

# Lecture #9: More Functions

## Another Tree Recursion: Hog Dice

- What are the odds of rolling at least  $k$  in hog with  $n$   $s$ -sided dice?  
( $n > 0$  and for us,  $s > 0$  is 4 or 6)

$$\frac{\text{\# rolls of } n \text{ } s\text{-sided dice totaling } \geq k}{s^n}$$

- If  $k \leq 1$ , then clearly the numerator is just  $s^n$ .
- For  $k > 1$ , we consider only rolls that include dice values 2- $s$ , since any 1-die “pigs out.” Let’s call this quantity `rolls2(k, n, s)`.
- The number of ways to score  $\geq k$  is 0 if \_\_\_\_\_. This is a base case.
- If  $n > 0$  then the number of ways to score at least  $k \leq 1$  with  $n$  dice none of which is 1 is \_\_\_\_\_. This is also a base case.
- If the first die comes up  $d$  ( $2 \leq d \leq s$ ), then there are \_\_\_\_\_ ways to throw the remaining  $n - 1$  dice to get a total of at least  $k$  with all  $n$  dice.
- This gives us a tree recursion. How would you modify it for the “swine swap” rule?

## Another Tree Recursion: Hog Dice

- What are the odds of rolling at least  $k$  in hog with  $n$   $s$ -sided dice?  
( $n > 0$  and for us,  $s > 0$  is 4 or 6)

$$\frac{\text{\# rolls of } n \text{ } s\text{-sided dice totaling } \geq k}{s^n}$$

- If  $k \leq 1$ , then clearly the numerator is just  $s^n$ .
- For  $k > 1$ , we consider only rolls that include dice values 2- $s$ , since any 1-die “pigs out.” Let’s call this quantity `rolls2(k, n, s)`.
- The number of ways to score  $\geq k$  is 0 if  $ns < k$ . This is a base case.
- If  $n > 0$  then the number of ways to score at least  $k \leq 1$  with  $n$  dice none of which is 1 is \_\_\_\_\_. This is also a base case.
- If the first die comes up  $d$  ( $2 \leq d \leq s$ ), then there are \_\_\_\_\_ ways to throw the remaining  $n - 1$  dice to get a total of at least  $k$  with all  $n$  dice.
- This gives us a tree recursion. How would you modify it for the “swine swap” rule?

## Another Tree Recursion: Hog Dice

- What are the odds of rolling at least  $k$  in hog with  $n$   $s$ -sided dice?  
( $n > 0$  and for us,  $s > 0$  is 4 or 6)

$$\frac{\# \text{ rolls of } n \text{ } s\text{-sided dice totaling } \geq k}{s^n}$$

- If  $k \leq 1$ , then clearly the numerator is just  $s^n$ .
- For  $k > 1$ , we consider only rolls that include dice values 2- $s$ , since any 1-die “pigs out.” Let’s call this quantity `rolls2(k, n, s)`.
- The number of ways to score  $\geq k$  is 0 if  $ns < k$ . This is a base case.
- If  $n > 0$  then the number of ways to score at least  $k \leq 1$  with  $n$  dice none of which is 1 is  $(s - 1)^n$ . This is also a base case.
- If the first die comes up  $d$  ( $2 \leq d \leq s$ ), then there are \_\_\_\_\_ ways to throw the remaining  $n - 1$  dice to get a total of at least  $k$  with all  $n$  dice.
- This gives us a tree recursion. How would you modify it for the “swine swap” rule?

## Another Tree Recursion: Hog Dice

- What are the odds of rolling at least  $k$  in hog with  $n$   $s$ -sided dice?  
( $n > 0$  and for us,  $s > 0$  is 4 or 6)

$$\frac{\text{\# rolls of } n \text{ } s\text{-sided dice totaling } \geq k}{s^n}$$

- If  $k \leq 1$ , then clearly the numerator is just  $s^n$ .
- For  $k > 1$ , we consider only rolls that include dice values 2- $s$ , since any 1-die “pigs out.” Let’s call this quantity  $\text{rolls2}(k, n, s)$ .
- The number of ways to score  $\geq k$  is 0 if  $ns < k$ . This is a base case.
- If  $n > 0$  then the number of ways to score at least  $k \leq 1$  with  $n$  dice none of which is 1 is  $(s - 1)^n$ . This is also a base case.
- If the first die comes up  $d$  ( $2 \leq d \leq s$ ), then there are  $\frac{\text{rolls2}(k - d, n - 1, s)}{s}$  ways to throw the remaining  $n - 1$  dice to get a total of at least  $k$  with all  $n$  dice.
- This gives us a tree recursion. How would you modify it for the “swine swap” rule?

