Lecture #11: Sequences

Announcements

- HKN review session for Midterm 1 in 145 Dwinelle from 5-8 PM TONIGHT.
- Rooms for midterm to be assigned by login. Please watch website and Piazza.
- Please watch Piazza for news about TA review session on Monday.
- Alternative exams will be given in the labs on Wednesday.
- No labs next week. Also no Wednesday lecture.

Sequences

- The term *sequence* refers generally to a data structure consisting of an *indexed collection of values.*
- That is, there is a first, second, third value (which CS types call #0, #1, #2, etc.
- A sequence may be *finite* (with a length) or *infinite.*
- As an object, it may be *mutable* (elements can change) or *immutable*.
- There are numerous alternative interfaces (i.e., sets of operations) for manipulating it.
- And, of course, numerous alternative implementations.
- Today: immutable, finite sequences, recursively defined.

A Recursive Definition

- A possible definition: A sequence consists of
  - An empty sequence, or
  - A first element and a sequence consisting of the rest of the elements of the sequence other than the first (its *tail*).
- The definition is clearly recursive ("a sequence consists of ...and a sequence ..."), so let's call it an rlist for now.
- Suggests the following ADT interface:

```
# The empty rlist (unique).
empty_rlist = ...
def rlist(first, rest = empty_rlist):
    """A recursive list, r, such that first(R) is FIRST and
    rest(R) is REST, which must be an rlist."""
def first(r):
    """The first item in R."""
def rest(r):
    """The tail of R: the sequence consisting of items 1, 2,...,
    renumbered from 0."""
```

Implementation With Pairs

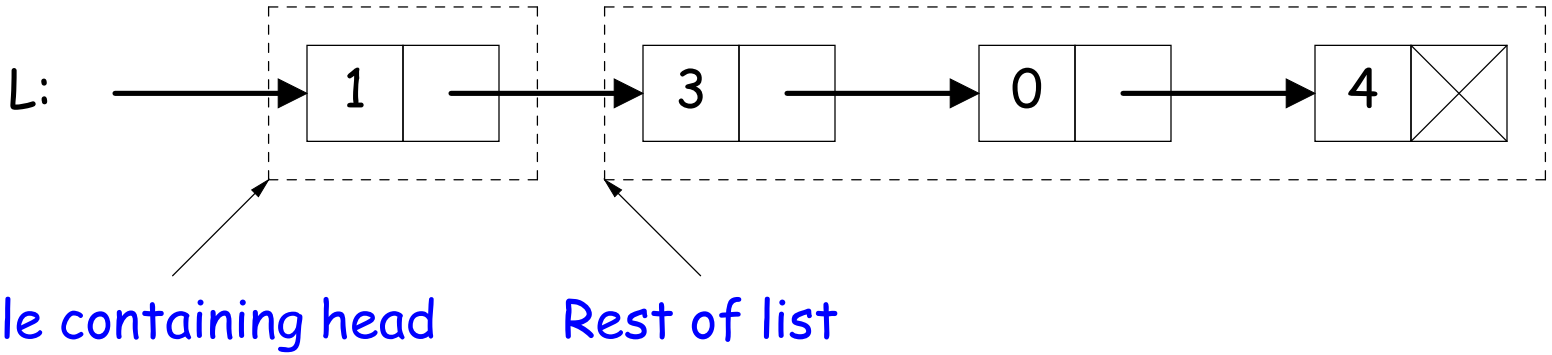
- An obvious implementation uses two-element tuples (pairs). The result is called a *linked list.*

```
empty_rlist = None
def rlist(first, rest = empty_rlist):
    return first, rest
def first(r):
    return r[0]
def rest(r):
    return r[1]
```

Box-and-Pointer Diagrams for Linked Lists

- Diagrammatically, one gets structures like this:

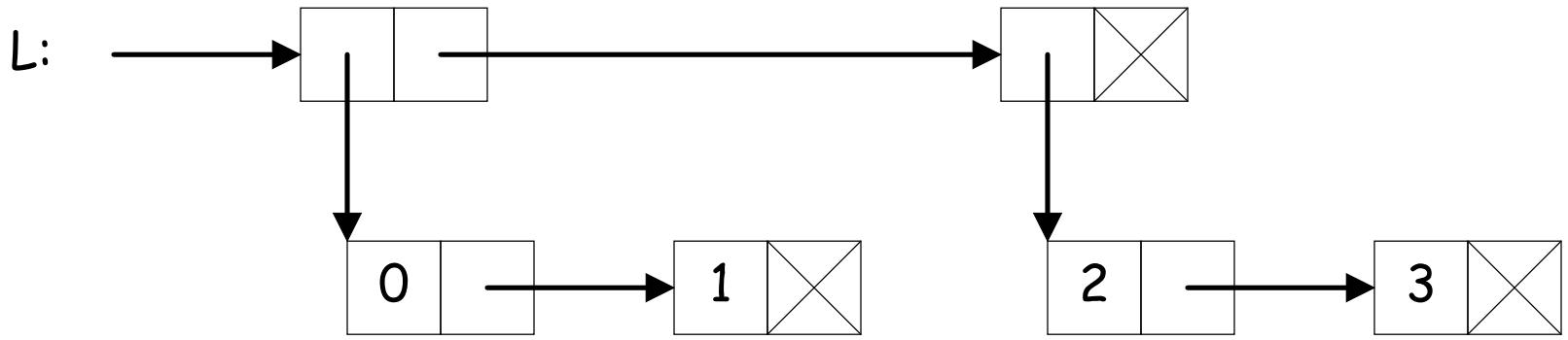
```
# The sequence 1, 3, 0, 4
L = rlist(1, rlist(3, rlist(0, rlist(4, empty_rlist))))
```



Adding Dimensions

Our rlists can contain anything, including other rlists:

```
# The sequence containing sequences (0, 1) and (2, 3)
L = rlist(rlist(0, rlist(1, empty_list)),
          rlist(rlist(2, rlist(3, empty_list)),
                empty_rlist))
```



Recursive Lists vs. Python Tuples

- In Python, tuples are not limited to pairs.
- Could have used (1, 3, 0, 4) or ((0, 1), (2, 3)).
- But there are advantages to rlists:
  - For tuples, rest(L) corresponds to L[1:].
  - The time and spaced required for this operation increases linearly with the length of L.
  - But rest(L) on an rlist takes constant time and no additional space.
- On the other hand,
  - Computing the length or the kth element of an rlist takes time proportional to the length of the sequence,
  - But for tuples, these are constant-time operations.

From Recursive Structure to Recursive Algorithm

- The cases in the recursive definition of list often suggest a recursive approach to implementing functions on them.
- Example: length of an rlist:

```
def len_rlist(s):                    # A sequence is:
    """The length of rlist S."""
    if s == empty_rlist:             # Empty or...
        return 0
    else:
        return 1 + len_rlist(rest(s))
                                     # A first element and
                                     # the rest of the list
```

- **Q:** Why do we know the comment is accurate?
- ***A*:** Recursive thinking: Because we assume the comment is accurate! (For "smaller" arguments, that is).
- Not tail recursive: can't directly make len_rlist iterative.

Tail Recursion (Again)

- But a slight modification makes iteration possible:

```
def len_rlist(s):
    def len(sofar, s):
        """Return SOFAR + the length of rlist S."""
        if s == empty_rlist:
            return sofar
        else:
            return len(sofar + 1, rest(s))
    len(0, s)
```

- We simply return the value of the recursive call to len directly, so this version is *tail recursive,* and can become a loop:

```
def len_rlist(s):
    sofar = 0
    while s != empty_rlist:
        sofar, s = sofar+1, rest(s)
    return sofar
```

Another Example: Selection

- Want to extract item #k from an rlist (number from 0).
- Recursively:

```
def getitem_rlist(s, k):
    """Return the element at index K of recursive list S.
    Assumes K >= 0.
    >>> getitem_rlist(rlist(2, rlist(3, rlist (4))), 1)
    3"""

    if ______:
        return _________
    else:
        return ____________________________________
```

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    if k == 0:
        return first(s)
    else:
        return _____________________________
```

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    if k == 0:
        return first(s)
    else:
        return getitem_rlist(rest(s), k-1)
```

