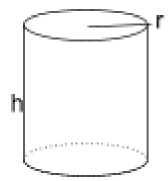
6.2) Volumes:

-How do you find the volume of a **right cylinder**? That is, a **cylinder** is an object consisting of two congruent faces B_1 and B_2 (most famously two circle faces, but they can be a number of shapes) that are parallel to one another and connected by all perpendicular line segments.



-Like this cylinder which has a **base** (that is one of the faces) that is in the shape of a circle of radius r, and another base in the shape of an identical circle h units above it. The cylinder is all space between these two circles bases, including the surface enclosing this space. So the volume that a shape like this takes up would be the space in-between the two circles, which means the volume should be the area of the either the top or bottom base multiplied by the distance between the two bases.

-The base has the shape of a circle, and the area of a circle is πr^2, meaning that the volume of this cylinder is:

$$Volume = \pi r^2 h$$

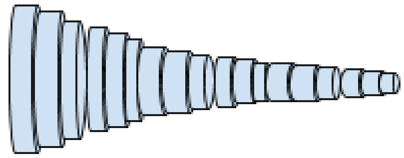
-This is only the case for a cylinder of this shape however. The bases don't have to be circles, they could be any shape they want. They could even be rectangles, in which case the cylinder is more a box than a cylinder, and has a volume of $V = wlh$. But even that volume is found the same way: find the area of one of the bases, multiply it by the height of the object, and you have the volume. That gives us a general formula for area of cylinders:

$$V = Ah$$

where A is the area of the base, and h is the height or distance between the two identical bases.

-But what if we wanted to find the volume of a non-cylindrical object? Remember how we find areas under curves by basically filling up the area with rectangles that have areas that add up to the area under the curve? We do the same here, but in three-dimensions: fill the space with smaller objects that we can more easily find the volumes of and add up the volumes of all these smaller objects, which should be approximately the volume of the space they are occupying.

-You may think that the smaller objects we use here are boxes or blocks, considering the rectangles we used in previous sections, but instead we use **cylinders** to fill up the object we want to find the volume of. Cylinders are just about as easy to find the volume of as boxes are: height times area of the face. So if you wanted the volume of a cone for example, imagine filling the cone with many small cylinders which you can then find the volume of and add up to get the volume of the cylinder.



-Let S be a solid in 3-dimensions that lies between x = a and x = b on the x-axis. To find its volume we “cut” S up into pieces (similar to the strips we used in 2D) along the x-axis, and the volumes of these pieces will be approximated using cylinders, and then all these cylinder volumes are combined to get the volume. These pieces are created from **cross-sections** found by intersecting S with a plane parallel to the bases. Call this plane P_x, since the plane is perpendicular to the x-axis and thus can be characterized by what value of x it passes through on the x-axis.

-If A(x) is the area of the cross-section of S on P_x, then the shape of the cross section has the potential to change from one value of x to another, and thus A(x) varies from one x to another between x=a to x=b. It’s almost like carving a ham into slices from one end to the other; even if you cut them all with equal thickness, some slabs of the ham will have different volumes than others due to the shape of the cross-section. Thus we can find the volume of this solid by adding up the volumes of the individual slabs.

-The slabs will be cut up on subintervals on the x-axis between x=a to x=b by using the planes $P_{x_1}, P_{x_2}, \ldots$ to slice the solid. If we choose the point x_i^* in $\left[x_{i-1}, x_i\right]$ then we can approximate the ith slab S_i by a cylinder with base area $A(x_i^*)$ and “height” Δx.

-Thus the volume of slab S_i is $V(S_i) \approx A(x_i^*)\Delta x$, which makes the volume of the entire solid after it has been sliced into n slabs equal to:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x,$$

-As was the case with the rectangles under a 2D curve, the volume of the solid is better approximated the more slabs there are, so we define the volume of the solid S by the limit of the above sum as n goes to infinity:

Volume:

Let S be a solid that lies between x= a and x= b . If the cross-sectional area of S in the plane P_x passing through x and perpendicular to the x-axis, is A(x), where A is a continuous function, then the **volume** of S is:

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx$$

-In case you are wondering, it doesn’t matter how the solid is situated on the x-axis (whether it is centered at the origin or not), all that matters is that A(x) is the area of a cross-section obtained by slicing through x perpendicular on the x-axis as opposed to another axis in three dimensions.