## Back to Numeric Pairs: Find the Number

- A *numeric pair* is either an empty tuple, an integer, or a tuple consisting of two numeric pairs (slight revision from last time).
- Problem: does the number  $x$  occur in a given numeric pair?

```
def occurs(x, pair):  
    """X occurs at least once in numeric pair PAIR.  
>>> occurs(3, ((2, 1), ((), (3, ())))  
True  
>>> occurs(5, ((2, 1), ((), (3, ())))  
False  
"""  
    if _____:  
        return True  
    elif _____:  
        return False  
    else:  
        return _____
```

- What is the time required by this function proportional to? A:

## Back to Numeric Pairs: Find the Number

- A *numeric pair* is either an empty tuple, an integer, or a tuple consisting of two numeric pairs (slight revision from last time).
- Problem: does the number  $x$  occur in a given numeric pair?

```
def occurs(x, pair):  
    """X occurs at least once in numeric pair PAIR.  
>>> occurs(3, ((2, 1), ((), (3, ())))  
True  
>>> occurs(5, ((2, 1), ((), (3, ())))  
False  
"""  
  
    if x == pair:  
        return True  
    elif _____:  
        return False  
    else:  
        return _____
```

- What is the time required by this function proportional to? A:

## Back to Numeric Pairs: Find the Number

- A *numeric pair* is either an empty tuple, an integer, or a tuple consisting of two numeric pairs (slight revision from last time).
- Problem: does the number  $x$  occur in a given numeric pair?

```
def occurs(x, pair):  
    """X occurs at least once in numeric pair PAIR.  
>>> occurs(3, ((2, 1), ((), (3, ())))  
True  
>>> occurs(5, ((2, 1), ((), (3, ())))  
False  
"""  
  
    if x == pair:  
        return True  
    elif pair == () or type(pair) is int:  
        return False  
    else:  
        return _____
```

- What is the time required by this function proportional to? A:



## Back to Numeric Pairs: Find the Number

- A *numeric pair* is either an empty tuple, an integer, or a tuple consisting of two numeric pairs (slight revision from last time).
- Problem: does the number  $x$  occur in a given numeric pair?

```
def occurs(x, pair):
    """X occurs at least once in numeric pair PAIR.
    >>> occurs(3, ((2, 1), ((), (3, ())))
    True
    >>> occurs(5, ((2, 1), ((), (3, ())))
    False
    """
    if x == pair:
        return True
    elif pair == () or type(pair) is int:
        return False
    else:
        return occurs(x, pair[0]) or occurs(x, pair[1])
```

- What is the time required by this function proportional to? A:

## Back to Numeric Pairs: Find the Number

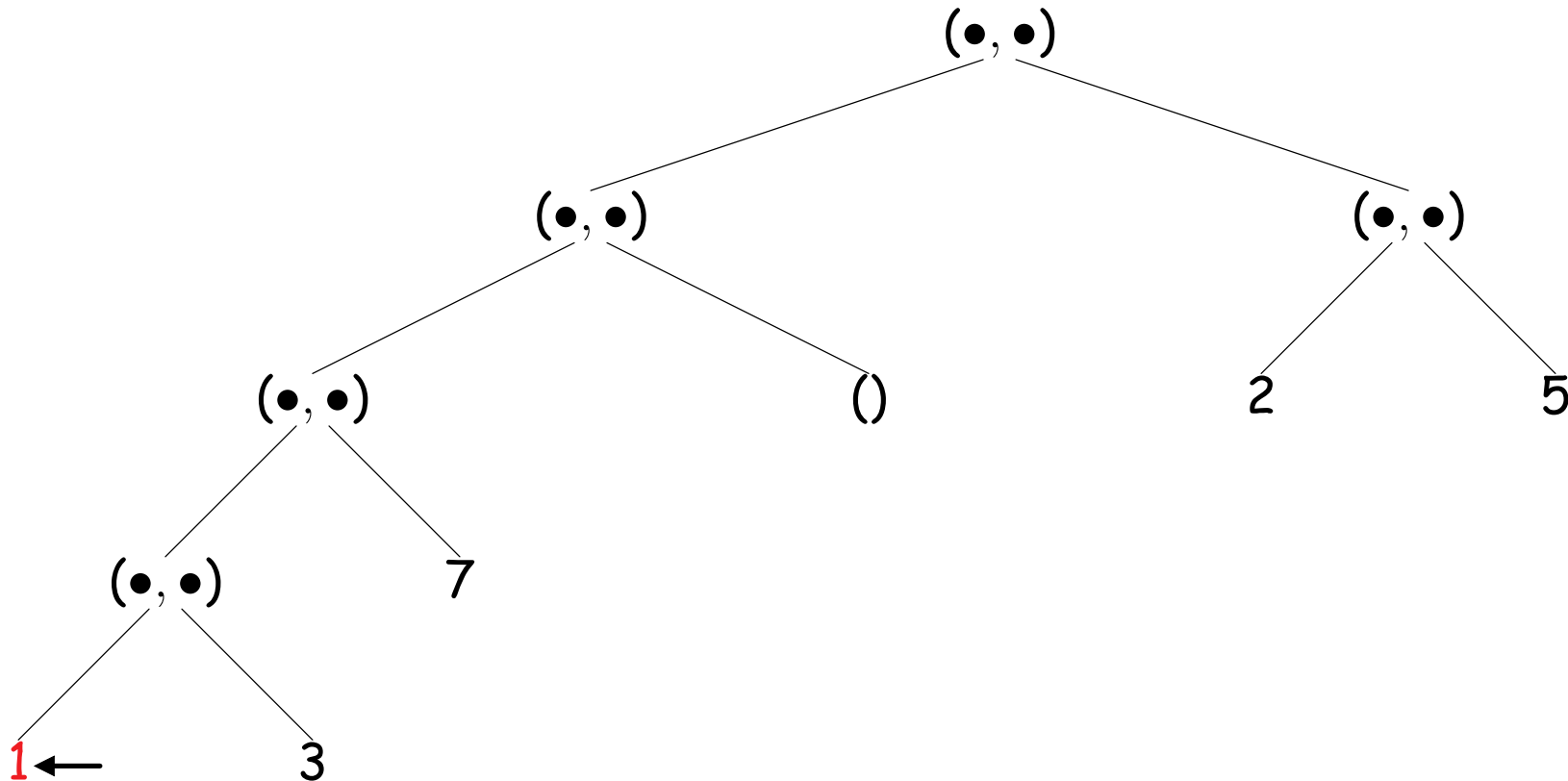
- A *numeric pair* is either an empty tuple, an integer, or a tuple consisting of two numeric pairs (slight revision from last time).
- Problem: does the number  $x$  occur in a given numeric pair?

```
def occurs(x, pair):
    """X occurs at least once in numeric pair PAIR.
    >>> occurs(3, ((2, 1), ((), (3, ())))
    True
    >>> occurs(5, ((2, 1), ((), (3, ())))
    False
    """
    if x == pair:
        return True
    elif pair == () or type(pair) is int:
        return False
    else:
        return occurs(x, pair[0]) or occurs(x, pair[1])
```

- What is the time required by this function proportional to? A:  
The total number of tuples and integers in pair.

# Numeric Pairs: First Leaf

- A *leaf* in a numeric pair is the empty tuple or an integer.
- Define the *first leaf* as the leftmost leaf in the Python expression that denotes a tree.
- Example: the first leaf of  $((((1, 3), 7), ()), (2, 5))$  is 1:



# First Leaf Code

```
def first_leaf(pair):  
    """The first leaf in PAIR, reading left to right.  
>>> first_leaf(())  
()  
>>> first_leaf(5)  
5  
>>> first_leaf(((3, ()), (2, 1)), ()))  
3  
>>> first_leaf(((((), 3), (2, 1)), ()))  
()  
"""  
if _____:  
    return pair  
else:  
    return _____
```

What kind of a recursive process is this? A: \_\_\_\_\_

# First Leaf Code

```
def first_leaf(pair):  
    """The first leaf in PAIR, reading left to right.  
>>> first_leaf(())  
()  
>>> first_leaf(5)  
5  
>>> first_leaf(((3, ()), (2, 1)), ()))  
3  
>>> first_leaf(((((), 3), (2, 1)), ()))  
()  
"""  
  
if type(pair) is int or pair == ():  
    return pair  
else:  
    return _____
```

What kind of a recursive process is this? A: \_\_\_\_\_

# First Leaf Code

```
def first_leaf(pair):  
    """The first leaf in PAIR, reading left to right.  
    >>> first_leaf(())  
    ()  
    >>> first_leaf(5)  
    5  
    >>> first_leaf(((3, ()), (2, 1)), ()))  
    3  
    >>> first_leaf(((((), 3), (2, 1)), ()))  
    ()  
    """  
    if type(pair) is int or pair == ():  
        return pair  
    else:  
        return first_leaf(pair[0])
```

