HEAPSORT & PRIORITY QUEUES

Juan Mendivelso

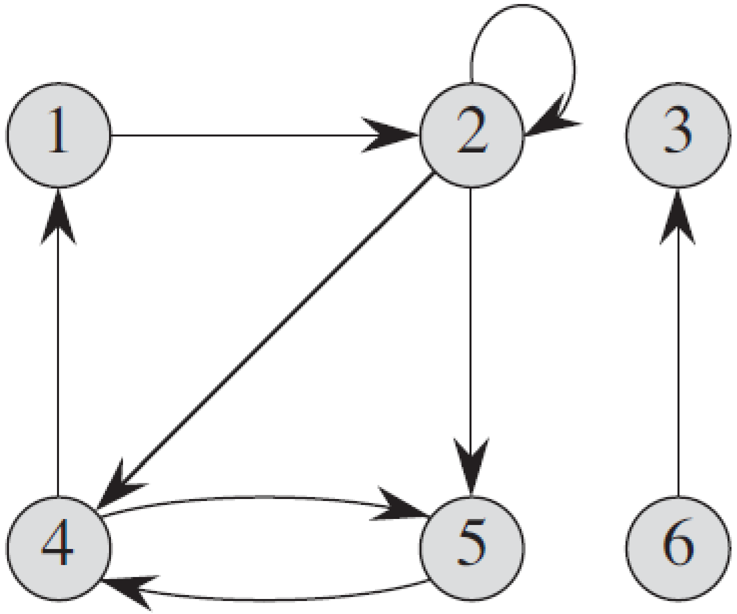
Presentation made by Juan Mendivelso. Contents and figures extracted from the book: Introduction to Algorithms, Third Edition. Cormen, Leiserson, Rivests and Stein. The MIT Press. 2009.

CONTENTS

1. TREES

Digraphs

- A directed graph (digraph) G is a pair (V,E), where V is a finite set and E is a binary relation on V.
  - V is the vertex set.
  - E is the edge set.
- Self-loops are permitted.



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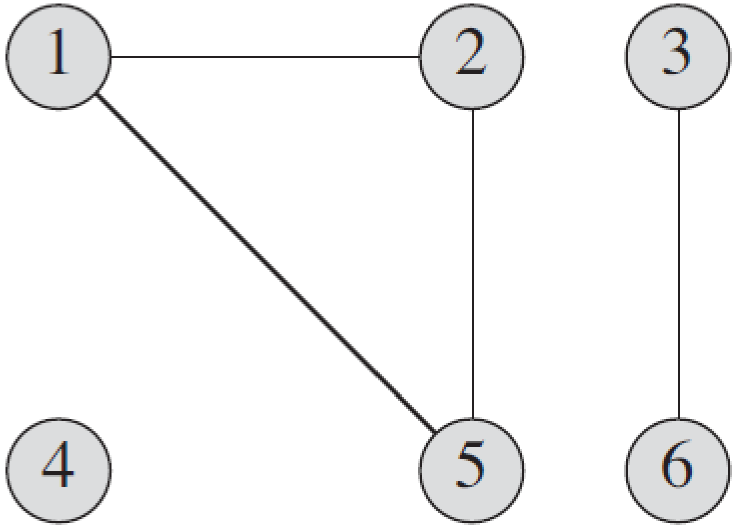
Undirected Graphs

- In an undirected graph G=(V,E), the edge set consists of unordered pairs of vertices rather than ordered sets.
- An edge is a set {u,v}, where u,v are in V and u ≠ v.
- But, by convention, we use the notation (u,v).
- However, (u,v) and (v,u) refer to the same edge.
- Self-loops are not permitted.



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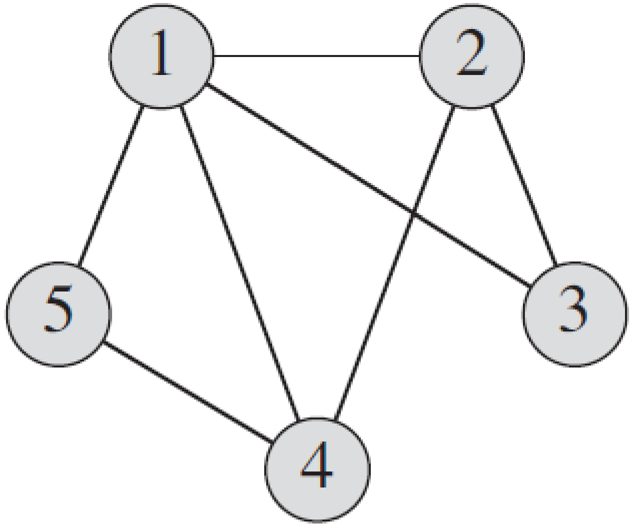
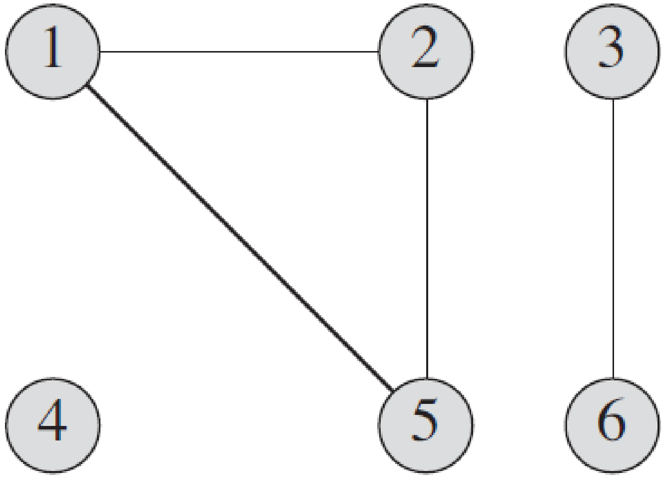
Paths & Cycles

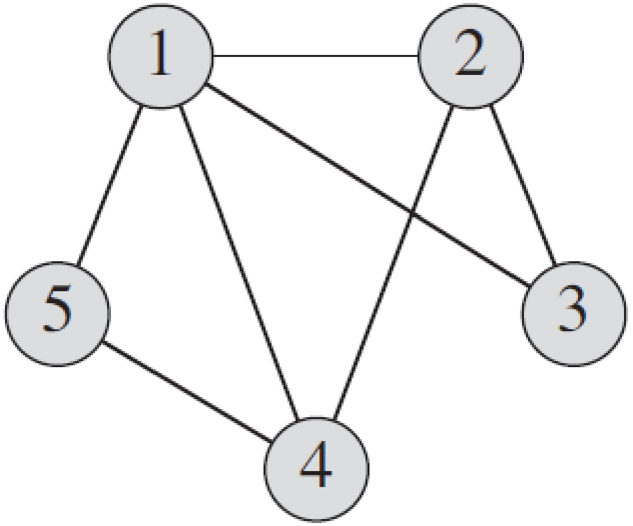
- A path of length k from vertex u to vertex u' in a graph $G=(V,E)$ is a sequence of vertices $v_0,v_1,\dots,v_k$ such that $u=v_0$, $u'=v_k$, and (v_{i-1},v_i) are in E for $i=1,2,\dots,k$.
- If there is a path p from u to u', we say that u' is reachable from u via p.
- A path forms a cycle if $v_0 = v_k$.
- A graph with no cycles is **acyclic**.



Connected Undirected Graph

- An undirected graph is connected if every vertex is reachable from all other vertices.





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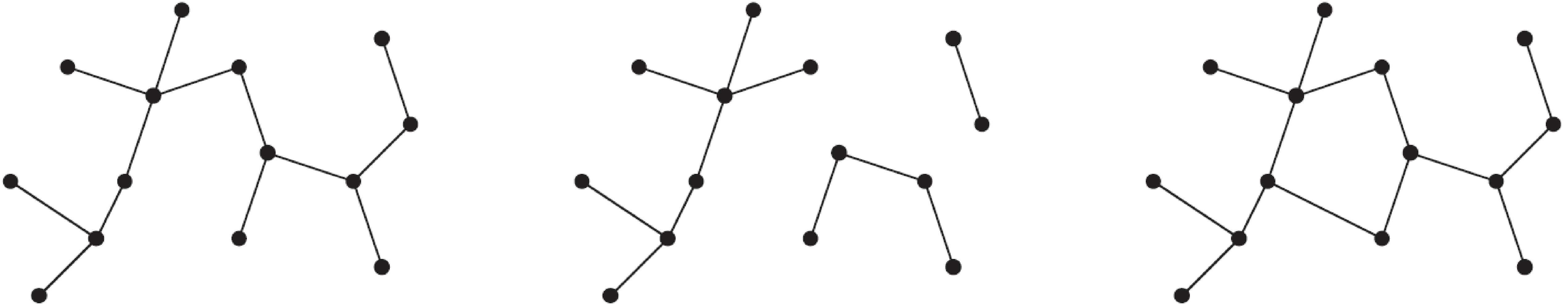
Free Trees & Forests

- A **free tree** is a connected, acyclic undirected graph.
- A **forest** is an acyclic undirected graph.



(a) (b) (c)

Properties of Free Trees

Theorem B.2 (Properties of free trees)
Let $G = (V, E)$ be an undirected graph. The following statements are equivalent.

