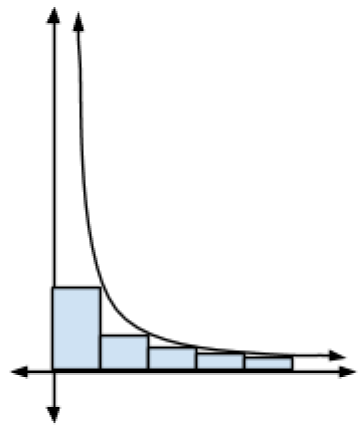
11.3) The Integral Test and Estimates of Sums:

-Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent? This is not as easy to answer as you may think with what we know so far. There is no formula for the partial sum of the first n terms of this series, and even the comparison test is difficult to apply here. Instead, we will attempt to determine this with geometry.



-The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be compared to the sum the area of rectangles under the curve $y = \frac{1}{x^2}$ where all the rectangles intersect the curve in the upper right corner, and the width of the rectangles are all $\Delta x = 1$. These rectangles all have heights of $\frac{1}{x^2}$ where x is an integer and widths of one, so the sum of the areas of this infinite number of rectangles is:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

-Okay, but how does this help us determine the value of the infinite series? Well, think of the sum of the areas of the rectangles in the picture above as being separated into two pieces: the biggest rectangle, and all the rest of the rectangles:

$$\left(\frac{1}{1^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

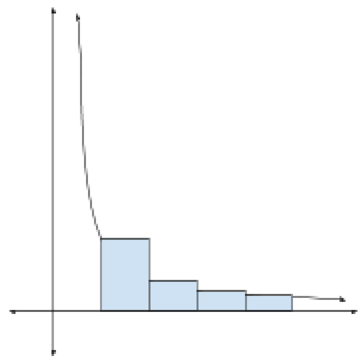
-The area of the first rectangle of course is 1. The sum of the areas of the rest of the rectangles, while not something we can compute (yet), is a value that we are certain is smaller than the area under the curve $y = \frac{1}{x^2}$ from x = 1 and to the right. We know this because $y = \frac{1}{x^2}$ is decreasing when x >0, and our rectangles all intersect the curve on the right corner, implying every rectangle is wholly under the curve.

-We can find the area under the curve $y = \frac{1}{x^2}$ from x =1 and to the right, and we can be sure that whatever it is, the sum of the area of the rectangles is smaller than this:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) < 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2} dx = 1 + \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1 - 0 + 1 = 2$$

-Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent? We have just shown that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2,$ implying that this always increasing sum of positive numbers is bounded above, so yes.

-What about $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, is this series convergent? We can use a similar strategy here, liken this infinite series to an integral. First we once again pretend that each term in the series is the area of rectangles of width 1 under the curve $y = \frac{1}{\sqrt{x}},$ but this time all the rectangles will intersect the curve on the left-endpoint:



-Unlike in the last example, we don't need to separate the first rectangle from the rest, (the only reason we did that in the last example was because it would have been too difficult to integrate $\int_0^{\infty} \frac{1}{x^2} dx$) so the sum of the rectangles this time is <u>greater</u> than the area under the curve $y = \frac{1}{\sqrt{x}}$ starting at x = 1 and to the right:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots > \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{\sqrt{x}} dx = \lim_{t \to \infty} (2\sqrt{t} - 2) \to \infty$$

-This time, the sum of the areas of the rectangles is greater than the expression that is infinitely large. That implies this sum is also infinitely large, so it cannot be a finite sum in this case.

-Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ convergent? We have just shown that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \infty,$ so no. (yes you could have also used the harmonic series and some comparison to get this conclusion, but you get the idea!)

-Using integrals and geometric reasoning to determine whether series are convergent or divergent is quite common, to the point that it has its own name, **the Integral Test**:

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty),$ and let $a_n = f(n).$ Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

-If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

-If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

-The integral test can be rewritten to start at values other than n=1 as well; as long as f is still continuous, positive, and decreasing on $[a, \infty)$, then you can use the improper integral $\int_a^\infty f(x)dx$ instead, and the same outcome rules apply.