Iterative getitem_rlist

- From the previous version:

```
def getitem_rlist(s, k):
    if k == 0:
        return first(s)
    else:
        return getitem_rlist(rest(s), k-1)
```

- Can transform into an iterative version:

```
def getitem_rlist(s, k):
    """Return the element at index K of recursive list S.
    Assumes K >= 0."""

    while _______:
        s, k = _______________
    return _________
```

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    """Return the element at index K of recursive list S.
    Assumes K >= 0."""

    while k != 0:
        s, k = rest(s), k-1
    return first(s)
```

Applying to All Elements

- Given an rlist, I'd like to create the list of the squares of its elements:

```
def square_rlist(s):
    """The list of squares of the elements of rlist S."""
    if ________________:
        return ___________:
    else:
        return ____________________________________________
```

Applying to All Elements

- Given an rlist, I'd like to create the list of the squares of its elements:

```
def square_rlist(s):
    """The list of squares of the elements of rlist S."""
    if s == empty_rlist:
        return ___________:
    else:
        return ____________________________________________
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Applying to All Elements

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def square_rlist(s):
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        return empty_rlist:
    else:
        return _____________________________________________
```

Applying to All Elements

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```
def square_rlist(s):
    """The list of squares of the elements of rlist S."""
    if s == empty_rlist:
        return empty_rlist:
    else:
        return rlist(first(s)**2, square_rlist(rest(s)))
```

On to Higher Orders!

```
def map_rlist(f, s):
    """The list of values F(x) for each element x of S in order."""
    if s == empty_rlist:
        return empty_rlist
    else:
        return rlist(f(first(s)), map_rlist(f, rest(s)))
```

- So square_rlist(L) is map_rlist(lambda x:x**2, L).
- [Python 3 produces a different kind of result from its map function; we'll get to it.]
- Iterative version difficult here!

Extending rlists

- Joining two lists together is called "appending" in most languages. Python uses "append" to mean "add an item," and uses the term "extend" for joining lists.

```
def extend_rlist(left, right):
    """The sequence of items of rlist 'left'
    followed by the items of 'right'."""
    if _____________________:
        return _____
    else:
        return ____________________________________________________
```

- Again, iterative version is difficult.

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def extend_rlist(left, right):
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    if left == empty_rlist:
        return right
    else:
        return rlist(first(left), extend_rlist(rest(left), right))
```
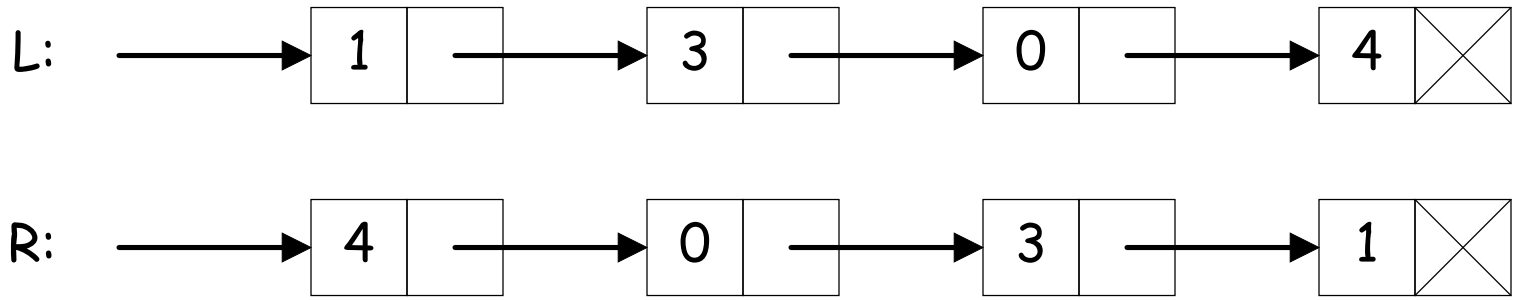

- Again, iterative version is difficult.

Reversing

- Given a sequence represented by an rlist L, how can I create the reverse sequence, reverse_rlist(L)?

