Commun Nonlinear Sci Numer Simulat 47 (2017) 292–307

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Analytic solution for American strangle options using Laplace–Carson transforms

Myungjoo Kang, Junkee Jeon*, Heejae Han, Somin Lee

Department of Mathematical Sciences, Seoul National University, Seoul 151–747, Republic of Korea

ARTICLE INFO

Article history:
Received 19 April 2016
Revised 27 November 2016
Accepted 28 November 2016
Available online 1 December 2016

Keywords:
American strangle options
Free boundary problems
Laplace–Carson transforms
Numerical Laplace inversion

ABSTRACT

A strangle has been important strategy for options when the trader believes there will be a large movement in the underlying asset but are uncertain of which way the movement will be. In this paper, we derive analytic formula for the price of American strangle options. American strangle options can be mathematically formulated into the free boundary problems involving two early exercise boundaries. By using Laplace–Carson Transform(LCT), we can derive the nonlinear system of equations satisfied by the transformed value of two free boundaries. We then solve this nonlinear system using Newton's method and finally get the free boundaries and option values using numerical Laplace inversion techniques. We also derive the Greeks for the American strangle options as well as the value of perpetual American strangle options. Furthermore, we present various graphs for the free boundaries and option values according to the change of parameters.

1. Introduction

With the development of the option pricing theory and the computational finance, there have been various option contracts to reduce risk. Barrier options [1], dynamic fund protection [2], and turbo warrant options [3] are typical example of option contracts proposed to reduce risk for option holders. There also have been many option strategies for reducing risk. One of the most famous strategies is strangle and straddle. A strangle is the options strategy where the investor holds both call and put options with different strike prices but with the same maturity and underlying asset. When a large movement of the underlying asset is expected, but uncertain of which way the movement will be, buying or selling them can reduce the risk proposed by having a single call or put option. A straddle is a strangle option whose strike price for call part is equal to that of put part.

In this paper, we study American strangle options. An American option is an option contract of which option holders can exercise their rights at any instant before maturity. Kim [4] derived the integral equation satisfied by the value of American put options. Peskir [5] researched on the analytic formula and the regularity of Russian options with finite time horizon, which can be classified as American path-dependent options. Jeon et al. [6] derived the general analytic solution for Russian options with finite time horizon by solving the inhomogeneous Black-Scholes equation with Neumann boundary conditions. Broadie and Detemple [7] studied American option pricing with multiple underlying assets. Jeon et al. [8] did analytic pricing for American quanto lookback options, which involve the stock price as well as the foreign currency exchange rate. Espe-

* corresponding author.
E-mail address: hollman@snu.ac.kr (J. Jeon).

http://dx.doi.org/10.1016/j.cnsns.2016.11.024
1007-5704/

cially for American strangle options, Chiarella and Ziogas [9] used Fourier transform method to derive the integral equation satisfied by American strangle option values. Also, Abdou and Moraux [10] priced American hybrid strangle options.

Generally, American type options have no closed form solution. Therefore, the valuation of American options heavily depends on numerical methods. Longstaff and Schwartz [11] proposed the Monte-Carlo simulation based on the least square approach for valuing American options. Ju [12] numerically solved the integral equation satisfied by the value of American options by multi piece exponential method. Carr [13] priced American put options using randomization approach, which has a great deal with Laplace–Carlson transform. Recently, many researchers have used the Laplace–Carlson transform in option pricing problems. Kimura priced Russian options with finite time horizon [14], American fractional floating strike lookback options [15], and American continuous installment options [16] using the Laplace–Carlson transform. Furthermore, Wong and Zhao [17] valued American options under the CEV model by Laplace–Carlson transform as well.

In this paper, we derive the value of American strangle options using Laplace–Carlson transform. The option value can be represented as the solution of PDE by using the variational inequality approach. After taking Laplace–Carlson transform, we convert the PDE into a simple Euler differential equation. By imposing the continuity and the smooth pasting conditions for the value function, we can derive nonlinear algebraic equations for the two early exercise free boundaries for American strangle options as well as the value of American strangle options in frequency domain. We numerically solve such nonlinear algebraic equations by using the Newton's method. Then, numerical Laplace inversion algorithms are applied to recover the option value in original time domain. Furthermore, we derive the Greeks for American strangle options as well. The option value obtained by using Laplace–Carlson transform indicates that it is more exact than the value obtained by other existing methods.

This paper consists of as follows. In Section 2, we formulate the free boundary problem of American strangle options using the standard variational inequality approach. In Section 3, we obtain the integral equation satisfied by the value of American strangle options. In Section 4, we derive the general solution for the value of American strangle options in frequency domain and nonlinear algebraic equation whose solution is the value of two early exercise free boundaries in frequency domain. From the general solution, we also derive the value of perpetual American strangle options. Furthermore, we obtain the Greek values of American strangle options in frequency domain. In Section 5, we finally get the two free boundaries and the value of American strangle options using numerical Laplace inversion algorithms to the result of Section 4. We also present the plot of the free boundaries and option values according to the change of various parameters.

2. Model formulation

In this paper, we adopt the usual assumptions for the Black–Scholes frameworks. Let $(S_t)_{t\,\in\,[0,\,T]}$ be the value of the underlying asset under a risk-neutral measure $\mathbb{P}$. For S_0 given, we assume that the stochastic dynamics of S_t is described by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \tag{2.1}$$

Here r represent the risk-free interest rate, $q > 0$ is the continuous dividend rate, and $\sigma > 0$ is the constant volatility of S_t. Also, $(W_t)_{t\,\geq\,0}$ is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by $\{W_t\}_{t\,\geq\,0}$.

Let $V(t, s)$ be the American strangle option price at time $t \in [0, T]$ when the value of underlying asset is s. In the absence of arbitrage opportunities, the value $V(t, s)$ is a solution of the optimal stopping problem

$$V(t,s) = \sup_{t\leq\eta_t\leq T} \mathbb{E}[e^{-r(\eta_t - t)}((K_1 - S_{\eta_t})^+ + (S_{\eta_t} - K_2)^+) \mid S_t = s] \tag{2.2}$$

where η_t is stopping time of the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and K_1, K_2 denote the strike price of put and call options respectively. All of the conditional expectation is calculated under the risk-neutral measure $\mathbb{P}$. **It should be noted that in this paper we consider *American strangle*, for which $K_1 < K_2$.**

Using the variational inequality approach, $V(t, s)$ satisfies the following linear complimentary form

$$\begin{aligned}
&\min\{\mathcal{L}V(t,s), V(t,s) - (K_1 - s)^+ - (s - K_2)^+\} = 0\\
&V(T,s) = (K_1 - s)^+ + (s - K_2)^+\\
&s > 0, 0 \leq t \leq T
\end{aligned}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2}{\partial s^2} + (r-q)s\frac{\partial}{\partial s} - r\mathcal{I}$$

Here, $\mathcal{I}$ is a identity operator.

Let

$$\mathcal{D} = \{(t,s) \in [0,T] \times \mathbb{R}_+\}$$

be the whole region, and let $\mathcal{E}$ and $\mathcal{C}$ denote the exercise region and the continuation region, respectively. In terms of the value function $V(t, s)$, the exercise region $\mathcal{E}$ is expressed by

$$\mathcal{E} = \{(t,s) \mid V(t,s) = (K_1 - s)^+ + (s - K_2)^+\}.$$

The continuation region $\mathcal{C}$ is the complement of $\mathcal{E}$ in $\mathcal{D}$, defined by

$$\mathcal{C} = \{(t,s) \mid V(t,s) > (K_1 - s)^+ + (s - K_2)^+\}.$$

For the exercise region $\mathcal{E}$, there exist two subregions $\mathcal{E}^A$, $\mathcal{E}^B$ of $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}^A \cup \mathcal{E}^B$, and

$$\mathcal{E}^A = \{(t,s) \mid V(t,s) = (K_1 - s)^+ > 0\}$$

$$\mathcal{E}^B = \{(t,s) \mid V(t,s) = (s - K_2)^+ > 0\}$$

Since $K_1 < K_2$, $\mathcal{E}^A \cap \mathcal{E}^B = \emptyset$.

The two boundaries that separate $\mathcal{E}$ from $\mathcal{C}$ are defined to be the *free boundaries*, and are expressed by

$$A(t) = \sup\{s \in \mathbb{R}_+ \mid (t,s) \in \mathcal{E}^A\}$$

$$B(t) = \inf\{s \in \mathbb{R}_+ \mid (t,s) \in \mathcal{E}^B\},$$

respectively.

In addition, at the free boundaries $s = A(t)$ and $s = B(t)$,

$$V(t,A(t)) = K_1 - A(t), \frac{\partial V}{\partial s}(t,A(t)) = -1 \tag{2.3}$$

$$V(t,B(t)) = B(t) - K_2, \frac{\partial V}{\partial s}(t,B(t)) = 1 \tag{2.4}$$

The boundary condition (2.4) is called *the smooth pasting condition*.

