On the Non-Identification of Revenue Production Functions

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Abstract

It is well-known that production functions are potentially misspecified when revenue is used as a proxy for output. In this paper, I formalize and strengthen this common knowledge by showing that neither the production function nor Hicks-neutral productivity can be identified when revenue is used as a proxy for physical output. This result holds under the standard assumptions used in the literature for a large class of production functions, including all commonly used parametric forms. Among the prevalent approaches to address this issue, I show that only those which impose assumptions on the underlying demand system can possibly identify the production function.

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1 Introduction

The identification and estimation of production functions are fundamental problems in economics, as they are essential for studying productivity, returns to scale, elasticities of substitution, and markups. Production functions describe how firms produce physical goods from inputs, but the empirical estimation of these functions often relies on sales revenue (output times prices) as a proxy for quantity because revenue is typically observed while quantity is not.

In this paper, I demonstrate that under the standard assumptions in the literature, the production function and Hicks-neutral productivity cannot be identified from such a "revenue production function" when there is imperfect competition. This result applies to a general class of "weakly separable" production functions, including all commonly used parametric forms such as the Cobb-Douglas and CES production functions. Importantly, I show that the Markov assumption, which has traditionally been used to identify production functions, does not provide identification for revenue production functions. The assumptions involved only require cost minimization, not profit maximization, so the result holds for a wide range of market structures. The implications of this result are significant, as it means that none of the objects of interest mentioned above can be identified from a revenue production function.

The issue of unobserved output prices has long been recognized in the production function literature. Early papers such as Abbott (1992); Basu and Fernald (1997) and Klette and Griliches (1996) demonstrated that value-added and revenue production functions contain an additional term related to marginal cost or the markup. More recent papers that use datasets where output prices are observed have empirically confirmed these findings by documenting biases in estimates based on revenue production functions, compared to quantity production functions (De Loecker, Goldberg, Khandelwal, and Pavcnik, 2016; De Ridder, Grassi, and Morzenti, 2022; Mairesse and Jaumandreu, 2005; Ornaghi, 2008).[2] Other papers have used ad-hoc approaches to address this bias without formally analyzing or quantifying it (Collard-Wexler and De Loecker, 2015; Smeets and Warzynski, 2013; Atalay, 2014; Allcott, Collard-Wexler, and O'Connell, 2016).[3] This article aims to advance our understanding of this issue by formally analyzing how the revenue proxy leads to non-identification of not only the production function but also productivity. Based on these derivations, I can make precise statements about which approaches are effective in overcoming this non-identification. Bond, Hashemi, Kaplan, and Zoch (2020) also consider the issue of identification, but they focus on the non-identification of the markup in a monopolistic competition setting. In contrast, this article establishes the non-identification of the entire production function and productivity under general conditions.

As mentioned, a number of papers have attempted to address or relax the issue of unobserved output prices. These approaches can be divided into three main groups: 1) using industry deflators, 2) controlling for the gap between input and output prices using observables (De Loecker, Eeckhout, and Unger, 2020), and 3) imposing assumptions on the underlying demand system to obtain an expression for the output price in terms of observables (De Loecker, 2011; Levinsohn and Melitz, 2002; De Loecker et al., 2016). My results suggest that, in the presence of cross-sectional output price variation (which

[2] De Loecker and Goldberg (2014) provide a summary of this literature and informally discuss the potential issues at play.

[3] See De Loecker (2021) for further references.

excludes perfect competition), only the third approach – imposing assumptions on the underlying demand system – can overcome non-identification. However, part of the appeal of production functions is that the same specification can be estimated across various industries, which may not be desirable with demand systems that ideally need to be tailored to specific industries.

This paper is also relevant for several other areas of literature. First, the literature on production function estimation and identification (Olley and Pakes, 1996; Blundell and Bond, 2000; Ackerberg, Caves, and Frazer, 2015; Levinsohn and Petrin, 2003; Demirer, 2020; Gandhi, Navarro, and Rivers, 2020). While these methods may be appropriate when physical output is observed or when perfect competition is assumed (resulting in homogeneous prices), my results imply that none of these methods can correctly identify the production function when only revenue is observed and there is imperfect competition.[4]

Second, several strands of literature in industrial organization (De Loecker and Syverson, 2021), trade (Keller and Yeaple, 2009; Amiti and Konings, 2007; Bloom, Draca, and Van Reenen, 2016; Brandt, Van Biesebroeck, Wang, and Zhang, 2017), and international economics (Halpern, Koren, and Szeidl, 2015) have aimed to estimate productivity (Syverson, 2011), returns to scale, and more recently, markups (De Loecker and Warzynski, 2012; De Loecker et al., 2020; Demirer, 2020), using production functions. Whenever these studies use revenue to proxy for output to estimate these objects of interest without making further restrictions on demand or competition, my results imply that they were not identified.

Third, several other papers have focused on further developing production function estimation methods when output *is* observed, which raises issues related to multi-product production (De Loecker, 2011; De Loecker et al., 2016; Dhyne, Petrin, Smeets, and Warzynski, 2020; Orr, 2022; De Loecker and Goldberg, 2014).[5] My results emphasize the importance of further developing these methods and the product-level production data they use.

Fourth, a strand of the macroeconomic literature, building on Hall (1990), has also dealt with the issue of unobserved output prices. Since the approaches in this literature impose strong assumptions on the production function, they can typically identify the production function (Basu and Fernald, 1997). However, my results imply that, in the absence of such strong assumptions, the way this literature has tried to address the problem (using the growth rate of value added) does not lead to identification.[6]

Notation. I refer to vectors in bold font, unknown variables in Greek letters, and observed ones in Latin letters. Levels of observed variables are written in capitals, while natural logarithms are written in lowercase.

2 Setting

I consider a general class of non-parametric production functions introduced by Shephard (1953). The production function inputs are separated into dynamic inputs $\mathbf{X}_{it}$ and freely

[4] In particular, Gandhi et al. (2020) establish the non-parametric identification of the production function under a model of price-taking, profit-maximizing firms.

[5] While I focus on the hitherto prevalent single-product case, my results generalize to multi-product settings where production is non-joint.

