Growth of Functions

Analysis of Algorithms

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1st Semester 2017

Outline

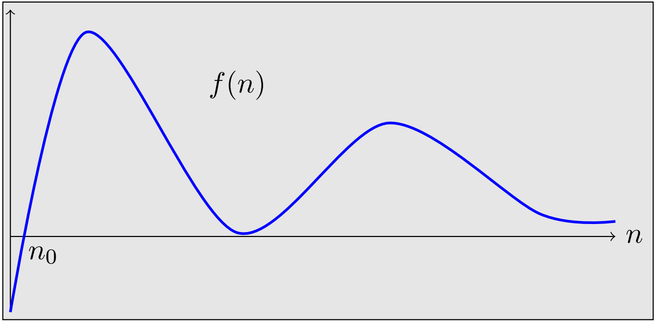
1. Asymptotic notation
2. Common functions
3. Examples
4. Master Theorem

Outline

Asymptotically No Negative Functions

Definition

$f(n)$ is asymptotically no negative if there exist $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $0 \leq f(n)$.

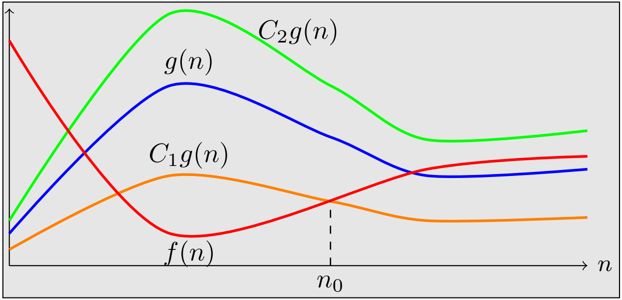


Theta Θ

Definition

$$\Theta\big(g(n)\big) = \big\{ f : \mathbb{N} \to \mathbb{R}^* : (\exists C_1, C_2 \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})$$
$$(\forall n \geq n_0)\big(0 \leq C_1 g(n) \leq f(n) \leq C_2 g(n)\big)\big\}$$

$$C_1 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C_2$$



"$f(n) = \Theta\big(g(n)\big)$" $\equiv$ "$f(n) \in \Theta\big(g(n)\big)$"

f is asymptotically tight bound for g

or

f is of the exact order of g

- Every member of $\Theta\big(g(n)\big)$ is asymptotically no negative.
- The function $g(n)$ must itself asymptotically no negative, or else $\Theta\big(g(n)\big) = \varnothing$.

Example

Lets show that

$$\frac{1}{2}n^2 - 3n = \Theta\left(n^2\right)$$

We have to find C_1, C_2 and n_0 such that

$$C_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq C_2 n^2$$

For all $n \geq n_0$, Dividing by n^2 yields

$$C_1 \leq \frac{1}{2} - \frac{3}{n} \leq C_2$$

We have that $\frac{3}{n}$ is a decreasing sequence

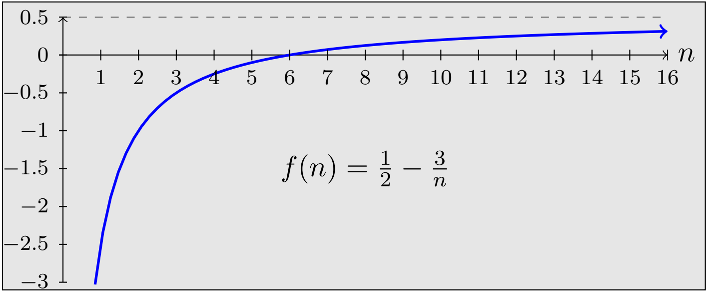
$$3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{3}{8}, \frac{1}{3}, \ldots$$

and then $\frac{1}{2} - \frac{3}{n}$ is an increasing sequence

$$-\frac{5}{2}, -1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{10}, 0, \frac{1}{14}, \ldots$$

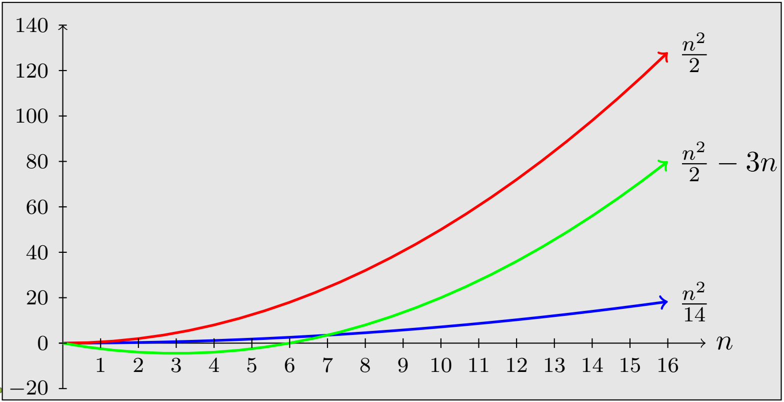
that is upper bounded by $\frac{1}{2}$.

