Learning Objective: Cartesian Coordinate, Vectors, and Operations

Problem 1. In a Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** points from P_1 to point $P_2 = (1, -2, 2)$. Review Table 1 for vector relations.

(a) Determine vector **A**, then calculate its magnitude $|A|$, and unit vector $\hat{\mathbf{A}}$.

$$\begin{aligned} \mathbf{A} &= 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 3\hat{\mathbf{z}} \\ |A| &= \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22} \\ \hat{\mathbf{A}} &= \frac{2}{\sqrt{22}}\hat{\mathbf{x}} + \frac{3}{\sqrt{22}}\hat{\mathbf{y}} + \frac{3}{\sqrt{22}}\hat{\mathbf{z}} \end{aligned}$$

(b) Determine the angle between **A** and the y-axis.

$$\begin{aligned} |A||\hat{y}|\cos(\theta) &= \mathbf{A} \cdot \hat{\mathbf{y}} \\ \theta &= \arccos\left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{|A||\hat{y}|}\right) \\ &= 50.2^\circ \end{aligned}$$

(c) Determine vector **B**, then calculate its magnitude $|B|$, and unit vector $\hat{\mathbf{B}}$.

$$\begin{aligned} \mathbf{B} &= (1\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}) - (2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 3\hat{\mathbf{z}}) \\ &= -\hat{\mathbf{x}} - 5\hat{\mathbf{y}} - \hat{\mathbf{z}} \\ |B| &= \sqrt{(-1)^2 + (-5)^2 + (-1)^2} = 3\sqrt{3} \\ \hat{\mathbf{B}} &= -\frac{1}{3\sqrt{3}}\hat{\mathbf{x}} - \frac{5}{3\sqrt{3}}\hat{\mathbf{y}} - \frac{1}{3\sqrt{3}}\hat{\mathbf{z}} \end{aligned}$$

(d) Find the dot product $\mathbf{A} \cdot \mathbf{B}$ and cross product $\mathbf{A} \times \mathbf{B}$ between vectors **A** and **B** in Cartesian, cylindrical and spherical coordinates.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= (2)(-1) + (3)(-5) + (3)(-1) = -20 \\ \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 3 & 3 \\ -1 & -5 & -1 \end{vmatrix} = 12\hat{\mathbf{x}} - \hat{\mathbf{y}} - 7\hat{\mathbf{z}} \end{aligned}$$

Learning Objective: Cylindrical Coordinate, Length, Surface Area, and Volume

Problem 2. A section of a cylinder is described as $0 \leq \rho \leq 3$, $60^\circ \leq \phi \leq 120^\circ$, and $-2 \leq z \leq 2$.

(a) Determine the perimeter of the enclosed area at $z = 0$.

$$C = 2\int_0^3 d\rho + \int_{\pi/3}^{2\pi/3} \rho d\phi = 2\int_0^3 d\rho + (3)\int_{\pi/3}^{2\pi/3} d\phi = 6 + \pi$$

(b) Determine the surface area of the cylinder section for $\rho = 3$.

$$\begin{aligned} d\hat{\mathbf{s}}_\rho &= \hat{\rho}\,\rho\, d\phi\, dz \\ \hat{\mathbf{s}}_\rho &= \hat{\rho}(3)\left(\int_{\pi/3}^{2\pi/3} d\phi\right)\left(\int_{-2}^{2} d\phi dz\right) = \hat{\rho}4\pi \end{aligned}$$

(c) Determine the enclosed volume of the cylinder section.

$$\begin{aligned} dV &= \rho d\rho d\phi dz \\ V &= \left(\int_0^3 \rho d\rho\right)\left(\int_{\pi/3}^{2\pi/3} d\phi\right)\left(\int_{-2}^{2} dz\right) \\ &= \left(\left[\frac{\rho^2}{2}\right]_0^3\right)\left(\frac{\pi}{3}\right)(4) \\ &= 6\pi \end{aligned}$$

Learning Objective: Spherical Coordinate, Length, Surface Area, and Volume

Problem 3. A section of a sphere is described as $0 \leq r \leq 2$, $0^\circ \leq \theta \leq 90^\circ$, and $30^\circ \leq \phi \leq 90^\circ$.

(a) Determine the perimeter of the enclosed surface area at $r = 2$.

$$\begin{aligned} C &= 2\int_0^{\pi/2} r d\theta + \int_{\pi/6}^{\pi/2} r\sin(\theta) d\phi \\ &= 2r\int_0^{\pi/2} d\theta + r\sin(\theta)\int_{\pi/6}^{\pi/2} d\phi \\ &= 2(2)\int_0^{\pi/2} d\theta + (2)\sin(\pi/2)\int_{\pi/6}^{\pi/2} d\phi \\ &= \frac{8}{3}\pi \end{aligned}$$

(b) Determine the surface area of the sphere section for $r = 2$.

$$\begin{aligned} d\hat{\mathbf{s}}_r &= \hat{r}r^2\sin(\theta) d\theta d\phi \\ \mathbf{s}_r &= \hat{r}r^2\left(\int_0^{\pi/2}\sin(\theta)d\theta\right)\left(\int_{\pi/6}^{\pi/2} d\phi\right) \\ &= \hat{r}(2)^2\left([-\cos(\theta)]_0^{\pi/2}\right)(\pi/3) \\ &= \hat{r}\frac{4}{3}\pi \end{aligned}$$

