Lecture #7: Recursion (and a data structure)

Announcements:

- A message from the AWE:

  > "The Association of Women in EECS is hosting a 61A party this Sunday (2/9) from 1–3PM in the Woz! Come hang out, befriend other girls in 61A and meet AWE members who have taken it before! There will be lots of food, games, and fun!"

- Guerrilla Sections this weekend. Extra, optional sections to practice HOF and Environment Diagrams this weekend. You'll be expected to work in groups on questions that range from basic to midterm-level. Details will be announced on Piazza.

Data Structures

- To date, we've dealt with numbers and functions for the most part.
- Although one can do just about anything with these, it's not exactly convenient.
- Example: encode a *pair of integers* as a single integer:

$$(x, y) \Leftrightarrow 2^x \cdot 3^y$$

- Every (x, y) pair can be encoded, but extracting x and y is a chore.
- So Python (like most languages) provides a set of additional *data structures* for representing *collections of values.*

Creating Tuples

- To create (*construct*) a tuple, use a sequence of expressions in parentheses:

```
()        # The tuple with no values
(1, 2)    # A pair: tuple with two items
(1, )     # A singleton tuple: use comma to distinguish from (1)
(1, "Hello", (3, 4)) # Any mix of values possible.
```

- When unambiguous, the parentheses are unnecessary:

```
x = 1, 2, 3          # Same as x = (1,2,3)
return True, 5       # Same as return (True, 5)
for i in 1, 2, 3:    # Same as for i in (1,2,3):
```

Selecting from Tuples

- Can compare, print, or *select* values from a tuple; little else.
- Selection is by explicit item number or "unpacking":

```
>>> x = (1, 7, 5)
>>> print(x[1], x[2])
7 5
>>> from operator import getitem
>>> print(getitem(x, 1), getitem(x, 2))
7 5
>>> x = (1, (2, 3), 5)
>>> print(len(x))
3
>>> a, b, c = x
>>> print(b, c)
(2, 3) 5
>>> d, (e, f), g = x
>>> print(e, g)
2, 5
>>> x, y = y, x
???
```

More Selection

Selecting subtuples (*slices*) is also possible:

```
>>> x = (1, 7, 5, 6)
>>> print(x[1:3], x[0:2], x[:2], x[1:4], x[1:], x[1:2])
(7, 5) (1, 7) (1, 7) (7, 5, 6) (7, 5, 6) (7,)
>>> from operator import getitem
>>> print(getitem(x, slice(1,3)), getitem(x, slice(0,2))
(7, 5) (1, 7)
>>> a, *b, c = x
>>> print(a, b, c)
1 (7, 5) 6
>>> a, *b = x
>>> print(a, b)
1 (7, 5, 6)
```

Multiple Returns

Tuples provide a useful way to return multiple things from a function:

```
>>> divmod(38, 5)   # Returns (38//5, 38%5)
(7, 3)

>>> def sumprod(x, y):
...     return x+y, x*y
>>> sumprod(3, 5)
(8, 15)
```

Tuple is a Recursive Type

- Tuple is one type of *value.*
- Values thus include integers, booleans, strings, and tuples (among others).
- Tuples are sequences of 0 or more *values.*
- Therefore, the definitions of "value" and "tuple" are is *recursive:* they refer to themselves.
- In this case, we'd say that their definitions are *mutually recursive*, since they each refers to the other.
- Recursive data types and recursive algorithms go together.

Example: How Many Numbers?

- Let's consider a restricted tuple (call it a "numeric pair") consisting of:
  - The empty tuple: (),
  - Or a tuple containing two values, each of which is an integer or a numeric pair (still more recursion!)
- Given such a numeric pair, how many numbers are in it?

Example: Code

```
def count_vals(pair):
    """Assuming PAIR is a numeric pair, the total number of integers
    contained in the pair.
    >>> count_vals(())
    0
    >>> count_vals( (1, ()) )
    1
    >>> count_vals( (1, 2) )
    2
    >>> count_vals( ((1, 2), ((3, 4), ())) )
    4
    """
    if __________:
        return 0
    elif type(pair) is int:
        return _
    else return __________________________________________
```

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    if pair == ():
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        return 1
    else return #ints in pair[0] + #ints in pair[1]
```

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    """
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        return 0
    elif type(pair) is int:
        return 1
    else return count_vals(pair[0]) + count_vals(pair[1])
```

The Recursive Leap of Faith

- To implement count_vals, we trusted its comment to be correct, even as we implemented it.
- This is the essence of *recursive thinking*.
- If we can show that
  - Our implementation is correct *given* that the comment is correct,
  - And if we can show that the process must terminate,

  then the comment (the specification of the function) is correct.
- For recursive data structures, showing termination involves using a form of *Noetherian induction*.

Noetherian Induction

(Source: http://en.wikipedia.org/wiki/Emmy_Noether)

- A relation on values is *well-founded* if there are no *infinite descending chains*:
- That is, if you start at some value and keep stepping to smaller values (according to the relation), then you must always get to a minimal value after finite steps.
- E.g., natural or positive numbers under $<$.
- Or numeric pairs under "is an element of."
- Principle of Noetherian induction (named after Emmy Noether):
  - If $P(x)$ is statement about values x from a well-founded set, and
  - If $P(x)$ is true whenever $P(y)$ is true for all $y < x$,
  - Then $P(x)$ is true for all x.

Induction and Recursion

- Recursive programs are justified (and constructed) by inductive reasoning.
- Basic structure:

```
def f(x):
    if There are no valid values ≺ x:
        # The ``base case''
        return   A value that's correct when x is minimal
    else:
        # Use ``The inductive hypothesis''
        return   A solution constructed using f(y) where y ≺ x
```

- The meaning of $\prec$ depends on the application.
- In place of "return" might also use side-effect-producing code.