The right hand inequality can be made to hold for $n \geq 1$ by choosing $C_2 \geq \frac{1}{2}$. Likewise the left hand inequality can be made to hold for $n \geq 7$ by choosing $C_1 \leq \frac{1}{14}$.



Thus, by choosing $C_1 = \frac{1}{14}$, $C_2 = \frac{1}{2}$ and $n_0 = 7$ then we can verify that

$$\frac{1}{2}n^2 - 3n = \Theta\left(n^2\right)$$

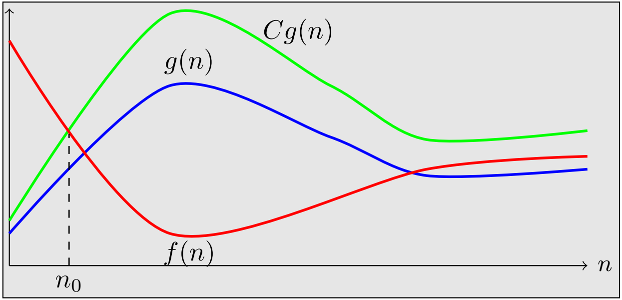


Big O (Omicron)

Definition

$$O\big(g(n)\big) = \big\{f : \mathbb{N} \to \mathbb{R}^* : (\exists C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N}) \\ (\forall n \geq n_0)\big(0 \leq f(n) \leq Cg(n)\big)\big\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C$$



$$\text{“}f(n) = O\big(g(n)\big)\text{”} \equiv \text{“}f(n) \in O\big(g(n)\big)\text{”}$$

f is asymptotically upper bound for g

or

g is an asymptotic upper bound for f

- $O\big(g(n)\big)$ is pronounced "big-oh of $g(n)$".

Example

Lets show that if $a, b > 0$ then

$$an + b = O(n)$$

We have to find C and n_0 such that

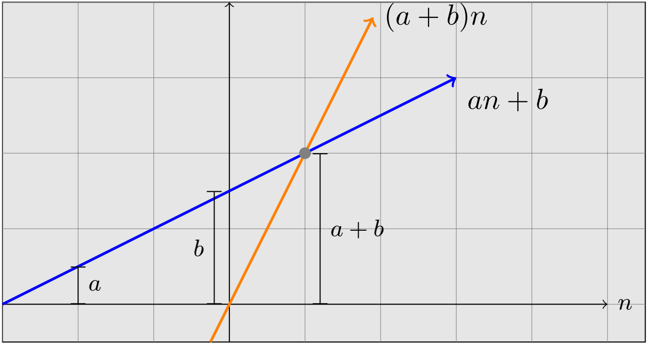
$$an + b \leq Cn$$

For all $n \geq n_0$, dividing by n yields

$$0 \leq a + \frac{b}{n} \leq C$$

The inequality can be made to hold for $n \geq 1$ by choosing $C \geq a + b$. Thus by choosing $C \geq a + b$ and $n_0 = 1$ then we can verify that $0 \leq an + b \leq (a + b)n$, that is to say

$$an + b = O(n)$$

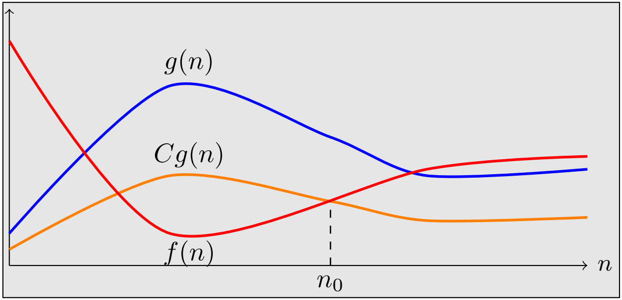


Big Omega Ω

Definition

$$\Omega\big(g(n)\big) = \big\{f : \mathbb{N} \to \mathbb{R}^* : (\exists C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N}) \\ (\forall n \geq n_0)\big(0 \leq Cg(n) \leq f(n)\big)\big\}$$

$$C \leq \lim_{n\to\infty} \frac{f(n)}{g(n)}$$



"$f(n) = \Omega\big(g(n)\big)$" $\equiv$ "$f(n) \in \Omega\big(g(n)\big)$"

f is asymptotically lower bounded for g
or
g is an asymptotic lower bound for f

- $\Omega\big(g(n)\big)$ is pronounced "big-omega of $g(n)$"

Example

Lets show that

$$5n^3 - 5n^2 - 2n - 3 = \Omega\left(n^2\right)$$

We have to find C and n_0 such that

$$0 \leq Cn^2 \leq 5n^3 - 5n^2 - 2n - 3$$

if $n \geq n_0$.