In terms of $A(t)$ and $B(t)$, the continuation region $\mathcal{C}$ can be represented by

$$\mathcal{C} = \{(t,s) \mid A(t) < s < B(t)\}.$$

For the free boundaries $A(t)$, $B(t)$, the optimal stopping problem (2.2) is equivalent to solve the following Black-scholes PDE.

$$\mathcal{L}V(t,s) = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, A(t) < s < B(t), \tag{2.5}$$

with the smooth pasting condition (2.4) and the terminal condition

$$V(t,s) = (K_1 - s)^+ + (s - K_2)^+ \tag{2.6}$$

For notational convenience, we introduce the time-reversed process

$$\tilde{V}(\tau,s) = V(T-\tau,s) = V(t,s), \tilde{A}(\tau) = A(T-\tau), \textit{ and } \tilde{B}(\tau) = B(T-\tau),$$

where $\tau = T - t$ is the remaining time to expiry.

3. Integral equation representation of American strangle options

The value of an American strangle options with an early exercise policy can be divided into two parts: the value of European strangle options and the early exercise premium of American strangle options. Let $V_E(t, s)$ and $V_P(t, s)$ denote the value function of European strangle options and early exercise premium of American strangle options, respectively. Then,

$$V(t,s) = V_E(t,s) + V_P(t,s)$$

Also, we consider the time reversed process of $V_E(t, s)$ and $V_P(t, s)$ as follows.

$$\tilde{V}_E(\tau,s) = V_E(T-\tau,s), \tilde{V}_P(\tau,s) = V_P(T-\tau,s)$$

The value of American strangle options can be represented by a solution of the integral equation. Chiarella and Ziogas in [9] derived the integral equation satisfied by the value of American strangle options using Fourier transform as follows.

Theorem 3.1 (Integral equation for the value of American strangle options)**.** *The value $\tilde{V}(\tau,s)$ of the time reversed process of American strangle options, has following premium decomposition*

$$\tilde{V}(\tau,s) = \tilde{V}_E(\tau,s) + \tilde{V}_P(\tau,s)$$

where

$$\tilde{V}_E(\tau,s) = K_1e^{-r\tau}\mathcal{N}(-d_2(\tau,s,K_1)) - se^{-q\tau}\mathcal{N}(-d_1(\tau,s,K_1)) \tag{3.1}$$

$$+se^{-q\tau}\mathcal{N}(d_1(\tau,s,K_2)) - K_2e^{-r\tau}\mathcal{N}(d_2(\tau,s,K_2)) \tag{3.2}$$

and

$$\tilde{V}_P(\tau,s) = \int_0^\tau [K_1 r e^{-r(\tau-\eta)}\mathcal{N}(-d_2(\tau-\eta,s,\tilde{A}(\eta))) - sqe^{-q(\tau-\eta)}\mathcal{N}(-d_1(\tau-\eta,s,\tilde{A}(\eta))) \tag{3.3}$$

$$+sqe^{-q(\tau-\eta)}\mathcal{N}(d_1(\tau-\eta,s,\tilde{B}(\eta))) - K_2 re^{-r(\tau-\eta)}\mathcal{N}(d_2(\tau-\eta,s,\tilde{B}(\eta)))]d\eta \tag{3.4}$$

with the free boundary $\tilde{A}(\tau)$ and $\tilde{B}(\tau)$ satisfying

$$K_1 - \tilde{A}(\tau) = \tilde{V}(\tau,\tilde{A}(\tau))$$
$$\tilde{B}(\tau) - K_2 = \tilde{V}(\tau,\tilde{B}(\tau))$$

and

$$d_1(\tau,s,x) = \frac{\log(s) - \log(x) + (r-q+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$$d_2(\tau,s,x) = \frac{\log(s) - \log(x) - (r-q+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

$\mathcal{N}(\cdot)$ is a standard normal cumulative distribution function.

Proof. The detailed proof for Theorem 3.1 is explained in [9]. □

Furthermore, Chiarella and Ziogas [9] described limiting behavior of the free boundaries as time to maturity goes to zero.

Corollary 3.2 (Limiting behavior of the free boundaries of American strangle options)**.** *The initial values of time-reversed early exercise boundaries are given by*

$$\tilde{A}(0) = K_1 \cdot \min\left(1, \frac{r}{q}\right), \tilde{B}(0) = K_2 \cdot \max\left(1, \frac{r}{q}\right)$$

4. The Valuation of American strangle options in frequency domain

In this section, we use the Laplace–Carlson transform(LCT) to get the option values for American strangle options. The LCT is essentially not different from the standard Laplace transform. However, the analysis of American strangle options is more convenient with LCT due to the initial and final value theorem for LT.

For $\lambda > 0$, define the Laplace–Carlson transform of the time-reversed American strangle option price $\tilde{V}(\tau,s)$ by

$$V^*(\lambda,s) = \int_0^\infty \tilde{V}(\tau,s)\lambda e^{-\lambda\tau}d\tau := \mathcal{LC}[\tilde{V}(\tau,s)](\lambda).$$

and also let

$$A^*(\lambda) = \mathcal{LC}[\tilde{A}(\tau)], \text{ and } B^*(\lambda) = \mathcal{LC}[\tilde{B}(\tau)].$$

Note that the standard Laplace transform(LT) of $\tilde{V}(\tau,s)$ defined by

$$\hat{V}(\lambda,s) = \int_0^\infty e^{-\lambda\tau}\tilde{V}(\tau,s)d\lambda,$$

it is clear that

$$V^*(\lambda,s) = \lambda\hat{V}(\lambda,s).$$

Hence, for arbitrary function $f(\tau)$ of τ, by the initial-value theorem of LT,

$$f(0) = \lim_{\lambda\uparrow\infty}\lambda\mathcal{L}[f](\lambda) = \lim_{\lambda\uparrow\infty}\mathcal{LC}[f](\lambda) = \lim_{\lambda\uparrow\infty} f^*(\lambda) \tag{4.1}$$

and by the final-value theorem of LT,

$$f(\infty) = \lim_{\lambda\downarrow 0}\lambda\mathcal{L}[f](\lambda) = \lim_{\lambda\downarrow 0}\mathcal{LC}[f](\lambda) = \lim_{\lambda\downarrow 0} f^*(\lambda) \tag{4.2}$$

We first get the value of European strangle options in frequency domain, i.e. $V^*(\lambda, s)$.

4.1. European strangle options

The value of European strangle options $V_E(t, s)$ can be decomposed into two parts, in a similar way to European vanilla call V_E^C and put V_E^P.

$$V_E(t,s) = V_E^C(t,s) + V_E^P(t,s)$$

Then, $V_E^C(t,s)$ and V_E^P satisfy the following Black–Scholes PDEs :

$$\begin{aligned} &\mathcal{L}V_E^C(t,s) = 0,\ V_E^C(T,s) = (s-K_2)^+ \\ &\lim_{s\downarrow 0} V_E^C(t,s) = 0, \lim_{s\uparrow\infty} \frac{\partial V_E^C}{\partial s} < \infty \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\mathcal{L}V_P^C(t,s) = 0, V_E^P(T,s) = (K_1-s)^+ \\ &\lim_{s\downarrow 0} V_E^P(t,s) = K_1 e^{-r(T-t)}, \lim_{s\uparrow\infty} V_E^P(t,s) = 0 \end{aligned} \tag{4.4}$$

For the time-reversed value $\tilde{V}_E(\tau,s) = V_E(T-\tau,s), (\tau \geq 0)$, define its LCT by $V_E^*(\lambda,s) = \mathcal{LC}[\tilde{V}_E(\tau,s)]$. Then, we have the following theorem.

Theorem 4.1 (Value of European strangle options in frequency domain)**.**

$$V_E^*(\lambda,s) = \begin{cases} -b_1\left(\dfrac{s}{K_1}\right)^{\xi_1} - c_1\left(\dfrac{s}{K_2}\right)^{\xi_1} + \dfrac{\lambda K_1}{\lambda+r} - \dfrac{\lambda s}{\lambda+q}, & 0 < s \leq K_1 \\ b_2\left(\dfrac{s}{K_1}\right)^{\xi_2} - c_1\left(\dfrac{s}{K_2}\right)^{\xi_1}, & K_1 < s < K_2 \\ b_2\left(\dfrac{s}{K_1}\right)^{\xi_2} + c_2\left(\dfrac{s}{K_2}\right)^{\xi_2} + \dfrac{\lambda s}{\lambda+q} - \dfrac{\lambda K_2}{\lambda+r}, & K_2 \leq s < \infty \end{cases} \tag{4.5}$$

where for $i = 1, 2$,

$$b_i = b_i(\lambda) = \frac{(-1)^i}{\xi_1-\xi_2}\cdot\frac{\lambda K_1}{\lambda+q}\left(1 - \frac{r-q}{\lambda+r}\xi_{3-i}\right), c_i = c_i(\lambda) = \frac{(-1)^i}{\xi_1-\xi_2}\cdot\frac{\lambda K_2}{\lambda+q}\left(1 - \frac{r-q}{\lambda+r}\xi_{3-i}\right)$$

and the parameter $\xi_1 = \xi_1(\lambda) > 1$ and $\xi_2 = \xi(\lambda) < 0$ are two real roots of the quadratic equation

$$\frac{1}{2}\sigma^2\xi^2 + \left(r - q - \frac{1}{2}\sigma^2\right)\xi - (\lambda+r) = 0$$

Proof. Let $V_E^{C*}(\lambda,s) = \mathcal{LC}[\tilde{V}_E^C(\tau,s)](\lambda)$. Then, from (4.3), V_E^{C*} satisfies the following ordinary differential equation(ODE).