(c) Determine the enclosed volume of the cylinder section.

$$
\begin{aligned}
dV &= r^2 \sin(\theta) dr d\theta d\phi \\
V &= \left(\int_0^2 r^2 dr\right)\left(\int_0^{\pi/2} \sin(\theta) d\theta\right)\left(\int_{\pi/6}^{\pi/2} d\phi\right) \\
&= \left(\left[\frac{r^3}{3}\right]_0^2\right)\left([-\cos(\theta)]_0^{\pi/2}\right)\left(\frac{\pi}{3}\right) \\
&= \frac{8}{9}\pi
\end{aligned}
$$

Learning Objective: Applications of Line, Surface Area, and Volume Integral

Problem 4. Determine the total charge of the following scenarios.

(a) Determine the total charge on a line for $0 \le y \le 5$ cm, where the line charge density is described as $\rho_L = 12y^2$ mC/cm.

$$
Q = \int_0^5 (12y^2) dy = \left[4y^3\right]_0^5 = 500 \text{ mC}
$$

(b) Determine the total charge on the surface of a cylinder for $\rho = 4$ cm, $0 \le z \le 2$ cm, where the surface charge density is described as $\rho_S = \rho z^2$ nC/cm^2.

$$
\begin{aligned}
dq &= (\rho z^2)\rho d\phi dz \\
Q &= \rho^2 \left(\int_0^{2\pi} d\phi\right)\left(\int_0^2 z^2 dz\right) \\
&= (4)^2 (2\pi)\left(\left[\frac{z^3}{3}\right]_0^2\right) \\
&= \frac{256}{3}\pi \text{ nC}
\end{aligned}
$$

(c) Determine the total charge enclosed by a sphere for $r = 4$ cm, where the volume charge density is described as $\rho_v = \frac{10}{r \cdot \sin(\theta)}$ C/cm^3

$$
\begin{aligned}
dV &= \left(\frac{10}{r \sin(\theta)}\right) r^2 \sin(\theta) dr d\theta d\phi \\
V &= \left(\int_0^4 10 r dr\right)\left(\int_0^{\pi} d\theta\right)\left(\int_0^{2\pi} d\phi\right) \\
&= \left(10\left[\frac{r^2}{2}\right]_0^4\right)(\pi)(2\pi) \\
&= 160\pi^2 \text{ C}
\end{aligned}
$$

Table 1: Summary of vector relations

	Cartesian Coordinates	**Cylindrical Coordinates**	**Spherical Coordinates**
Coordinate variables	x, y, z	ρ, ϕ, z	r, θ, ϕ
Vector Notation A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$
Magnitude $\lvert A\rvert =$	$\sqrt{A_x^2 + A_y^2 + A_z^2}$	$\sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$	$\sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$ for $P_1 = (x_1, y_1, z_1)$	$\hat{\rho}\rho_1 + \hat{\mathbf{z}}z_1$ for $P_1 = (\rho_1, \phi_1, z_1)$	$\hat{\mathbf{r}}r_1$ for $P_1 = (r_1, \theta_1, \phi_1)$
Base vector properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\rho} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\rho} = 0$ $\hat{\rho} \times \hat{\phi} = \hat{\mathbf{z}}$ $\hat{\phi} \times \hat{\mathbf{z}} = \hat{\rho}$ $\hat{\mathbf{z}} \times \hat{\rho} = \hat{\phi}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$ $\hat{\mathbf{r}} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\theta} = \hat{\phi}$ $\hat{\theta} \times \hat{\phi} = \hat{\mathbf{r}}$ $\hat{\phi} \times \hat{\mathbf{r}} = \hat{\theta}$
Dot product A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_\rho B_\rho + A_\phi B_\phi + A_zB_z$	$A_rB_r + A_\theta B_\theta + A_\phi B_\phi$
Cross product A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{\mathbf{z}} \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\theta} & \hat{\phi} \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\rho}\,d\rho + \hat{\phi}\rho\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\theta}r\,d\theta + \hat{\phi}r\sin(\theta)\,d\phi$
Differential surface areas	$d\hat{\mathbf{s}}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\hat{\mathbf{s}}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\hat{\mathbf{s}}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\hat{\mathbf{s}}_\rho = \hat{\rho}\,\rho\,d\phi\,dz$ $d\hat{\mathbf{s}}_\phi = \hat{\phi}\,d\rho\,dz$ $d\hat{\mathbf{s}}_z = \hat{\mathbf{z}}\,\rho\,d\rho\,d\phi$	$d\hat{\mathbf{s}}_r = \hat{\mathbf{r}}\,r^2\sin(\theta)\,d\theta\,d\phi$ $d\hat{\mathbf{s}}_\theta = \hat{\theta}\,r\sin(\theta)\,dr\,d\phi$ $d\hat{\mathbf{s}}_\phi = \hat{\phi}\,r\,dr\,d\theta$
Differential volume $d\mathcal{V}$	$dx\,dy\,dz$	$\rho\,d\rho\,d\phi\,dz$	$r^2\sin(\theta)\,dr\,d\theta\,d\phi$