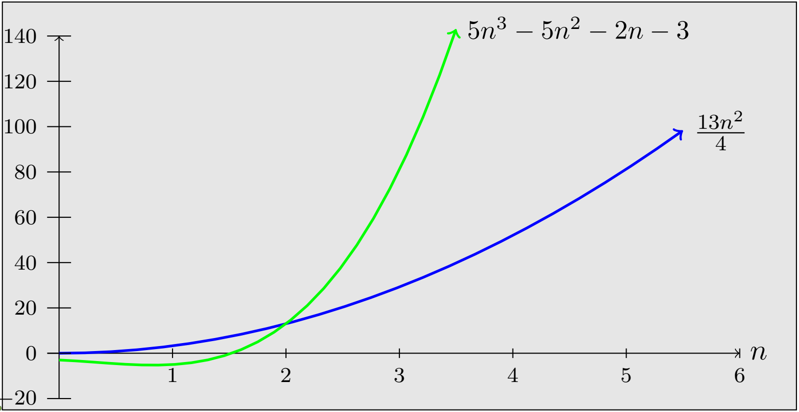
For all $n \geq n_0$, Dividing by n^2 yields

$$0 \leq C \leq 5n - 5 - \frac{2}{n} - \frac{3}{n^2} = 5n - \left(5 + \frac{2}{n} + \frac{3}{n^2}\right)$$

We have that $5 + \frac{2}{n} + \frac{3}{n^2}$ is a decreasing sequence that takes its maximum value 10 when $n = 1$; and therefore $5n - \left(5 + \frac{2}{n} + \frac{3}{n^2}\right)$ is an increasing sequence such that is non-negative for $n \geq 2$, whence if $n_0 = 2$ then it is lower bounded by $\frac{13}{4}$.

Thus, by choosing $C = \frac{13}{4}$ and $n_0 = 2$ then we can verify that

$$5n^3 - 5n^2 - 2n - 3 = \Omega\left(n^2\right)$$

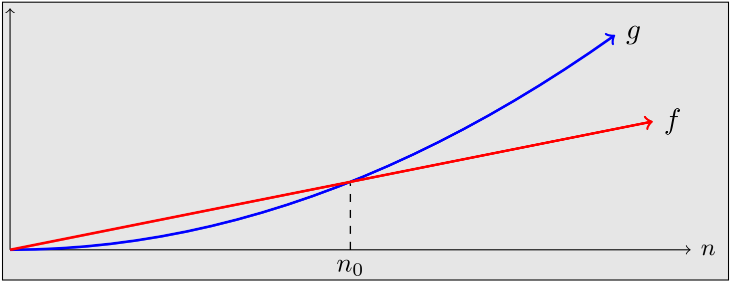


Little o

Definition

$$o(g(n)) = \{f : \mathbb{N} \to \mathbb{R}^* : (\forall C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})$$
$$(\forall n \geq n_0)\big(0 \leq f(n) < Cg(n)\big)\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$



$$\text{“}f(n) = o\big(g(n)\big)\text{”} \equiv \text{“}f(n) \in o\big(g(n)\big)\text{”}$$

f is asymptotically smaller than g

- $o\big(g(n)\big)$ is pronounced "little-oh of $g(n)$".
- $o\big(g(n)\big)$ is the set of functions that grow slower than g.

