Group assignment 1

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Algorithm ALG

ALG := Assign a random value from $\{1, 2, \ldots, k\}$ to each variable $x \in x_1, \ldots, x_n$.

Time Complexity

The algorithm has to iterate through each inequality and generate two random integers in the correct range $[1, k]$. Under the assumption that generating a random integer in the correct range takes $O(1)$ time, the *Time Complexity* of the algorithm is clearly $O(N)$

Lemma 1: $E[ALG] = (1 - 1/k) * m$

Proof:
Let $X_1, X_2, ..., X_m$ be random variables so that X_i takes the value 1 if inequality i is satisfied and 0 otherwise. Since the integer values given to each inequality are uniformly sampled from the set $\{1, 2, .., k\}$ we have that the probability that both integers are equal is $\frac{1}{k}$. Thus, we have $Pr(X = 0) = \frac{1}{k}$ from which it follows that $Pr(X = 1) = 1 - \frac{1}{k}$
The expected value of any of the random variables X_i can be computed by:

$$E[X_i] = 1 * Pr(X_i = 1) + 0 * Pr(X_i = 0)$$

$$E[X_i] = 1 * 1 - \tfrac{1}{k} + 0 * \tfrac{1}{k} = 1 - \tfrac{1}{k}$$

By the linearity of expectation, the expected value of the sum of all expected values is the sum of the expected value of each individual X_i. Since there are in total m inequalities with each one having the expected value $E[X_i] = \frac{1}{k}$ we have that the expected value of the sum is:

$$E[\textstyle\sum_{i=1}^{k} X_i] = m(1 - \frac{1}{k})$$

Lemma 2: $Opt \leq m$

Proof:
Since there are m inequalities, the maximum number of inequalities that can be satisfied is m. Thus, an optimal algorithm can satisfy at most is m inequalities.

Conclusion

Dividing both sides of the conclusion of Lemma 1 with $1 - \frac{1}{k}$ gives $\frac{E[Alg]}{1-\frac{1}{k}} = m$. Substituting this expression into the expression proved in Lemma 2 $Opt \leq m.$, we can conclude that $Opt \leq \frac{E[Alg]}{1-\frac{1}{k}} \implies Opt(1 - \frac{1}{k}) \leq E[Alg]$. Which was to be proved.