Math 217 List of Definitions

Vector Space[1]**.** A vector space is a set V, equipped with a rule for *addition* of any two vectors and for *scalar multiplication* of a vector by a scalar. The addition + must satisfy the following axioms.

(1) The set V is closed under addition: For any two vectors $\vec{v}$ and $\vec{w}$ of V, the sum $\vec{v}+\vec{w}$ is also in V.
(2) Addition is associative: For all $\vec{v}, \vec{w}, \vec{y} \in V$, $(\vec{v}+\vec{w})+\vec{y} = \vec{v}+(\vec{w}+\vec{y})$.
(3) Addition is commutative: For all $\vec{v}, \vec{w} \in V$, $\vec{v}+\vec{w} = \vec{w}+\vec{v}$.
(4) There is an additive identity: that is, there exists $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{0}+\vec{v} = \vec{v}$ for all $\vec{v} \in V$.
(5) Every element has an additive inverse: for every $\vec{v} \in V$, there exists a vector $\vec{y} \in V$ such that $\vec{v}+\vec{y} = \vec{y}+\vec{v} = \vec{0}$.

The scalar multiplication must satisfy the following axioms.

(1) The set V is closed under scalar multiplication: For any vector $\vec{v}$ in V and any scalar $\lambda \in \mathbb{R}$, the scalar multiple $\lambda\vec{v}$ is also in V.
(2) For two scalars $a, b \in \mathbb{R}$, we have $a(b\vec{v}) = (ab)\vec{v}$ for all vectors $\vec{v} \in V$.
(3) For $0 \in \mathbb{R}$, we have $0\vec{v} = \vec{0}$ for all $\vec{v} \in V$.
(4) For $1 \in \mathbb{R}$, we have $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$.

And finally, scalar multiplication distributes over addition:

(1) $\lambda(\vec{v}+\vec{w}) = \lambda(\vec{v})+\lambda(\vec{w})$ for all $\vec{v}, \vec{w} \in V$ and all $\lambda \in \mathbb{R}$.
(2) $(a+b)\vec{v} = a\vec{v}+b\vec{v}$ for all vectors $\vec{v} \in V$ and all scalars $a, b, \in \mathbb{R}$.

Subspace. If V is a vector space, a **subspace** of V is a subset W of V such that:

(1) $\vec{0} \in W$;
(2) for all vectors $\vec{x}$ and $\vec{y}$ in W, $\vec{x}+\vec{y} \in W$;
(3) for all $\vec{x} \in W$ and $k \in \mathbb{R}$, $k\vec{x} \in W$.

Linear Transformation. Let V and W be vector spaces. A **linear transformation** from V to W is a mapping $V \xrightarrow{T} W$ that satisfies:

(1) $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$ for all vectors $\vec{x}, \vec{y} \in V$;
(2) $T(k\vec{x}) = kT(\vec{x})$ for all vectors $\vec{x} \in V$ and all scalars $k \in \mathbb{R}$.

Source & Target. Let V and W be vector spaces. If $V \xrightarrow{T} W$ is a linear transformation from V to W, then V is called the **source** of T and W is called the **target** of T. The source T is also sometimes called the **domain** of T, and the target of T is sometimes called the **codomain** of T.

[1]Please note that while you should know the definition of vector space, we will not ask you to give the definition of a vector space as part of the "definitions" problem on any exam.

Standard Matrix. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The **standard matrix** of T is the unique matrix $A \in \mathbb{R}^{m\times n}$ such that $A\vec{x} = T(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.

Transpose. The **transpose** of the $m\times n$ matrix A is the $n\times m$ matrix A^T whose (i,j)-entry is the (j,i)-entry of A.

Symmetric and skew-symmetric matrices. The matrix A is **symmetric** if $A^\top = A$, and **skew-symmetric** if $A^\top = -A$.

Diagonal and Triangular matrices. Let A be an $n \times n$ matrix, and for each $1 \leq i, j \leq n$ let $A(i,j)$ be the (i,j)-entry of A. Then A is **diagonal** if $A(i,j) = 0$ whenever $i \neq j$, **upper-triangular** if $A(i,j) = 0$ whenever $i > j$, and lower-triangular if $A(i,j) = 0$ whenever $i < j$. Finally, A is **triangular** if A is either upper-triangular or lower-triangular.

Injective. The function $f : A \to B$ is **injective** if for all $x, y \in A$, if $x \neq y$ then $f(x) \neq f(y)$.