[6] In that case, one can also not identify the "revenue productivity" term studied in another strand of the macroeconomics literature (Foster, Haltiwanger, and Syverson, 2008; Hsieh and Klenow, 2009).

variable inputs $\mathbf{V}_{it}$. The vectors of associated input prices are denoted as $\mathbf{P}_{it}^X$ and $\mathbf{P}_{it}^V$. Hicks-neutral (total-factor) productivity is denoted as ω_{it}, and the unobserved error term capturing ex-post output shocks and/or measurement error in output is denoted as ε_{it}. The information set of firm i at time t, $\mathcal{F}_{it}$, includes ω_{it} and is assumed to be mean independent of ex-post output shocks. The ex-ante expected output shocks are denoted as $\mathcal{E}_{it} := E[\exp(\varepsilon_{it})|\mathcal{F}_{it}]$. The production technology is denoted as $Q_t(\mathbf{X}_{it}, \mathbf{V}_{it})$, the planned (ex-ante) output as $Q_{it}^* = Q_t(\cdot)\exp(\omega_{it})$, and the observed (ex-post) output as $Q_{it}^* := Q_t(\cdot)\exp(\omega_{it})\exp(\varepsilon_{it})$.

In the literature on production functions there are some fundamental assumptions that are typically used to identify the production function. I will impose these assumptions and show that when only revenue is observed, the production functions is not identifiable. The assumptions are the following,

Assumption 1 (Properties of the Production and Cost Functions)**.** *The production possibilities set satisfies the standard properties in Assumption 6 in Appendix such that the production function exists. Moreover, the production function satisfies,*

(i) If $h(\mathbf{X}_{it}, \mathbf{V}_{it}) > h(\mathbf{X}_{it}', \mathbf{V}_{it}')$ then $F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it})) > F(\mathbf{X}_{it}, h(\mathbf{X}_{it}', \mathbf{V}_{it}'))$ (strict monotonicity in variable inputs).

(ii) The production function $F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it}))$ is continuous and differentiable in variable inputs $\mathbf{V}_{it}$ (duality).

(iii) The cost function $C\left(Q_{it}^*, \mathbf{X}_{it}, \mathbf{P}_{it}^V, \omega_{it}\right)$ is continuous and differentiable in planned output Q_{it}^* (existence of the marginal cost function).

Assumption 1 compiles standard existence and regularity assumptions. (i) slightly strengthens the weak monotonicity necessary for the existence of the production function, but is still weaker than strict monotonicity in all inputs, which is a standard assumption (Chambers, 1988, p.9). The assumption guarantees that the inverse of $F(\cdot)$, given $\mathbf{X}_{it}$, exists. Assumptions 1 (ii) and (iii) are standard differentiability assumptions (Chambers, 1988; McFadden, 1978). The continuity in (ii), combined with the assumptions on the production possibilities set, implies duality between the variable cost function and the production function (Diewert, 2022, Thm. 2). Part (iii) implies that the marginal cost function, $\frac{\partial C(\cdot)}{\partial Q_{it}^*}$ exists.

Then, I assume the following,

Assumption 2 (Weak homothetic separability)**.**

(i) The production function of firm i at time t is of the form

$$Q_{it} = F_t(\mathbf{X}_{it}, h_t(\mathbf{X}_{it}, \mathbf{V}_{it}))\exp(\omega_{it})\exp(\varepsilon_{it}), \tag{2.1}$$

(ii) with $h_t(\mathbf{X}_{it}, \cdot)$ homogeneous of arbitrary degree (homothetic) for all $\mathbf{X}_{it}$.

Assumption 3 (Cost minimization)**.** *The firm minimizes its short-term cost of production with respect to its freely variable non-separable inputs $\mathbf{V}_{it}$, given variable input prices $\mathbf{P}_{it}^V$ and the productivity shocks ω_{it}.*

Assumption 2 was introduced by Shephard (1953) and has recently been used by Demirer (2020) in the context of non-parametric identification of quantity production functions. While it is not a standard assumption, it encompasses a large class of production functions, including all commonly used parametric ones.[7] In particular, Assumption 2 (i) formulates a non-parametric production function and does not restrict the production function in the absence of further restrictions on $h(\cdot)$.[8] Assumption 2 (ii) implies that ratio of the non-separable inputs' marginal product depends only on the inputs through their ratio. This assumption only restricts my model insofar as Assumption 3 requires that the inputs with respect to which the firm minimizes its short-term cost are homothetic in $h(\cdot)$.[9] This does not mean that the entire production function has to satisfy homotheticity. Moreover, if only one input enters the firm's short-term cost, then (ii) can always be satisfied by rewriting the production function appropriately. For example, consider a standard three-input setup, with capital K_{it}, labor L_{it}, and material (or intermediate) inputs M_{it}. If labor L_{it} is subject to some adjustment cost and therefore dynamically chosen, we have $\mathbf{V}_{it} = M_{it}$, and homotheticity of $h(\cdot)$ can always be satisfied by defining $h(\cdot)$ as the identity function. When both labor and material inputs are freely variable, however, the restriction on $h(\cdot)$ becomes meaningful.

Assumption 3 is a commonly used assumption in the literature and allows for most forms of imperfect competition. It is therefore much weaker than the common assumption of perfect competition. Though the latter assumption implies that the unobserved prices are common across firms and can hence be absorbed by industry price deflators, it is very strong and rarely realistic in practice.

The following two assumptions are ubiquitous in the literature and have been instrumental to establishing identification of quantity production functions. Even though they are rather strong and as such should aid identification, I will show that they still fail to identify the production function in the context of revenue production functions.

Assumption 4 (Scalar Unobservable)**.** *Demand for (one of) the firm's inputs v_{it} is given by,*

$$v_{it} = s_t(x_{it}, \omega_{it}).$$

This is simply a general way to write the scalar unobservable assumption used in most of the production function literature. Olley and Pakes (1996) first used this assumption for investment, not input demand. Levinsohn and Melitz (2002) later proposed an alternative version for labor demand, and Ackerberg et al. (2015) for material inputs. The assumption plays a crucial role in the identification arguments of these papers, by allowing one to substitute out for unobserved ω_{it} in the production function, predict target output Q^*_{it}, and form moments on the ex-post shock to productivity by leveraging the following assumption,

Assumption 5 (Markov)**.** *Hicks-neutral productivity follows a first-order Markov process,*

$$\mathsf{P}(\omega_{it} \mid \mathcal{F}_{it-1}) = \mathsf{P}(\omega_{it} \mid \omega_{it-1}). \tag{2.2}$$

[7] See Demirer (2020) for parametric examples.

