Learning Objective: Electric Field and Potential Due to Point Charges

Problem 1. Two point charges with $q_1 = 20\ \mu$C and $q_2 = -40\ \mu$C are located in a free space at points with Cartesian coordinates $(1, 3, -1)$ m and $(-3, 1, -2)$ m, respectively.

(a) Determine the force $\mathbf{F_1}$ acting on charge q_1.

$$\begin{aligned}
\mathbf{F_1} &= \frac{q_1 q_2 (\mathbf{r}_1 - \mathbf{r}_2)}{4\pi\epsilon |\mathbf{r}_1 - \mathbf{r}_2|^3} \\
&= \frac{(20 \times 10^{-6})(-40 \times 10^{-6})(-4\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + \hat{\mathbf{z}})}{4\pi(8.854 \times 10^{-12})(\sqrt{21})^3} \\
&= -0.2988\hat{\mathbf{x}} - 0.1494\hat{\mathbf{y}} - 0.07471\hat{\mathbf{z}}\ \text{N}
\end{aligned}$$

(b) Find the electric field $\mathbf{E}$ at $(3, 1, -2)$ m.

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \\
&= \frac{1}{4\pi\epsilon} \left[\frac{q_1 (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} + \frac{q_2 (\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3} \right] \\
&= \frac{1}{4\pi(8.854 \times 10^{-12})} \left[\frac{(20 \times 10^{-6})(2\hat{\mathbf{x}} - 2\hat{\mathbf{y}} - \hat{\mathbf{z}})}{(3)^3} + \frac{(-40 \times 10^{-6})(6\hat{\mathbf{x}})}{(6)^3} \right] \\
&= 3.328\hat{\mathbf{x}} - 13.32\hat{\mathbf{y}} - 6.658\hat{\mathbf{z}}\ \text{kV/m}
\end{aligned}$$

(c) Suppose a new point charge $q_3 = 80\ \mu$C is placed at $(3, 1, -2)$ m, determine the (i) force $\mathbf{F_3}$ acting on charge q_3.

$$\begin{aligned}
\mathbf{F_3} &= q_3 \mathbf{E} \\
&= (80 \times 10^{-6}) \left[3.328\hat{\mathbf{x}} - 13.32\hat{\mathbf{y}} - 6.658\hat{\mathbf{z}}\right] \times 10^3 \\
&= 0.2662\hat{\mathbf{x}} - 1.065\hat{\mathbf{y}} - 0.5325\hat{\mathbf{z}}\ \text{N}
\end{aligned}$$

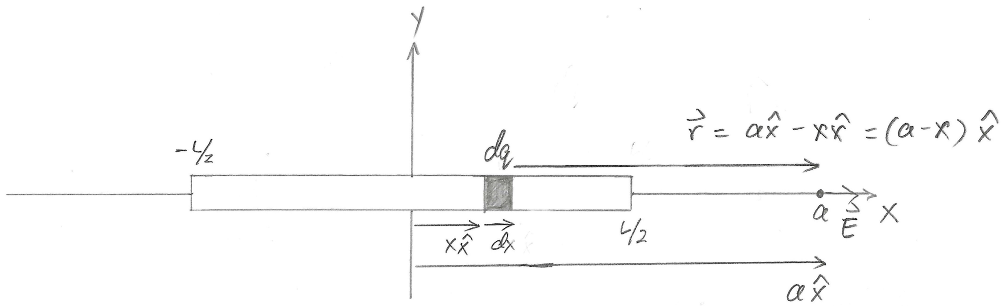
Learning Objective: Electric Field and Potential Due to Charge Distribution

Problem 2. A charge $+Q$ is evenly spread along the x-axis from $x = -L/2$ to $x = L/2$.

