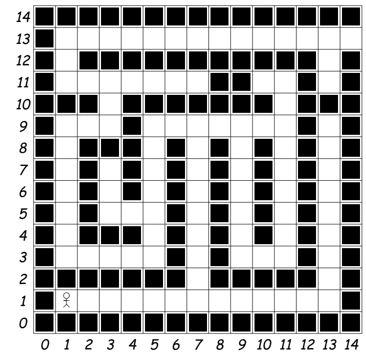
Lecture #20: Recursive Processes, Memoization, Tree Structures

Example: Escape from a Maze

- Consider a rectangular maze consisting of an array of squares some of which are occupied by large blocks of concrete:



- Given the size of the maze and locations of the blocks, prisoner, and exit, how does the prisoner escape?

Maze Program (Incorrect)

```
def solve_maze(row0, col0, maze):
    """Assume that MAZE is a rectangular 2D array (list of lists) where
    maze[r][c] is true iff there is a concrete block occupying
    column c of row r.  ROW0 and COL0 are the initial row and column
    of the prisoner.  Returns true iff there is a path of empty
    squares that are horizontally or vertically adjacent to each other
    starting with (ROW0, COL0) and ending outside the maze."""
    if row0 not in range(len(maze)) or col0 not in range(len(maze[row])):
        return True
    elif maze[row0][col0]:  # In wall
        return False
    else:
        return solve_maze(row0+1, col0, maze) or solve_maze(row0-1, col0, maze) \
            or solve_maze(row0, col0+1, maze) or solve_maze(row0, col0-1, maze) \
    # What's wrong?
```


Maze Program (Corrected)

To fix the problem, remember where we've been:

```
def solve_maze(row0, col0, maze):
    """Assume that MAZE is a rectangular 2D array (list of lists) where
    maze[r][c] is true iff there is a concrete block occupying
    column c of row r.  ROW0 and COL0 are the initial row and column
    of the prisoner.  Returns true iff there is a path of empty
    squares that are horizontally or vertically adjacent to each other
    starting with (ROW0, COL0) and ending outside the maze."""
    visited = set()   # Set of visited cells
    cols, rows = range(len(maze[0])), range(len(maze))
    def escapep(r, c):
        """True iff is a path of empty, unvisited cells from (R, C) out of maze."""
        if r not in rows or c not in cols:
            return True
        elif maze[r][c] or (r, c) in visited:
            return False
        else:
            visited.add((r,c))
            return escapep(r+1, c) or escapep(r-1, c) \
                or escapep(r, c+1) or escapep(r, c-1)
    return escapep(row0, col0)
```


Example: Making Change

```
def count_change(amount, denoms = (50, 25, 10, 5, 1)):
    """The number of ways to change AMOUNT cents given the
    denominations of coins and bills in DENOMS.
    >>> # 9 cents = 1 nickel and 4 pennies, or 9 pennies
    >>> count_change(9)
    2
    >>> # 12 cents = 1 dime and 2 pennies, 2 nickels and 2 pennies,
    >>> # 1 nickel and 7 pennies, or 12 pennies
    >>> count_change(12)
    4
    """
    if amount == 0:        return 1
    elif len(denoms) == 0:  return 0
    elif amount >= denoms[0]:
        return count_change(amount-denoms[0], denoms) \
              + count_change(amount, denoms[1:])
    else:
        return count_change(amount, denoms[1:])
```


Avoiding Redundant Computation

- In the (tree-recursive) maze example, a naive search could take us in circles, resulting in infinite time.
- Hence the visited set in the escapep function.
- This set is intended to catch redundant computation, in which reprocessing certain arguments cannot produce anything new.
- We can apply this idea to cases of finite but redundant computation.
- For example, in count_change, we often revisit the same subproblem:
  - E.g., Consider making change for 87 cents.
  - When choose to use one half-dollar piece, we have the same subproblem as when we choose to use no half-dollars and two quarters.
- Saw an approach in Lecture #16: memoization.

Memoizing

- Idea is to keep around a table ("memo table") of previously computed values.
- Consult the table before using the full computation.
- Example: count_change:

```
def count_change(amount, denoms = (50, 25, 10, 5, 1)):
    memo_table = {} # Indexed by pairs (row, column)
    # Local definition hides outer one so we can cut-and-paste
    # from the unmemoized solution.
    def count_change(amount, denoms):
        if (amount, denoms) not in memo_table:
            memo_table[amount,denoms] \
              = full_count_change(amount, denoms)
        return memo_table[amount,denoms]
    def full_count_change(amount, denoms):
        unmemoized original solution goes here verbatim
    return count_change(amount,denoms)
```

- Question: how could we test for infinite recursion?