[8] The only purpose of the exponential functions is to facilitate taking logs later on, and to align with the notation in the literature.

[9] Also, note that these restrictions allow for any additional type of functional dependence between the separable inputs X^S_{it}.

Together with the mean independence of ex-post output shocks, this assumption implies that one can write firms' Hicks-neutral productivity process as $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$, where $E[\xi_{it}|\mathcal{F}_{it-1}] = 0$.

3 Non-Identification of Revenue Production Functions

3.1 Main Result

We are now equipped to prove the main result.[10] First, we prove an auxiliary Lemma, which establishes a variant of the intuitive fact that both component functions $F(\cdot), h(\cdot)$ of the production function need to be identified for the production function to be. Denote a firm's target revenue $R^*_{it} := P_{it} \cdot Q^*_{it}$ where P_{it} indicates firm-level output prices. Also, write $S^{*V}_{it} := \frac{P^V_{it} V_{it}}{R^*_{it}}$ to indicate the revenue share of the input $V_{it} \in \mathbf{V}_{it}$. Then we have,

Lemma 1. *Let $h : W \to Y_1, F : X \times Y_1 \to Z, Q^* : (w, y_2) \in W \times Y_2 \to F(y_2, h(w))$, with $W := (X, V)$ for some $F \in \mathcal{F}, h \in \mathcal{H}, Q^* \in \mathcal{Q}$ where $\mathcal{F}, \mathcal{H}, \mathcal{Q}$ are spaces of composite and composing functions for which Assumptions 1 and 2 are satisfied. Then, for a given $h \in \mathcal{H}$ and a corresponding $q \in \mathcal{Q}, f \in \mathcal{F}$ such that $q = f(x, h(w))$ for all $w \in W, x \in X$; if h is given but f not, then q can in general not be uniquely determined.*

Proof. Suppose not. Then for any $\tilde{h} \in \mathcal{H}, \exists \tilde{f} \in \mathcal{F}$ such that, $\forall Q^* \in \mathcal{Q}, Q^* = \tilde{f}(y_2, \tilde{h}(x)), \forall w \in W, x \in X$. Since $\mathcal{F}$ is defined as the set of component functions which, composed with some function in $\mathcal{H}$, give a function in $\mathcal{Q}$, this means that $\mathcal{F}$ must be a singleton set of functions $\{\tilde{f}\}$. In general, it is clear that the set of functions $\mathcal{F}$ that satisfy Assumption 2 and Proposition 1 is not a singleton.

For example, consider any two functions F_1, F_2, where $Q_i(y_2, w) = F_i(y_2, h(v)) = \alpha_i \cdot y_2 + h(v), i = 1, 2, \alpha_1 \neq \alpha_2, \alpha_1, \alpha_2 > 0$, and where $h(\cdot)$ is differentiable and satisfies homothetic separability. Then these functions satisfy Proposition 1: (i) take $y'_2 > y_2, w$, then we have $Q_i(y'_2, w) - Q_i(y_2, w) = \alpha_i(y'_2 - y_2) > 0$; (ii) $h(v)$ is normalized to be homogeneous of degree one and hence concave and $\alpha_i \cdot y_2$ is concave, hence F_i is concave and thus quasi-concave; (iii) is satisfied by the fact that F_i is the sum of a linear function and a homogeneous function; (iv) is satisfied as for any $v \in V$, we can always find a y_2 such that $Q_i = y$ and the set is closed, and a y'_2 large enough such that the set is non-empty. They also satisfy Assumption 1: (i) by homotheticity of $h(\cdot)$; (ii) by assumption; and (iii) can easily be shown by deriving the cost function from cost minimization. But, for a given $h(v)$, the set of functions F_i of this form is infinitely large. Contradiction. ∎

This Lemma allows us to prove the main result,

Theorem 1 (Non-Identification of the Production Function)**.** *Let Assumptions 1, 2, 3, 4, and 5 hold. Furthermore, assume that production inputs $(\mathbf{X}_{it}, \mathbf{V}_{it})$, input prices $\mathbf{P}^X_{it}, \mathbf{P}^V_{it}$, and revenue R_{it} are observed, but output prices P_{it} and quantities Q_{it} are not. Then the revenue production function can be written in terms of observables as,*

$$R_{it} = G(\mathbf{X}_{it}, \mathbf{V}_{it}, \mathbf{P}^V_{it}, \mathbf{S}^{*\mathbf{V}}_{it})\mathcal{E}^{-1}_{it} \exp(\varepsilon_{it}), \tag{3.1}$$

and identification of $G(\cdot)$ is insufficient for identification of the production function.

[10] I will drop the time subscripts on all functions from here on to ease notation, though they are allowed to be time-varying in general. In practice, one often estimates separate functions for rolling time windows.

Proof. The cost function for the class of production functions satisfying weak homothetic separability can be obtained from the general cost minimization problem, where the firm minimizes short-run costs given its level of planned output Q_{it}^*

$$\begin{aligned} &\min_{\mathbf{V}_{it}} \mathbf{P}_{it}^V \cdot \mathbf{V}_{it} \\ &\text{s.t. } E[F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it})) \exp(\omega_{it}) \exp(\varepsilon_{it}) | \mathcal{F}_{it}] \geqslant Q_{it}^*, \end{aligned} \tag{3.2}$$