(a) Determine the line charge density ρ_l.

$$\rho_l = \frac{Q}{L}$$

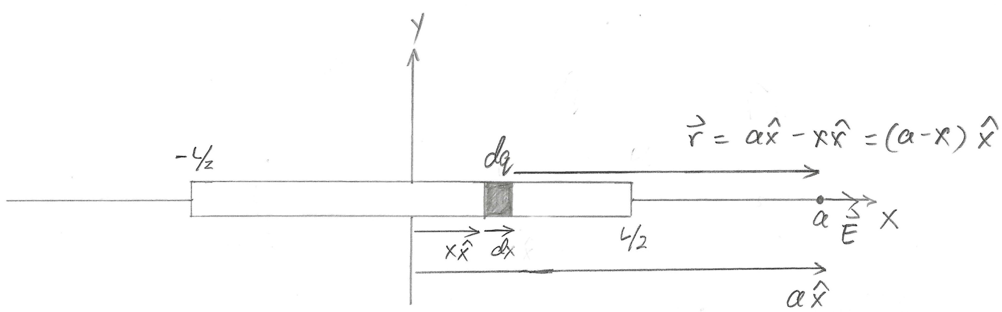
(b) Derive an expression of the electric field $\mathbf{E}$ at $x = a$ where $a > L/2$.



$$d\mathbf{E} = \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_l}{4\pi\epsilon}\frac{(a-x)\hat{\mathbf{x}}}{(a-x)^3}dx = \frac{\rho_l}{4\pi\epsilon}\frac{\hat{\mathbf{x}}}{(a-x)^2}dx \qquad (dq/dx = \rho_l)$$

$$\begin{aligned}\mathbf{E} &= \int_{-L/2}^{L/2}\frac{\rho_l}{4\pi\epsilon}\frac{\hat{\mathbf{x}}}{(a-x)^2}dx = \hat{\mathbf{x}}\frac{\rho_l}{4\pi\epsilon}\int_{-L/2}^{L/2}\frac{1}{(a-x)^2}dx \\ &= \hat{\mathbf{x}}\frac{\rho_l}{4\pi\epsilon}\left[-\frac{1}{a-x}\right]_{-L/2}^{L/2} \\ &= \hat{\mathbf{x}}\frac{\rho_l}{4\pi\epsilon}\left(\frac{1}{a+L/2} - \frac{1}{a-L/2}\right)\end{aligned}$$

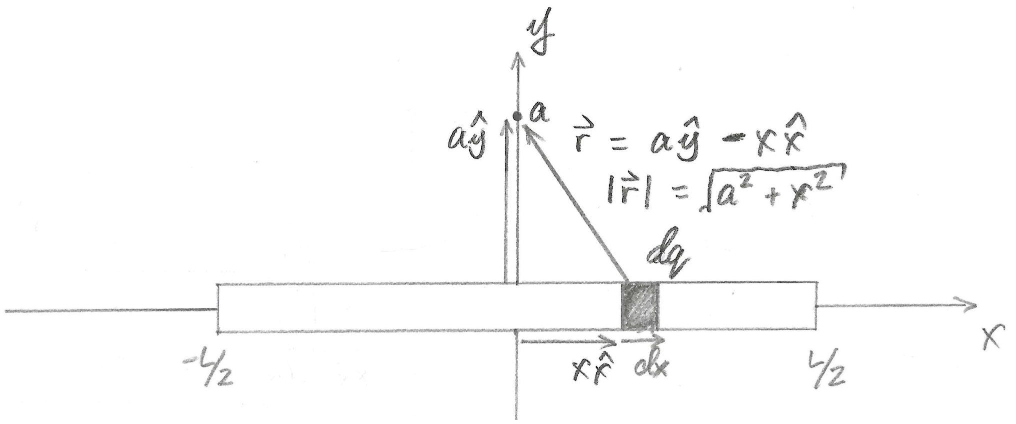
(c) Derive an expression of the electric potential V at $x = a$ where $a > L/2$.



$$dV = \frac{dq}{4\pi\epsilon}\frac{1}{|\mathbf{r}|} = \frac{\rho_l}{4\pi\epsilon}\frac{1}{(a-x)}dx \qquad (dq/dx = \rho_l)$$

$$\begin{aligned}V &= \int_{-L/2}^{L/2}\frac{\rho_l}{4\pi\epsilon}\frac{1}{(a-x)}dx = \frac{\rho_l}{4\pi\epsilon}\int_{-L/2}^{L/2}\frac{1}{(a-x)}dx \\ &= \frac{\rho_l}{4\pi\epsilon}\left[-\ln|a-x|\right]_{-L/2}^{L/2} \\ &= \frac{\rho_l}{4\pi\epsilon}\left(\ln|a+L/2| - \ln|a-L/2|\right)\end{aligned}$$