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2V_E^{C*}}{ds^2} + (r-q)s\frac{dV_E^{C*}}{ds} - (\lambda+r)V_E^{C*} + \lambda(s-K_2)^+ = 0, s > 0 \tag{4.6}$$

with boundary condition

$$\lim_{s\downarrow 0} V_E^{C*}(\lambda,s) = 0 \text{and} \lim_{s\uparrow\infty}\frac{dV_E^{C*}}{ds} < \infty, \tag{4.7}$$

By using the continuity conditions of V_E^{C*} and its first derivative at $s = K_2$, we can obtain a solution of (4.6) with the boundary condition (4.7),

$$V_E^{C*}(\lambda,s) = \begin{cases} -c_1\left(\dfrac{s}{K_2}\right)^{\xi_1}, & s \leq K_2 \\ c_2\left(\dfrac{s}{K_2}\right)^{\xi_2} + \dfrac{\lambda s}{\lambda+q} - \dfrac{\lambda K_2}{\lambda+r}, & K_2 < s \end{cases} \tag{4.8}$$

Similarly, $V_E^{P*}(\lambda,s) := \mathcal{LC}[\tilde{V}_E^P(\tau,s)](\lambda)$ is the solution of the following ODE:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2V_E^{P*}}{ds^2} + (r-q)s\frac{dV_E^{P*}}{ds} - (\lambda+r)V_E^{P*} + \lambda(K_1-s)^+ = 0,\ s > 0 \tag{4.9}$$

with boundary condition

$$\lim_{s\downarrow 0} V_E^{P*}(\lambda,s) = \frac{\lambda K_1}{\lambda+r} \text{ and } \lim_{s\uparrow\infty} V_E^{P*}(\lambda,s) = 0, \tag{4.10}$$

By using the continuity conditions of V_E^{P*} and its first derivative at $s = K_1$,

$$V_E^{P*}(\lambda, s) = \begin{cases} -b_1\left(\dfrac{s}{K_1}\right)^{\xi_1} + \dfrac{\lambda K_1}{\lambda + r} - \dfrac{\lambda s}{\lambda + q}, & s \le K_1 \\ b_2\left(\dfrac{s}{K_1}\right)^{\xi_2}, & K_1 < s \end{cases} \tag{4.11}$$

Since $V_E{}^*(\lambda, s) = V_E^{C*}(\lambda, s) + V_E^{P*}(\lambda, s)$, we have proved the desired result (4.5). □

4.2. American strangle options

The value of American strangle options in frequency domain can be also obtained by a similar procedure used in Section 4.1. Especially, we can obtain two free boundaries in frequency domain using smooth pasting conditions.

Theorem 4.2 (Premium decomposition of American strangle options). *In frequency domain, the value of the American strangle option $V^*(\lambda, s)$ is represented by*

$$V^*(\lambda, s) = \begin{cases} K_1 - s, & 0 < s \le A^*(\lambda) \\ V_E^*(\lambda, s) + V_P^*(\lambda, s), & A^*(\lambda) < s < B^*(\lambda) \\ s - K_2, & B^*(\lambda) \le s < \infty \end{cases} \tag{4.12}$$

where $V_P^*(\lambda, s)$ is defined by

$$\begin{aligned} V_P^*(\lambda, s) = & \frac{\xi_1\xi_2}{\xi_1 - \xi_2} \cdot \left(\frac{qA^*}{\lambda + q} - \frac{rK_1}{\lambda + r}\right) \cdot \left\{\frac{1}{\xi_1}\left(\frac{s}{A^*}\right)^{\xi_1} - \frac{1}{\xi_2}\left(\frac{s}{A^*}\right)^{\xi_2}\right\} + \frac{1}{\xi_1 - \xi_2} \cdot \frac{qA^*}{\lambda + q} \cdot \left\{\left(\frac{s}{A^*}\right)^{\xi_2} - \left(\frac{s}{A^*}\right)^{\xi_1}\right\} \\ & + b_1\left(\frac{s}{K_1}\right)^{\xi_1} + c_1\left(\frac{s}{K_2}\right)^{\xi_1} \end{aligned} \tag{4.13}$$

and A^* and B^* satisfy the following a pair of equations

$$\begin{aligned} & \frac{\xi_1\xi_2}{\xi_1 - \xi_2} \cdot \frac{qA^*}{\lambda + q} \cdot \left\{\frac{1}{\xi_2}\left(\frac{B^*}{A^*}\right)^{\xi_1} - \frac{1}{\xi_1}\left(\frac{B^*}{A^*}\right)^{\xi_2}\right\} + \frac{\xi_1\xi_2}{\xi_1 - \xi_2} \cdot \left(\frac{qA^*}{\lambda + q} - \frac{rK_1}{\lambda + r}\right) \cdot \left\{\left(\frac{B^*}{A^*}\right)^{\xi_2} - \left(\frac{B^*}{A^*}\right)^{\xi_1}\right\} = h(B^*) - \frac{qB^*}{\lambda + q} \\ & \frac{\xi_1\xi_2}{\xi_1 - \xi_2} \cdot \frac{qB^*}{\lambda + q} \cdot \left\{\frac{1}{\xi_2}\left(\frac{A^*}{B^*}\right)^{\xi_1} - \frac{1}{\xi_1}\left(\frac{A^*}{B^*}\right)^{\xi_2}\right\} + \frac{\xi_1\xi_2}{\xi_1 - \xi_2} \cdot \left(\frac{qB^*}{\lambda + q} - \frac{rK_2}{\lambda + r}\right) \cdot \left\{\left(\frac{A^*}{B^*}\right)^{\xi_2} - \left(\frac{A^*}{B^*}\right)^{\xi_1}\right\} \\ & \quad = h(A^*) - \frac{qA^*}{\lambda + q} \end{aligned} \tag{4.14}$$

where

$$h(x) = \xi_1 b_1\left(\frac{x}{K_1}\right)^{\xi_1} + \xi_2 b_2\left(\frac{x}{K_1}\right)^{\xi_2} + \xi_1 c_1\left(\frac{x}{K_2}\right)^{\xi_1} + \xi_2 c_2\left(\frac{x}{K_2}\right)^{\xi_2}.$$

The constants b_1, b_2, c_1 and c_2 are defined in Theorem 4.1.

Proof. Taking LCTs on Eq. (2.5) give us

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2V^*}{ds^2} + (r - q)s\frac{dV^*}{ds} - (\lambda + r)V^* + \lambda(K_1 - s)^+ + \lambda(s - K_2)^+ = 0, A^*(\lambda) < s < B^*(\lambda)$$

together with the boundary conditions

$$\begin{cases} \lim\limits_{s \downarrow A^*} V^*(\lambda, s) = K_1 - A^*, \lim\limits_{s \downarrow A^*} \dfrac{dV^*}{ds} = -1 \\ \lim\limits_{s \uparrow B^*} V^*(\lambda, s) = B^* - K_2, \lim\limits_{s \uparrow B^*} \dfrac{dV^*}{ds} = 1 \end{cases} \tag{4.15}$$

We can let the general solution for (4.15) in the form of

$$V^*(\lambda,s)=\begin{cases}\displaystyle\sum_{i=1}^{2}\alpha_i\left(\frac{s}{K_1}\right)^{\xi_i}+\frac{\lambda K_1}{\lambda+r}-\frac{\lambda s}{\lambda+q}, & A^*(\lambda)<s\le K_1\\ \displaystyle\sum_{i=1}^{2}(\alpha_i+\beta_i)\left(\frac{s}{K_1}\right)^{\xi_i}, & K_1<s<K_2\\ \displaystyle\sum_{i=1}^{2}(\alpha_i+\beta_i)\left(\frac{s}{K_1}\right)^{\xi_i}+\sum_{i=1}^{2}\gamma_i\left(\frac{s}{K_2}\right)^{\xi_i}+\frac{\lambda s}{\lambda+q}-\frac{\lambda K_2}{\lambda+r}, & K_2\le s<B^*(\lambda)\end{cases}\tag{4.16}$$

Based on the continuity of the solution $V^*(\lambda,\ s)$ and its first derivative at $s=K_1,\ \beta_i(i=1,2)$ is given by

$$\beta_i=\frac{(-1)^i}{\xi_1-\xi_2}\cdot\frac{\lambda K_1}{\lambda+q}\left(1-\frac{r-q}{\lambda+r}\xi_{3-i}\right)$$

Similarly, from the continuity conditions of $V^*(\lambda,\ s)$ and its first derivative at $s=K_2,\ \gamma_i(i=1,2)$ is expressed by

$$\gamma_i=\frac{(-1)^i}{\xi_1-\xi_2}\cdot\frac{\lambda K_2}{\lambda+q}\left(1-\frac{r-q}{\lambda+r}\xi_{3-i}\right)$$