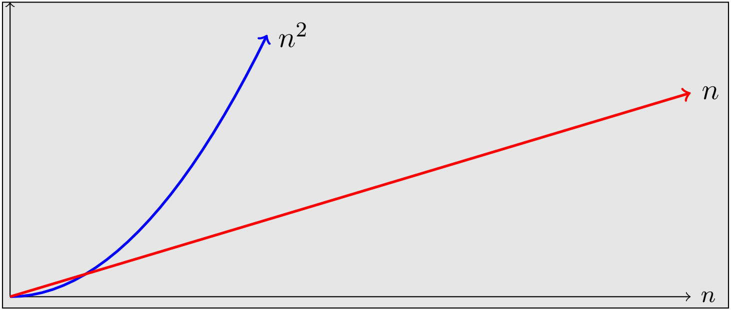
Example

Lets show that

$$n = o\left(n^2\right)$$

we only have to show

$$\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$



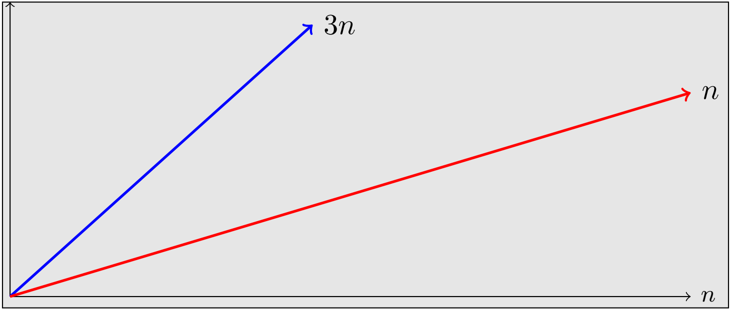
Example

Lets show that

$$n \notin o(3n)$$

we have

$$\lim_{n\to\infty} \frac{n}{3n} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$$

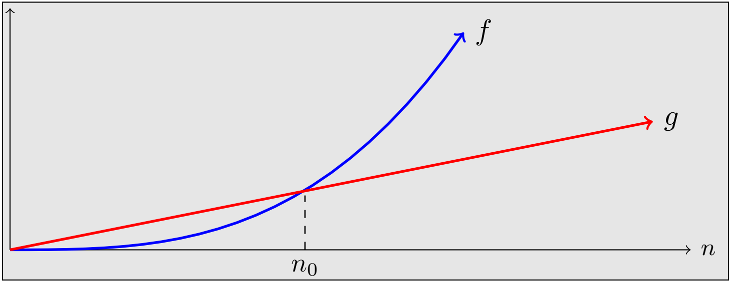


Little omega ω

Definition

$$\omega\big(g(n)\big) = \{f : \mathbb{N} \to \mathbb{R}^* : (\forall C \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})$$
$$(\forall n \geq n_0)\big(0 \leq Cg(n) < f(n)\big)\}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$



$$\text{“}f(n) = \omega\big(g(n)\big)\text{”} \equiv \text{“}f(n) \in \omega\big(g(n)\big)\text{”}$$

f is asymptotically larger than g

- $\omega\big(g(n)\big)$ is pronounced “little-omega of $g(n)$”.
- $\omega\big(g(n)\big)$ is the set of functions that grow faster than g.

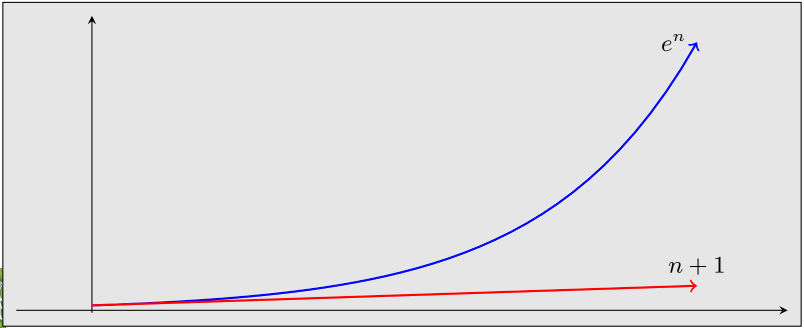
Example

Lets show that

$$e^n = \omega(n+1)$$

we only have to show

$$\lim_{n\to\infty} \frac{e^n}{n+1} = \lim_{n\to\infty} e^n = \infty$$



Analogy with the comparison of two real numbers

Asymptotic notation	Real numbers
$f(n) \in O\big(g(n)\big)$	$f \leq g$
$f(n) \in \Omega\big(g(n)\big)$	$f \geq g$
$f(n) \in \Theta\big(g(n)\big)$	$f = g$
$f(n) \in o\big(g(n)\big)$	$f < g$
$f(n) \in \omega\big(g(n)\big)$	$f > g$

Trichotomy does not hold!

Not all functions are asymptotically comparable

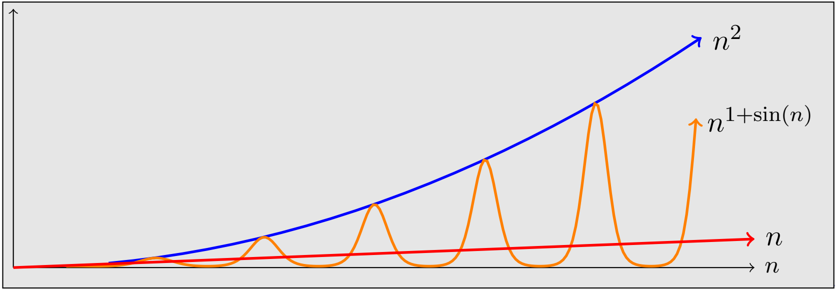
Trichotomy does not hold

Example

Following functions are asymptotically non-negative

- $f(n) = n$
- $g(n) = n^{1+\sin(n)}$

but, they are not comparable because $1 + \sin(n) \in [0, 2]$, the function g varies between 1 and n^2, when $n \to \infty$.