Since $\mathcal{F}_{it}$ includes inputs and productivity shocks, we can rewrite the constraint as,

$$F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it})) \exp(\omega_{it}) \geqslant \tilde{Q}_{it}^*, \tag{3.3}$$

with $\tilde{Q}_{it}^* := \frac{Q_{it}^*}{\mathcal{E}_{it}} := \frac{Q_{it}^*}{E[\exp(\varepsilon_{it})|\mathcal{F}_{it}]}$. Then, we can write the general cost function as (Chambers, 1988; Demirer, 2020),

$$\begin{aligned} &C\left(\tilde{Q}_{it}^*, \mathbf{X}_{it}, \mathbf{P}_{it}^V, \omega_{it}\right) \\ &= \min_{\mathbf{V}_{it}} \left\{ \mathbf{P}_{it}^V \cdot \mathbf{V}_{it} : \tilde{Q}_{it}^* \leqslant F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it})) \exp(\omega_{it}) \right\} \\ &= \min_{\mathbf{V}_{it}} \left\{ \mathbf{P}_{it}^V \cdot \mathbf{V}_{it} : F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) \leqslant h(\mathbf{X}_{it}, \mathbf{V}_{it}) \right\} \\ &= \min_{\mathbf{V}_{it}} \left\{ \mathbf{P}_{it}^V \cdot \mathbf{V}_{it} : 1 \leqslant h\left(\mathbf{X}_{it}, \frac{\mathbf{V}_{it}}{F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right)}\right) \right\} \\ &= \min_{\mathbf{V}_{it}} \left\{ F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) (\mathbf{P}_{it}^V \cdot \mathbf{V}_{it}) : 1 \leqslant h(\mathbf{X}_{it}, \mathbf{V}_{it}) \right\} \\ &= F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) \min_{\mathbf{V}_{it}} \left\{ \mathbf{P}_{it}^V \cdot \mathbf{V}_{it} : 1 \leqslant h(\mathbf{X}_{it}, \mathbf{V}_{it}) \right\} \\ &:= F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) C_2\left(\mathbf{X}_{it}, \mathbf{P}_{it}^V\right) \end{aligned} \tag{3.4}$$

where line 2 follows from Assumption 1 (strict monotonicity), line 3 and 4 follow from the assumption that $h(\cdot)$ is homothetic (since we can always redefine $F(\cdot)$ and $h(\cdot)$ to make $h(\cdot)$ homogeneous of degree one), and the last line defines a new function $C_2(\cdot)$.[11] To obtain the marginal cost, take the derivative of $C(\cdot)$ in Eq. (3.4) with respect to target output, which exists by Assumption 1,

$$\begin{aligned} \lambda_{it} &= \frac{\partial C\left(\tilde{Q}_{it}^*, \mathbf{X}_{it}, \mathbf{P}_{it}^V, \omega_{it}\right)}{\partial Q_{it}^*} \\ &= \frac{\partial F^{-1}}{\partial x_2}\left(\mathbf{X}_{it}, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) (\exp(\omega_{it})\mathcal{E}_{it})^{-1} C_2\left(\mathbf{X}_{it}, \mathbf{P}_{it}^V\right) \\ &= \frac{\partial F^{-1}}{\partial x_2}(\mathbf{X}_{it}, F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it}))) (\exp(\omega_{it})\mathcal{E}_{it})^{-1} C_2\left(\mathbf{X}_{it}, \mathbf{P}_{it}^V\right) \\ &= \frac{C_2\left(\mathbf{X}_{it}, \mathbf{P}_{it}^V\right)}{\frac{\partial F}{\partial x_2}(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it})) \exp(\omega_{it})\mathcal{E}_{it}} \end{aligned} \tag{3.5}$$

[11]The subscript is chosen to be consistent with notation in Demirer (2020, Thm. 2.1).

where $\frac{\partial F(\cdot)}{\partial x_i}$ denotes the derivative of $F(\cdot)$ with respect to its ith argument, line 3 imposes that the constraint of the cost minimization problem is binding at the optimum such that $Q^*_{it} = F(\cdot)\exp(\omega_{it})$ and line 4 uses the inverse function theorem for the $\mathbb{R} \to \mathbb{R}$ function obtained when $\mathbf{X}_{it}$ is held fixed, for all values of $\mathbf{X}_{it}$.

Finally, rewrite the output elasticity with respect to any of the (log) variable inputs $v_{it} \in \mathbf{v}_{it}$,

$$\frac{\partial f\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right)}{\partial v_{it}} = \frac{\partial F\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right)}{\partial x_2} \frac{1}{F\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right)} \frac{\partial h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)}{\partial v_{it}}, \tag{3.6}$$

where $f(\cdot)$ indicates the log production function.

Under imperfect competition, price equals markup, denoted as μ_{it}, times marginal cost. Eq. (3.5) gives an expression for marginal cost. The markup can be written as the ratio of variable output elasticity to revenue share: $\mu_{it} = \frac{\partial f(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it}))}{\partial v_{it}} (S^{*V}_{it})^{-1}$ (De Loecker and Warzynski, 2012; Klette, 1999). We can plug this into the RHS of the production function in Eq. (2.1) and multiply the LHS by P_{it} to get,

$$\begin{aligned} R_{it} &= F\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right) \exp(\omega_{it}) \lambda_{it} \frac{\partial f\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right)}{\partial v_{it}} (S^{*V}_{it})^{-1} \exp(\varepsilon_{it}) \\ &= \frac{F\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right) \exp(\omega_{it}) C_2\left(\mathbf{X}_{it}, \mathbf{P}^V_{it}\right) \frac{\partial F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it}))}{\partial x_2} \frac{\partial h(\mathbf{X}_{it}, \mathbf{V}_{it})}{\partial v_{it}}}{\frac{\partial F(\mathbf{X}_{it}, h(\mathbf{X}_{it}, \mathbf{V}_{it}))}{\partial x_2} \exp(\omega_{it}) \mathcal{E}_{it} S^{*V}_{it} F\left(\mathbf{X}_{it}, h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)\right)} \exp(\varepsilon_{it}) \\ &= C_2\left(\mathbf{X}_{it}, \mathbf{P}^V_{it}\right) \frac{\partial h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)}{\partial v_{it}} (S^{*V}_{it} \mathcal{E}_{it})^{-1} \exp(\varepsilon_{it}) \end{aligned} \tag{3.7}$$

which gives M equations for R_{it}, one for each flexible input V.

