Group assignment 2

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Problem description

In some graphs the minimum cut is not unique, i.e., there can be more than one minimum cut. Design and analyze a polynomial time algorithm that finds all minimum cuts in a graph. The algorithm should output a correct answer with probability p for some constant (independent of the size of the graph) p > 0.

Hint: look closely at the analysis of Karger's random contraction algorithm. What is the probability that this algorithm outputs any particular minimum cut?

Algorithm ALG

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Require: $Graph G = (V, E)$ & $iterations$
Ensure: Return all minimum cuts in G with probability $p > 0$

```
solutions ← {}
minCut ← ∞
counter ← 1
while counter ≤ iterations do
    (cut, A, B) ← KargerAlg(G)                    ▷ Running Karger's algorithm
    if cut < minCut then
        solutions ← ∅        ▷ Empty the list of solutions when a new minimum cut is found
        minCut ← cut
    else
        if cut == minCut then
            if Not (A, B) in solutions then
                solutions ← solutions ∪ (A, B)
            end if
        end if
    end if
    counter ← counter + 1
end while
return (minCut, solutions)
```

Time Complexity: $\mathcal{O}(n^4 \ln n)$

The first loop will be based on how many iterations we want to run. In this context the number of iterations is chosen to be $4n^2 \ln n$ giving us $\mathcal{O}(n^2 \ln n)$. Furthermore, when running Karger's Algorthim which is the second loop we get $\mathcal{O}(n^2)$ proven from class [1]. Hence, the overall time complexity of the algorithm would then be $\mathcal{O}(n^4 \ln n)$ which is polynomial time.

Lemma 1: Probability of Karger's algorithm missing a particular mincut $< 1 - \frac{1}{n^2}$

Proof: From the lecture [2], we have that the probability of Kargers algorithm finding a particular mincut when running it once to be $< \frac{1}{n^2}$. Thus, the probability that the algorithm misses that particular min cut is $< 1 - \frac{1}{n^2}$

Lemma 2: $(1 - \frac{1}{x})^x \leq \frac{1}{e}$

Proof: Also from the lecture [2], we have that $1 - x \leq e^{-x}$. We can use this as follows

$$(1 - \frac{1}{x})^x \leq e^{-\frac{1}{x}x} = e^{-1} = \frac{1}{e}$$

Lemma 3: The number of possible min cuts in a graph is $O(n^2)$

Proof There are n vertices in the graph. The number of possible different pairs of vertices is thus

$$\binom{n}{2} = \frac{1}{2}n(n-1)$$

As the algorithm outputs 2 vertices the number of possible min cuts is bounded by the number of pairs, and from the expression above we get that the number of min cuts is thus $O(n^2)$

Lemma 4: if $iterations == 4n^2 \ln n$ then Pr[finding all minimum cuts] $> \frac{8}{9}$

Proof: According to lemma1, if we run the Karger's algorithm $4n^2 \ln n$ times, we have:

$$Pr[\text{missing a particular mincut}] < (1 - \frac{1}{n^2})^{4n^2 \ln n}$$

And by lemma2, we have:

$$(1 - \frac{1}{n^2})^{4n^2 \ln n} = ((1 - \frac{1}{n^2})^{n^2})^{4 \ln n} \leq (\frac{1}{e})^{4 \ln n} = \frac{1}{e^{4 \ln n}} = \frac{1}{e^{\ln n^4}} = \frac{1}{n^4}$$

therefore:

$$Pr[\text{missing a particular mincut}] < \frac{1}{n^4}$$

And that is the same for all minimum cuts, because finding a minimum cut in every execution is independent from other executions, meaning that the probability of not finding 2 minimum cuts, would be the same as multiplying the probability of not finding a minimum cut by 2.

Also, according to lemma3, the total number of minimum cuts are $< n^2$, so we have:

$$Pr[\text{missing X mincuts}] < n^2(\frac{1}{n^4}) = \frac{1}{n^2} \quad \text{where } 1 \leq X < n^2$$

Assuming $n > 2$, we have:

$$Pr[\text{missing X mincuts}] < \frac{1}{9} \quad \text{where } 1 \leq X < n^2$$

Therefore,

$$Pr[\text{finding all minimum cuts}] > 1 - \frac{1}{9} > \frac{8}{9}$$

Conclusion

We have shown that the probability p of finding all minimum cuts is constant with $p > \frac{8}{9}$ when running Karger's algorithm $4n^2 \ln n$ times and saving all the minimum cuts found. Also, this algorithm has a polynomial time complexity of $O(n^4 \ln n)$.

References

[1] Austrin P., 2022, Chapter 5: Minimum Cut - Lecture 8. p. 19 - 26

[2] Austrin P., 2022, Chapter 5: Minimum Cut - Lecture 8. p. 18