Properties

$$\Theta\big(f(n)\big) = O\big(f(n)\big) \cap \Omega\big(f(n)\big)$$

The running time of an algorithm is $\boldsymbol{\Theta\big(f(n)\big)}$

if and only if

1. Its worst-case running time is $O\big(f(n)\big)$, and
2. Its best-case running time is $\Omega\big(f(n)\big)$.

Properties (conti.)

Given f, g and h asymptotically no negative functions, we have:

Transitivity of O, Ω, Θ $f(n) \in \Delta\big(g(n)\big)$ and $g(n) \in \Delta\big(h(n)\big)$ then $f(n) \in \Delta\big(h(n)\big)$, for $\Delta \in \{O, \Omega, \Theta\}$.

Reflexivity of O, Ω, Θ $f(n) \in \Delta\big(f(n)\big)$, for $\Delta \in \{O, \Omega, \Theta\}$.

Symmetry of Θ $f(n) \in \Theta\big(g(n)\big) \iff g(n) \in \Theta\big(f(n)\big)$.

Anti-symmetry of O, Ω $\forall f(n) \notin \Theta\big(g(n)\big)$, $f(n) \in \Delta\big(g(n)\big) \implies g(n) \notin \Delta\big(f(n)\big)$, for $\Delta \in \{O, \Omega\}$.

Transpose Symmetry

$$f(n) \in O\big(g(n)\big) \iff g(n) \in \Omega\big(f(n)\big)$$
$$f(n) \in o\big(g(n)\big) \iff g(n) \in \omega\big(f(n)\big)$$

Properties (conti.)

- $f \leq g \iff f(n) \in O\big(g(n)\big)$ **order relation**
  - reflexive
  - anti-symmetric
  - transitive
- $f \geq g \iff f(n) \in \Omega\big(g(n)\big)$ **order relation**
  - reflexive
  - anti-symmetric
  - transitive
- $f = g \iff f(n) \in \Theta\big(g(n)\big)$ **equivalence relation**
  - reflexive
  - symmetric
  - transitive

Properties (conti.)

$$o\big(f(n)\big) \cap \omega\big(f(n)\big) = \varnothing$$

Relation between o and O

$$f(n) \in o\big(g(n)\big) \Longrightarrow f(n) \in O\big(g(n)\big)$$

Relation between ω and Ω

$$g(n) \in \omega\big(f(n)\big) \Longrightarrow g(n) \in \Omega\big(f(n)\big)$$

Asymptotic notation two variables

Definition

$$
\begin{aligned}
O\big(g(m,n)\big) = \big\{ f : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^{*} : (\exists C \in \mathbb{R}^{+})(\exists m_0, n_0 \in \mathbb{N}) \\
(\forall m \geq m_0)(\forall n \geq n_0)\big(f(m,n) \leq C g(m,n)\big)\big\}
\end{aligned}
$$

Outline

Monotonicity

f is monotonically increasing if: $\forall x, y \in \mathbb{R}, x < y \Longrightarrow f(x) \leq f(y)$

f is monotonically decreasing if: $\forall x, y \in \mathbb{R}, x < y \Longrightarrow f(x) \geq f(y)$

f is strictly increasing if: $\forall x, y \in \mathbb{R}, x < y \Longrightarrow f(x) < f(y)$

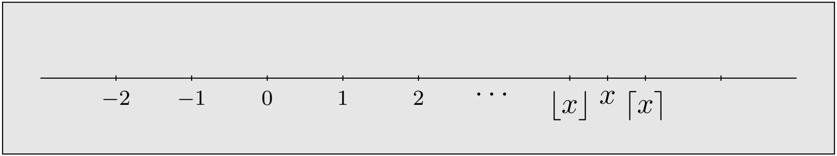
f is strictly decreasing if: $\forall x, y \in \mathbb{R}, x < y \Longrightarrow f(x) > f(y)$

Floors and Ceilings

Definition

$\lfloor x \rfloor$ floor of x: The greatest integer less than or equal to x.