Thus, we obtain an expression for the revenue production function in terms of observables. To establish that the production function $Q(\cdot)$ cannot be identified from this equation, assume without loss of generality that $\frac{\partial h(\cdot)}{\partial v_{it}}$ and $C_2(\cdot)$ can be identified from the last line of Eq. (3.7).[12] Then $h(\cdot)$ is identified up to a constant. Now, since $h(\cdot)$ is a component of the composite function $Q(\cdot)$, it is intuitive that identification of $h(\cdot)$ alone is not sufficient for identification of $Q(\cdot)$. For the sake of completeness, this was proved in Lemma 1. Moreover, since the cost function was shown to equal $F^{-1}\left(\mathbf{X}_{it}, \frac{\tilde{Q}^*_{it}}{\exp(\omega_{it})}\right) \cdot C_2\left(\mathbf{X}_{it}, \mathbf{P}^V_{it}\right)$ in Eq. (3.4), the proof of Lemma 1 also immediately implies that identification of $C_2(\cdot)$ is not sufficient for identification of the cost function.[13] But by Shephard's Duality Theorem (Diewert, 2022, Thm. 2), the cost function is dual to the production function.[14] As a result, identification of $C_2(\cdot)$ is also not sufficient for identification of $Q(\cdot)$. Furthermore, it is clear that no composition of $C_2(\cdot)$ and $h(\cdot)$ can identify $Q(\cdot)$ either, since obtaining $Q(\cdot)$ from a composition of either function requires that the composing functions equal either $F(\cdot)$ or $F^{-1}(\cdot)$, which by the preceding arguments clearly do not coincide with $C_2(\cdot)$ or $h(\cdot)$, respectively.

Finally, note that combining the M versions of Eq. (3.7) gives,

$$R_{it} = C_2\left(\mathbf{X}_{it}, \mathbf{P}^V_{it}\right) \left(\frac{1}{M} \sum_{V \in \mathbf{V}} \frac{\partial h\left(\mathbf{X}_{it}, \mathbf{V}_{it}\right)}{\partial \log V_{it}} (S^{*V}_{it})^{-1} \right) \mathcal{E}^{-1}_{it} \cdot \exp(\varepsilon_{it})$$

[12]This is without loss of generality as this is the best-case identification of the functions involved.

[13]This follows because it is shown in the Lemma that the class of functions F, $\mathcal{F}$ that satisfy the stated assumptions is not a singleton set, and hence the class of inverse functions F^{-1} is neither.

[14]That is, the cost function is necessary and sufficient for the production function, under the maintained assumption of cost minimization and firms being input price-takers. This can also be seen immediately from Eq. (3.4)

$$:= G(\mathbf{X}_{it}, \mathbf{V}_{it}, \mathbf{P}^V_{it}, \mathbf{S}^{*\mathbf{V}}_{it}) \cdot \mathcal{E}^{-1}_{it} \cdot \exp(\varepsilon_{it}). \tag{3.8}$$

By the preceding arguments about $C_1(\cdot), h(\cdot)$, clearly, identification of $G(\cdot)$ is in general also not sufficient to identify $Q(\cdot)$. Moreover, $G(\cdot)$ takes all observable production-side input variables (prices, inputs, and revenue shares) as its arguments. Hence, there is no alternative way of rewriting $G(\cdot)$ as some function $H(\mathbf{X}_{it}, \mathbf{V}_{it}, \mathbf{P}^V_{it}, \mathbf{S}^{*\mathbf{V}}_{it})$ using production-side model equations such that identification of $H(\cdot)$ would be sufficient for identification of $Q(\cdot)$, as any such $H(\cdot)$ would have to be a composite function of $G(\cdot)$.

Finally, we need to argue that Assumption 5 cannot deliver identification of $Q(\cdot)$ either. This is intuitive, since the Markov assumption was introduced in the literature to handle the unobserved productivity term ω_{it} in the quantity production function – but this term drops out in the revenue production function. Indeed, the way the preceding literature has used this assumption for identification is by rewriting ω_{it} as a function of model parameters and "observables",

$$\omega_{it} = q^*_{it-1} - q(\mathbf{X}_{it-1}, \mathbf{V}_{it-1}) \tag{3.9}$$

where (log) target output $q^*_{it} = q(\mathbf{X}_{it}, \mathbf{V}_{it}) + \omega_{it}$ would be identified (and thus "observed") by substituting out ω_{it} in the (quantity) production function using the scalar unobservable Assumption 4 and projecting out the ex-post output shock,

$$q_{it} = q^*(\mathbf{X}_{it}, \mathbf{V}_{it}) + \varepsilon_{it}. \tag{3.10}$$

Then, identification would obtain from forming moments on the ex-post shock to ω_{it}, ξ_{it} (Ackerberg et al., 2015; Gandhi et al., 2020),

$$E\big[\xi_{it}|\mathcal{F}_{it-1}\big] = E\big[(q^*_{it} - q(\mathbf{X}_{it}, \mathbf{V}_{it})) - g\left(q^*_{it-1} - q(\mathbf{X}_{it-1}, \mathbf{V}_{it-1})\right)|\mathcal{F}_{it-1}\big] = 0 \tag{3.11}$$

Since a firm's output, Q_{it} is, however, not observed, the target output function $Q^*(\cdot)$ cannot be identified through the usual conditional moment equation for Eq. (3.10), $E[\varepsilon_{it}|\mathcal{F}_{it-1}] = 0$. Even if one could identify target revenue

$$R^*_{it} = P_{it} \cdot Q^*_{it} = G(\mathbf{X}_{it}, \mathbf{V}_{it}, \mathbf{P}^V_{it}, \mathbf{S}^{*\mathbf{V}}_{it}) \cdot \mathcal{E}^{-1}_{it}$$

from the revenue production function,[15] one could still not identify Q^*_{it} from this as P_{it} is, of course, unobserved by assumption. Moreover, since we have shown that $Q(\cdot)$ cannot be identified from the revenue production function, and since $Q^*_{it} = Q(\cdot)\exp(\omega_{it})$, we conclude that Q^*_{it} cannot be identified from the revenue production function. Hence, Eq. (3.11) depends on an unobservable, q^*_{it}, and cannot deliver identification of $q(\cdot)$ either. ■

We immediately obtain the following three corollaries.

Corollary 1 (Non-Identification From Markov)**.** *The Markov assumption 5 does not lead to identification of the production function when output quantities are unobserved.*

This corollary is embedded in the main theorem, but is worth emphasizing as the Markov assumption has been the main identifying assumption in the production function literature thus far.

[15] This could, for example, be done by imposing a full independence assumption to make $E[\mathcal{E}_{it}|\mathcal{F}_{it}]$ a constant as in Gandhi et al. (2020, p.2979).