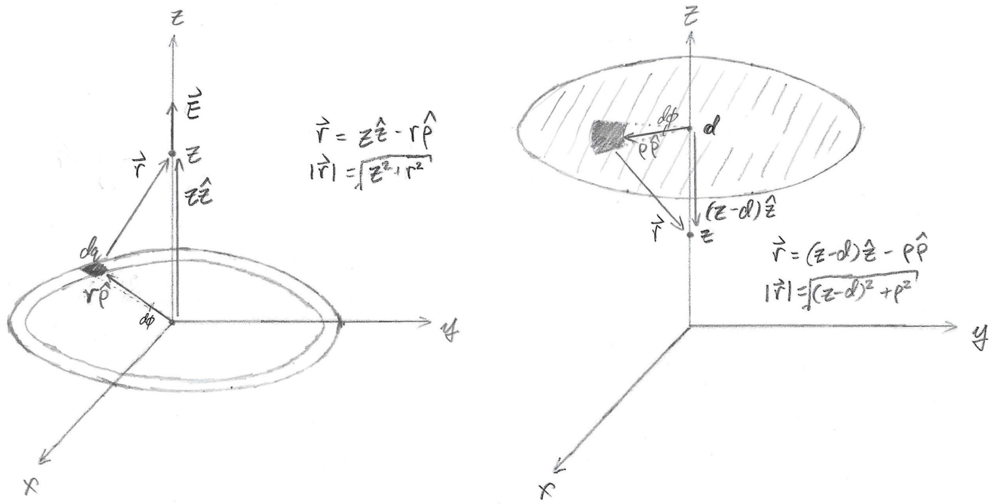
(d) Derive an expression of the electric field $\mathbf{E}$ at $y = a$ where $a > 0$.



$$
\begin{aligned}
d\mathbf{E} &= \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_l}{4\pi\epsilon}\frac{a\hat{\mathbf{y}} - x\hat{\mathbf{x}}}{(a^2+x^2)^{3/2}}dx \qquad (dq/dx = \rho_l) \\
&= \frac{\rho_l}{4\pi\epsilon}\left[\frac{a\hat{\mathbf{y}}}{(a^2+x^2)^{3/2}} - \frac{x\hat{\mathbf{x}}}{(a^2+x^2)^{3/2}}\right]dx \\
\mathbf{E} &= \frac{\rho_l}{4\pi\epsilon}\left[\int_{-L/2}^{L/2}\frac{a\hat{\mathbf{y}}}{(a^2+x^2)^{3/2}}dx - \int_{-L/2}^{L/2}\frac{x\hat{\mathbf{x}}}{(a^2+x^2)^{3/2}}dx\right] \\
&= \frac{\rho_l}{4\pi\epsilon}\left(\hat{\mathbf{y}}\left[\frac{x}{a\sqrt{a^2+x^2}}\right]_{-L/2}^{L/2} - \hat{\mathbf{x}}\left[-\frac{1}{\sqrt{a^2+x^2}}\right]_{-L/2}^{L/2}\right) \\
&= \hat{\mathbf{y}}\frac{\rho_l}{4\pi\epsilon}\left(\frac{L/2}{a\sqrt{a^2+(L/2)^2}} - \frac{-L/2}{a\sqrt{a^2+(-L/2)^2}}\right) \\
&= \hat{\mathbf{y}}\frac{\rho_l}{4\pi\epsilon}\frac{L}{a\sqrt{a^2+(L/2)^2}}
\end{aligned}
$$

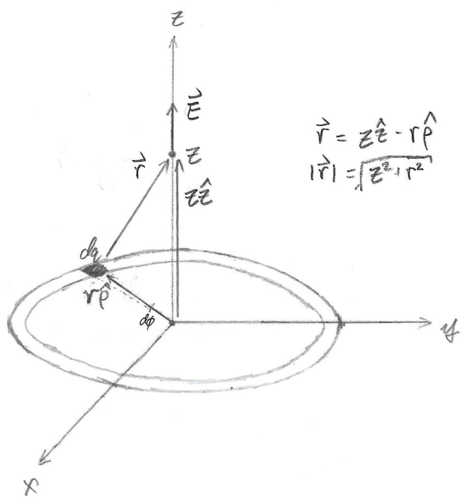
Problem 3. Consider a ring of radius $r = a$ in the $z = 0$ plane, centered at the origin, has a uniform line charge density ρ_l. Another circular disk of radius $r = a$ in the $z = d$ plane ($d > 0$), centered at the origin, has a uniform surface charge density ρ_s.