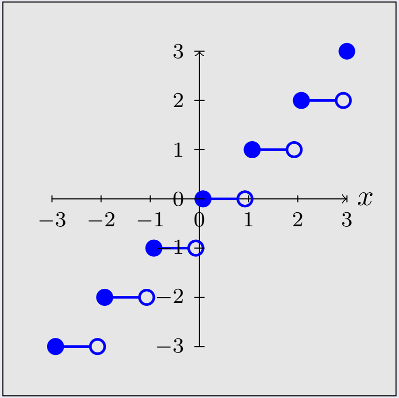
$\lceil x \rceil$ ceiling of x: The smallest integer greater than or equal to x.



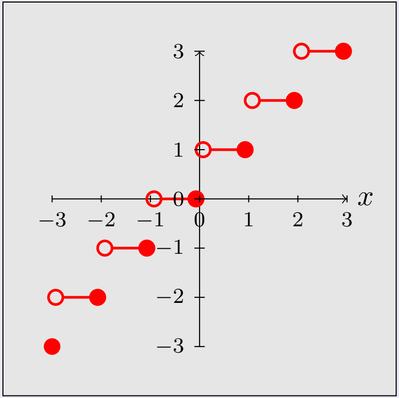
$$\forall x \in \mathbb{R}, \qquad x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$$

$$\forall n \in \mathbb{Z}, \qquad \lfloor n \rfloor = n = \lceil n \rceil \text{ and } \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$$

Graph of the function $f(x) = \lfloor x \rfloor$



Graph of the function $f(x) = \lceil x \rceil$



Properties

$\forall x \in \mathbb{R}$ and $n, m \in \mathbb{Z}^+$

$$\begin{aligned}
\lfloor \lfloor x/n \rfloor /m \rfloor &= \lfloor x/nm \rfloor \\
\lceil \lceil x/n \rceil /m \rceil &= \lceil x/nm \rceil \\
\lfloor n/m \rfloor &\leq (n + (m-1))/m \\
\lceil n/m \rceil &\geq (n - (m-1))/m
\end{aligned}$$

$\lfloor x \rfloor$ and $\lceil x \rceil$ are monotonically increasing.

Modular arithmetic

For every integer a and any possible positive integer n,

$$a \bmod n$$

is the **remainder** (or **residue**) of the quotient a/n

$$a \bmod n = a - \lfloor a/n \rfloor n$$

congruency or equivalence $\bmod\ n$

If $(a \bmod n) = (b \bmod n)$ we write

$$a \equiv b \pmod{n}$$

and we say that a is **equivalent** to b module n or that a is **congruent** to b module n.

In other words $a \equiv b \pmod{n}$ if a and b have the same remainder when they are divided by n.

Also $a \equiv b \pmod{n}$ if and only if n is a divisor of $b - a$

$\equiv \pmod{n}$

defines a equivalence relation in $\mathbb{Z}$ and produces a partitioned set called $\mathbb{Z}_n = \mathbb{Z}_{/n} = \{0, 1, 2, \dots, n-1\}$ in which can be defined arithmetic operations

$$a + b \pmod{n}$$

$$a * b \pmod{n}$$

Example $\equiv \pmod{4}$

$$\mathbb{Z}_{/4} = \{[0], [1], [2], [3]\} = \{0, 1, 2, 3\}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-12	-11	-10	-9
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
[0]	**[1]**	**[2]**	**[3]**

$$4 + 1 \pmod{4} = 1$$

$$5 * 2 \pmod{4} = 2$$

Polynomials

Given a no negative integer d, a **polynomial in n of degree d** is a function $p(n)$ of the form:

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

Where $a_0, a_1, a_2, \ldots, a_d$ are the **coefficients** and $a_d \neq 0$, a_d is called the **main coefficient** and a_0 is called the **independent term**.

Properties

- A polynomial $p(n)$ es asymptotically positive if and only if $a_d > 0$.
- If $p(n)$, of degree d is asymptotically positive, we have $p(n) = \Theta\left(n^d\right)$.
- $\forall a \in \mathbb{R}$, $a > 0$, n^a es monotonically increasing.
- $\forall a \in \mathbb{R}$, $a < 0$, n^a es monotonically decreasing.
- A function $f(n)$ is **polynomially bounded** if $f(n) = O\left(n^d\right)$ for some constant d.

Exponentials

For all reals $a > 0$, m and n, we have the following identities:

- $a^0 = 1$
- $a^1 = a$
- $a^{-1} = 1/a$
- $(a^m)^n = a^{mn}$
- $(a^m)^n = (a^n)^m$
- $a^m a^n = a^{m+n}$
- $\dfrac{a^n}{a^m} = a^{n-m}$
- $\frac{d}{dx}(a^u) = a^u \cdot \ln a \frac{du}{dx}$

- If $a > 1$, for all $n \in \mathbb{Z}^+$,

  $$a^n \text{ is monotonically increasing}$$

- If $0 < a < 1$, for all $n \in \mathbb{Z}^+$,

  $$a^n \text{ is monotonically decreasing}$$

- $\forall a \in \mathbb{R}$ with $a > 1$, as:

  $$\lim_{n \to \infty} \frac{n^d}{a^n} = 0$$

  then $n^d = o(a^n)$.

