# Lecture #4: Higher-Order Functions

**Announcements:**

- Theta Tau rush events, starting 1/31. See Piazza post.
- CSUA Unix/Emacs help sessions: Thursday 1/26, Tuesday 1/31, Thursday 2/2 in 310 Soda, 6–8PM.



## A Simple Recursion

- The Fibonacci sequence is defined

$$F_k = \begin{cases} k, & \text{for } k = 0, 1 \\ F_{k-2} + F_{k-1}, & \text{for } k > 1 \end{cases}$$

- . . . which translates easily into Python:

```
def fib(n):
    """The Nth Fibonacci number, N>=0."""
    assert n >= 0
    if n <= 1:
        return n
    else:
        return fib(n-2) + fib(n-1)
```

- This definition works, but why is it so slow?



## Redundant Calculation

- Consider the computation of fib(10).
- This calls fib(9) and fib(8), but then fib(9) calls fib(8) again and both fib(9) and the two calls to fib(8) call fib(7), so that fib(7) is called 3 times.
- Likewise, fib(6) is called 5 times, fib(7) is called 8 times, and so forth in increasing Fibonacci sequence, interestingly enough.
- Therefore, the time required (proportional to the number of calls) grows exponentially:
- As it turns out, fib($N$) requires time roughly proportional to $\Phi^N$, where the golden ratio $\Phi = (1 + \sqrt{5})/2$.



## Avoiding Recalculation

- To compute the next Fibonacci number, we need the preceding two.
- Let's generalize and consider what it takes to compute $N$ more:

```
def fib2(fk1, fk, k, n):
    """Assuming FK1 and FK F[K-1] and F[K] in the Fibonacci
    sequence numbers and N>=K, return F[N]."""
    if n == k:
        return fk
    else:
        return fib2(fk, fk1+fk, k+1, n)
def fib(n):
    if n <= 1:
       return n
    else:
       return fib2(0, 1, 1, n)
```



## Tail Recursion and Repetition

- In this last version, whenever fib2 is called recursively, the value of that call is immediately returned.
- This property is called *tail recursion*.

```
def fib2(fk1, fk, k, n):
    if n == k: return fk
    else:      return fib2(fk, fk1+fk, k+1, n)
def fib(n):
    if n <= 1: return n
    else:      return fib2(0, 1, 1, n)
```

- It is this sort of process that is easily expressed as a repetition.
- Parameters become variables; initial call on fib2 inside fib initializes them; each tail-recursive call updates them. Iterative equivalent:

```
def fib3(n):
    if n <= 1: return n
    fk1, fk, k = 0, 1, 1
    while n != k:
        fk1, fk, k = fk, fk1+fk, k+1
    return fk
```



## Nested Functions

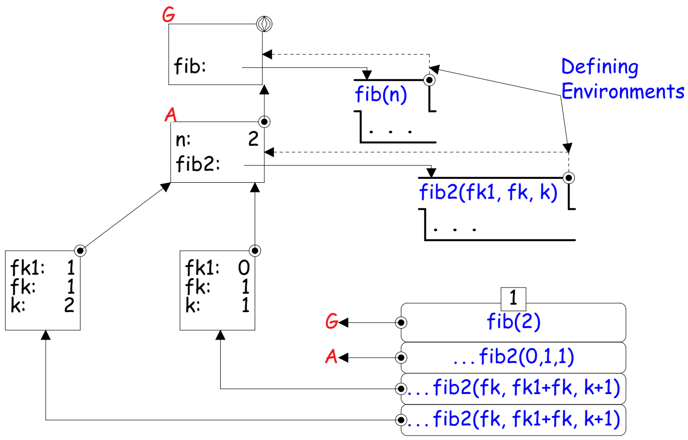
- In the last recursive version, fib2 function is an auxiliary function, used only by fib.
- It makes sense to tuck it away inside fib, like this:

```
def fib(n):
    def fib2(fk1, fk, k):
        if n == k: return fk
        else:      return fib2( fk, fk1+fk, k+1)

    if n <= 1: return n
    else:      return fib2(0, 1, 1)
```

- I've taken the liberty here of removing the parameter n from fib2: it's always the same as the outer n and never changes.
- But to explain how this works, we'll have to extend the environment model just a bit.

## Nested Functions and Environments






## Defining Environments

- Each function value is attached to the environment frame in which the **def** statement that created it was evaluated.
- Since the **def** for `fib` was evaluated in the global frame, the resulting function value bound to `fib` is attached to the global frame.
- Since the **def** for `fib2` was evaluated in the local frame of an execution of `fib`, the resulting function value is attached to that local frame.
- When a user-defined function value is called, the local frame that is created for that call is attached to the defining frame of the function.