Optimizing Memoization

- Used a dictionary to memoize count_change, which is highly general, but can be relatively slow.
- More often, we use arrays indexed by integers (lists in Python), but the idea is the same.
- For example, in the count_change program, we can index by amount and by the portion of denoms that we use, which is always a slice that runs to the end.

```
def count_change(amount, denoms = (50, 25, 10, 5, 1)):
    # memo_table[amt][k] contains the value computed for
    #    count_change(amt, denoms[k:])
    memo_table = [ [-1] * (len(denoms)+1) for i in range(amount+1) ]
    def count_change(amount, denoms):
        if memo_table[amount][len(denoms)] == -1:
              memo_table[amount][len(denoms)] \
                = full_count_change(amount, denoms)
        return memo_table[amount][len(denoms)]
    ...
```


Order of Calls

- Going one step further, we can analyze the order in which our program ends up filling in the table.
- So consider adding some tracing to our memoized count_change program:

```
memo_table = {}
def count_change(amount, denoms):
    ... full_count_change(amount, denoms) ...
    return memo_table[amount,denoms]
@trace
def full_count_change(amount, denoms):
    if amount == 0: return 1
    elif not denoms: return 0
    elif amount >= denoms[0]:
        return count_change(amount, denoms[1:]) \
               + count_change(amount-denoms[0], denoms)
    else:
        return count_change(amount, denoms[1:])
return count_change(amount,denoms)
```


Result of Tracing

- Consider count_change(57) (returns only):

```
full_count_change(57, ()) -> 0
full_count_change(56, ()) -> 0
...
full_count_change(1, ()) -> 0
full_count_change(0, (1,)) -> 1
full_count_change(1, (1,)) -> 1
...
full_count_change(57, (1,)) -> 1
full_count_change(2, (5, 1)) -> 1
full_count_change(7, (5, 1)) -> 2
...
full_count_change(57, (5, 1)) -> 12
full_count_change(7, (10, 5, 1)) -> 2
full_count_change(17, (10, 5, 1)) -> 6
...
full_count_change(32, (10, 5, 1)) -> 16
full_count_change(7, (25, 10, 5, 1)) -> 2
full_count_change(32, (25, 10, 5, 1)) -> 18
full_count_change(57, (25, 10, 5, 1)) -> 60
full_count_change(7, (50, 25, 10, 5, 1)) -> 2
full_count_change(57, (50, 25, 10, 5, 1)) -> 62
```


Dynamic Programming

- Now rewrite count_change to make the order of calls explicit, so that we needn't check to see if a value is memoized.
- Technique is called *dynamic programming* (for some reason).
- We start with the base cases, and work backwards.

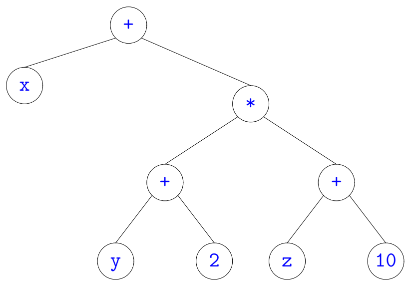
```
def count_change(amount, denoms = (50, 25, 10, 5, 1)):
    memo_table = [ [-1] * (len(denoms)+1) for i in range(amount+1) ]
    def count_change(amount, denoms):
        return memo_table[amount][len(denoms)]
    def full_count_change(amount, denoms):
        # How often is this called?
        ... # (calls count_change for recursive results)

    for a in range(0, amount+1):
        memo_table[a][0] = full_count_change(a, ())
    for k in range(1, len(denoms) + 1):
        for a in range(1, amount+1):
            memo_table[a][k] = full_count_change(a, denoms[-k:])
    return count_change(amount, denoms)
```


New Topic: Tree-Structured Data

- 1 Linear-recursive and tail-recursive functions make a single recursive call in the function body. Tree-recursive functions can make more.
- Linear recursive data structures (think rlists) have single embedded recursive references to data of the same type, and usually correspond to linear- or tail-recursive programs.
- To model some things, we need mulitple recursive references in objects.
- In the absence of circularity (paths from an object eventually leading back to it), such objects form data structures called *trees*:
  - The objects themselves are called *nodes* or *vertices*.
  - Tree objects that have no (non-null) pointers to other tree objects are called *leaves*.
  - Those that do have such pointers are called *inner nodes*, and the objects they point to are *children* (or *subtrees* or (uncommonly) *branches*).
  - A collection of disjoint trees is called a *forest*.

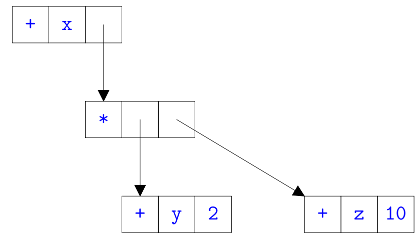
Example: Expressions

- An expression (in Python or other languages) typically has a recursive structure. It is either
  - A literal (like 5) or symbol (like x)—a leaf—or
  - A compound expression consisting of an operator and zero or more operands, each of which is itself an expression.
- For example, the expression x + (y+2)*(z+10) can be thought of as a tree (what happened to the parentheses?):



Expressions as Tuples or Lists

- We can represent the abstract structure of the last slide with Python objects we've already seen:




```
("+", "x", ("*", ("+", "y", "2"), ("+", "z", "10")))
```