Hence, from the definition of b_i and c_i in Theorem 4.1,

$$\beta_i=b_i,\gamma_i=c_i,\ (i=1,2),$$

Also, the value of $\alpha_i(i=1,2)$ can be determined from the *smooth pasting condition.*

$$\frac{dV^*}{ds}(\lambda,A^*(\lambda))=-1,\ \frac{dV^*}{ds}(\lambda,B^*(\lambda))=1$$

The value of $\alpha_i(i=1,2)$ is as follows.

$$\begin{aligned}\alpha_1&=\frac{-\dfrac{A^*q}{\lambda+q}\dfrac{K_1{}^{\xi_1}}{\xi_1}\cdot(B^*)^{\xi_2}-\dfrac{B^*q}{\lambda+q}\dfrac{K_1{}^{\xi_1}}{\xi_1}\cdot(A^*)^{\xi_2}+h(B^*)\dfrac{K_1{}^{\xi_1}}{\xi_1}(A^*)^{\xi_2}}{(A^*)^{\xi_1}(B^*)^{\xi_2}-(A^*)^{\xi_2}(B^*)^{\xi_1}}\\ \alpha_2&=\frac{\dfrac{A^*q}{\lambda+q}\dfrac{K_1{}^{\xi_2}}{\xi_2}\cdot(B^*)^{\xi_1}+\dfrac{B^*q}{\lambda+q}\dfrac{K_1{}^{\xi_2}}{\xi_2}\cdot(A^*)^{\xi_1}-h(B^*)\dfrac{K_1{}^{\xi_2}}{\xi_2}(A^*)^{\xi_1}}{(A^*)^{\xi_1}(B^*)^{\xi_2}-(A^*)^{\xi_2}(B^*)^{\xi_1}}\end{aligned}\tag{4.17}$$

where

$$h(x)=\xi_1b_1\left(\frac{x}{K_1}\right)^{\xi_1}+\xi_2b_2\left(\frac{x}{K_1}\right)^{\xi_2}+\xi_1c_1\left(\frac{x}{K_2}\right)^{\xi_1}+\xi_2c_2\left(\frac{x}{K_2}\right)^{\xi_2}.$$

Finally, we can use the condition $\lim_{s\downarrow A^*}V^*(\lambda,s)=K_1-A^*$ and $\lim_{s\uparrow B^*}V^*(\lambda,s)=B^*-K_2$ in (4.15). Then,

$$\begin{aligned}K_1-A^*&=\sum_{i=1}^{2}\alpha_i\left(\frac{A^*}{K_1}\right)^{\xi_i}+\frac{\lambda K_1}{\lambda+r}-\frac{\lambda A^*}{\lambda+q}\\ B^*-K_2&=\sum_{i=1}^{2}(\alpha_i+b_i)\left(\frac{B^*}{K_1}\right)^{\xi_i}+\sum_{i=1}^{2}c_i\left(\frac{B^*}{K_2}\right)^{\xi_i}+\frac{\lambda B^*}{\lambda+q}-\frac{\lambda K_2}{\lambda+r}\end{aligned}\tag{4.18}$$

By rearranging the terms in (4.18),

$$\begin{aligned}&\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left\{\frac{1}{\xi_2}\left(\frac{B^*}{A^*}\right)^{\xi_1}-\frac{1}{\xi_1}\left(\frac{B^*}{A^*}\right)^{\xi_2}\right\}+\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left\{\left(\frac{B^*}{A^*}\right)^{\xi_2}-\left(\frac{B^*}{A^*}\right)^{\xi_1}\right\}=h(B^*)-\frac{qB^*}{\lambda+q}\\ &\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\frac{qB^*}{\lambda+q}\cdot\left\{\frac{1}{\xi_2}\left(\frac{A^*}{B^*}\right)^{\xi_1}-\frac{1}{\xi_1}\left(\frac{A^*}{B^*}\right)^{\xi_2}\right\}+\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qB^*}{\lambda+q}-\frac{rK_2}{\lambda+r}\right)\cdot\left\{\left(\frac{A^*}{B^*}\right)^{\xi_2}-\left(\frac{A^*}{B^*}\right)^{\xi_1}\right\}\\ &\quad=h(A^*)-\frac{qA^*}{\lambda+q}\end{aligned}\tag{4.19}$$

For the second equation, we used the following relationship between $h(A^*)$ and $h(B^*)$, which can be established by expanding right hand side and cancelling the terms of which the power of $\frac{A^*}{B^*}, \frac{B^*}{K_i}$ do not match.

$$(\xi_1-\xi_2)h(A^*)=h(B^*)\left[\xi_1\left(\frac{A^*}{B^*}\right)^{\xi_1}-\xi_2\left(\frac{A^*}{B^*}\right)^{\xi_2})\right]+\xi_1\xi_2\left[\left(\frac{A^*}{B^*}\right)^{\xi_1}-\left(\frac{A^*}{B^*}\right)^{\xi_2}\right)\right]\left[\sum_{i=1}^{2}b_i\left(\frac{B^*}{K_1}\right)^{\xi_i}+\sum_{i=1}^{2}c_i\left(\frac{B^*}{K_2}\right)^{\xi_i}\right]$$

By Theorem 4.1, $V^*(\lambda,\ s)$ can be decomposed into

$$V^*(\lambda,s)=V_E^*(\lambda,s)+V_P^*(\lambda,s) \tag{4.20}$$

where

$$V_P^*(\lambda,s)=\sum_{i=1}^{2}\alpha_i\left(\frac{s}{K_1}\right)^{\xi_i}+b_1\left(\frac{s}{K_1}\right)^{\xi_1}+c_1\left(\frac{s}{K_2}\right)^{\xi_1} \tag{4.21}$$

Now, we can compute $V_P^*(\lambda,s)$ using (4.17) and (4.21).

To make the formula in simpler form, we explicitly compute first two terms of $V_P^*(\lambda,s)$.

Note that

$$\begin{aligned}\alpha_1\left(\frac{s}{K_1}\right)^{\xi_1} &= \frac{-\frac{A^*q}{\lambda+q}\frac{s^{\xi_1}}{\xi_1}(B^*)^{\xi_2}+\frac{s^{\xi_1}}{\xi_1}(A^*)^{\xi_2}\left(h(B^*)-\frac{qB^*}{\lambda+q}\right)}{(A^*)^{\xi_1}(B^*)^{\xi_2}-(A^*)^{\xi_2}(B^*)^{\xi_1}}\\ &= \frac{-\frac{1}{\xi_1}\frac{A^*q}{\lambda+q}\left(\frac{B^*}{A^*}\right)^{\xi_2}+\frac{1}{\xi_1}\left(h(B^*)-\frac{qB^*}{\lambda+q}\right)}{\left(\frac{B^*}{A^*}\right)^{\xi_2}-\left(\frac{B^*}{A^*}\right)^{\xi_1}}\left(\frac{s}{A^*}\right)^{\xi_1}\\ &= \frac{\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left(\frac{s}{A^*}\right)^{\xi_1}-\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left(\frac{s}{A^*}\right)^{\xi_1}\end{aligned}$$

For the last equality, we used (4.19).

Similarly,

$$\alpha_2\left(\frac{s}{K_1}\right)^{\xi_2}=\frac{-\xi_1}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left(\frac{s}{A^*}\right)^{\xi_2}+\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left(\frac{s}{A^*}\right)^{\xi_2}$$

Therefore,

$$\begin{aligned}V_P^*(\lambda,s)=&\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left\{\frac{1}{\xi_1}\left(\frac{s}{A^*}\right)^{\xi_1}-\frac{1}{\xi_2}\left(\frac{s}{A^*}\right)^{\xi_2}\right\}+\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left\{\left(\frac{s}{A^*}\right)^{\xi_2}-\left(\frac{s}{A^*}\right)^{\xi_1}\right\}\\ &+b_1\left(\frac{s}{K_1}\right)^{\xi_1}+c_1\left(\frac{s}{K_2}\right)^{\xi_1}\end{aligned}$$

□

4.3. Perpetual American strangle options

In this subsection, we consider the perpetual American strangle options. There are many studies on perpetual American strangle options. For example, in [18], Yuan-Chung Sheu and Ming-Chi Chang considered the problem of perpetual American strangle options under a jump-diffusion model. Due to the final value theorem for Laplace transform, it is easy to derive pricing formula for the perpetual American strangle options once we have the value of American strangle options in frequency domain.