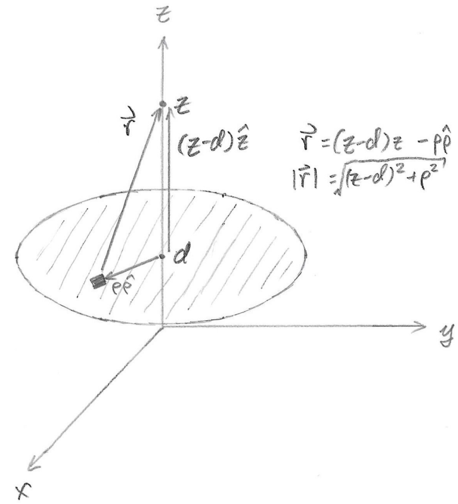
(a) Derive an expression of the electric field $\mathbf{E}$ along the z axis where $0 < z < d$.



$$
\begin{aligned}
d\mathbf{E}_{\text{ring}} &= \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_l a}{4\pi\epsilon(a^2+z^2)^{3/2}}(z\hat{\mathbf{z}} - a\hat{\rho})d\phi \\
d\mathbf{E}_{\text{disk}} &= \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_s}{4\pi\epsilon}\frac{(z-d)\hat{\mathbf{z}} - \rho\hat{\rho}}{[(z-d)^2+\rho^2]^{3/2}}\rho d\rho d\phi \\
\mathbf{E} &= \mathbf{E}_{\text{ring}} + \mathbf{E}_{\text{disk}} \\
&= \frac{\rho_l a}{4\pi\epsilon(a^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}(z\hat{\mathbf{z}} - a\hat{\rho})d\phi + \frac{\rho_s}{4\pi\epsilon}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(z-d)\hat{\mathbf{z}} - \rho\hat{\rho}}{[(z-d)^2+\rho^2]^{3/2}}\rho d\rho d\phi \\
&= \hat{\mathbf{z}}\frac{\rho_l a z}{4\pi\epsilon(a^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}d\phi + \hat{\mathbf{z}}\frac{\rho_s(z-d)}{4\pi\epsilon}\int_{\rho=0}^{a}\frac{\rho}{[(z-d)^2+\rho^2]^{3/2}}d\rho\int_{\phi=0}^{2\pi}d\phi \\
&= \hat{\mathbf{z}}\frac{\rho_l a z(2\pi)}{4\pi\epsilon(a^2+z^2)^{3/2}} + \hat{\mathbf{z}}\frac{\rho_s(z-d)(2\pi)}{4\pi\epsilon}\left[-\frac{1}{\sqrt{(z-d)^2+\rho^2}}\right]_{\rho=0}^{a} \\
&= \hat{\mathbf{z}}\frac{\rho_l a}{2\epsilon}\frac{z}{(a^2+z^2)^{3/2}} + \hat{\mathbf{z}}\frac{\rho_s(z-d)}{2\epsilon}\left(\frac{1}{|z-d|} - \frac{1}{\sqrt{(z-d)^2+a^2}}\right) \\
&= \hat{\mathbf{z}}\left[\frac{\rho_l a}{2\epsilon}\frac{z}{(a^2+z^2)^{3/2}} + \frac{\rho_s(z-d)}{2\epsilon}\left(\frac{1}{|z-d|} - \frac{1}{\sqrt{(z-d)^2+a^2}}\right)\right]
\end{aligned}
$$

(b) Derive an expression of the electric field $\mathbf{E}$ along the z axis where $z > d$.