1. G is a free tree.
2. Any two vertices in G are connected by a unique simple path.
3. G is connected, but if any edge is removed from E, the resulting graph is disconnected.
4. G is connected, and $|E| = |V| - 1$.
5. G is acyclic, and $|E| = |V| - 1$.
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

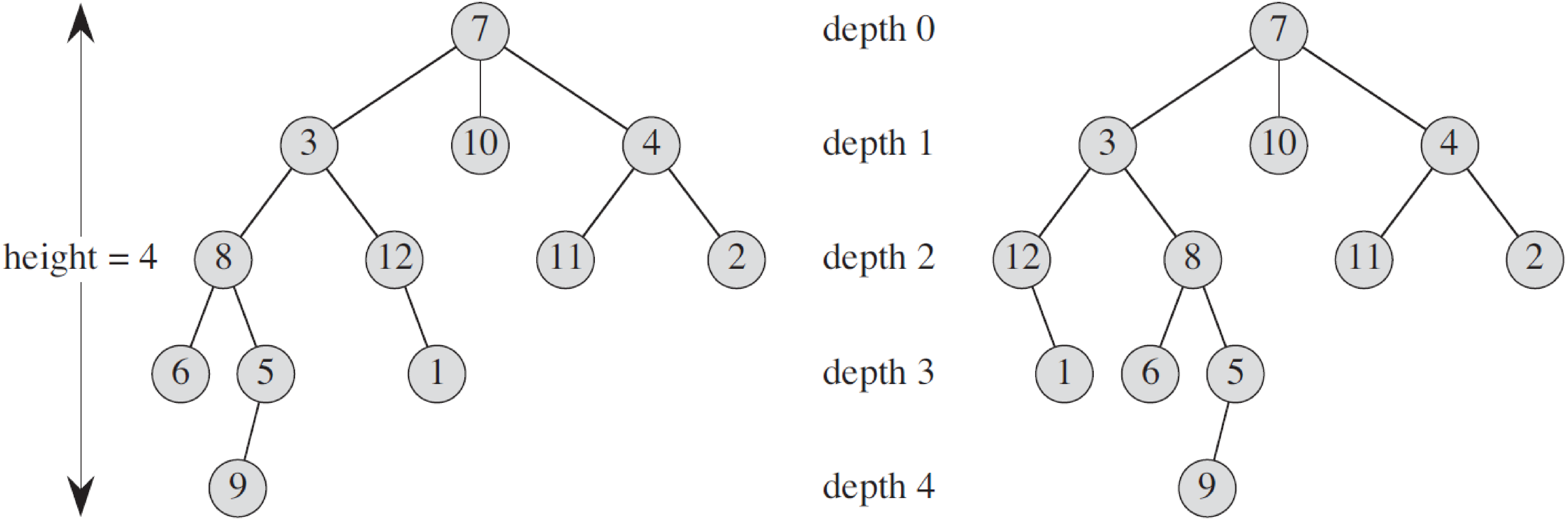
Rooted Trees

- A **rooted tree** is a free tree in which one of the vertices is distinguished from the others; it is called the **root**.
- A vertex in a rooted tree is called a **node**.
- Consider a node x in a rooted tree T with root r.
  - We call any node y on the unique simple path from r to x an **ancestor** of x; x is a **descendant** of y.
  - If the last edge on the simple path from r to x is (y, x), y is the **parent** of x and x is the **child** of y. The root is the only node with no parent.
- If two nodes have the same parents, they are **siblings**.

Rooted Trees

- A node with no children is a **leaf** (**external node**).
- A nonleaf node is an **internal node**.
- The **degree** of x is its number of children.
- The length of the simple path from r to x is the **depth** of x in T.
- A **level** of a tree consists of all the nodes at the same depth.
- The **height** of x is the number of edges on the longest simple downward path from x to a leaf. The height of the tree is the height of the root.
- An **ordered tree** is a rooted tree in which the children of each node are ordered.

Rooted Trees



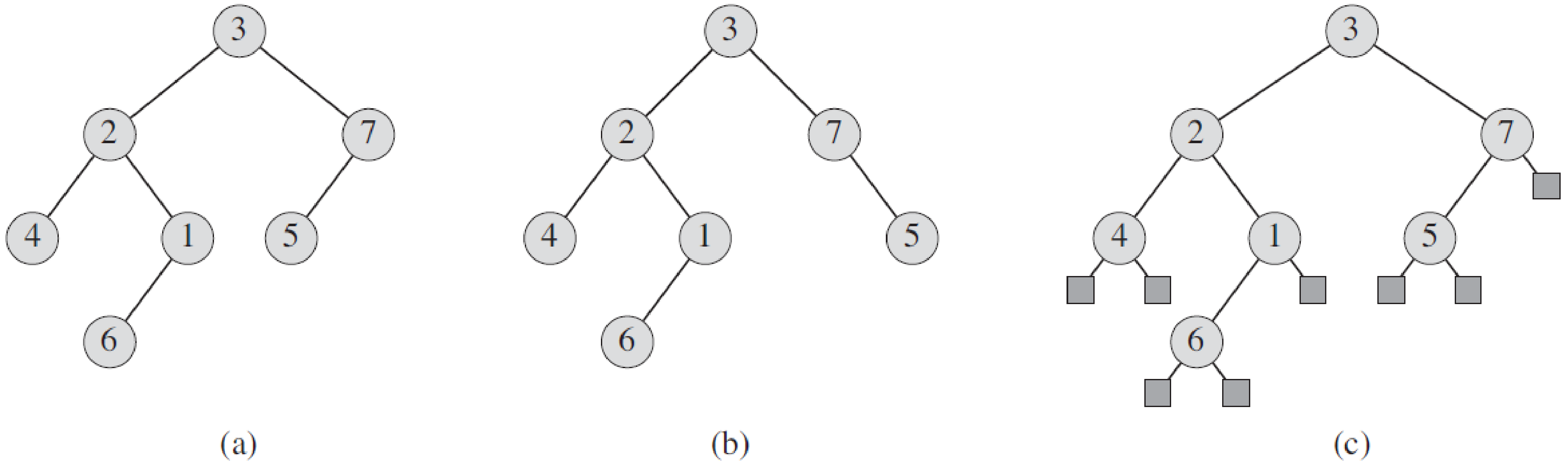
(a)
(b)

Binary Tree

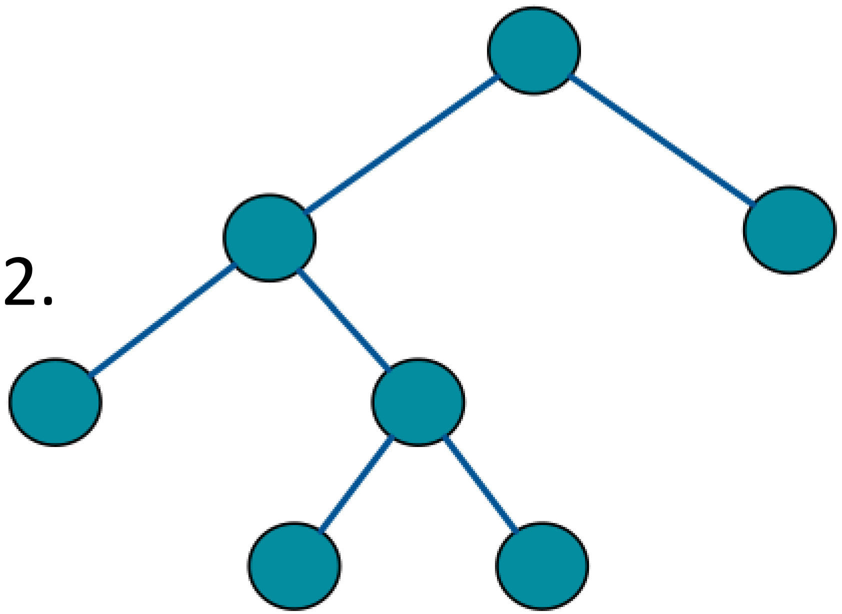
- A **binary tree** T is a structure defined on a finite set of nodes that either:
  - contains no nodes, or
  - is composed of three disjoint sets of nodes: a **root** node, a binary tree called its **left subtree**, and a binary tree called its **right subtree**.
- The binary tree that contains no nodes is called **empty tree** (**null tree**). It's often denoted by **NIL**.
- If the left (right) subtree is not empty, it's called **left** (**right**) **child**.
- If a subtree is the null tree, we say that the child is **absent** or **missing**.

Binary Tree

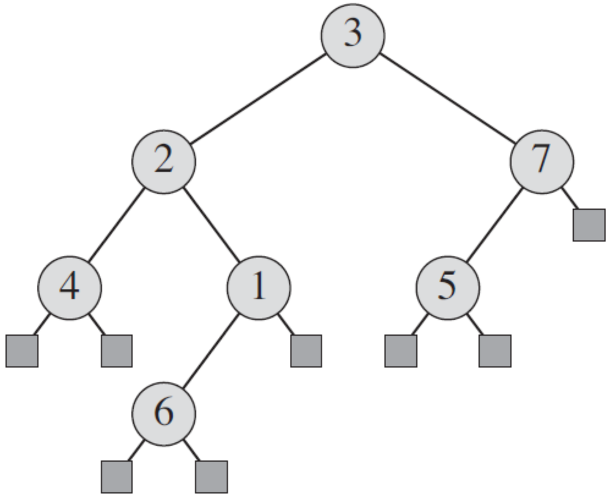
- It’s not simply an ordered tree where each node has degree at most 2: if a node has one child, the position of the child (left or right) matters!
- We can represent the positioning information in a binary tree by the internal nodes of an ordered tree.



Full Binary Tree

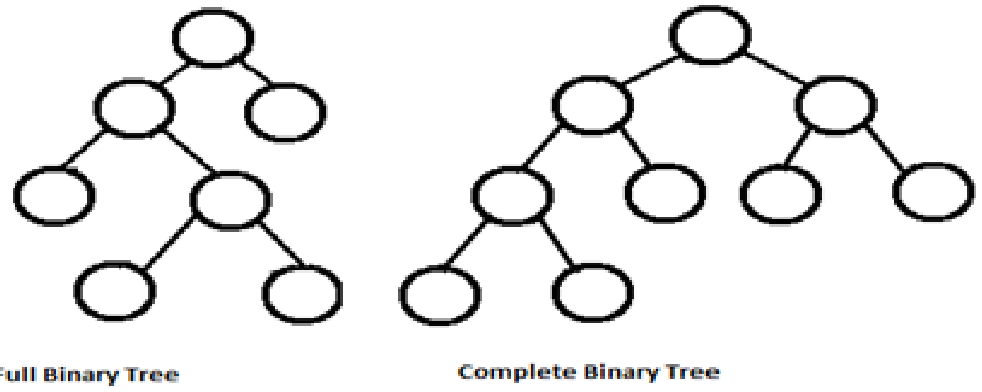


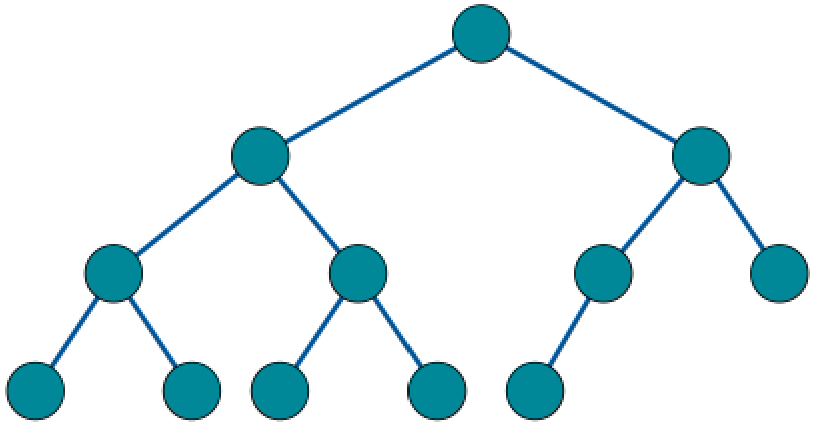
- Each node is either a leaf or has degree exactly 2.
- Because there is no degree-1 nodes, the order of the children of a node preserves positioning information.