For all $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

- $\forall x \in \mathbb{R}$, $e^x \geq 1 + x$, equality holds for $x = 0$.
- If $|x| \leq 1$, $\quad 1 + x \leq e^x \leq 1 + x + x^2$.
- When $x \to 0$, $e^x = 1 + x + O\left(x^2\right)$.
- $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

Logarithms

Notations:

- $\lg n = \log_2 n$
- $\ln n = \log_e n$
- $\lg^k n = (\lg n)^k$
- $\lg \lg n = \lg(\lg n)$

Logarithm function will only apply to next term in the formula:

$$\lg n + k = (\lg n) + k$$

For $b > 1$ constant and $n > 0$,

$$\log_b n$$

is strictly increasing.

For all reals $a > 0$, $b > 0$, $c > 0$ and n, we have the following identities:

- $a = b^{\log_b a}$
- $\log_c(ab) = \log_c a + \log_c b$
- $\log_c(a/b) = \log_c(a) - \log_c(b)$
- $\log_b a^n = n \log_b a$
- $\log_b a = \frac{\log_c a}{\log_c b}$
- $\log_b(1/a) = -\log_b a$
- $\log_b a = \frac{1}{\log_a b}$
- $a^{\log_b c} = c^{\log_b a}$
- $\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$

For $x \in \mathbb{R}$, if $|x| < 1$ then:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}x^i}{i}$$

- When $x > -1$, $\quad \frac{x}{1+x} \leq \ln(1+x) \leq x$.
- For $x > -1$, equality holds for $x = 0$.

A function $f(n)$ is **polylogaritmically** bounded if

$$f(n) = O(\lg^k n) \text{ for some constant } k$$

We have the following relation between polynomials and polylogarithms:

$$n^d = 2^{\lg n^d} = 2^{d(\lg n)} = \left(2^d\right)^{\lg n}$$

$$\lim_{n\to\infty} \frac{\lg^k n}{n^d} = \lim_{n\to\infty} \frac{\lg^k n}{(2^d)^{\lg n}} = 0$$

then $\lg^k n = o\left(n^k\right)$.

Factorials

Given $n \in \mathbb{N}$, factorial of n is defined as:

Definition (No recursive)

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ \prod_{i=1}^{n} i, & \text{if } n > 0. \end{cases}$$

Definition (Recursive)

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ n \cdot (n-1)!, & \text{if } n > 0. \end{cases}$$

Weak upper bound

$$\boxed{n! \leq n^n}$$

Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + \Theta\left(\frac{1}{n}\right)\right]$$

then

$$\begin{aligned}
n! &= o(n^n) \\
n! &= \omega(2^n) \\
\lg(n!) &= \Theta(n \lg n) \\
n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}, \qquad \text{where } \frac{1}{12n+1} \leq \alpha_n \leq \frac{1}{12n}
\end{aligned}$$

Functional iteration

Definition

Given a function $f(n)$ the i-th functional iteration of f is defined as:

$$f^i = \begin{cases} I, & \text{if } i = 0; \\ f \circ f^{(i-1)}, & \text{if } i > 0. \end{cases}$$

with I the identity function.
For a particular n, we have:

$$f^i(n) = \begin{cases} n, & \text{if } i = 0; \\ f\left(f^{(i-1)}(n)\right), & \text{if } i > 0. \end{cases}$$

Examples

1. $f(n) = 2n$ then $f^{(i)}(n) = 2^i n$
2. $f(n) = n^2$ then:

$$
\begin{aligned}
f^{(2)}(n) &= \left(n^2\right)^2 = \left(n^2\right)\left(n^2\right) = n^{2*2} = n^4 \\
f^{(3)}(n) &= \left(n^{2*2}\right)^2 = n^{2*2*2} = n^8 \\
f^{(4)}(n) &= \left(n^{2*2*2}\right)^2 = n^{2*2*2*2} = n^{16} \\
&\vdots \\
f^{(i)}(n) &= n^{2^i}
\end{aligned}
$$

Example

$f(n) = n^n$ then

$$f^{(2)}(n) = n^{n^n}$$

$$f^{(3)}(n) = n^{n^{n^n}}$$

$$f^{(4)}(n) = n^{n^{n^{n^n}}}$$

$$\vdots$$

$$f^{(i)}(n) = n^{\overbrace{n^{n^{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}}}^{i}}$$

