11.10) Taylor Series and Maclaurin Series:

-What other functions can be expressed by a power series? All the examples from last section were rational functions or derivatives and integrals of them. There has to be others, right?

-Well, suppose f is a general function that can be expressed by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \qquad |x-a| < R$$

-The interval of convergence is given too. Can we solve for the missing coefficients, c_0, c_1, and so on? The nice thing about a function that looks like this is that since all the terms have a factor of (x-a) attached, if you plug in x=a, those terms all disappear, except one.

$$f(a) = c_0$$

-So the constant is f(a). What about the next coefficient? To solve we need another function, so we will get one by taking the derivative of the power series expansion:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$$

-If we plug a into the first derivative, everything disappears again, except one term:

$$f'(a) = c_1$$

-You may have spotted the pattern by now. We can find all the derivatives we need by taking the derivative again and again and plugging x=a into those derivatives again and again:

$$f''(a) = 2c_2$$
$$f'''(a) = 6c_3$$
$$f^{(4)}(a) = 24c_4$$
$$\ldots\ldots\ldots$$
$$f^{(n)}(a) = (n!)c_n$$

-This means we can find the coefficients of this general power series expansion by the following:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Therefore, if f has a power series expansion at a, which is to say:

$$\text{If } f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad \text{where } |x-a| < R$$

Then its coefficients are given by the formula:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

-In other words, if f has a power series expansion at x=a, then it will look like this:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \ldots$$

-This series is commonly used to the point that it has its own name: **a Taylor series of the function f at a** (or **about a** or **centered at a**). If the Taylor series is centered at a = 0, it has another special number, **a Maclaurin series**:

$$f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)(x) + \frac{f''(0)}{2}(x)^2 + \frac{f'''(0)}{6}(x)^3 + \ldots \qquad \text{where } |x| < 1$$

Example: Confirm that the power series expansion of $\frac{1}{1-x}$ at x = 0 is $\sum_{n=0}^{\infty} x^n$ using a Maclaurin series.

Solution: We know already that $\sum_{n=0}^{\infty} x^n$ is the power series expansion, but we need to prove it so by finding the Maclaurin series for $f(x) = \frac{1}{1-x}$ at x = 0. The Maclaurin series expansion requires us to know the coefficients, and the coefficients requires the derivatives of f(x), which thankfully are not too hard to find here:

$$f(x) = (1-x)^{-1}, \qquad f'(x) = (1)(1-x)^{-2}, \qquad f''(x) = (2)(1-x)^{-3}, \qquad f'''(x) = (6)(1-x)^{-4}, \ldots$$

You have probably figured out the pattern for the nth derivative is $f^{(n)}(x) = (n!)(1-x)^{-(n+1)}$

More importantly though, we need the values of these derivatives when x = 0. When x = 0, all these derivatives become:

$$f^{(n)}(0) = (n!)(1)^{-(n+1)} = n!$$

So this means that the Maclaurin series for $\frac{1}{1-x}$ is:

$$f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = \sum_{n=0}^{\infty} \frac{n!}{n!}(x)^n = \sum_{n=0}^{\infty} x^n$$

Therefore we have confirmed that the power series expansion for $\frac{1}{1-x}$ at x = 0 is $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

-As you can imagine, Maclaurin series are not particularly interesting when created for functions that have derivatives that disappear after a finite order. So you can expect most Maclaurin series examples to involve functions with derivatives that never disappear. Let's take a look at arguably the most famous example of such:

Example: For the function $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

Solution: In this case, since $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for every derivative of f(x), and so that means we have:

$$f^{(n)}(0) = e^0 = 1$$

Therefore, the Maclaurin series is:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = \sum_{n=0}^{\infty} \frac{(x)^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Now that we know the Maclaurin series for e^x, we must find the radius of convergence, which requires the Ratio test. Since $e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$, we let $a_n = \frac{x^n}{n!}$, so by the Ratio test we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \left|\frac{x^{n+1}}{x^n} * \frac{n!}{(n+1)!}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} < 1$$

Since n+1 is going to infinity, no matter what finite value you choose for x, this ratio will go to 0 and be less than 1. Therefore, the radius of convergence is infinite ($R = \infty$).