Surjective. The function $f : A \to B$ is **surjective** if for all $y \in B$ there exists $x \in A$ such that $f(x) = y$.

Bijective. The function $f : A \to B$ is **bijective** if f is both injective and surjective.

Isomorphism. If V and W are vector spaces, an **isomorphism** from V to W is a bijective linear transformation from V to W. We say that V is **isomorphic** to W if there exists an isomorphism from V to W.

Invertibility (Transformations). A linear transformation $V \xrightarrow{T} W$ from V to W is *invertible* if there exists a linear transformation $W \xrightarrow{S} V$ such that $S(T(\vec{v})) = \vec{v}$ for all $\vec{v} \in V$ and $T(S(\vec{w})) = \vec{w}$ for all $\vec{w} \in W$. If T is invertible, then the map S with this property is unique and is called the **inverse** of T, written T^{-1}.

Invertibility (Matrices). An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix. If A is invertible, then the matrix B such that $AB = BA = I_n$ is unique and is called the **inverse** of A, written A^{-1}.

Kernel. The **kernel** of the linear transformation $V \xrightarrow{T} W$ is the set of all vectors $\vec{v} \in V$ such that $T(\vec{v})$ is the zero vector in W. That is,

$$\ker(T) = \{\vec{v} \in V \ : \ T(\vec{v}) = \vec{0}\}.$$

Similarly, the **kernel** of the $n \times m$ matrix A is the set

$$\ker(A) = \{\vec{v} \in \mathbb{R}^m \ : \ A\vec{v} = \vec{0}\}.$$

Image. The **image** of the linear transformation $V \xrightarrow{T} W$ is the set of all vectors $\vec{w} \in W$ for which there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. That is,

$$\operatorname{im}(T) = \{\vec{w} \in W \ : \ T(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V\}.$$

Similarly, the **image** of the $n \times m$ matrix A is the set

$$\operatorname{im}(A) = \{A\vec{v} \ : \ \vec{v} \in \mathbb{R}^m\}.$$

Linear combination. A **linear combination** of the finitely many vectors $\vec{v}_1, \ldots, \vec{v}_n$ in V is an expression of the form

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

where each c_i is a scalar.

Span (of a finite list). Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a finite list of vectors in the vector space V. The **span** of $(\vec{v}_1, \ldots, \vec{v}_n)$ is the set

$$\operatorname{Span}(\vec{v}_1, \ldots \vec{v}_n) = \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \ : \ c_1, \ldots, c_n \in \mathbb{R}\}.$$

Span (of a set). Let $\mathcal{S}$ be a (possibly infinite) subset of V. The **span** of $\mathcal{S}$ is the set of all vectors in V that can be expressed as a linear combination of some finite list of vectors in $\mathcal{S}$. That is,

$$\operatorname{Span}(\mathcal{S}) \ = \ \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \ : \ n \in \mathbb{N} \text{ and for each } 1 \leq i \leq n, \ c_i \in \mathbb{R} \text{ and } \vec{v}_i \in \mathcal{S}\}.$$

If $\operatorname{Span}(\mathcal{S}) = V$, then $\mathcal{S}$ is said to **span** V or is called a **spanning set** for V.

Linear relation. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a finite list of vectors in the vector space V. A **linear relation** on $(\vec{v}_1, \ldots, \vec{v}_n)$ is an equation of the form

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0},$$

where $c_1, \ldots, c_n \in \mathbb{R}$. Such a relation is said to be **trivial** if $c_i = 0$ for each $1 \leq i \leq n$.

Linearly dependent (finite list). Let V be a vector space and let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a finite list of vectors in V. Then the list $(\vec{v}_1, \ldots, \vec{v}_n)$ is **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ that are not all zero such that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$.

Linearly dependent (set). Let V be a vector space and let $\mathcal{S}$ be a (possibly infinite) subset of V. Then $\mathcal{S}$ is **linearly dependent** if there is a finite list of distinct vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathcal{S}$ and scalars $c_1, \ldots, c_n$ that are not all zero such that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$.

Linearly independent (finite list). Let V be a vector space and let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a finite list of vectors in V. Then the list $(\vec{v}_1, \ldots, \vec{v}_n)$ is **linearly independent** if for all scalars $c_1, \ldots, c_n \in \mathbb{R}$, if $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ then $c_1 = \cdots = c_n = 0$.

Linearly independent (set). Let V be a vector space and let $\mathcal{S}$ be a (possibly infinite) subset of V. Then $\mathcal{S}$ is **linearly independent** if for every finite list of distinct vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathcal{S}$ and for all scalars $c_1, \ldots, c_n \in \mathbb{R}$, if $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ then $c_1 = \cdots = c_n = 0$.