Iterated logarithm

Definition

The iterated logarithm of n, denoted $\lg^* n$ ("log star of n") is defined as:

$$\lg^* n = \min\left\{i \geq 0 : \lg^{(i)} n \leq 1\right\}$$

$\lg^* n$, is a very slowly growing function

$$\begin{aligned} \lg^* 1 &= 0 \\ \lg^* 2 &= 1 \\ \lg^* 4 &= 2 \\ \lg^* 16 &= 3 \\ \lg^* 65536 &= 4 \\ \lg^* (65536)^2 &= 5 \end{aligned}$$

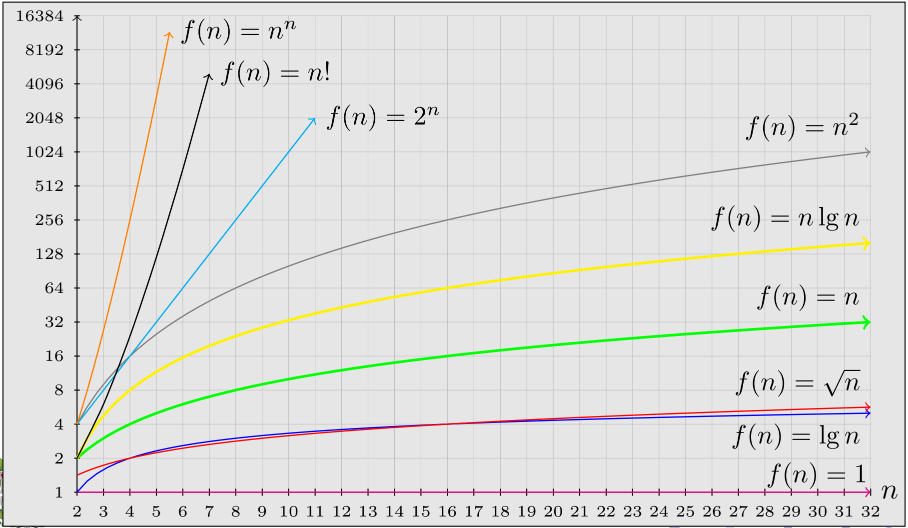
In general

$$\lg^* \underbrace{2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}}_{k} = k$$

Summary

$$1 \in O(\lg n) \in O(\sqrt{n}) \in O(n) \in O(n \lg n) \in O(n^2) \in O(2^n) \in O(n!) \in O(n^n)$$

Logarithmic graph of main functions into Computer Sciences



Outline

Example

A	B	
$5n^2 + 100n$	$3n^2 + 2$	$\mathrm{A} \in \Theta(\mathrm{B})$
$\log_3 \left(n^2\right)$	$\log_2 \left(n^3\right)$	$\mathrm{A} \in \Theta(\mathrm{B})$
$n^{\lg 4}$	$3^{\lg n}$	$\mathrm{A} \in \omega(\mathrm{B})$
$\lg n$	$n^{1/2}$	$\mathrm{A} \in o(\mathrm{B})$

$\lg n$ denotes $\log_2 n$.

A	B	
$5n^2 + 100n$	$3n^2 + 2$	$\mathrm{A} \in \Theta(\mathrm{B})$

$\mathrm{A} \in \Theta(n^2), n^2 \in \Theta(\mathrm{B}) \implies \mathrm{A} \in \Theta(B)$

$\log_3(n^2)$	$\log_2(n^3)$	$\mathrm{A} \in \Theta(\mathrm{B})$

$\log_b a = \log_c a / \log_c b$; $\mathrm{A} = 2\lg n / \lg 3$, $\mathrm{B} = 3 \lg n$; $\mathrm{A}/\mathrm{B} = 2/(3\lg 3)$

$n^{\lg 4}$	$3^{\lg n}$	$\mathrm{A} \in \omega(\mathrm{B})$

$a^{\log b} = b^{\log a}$; $B = 3^{\lg n} = n^{\lg 3}$; $\mathrm{A}/\mathrm{B} = n^{\lg(4/3)} \to \infty$ as $n \to \infty$

$\lg n$	$n^{1/2}$	$\mathrm{A} \in o(\mathrm{B})$

$\lim_{n\to\infty} \left(\log_a n / n^b\right) = 0$, here $a = 2$ and $b = 1/2 \implies \mathrm{A} \in o(\mathrm{B})$

Outline

Master Theorem I

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a > 1, \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2,$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Figure: Master Theorem first version.

Master Theorem II

MASTER THEOREM Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Figure: Master Theorem second version.

Master Theorem III

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)\ ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

Figure: Master Theorem full version.