6.4) Work:

-**Work** has many meanings in everyday language, but in physics its meaning depends on the idea of **force**. Force can be used to describe a push or pull on an object, like say a push of a door swinging open. An object moving in a straight line with a position object $s(t)$, then the **force** on that object is the product of its mass (m) and acceleration (a). This is also known as Newton's famous Second Law of Motion:

$$F = ma = m\frac{d^2s}{dt^2}$$

-You probably remember that acceleration is the second derivative of position, so acceleration can be replaced with $s''(t)$. For constant acceleration scenarios (like say, any physics problem where the acceleration is gravity) the force would also be constant (since objects don't tend to change mass rapidly) and the **work** done is the product of force F and distance the object moves as a result of that force, d:

$$W = Fd, \qquad \text{Work} = \text{(Force)(Distance)}$$

-The unit of measure for force is newtons $(N = kg \ * \ m/s^2)$ and when newtons are multiplied by a unit of distance, you have newtons times meters which is Joules, the unit for work, also known as a newton-meter. Force can also be measured in pounds, and if the force is multiplied by distance in feet (match US standard units) the work unit is a foot-pound, which is worth 1.36 Joules.

Example: a) How much work is done lifting a 1.2 kg book off the ground to put on a 0.7 m high shelf?
b) How much work is done lifting a 20 pound weight 6 feet off the ground?

Solution: a) We need force before we can find work, and force is mass times acceleration. We have a mass of 1.2 kg, and the acceleration is gravity, which in metric units is $9.8\ m/s^2$, (remember this!) so the force is:

$$F = 1.2 * 9.8 = 11.76\ N$$

Which means the work done is going to be this force times the distance: $W = 11.76 * 0.7 \ = \ 8.2\ J$.

b) For this case the force is given already. Remember, pounds are a unit of weight, which is a force (unlike mass which needs to be multiplied by gravity first to become a weight), so we have $F = 20\ lbs$ already. All we need is to multiply this by the distance traveled: $W = 20 \ * 6 \ = \ 120\,ft \ - lbs$.

-If force is constant then W=Fd, but force can be a variable ; as an object travels from one point a to another point b, forces can act upon that object as it travels. Ever drive next to a semi-truck and it feels like it is drawing you in? That's the gravitational force of a large object attracting a smaller object. So if force is a function depending on position, x, we call it f(x), a continuous function. Can you find the work done to move an object from one point to another if the force being applied to it is changing from one point a to another point b?

-Since force can change from one point to another, we divide the interval [a,b] into subintervals $[x_0, x_1]$, $[x_1, x_2]$, and so on, choose a sample point x_i^* in the subinterval $[x_{i-1}{}^*, x_i]$. The work being done on the object in this particular interval (call it W_i) can be approximated by multiplying the distance from one end of the interval to another (call it Δx) by the force exerted on the object at x_i^*.

$$W_i \approx f(x_i^*)\Delta x$$

-If we do this for every interval and add up all these amounts of work, we get an approximate amount of work done to get the object from point a to point b:

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

-As the intervals get smaller and smaller (you guessed it) the approximate work done approaches the actual work done to get the object from point a to point b:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

-Finally, this means the work done to get an object from point a to point b is in fact an integral. A definite integral from a to b of the continuous force function f(x):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

Example: When a particle is located a distance x feet from an origin of force, $f(x) = x^2 + 2x$ pounds acts on it. How much work is done moving it from x = 1 foot to x = 3 feet?

Solution: The work done is found by integrating the force exerted on it from one point to another:

$$W = \int_1^3 (x^2 + 2x)dx = \tfrac{1}{3}x^3 + x^2|^3{}_1 = (18) - (\tfrac{4}{3}) = \tfrac{50}{3} \text{ foot-lbs.}$$

-A famous physics law, **Hooke's Law** states that the force required to maintain a spring stretch x units beyond its resting point (also called its natural length) is proportional to x. In other words:

$f(x) = kx$ for some constant k called the **spring constant**

Example: A spring needs a force of 40 N to hold a spring at a length of 15 cm when its natural length is only 10 cm. How much work is needed to stretch that same spring from 15 cm to 18 cm?