Exercise: For the function $f(x) = e^{2x}$, find the Maclaurin series and its radius of convergence.

-Remember though, just because you can find a power series representation at x=0 for a function, that does not mean the function actually is equal to that representation. Aside from the fact that there may be rounding errors that make the representation equal a value slightly different from what the function equals at the same input, a power series representation is meant to be a merely theoretical equivalent representation of a function.

-So how do you determine if a function has a power series representation, or even whether the function is equal to the sum of its Taylor series?

-A Taylor series is an infinite sum, so if $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then f(x) is the result of taking the limit as n goes to infinity of the nth partial sum of this series. We denote the partial sum of a Taylor series as $T_n(x)$:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

-We call this partial sum the **nth-degree Taylor polynomial of f at a**. If you take the limit of the nth-degree Taylor polynomial of f at a, you should in theory get f(x):

$$\lim_{n\to\infty} T_n(x) = f(x)$$

-Like all partial sums that converge, if you take the nth-degree Taylor polynomial of f at a and subtract it from f(x), you should get a remainder. We will call this **remainder** $R_n(x)$.

$$R_n(x) = f(x) - T_n(x)$$

-Of course, this can also be rewritten as: $f(x) = R_n(x) + T_n(x)$

-What happens if you take the limit on each side of this equation?

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} (R_n(x) + T_n(x))$$

$$f(x) = \lim_{n\to\infty} R_n(x) + \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} R_n(x) + f(x)$$

However, this would imply: $0 = \lim_{n\to\infty} R_n(x)$

-Since $R_n(x)$ is the remainder of the series, if you can show that this remainder expression, $R_n(x)$ has a limit of 0 as n goes to infinity, then you have shown that f and its Taylor series are equivalent.

-If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f at a. Thus, if:

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

-Trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f can be tricky, so we usually use the following theorem:

Taylor's Inequality:

If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ for the Taylor series satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

-The proof for this theorem is in the ebook and textbook, but we'll skip it here.

-As we will see, Taylor's Inequality is useful for showing that $|R_n(x)|$ is bounded above by another expression, $\frac{M}{(n+1)!}|x - a|^{n+1}$, and if you can show that this expression goes to zero as n goes to infinity, then you have shown that $R_n(x)$ goes to 0 as n goes to infinity too.

-There are alternatives to Taylor's Inequality too, as the remainder for a Taylor series can be rewritten:

If $f^{(n+1)}$ is continuous on an interval I and x is on that interval, then $R_n(x)$ can be rewritten as **the integral form of the remainder term**:

$$R_n(x) = \frac{1}{n!}\int_a^x (x - t)^n f^{(n+1)}(t)dt$$

Another formula, **Lagrange's form of the remainder term**,states that there is a number between x and a such that:

$$R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x - a)^{n+1}$$

-Regardless of what method you want to use to show $\lim_{n\to\infty} R_n(x) = 0$, you may want to use the following fact:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for every real number x}$$

-No matter how large x is, the factorial on bottom will eventually cross over the threshold of scaling up by a factor larger than x, and continue to do so an infinite number of times, which will cause the fraction to go to 0.

Example: Prove that e^x is equal to the sum of its Maclaurin series.

Solution: If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ as well, and we need a number M that is larger than $f^{(n+1)}(x)$ on the interval $|x| \leq d$. The largest value of e^x given any x on[-d, d] is e^d, the largest value on the interval. So that means we can let our designated value M be equal to e^d. Thus, by Taylor's Inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

becomes

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for } |x| \leq d$$

So if we take the limit of $\frac{e^d}{(n+1)!}|x|^{n+1}$ as n goes to infinity, we should get:

$$\lim_{n\to\infty} e^d \frac{|x|^{n+1}}{(n+1)!} = e^d \left(\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}\right) = e^d(0) = 0$$

Since the limit of our expression goes to 0 as n goes to infinity, that means the remainder of the Maclaurin series also goes to zero:

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} e^d \frac{|x|^{n+1}}{(n+1)!} = 0$$

Therefore the remainder of the Maclaurin series goes to 0 for all x, and so that means

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} (T_n(x) + R_n(x))$$

$$f(x) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} T_n(x)$$

and e^x is equal to its own Maclaurin series for all x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

-You can even plug values in for x on both sides (like x = 1) to find values of powers of e using power series.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

-Of course, not every power series has to be a Maclaurin series. You could also create a Taylor series for $f(x) = e^x$ at x = 2 for example....