Theorem 4.3 (Value of the Perpetual American Strangle Options)**.** *For $s\in(A_\infty,\ B_\infty)$, the value $V_\infty(s)$ of perpetual American strangle option is given by*

$$V_\infty(s)=\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot(A_\infty-K_1)\cdot\left\{\frac{1}{\xi_1^0}\left(\frac{s}{A_\infty}\right)^{\xi_1^0}-\frac{1}{\xi_2^0}\left(\frac{s}{A_\infty}\right)^{\xi_2^0}\right\}+\frac{1}{\xi_1^0-\xi_2^0}\cdot A_\infty\cdot\left\{\left(\frac{s}{A_\infty}\right)^{\xi_2^0}-\left(\frac{s}{A_\infty}\right)^{\xi_1^0}\right\} \tag{4.22}$$

where $\xi_1^0>1$ and $\xi_2^0<0$ are two real roots of the quadratic equation

$$\frac{1}{2}\sigma^2\xi^2+(r-q-\frac{1}{2}\sigma^2)\xi-r=0$$

and the constant A_∞ and B_∞ are the solutions of the following equations.

$$\begin{aligned}
&\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot A_\infty \cdot\left\{\frac{1}{\xi_2^0}\left(\frac{B_\infty}{A_\infty}\right)^{\xi_1^0}-\frac{1}{\xi_1^0}\left(\frac{B_\infty}{A_\infty}\right)^{\xi_2^0}\right\}+\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot(A_\infty-K_1)\cdot\left\{\left(\frac{B_\infty}{A_\infty}\right)^{\xi_2^0}-\left(\frac{B_\infty}{A_\infty}\right)^{\xi_1^0}\right\}+B_\infty=0\\
&\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot B_\infty\cdot\left\{\frac{1}{\xi_2^0}\left(\frac{A_\infty}{B_\infty}\right)^{\xi_1^0}-\frac{1}{\xi_1^0}\left(\frac{A_\infty}{B_\infty}\right)^{\xi_2^0}\right\}+\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot(B_\infty-K_2)\cdot\left\{\left(\frac{A_\infty}{B_\infty}\right)^{\xi_2^0}-\left(\frac{A_\infty}{B_\infty}\right)^{\xi_1^0}\right\}+A_\infty=0
\end{aligned}\tag{4.23}$$

Proof. By the final-value theorem of LT in (4.2), the value of American strangle options is given by

$$V_\infty(s)=\lim_{\lambda\downarrow 0}V^*(\lambda,s)$$

For b_i and c_i $(i=1,2)$ defined in Theorem 4.1,

$$\lim_{\lambda\downarrow 0}b_i=\lim_{\lambda\downarrow 0}c_i=0$$

Hence, for $A_\infty=\lim_{\lambda\downarrow 0}A^*(\lambda)$, $B_\infty=\lim_{\lambda\downarrow 0}B^*(\lambda)$ and $\xi_i^0=\lim_{\lambda\downarrow 0}\xi_i(\lambda)$,

$$V_\infty(s)=\lim_{\lambda\downarrow 0}V^*(\lambda,s)=\frac{\xi_1^0\xi_2^0}{\xi_1^0-\xi_2^0}\cdot(A_\infty-K_1)\cdot\left\{\frac{1}{\xi_1^0}\left(\frac{s}{A_\infty}\right)^{\xi_1^0}-\frac{1}{\xi_2^0}\left(\frac{s}{A_\infty}\right)^{\xi_2^0}\right\}+\frac{1}{\xi_1^0-\xi_2^0}\cdot A_\infty\cdot\left\{\left(\frac{s}{A_\infty}\right)^{\xi_2^0}-\left(\frac{s}{A_\infty}\right)^{\xi_1^0}\right\}$$

where $A_\infty<s<B_\infty$. Clearly, (4.23) holds by letting $\lambda\to 0^+$ in (4.14). □

4.4. Greeks of American strangle options

Computing the Greeks of options is as important as computing the value of options. Δ, Γ, Θ are important parameters for determining sensibility of the option price, which should be considered when one proposes hedging strategies for options. Using the value of American strangle options in frequency domain, we can easily derive the Greeks for American strangle options in frequency domain as well.

In Theorem 4.2, the LCT of American strangle options $V^*(\lambda,s)$ is decomposed into

$$V^*(\lambda,s)=V_E^*(\lambda,s)+V_P^*(\lambda,s)$$

Hence, the corresponding decomposition of American strangle options is given by

$$V(t,s)=\tilde V(\tau,s)=V_E(\tau,s)+\mathcal{LC}^{-1}[V_P^*(\lambda,s)]$$

Since we know the value of $V_P^*(\lambda,s)$ from Theorem 4.2, we can derive the Greeks of American strangle options easily.

Theorem 4.4 (Greeks of American strangle options)**.** *The Greeks of American strangle options is given by*

$$\begin{aligned}
\Delta_V&=\frac{\partial V}{\partial s}=\Delta_{V_E}+\mathcal{LC}^{-1}\left[\Delta_{V_P^*}\right]\\
\Gamma_V&=\frac{\partial^2 V}{\partial s^2}=\Gamma_{V_E}+\mathcal{LC}^{-1}\left[\Gamma_{V_P^*}\right]\\
\Theta_V&=-\frac{\partial V}{\partial \tau}=\Theta_{V_E}+\mathcal{LC}^{-1}\left[\Theta_{V_P^*}\right]
\end{aligned}$$

where d_1, d_2 are defined in Theorem 3.1 and $n(\cdot)$ is a standard normal probability density function , $N(\cdot)$ is a standard normal cumulative density function. Also,

$$\begin{aligned}
\Delta_{V_E}&=e^{-q\tau}(\mathcal{N}(d_1(\tau,s,K_2))-\mathcal{N}(-d_1(\tau,s,K_1)))\\
\Gamma_{V_E}&=\frac{e^{-q\tau}}{s\sigma\sqrt{\tau}}(n(d_1(\tau,s,K_1))+n(d_1(\tau,s,K_2)))\\
\Theta_{V_E}&=-\frac{s\sigma e^{-q\tau}}{\sqrt{\tau}}n(d_1(\tau,s,K_2))+qse^{-q\tau}\mathcal{N}(d_1(\tau,s,K_2))-rK_2e^{-r\tau}\mathcal{N}(d_2(\tau,s,K_2))\\
&\quad+qse^{-q\tau}\mathcal{N}(-d_1(\tau,s,K_1))+rK_2e^{-r\tau}\mathcal{N}(-d_2(\tau,s,K_1))
\end{aligned}$$

denote the Greeks for European strangle options and

$$\begin{aligned}
\Delta_{V_P^*}=\frac{1}{s}\Bigg[&\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left\{\left(\frac{s}{A^*}\right)^{\xi_1}-\left(\frac{s}{A^*}\right)^{\xi_2}\right\}+\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left\{\xi_2\left(\frac{s}{A^*}\right)^{\xi_2}-\xi_1\left(\frac{s}{A^*}\right)^{\xi_1}\right\}\\
&+\xi_1 b_1\left(\frac{s}{K_1}\right)^{\xi_1}+\xi_2 c_1\left(\frac{s}{K_2}\right)^{\xi_1}\Bigg]
\end{aligned}$$

$$\Gamma_{V_P^*} = \frac{1}{s^2}\left[\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left\{\xi_1\left(\frac{s}{A^*}\right)^{\xi_1}-\xi_2\left(\frac{s}{A^*}\right)^{\xi_2}\right\}+\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left\{\xi_2(\xi_2-1)\left(\frac{s}{A^*}\right)^{\xi_2}\right.\right.$$
$$\left.\left.-\xi_1(\xi_1-1)\left(\frac{s}{A^*}\right)^{\xi_1}\right\}+\xi_1(\xi_1-1)b_1\left(\frac{s}{K_1}\right)^{\xi_1}+\xi_2(\xi_2-1)c_1\left(\frac{s}{K_2}\right)^{\xi_1}\right]$$
$$\Theta_{V_P^*} = -\lambda\left[\frac{\xi_1\xi_2}{\xi_1-\xi_2}\cdot\left(\frac{qA^*}{\lambda+q}-\frac{rK_1}{\lambda+r}\right)\cdot\left\{\frac{1}{\xi_1}\left(\frac{s}{A^*}\right)^{\xi_1}-\frac{1}{\xi_2}\left(\frac{s}{A^*}\right)^{\xi_2}\right\}+\frac{1}{\xi_1-\xi_2}\cdot\frac{qA^*}{\lambda+q}\cdot\left\{\left(\frac{s}{A^*}\right)^{\xi_2}-\left(\frac{s}{A^*}\right)^{\xi_1}\right\}\right.$$
$$\left.+b_1\left(\frac{s}{K_1}\right)^{\xi_1}+c_1\left(\frac{s}{K_2}\right)^{\xi_1}\right]$$

Proof. The Greeks of European strangle option can be obtained from (3.2) in Theorem 3.1. Also,

$$\Delta_{V_P^*} = \frac{\partial V_P^*}{\partial s}, \Gamma_{V_P^*} = \frac{\partial^2 V_P^*}{\partial s^2}, \Theta_{V_P^*} = -\frac{\partial V_P^*}{\partial \tau}$$

Also,

$$\Theta_{V_P^*} = -\frac{\partial V_P^*}{\partial \tau} = -\mathcal{LC}\left[\frac{\partial \tilde{V}_P}{\partial \tau}\right] = -\lambda\mathcal{L}\left[\frac{\partial \tilde{V}_P}{\partial \tau}\right] = -\lambda\left\{V_P^*(\lambda,s)-V_P^*(0,s)\right\}$$

Since

$$\tilde{V}_P(0,s) = V_P(T,s) = V(T,s)-V_E(T,s) = 0,$$

We get

$$\Theta_{V_P^*} = -\lambda V_P^*(\lambda,s).$$

□

5. Numerical experiment

We derived the value, the free boundaries, and the Greeks for American strangle options in frequency domain in previous section. To obtain the option price and the two early exercise free boundaries in time domain, we need to take Laplace–Carlson inversion of these quantities. However, one can see that the two free boundaries $A^*(\lambda)$, $B^*(\lambda)$ cannot be expressed explicitly as a function of λ from the system of equations (4.14). Therefore, we need a numerical method for Laplace–Carlson inversion.