What kind of a recursive process is this? A: \_\_\_\_\_

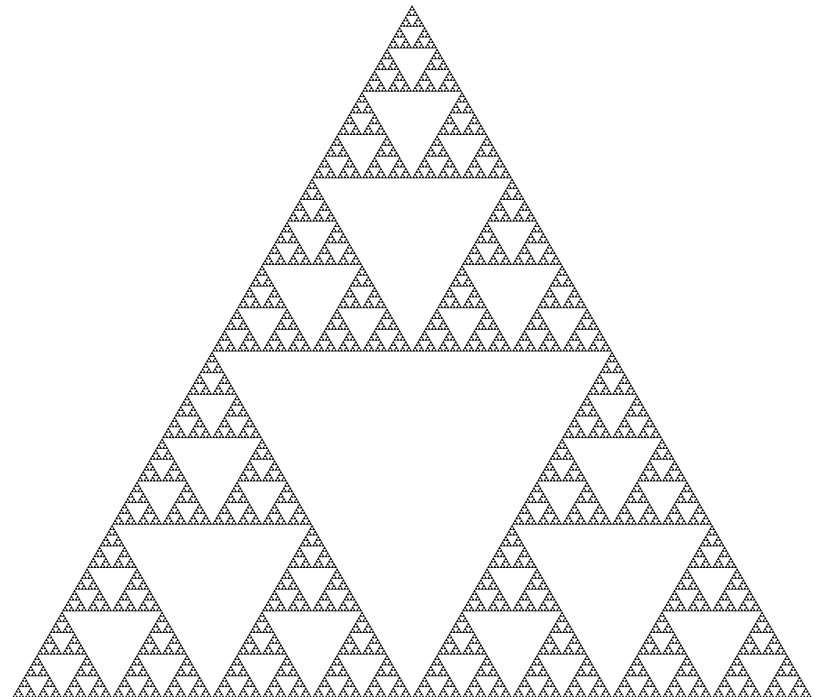
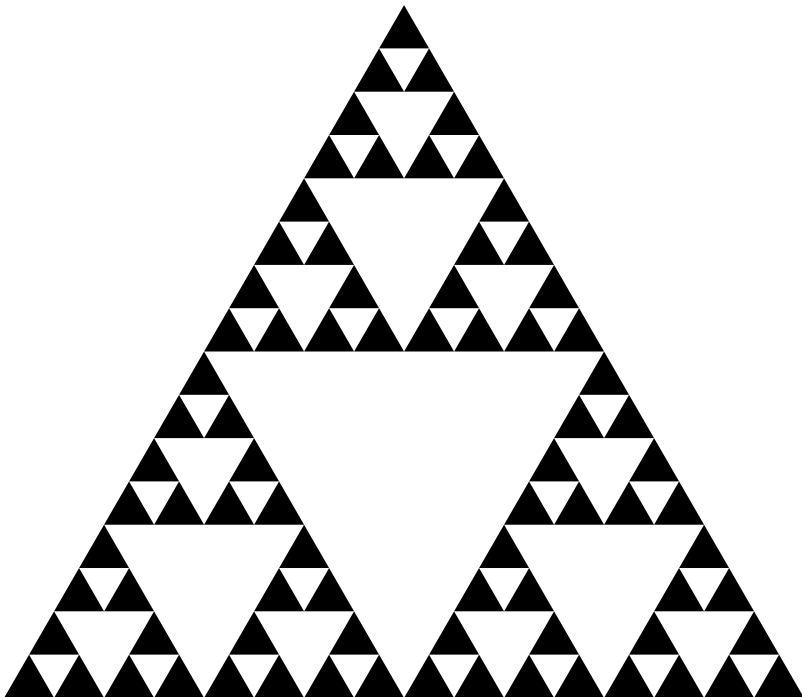
# First Leaf Code

```
def first_leaf(pair):  
    """The first leaf in PAIR, reading left to right.  
>>> first_leaf(())  
()  
>>> first_leaf(5)  
5  
>>> first_leaf(((3, ()), (2, 1)), ()))  
3  
>>> first_leaf(((((), 3), (2, 1)), ()))  
()  
"""  
  
if type(pair) is int or pair == ():  
    return pair  
else:  
    return first_leaf(pair[0])
```

What kind of a recursive process is this? A: Iterative process (tail recursion)