Complete Binary Tree

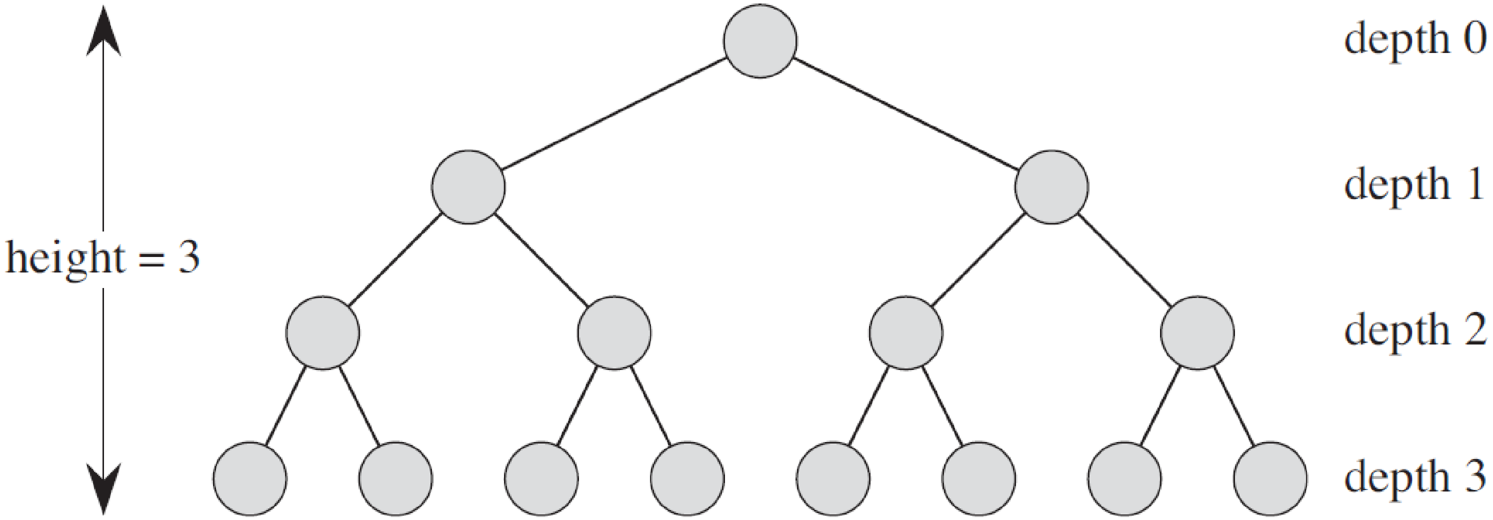
- Every level, except possibly the last, is completely filled and all the nodes in the last level are as far left as possible.





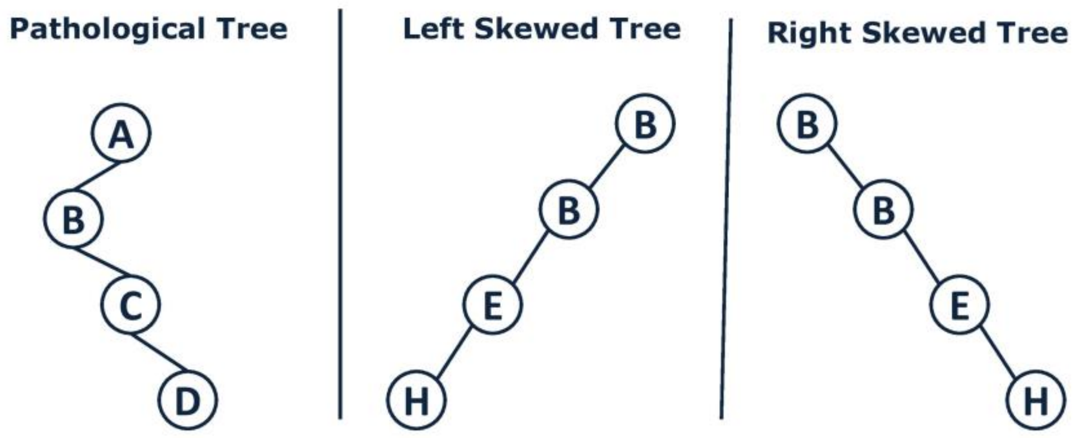
Perfect Binary Tree

- All internal nodes have degree two and all leaves have the same depth or (same level).
- Its number of internal nodes is 2^h-1, where h is its height. Why?



Degenerate (Pathological) Binary Tree

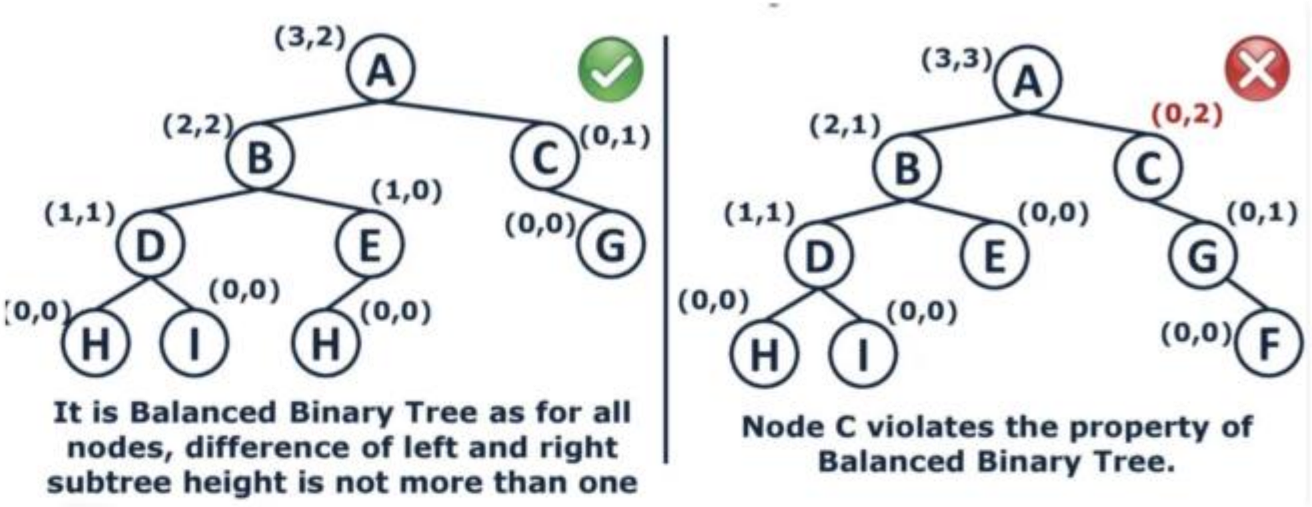
- In a **degenerate tree**, each parent node has only one associated child node. It will behave like a linked list data structure.
- A **skewed binary tree** is a degenerate tree which is solely dominated by left or right child nodes. All skewed binary trees are degenerate but not all degenerate trees are skewed.



https://anilgrgkarma.medium.com/binary-tree-and-its-types-8f2373b40837 Source: TarunChawla Tech

Balanced Binary Tree

- The left and right subtrees of every node differ in height by no more than 1.



Balanced Binary Tree. (x,y): x is left sub tree height and y is right sub tree height.