There are plenty of Laplace inversion algorithms, and they are widely used in various applications including option pricing problems. In this paper, we use the Gaver–Stehfest method, the Euler method, and the Talbot method.

The Gaver–Stehfest method has an advantage in that it does not involve any complex-valued computations. Gaver first proposed an algorithm for computing Laplace inversion asymptotically by defining a recursive sequence which converges to Laplace inversion. However, the convergence of this recursive sequence is very slow. Accordingly, Stehfest proposed the acceleration scheme using Richardson extrapolation scheme. The interested reader can refer to Abate and Whitt [19] for details.

The Euler and Talbot algorithm are widely used for computing the Bromwich inversion integral. Since the Laplace inversion can be considered as a special case of the Bromwich inversion, they apply to the Laplace inversion as well. Different from the Gaver–Stehfest method, they both involve complex-valued integrals. The Euler method approximates the Bromwich inversion integral by a Fourier-series method, and accelerate the convergence using Euler summation. The Talbot method computes the Bromwich inversion integral by cleverly deforming the integral contour. One can refer to Abate et al. [20] and Talbot [21] for details of each algorithm.

The following Table 1,2,3 shows the value of American strangle options according to the change of parameters. We simulate three cases which is based on the relative value of r, q. For case 1, we choose $r = 0.1, q = 0.05$ which corresponds to the case of $r > q$. For case 2, we choose $r = q = 0.05$ which corresponds to the case of $r = q$. For the last case, we choose $r = 0.03, q = 0.05$ which corresponds to the case of $r < q$. In all cases, we fix $K_1 = 1, \sigma = 0.2$ and we obtain option values numerically according to the change of K_2, s, T.

For each table, the 4th column Binomial(20,000) is a benchmark value of American strangle options obtained by applying binomial tree method with 20,000 equally spaced time grid. The 5th column Monte Carlo(100,000) is the option value obtained by the Monte-Carlo simulation using least-square approach of Longstaff(2001) 100,000 times. Finally, 6,7,8 th column is the value of American strangle options obtained by applying three numerical Laplace inversion methods we described above. For example, Talbot(16) denotes the Talbot method with 16 quadrature points.

The result of Table 1,2,3 shows that the valuation of American strangle options based on numerical Laplace inversion method is fairly exact. Also, from the computed RSME, we conclude that it is more exact than the value obtained by using the Monte-Carlo method.

Table 1
Values of American strangle optionS with $r = 0.10, q = 0.05, \sigma = 0.2$ and $K_1 = 1$.

K_2	T(yr)	s	Binomial (20,000)	Monte Carlo (100,000)	(G-S) (6)	Euler (6)	Talbot (16)
	0.3333	1	0.05632	0.05420	0.05558	0.05556	0.05558
		1.1	0.06957	0.06654	0.06942	0.06944	0.06942
		1.2	0.13144	0.12726	0.13141	0.13143	0.13141
		1.3	0.21810	0.21355	0.21810	0.21813	0.21810
1.1	0.5833	1	0.08208	0.07886	0.08079	0.08080	0.08079
		1.1	0.09945	0.09539	0.09906	0.09908	0.09906
		1.2	0.15573	0.14930	0.15563	0.15565	0.15563
		1.3	0.23503	0.22712	0.23499	0.23503	0.23499
	1	1	0.11605	0.11056	0.11395	0.11397	0.11395
		1.1	0.13803	0.13185	0.13716	0.13719	0.13716
		1.2	0.19041	0.18125	0.19008	0.19010	0.19008
		1.3	0.26256	0.25152	0.26243	0.26247	0.26243
	0.3333	1	0.03972	0.03918	0.03914	0.03915	0.03914
		1.1	0.01682	0.01646	0.01669	0.01669	0.01669
		1.2	0.02791	0.02684	0.02789	0.02790	0.02789
		1.3	0.06989	0.06891	0.06988	0.06990	0.06988
1.3	0.5833	1	0.05330	0.05259	0.05238	0.05239	0.05238
		1.1	0.03630	0.03507	0.03601	0.03602	0.03601
		1.2	0.05269	0.05134	0.05260	0.05262	0.05260
		1.3	0.09714	0.09399	0.09712	0.09714	0.09712
	1	1	0.07481	0.07337	0.07333	0.07335	0.07333
		1.1	0.06709	0.06501	0.06649	0.06650	0.06649
		1.2	0.08910	0.08570	0.08886	0.08888	0.08886
		1.3	0.13490	0.12947	0.13480	0.13483	0.13480
	0.3333	1	0.03896	0.03875	0.03839	0.03839	0.03839
		1.1	0.01098	0.01083	0.01085	0.01086	0.01085
		1.2	0.00439	0.00434	0.00437	0.00438	0.00437
		1.3	0.01122	0.01086	0.01122	0.01123	0.01122
1.5	0.5833	1	0.04910	0.04857	0.04825	0.04826	0.04825
		1.1	0.02128	0.02107	0.02100	0.02100	0.02100
		1.2	0.01580	0.01544	0.01572	0.01573	0.01572
		1.3	0.02878	0.02832	0.02876	0.02877	0.02876
	1	1	0.06270	0.06250	0.06151	0.06152	0.06150
		1.1	0.03967	0.03900	0.03915	0.03916	0.03915
		1.2	0.03970	0.03871	0.03947	0.03949	0.03950
		1.3	0.05960	0.05776	0.05951	0.05953	0.05951
		RSME		3.8238e-03	6.3523e-04	6.2559e-04	6.3534e-04

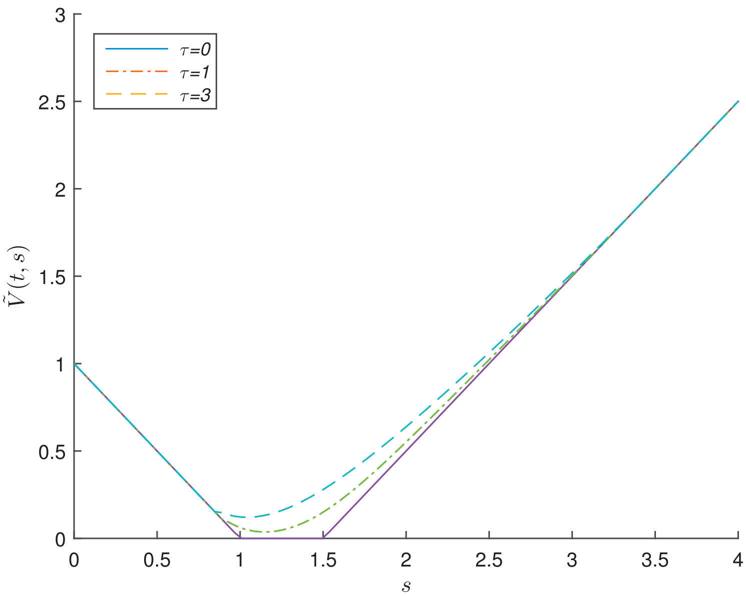


Fig. 1. Values of $\tilde{V}(\tau, s)$ with respect to s ($K_1 = 1, K_2 = 1.5$ and $r = 0.1, q = 0.05, \sigma = 0.2$).

Table 2
Values of American strangle optionS with $r = 0.05, q = 0.05, \sigma = 0.2$ and $K_1 = 1$.