Solution: Since this is a spring, we know that the force function is $f(x) = kx$, but can we find k? Remember, x should be the meters past the springs natural length, so if it takes 40 N to keep the spring at 0.15 m, which is 0.05 m more than its natural length, we have:

$$40 = k(0.05)$$

So k is 800 (don't worry about units), and so the work needed to stretch the spring from 0.15 cm to 0.18 cm will be the integral of f(x) = 800x from 0.05 m past resting length to 0.08 m past resting length:

$$W = \int_{0.05}^{0.08} 800x dx = 400x^2|^{0.08}{}_{0.05} = 400(0.0064 - 0.0025) = 1.56\ J$$

-Remember, the standard length is meters, and the standard mass is kg. You may need to adjust your units of measurement here and there.

Exercise: A spring needs a force of 60 N to hold a spring at a length of 8 cm when its natural length is 12cm. How much work is needed to push that same spring from 10 cm to 4 cm?

Example: A 200-lb cable is 100 feet long and hangs vertically from the top of the building.

a) How much work is required to lift the cable to the top of the building?
b) How much work is required to pull up 20 feet of the cable alone?

Solution: Unlike the weight example from earlier, the force of the cable as it moves up the building is not constant. The cable may be 200 pounds of weight/force, but depending on how much of the cable still needs to be pulled up, the amount of weight/force is going to change.

a) Pretend that the origin in 2-dimensional space is at the top of the building and the x-axis is pointing downward where the cable is going, and that the cable is divided into subintervals of length Δx each, so if x_i^* is a point on the ith such interval, all points in that interval are lifted by about x_i^* feet in order to get the cable up the building entirely.

Since the cable weighs 2 pounds per foot, the weight that each subinterval weighs is $2\Delta x$ pounds, and so the work needed to lift this subinterval up the building is the force $2\Delta x$ times the distance, x_i^*, or if you wish, 2xdx. If we integrate this from x = 0 to x = 100, we will have the total work needed to lift the entire cable:

$$W = \int_0^{100} 2x dx = x^2|_0^{100} = 10000 \text{ ft - lbs}$$

Keep in mind, the procedure worked out this way due to the fact that we left x be the distance each part of the cable was from the top of the building. You could have also let x be the distance each part of the cable was from the bottom of the cable, in which case the integral would have looked like this:

$$W = \int_0^{100} 2(100 - x)dx$$

This would give you the same answer, so there's more than one way to answer the question.

b) For this part of the problem we will still let x be the distance each part of the cable is from the top of the building. This time however, we are only pulling 20 feet of the cable up. The entire 100 foot cable has to move, but only 20 of those feet will reach the top. How much work does it take for just those 20 feet to reach the top of the building? We will find the work needed for this first 20 feet and call it W_1, we'll do the rest later.

For that, we can use the same integral that we used in part a, but the limit will only go from the top of the cable, where x = 0, to 20 feet down the cable, where x = 20, since this makes up the 20 feet that reach the top:

$$W = \int_0^{20} 2x dx = x^2|_0^{20} = 400 \text{ ft - lbs}$$

Why separate the 20 feet that reach the top of the building from the other 80 feet? Since this is the section of the cable that is reaching the top, the force on these first 20 feet vary as x goes from 0 to 20. The other 80 feet do not reach the top however, so the force for them is a constant number.

How much force will it take to move every other point of the cable? The res of the cable all moves 20 feet upwards, and the cable still weighs 2 pounds per foot, so they all have a force of $20 * 2 = 40$ pounds.

That means the work (call it W_2) needed to lift the rest of the cable, from x = 20 to x = 100, 20 feet upwards is:

$$W_2 = \int_{20}^{100} (20) * 2dx = 40x|_{20}^{100} = 3200 \text{ ft - lbs}$$

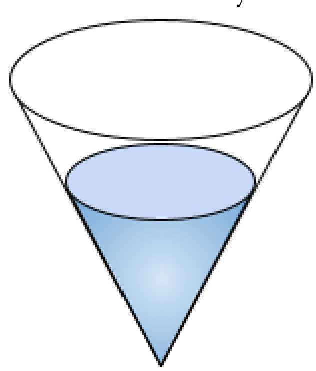
So the total work needed to lift the cable 20 feet is $W_1 + W_2 = 3600$ ft - lbs. Consider that the amount of work needed to lift the cable is not constantly increasing from start to finish; the first 20 feet of cable will be harder to pull up than the next 20 feet, or the last 20 feet (makes sense, at the start there's more cable to lift, so it's heavier).