# Sierpinski Triangle

- No discussion of recursion is complete without a mention of *fractal patterns*, which exhibit self-similarity when scaled.
- We'll define a "Sierpinski Triangle of depth  $k$  and side  $s$ " to be
  - A filled equilateral triangle with sides of length  $s$ , if  $k = 0$ , else
  - Three Sierpinski Triangles of depth  $k - 1$  and side  $s/2$  arranged in the three corners of an equilateral triangle with side  $s$ .
- Here are triangles of degree 4 and 8:





# Drawing Sierpinski Triangles

- Assume the existence of the function `triangle`:

```
def triangle(x, y, side):  
    """Draw a filled equilateral triangle with its lower-left corner  
    at (X, Y) and with given SIDE. The base is aligned with the x-axis."""
```

- We can now read off the definition of the triangle:

```
def sierpinski(x, y, side, depth):  
    """Draw a Sierpinski triangle of given DEPTH with given SIDE and  
    lower-left corner at (X, Y)."""
```

```
    if depth == 0:
```

```
        _____  
    else:
```

```
        height = 0.25 * sqrt(3) * side
```

```
        _____  
        _____  
        _____
```

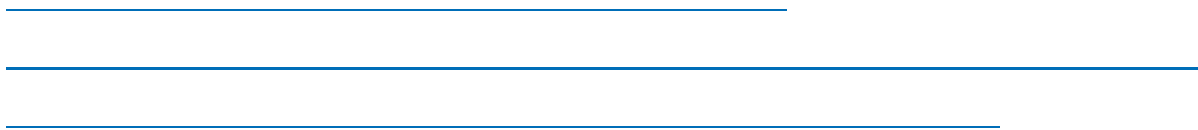
# Drawing Sierpinski Triangles

- Assume the existence of the function `triangle`:

```
def triangle(x, y, side):  
    """Draw a filled equilateral triangle with its lower-left corner  
    at (X, Y) and with given SIDE. The base is aligned with the x-axis."""
```

- We can now read off the definition of the triangle:

```
def sierpinski(x, y, side, depth):  
    """Draw a Sierpinski triangle of given DEPTH with given SIDE and  
    lower-left corner at (X, Y)."""  
  
    if depth == 0:  
        triangle(x, y, side)  
    else:  
        height = 0.25 * sqrt(3) * side
```



# Drawing Sierpinski Triangles

- Assume the existence of the function `triangle`:

```
def triangle(x, y, side):  
    """Draw a filled equilateral triangle with its lower-left corner  
    at (X, Y) and with given SIDE. The base is aligned with the x-axis."""
```

- We can now read off the definition of the triangle:

```
def sierpinski(x, y, side, depth):  
    """Draw a Sierpinski triangle of given DEPTH with given SIDE and  
    lower-left corner at (X, Y)."""
```

```
    if depth == 0:  
        triangle(x, y, side)  
    else:  
        height = 0.25 * sqrt(3) * side  
  
        sierpinski(x, y, side/2, depth-1)
```

---

---

# Drawing Sierpinski Triangles

- Assume the existence of the function `triangle`:

```
def triangle(x, y, side):  
    """Draw a filled equilateral triangle with its lower-left corner  
    at (X, Y) and with given SIDE. The base is aligned with the x-axis."""
```

- We can now read off the definition of the triangle:

```
def sierpinski(x, y, side, depth):  
    """Draw a Sierpinski triangle of given DEPTH with given SIDE and  
    lower-left corner at (X, Y)."""
```

```
    if depth == 0:  
        triangle(x, y, side)  
    else:  
        height = 0.25 * sqrt(3) * side  
  
        sierpinski(x, y, side/2, depth-1)  
        sierpinski(x + side/4, y + height, side/2, depth-1)
```

---

# Drawing Sierpinski Triangles

- Assume the existence of the function `triangle`:

```
def triangle(x, y, side):  
    """Draw a filled equilateral triangle with its lower-left corner  
    at (X, Y) and with given SIDE. The base is aligned with the x-axis."""
```

- We can now read off the definition of the triangle:

```
def sierpinski(x, y, side, depth):  
    """Draw a Sierpinski triangle of given DEPTH with given SIDE and  
    lower-left corner at (X, Y)."""  
  
    if depth == 0:  
        triangle(x, y, side)  
    else:  
        height = 0.25 * sqrt(3) * side  
  
        sierpinski(x, y, side/2, depth-1)  
        sierpinski(x + side/4, y + height, side/2, depth-1)  
        sierpinski(x + side/2, y, side/2, depth-1)
```