Example: Find the Taylor series for $f(x) = e^x$ at a = 2.

Solution: This time, we are centering the series at 2 instead of zero, so our series will have this form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

Since $f(x) = e^x$ we still have $f^{(n)}(x) = e^x$, so $f^{(n)}(2) = e^2$. That makes our series:

$$f(x) = \sum_{n=0}^{\infty} \tfrac{e^2}{n!}(x-2)^n$$

-You can prove in a manner similar to the previous example that this series has an infinite radius of convergence and that $R_n(x)$ approaches 0 as n goes to infinity, which means

$$e^x = \sum_{n=0}^{\infty} \tfrac{e^2}{n!}(x-2)^n$$

-So is this Taylor series the better series for approximating values of e^x, or is the previous Maclaurin series better? As a general rule, it's better to use a Taylor series that is centered closer to the value of the input. If you had to predict what e^3 was equal to for example, the Taylor series is better because it is centered at 2, while the Maclaurin series is centered at 0, and 2 is closer to 3 than 0 is.

-Let's take a look at some other famous Taylor and Maclaurin series of important functions:

Example: Find the Maclaurin series for sin(x) and prove that it represents sin(x) for all x.

Solution: To find the Maclaurin series for f(x)=sin(x), we will center the series at 0 and we will need the derivatives of f(x)=sin(x) at x=0:

$$\begin{array}{cc} f(x) = sin(x) & f(0) = 0 \\ f'(x) = cos(x) & f'(0) = 1 \\ f''(x) = -sin(x) & f''(0) = 0 \\ f'''(x) = -cos(x) & f'''(0) = -1 \\ f^{(4)}(x) = sin(x) & f^{(4)}(0) = 0 \end{array}$$

....

The derivatives repeat after every 4th order, and half of these derivatives are 0 at x = 0. The derivatives that are not zero are of odd order, and alternate in sign. This means we can think of the series as:

$$f(x) = f(0) + \tfrac{f'(0)}{1!}x + \tfrac{f''(0)}{2!}x^2 + \ldots = \tfrac{1}{1!}x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

So we have a power series representation for f(x)=sin(x). But does it represent sin(x) for all x? We need Taylor's Inequality for this.

$$|R_n(x)| \le \tfrac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

We have that a = 0, but we need M which is greater than any derivative of sin(x). Since all the derivatives of sin(x) are positive or negative sine and cosine, which are bounded above by 1, we can simply let M equal 1.

$$\left|f^{(n)}(x)\right| \le 1$$

That means that the expression in Taylor's inequality is $\frac{|x|^{n+1}}{(n+1)!}$, and as n goes to infinity, this expression goes to 0 for any value of x. Therefore, the remainder $R_n(x)$ goes to zero for any x, which means f(x)=sin(x) is equal to its Maclaurin series.

Example: Find the Maclaurin series for cos(x).

Solution: You could proceed here the way we did in the last example, but remember, g(x)=cos(x) is the derivative of f(x)=sin(x), so we can find the Maclaurin series here by just taking the derivative of our last Maclaurin series.

$$f(x) = sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$f'(x) = \frac{d}{dx}[sin(x)] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$
$$g(x) = cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This series has the same radius of convergence as the last series did, which was infinite. You can even use the same x=a, M=1, and logic to show that the remainder in Taylor's Inequality goes to 0 for all x just like in the last example.