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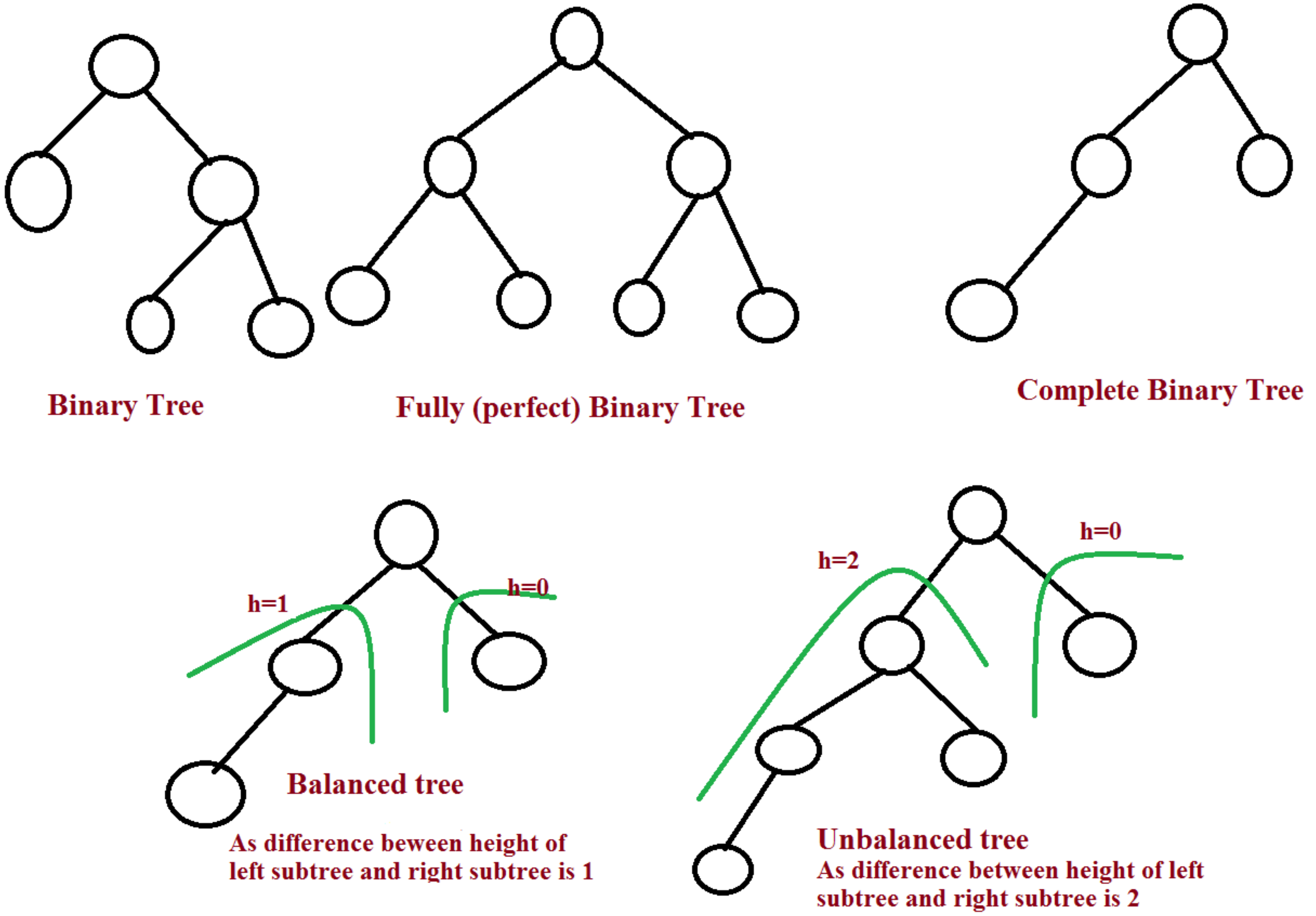
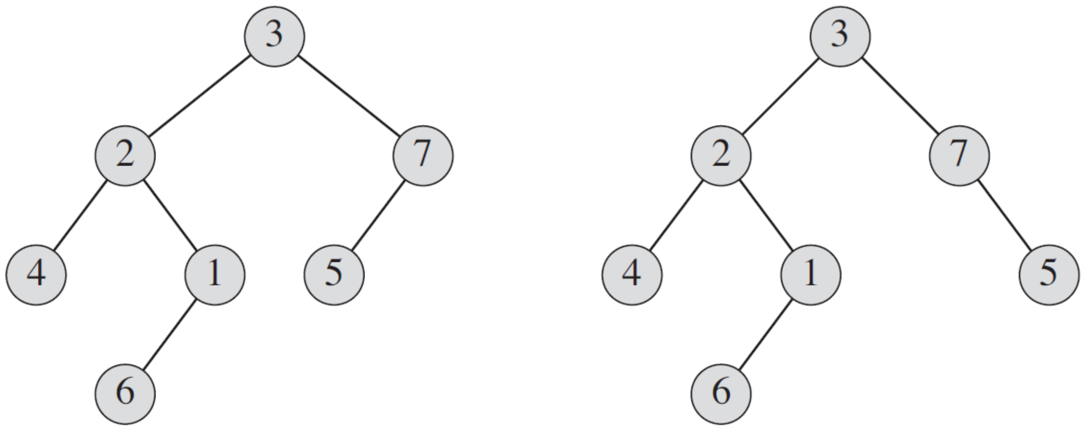
Binary Tree Types



Positional Tree & k-ary Tree

- In a **positional tree**, the children of a node are labeled with distinct positive integers. The i-th child of a node is absent if no child is labeled with integer i.
- A **k-ary tree** is a positional tree in which, for every node, all children with labels greater than k are missing. Example: binary tree.



(a) (b)

Positional Tree & k-ary Tree

- A **perfect k-ary tree** is a k-ary tree in which all leaves have the same depth and all the internal nodes have degree k.
- The number of internal nodes of a perfect k-ary tree is

$$
\begin{aligned}
1 + k + k^2 + \cdots + k^{h-1} &= \sum_{i=0}^{h-1} k^i \\
&= \frac{k^h - 1}{k - 1}
\end{aligned}
$$

- A perfect binary tree has 2^h -1 internal nodes.

Properties of a Binary Tree

1. Every binary tree with n nodes, $n>0$, has exactly $n-1$ edges.

   Proof: Each element (except the root) has one parent. There is exactly one edge between each child and its parent. Hence, there are $n-1$ edges.

2. The number of nodes at level i is $\leq 2^i$, $i \geq 0$.

   Proof (By induction on i)

   Basis: The root is the only node at level $i=0$ and $2^0=1$.

   Inductive hypothesis: The number of nodes at level $i=k-1$ is $\leq 2^{k-1}$.

   We need to prove that at level $i=k$, we have $\leq 2^k$ nodes.

   The number of nodes at level k is $\leq 2 * 2^{k-1} = 2^k$ because each node at level $k-1$ has at most 2 children.

Properties of a Binary Tree

3. A binary tree of height h, h ≥ 0, has at least h+1 and at most 2^{h+1} -1 nodes, i.e. h < n < 2^{h+1}.

Proof: Let n be the number of nodes. There must be at least one node at each level and there are h+1 levels of nodes. This is the case of a degenerate tree. Therefore, n ≥ h+1.

Because of P2, there are at most 2^i nodes at level i. Thus, the maximum number of nodes in the whole tree is:

$$n \leq \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$$

This is the number of nodes in a perfect binary tree of height h.

Properties of a Binary Tree

4. Let h be the height of a binary tree with n nodes, n ≥ 0.
   Then, $\lceil \lg(n+1) \rceil < h < n$.

   Proof: There must be at least one node at each level. Because there are h+1 levels, the height should not exceed n-1, i.e. the number of nodes of a degenerate tree.

   The minimum possible height occurs when all the levels are completely filled, i.e. a perfect binary tree. Because such tree has 2^{h+1} -1 nodes (P3),

$$n \leq 2^{h+1} - 1$$
$$n + 1 \leq 2^{h+1}$$
$$\lceil \lg(n+1) \rceil \leq h + 1$$
$$h \geq \lceil \lg(n+1) \rceil - 1$$
$$h > \lceil \lg(n+1) \rceil$$

Indices in a Complete Binary Tree

Let i be the index assigned to a node u of a complete binary tree. Then, $1 \leq i \leq n$, is

a) If i=1, then u is the root.
b) If 2i > n, then u has no left child. Otherwise, its left child is labeled as 2i.
c) If 2i+1 > n, then u has no right child. Otherwise, its right child has been labeled as 2i+1.
d) If i>1, the parent of i is $\lfloor i/2 \rfloor$.

```
PARENT(i)
1  return ⌊i/2⌋

LEFT(i)
1  return 2i

RIGHT(i)
1  return 2i + 1
```

Properties of a Complete Binary Tree

1. The number of internal nodes of a length-n complete binary tree is $\lfloor n/2 \rfloor$ and its number of leaves is $\lceil n/2 \rceil$.

   Proof: Because of the way indices are assigned, the parent with highest index is $\lfloor n/2 \rfloor$. Thus, the nodes with indices 1, 2, …, $\lfloor n/2 \rfloor$ are the internal nodes. Then, the number of leaves is $n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$.

2. The height h of a length-n complete binary tree is $\lfloor \lg n \rfloor$.

   Proof:

   - A perfect binary tree of height h-1 has 2^h-1 nodes.
   - A perfect binary tree of height h has 2^{h+1}-1 nodes.
   - Then, a complete binary tree of height h has $2^h \le n < 2^{h+1}$ nodes.
   - Consequently, h ≤ lg n < h+1, i.e. $\lfloor \lg n \rfloor = h$.

Properties of a Complete Binary Tree

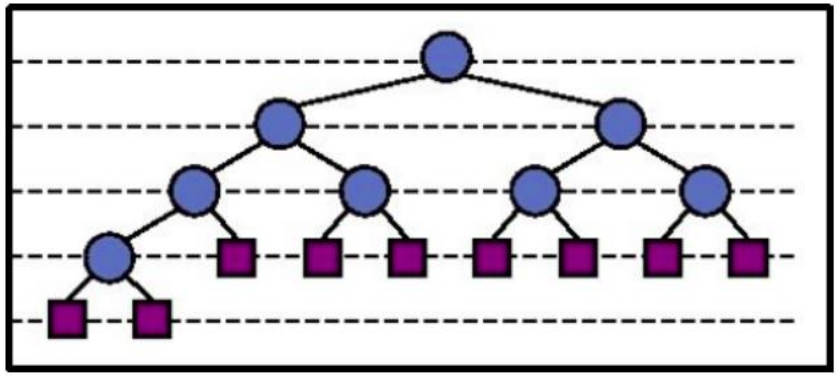
3. There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of height h in a length-n tree.

   Proof (by induction)

   Basis: Because of P1, the number of leaves (h=0) is $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n}{2^{0+1}} \right\rceil$.

   Inductive step: We assume it holds for h-1 and prove it holds for h.

   Even though all nodes of depth h have height 0, the nodes of depth h-1 can have height 0 or 1.



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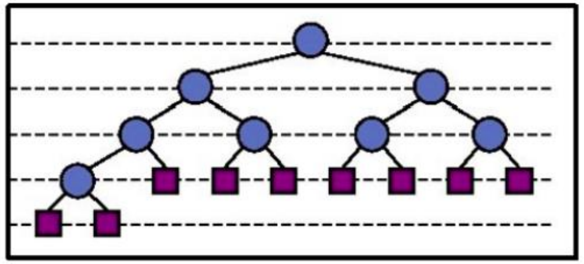
Properties of a Complete Binary Tree

3. There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of height h in a length-n tree.

Proof (by induction)

- Let T be the original tree and let T' be the tree resulting from removing the leaves of T.
- Let n (n') be the number of nodes in T (T').
- Let n_h (n_h') be the number of nodes of height h in T (T').
- Note that $n_h = n_{h-1}'$ and $n' = n - n_0 = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$.