## Do You Understand the Machinery? (I)

What is printed (0, 1, or error) and why?

```
def f():
    return 0

def g():
    print(f())

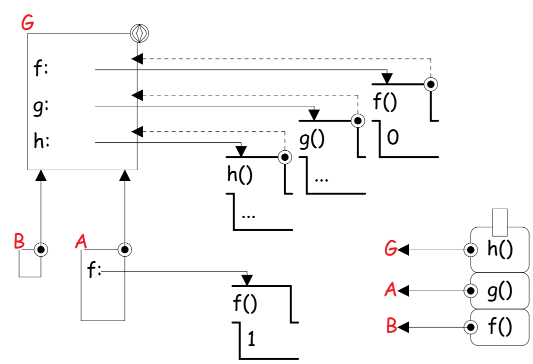
def h():
    def f():
        return 1
    g()

h()
```



## Answer (I)

The program prints `0`. At the point that `f` is called, we are in the situation shown below:




So we evaluate f in an environment (B) where it is bound to a function that returns 0.



## Do You Understand the Machinery? (II)

What is printed (0, 1, or error) and why?

```
def f():
    return 0

g = f

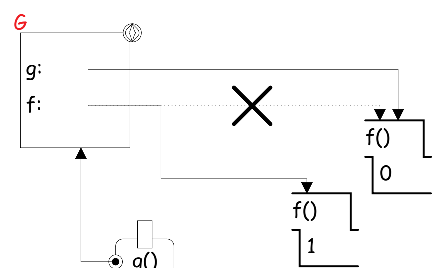
def f():
    return 1

print(g())
```



## Answer (II)

The program prints `0` again:




At the time we evaluate f to assign it to g, it has the value indicated by the crossed-out dotted line, so that is the value g gets. The fact that we change `f`'s value later is irrelevant, just as x = 3; y = x; x = 4; print(y) prints `3` evan though x changes: y doesn't remember where its value came from.

## Do You Understand the Machinery? (III)

What is printed (0, 1, or error) and why?

```
def f():
    return 0

def g():
    print(f())

def f():
    return 1

g()
```



## Answer (III)

This time, the program prints 1. When g is executed, it evaluates the name 'f'. At the time that happens, f's value has been changed (by the third **def**), and that new value is therefore the one the program uses.



## Functions As Templates

- If we think of a function body as a template for a computation, parameters are "blanks" in that template.
- For example:

```
def sum_squares(N):
    k, sum = 0, 0
    while k <= N:
        sum, k = sum+k**2, k+1
    return sum
```

is a template for an infinite set of computations that add squares of numbers up to 0, 1, 2, 3, ..., in place of the N.



## Functions on Functions

- Likewise, function parameters allow us to have templates with slots for computations:

```
def summation(N, f):
    k, sum = 1, 0
    while k <= N:
        sum, k = sum+f(k), k+1
    return sum
```

- Generalizes sum_squares. We can write `sum_squares(5)` as:

```
def square(x): return x*x
summation(5, square)
```

- or (if we don't really need a "square" function elsewhere), we can create the function argument anonymously on the fly:

```
summation(5, lambda x: x*x)
```



## Lambda

- In Python, **lambda** is just an abbreviation.
- Writing lambda *PARAMS*: *EXPRESSION* is the same as writing NAME, where NAME is a name that appears nowhere else in the program and is defined by

```
def NAME(PARAMS):
    return EXPRESSION
```

evaluated in the same environment in which the original **lambda** was.

- Now we can write any number of summations succinctly:

```
summation(10, lambda x: x**3)       # Sum of cubes
summation(10, lambda x: 1 / x)      # Harmonic series
summation(10, lambda k: x**(k-1) / factorial(k-1))
                                    # Approximate e**x
```



## Functions that Produce Functions

- Functions are *first-class values*, meaning that we can assign them to variables, pass them to functions, and return them from functions.
- Example:

```
def add_func(f, g):
    """Return function that returns f(x)+g(x) for argument x."""
    def adder(x):            #
        return f(x) + g(x)   # or return lambda x: f(x) + g(x)
    return adder             #

h = add_func(abs, lambda x: -x)
>>> print(h(-5))
10
```

- Generalize the example:

```
def combine_funcs(op, f, g):
    return lambda x: op(f(x), g(x))
# Now add_func = ____________________
```

- What do the environments look like here?

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    return adder              #

h = add_func(abs, lambda x: -x)
>>> print(h(-5))
10
```

- Generalize the example:

```
def combine_funcs(op, f, g):
    return lambda x: op(f(x), g(x))
# Now add_func = lambda f, g:  combine_funcs(sum
```

- What do the environments look like here?