Example: Find the Taylor series for f(x)=sin(x) centered at $x = \frac{\pi}{3}$

Solution: This might not sound too different from the last example, but the fact that we are centered at $x = \frac{\pi}{3}$ instead makes the coefficients quite different for the series:

$$\begin{aligned} f(x) &= sin(x) & f(\tfrac{\pi}{3}) &= \tfrac{\sqrt{3}}{2} \\ f'(x) &= cos(x) & f'(\tfrac{\pi}{3}) &= \tfrac{1}{2} \\ f''(x) &= -sin(x) & f''(\tfrac{\pi}{3}) &= -\tfrac{\sqrt{3}}{2} \\ f'''(x) &= -cos(x) & f'''(\tfrac{\pi}{3}) &= -\tfrac{1}{2} \\ f^{(4)}(x) &= sin(x) & f^{(4)}(\tfrac{\pi}{3}) &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

....

The derivatives repeat after every 4th order. Half of these derivatives are a variation on $\frac{1}{2}$ and the other half are a variation on $\frac{\sqrt{3}}{2}$. So we need to break the summation up into two halves:

$$f(x) = f(\tfrac{\pi}{3}) + \frac{f'(\frac{\pi}{3})}{1!}(x-\tfrac{\pi}{3}) + \frac{f''(\frac{\pi}{3})}{2!}(x-\tfrac{\pi}{3})^2 + \ldots = \frac{\sqrt{3}}{2} + \frac{1}{2(1!)}(x-\tfrac{\pi}{3}) - \frac{\sqrt{3}}{2(2!)}(x-\tfrac{\pi}{3})^2 - \frac{1}{2(3!)}(x-\tfrac{\pi}{3})^3 + \frac{\sqrt{3}}{2(4!)}(x-\tfrac{\pi}{3})^4$$
$$f(x) = \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2(2!)}(x-\tfrac{\pi}{3})^2 + \frac{\sqrt{3}}{2(4!)}(x-\tfrac{\pi}{3})^4 - \ldots\right) + \left(\frac{1}{2(1!)}(x-\tfrac{\pi}{3}) - \frac{1}{2(3!)}(x-\tfrac{\pi}{3})^3 + \frac{1}{2(5!)}(x-\tfrac{\pi}{3})^5 - \ldots\right)$$

$$f(x) = \frac{\sqrt{3}}{2}\left(1 - \frac{(x-\frac{\pi}{3})^2}{2!} + \frac{(x-\frac{\pi}{3})^4}{4!} - \ldots\right) + \frac{1}{2}\left(\frac{(x-\frac{\pi}{3})^1}{1!} - \frac{(x-\frac{\pi}{3})^3}{3!} + \frac{(x-\frac{\pi}{3})^5}{5!} - \ldots\right) = \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\frac{\pi}{3})^{2n}}{(2n)!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\frac{\pi}{3})^{2n+1}}{(2n+1)!}$$

You can replace x with $x - \frac{\pi}{3}$ in Taylor's inequality and use the same M as we did two examples ago if you wish to show that this series represents sin(x) for all x. Therefore we have:

$$sin(x) = \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\frac{\pi}{3})^{2n}}{(2n)!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\frac{\pi}{3})^{2n+1}}{(2n+1)!}$$

Exercise: Find the Taylor series for f(x)=sin(x) centered at $x = \frac{\pi}{4}$

Example: Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution: The derivatives will all use the power rule:

$$f(x) = (1+x)^k \qquad f(0) = 1$$

$$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^k \qquad f(0) = k(k-1)$$

$$\ldots$$

$$f^{(n)}(x) = k(k-1)(k-2)\ldots(k-n+1)(1+x)^k \qquad f(0) = k(k-1)(k-2)\ldots(k-n+1)$$

Don't use factorial notation here, there is no indication that k is an integer. So the Maclaurin series for $f(x) = (1+x)^k$ is:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}x^n$$

-In general we call this series the **binomial series**. This series will terminate if k is a nonnegative integer, which makes the series finite and convergent. What if k wasn't a nonnegative integer though, when does it converge (if ever)?