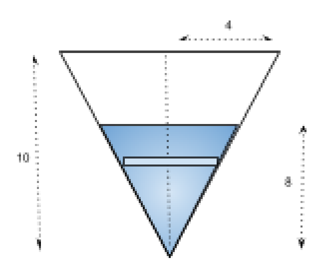
Exercise: A 30-lb rope is 20 feet long and hangs vertically down a well, fully extended. (there's no bucket at the end)

a) How much work is required to pull the rope up completely?
b) How much work is required to pull only half of the rope up?

-This next example is a classic:

Example: A tank has the shape of an inverted circular cone with height 10 meters and base radius 4 meters. It is filled with water to a height of 8 meters. Find the work required to empty the tank b pumping all the water out the top of the tank. Note: The density of water is $1000\ kg/m^3$





Solution: The pictures above are the conical tank on the left and a side view on the right with dimensions of labeled and one "disk" of water represented by the rectangle.

The distance the water inside the tank has to travel to get out the top ranges from 2 m to 10 m, so imagine the interval [2, 10] divided into subintervals, with a generic point x_i^* in the ith interval. This in turn divides this "cone" of water that has a height of 8 meters into layers or disks of water. How much work does it take to transport any individual disk of water out the top of the tank? We'll call this disk of water the ith disk, with radius r_i and height Δx.

What is important here is what the radius of this disk of water is. This radius will depend on exactly how deep this layer of water is. The disk is x_i^* meters from the top, so that means it is $10 - x_i^*$ m from the bottom. So since the entire cross-section has a height of 10 m and a radius of 4 m, by similar triangles, we have the following ratio:

$$\frac{4}{10} = \frac{r_i}{10 - x_i^*}$$

The radius therefore, in terms of x_i^*, is $r_i = \frac{2}{5}(10 - x_i^*)$. So now that we have a radius for this disk of water and a height of Δx, we know that the volume of this disk of water is:

$$V_i = (\tfrac{2}{5}(10 - x_i^*))^2 \pi * \Delta x = \tfrac{4\pi}{25}(10 - x_i^*)^2 \Delta x$$

However, what we need is force/weight. To determine the weight of this disk of water, we first need the mass of this disk of water, call it m_i. We find that by multiplying the volume of this disk of water by the density of the water, which is equal to $1000\ kg/m^3$, so:

$$m_i = V_i * \rho = \tfrac{4\pi}{25}(10 - x_i^*)^2 \Delta x * 1000 = 160\pi(10 - x_i^*)^2 \Delta x$$

Now we need the force/weight, which is still mass times acceleration, or in this case, gravity:

$$F_i = m_i a = 160\pi(10 - x_i^*)^2 \Delta x \ * \ 9.8 = 1568\pi(10 - x_i^*)^2 \Delta x$$

So how much work does it take to get this disk out of the top of the tank? This weight/force now needs to be multiplied by the distance it has to travel to get out the top of the tank, which was x_i^*. Therefore the work for this disk, W_i is approximately equal to:

$$W_i \approx F_i * x_i^* = 1568\pi x_i^*(10 - x_i^*)^2 \Delta x$$

The total work to get all the disks out will therefore be the sum of all the work for each disk, and if there are an infinite number of disks, the sum will equal the total work:

$$W \approx \lim_{n\to\infty} \sum_{i=1}^{n} F_i * x_i^* = \lim_{n\to\infty} \sum_{i=1}^{n} 1568\pi x_i^*(10 - x_i^*)^2 \Delta x$$

By Riemann sums, this will be an integral from x = 2 to x = 10:

$$W = \int_2^{10} 1568\pi x(10 - x)^2 dx = 1568\pi \int_2^{10} (100x - 20x^2 + x^3) dx$$

$$W = \int_2^{10} 1568\pi x(10 - x)^2 dx = 1568\pi(50x^2 - \tfrac{20}{3}x^3 + \tfrac{x^4}{4}\Big|_2^{10}) = 1568\pi * (\tfrac{2500}{3} - \tfrac{452}{3})$$

$$W = \int_{2}^{10} 1568\pi x(10 - x)^2 dx = 1568\pi * (\frac{2048}{3}) \ = \ 3363837.797 \ J$$

Exercise: A tank has the shape of an inverted circular cone with height 6 meters and base radius 4 meters. It is filled with water to a height of 5 meters. Find the work required to empty the tank b pumping all the water out the top of the tank. Note: The density of water is $1000 \ kg/m^3$