K_2	T(yr)	s	Binomial (20,000)	Monte Carlo (100,000)	(G-S) (6)	Euler (6)	Talbot (16)
	0.3333	1	0.05911	0.05735	0.05901	0.05902	0.05901
		1.1	0.06365	0.06174	0.06354	0.06355	0.06354
		1.2	0.11898	0.11746	0.11862	0.11901	0.11862
		1.3	0.20342	0.20269	0.20003	0.20003	0.20000
1.1	0.5833	1	0.08477	0.08163	0.08451	0.08453	0.08451
		1.1	0.09072	0.08782	0.09044	0.09046	0.09044
		1.2	0.13765	0.13595	0.13703	0.13704	0.13703
		1.3	0.21220	0.21033	0.21087	0.21382	0.21479
	1	1	0.11776	0.11317	0.11717	0.11719	0.11717
		1.1	0.12542	0.12197	0.12479	0.12481	0.12479
		1.2	0.16550	0.16179	0.16443	0.16446	0.16443
		1.3	0.22991	0.22647	0.22800	0.22691	0.22785
	0.3333	1	0.04593	0.45412	0.04585	0.04585	0.04585
		1.1	0.01819	0.01786	0.01817	0.01817	0.01817
		1.2	0.02356	0.02327	0.02353	0.02353	0.02353
		1.3	0.05955	0.05944	0.05944	0.05946	0.05944
1.3	0.5833	1	0.06244	0.06120	0.06226	0.06227	0.06225
		1.1	0.03763	0.03684	0.03756	0.03757	0.03756
		1.2	0.04463	0.04408	0.04454	0.04455	0.04454
		1.3	0.08028	0.07985	0.08004	0.08006	0.08004
	1	1	0.08627	0.08412	0.08587	0.08589	0.08587
		1.1	0.06692	0.06479	0.06669	0.06670	0.06669
		1.2	0.07562	0.07479	0.07534	0.07536	0.07534
		1.3	0.10925	0.10775	0.10875	0.10877	0.10875
	0.3333	1	0.04542	0.04554	0.04534	0.04535	0.04534
		1.1	0.01356	0.01385	0.01384	0.01385	0.01384
		1.2	0.00458	0.00460	0.00458	0.00458	0.00458
		1.3	0.00871	0.00853	0.00871	0.00872	0.00871
1.5	0.5833	1	0.05970	0.05976	0.05952	0.05953	0.05952
		1.1	0.02678	0.02629	0.02673	0.02674	0.02673
		1.2	0.01562	0.01563	0.01560	0.01560	0.01560
		1.3	0.02243	0.02218	0.02239	0.02240	0.02239
	1	1	0.07846	0.07773	0.07810	0.07811	0.07810
		1.1	0.04753	0.04701	0.04736	0.04737	0.04736
		1.2	0.03756	0.03702	0.03745	0.03746	0.03745
		1.3	0.04676	0.04637	0.04661	0.04663	0.04661
		RSME		1.7465e-03	5.2914e-04	8.5296e-04	8.5309e-04

Fig. 1 is the graph for option values according to the value of underlying asset. Here the parameters are $K_1 = 1, K_2 = 1.15, r = 0.1, q = 0.05, \sigma = 0.2$. As you can see from the figure, the option value increases as the time to maturity increases.

Fig. 2 represents two early exercise free boundaries for American strangle options when the parameters are $K_1 = 1, K_2 = 1.1, r = q = 0.05$. Here, we vary the volatility σ. The region surrounded by two free boundaries is the continuation region for American strangle options and the complement of the region is the exercise region. As you can see from the figure, the exercise region becomes wider as the volatility σ increases.

Fig. 3 represents the behavior of free boundary for each side according to the value of call strike price K_2. Here, the parameters are chosen as $r = 0.1, q = 0.05$ when the parameters are $K_1 = 1, K_2 = 1.1, r = q = 0.05$. In Fig. 3(a), we plot the call side free boundary $B^*(\lambda)$, and the put side free boundary $A^*(\lambda)$ is plotted in Fig. 3(b). As you can see from the figure, the n becomes wider as the volatility σ increases.

Fig. 4 also represents the behavior of free boundary for each side according to the value of call strike price K_2. Here, the parameters are chosen as $r = 0.05, q = 0.05$. As in Fig. 3, the value of free boundaries increase as the call side strike price K_2 increases.

6. Concluding remarks

In this paper, we obtained the value, the free boundary, and the Greeks of American strangle options using the Laplace–Carlson transform method. An American strangle option involve buying or selling of call and put options of the same underlying asset simultaneously. Hence, it has two free boundaries, one for call options and the other for put options. Using the variational inequality approach, we get the PDE satisfied by the value of American strangle options. Then we apply the Laplace–Carlson transform to convert the PDE problem into the ODE problem. By solving the ODE and using the smooth

Table 3
Values of American strangle options with $r = 0.03, q = 0.05, \sigma = 0.2$ and $K_1 = 1$.

K_2	T(yr)	s	Binomial (20,000)	Monte Carlo (100,000)	(G-S) (6)	Euler (6)	Talbot (16)
	0.3333	1	0.06128	0.05843	0.06120	0.06122	0.06120
		1.1	0.06229	0.06044	0.06190	0.06190	0.06190
		1.2	0.11573	0.11510	0.11444	0.11270	0.11428
		1.3	0.20342	0.20269	0.20003	0.20003	0.20000
1.1	0.5833	1	0.08802	0.08346	0.08782	0.08783	0.08782
		1.1	0.08934	0.08586	0.08863	0.08864	0.08863
		1.2	0.13320	0.13154	0.13138	0.13096	0.13136
		1.3	0.20748	0.20612	0.20635	0.20003	0.20000
	1	1	0.12268	0.11695	0.12220	0.12222	0.12220
		1.1	0.12455	0.11970	0.12333	0.12335	0.12333
		1.2	0.16021	0.15749	0.15763	0.15750	0.15763
		1.3	0.22279	0.22048	0.21815	0.22084	0.21751
	0.3333	1	0.04923	0.04807	0.04923	0.04924	0.04923
		1.1	0.01916	0.01881	0.01915	0.01915	0.01915
		1.2	0.02229	0.02194	0.02218	0.02219	0.02218
		1.3	0.05628	0.05584	0.05586	0.05587	0.05586
1.3	0.5833	1	0.06783	0.06558	0.06782	0.06783	0.06782
		1.1	0.03944	0.03815	0.03937	0.03938	0.03937
		1.2	0.04273	0.04201	0.04246	0.04248	0.04246
		1.3	0.07530	0.07529	0.07549	0.07460	0.07459
	1	1	0.09447	0.09088	0.09440	0.09442	0.09440
		1.1	0.07003	0.06681	0.06981	0.06983	0.06981
		1.2	0.07342	0.07180	0.07286	0.07288	0.07286
		1.3	0.10260	0.10143	0.10141	0.10143	0.10141
	0.3333	1	0.04880	0.04845	0.04880	0.04881	0.04880
		1.1	0.01532	0.01498	0.01532	0.01533	0.01532
		1.2	0.00482	0.00489	0.00482	0.00482	0.00481
		1.3	0.00795	0.00783	0.00791	0.00792	0.00791
1.5	0.5833	1	0.06552	0.06448	0.06552	0.06554	0.06552
		1.1	0.02990	0.02941	0.02990	0.02991	0.02990
		1.2	0.01626	0.01620	0.01622	0.01623	0.01622
		1.3	0.02072	0.02079	0.02059	0.02060	0.02059
	1	1	0.08795	0.08590	0.08794	0.08796	0.08794
		1.1	0.05313	0.05158	0.05309	0.05310	0.05309
		1.2	0.03895	0.03792	0.03883	0.03884	0.03883
		1.3	0.04401	0.04350	0.04370	0.04371	0.04370
		RSME		2.1368e-03	1.1776e-04	1.5532e-04	1.6854e-04

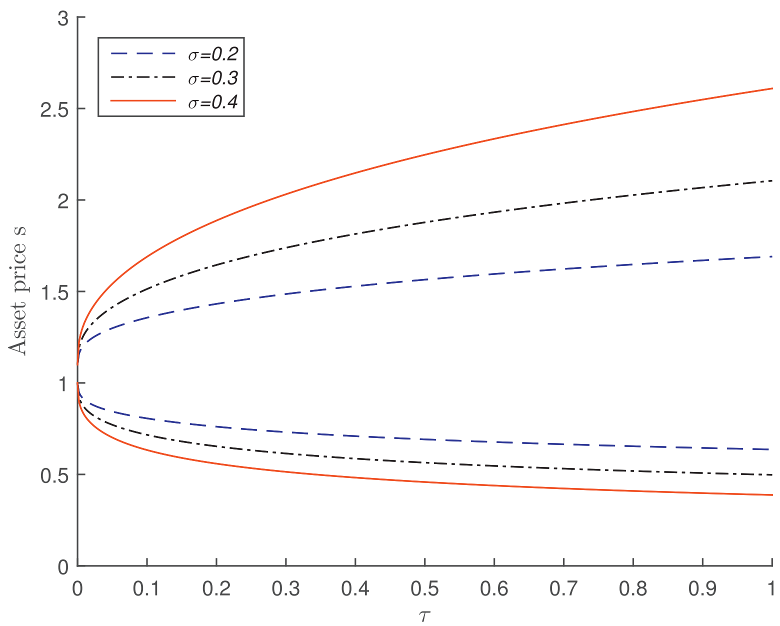
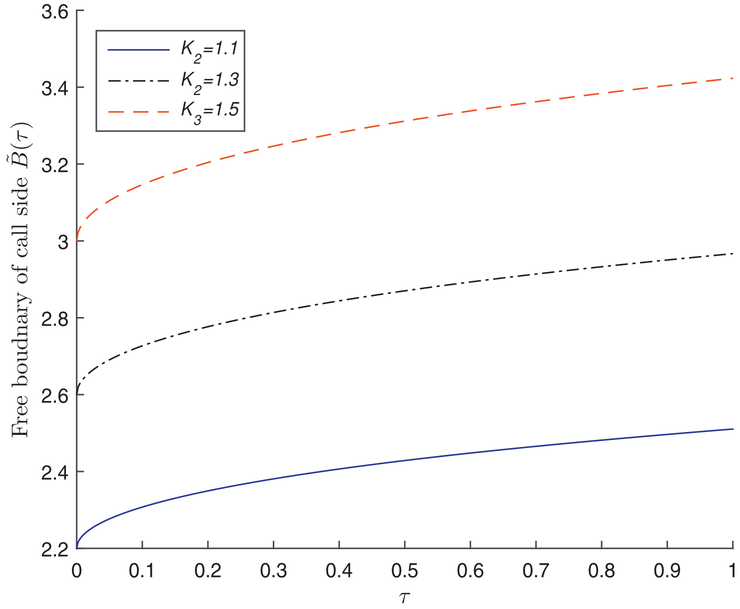
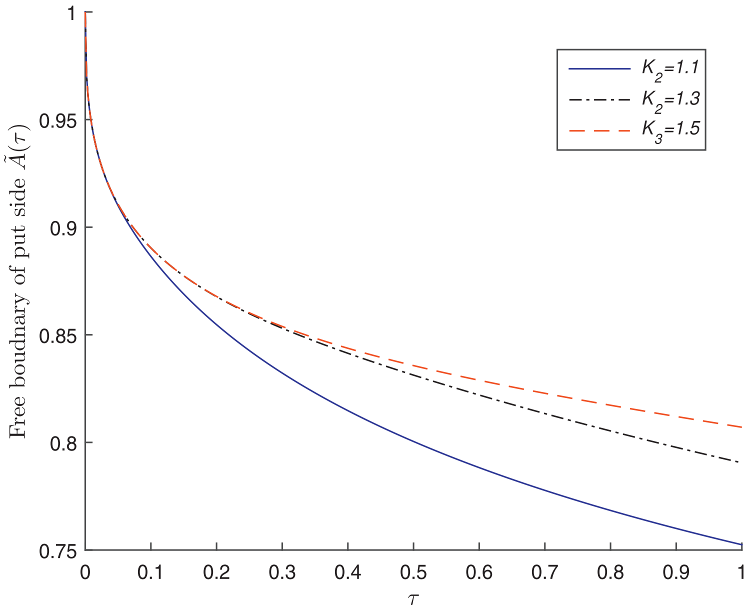