-We will use the Ratio Test to figure this out:

$$\left|\frac{\frac{k(k-1)(k-2)\ldots(k-n)}{(n+1)!}x^{n+1}}{\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}x^n}\right|$$

$$\left|\frac{k(k-1)(k-2)\ldots(k-n)}{(n+1)!} * \frac{n!}{k(k-1)(k-2)\ldots(k-n+1)}x^n * x\right| = \left|\frac{(k-n)}{n+1}\right| * |x| < 1$$

$\lim_{n\to\infty}\left|\frac{(k-n)}{n+1}\right| = 1$, so the series will converge if $|x| < 1$, and will diverge if $|x| > 1$. As for the endpoints, we'll talk about those later.

-The notation for the coefficients in the binomial series is:

$$\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$$

we call these numbers the **binomial coefficients** (again, we avoid using factorials on top since k does not have to be a positive integer).

The Binomial Series:

If k is any real number, and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \dots$$

-The interval of convergence will always include (-1, 1), but what about x = 1 and x = -1?

-If $-1 < k \le 0$, then the series converges at x = 1.

-If $k > 0$, the series converges at x = -1 and x = 1.

-If k is a positive integer and n > k, the series will terminate since $\binom{k}{n} = 0$ if n>k, so we get an expansion due to the binomial theorem.

Example: For the function $f(x) = \frac{1}{\sqrt{4-x}}$, find the Mclaurin series and its radius of convergence.

Solution: We can use the binomial series procedure to get the series, but we may want to rewrite it a bit first:

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$$

So use the binomial series equation with x replaced with $-\frac{x}{4}$ and $k = -\frac{1}{2}$.

$$\frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-0.5}{n}\left(-\frac{x}{4}\right)^n = \frac{1}{2}\left(1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \left(\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\right)\left(-\frac{x}{4}\right)^2 + \left(\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\right)\left(-\frac{x}{4}\right)^3 + \dots\right)$$

$$\frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-0.5}{n}\left(-\frac{x}{4}\right)^n = \frac{1}{2}\left(1 + \left(\frac{1}{8}\right)x + \left(\frac{1*3}{2!8^2}\right)x^2 + \left(\frac{1*3*5}{3!8^3}\right)x^3 + \dots + \left(\frac{1*3*5*\dots(2n-1)}{n!8^n}\right)x^n + \dots\right)$$

Try not to abbreviate here, it can be a little difficult to write in Riemann summation notation. The interval of convergence contains $\left|\frac{x}{4}\right| < 1$, or $|x| < 4$. The radius therefore is R=4.

Exercise: For the function $f(x) = \frac{1}{\sqrt{1-x}}$, find the Mclaurin series and its radius of convergence.

-Taylor series are fairly arithmetic in their combinations, so it's fairly straightforward to create new Taylor series by adding, subtracting, multiplying, dividing, and of course integrating/differentiating Taylor series. If you need a refresher on the Maclaurin series we have talked about so far, there's a table in the 11.10 section of the ebook.

Example: Find the Maclaurian series for:

a) $f(x) = xcos(x)$ b) $f(x) = ln(1+3x^2)$

Solution: a) f(x) = xcos(x) is merely a previous Maclaurin series multiplied by another x:

$$f(x) = xcos(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$$

The radius of convergence is infinite.

b) Replace x with $3x^2$ in the Maclaurin series for ln(1+x):

$$f(x) = ln(1 + 3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{2n}}{n}$$

For the radius of convergence this time we substitute $3x^2$ for x in $|x| < 1$:

$$\left|3x^2\right| < 1$$
$$\left|x^2\right| < \frac{1}{3}$$
$$|x| < \frac{\sqrt{3}}{3}$$

We have a radius of convergence of $R = \frac{\sqrt{3}}{3}$.

-What about going in reverse? This will simply be a matter of recognizing which power series fits what is given, but of course you may need to substitute an expression in place of x.

Example: Find the function represented by the Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$

Solution: We can rewrite the expression as $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$ due to the matching power. So which Maclaurin series does this appear to match?

The closest is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which is the Maclaurin series for e^x. In our case, we will replace the x with -2x. So this is a Maclaurin series for $f(x) = e^{-2x}$.

