Learning Objective: Ampere's Law

Problem 1. Consider an infinite line carrying current I_1 in the $+z$ direction is placed along the z-axis. Another infinite line carrying current I_2, in the same direction as I_1, is located at $x = a$ across the x-axis, where $a > 0$.

(a) Determine the magnetic field $\mathbf{H}$ as a function of x using Cartesian coordinate.

The magnetic field of a infinite line along the z-axis can be derived using Ampere's Law:

$$\oint_c \mathbf{H} \cdot dl = I_{\text{encl}}$$
$$\mathbf{H}(2\pi\rho) = I_{\text{encl}}$$
$$\mathbf{H} = \hat{\phi}\frac{I_{\text{encl}}}{2\pi\rho}$$

For $x < 0$, the magnetic field by both currents are pointing at $\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(-x)} + \frac{I_2}{2\pi(a-x)}\right)$$

For $x = 0$, the magnetic field is only affected by I_2 going $\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\frac{I_2}{2\pi(a)}$$

For $0 < x < a$, the magnetic field by I_1 and I_2 are pointing at $\hat{\phi}$ and $-\hat{\phi}$ direction, and so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(x)} - \frac{I_2}{2\pi(a-x)}\right)$$

For $x = a$, the magnetic field is only affected by I_1 going $\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\frac{I_1}{2\pi(a)}$$

For $x > a$, the magnetic field by both currents are pointing at $\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(x)} + \frac{I_2}{2\pi(x-a)}\right)$$

Therefore:

$$\mathbf{H} = \begin{cases} \hat{\phi}\frac{I_2}{2\pi(a)} & x = 0 \\ \hat{\phi}\left(\frac{I_1}{2\pi(x)} - \frac{I_2}{2\pi(a-x)}\right) & 0 < x < a \\ \hat{\phi}\frac{I_1}{2\pi(a)} & x = a \\ \hat{\phi}\left(\frac{I_1}{2\pi|x|} + \frac{I_2}{2\pi|x-a|}\right) & \text{otherwise} \end{cases}$$

(b) Determine the magnetic flux density $\mathbf{B}$ as a function of x using Cartesian coordinate.

$$\mathbf{B} = \frac{1}{\mu}\mathbf{H}$$
$$= \begin{cases} \hat{\phi}\frac{1}{\mu}\frac{I_2}{2\pi(a)} & x = 0 \\ \hat{\phi}\frac{1}{\mu}\left(\frac{I_1}{2\pi(x)} - \frac{I_2}{2\pi(a-x)}\right) & 0 < x < a \\ \hat{\phi}\frac{1}{\mu}\frac{I_1}{2\pi(a)} & x = a \\ \hat{\phi}\frac{1}{\mu}\left(\frac{I_1}{2\pi|x|} + \frac{I_2}{2\pi|x-a|}\right) & \text{otherwise} \end{cases}$$

(c) Suppose I_2 flows in the opposite direction as I_1, determine the magnetic field $\mathbf{H}$ as a function of x using Cartesian coordinate.

For $x < 0$, the magnetic field by I_1 and I_2 are pointing at $\hat{\phi}$ and $-\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(-x)} - \frac{I_2}{2\pi(a-x)}\right)$$

For $x = 0$, the magnetic field is only affected by I_2 going $-\hat{\phi}$ direction, so

$$\mathbf{H} = -\hat{\phi}\frac{I_2}{2\pi(a)}$$

For $0 < x < a$, the magnetic field by both currents are pointing at $\hat{\phi}$ direction, and so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(x)} + \frac{I_2}{2\pi(a-x)}\right)$$

For $x = a$, the magnetic field is only affected by I_1 going $\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\frac{I_1}{2\pi(a)}$$

For $x > a$, the magnetic field by I_1 and I_2 are pointing at $\hat{\phi}$ and $-\hat{\phi}$ direction, so

$$\mathbf{H} = \hat{\phi}\left(\frac{I_1}{2\pi(x)} - \frac{I_2}{2\pi(x-a)}\right)$$

Therefore:

$$\mathbf{H} = \begin{cases} -\hat{\phi}\frac{I_2}{2\pi(a)} & x = 0 \\ \hat{\phi}\left(\frac{I_1}{2\pi(x)} + \frac{I_2}{2\pi(a-x)}\right) & 0 < x < a \\ \hat{\phi}\frac{I_1}{2\pi(a)} & x = a \\ \hat{\phi}\left(\frac{I_1}{2\pi|x|} - \frac{I_2}{2\pi|x-a|}\right) & \text{otherwise} \end{cases}$$

Problem 2. Consider an infinitely long cylindrical conductor of radius $r = a$ cm is placed along the z-axis. A current in the $+z$ direction flows through the cylindrical conductor, where the current density is given by

$$J(\rho) = -2\exp\left(-\frac{\rho}{z}\right)\ \text{A/cm}^2$$

where $\rho > 0$ and $z = 2$ cm.