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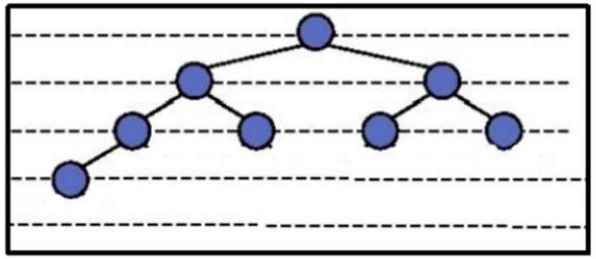

T'

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Properties of a Complete Binary Tree

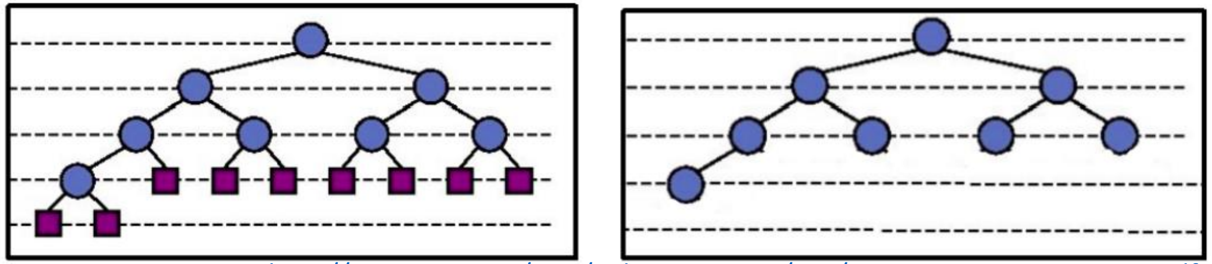
3. There are at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of height h in a length-n tree.

Proof (by induction)

Recall that $n_h = n_{h-1}'$ and $n' = n - n_0 = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$.

Inductive hypothesis: A tree contains at most $\left\lceil \frac{n}{2^{(h-1)+1}} \right\rceil = \left\lceil \frac{n}{2^h} \right\rceil$ nodes of height h-1.

$$n_h = n_{h-1}' = \left\lceil \frac{n'}{2^h} \right\rceil = \left\lceil \frac{\lfloor n/2 \rfloor}{2^h} \right\rceil < \left\lceil \frac{n/2}{2^h} \right\rceil = \left\lceil \frac{n}{2^{h+1}} \right\rceil.$$



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T T'

Properties of a Complete Binary Tree

4. A length-n complete binary tree with its last level half full has at most 2n/3 nodes in its left subtree. It is the greatest unbalance such tree can have.

Proof: Let h be the height of such tree, and n_L (n_R) be the number of nodes in the left (right) subtree. Then,

$$n_L = \sum_{i=0}^{h-1} 2^i = 2^h - 1 = 2.2^{h-1} - 1\,. \qquad n_R = \sum_{i=0}^{h-2} 2^i = 2^{h-1} - 1.$$

$$n = n_L + n_R + 1 = 3.\,2^{h-1} - 1$$

$$\frac{n_L}{n} = \frac{2.2^{h-1} - 1}{3.2^{h-1} - 1} < \frac{2}{3}.$$

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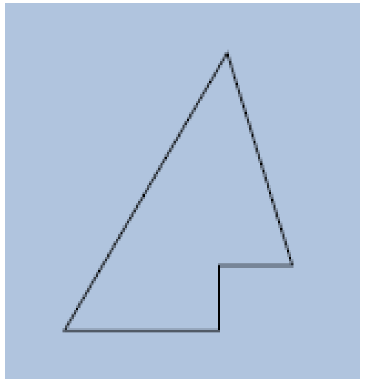
2. HEAPS & HEAPSORT

Max Heaps & Min Heaps

- A **max heap** is a complete binary tree such that for every node i>1, A[Parent(i)] ≥ A[i].
- A **min heap** is a complete binary tree such that for every node i>1, A[Parent(i)] ≤ A[i].
- It's optimal to represent heaps as arrays.



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PARENT(i)

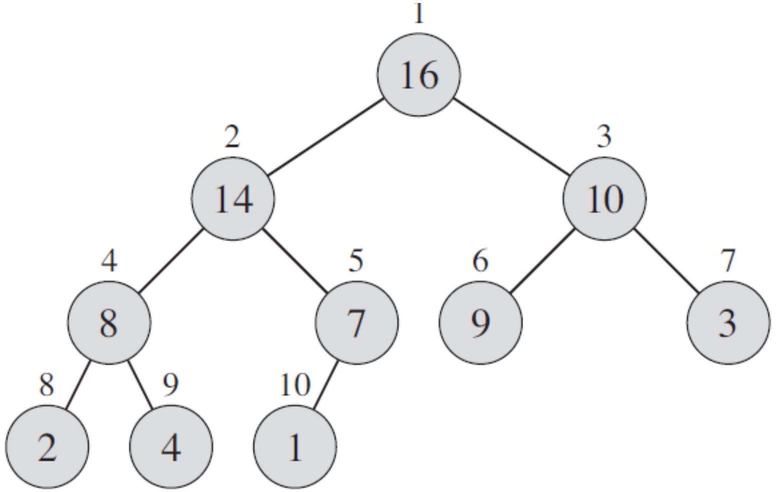
1 **return** $\lfloor i/2 \rfloor$

LEFT(i)

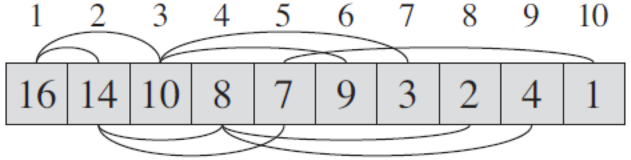
1 **return** $2i$

RIGHT(i)

1 **return** $2i + 1$



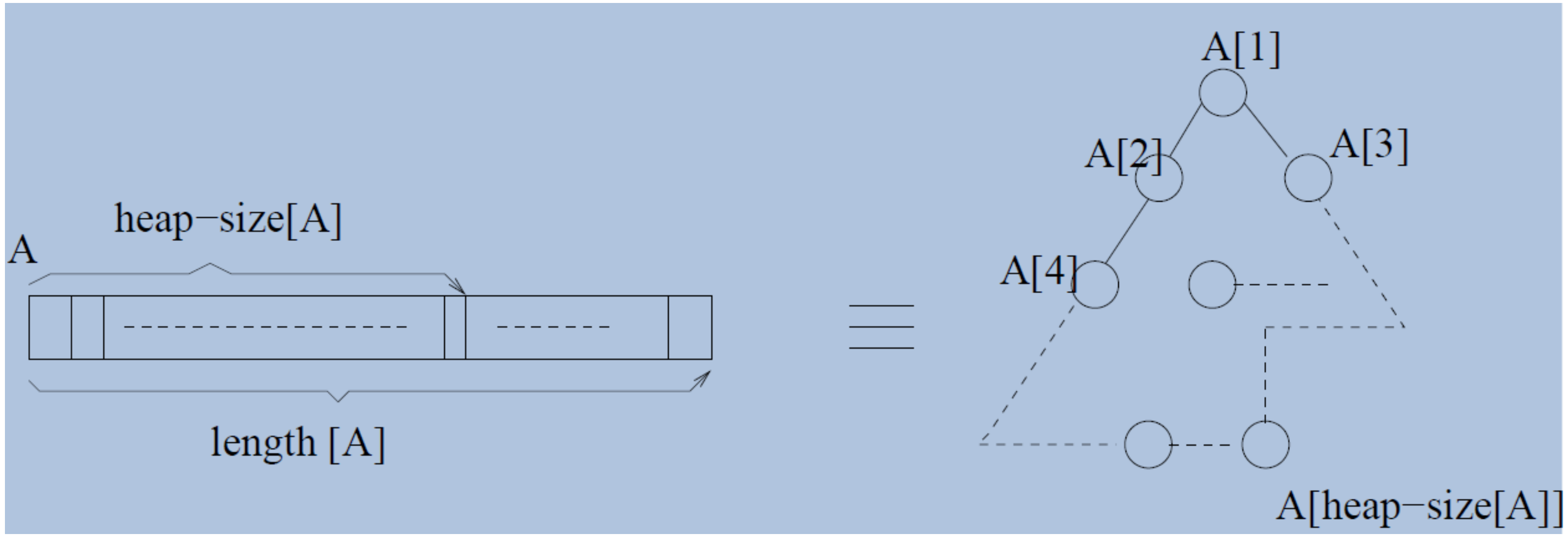
(a)



(b)

Max Heaps Representation

- Array A.
- A.length: size of the array.
- heapsize number of the current elements in the heap.
- Particularly, the heap is stored in A[1..heapsize].

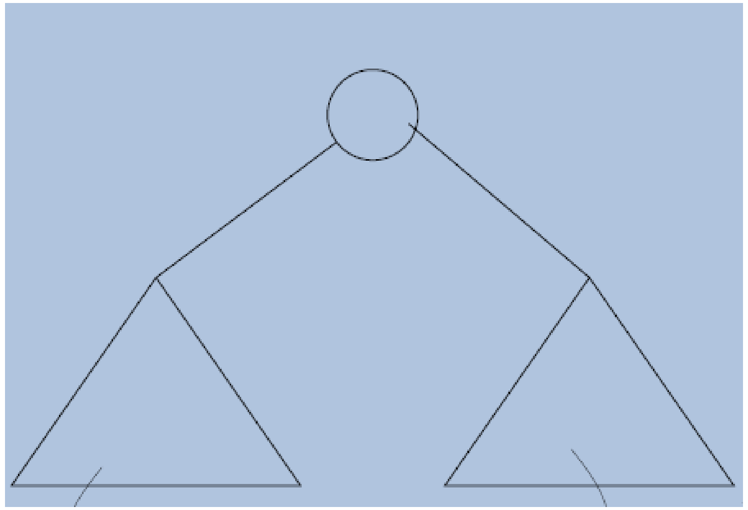


Important Routines for Max-Heaps

- Max-Heapify(A,i): O(lg n).
- Build-Heap(A): O(n).
- Heapsort: O(n lg n).
- Priority Queue Routines (O (lg n)):
  - Max-Heap-Insert()
  - Max-Heap-Extract-Max()
  - Max-Heap-Increase-Key()
  - Max-Heap-Maximum()

Max-Heapify

- Preconditions: The subtrees Left(i) and Right(i) are max heaps.
- Postconditions: The tree with root at i is a maxheap.