Fig. 2. Free boundaries of American strangle options ($K_1 = 1, K_2 = 1.1$ and $r = 0.05, q = 0.05$).



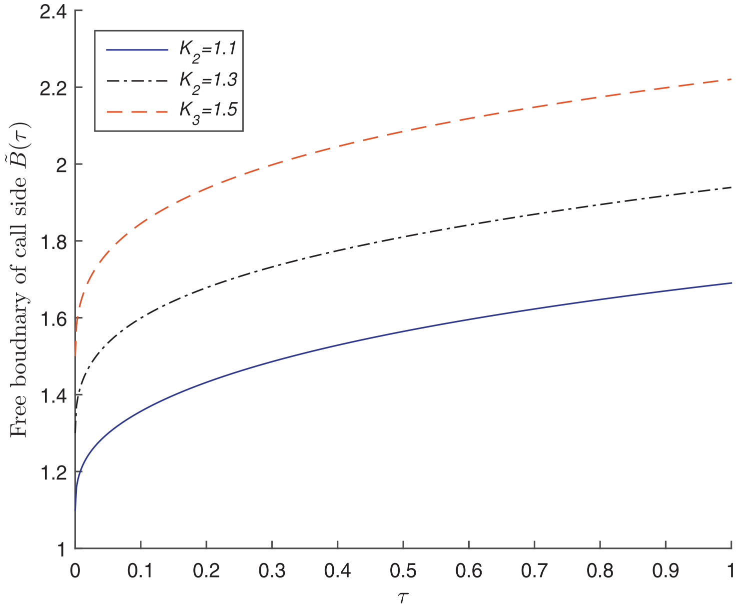
(a) Free boundary of call side



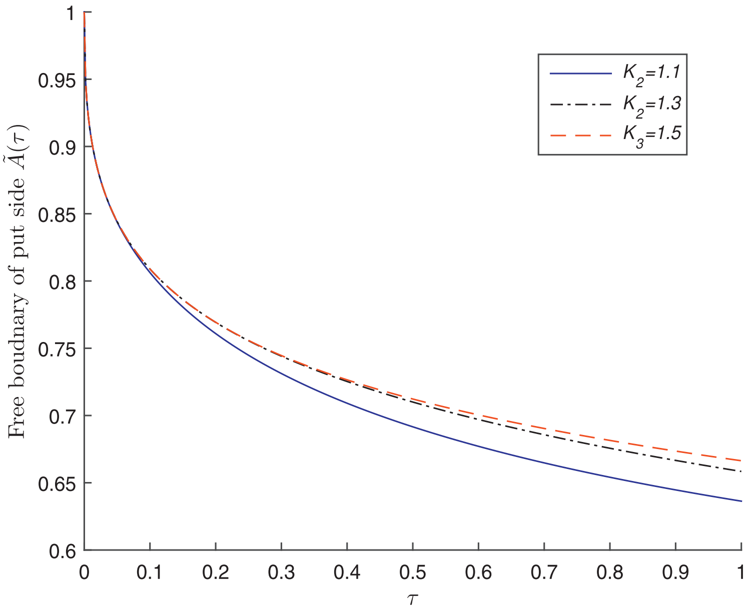
(b) Free boundary of put side

Fig. 3. Free boundaries of American strangle options ($K_1 = 1$ and $r = 0.1, q = 0.05, \sigma = 0.2$).

pasting conditions, we get the nonlinear algebraic equations for the two free boundaries in frequency domain. Nonlinear numerical solver such as Newton method is used for finding the roots of such equations. Finally, we used numerical Laplace inversion algorithms such as the Gaver-stehfest method, the Euler method, and the Talbot method to get the value of two free boundaries. Then the value of American strangle options are computed from the value of free boundaries. Our results show that the valuation of American strangle options using Laplace–Carlson transform method is more accurate than using the Monte-Carlo simulation.



(a) Free boundary of call side



(b) Free boundary of put side

Fig. 4. Free boundaries of American strangle options ($K_1 = 1$ and $r = 0.05, q = 0.05, \sigma = 0.2$).

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the paper. The researcher of Myungjoo Kang is supported by NRF Grant(2014R1A2A1A10050531, 2015R1A5A1009350), and the researcher of Junkee Jeon is supported by BK21 PLUS SNU Mathematical Sciences Division.

References

[1] Reiner E, Rubinstein M. Breaking down the barriers. Risk 1991;4(8):28–35.
[2] Gerber HU, Pafumi G. Pricing dynamic investment fund protection. N Am Actuar J 2000;4(2):28–41.
[3] Wong HY, Chan CM. Turbo warrants under stochastic volatility. Quant Financ 2008;8(7). 739–51
[4] Kim I. The analytic valuation of american options. Rev Financ Stud 1990;3. 547–72
[5] Peskir G. The russian option: finite horizon. Finance Stoch 2005;9. 251–67
[6] Jeon J, Han H, Kim H, Kang M. An integral equation representation approach for valuing russian options with a finite time horizon. Commun Nonlinear Sci 2016;36:496–516.
[7] Broadie M, Detemple J. Valuation of american options on multiple assets. Math Finance 1997;7. 241–86
[8] Jeon J., Han H. Analytic valuation of american maximum exchange rate quanto lookback options. submitted.
[9] Chiarella C, Ziogas A. Evaluation of american strangles. J Econ Dyn Control 2009;29:31–62.
[10] Abdou Sl, Moraux F. Pricing and hedging american and hybrid strangles with finite maturity. J Bank Financ 2016;62. 112–25
[11] Longstaff FA, Schwartz ES. Valuing american options by simulation: a simple least-squares approach. Rev Financ Stud 2001;14. 113–47
[12] Ju N. Pricing an american option by approximating its early exercise boundary as a multipiece exponential function. Rev Financ Stud 1998;11. 627–46
[13] Carr P. Randomization and the american put. Rev Financ Stud 1998;11(3):597–626.
[14] Kimura T. Valuing finite-lived russian options. Eur J Oper Res 2008;189(2). 363–74
[15] Kimura T. American fractional lookback options: valuation and premium decomposition. SIAM J Appl Math 2011;71. 517–39
[16] Kimura T. American continuous-installment options: valuation and premium decomposition. SIAM J Appl Math 2009;70. 803–24
[17] Wong HY, Zhao J. Valuing american options under the CEV model by Laplace–Carson transforms. Oper Res Lett 2010;38. 474–81
[18] Sheu YC, Chang MC. Free boundary problems and perpetual american strangles. Quant Finance 2013;08:1149–55.
[19] Abate J, Whitt W. The fourier-series method for inverting transforms of probability distributions. Queueing Syst 1992;10:5–88.
[20] Abate J, Choudhury L, Whitt W. In: Grassman W, editor. An introduction to numerical inversion and its application to probability models. Computational Probability, Kluwer, Boston, MA; 1999. p. 257–323.
[21] Talbot A. The accurate numerical inversion of Laplace transforms. J Inst Maths Appl 1979;23:97–120.