7.2) Trigonometric Integrals

-We continue our work from the end of the last section by taking a look at some more trigonometric function powers and their integrals:

Example: Integrate $\int cos^3(x)dx$

Solution: Using substitution will not work here, since letting u = cos(x) gives us the differential du= -sin(x)dx, which adds a factor of sin(x) that will not disappear. Instead, a different kind of substitution can work here:

$$sin^2(x) + cos^2(x) = 1, \qquad tan^2(x) + 1 = sec^2(x),$$
$$1 + cot^2(x) = csc^2(x)$$

We will be making use of these trigonometric identities (and more) in this section, so you will want to remember these. Since $cos^3(x) = cos(x)(cos^2(x))$, we can replace $cos^2(x)$ with $1 - sin^2(x)$ and integrate from there:

$$\int cos^3(x)dx = \int cos(x)cos^2(x)dx = \int cos(x)(1 - sin^2(x))dx = \int cos(x)dx - \int cos(x)sin^2(x)dx$$

We will use integration by substitution for the second integral, where $u = sin(x)$, $du = cos(x)dx$

$$\int cos^3(x)dx = sin(x) - \int u^2du = sin(x) - \frac{u^3}{3} + C = sin(x) - \frac{sin^3(x)}{3} + C$$

-Remember, when you have an integral of the form $\int sin(x)cos^n(x)dx$ or $\int cos(x)sin^n(x)dx$, you can

integrate by substitution to get $\int sin(x)cos^n(x)dx = -\frac{cos^{n+1}(x)}{n+1} + C$ and

$\int cos(x)sin^n(x)dx = \frac{sin^{n+1}(x)}{n+1} + C$. By using proper substitution with the Pythagorean identities above

allows you to integrate odd powers of sine and cosine, as well as odd powers times other powers:

Example: Integrate $\int cos^2(x)sin^5(x)dx$.

Solution: You may think the proper procedure here is to replace $cos^2(x)$ so that the expression inside is all in terms of sine, however integration by substitution when it comes to sine and cosine requires both. In this case, try to use trigonometric identities on $sin^5(x)$:

$$\int cos^2(x)sin^5(x)dx = \int cos^2(x)(sin^2(x))^2(sin(x))dx = \int cos^2(x)(1 - cos^2(x))^2(sin(x))dx$$

$$\int cos^2(x)sin^5(x)dx = \int cos^2(x)(1 - 2cos^2(x) + cos^4(x))(sin(x))dx = \int (cos^2(x) - 2cos^4(x) + cos^6(x))(sin(x))dx$$

The odd power of sine is advantageous here as it leaves behind a factor of sin(x) we can use for integration by substitution:

$$\int cos^2(x)sin^5(x)dx = \int(cos^2(x)sin(x) - 2cos^4(x)sin(x) + cos^6(x)(sin(x))dx$$

$$\int cos^2(x)sin^5(x)dx = \frac{cos^3(x)}{3} - \frac{2cos^5(x)}{5} + \frac{cos^7(x)}{7} + C$$

Exercise: Integrate $\int sin^4(x)cos^3(x)dx$.

-Odd powers of sine and cosine allow us to use Pythagorean identities that will leave behind a single factor that can be our integration by substitution factor. But what if we are integrating even powers of sine or cosine? In these cases, you may want to use another trigonometric identity you hopefully remember from precalculus:

$$sin^2(x) = \frac{1}{2}(1 - cos(2x)) \qquad cos^2(x) = \frac{1}{2}(1 + cos(2x))$$

Example: Evaluate $\int_0^{\pi/2} cos^2(x)dx$

Solution: Using a Pythagorean identity will not help here. You will simply be trading one square of a trigonometric function for another, and there would still be no single powers of sine or cosine left for substitution. So instead, why not use one of the formulas for reducing powers seen above?

$$\int_0^{\pi/2} cos^2(x)dx = \int_0^{\pi/2} \frac{1}{2}(1 + cos(2x))dx = \frac{1}{2}(x + \frac{1}{2}sin(2x) \mid_0^{\pi/2}) = \frac{1}{2}((\frac{\pi}{2} + 0) - (0)) = \frac{\pi}{4}$$

-Integration by substitution is still used here, but unlike odd powers of trigonometric functions, the substitution factors are constant, so the procedure is much easier. That said, these questions can be a lot more difficult as the powers get larger:

Example: Integrate $\int sin^4(x)dx$

Solution: Here we will have to employ the formula for reducing powers multiple times:

$$\int sin^4(x)dx = \int (sin^2(x))^2dx = \int(\frac{1-cos(2x)}{2})^2dx = \int \frac{1-2cos(2x)+cos^2(2x)}{4}dx$$

$$\int sin^4(x)dx = \frac{1}{4}\int dx - \frac{2}{4}\int cos(2x)dx + \frac{1}{4}\int cos^2(2x)dx$$