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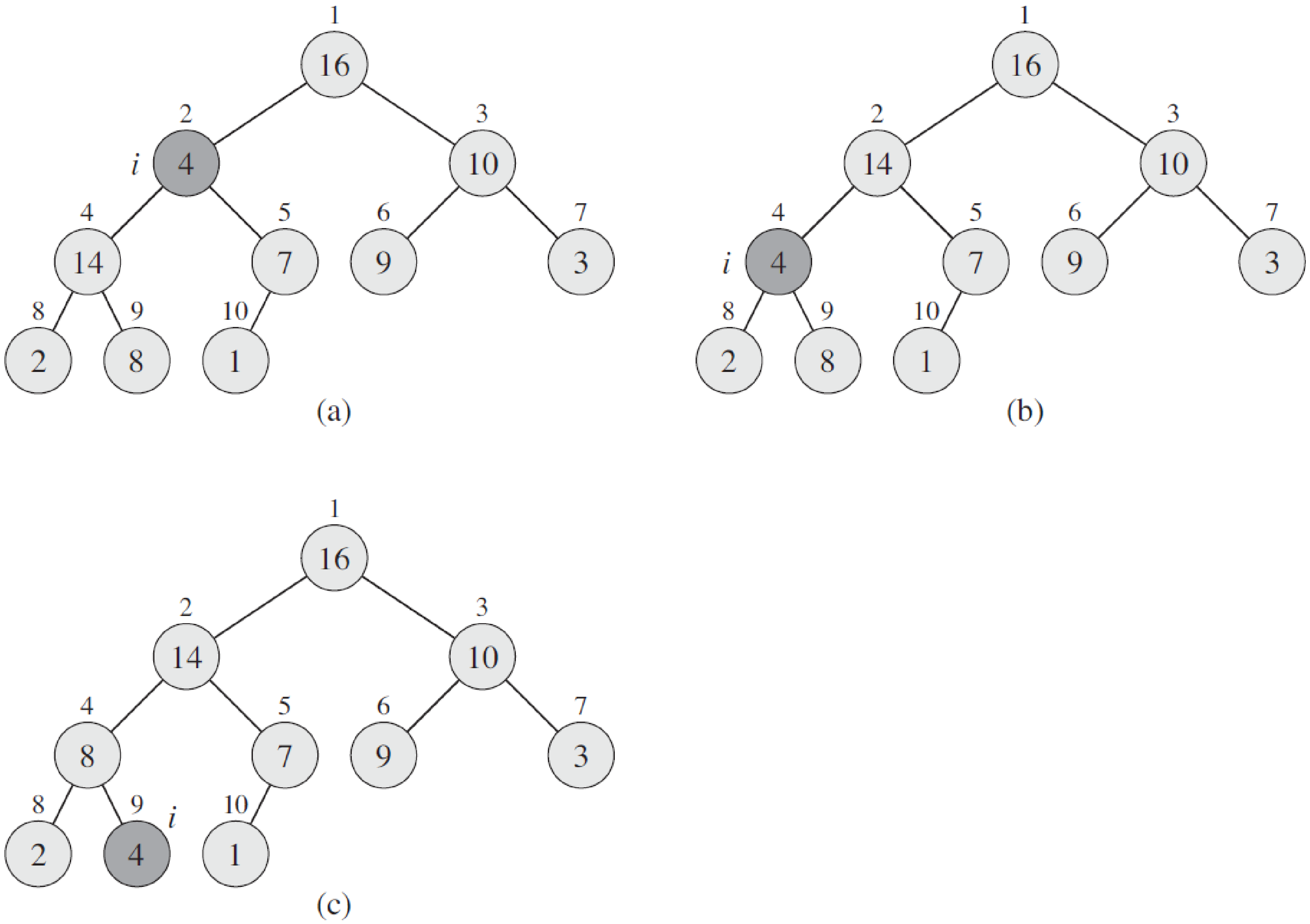
```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    largest = r
if largest ≠ i
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```

Max-Heapify

- Max-Heapify(A,2)

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    largest = r
if largest ≠ i
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```



Max-Heapify

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MAX-HEAPIFY(A, i)
l = LEFT(i)
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else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    largest = r
if largest ≠ i
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
```

- In the worst case, the max-heap is a complete binary tree with its last level half full, i.e. the most unbalanced a complete binary tree can get.
- Because of the corresponding property:

$$T(n) \leq T(2n/3) + \Theta(1)$$

$$T(n) = O(\lg n)$$

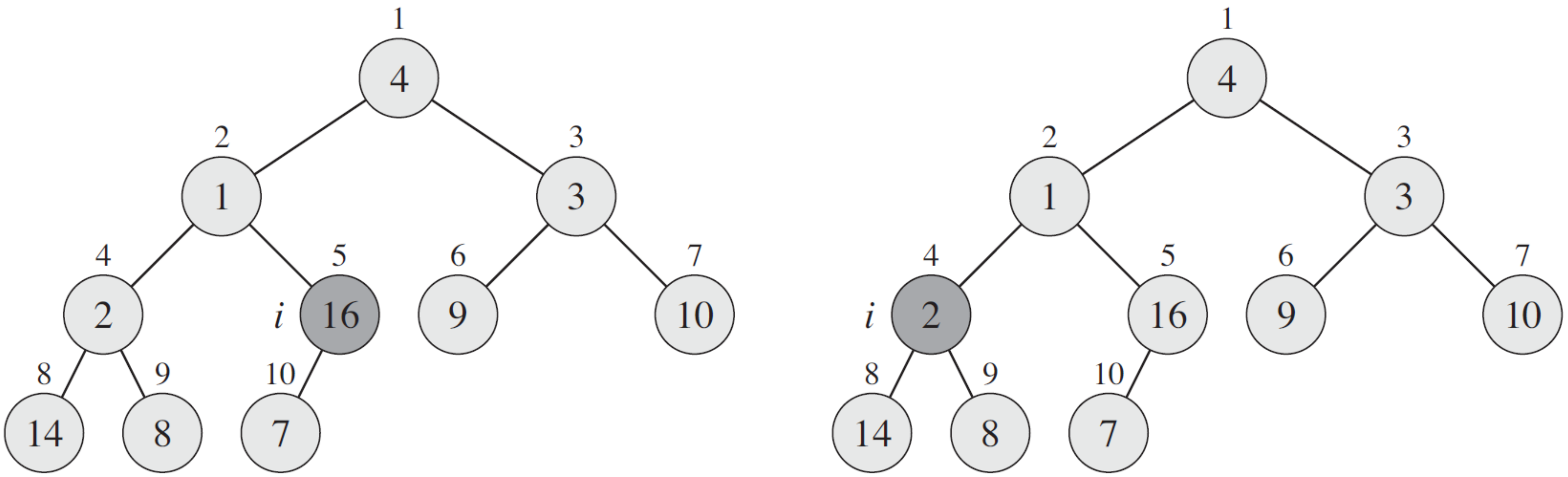
Build-Heap

BUILD-MAX-HEAP(A)

```
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

At the start of each iteration of the **for** loop of lines 2–3, each node $i + 1, i + 2, \ldots, n$ is the root of a max-heap.

A									
4	1	3	2	16	9	10	14	8	7



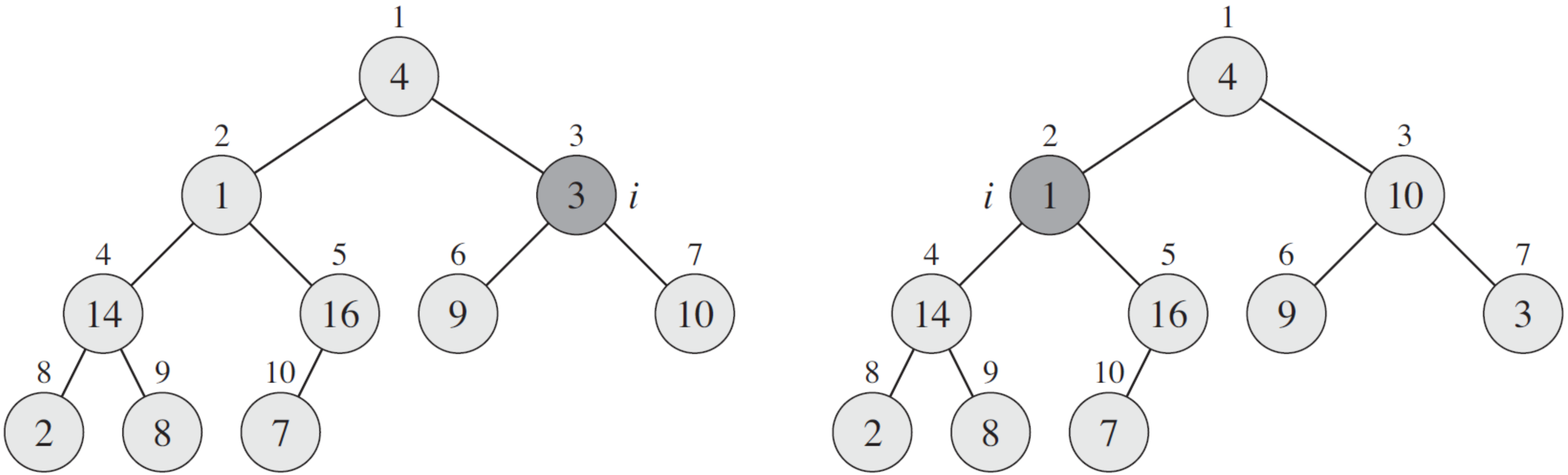
(a) (b)

Build-Heap

BUILD-MAX-HEAP(A)

1 $A.heap\text{-}size = A.length$
2 **for** $i = \lfloor A.length/2 \rfloor$ **downto** 1
3   MAX-HEAPIFY(A, i)

At the start of each iteration of the **for** loop of lines 2–3, each node $i + 1$, $i + 2, \ldots, n$ is the root of a max-heap.



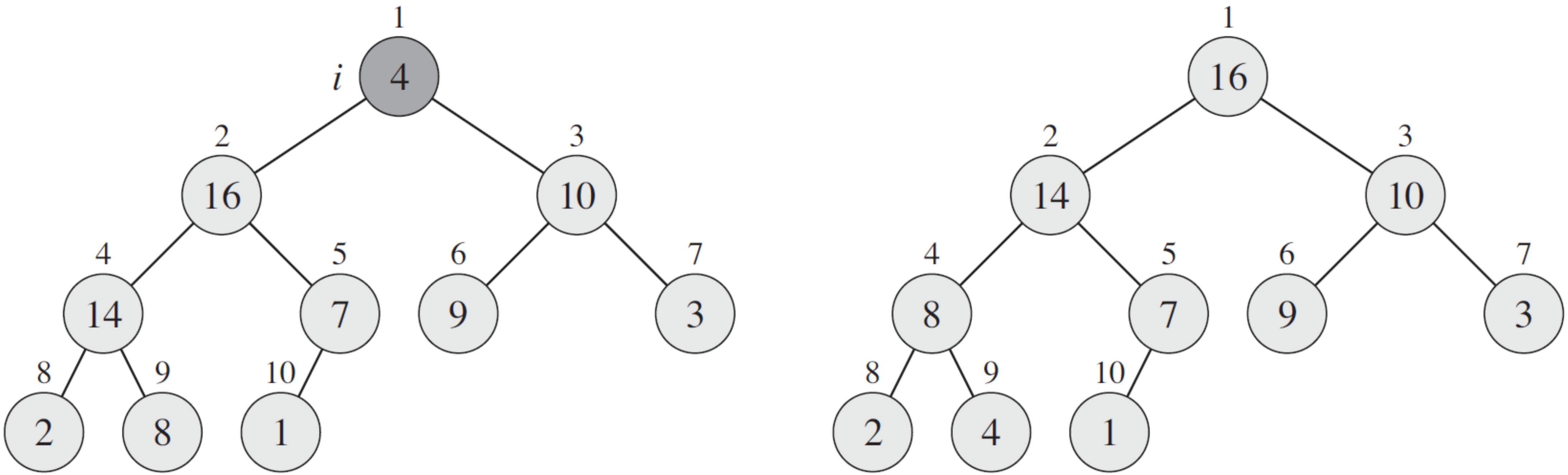
(c) (d)

Build-Heap

BUILD-MAX-HEAP(A)