Basis. A **basis** of the vector space V is a linearly independent subset of V that spans V.

Dimension. The **dimension** of the vector space is the number of elements in any basis of V.[2] A vector space is **finite-dimensional** if it has a finite spanning set, and **infinite-dimensional** otherwise.

Rank & Nullity. The **rank** of a linear transformation T is the dimension of $\mathrm{im}(T)$. The **nullity** of T is the dimension of $\ker(T)$. If A is an $m \times n$ matrix, then the **rank** and **nullity** of A are, respectively, the rank and nullity of T_A where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation defined by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.[3]

Coordinates. Suppose $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ is an ordered basis of the finite dimensional vector space V, and let $\vec{v} \in V$. The **$\mathcal{B}$-coordinate vector** of $\vec{v}$, written $[\vec{v}]_{\mathcal{B}}$, is the unique vector

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$$

such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. The map $L_{\mathcal{B}} : V \to \mathbb{R}^n$ defined by $L_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ is called the **$\mathcal{B}$-coordinate isomorphism**.

Change-of-coordinates matrix. If $\mathcal{B} = (\vec{b}_1, \dots, \vec{b}_n)$ and $\mathcal{C} = (\vec{c}_1, \dots, \vec{c}_n)$ are two ordered bases of the vector space V, the **change-of-coordinates matrix** from $\mathcal{B}$ to $\mathcal{C}$ is the standard matrix of $L_{\mathcal{C}} \circ L_{\mathcal{B}}^{-1}$, where $L_{\mathcal{B}}$ and $L_{\mathcal{C}}$ are the $\mathcal{B}$- and $\mathcal{C}$-coordinate isomorphisms, respectively.

$\mathcal{B}$-matrix. If $T : V \to V$ is a linear transformation of the vector space V and if $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ is an ordered basis of V, then the **$\mathcal{B}$-matrix** of T is the standard matrix of $L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1}$, where $L_{\mathcal{B}} : V \to \mathbb{R}^n$ is the $\mathcal{B}$-coordinate isomorphism.

Similar. Let A and B be $n \times n$ matrices. Then A is **similar** to B if there exists an invertible $n \times n$ matrix S such that $A = S^{-1}BS$.

Inner Product & Inner Product Space. An **inner product** on a vector space V is a function

$$V \times V \longrightarrow \mathbb{R}$$

which assigns to each pair of vectors $f, g \in V$ some scalar $\langle f, g \rangle$, called their **inner product.** The inner product must satisfy the following axioms:

(1) *Symmetry*: $\langle f, g \rangle = \langle g, f \rangle$ for all vectors $f, g \in V$;
(2) *Linearity in each argument*: $\langle (f + g), h \rangle = \langle f, h \rangle + \langle g, h \rangle$ for all vectors $f, g, h \in V$ and $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$ for all scalars λ and all $f, g \in V$.

[2] This makes sense because every vector space has a basis and any two bases of a vectors space have the same number of elements in them. The number of vectors in a basis of V *could* be infinite, in which case V is infinite-dimensional.

[3] The textbook defines the rank of the matrix A to be the number of leading 1's in the reduced row echelon form of A. This definition is equivalent to ours — can you explain why?

(3) *Positive Definiteness*: $\langle f, f\rangle > 0$ for all nonzero $f \in V$.

A vector space V together with a choice of an inner product is called an **inner product space.**

Magnitude & Orthogonality of vectors in an inner product space. Let $\vec{v}$ and $\vec{w}$ be vectors in an inner product space. The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is the scalar $\langle \vec{v}, \vec{v}\rangle^{1/2}$. We say that $\vec{w}$ is perpendicular or **orthogonal** to $\vec{v}$ if $\langle \vec{v}, \vec{w}\rangle = 0$.

Orthonormal Set. A set of vectors $\vec{u}_1, \ldots, \vec{u}_n$ in the inner product space V is **orthonormal** if $\langle u_i, u_j\rangle = 0$ whenever $i \neq j$, and $\langle u_i, u_i\rangle = 1$ for all i. An **orthonormal basis** of V is an orthonormal set in V that also a basis of V.

Orthogonal Transformation. An orthogonal transformation of $\mathbb{R}^n$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Orthogonal Matrix. The $n \times n$ matrix A is *orthogonal* if $A^\top A = I_n = AA^\top$.