Corollary 2 (Non-Identification From First-Order Conditions)**.** *The cost minimization first-order conditions can be rewritten in terms of observables as,*

$$P_{it}^V = C_2\left(\mathbf{X}_{it}, \mathbf{P}_{it}^V\right) \frac{\partial h(\mathbf{X}_{it}, \mathbf{V}_{it})}{\partial V_{it}} \cdot \mathcal{E}_{it}^{-1} \quad \forall\, V \in \mathbf{V}, \tag{3.12}$$

and hence contain no additional information relative to the revenue production function.

Proof. Impose equality on the constraint in Eq. (3.2) by the strict monotonicity assumption of the production function with respect to $\mathbf{V}_{it}$, take the first derivative, and plug in the expression for λ_{it} from Eq. (3.5). This gives Eq. (3.12), which is equivalent to the revenue production function. ■

Thus, the first-order conditions depend on the same functions as the revenue production function does. As a result, neither can they be used to identify $F(\cdot)$. This is of course, not surprising, as we used these first-order conditions to derive the result in Theorem 1. Yet, it is also worth emphasizing, given that many papers have used first-order conditions for the estimation of production functions (Gandhi et al., 2020, VI.B).

Corollary 3 (Non-Identification of Productivity)**.** *Hicks-neutral productivity cannot be identified from a revenue production function.*

Again, this corollary follows directly from the main theorem, but is worth emphasizing given the fact that production function estimation has been widely used for the estimation of productivity (De Loecker and Syverson, 2021). The intuition behind this corollary is that marginal cost – and, hence, output price – depends inversely and one-to-one on Hicks-neutral productivity, since the latter linearly scales the production function. That is, an increase in Hicks-neutral by construction increases output one-to-one. As a result, it also decreases marginal cost one-to-one, and these two effects cancel out in the revenue production function.

Discussion

Having established the main result, I now discuss two open questions: what does non-identification mean in this context, and how could it be resolved?

There is a simple intuition behind Theorem 1. Under the assumption of variable cost minimization, and the assumption of price-setting implicit in the markup formula (markup equals price times marginal cost), the dynamic inputs $\mathbf{X}_{it}$ will only affect a firm's price through the *level* of the production function, while the variable inputs $\mathbf{V}_{it}$ also affect the price through the gradient of the production function. The idea here is that small changes in variable inputs should affect a firm's price because both variable inputs and price can be adjusted instantaneously under the given assumptions.[16] Changes in the dynamic inputs, however, only affect overall, not marginal production capacity, and hence only affect prices in that way as well. As a result, if we re-express output prices in terms of production-side variables, the part of the production function associated with the dynamic inputs, which is assumed to be separable, exactly cancels out, because the effect of these dynamic inputs on output feeds through one-to-one to output prices.

A remarkable result driving this mechanism is that we can express output prices, which are clearly driven by the demand side as well, in terms of production-side observables

[16] This follows by the assumption that markup equals price over marginal cost, which rules out dynamic pricing, as was also noted in De Loecker and Warzynski (2012, p.2443).

only. This is the insight that motivated the so-called "production approach" to markup estimation, which relies on the fact that the markup can be expressed as the ratio of output elasticity to revenue shares (De Loecker, 2011; Klette, 1999). The intuition behind this fact rests on a simple cost-benefit condition. The output elasticity tells us the percent increase in output – and thus, for fixed prices, revenue – for a percent increase in material inputs. The revenue share, on the other hand, tells us what percent of revenue is spent on input costs for each percent increase in input (since $\frac{\partial s_{it}^{*V}}{\partial m_{it}} = S_{it}^{*V}$). As long as the output elasticity is larger than the revenue share, the firm is making a profit. In fact, the gap between both is exactly equal to the markup.

The literature has proposed several solutions to the lack of data on output prices. First, as is well-known, the use of industry deflators cannot resolve the issue when there is cross-sectional output price variation, so this approach only supports a highly restrictive set of models of imperfect competition (De Loecker, 2021) or perfect competition.[17] Second, the introduction of input prices on the right-hand side of the revenue production function can also not alleviate the issue (De Loecker and Goldberg, 2014; De Loecker et al., 2020; De Loecker, 2021). This follows by the same logic as Corollary 2, as Eq. (3.12) gives an expression for input prices that depends only on the components of the revenue production function. This shows that input prices cannot introduce any additional identifying information into the revenue production function.

This leaves, third, the imposition of assumptions on the underlying demand system.[18] When a parametric demand system is specified, this approach can provide an explicit expression for output prices that does not directly depend on the production function (De Loecker, 2011; Gandhi et al., 2020). In some cases, non-parametric restrictions on the demand system may alternatively require an exact set of additional demand-side variables to be included in the model (De Loecker et al., 2016).[19] In general, insofar as Theorem 1 establishes non-identification under a general non-parametric revenue production function that depends on all production-side observables, it is clear that identification can only be established by introducing additional non-production (demand-side) variables.[20] This requires restrictions on the underlying demand system. As discussed in the introduction, however, the appeal of production function estimation lies precisely in the fact that the same estimator can be applied uniformly across industries. Since demand estimation typically requires industry-specific modeling assumptions, it is not clear that this is a satisfactory solution.

Finally, the growth rate of revenue (Basu and Fernald, 1997) provides no further information about the production function. This follows directly from the fact that the level of revenue does not identify the production function in a single period, so neither will it do so across periods.

The importance of these non-identification results is twofold. First, most parametric production functions satisfy weak homothetic separability, which implies that previous

[17] Indeed, as noted, perfect competition implies homogenous prices, in which case industry-level price deflators equal firm-level prices up to some constant.

[18] Some approaches (Kirov and Traina, 2021; De Loecker et al., 2016) have also tried to control for the markup through observables. Since these make implicit assumptions on the structure of underlying demand, I consider them to fall under this group of solutions.

[19] In particular, this paper leverages the insight of Berry (1994) that in a simple logit demand model, quality is an invertible function of output prices and market shares. If one imposes that model and one has a good proxy for quality, it is sufficient to control for market shares and the proxy to control for P_{it}.

[20] This does, of course, does not guarantee identification – it will need to be established in the context of the particular assumptions imposed.

studies that estimated productivity, returns to scale, or markups using revenue as a proxy for output without imposing further structure are likely unidentified. Second, since these results apply to a general class of production functions and market structures, and demonstrate not just the misspecification but the non-identification of revenue production functions, they underscore the importance of developing suitable estimators and datasets that address these issues (Dhyne et al., 2020). Finally, in the absence of explicit consideration of this non-identification, these results suggest that practitioners should avoid estimating production functions and productivity with revenue data. Below, I further illustrate the results in light of the two most commonly used parametric functional forms.