```
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

At the start of each iteration of the **for** loop of lines 2–3, each node $i + 1, i + 2, \ldots, n$ is the root of a max-heap.



(e) (f)

Build-Heap Complexity

BUILD-MAX-HEAP(A)

```
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

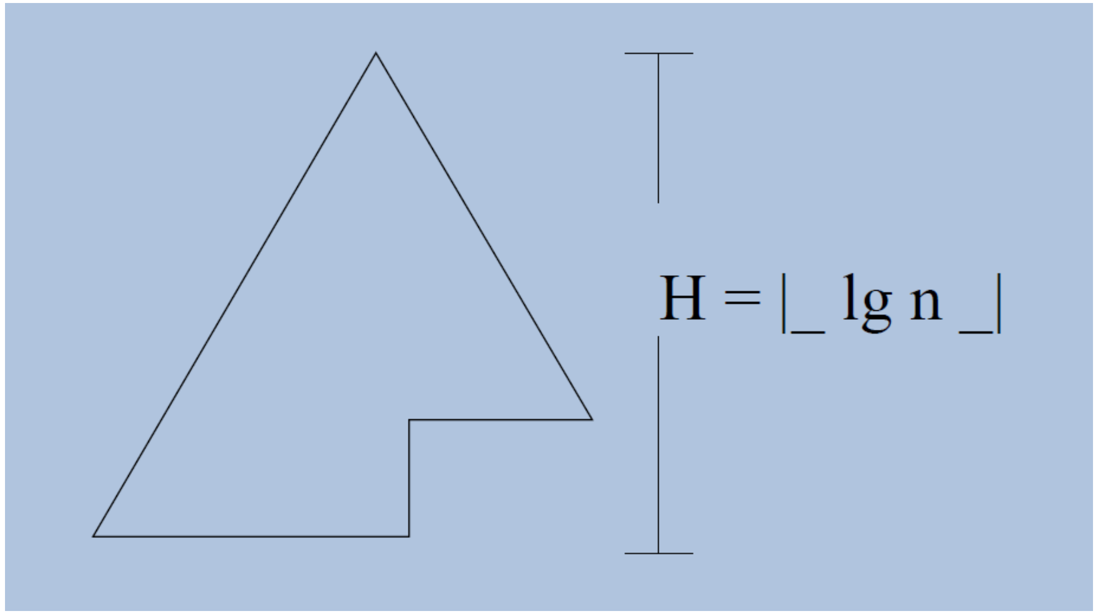
- The complexity of Max-Heapify(A,i) is O(h) where h is the height of node i.
- Because, the largest height, the root's height, is lg n, each call to Max-Heapify is O(lg n).
- There are n/2 calls of this procedure.
- Thus, Build-Heap is O(n lg n).
- However, we can find a tighter upper bound.

Build-Heap Complexity

BUILD-MAX-HEAP(A)

```
A.heap-size = A.length
for i = ⌊A.length/2⌋ downto 1
        MAX-HEAPIFY(A, i)
```

- If the heap has n nodes,



- The number of nodes of a given height h is at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$.

Build-Heap Complexity

BUILD-MAX-HEAP(A)

```
A.heap-size = A.length
for i = ⌊A.length/2⌋ downto 1
      MAX-HEAPIFY(A, i)
```

- So we can sum the complexity of each possible of the height h multiplied by the number of nodes of such height, i.e. $\left\lceil \frac{n}{2^{h+1}} \right\rceil$:

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n)\,.$$

- This is because

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

for $|x| < 1$.

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2\,.$$

Presentation made by Juan Mendivelso. Contents and figures extracted from the book: Introduction to Algorithms, Third Edition. Cormen, Leiserson, Rivests and Stein. The MIT Press. 2009.

Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```

At the start of each iteration of the **for** loop of lines 2–5, the subarray $A[1..i]$ is a max-heap containing the i smallest elements of $A[1..n]$, and the subarray $A[i+1..n]$ contains the $n-i$ largest elements of $A[1..n]$, sorted.

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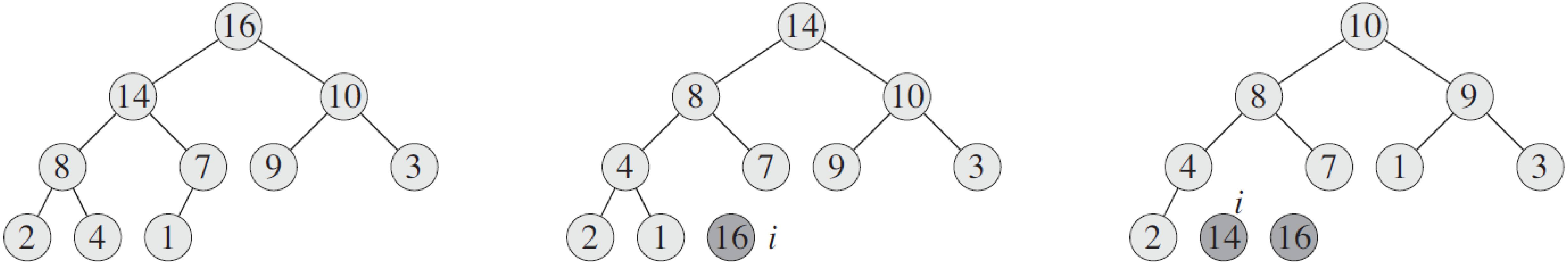

(a) (b) (c)

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Heapsort

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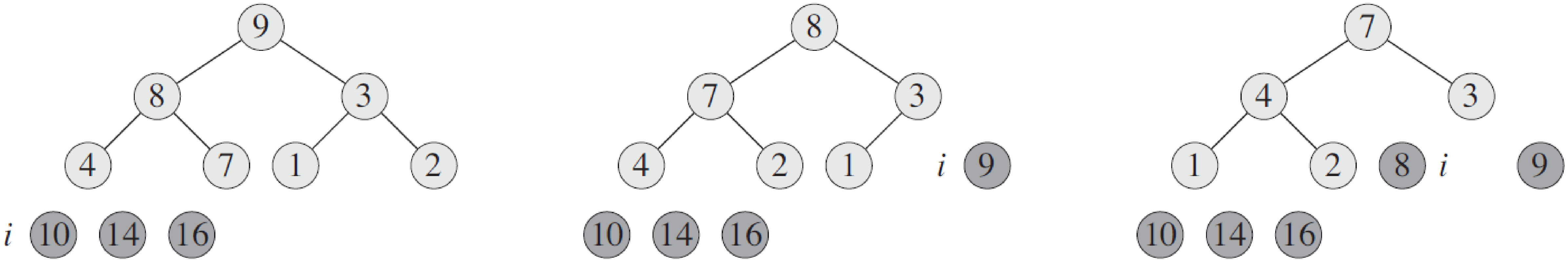

(d) (e) (f)

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Heapsort

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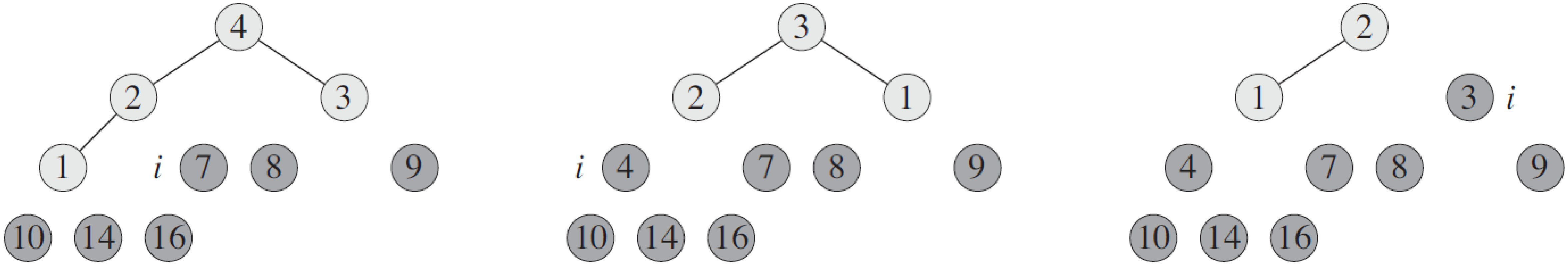
```

At the start of each iteration of the **for** loop of lines 2–5, the subarray $A[1..i]$ is a max-heap containing the i smallest elements of $A[1..n]$, and the subarray $A[i+1..n]$ contains the $n-i$ largest elements of $A[1..n]$, sorted.