(a) Determine the magnetic field $\mathbf{H}$ as a function of ρ using cylindrical coordinate.

For $\rho > a$:

$$\begin{aligned}
\oint_c \mathbf{H}\cdot dl &= I_{\text{encl}} = \int_s J ds \\
\mathbf{H}(2\pi\rho) &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} -2\exp\left(-\frac{\rho}{z}\right)\rho d\rho d\phi \\
&= 4\pi\left[az\exp\left(-\frac{a}{z}\right) + z^2\exp\left(-\frac{a}{z}\right) - z^2\right] \\
\mathbf{H} &= \hat{\phi}\frac{2}{\rho}\left[az\exp\left(-\frac{a}{z}\right) + z^2\exp\left(-\frac{a}{z}\right) - z^2\right]
\end{aligned}$$

For $\rho \leq a$:

$$\begin{aligned}
\oint_c \mathbf{H}\cdot dl &= I_{\text{encl}} = \int_s J ds \\
\mathbf{H}(2\pi\rho) &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} -2\exp\left(-\frac{\rho}{z}\right)\rho d\rho d\phi \\
&= 4\pi\left[\rho z\exp\left(-\frac{\rho}{z}\right) + z^2\exp\left(-\frac{\rho}{z}\right) - z^2\right] \\
\mathbf{H} &= \hat{\phi}\frac{2}{\rho}\left[\rho z\exp\left(-\frac{\rho}{z}\right) + z^2\exp\left(-\frac{\rho}{z}\right) - z^2\right]
\end{aligned}$$

Therefore:

$$\mathbf{H} = \begin{cases} \hat{\phi}\frac{2}{\rho}\left[az\exp\left(-\frac{a}{z}\right) + z^2\exp\left(-\frac{a}{z}\right) - z^2\right] & \rho > a \\ \hat{\phi}\frac{2}{\rho}\left[\rho z\exp\left(-\frac{\rho}{z}\right) + z^2\exp\left(-\frac{\rho}{z}\right) - z^2\right] & \text{otherwise} \end{cases}$$

(b) Suppose the same current flows in the $-z$ direction, determine the magnetic field $\mathbf{H}$ as a function of ρ using cylindrical coordinate.

By switching the direction of the current, the magnitude of magnetic field doesn't change but the direction flips, therefore:

$$\mathbf{H} = \begin{cases} -\hat{\phi}\frac{2}{\rho}\left[az\exp\left(-\frac{a}{z}\right) + z^2\exp\left(-\frac{a}{z}\right) - z^2\right] & \rho > a \\ -\hat{\phi}\frac{2}{\rho}\left[\rho z\exp\left(-\frac{\rho}{z}\right) + z^2\exp\left(-\frac{\rho}{z}\right) - z^2\right] & \text{otherwise} \end{cases}$$

Learning Objective: Inductance of a Straight Coil

Problem 3. Consider a solenoid with 10 cm long, 1 mm in diameter, and 1000 terns of wire. The core of the solenoid consists a ferromagnetic material with a relative permeability of 500.

(a) Determine the magnitude of magnetic field $|\mathbf{H}|$ within the solenoid core.

$$\begin{aligned}\oint_c \mathbf{H} \cdot dl &= I_{\text{encl}} \\ |\mathbf{H}|L &= NI \\ |\mathbf{H}| &= \frac{NI}{L} \\ &= \frac{(1000)I}{0.1} = 100I \text{ A/m}\end{aligned}$$

(b) Determine the magnitude of the magnetic flux Φ within the solenoid core.

$$\begin{aligned}\Phi &= \int_s \mathbf{B} \cdot ds \\ &= \frac{1}{\mu}\int_s \mathbf{H} \cdot ds \\ &= \frac{NI}{\mu L}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{r} \rho d\rho d\phi \\ &= \frac{NI}{\mu L}\pi r^2\end{aligned}$$

(c) Determine the inductance of the solenoid.

$$\begin{aligned}L &= \frac{\mu N^2 A}{l} \\ &= \frac{\mu_r \mu_0 N^2 \pi r^2}{l} \\ &= 4.93 \text{ mH}\end{aligned}$$

(d) Suppose this solenoid is broke from the middle, determine the inductance of this solenoid.

$$\begin{aligned}L' &= \frac{\mu (N/2)^2 A}{l/2} \\ &= \frac{L}{2} \\ &= 2.465 \text{ mH}\end{aligned}$$