The first two integrals can be integrated now, but the third requires yet another use of the formula:

$$\int sin^4(x)dx = \frac{1}{4}\int dx - \frac{2}{4}\int cos(2x)dx + \frac{1}{4}\int \frac{1+cos(4x)}{2}dx$$

$$\int sin^4(x)dx = \frac{1}{4}\int dx - \frac{2}{4}\int cos(2x)dx + \frac{1}{8}\int dx + \frac{1}{8}\int cos(4x)dx$$

No more higher powers, so we can finally integrate now:

$$\int sin^4(x)dx = \frac{1}{4}x - \frac{1}{4}sin(2x) + \frac{1}{8}x + \frac{1}{8}\frac{sin(4x)}{4} + C = \frac{3}{8}x - \frac{1}{4}sin(2x) + \frac{1}{32}sin(4x) + C$$

Strategies for Integrating $\int sin^m(x)cos^n(x)dx$:

-m is odd and equal to $2k + 1$ for some k

Save one factor of sin(x) by using $sin^2(x) = 1 - cos^2(x)$ and express the remaining factors in terms of cos(x).

$$\int sin^m(x)cos^n(x)dx = \int sin(x)(sin^2(x))^k cos^n(x)dx = \int sin(x)(1 - cos^2(x))^k cos^n(x)dx$$

-n is odd and equal to $2k + 1$ for some k

Save one factor of cos(x) by using $cos^2(x) = 1 - sin^2(x)$ and express the remaining factors in terms of cos(x).

$$\int sin^m(x)cos^n(x)dx = \int sin^m(x)(cos^2(x))^k cos(x)dx = \int sin(x)(1 - sin^2(x))^k cos^n(x)dx$$

-n and m are both even

Use the formulas for reducing powers and the half-angle identities:

$sin^2(x) = \frac{1}{2}(1 - cos(2x))$ $\qquad$ $cos^2(x) = \frac{1}{2}(1 + cos(2x))$

$sin(x)cos(x) = \frac{1}{2}sin(2x))$

-If both m and n are odd, then you can choose which factor to keep (sin(x) or cos(x)), but it helps to save a factor for the function that has a smaller power.

Exercise: Integrate the following:

a) $\int cos^4(x)sin^2(x)dx$ $\qquad$ b) $\int sin^5(x)cos^3(x)dx$.

-With all the strategies we have seen for sin(x) and cos(x), what about the other trigonometric functions? We will take a look at how tan(x) and sec(x) are also linked and have integration strategies next. cot(x) and csc(x) have similar integration strategies to tan(x) and sec(x), so we won't spend a lot of time on them.

-First, we will take a moment to remind ourselves of the derivatives and antiderivatives of sec(x) and tan(x):

$\frac{d}{dx}[tan(x)] = sec^2(x)$ $\qquad$ $\frac{d}{dx}[sec(x)] = sec(x)tan(x)$

$$\int tan(x) = ln(sec(x)) + C \qquad \int sec(x) = ln(sec(x) + tan(x)) + C$$

Example: Integrate $\int tan^6(x)sec^4(x)dx$.

Solution: The Pythagorean identity for tangent and secant you might recall is: $tan^2 + 1 = sec^2(x)$. So you have the option of rewriting secant or tangent in terms of the other. In this case, consider the following:

$$\int tan^6(x)sec^4(x)dx = \int tan^6(x)(tan^2(x) + 1)sec^2(x)dx = \int tan^8(x)sec^2(x)dx + \int tan^6(x)sec^2(x)dx$$

Why keep the factor of $sec^2(x)$? Because $sec^2(x)$ is the derivative of tan(x), so by integration by substitution, if $u = tan(x)$, then $du = sec^2(x)dx$. That makes the integrals look like:

$$\int tan^6(x)sec^4(x)dx = \int tan^8(x)sec^2(x)dx + \int tan^6(x)sec^2(x)dx = \int u^8du + \int u^6du = \frac{u^9}{9} + \frac{u^7}{7} + C = \frac{tan^9(x)}{9} + \frac{tan^7(x)}{7} + C$$

-Keep track of factors of $sec^2(x)$ and $sec(x)tan(x)$, as these are the derivatives of tan(x) and sec(x) respectively, and can serve as differential factors for integration by substitution.