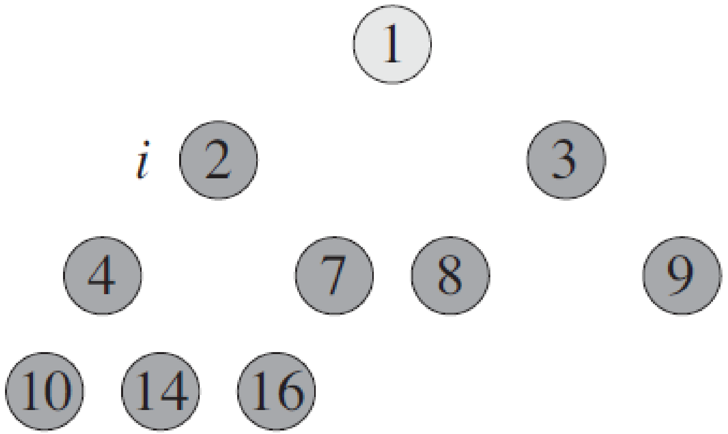


(g) (h) (i)

Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```

At the start of each iteration of the **for** loop of lines 2–5, the subarray $A[1..i]$ is a max-heap containing the i smallest elements of $A[1..n]$, and the subarray $A[i+1..n]$ contains the $n-i$ largest elements of $A[1..n]$, sorted.



(j)

A	1	2	3	4	7	8	9	10	14	16

(k)

Heapsort Complexity

$$\underbrace{\mathcal{O}(n)}_{\text{construir}} + (n-1)\underbrace{\mathcal{O}(\lg n)}_{Ordenar} = \underbrace{\mathcal{O}(n \lg n)}_{\text{HeapSort}}$$

```
HEAPSORT(A)
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2  for i = A.length downto 2
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3. PRIORITY QUEUES

Priority Queues

- Heapsort is excellent but Quicksort is the usual choice because of its practical efficiency.
- However, heaps are very useful to implement priority queues.
- A **priority queue** is a data structure that allows the insertion and deletion of elements regarding priorities assigned to them.
- When an element is inserted, its priority is established. However, it can be updated later.
- The element that is deleted is the one with highest (lowest) priority in a **max-priority queue** (**min-priority queue**). Therefore, max heaps (**min-heaps**) are a good choice for this operation.

Operations for Max-Priority Queues

- Insert(S,x): Insert element x in the priority queue S.
- Maximum(S): It returns the element of S with the largest key.
- Extract-Max(S): It removes and returns the element of S with the largest key.
- Increase-Key(S,x,k): It increases the value of element x's key to the new value k, which is assumed to be at least as large as x's current key value.

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Operations for Min-Priority Queues

- Insert(S,x): Insert element x in the priority queue S.
- Minimum(S): It returns the element of S with the smallest key.
- Extract-Min(S): It removes and returns the element of S with the smallest key.
- Decrease-Key(S,x,k): It increases the value of element x's key to the new value k, which is assumed to be at most as large as x's current key value.

Applications of Priority Queues

- Schedule jobs on a shared computer.
- Event-Driven Simulation.
- Agenda administration.

They allow to store a handle to the corresponding application object in each heap element.

Applications of Priority Queues

HEAP-MAXIMUM(A)

```
return A[1]
```

HEAP-EXTRACT-MAX(A)

```
if A.heap-size < 1
    error "heap underflow"
max = A[1]
A[1] = A[A.heap-size]
A.heap-size = A.heap-size - 1
MAX-HEAPIFY(A, 1)
return max
```

Complexity?

Applications of Priority Queues

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

Applications of Priority Queues

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HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

```
MAX-HEAP-INSERT(A, key)
A.heap-size = A.heap-size + 1
A[A.heap-size] = -∞
HEAP-INCREASE-KEY(A, A.heap-size, key)
```

Applications of Priority Queues

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
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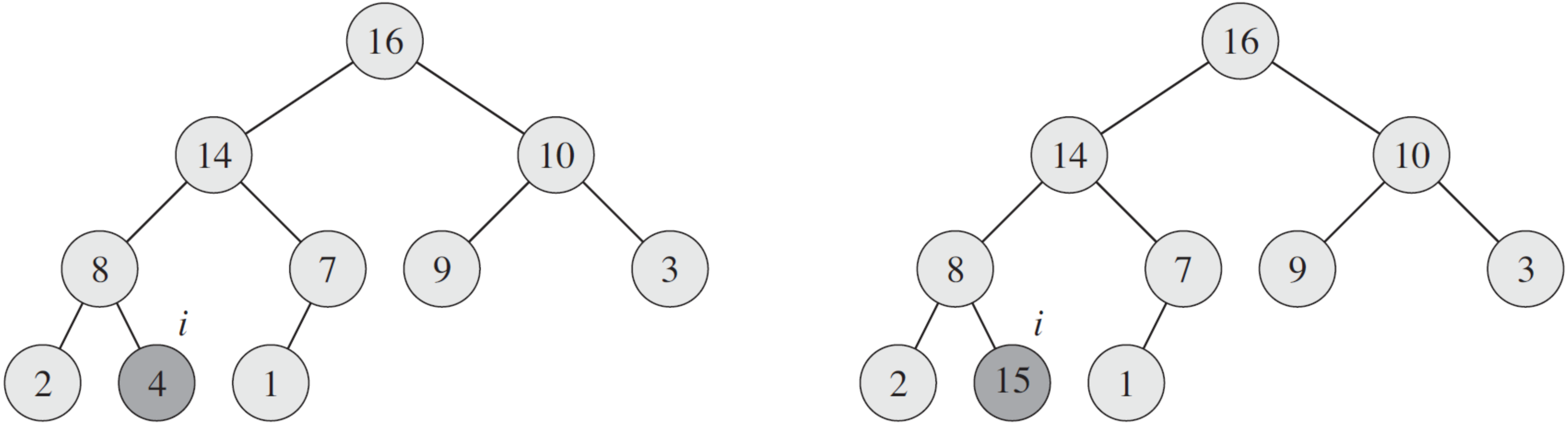

(a) (b)

Applications of Priority Queues

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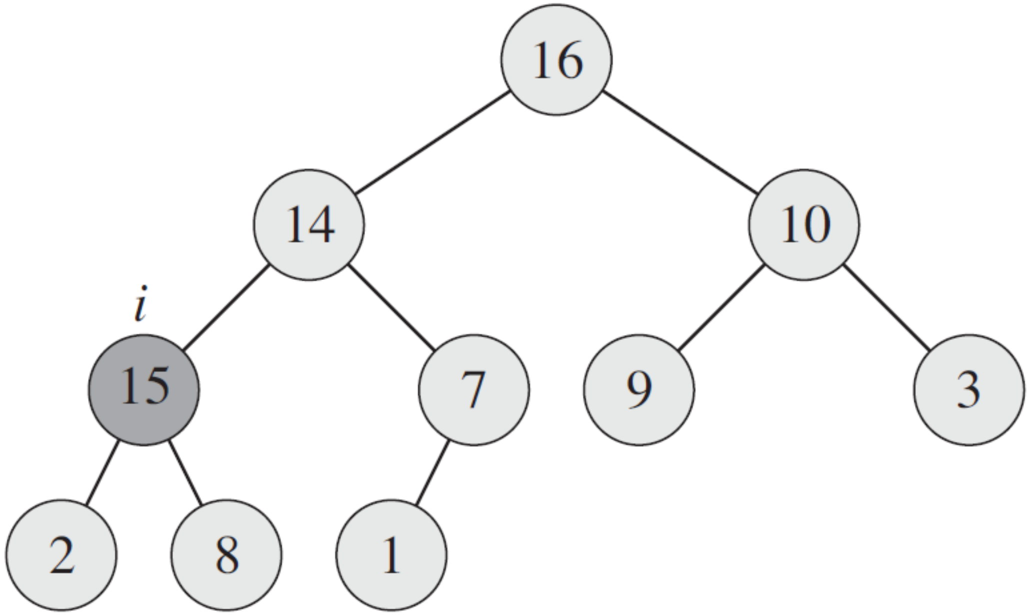
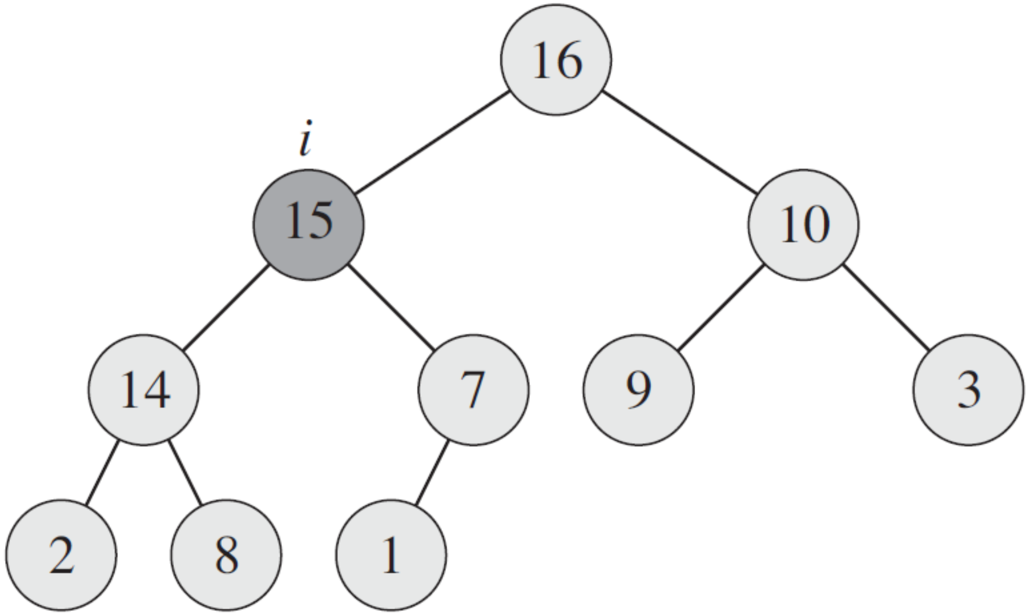


(c)



(d)

BIBLIOGRAPHY

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein. Introduction to Algorithms, Third Edition. The MIT Press. 2009.
- https://codepumpkin.com/binary-tree-types-introduction/#FullBinaryTree
- https://anilgrgkarma.medium.com/binary-tree-and-its-types-8f2373b40837
- http://www.iitg.ac.in/psm/indexing ma252/y12/LectureNoteMA252Jan23.pdf
- https://www2.cs.sfu.ca/CourseCentral/307/petra/2009/SLN 2.pdf
- Images of Julio Cesar Lopez.