3.2 What Can Be Identified In Leading Parametric Cases?

Next, I demonstrate Theorem 1 for two leading parametric production functions, the Cobb-Douglas and CES production function. Since this requires a parametrization, I let $\mathbf{X}_{it} = K_{it}, \mathbf{V}_{it} = (L_{it}, M_{it})$, but the non-identification does not depend on these particular choices, as long as there is at least one dynamic input.

3.2.1 Cobb-Douglas

For the Cobb-Douglas production function,

$$Q_{it} = K_{it}^{\beta_K} \, L_{it}^{\beta_L} M_{it}^{\beta_M} \exp(\omega_{it}) \exp(\varepsilon_{it}), \tag{3.13}$$

the (log) revenue production function can be written as,

$$r_{it} = \theta_0 + \frac{\beta_L}{\beta_L + \beta_M}(l_{it} + p_{it}^L) + \frac{\beta_M}{\beta_L + \beta_M}(m_{it} + p_{it}^M) - s_{it}^{*V} - \log \mathcal{E}_{it} + \varepsilon_{it}; \tag{3.14}$$

where $\theta_0 = -\log(\beta_L + \beta_M) + \frac{\beta_M - \beta_L}{\beta_L + \beta_M} (\log \beta_L - \log \beta_M) + \log \beta_V$ for some $V \in (L, M)$.[21] The equation illustrates Theorem 1, as the coefficient on capital, which determines the non-separable part of the production function, is not identified. Moreover, identification of the separable part, $h(L_{it}, M_{it}) = L_{it}^{\frac{\beta_L}{\beta_L + \beta_M}} M_{it}^{\frac{\beta_M}{\beta_L + \beta_M}}$, only allows us to identify the ratio of short-run output elasticities. Corollary 3 is also illustrated since ω_{it} does not show up in these expressions. We obtain the following,

Corollary 4 (Non-Identification of the Revenue Cobb-Douglas)**.** *The revenue Cobb-Douglas production function* (3.14) *only identifies the ratio of variable output elasticities,* β_L, β_M*. Neither the dynamic output elasticity* β_K *nor productivity* ω_{it} *are identified.*

3.2.2 CES Production Function

Write the CES production function as,

$$Q_{it} = ((1 - \beta_L - \beta_M) K_{it}^{\sigma} + \beta_L L_{it}^{\sigma} + \beta_M M_{it}^{\sigma})^{\frac{v}{\sigma}} \exp(\omega_{it}) \exp(\varepsilon_{it}), \tag{3.15}$$

where β_L, β_M are the share parameters, v is the return to scale parameter, and σ is the elasticity of substitution parameter. The (log) revenue equivalent is,

$$r_{it} = \log \beta_V + \sigma v_{it} + \frac{1 - \sigma}{\sigma} \log \left(\beta_L L_{it}^{\sigma} + \beta_M M_{it}^{\sigma}\right) + \frac{\sigma - 1}{\sigma} \log B - s_{it}^{*V} - \log \mathcal{E}_{it} + \varepsilon_{it}, \tag{3.16}$$

[21] See Appendix C for the full derivations.

where the unit cost function $B := (P_{it}^L)^{\frac{\sigma}{\sigma-1}} \beta_L^{-\frac{1}{\sigma-1}} + (P_{it}^M)^{\frac{\sigma}{\sigma-1}} \beta_M^{-\frac{1}{\sigma-1}}$. Again, following Theorem 1, we can see that the non-separable part of the production function is not identified, as the returns to scale v do not show up in this expression. Similarly, ω_{it} again drops out, in line with Corollary 3. The below proposition establishes what can be identified from a (potentially nested) revenue CES production function,

Corollary 5 (Non-Identification of the Revenue CES)**.** *The revenue CES production function (Eq. (3.16)) can at most identify the elasticity of substitution σ and the ratio of short-run distribution parameters $\frac{\beta_L}{\beta_M}$. Neither the returns to scale v, the output elasticities, nor Hicks-neutral productivity ω_{it} are identified.*
Proof. *See Appendix A.*

4 Conclusion

This paper has shown that neither the production function nor Hicks-neutral productivity can be identified when using revenue as a proxy for output. This holds true for a general class of production functions and under standard assumptions. There is no way to rewrite the revenue production function in terms of observable production-side variables to break this non-identification. The only approach that can obtain identification in this scenario is to impose restrictions on the underlying demand system, which could introduce demand-side observables into the revenue production function. The non-identification was demonstrated parametrically for the Cobb-Douglas and CES production functions. This work formalizes and generalizes previous findings that revenue production functions are misspecified and generate biased estimates.

The implications for practitioners are significant. In the absence of observed output prices, perfect competition, or further restrictions on demand, practitioners should avoid estimating production functions using revenue as a proxy for output. Most objects of interest such as productivity, returns to scale, and markups cannot be identified. However, markups may still be estimated without production functions using the production approach to markup estimation, subject to additional restrictions (De Loecker, 2021).

Future research may focus on establishing partial identification of the production function when output prices are unobserved (Flynn, Gandhi, and Traina, 2019), or on further developing production function theory for when they *are* observed (Dhyne et al., 2020; De Loecker et al., 2016).