-This backwards thinking can help when trying to determine the value of an infinite series:

Example: Find the sum of the series $\frac{1}{1*2} - \frac{1}{2*2^2} + \frac{1}{3*2^3} - \frac{1}{4*2^4} + \ldots$

Solution: If we can recognize that this is an expansion of a Maclaurin series at a particular value of x, we can evaluate the function represented by this Maclaurin series at that value of x to find the sum of the series.

You may be able to spot the rule of this series overall by the alternating terms, and the denominator consisting of two factors, an increasing power of 2, and an arithmetically increasing factor:

$$\frac{1}{1*2} - \frac{1}{2*2^2} + \frac{1}{3*2^3} - \frac{1}{4*2^4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n*2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} * \left(\frac{1}{2}\right)^n$$

The Maclaurin series this matches up with is ln(1+x), but we can recognize that instead of x, we have $x = \frac{1}{2}$.

Therefore, we know this series is ln(1+x) where $x = \frac{1}{2}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} * \left(\frac{1}{2}\right)^n = ln(1 + \frac{1}{2}) = ln(\frac{3}{2})$$

-Taylor series are essentially a means to rewrite functions in different ways, and that can be very helpful in situations where we were unable to progress earlier. Not only did the last example show how a Taylor series can be used to sum up a series that might have seemed impossible before, but we shall see that it gives us the means to integrate expressions we might have thought impossible before.

Example: a) Evaluate $\int e^{-x^2} dx$ as an infinite series.

b) Evaluate $\int_0^1 e^{-x^2} dx$ correct to within an error of 0.001.

Solution: a) We have seen already we do not have the means to integrate this expression, but if we wrote it as a Maclaurin series first, then we would be able to integrate. What is the Maclaurin series for e^{-x^2}?

Start with the Maclaurin series for e^x, and then substitute $-x^2$ in for x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{n!}$$

So if we integrate term-by-term, we get the following:

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{n!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)n!}$$

We will leave C there as there is no initial value condition here. This series equals e^{-x^2} for all x, just like e^x does.

b) To evaluate $\int_0^1 e^{-x^2} dx$ we will use the Fundamental Theorem of Calculus (thankfully, our summation equals 0 when we plug x = 0 in, so we only have one round of evaluations!):

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)n!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} = 1 - \frac{1}{3(1!)} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + ...$$

What will this equal within an error of 0.001? First we need to use the correct number of terms. This is an alternating series, so use every term before the first term that is smaller than the error, $0.001 = \frac{1}{1000}$.

$$1, \frac{1}{3}, \frac{1}{10}, \frac{1}{42}, \frac{1}{216}, \frac{1}{1320}, \ldots$$

Stop at $\frac{1}{1320}$ and add up the numbers that came before it (though they are alternating):

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.7475$$

Exercise: a) Evaluate $\int e^{-x^3}dx$ as an infinite series.

b) Evaluate $\int_0^1 e^{-x^3}dx$ correct to within an error of 0.00001.

Example: Evaluate $\lim\limits_{x\to 0} \frac{e^x-1-x}{x^2}$

Solution: You can use L'Hospital's Rule for this, but it turns out that you can use Maclaurian series here too. After all, e^x has the series $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, so if you substitute that in you will get:

$$\lim_{x\to 0} \frac{\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x}{x^2} = \lim_{x\to 0} \frac{(1+x+\frac{x^2}{2}+\frac{x^3}{6}+..)-1-x}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+...}{x^2} = \lim_{x\to 0} \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + ... = \frac{1}{2}$$

No derivatives necessary!

-As for multiplying or dividing Taylor series, it is essentially very long-winded polynomial multiplication and division. Not the most interesting procedure overall so if you are asked to multiply or divide a Taylor series by another Taylor series, we will usually only ask for the first few terms at the most, as they tend to pull the most weight for the series (they have the lowest power and the smallest denominators, so they larger than the rest).

-One thing to remember however, is that when you multiply two Taylor series, f(x) and g(x) that both converge for the same radius of convergence, $|x| < R$, their product will converge over that same interval, $|x| < R$. As for representation, if the Taylor series for two different functions each represent their functions over $|x| < R$, then the product of those Taylor series will represent the product of those two functions over $|x| < R$ as well. For division this is all true as well, as long as the leading term in the series is not 0.