```
L = rlist(1, rlist(3, rlist(0, rlist(4, empty_rlist))))
R = reverse_rlist(L)
```



- What is the reverse of empty_rlist? _____.
- Given an rlist L, what is the relationship between first(L), rest(L), and R=reverse_rlist(L)?

_____

_____

Reversing

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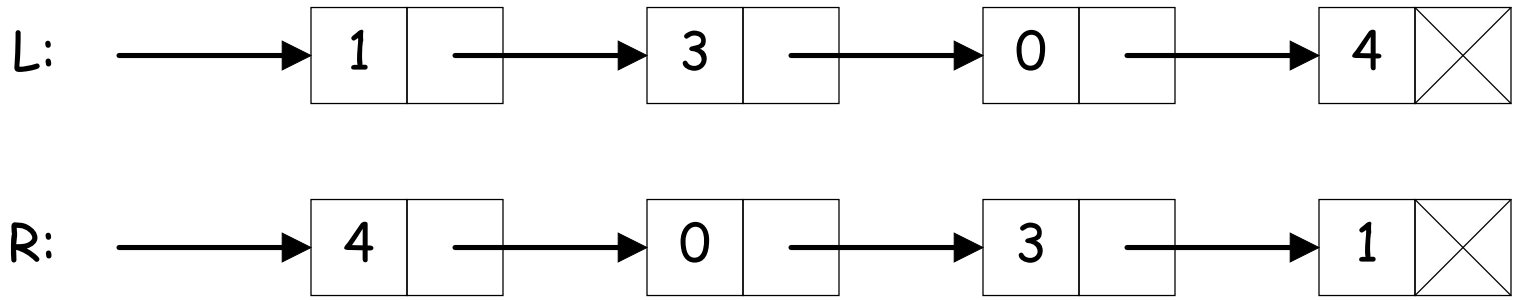

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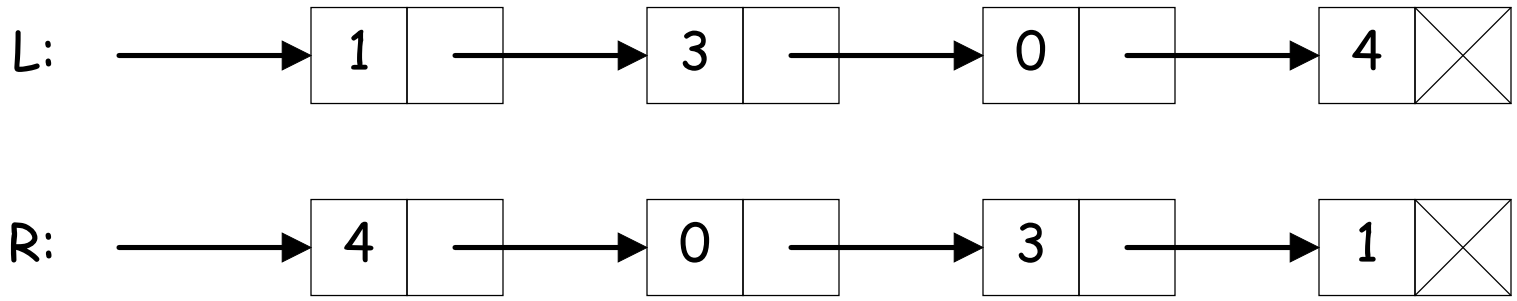

- What is the reverse of empty_rlist? empty_rlist.
- Given an rlist L, what is the relationship between first(L), rest(L), and R=reverse_rlist(L)?

```
R = extend_rlist(reverse_rlist(rest(L)),
                 rlist(first(L),empty_rlist))
```

Iterative Reversing

- The iterative version of rlist_reverse is actually not bad.
- Rlists are most conveniently build from the end (because a tuple, once created, can't be changed).
- The *last* item of a reversed list is the *first* item of the original list.
- This leads to the following tail recursion:

```
def reverse_rlist(L):
    def reverse_extend(to_do, already_done):
        """The result of extending ALREADY_DONE with
        the reverse of TO_DO."""
        if to_do == empty_rlist:
            return empty_rlist
        else:
            return reverse_extend(rest(to_do),
                                  rlist(first(to_do), already_done))
    reverse_extend(L, empty_rlist)
```

- Iterative version?