-There are other rules that can be bent here too. For example, the function does not have to be decreasing **everywhere** on $[1, \infty)$, it just has to be decreasing *eventually* on that interval (find some integer on $[1, \infty)$, say A, at which f(x) is decreasing on $[A, \infty)$, and then break up $\sum_{n=1}^{\infty} a_n$ into $\sum_{n=1}^{A} a_n + \sum_{n=A}^{\infty} a_n$ and find the integral of $\int_A^\infty f(x)dx$).

Example: Test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution: The function $f(x) = \frac{1}{x^2+1}$ is decreasing, positive, and continuous on $[1, \infty)$. Therefore, we can use the Integral Test on this series:

$$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty} (tan^{-1}(t) - tan^{-1}(1\ ) = \lim_{t\to\infty} tan^{-1}(t) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

This integral evaluate was finite, so the integral is convergent, and therefore the original series is also convergent.

Exercise: a) Test $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ for convergence or divergence.

b) Test $\sum_{n=1}^{\infty} \frac{1}{n^3}$ for convergence or divergence.

-Now let's try to apply the Integral Test to a general situation:

Example: For what values of r is $\sum_{n=1}^{\infty} \frac{1}{n^r}$ convergent?

Solution: If r < 0, then $\frac{1}{n^r}$ is increasing on the interval $[1, \infty)$, which implies $\sum_{n=1}^{\infty} \frac{1}{n^r}$ is divergent. If r = 0, then the sum is nothing but 1's, so it is clearly divergent. What if p>0?

If r>0, $\frac{1}{x^r}$ is decreasing, continuous, and positive on for x>1, and we have seen from a previous section that $\int_1^\infty \frac{1}{x^r}dx$ is divergent if $0 < r \leq 1$, and convergent if 1 < r.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^r}$ is convergent if r > 1, and divergent if $r \leq 1$.

-A word of warning though, the integral test is only good for determining whether an infinite series is convergent or divergent. It <u>does not</u> tell you what the infinite sum equals, should it be convergent.

-For the record, we saw that $\int_1^\infty \frac{1}{x^2}dx = 1$, which means $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, but $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example: Test $\sum_{n=1}^{\infty} \frac{ln(n)}{n}$ for convergence or divergence.

Solution: The function $f(x) = \frac{ln(x)}{x}$ is continuous and position for all x >1, but is it decreasing? Let's find out:

$$f'(x) = \frac{x(\frac{1}{x})-(1)ln(x)}{x^2} = \frac{1-ln(x)}{x^2}$$

Again, it's not important if it is decreasing everywhere, just that it decreases eventually. $f'(x)$ is negative if x is greater than e, so the expression does eventually decrease. Thus we can use the integral test:

$$\int_1^\infty \frac{ln(x)}{x}dx = \lim_{t\to\infty}\int_1^t \frac{ln(x)}{x}dx = \lim_{t\to\infty}\left(\frac{ln(t^2)}{2} - 0\right) \to \infty$$

The integral is divergent, so the series is divergent too.

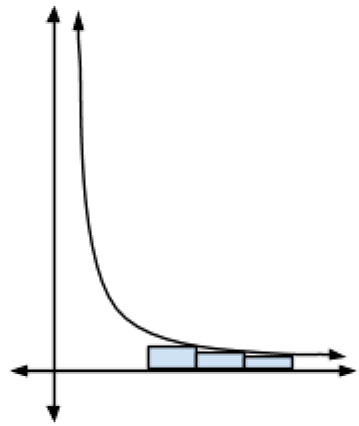
-Suppose that a series $\sum_{n=1}^{\infty} a_n$ is convergent and we would like to know what the sum of the series is. If we do not have the means of being able to find the exact value of $\sum_{n=1}^{\infty} a_n$, can we at least approximate the value of the sum? The best way to approximate is to merely find a partial sum of the series.