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A Proofs

A.1 Proof of Corollary 5

Slightly rewriting Eq. (3.16) with e.g. $V = M$, we get,

$$r_{it} = \sigma m_{it} + \frac{1-\sigma}{\sigma}\log\left(\frac{\beta_L}{\beta_M}\left(\frac{L_{it}}{M_{it}}\right)^{\sigma} + M_{it}^{\sigma}\right) + \frac{\sigma-1}{\sigma}\log B - s_{it}^{*M} - \log\mathcal{E}_{it} + \varepsilon_{it}, \quad \text{(A.1)}$$

where $\bar{B} = (P_{it}^{L})^{\frac{\sigma}{\sigma-1}}\left(\frac{\beta_L}{\beta_M}\right)^{\frac{1}{\sigma-1}} + (P_{it}^{M})^{\frac{\sigma}{\sigma-1}}$. It is immediate that one can *at most* identify σ and the ratio $\frac{\beta_L}{\beta_M}$ from these revenue production functions. Non-identification of v and ω_{it} is immediate. Non-identification of the output elasticities follows from the fact that these depend on $(1-\beta_L-\beta_M)$ as well (see Appendix C), which cannot be identified. ■

B Properties of the Production Function

For ease of reference, I restate the standard assumptions that the production function $F(\cdot)$ and the production possibilities set need to satisfy. Denote x the n-dimensional vector of inputs and y the scalar output. When comparing two vectors x, y, let $x \geq y$ indicate that all elements of x are at least as great as the corresponding elements of y, and at least one element of x is strictly greater than the corresponding elements of y. The production possibilities set T is,

$$T := \{(x,y) : F(x) \geq y, x \geq 0\}, \quad \text{(B.1)}$$

and it is assumed to satisfy (Chambers, 1988, p.252),

Assumption 6 (Properties of T)**.**

(i) *T is nonempty*

(ii) *T is a closed set*

(iii) *T is a convex set*

(iv) *if $(x,y) \in T, x^1 \geq x$, then $(x^1, y) \in T$ (free disposability of x)*

(v) *if $(x,y) \in T, y^1 \leq y$ then $(x, y^1) \in T$ (free disposability of y)*

(vi) *for every finite x, T is bounded from above*

(vii) *$(x, 0) \in T$, but if $y \geq 0, (0_n, y) \notin T$ (weak essentiality),*

where 0_n is the n-dimensional zero vector.

Based on these assumptions, the production function that is the solution to,

$$F(x) = \max\{y : (x,y) \in T\}, \quad \text{(B.2)}$$

exists and has the following properties (Shephard, 1970; Chambers, 1988),

Proposition 1 (Properties of F)**.**

(i) If $x' \geq x$ then $F(x') \geq F(x)$ (monotonicity)

(ii) $V(y) := \{x : F(x) \geq y\}$ is a convex set (quasi-concavity)

(iii) $F(0_n) = 0$ (weak essentiality)

(iv) The set $V(y)$ is closed and non-empty for all $y > 0$.

Moreover, Proposition 1 (iv) implies that the cost function exists (McFadden, 1978). If, in addition, we assume that that $F(x)$ is finite, non-negative and real-valued for all non-negative and finite x, then the cost function possesses standard properties (Chambers, 1988, p.52).

C Derivation of Parametric Revenue Production Functions

C.1 Cobb-Douglas Revenue Production Function

The Cobb-Douglas cost function with K_{it} as dynamic input and L_{it}, M_{it} as variable inputs is,

$$C_{\text{CD}}\left(K_{it}, \mathbf{P}_{it}^V, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) = \frac{1}{\nu}\left(\frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right)^{\frac{1-\nu}{\nu}} K_{it}^{-\frac{\beta_K}{\nu}} (P_{it}^M)^{\frac{\beta_M}{\nu}} B, \tag{C.1}$$

where the unit cost function is $B = (P_{it}^L)^{\frac{\beta_L}{\nu}} \cdot (P_{it}^M)^{\frac{\beta_M}{\nu}} \cdot \left(\left(\frac{\beta_M}{\beta_L}\right)^{\frac{\beta_L}{\nu}} + \left(\frac{\beta_L}{\beta_M}\right)^{\frac{\beta_M}{\nu}}\right)$. Hence, the marginal cost is,

$$\lambda_{it}^{\text{CD}} = \frac{1}{\beta_L + \beta_M} F(K_{it}, h(K_{it}, L_{it}, M_{it}))^{\frac{1}{\beta_L + \beta_M} - 1} K_{it}^{\frac{\beta_K}{\nu}} c(\mathbf{P}_{it}^V) \mathcal{E}_{it}^{-1}. \tag{C.2}$$

The output elasticities with respect to the variable inputs are β_L and β_M. Plugging these into the RHS of the production function, we get Eq. (3.14).

C.2 CES Revenue Production Function

Similarly, the CES cost function is,

$$C_{\text{CES}}\left(K_{it}, \mathbf{P}_{it}^V, \frac{\tilde{Q}_{it}^*}{\exp(\omega_{it})}\right) = \left(\left(\frac{\tilde{Q}^*}{\exp(\omega_{it})}\right)^{\frac{\sigma}{v}} - \beta_K K_{it}^{\sigma}\right)^{\frac{1}{\sigma}} B^{\frac{\sigma-1}{\sigma}}, \tag{C.3}$$

where $B := (\tilde{P}_{it}^L)^{\frac{\sigma}{\sigma-1}} \beta_L^{-\frac{1}{\sigma-1}} + (P_{it}^M)^{\frac{\sigma}{\sigma-1}} \beta_M^{-\frac{1}{\sigma-1}}$. Hence, the marginal cost is,

$$\begin{aligned}
\lambda_{it}^{\text{CES}} &= \frac{1}{\sigma}\left(\left(\frac{Q_{it}^*}{\exp(\omega_{it})}\right)^{\frac{\sigma}{v}} - \beta_K K_{it}^{\sigma}\right)^{\frac{1}{\sigma}-1} \exp(\omega_{it})^{-1} B^{\frac{-1}{\sigma}} \left((P_{it}^L)^{\frac{\sigma}{\sigma-1}} \beta_L^{\frac{-1}{\sigma-1}} + (P_{it}^M)^{\frac{\sigma}{\sigma-1}} \beta_M^{\frac{-1}{\sigma-1}}\right) \mathcal{E}_{it}^{-1} \\
&= \frac{1}{v}\left(\beta_L L_{it}^{\sigma} + \beta_M M_{it}^{\sigma}\right)^{\frac{1}{\sigma}-1} F(K_{it}, L_{it}, M_{it})^{\frac{\sigma}{v}-1} B^{\frac{\sigma-1}{\sigma}} \exp(\omega_{it})^{-1} \mathcal{E}_{it}^{-1},
\end{aligned} \tag{C.4}$$

The output elasticity with respect to variable input $V_{it} \in \mathbf{V}_{it}$ is,

$$\frac{\partial f(\cdot)}{\partial v_{it}} = vF(K_{it}, L_{it}, M_{it})^{\frac{-\sigma}{v}} \beta_V V_{it}^{\sigma}. \tag{C.5}$$

Combining these and rewriting gives Eq. (3.16).