$$
\begin{aligned}
d\mathbf{E}_{\text{ring}} &= \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_l a}{4\pi\epsilon(a^2+z^2)^{3/2}}(z\hat{\mathbf{z}} - a\hat{\rho})d\phi \\
d\mathbf{E}_{\text{disk}} &= \frac{dq}{4\pi\epsilon}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\rho_s}{4\pi\epsilon}\frac{(z-d)\hat{\mathbf{z}} - \rho\hat{\rho}}{[(z-d)^2+\rho^2]^{3/2}}\rho d\rho d\phi \\
\mathbf{E} &= \mathbf{E}_{\text{ring}} + \mathbf{E}_{\text{disk}} \\
&= \frac{\rho_l a}{4\pi\epsilon(a^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}(z\hat{\mathbf{z}} - a\hat{\rho})d\phi + \frac{\rho_s}{4\pi\epsilon}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{(z-d)\hat{\mathbf{z}} - \rho\hat{\rho}}{[(z-d)^2+\rho^2]^{3/2}}\rho d\rho d\phi \\
&= \hat{\mathbf{z}}\frac{\rho_l a z}{4\pi\epsilon(a^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}d\phi + \hat{\mathbf{z}}\frac{\rho_s(z-d)}{4\pi\epsilon}\int_{\rho=0}^{a}\frac{\rho}{[(z-d)^2+\rho^2]^{3/2}}d\rho\int_{\phi=0}^{2\pi}d\phi \\
&= \hat{\mathbf{z}}\frac{\rho_l a z(2\pi)}{4\pi\epsilon(a^2+z^2)^{3/2}} + \hat{\mathbf{z}}\frac{\rho_s(z-d)(2\pi)}{4\pi\epsilon}\left[-\frac{1}{\sqrt{(z-d)^2+\rho^2}}\right]_{\rho=0}^{a} \\
&= \hat{\mathbf{z}}\frac{\rho_l a}{2\epsilon}\frac{z}{(a^2+z^2)^{3/2}} + \hat{\mathbf{z}}\frac{\rho_s(z-d)}{2\epsilon}\left(\frac{1}{|z-d|} - \frac{1}{\sqrt{(z-d)^2+a^2}}\right) \\
&= \hat{\mathbf{z}}\left[\frac{\rho_l a}{2\epsilon}\frac{z}{(a^2+z^2)^{3/2}} + \frac{\rho_s(z-d)}{2\epsilon}\left(\frac{1}{|z-d|} - \frac{1}{\sqrt{(z-d)^2+a^2}}\right)\right]
\end{aligned}
$$

Learning Objective: Gauss's Law

Problem 4. Consider a sphere of radius $r = a$ centered at the origin has a uniform volume charge density ρ_v in a spherical coordinate.

(a) Determine the electric field $\mathbf{E}$ for $r > a$.

$$\begin{aligned}
\oint \mathbf{D} \cdot ds &= \oint \epsilon \mathbf{E} \cdot ds = Q_{\text{encl}} \\
\oint \mathbf{E} \cdot ds &= \frac{Q_{\text{encl}}}{\epsilon} \\
\mathbf{E}(4\pi r^2) &= \frac{\frac{4}{3}\pi a^3}{\epsilon} \\
\mathbf{E} &= \hat{\mathbf{r}} \frac{a^3}{3\epsilon r^2}
\end{aligned}$$

(b) Determine the electric field $\mathbf{E}$ for $0 < r < a$.

$$\begin{aligned}
\oint \mathbf{D} \cdot ds &= \oint \epsilon \mathbf{E} \cdot ds = Q_{\text{encl}} \\
\oint \mathbf{E} \cdot ds &= \frac{Q_{\text{encl}}}{\epsilon} \\
\mathbf{E}(4\pi r^2) &= \frac{\frac{4}{3}\pi r^3}{\epsilon} \\
\mathbf{E} &= \hat{\mathbf{r}} \frac{r}{3\epsilon}
\end{aligned}$$

Learning Objective: Capacitance

Problem 5. Consider an n-channel MOSFET with a gate oxide thickness of 10 nm, gate width of 25 μm and gate length of 1 μm. A gate-to-source voltage $V_{\text{GS}} = 1.5V$ is applied. Use $\epsilon_{ox} = 3.9\epsilon_0$, $n_i = 1.5 \times 10^{10}$ cm^{-3} if needed.

(a) Determine the gate capacitance of the MOSFET.

$$\begin{aligned} C_{\text{gate}} &= \frac{\epsilon A}{d} = \frac{(3.9)(8.85 \times 10^{-12})(25 \times 10^{-6})(1 \times 10^{-6})}{10 \times 10^{-9}} \\ &= 86.28 \text{ fF} \end{aligned}$$

(b) Determine the total charge at the gate.

$$Q = CV = (86.28 \times 10^{-15})(1.5) = 0.129 \text{ pC}$$

(c) Determine the electric field intensity inside the oxide.

$$E = \frac{V}{L} = \frac{2.5}{10 \times 10^{-9}} = 2.5 \times 10^{8} \text{ V/m}$$