Orthogonal Complement. If W is a subspace of the inner product space V, the **orthogonal complement** of W in V is the set

$$W^\perp \;=\; \{\vec{v} \in V \;:\; \langle \vec{v}, \vec{w}\rangle = 0 \text{ for all } \vec{w} \in W\}.$$

Orthogonal Projection. If W is a subspace of an inner product space V and if $\vec{v} \in V$, the **orthogonal projection** of $\vec{v}$ onto W is the unique vector $\vec{w} \in W$ such that $\vec{v} - \vec{w} \in W^\perp$. The orthogonal projection of $\vec{v}$ onto W is sometimes denoted $\text{proj}_W(\vec{v})$.

Reflection. If V is a subspace of $\mathbb{R}^n$, then *reflection* of $\mathbb{R}^n$ through V is the linear transformation $\text{refl}_V : \mathbb{R}^n \to \mathbb{R}^n$ given by $\text{refl}_V = 2\text{proj}_V - I_n$.

Least-squares solution. If A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$, the vector $\vec{x}^* \in \mathbb{R}^n$ is a **least-squares solution** of the linear system $A\vec{x} = \vec{b}$ if $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$ for all $\vec{x} \in \mathbb{R}^n$.

Determinant of a Linear Transformation. Let V be a finite dimensional vector space. The **determinant** $\det(T)$ of a linear transformation $T : V \to V$ is the determinant of A where A is the matrix of T relative to any basis of V.[4]

Eigenvector & Eigenvalue. An **eigenvector** of the linear transformation $V \xrightarrow{T} V$ is any non-zero vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda\vec{v}$ for some scalar λ. The scalar λ is called the **eigenvalue** of T corresponding to the eigenvector $\vec{v}$.

Eigenbasis. Let $V \xrightarrow{T} V$ be a linear transformation. An **eigenbasis** of V for T is a basis of V consisting of eigenvectors of T.

[4]This makes sense because for any two bases $\mathcal{B}$ and $\mathcal{C}$ of V, the $\mathcal{B}$-matrix of T and the $\mathcal{C}$-matrix of T have the same determinant, since they are similar to each other and similar matrices have the same determinant.

Diagonalizable. The linear transformation $V \xrightarrow{T} V$ of the vector space V is **diagonalizable** if there is a basis $\mathcal{B}$ of V such that the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T is diagonal. The matrix A is **diagonalizable** if A is similar to a diagonal matrix.

Eigenspace. Suppose $V \xrightarrow{T} V$ is a linear transformation of the vector space V. If λ is an eigenvalue of T, then the subset

$$E_\lambda = \{\vec{v} \in V \ : \ T(\vec{v}) = \lambda\vec{v}\}$$

consisting of all λ-eigenvectors (together with $\vec{0}$) is called the **λ-eigenspace** of T, or the **eigenspace of T corresponding to λ**.

Characteristic Polynomial of a Matrix. The **characteristic polynomial** of the $n \times n$ matrix A is the polynomial f_A in the variable x given by[5]

$$f_A(x) = \det(A - xI_n).$$

Characteristic Polynomial of a Linear Transformation. Let $V \xrightarrow{T} V$ be a linear transformation on the vector space V of finite dimension n. The **characteristic polynomial** of T is the polynomial f_T in the variable x given by

$$f_T(x) = \det(A - xI_n),$$

where A is the matrix of T relative to *any* basis of V.

Algebraic & Geometric Multiplicity. Let $V \xrightarrow{T} V$ be a linear transformation of the finite-dimensional vector space V, and let λ be an eigenvalue of T. The **algebraic multiplicity** of λ is the largest integer power r such that $(x-\lambda)^r$ is a factor of the characteristic polynomial of T. The **geometric multiplicity** of λ is $\dim(E_\lambda)$, the dimension of the λ-eigenspace of T.

Orthogonally Diagonalizable. The $n \times n$ matrix A is **orthogonally diagonalizable** if there is an $n \times n$ orthogonal matrix Q such that $Q^T AQ$ is a diagonal matrix. The linear transformation $V \xrightarrow{T} V$ of the finite-dimensional inner product space V is **orthogonally diagonalizable** if there is an ordered orthonormal basis $\mathcal{U}$ of V such that the $\mathcal{U}$-matrix of T is diagonal.

[5]Some authors define the characteristic polynomial of A to be $\det(xI_n - A)$ rather than $\det(A - xI_n)$. The difference between these definitions is usually not important, since both polynomials have the same roots.