Example: Integrate $\int tan^7(x)sec^3(x)dx$

Solution: You can save one factor of tan(x) and one factor of sec(x) for integration by substitution later, and the remaining terms and factors should all be written in terms of sec(x):

$$\int tan^7(x)sec^3(x)dx = \int (tan^2(x))^3sec^2(x)(sec(x)tan(x))dx = \int(sec^2x - 1)^3sec^2(x)sec(x)tan(x)dx$$

$$\int tan^7(x)sec^3(x)dx = \int(sec^6(x) - 3sec^4(x) + 3sec^2(x) - 1)sec^2(x)sec(x)tan(x)dx$$

$$\int tan^7(x)sec^3(x)dx = \int(sec^8(x) - 3sec^6(x) + 3sec^4(x) - sec^2(x))sec(x)tan(x)dx$$

It was important that we write in terms of sec(x), not tan(x), since the leftover factor is the derivative of sec(x), not tan(x). Thus, we now let u = sec(x), and du=sec(x)tan(x)dx:

$$\int tan^7(x)sec^3(x)dx = \int(u^8 - 3u^6 + 3u^4 - u^2)du = \frac{u^9}{9} - \frac{3u^7}{7} + \frac{3u^5}{5} - \frac{u^3}{3} + C$$

Strategies for Integrating $\int tan^m(x)sec^n(x)dx$:

-m is odd and equal to $2k + 1$ for some k

Save one factor of tan(x) and one factor of sec(x) by using $tan^2(x) = sec^2(x) - 1$ and express the remaining factors in terms of sec(x).

$$\int tan^m(x)sec^n(x)dx = \int (tan^2(x))^k sec^{n-1}(x)tan(x)sec(x)dx = \int (sec^2(x) - 1)^k sec^{n-1}(x)tan(x)sec(x)dx$$

-n is even and equal to 2k for some k:

Save one factor of $sec^2(x)$ by using $sec^2(x) = tan^2(x) + 1$ and express the remaining factors in terms of tan(x).

$$\int tan^m(x)sec^n(x)dx = \int tan^m(x)sec^{2k-2}(x)sec^2(x)dx = \int tan^m(x)(1 + tan^2(x))^{k-1}sec^2(x)dx$$

-Scenarios not listed above can still be integrated, but require other tools, like integration by parts:

Example: Integrate $\int sec^3(x)dx$.

Solution: It's actually easiest here to integrate by parts:

$u = sec(x)$ $\quad du = sec(x)tan(x)dx$ $\quad dv = sec^2(x)dx$ $\quad v = tan(x)$

$$\int sec^3(x)dx = sec(x)tan(x) - \int sec(x)tan^2(x)dx$$

Now you can use the Pythagorean identities:

$$\int sec^3(x)dx = sec(x)tan(x) - \int sec(x)(sec^2(x) + 1)dx$$

$$\int sec^3(x)dx = sec(x)tan(x) - \int sec^3(x)dx - \int sec(x)dx$$

$$2\int sec^3(x)dx = sec(x)tan(x) - \int sec(x)dx$$

$$\int sec^3(x)dx = \frac{1}{2}\left(sec(x)tan(x) - \int sec(x)dx\right)$$

$$\int sec^3(x)dx = \frac{1}{2}(sec(x)tan(x) - ln|sec(x) + tan(x)|) + C$$

-Using pythagorean identities from the start would have worked here too, but might have resulted in more circular logic and would have taken longer to complete. That's not to say it's never a bad idea to start off with that idea:

Example: Integrate $\int tan^3(x)dx$

Solution: Here we will start by using $tan^2(x) = sec^2(x) - 1$.

$$\int tan^3(x)dx = \int tan(x)(tan^2(x)dx = \int(tan(x)sec^2(x) - tan(x))dx$$

$$\int tan^3(x)dx = \int sec(x)(tan(x)sec(x))dx - \int tan(x)dx$$

From here, the first integral can be integrated with substitution, let u = sec(x) and du=sec(x)tan(x):

$$\int tan^3(x)dx = \int udu - \int tan(x)dx$$

$$\int tan^3(x)dx = \frac{1}{2}u^2 - ln|sec(x)| + C = \frac{1}{2}(sec(x))^2 - ln|sec(x)| + C$$

-If you were integrating anything that had to do with powers of cot(x) or csc(x), you may want to remember the following:

$$\frac{d}{dx}[cot(x)] = - csc^2(x) \qquad \frac{d}{dx}[csc(x)] = - csc(x)cot(x)$$

$$\int cot(x) = ln(sin(x)) + C \qquad \int csc(x) = - ln(csc(x) + cot(x)) + C$$

-One last topic for this section: the product identities are often glossed over in precalculus or not covered at all, but they can be of great help when rewriting integral expressions:

$$sin(A)cos(B) = \frac{1}{2}(sin(A - B) + sin(A + B))$$

$$sin(A)sin(B) = \frac{1}{2}(cos(A - B) - cos(A + B))$$

$$cos(A)cos(B) = \frac{1}{2}(cos(A + B) - cos(A - B))$$

Exercises: Integrate the following:

a) $\int csc^4(x)cot^3(x)dx$ b) $\int tan^5(x)dx$ c) $\int cos(5x)sin(3x)dx$