-The partial sums approach the value of the infinite sum as more terms are used, and that's why $\lim_{n\to\infty} s_n = s$. However, how good an approximation would we get from a partial sum? That is determined by the value of the **remainder** of a partial sum approximation. A remainder, or **error** of a partial sum is merely:

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots.$$

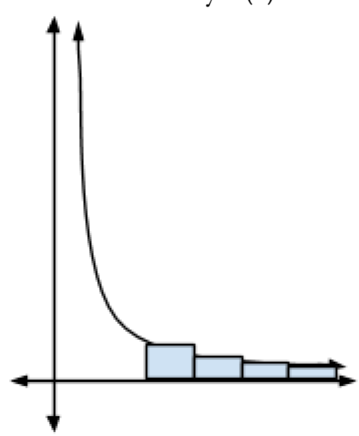
A partial sum is the sum of the first n terms, so the remainder is all the other terms after the nth term added together.

-If the function f that defines the series is decreasing on $[n, \infty)$, then the remainder would be less than the area under the curve y = f(x) from x = n and to the right if we imagine the terms as area of rectangles intersecting the curve on the right endpoint each time:



$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + ... \leq \int_n^\infty f(x)dx$$

-However, if we imagine the terms as area of rectangles intersecting the curve on the left endpoint each time, then the remainder would be greater than the area under the curve y=f(x) from x=n and to the right:



$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + ... \geq \int_{n+1}^\infty f(x)dx$$

-This implies that the remainder between the sum of an infinite series and the sum of the first n terms is sandwiched in-between these two integrals.

The Remainder Estimate for the Integral Test:

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent.

If $R_n = s - s_n$, then:

$$\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx$$

Example: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the first 10 terms. Estimate the error involved in this approximation.
b) Approximate the sum of the series using your error and your approximation.
c) How many terms are needed to ensure the sum is accurate to 3 decimal places?

Solution: We will need to find out what $\int_n^{\infty} \frac{1}{x^3} dx$ is for each part of this problem, so might as well find out now:

$$\int_n^{\infty} \frac{1}{x^3} dx = \lim_{t \to \infty} \int_n^{t} \frac{1}{x^3} dx = \lim_{t \to \infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

a) First off, we will need an approximation of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ using the first 10 terms:

$$\sum_{n=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{9^3} + \frac{1}{10^3} \approx 1.1975$$

Remember, we don't need to know the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ overall to find the remainder, we are just trying to estimate the remainder. According to the remainder theorem, the remainder is between:

$$\int_{11}^{\infty} \frac{1}{x^3} dx \leq R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx$$

$$\frac{1}{2(11)^2} \leq R_{10} \leq \frac{1}{2(10)^2}$$

$$\frac{1}{2(11)^2} \leq R_{10} \leq \frac{1}{2(10)^2}$$

$$0.00413 \leq R_{10} \leq 0.005$$

You can also simply say that the error is less than 0.005.

b) This series is positive, so any estimation you find will always be an underestimation. So if we want a range of values that the actual sum of the entire series (s) could possibly equal, you don't need to subtract your possible errors from your approximation (s_n), the actual value will be some value in:

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$$

For for our case, since our approximation was a partial sum of the first 10 terms, $s_{10} = 1.1975$ is our approximation, along with our error boundaries from part a):

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$$

$$1.1975 + 0.00413 \leq s \leq 1.1975 + 0.005$$

$$1.2016 \leq s \leq 1.2025$$

The entire series sum is somewhere between 1.2016 and 1.2025.

c) In this case we want to know how large n would have to be for R_n to be less than 0.0005 (remember, correct to n decimal places translates to the error being less than $0.5 * 10^{-n}$). So that means since:

$$R_n \leq \int_n^\infty \frac{1}{x^3} dx = \frac{1}{2n^2}$$

We want n such that:

$$\frac{1}{2n^2} \leq 0.0005$$

Solving this inequality gives us:

$$1 \leq 0.001n^2$$
$$1000 \leq n^2$$
$$31.62 \leq n$$

You need 32 terms or more to get an approximation that is within 3 decimal places.

Exercise: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 10 terms. Estimate the error involved in this approximation.

b) Approximate the sum of the series using your error and your approximation.

c) How many terms are needed to ensure the sum is accurate to 5 decimal places?