this set of 128 points

```
time = np.linspace(0, 1.0, 129)[0:-1]
```

to this very similar set of 128 points

```
ttime = np.linspace(0, 1.0, 128)
```

then generate a new complex harmonic and see how the FFT changes. Discuss.

```
In [ ]:
```