Class Representation

- . . . or we can introduce a Python class:

```
class ExprTree:
    def __init__(self, operator):
        self.__operator = operator

    @property
    def operator(self):
        return self.__operator

    @property
    def left(self):
        raise NotImplementedError

    @property
    def right(self):
        raise NotImplementedError
```

```
class Leaf(ExprTree):
    pass

class Inner(ExprTree):
    def __init__(self, operator,
                 left, right):
        ExprTree.__init__(self, operator)
        self._left = left;
        self._right = right
    @property
    def left(self):
        return self._left
    @property
    def right(self):
        return self._right
```

```
Inner("+", Leaf("x"),
      Inner("*", Inner("+", Leaf("y"), Leaf("2")),
            Inner("+", Leaf("z"), Leaf("10"))))
```


A General Tree Type

- Trees don't quite lend themselves to being captured with standard syntax like tuples or lists, because they get accessed in various ways, with slightly varying interfaces.
- To start with, we'll use this type, which has no empty trees:

```
class Tree:
    """A Tree consists of a label and a sequence
    of 0 or more Trees, called its children."""

    def __init__(self, label, *children):
        """A Tree with given label and children.
        For convenience, if children[k] is not a Tree,
        it is converted into a leaf whose operator is
        children[k]."""
        self._label = label;
        self._children = \
          [ c if type(c) is Tree else Tree(c)
              for c in children]
```


A General Tree Type: Accessors

```
# class Tree:
    @property
    def is_leaf(self):
        return self.arity == 0

    @property
    def label(self):
        return self._label

    @property
    def arity(self):
        """The number of my children."""
        return len(self._children)

    def __iter__(self):
        """An iterator over my children."""
        return iter(self._children)

    def __getitem__(self, k):
        """My kth child."""
        return self._children[k]
```


A Simple Recursion

- Since trees are recursively defined, recursion generally figures in algorithms on them.
- Example: number of leaf nodes.

```
def leaf_count(T):
    """Number of leaf nodes in the Tree T."""
    if T.is_leaf:
        return 1
    else:
        s = 0
        for child in T:
            s += leaf_count(child)
        return s
        # Can you put the else clause in one line instead?
        return functools.reduce(operator.add, map(leaf_count, T), 0)
```

- How long does this take (for a tree with N leaves)?

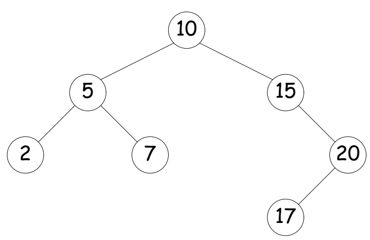
Tree to List

- Another example with no explicit base cases:

```
from functools import reduce
from operator import add
def tree_to_list_preorder(T):
    """The list of all labels in T, listing the labels
    of trees before those of their children, and listing their
    children left to right (preorder)."""
    return (T.label, ) + \
           reduce(add, map(tree_to_list_preorder, T, ()))
```


Search Trees

- The book talks about *search trees* as implementations of sets of values.
- Here, the purpose of the tree is to divide data into smaller parts.
- In a *binary search tree*, each node is either empty or has two children that are binary search trees such that all labels in the first (left) child are less than the node's label and all labels in the second (right) child are greater.



Search Tree Class

- To work on search trees, it is useful to have a few more methods on trees:

```
class BinaryTree(Tree):
    @property
    def is_empty(self):
        """This tree contains no labels or children."""

    @property
    def left(self):
        return self[0]

    @property
    def right(self):
        return self[1]

    """The empty tree"""
    empty_tree = ...
```


Tree Search Program

```
def tree_find(T, x):
    """True iff x is a label in set T, represented as a search tree.
    That is, T
       (a) Represents an empty tree if its label is None, or
       (b) has two children, both search trees, and all labels in
           T[0] are less than T.label, and all labels in T[1] are
           greater than T.label."""
    if T.is_empty:
        return False
    if x == T.label:
        return True
    if x < T.label:
        return tree_find(T.left, x)
    else:
        return tree_find(T.right, x)
```


Timing

- How long does the tree_find program (search binary tree) take in the worst case
  - 1. As a function of D, the depth of the tree?
  - 2. As a function of N, the number of keys in the tree?
  - 3. As a function of D if the tree is as shallow as possible for the amount of data?
  - 3. As a function of N if the tree is as shallow as possible for the amount of data?

Timing

- How long does the tree_find program (search binary tree) the worst case
  - 1. As a function of D, the depth of the tree? $\Theta(D)$
  - 2. As a function of N, the number of keys in the tree?
  - 3. As a function of D if the tree is as shallow as possible amount of data?
  - 3. As a function of N if the tree is as shallow as possible amount of data?

Timing

- How long does the tree_find program (search binary tree) the worst case
  - 1. As a function of D, the depth of the tree? $\Theta(D)$
  - 2. As a function of N, the number of keys in the tree? Θ
  - 3. As a function of D if the tree is as shallow as possible amount of data?
  - 3. As a function of N if the tree is as shallow as possible amount of data?

Timing

- How long does the tree_find program (search binary tree) the worst case
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Timing

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