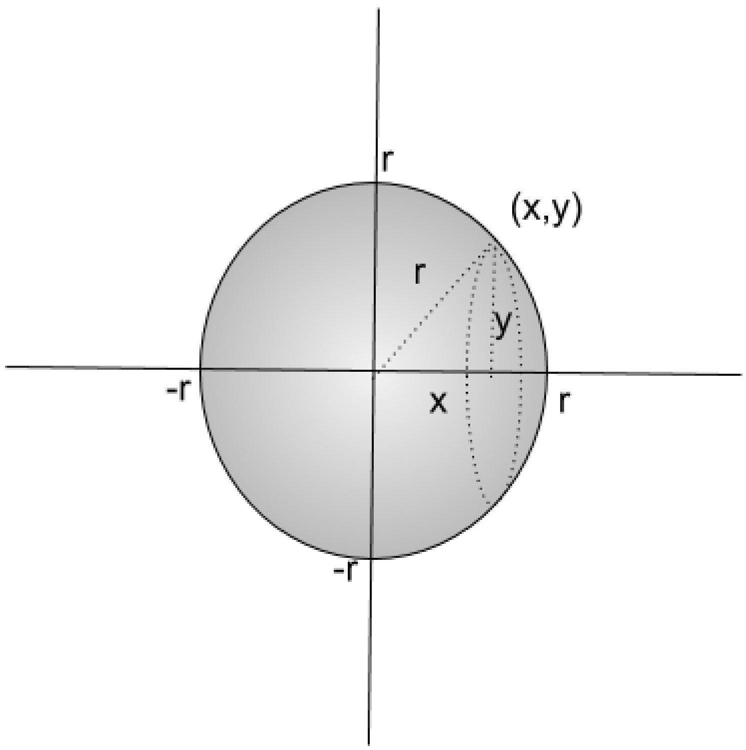
-This serves as a proof why the volume of a cylinder is $V = Ah$. If A is the area of the base, then it is also the constant area of every cross-section from x=a to x=b. Thus, if the height of the cylinder is merely b-a, we have:

$$V = \int_a^b A(x)dx = \int_a^b Adx = Ax\,|^b{}_a = A(b-a) \;=\; Ah$$

-In fact, this integral can explain why many famous volume formulas are what they are. Of course, you will need to find a means of determining the area of the cross-section of any value of x from x = a to x = b.

Example: Show that the volume of a sphere is $V = \frac{4}{3}r^3\pi$.

Solution: What does the cross-section of a sphere look like? It's a round ball, so every cross-section is a circle. What would the radius of the circle in a given cross-section be?



The sphere has a radius of r, and so if you wanted the cross-section at x, you can imagine that the plane P_x intersects the sphere in a circle that has a radius of y. What is y? Since (x,y) is a point on the circle of radius r, y must satisfy the equation $r^2 = x^2 + y^2$. This means $y = \sqrt{r^2 - x^2}$ is the radius of the cross-section circle at x.

However, this also means that the area of this cross-section is $A(x) = \pi y^2 = \pi(r^2 - x^2)$.

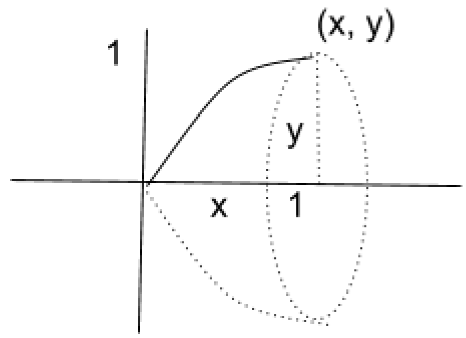
We have an expression for the cross-section area at x, so the volume will be the integral of these cross-section area from x = -r to x = r:

$$V = \int_{-r}^{r} \pi(r^2 - x^2)dx = 2\pi\int_0^r (r^2 - x^2)dx = 2\pi\left(r^2x - \tfrac{1}{3}x^3|^r{}_0\right) = 2\pi(r^3 - \tfrac{1}{3}r^3) = \tfrac{4}{3}\pi r^3$$

-Instead of well-known solids, what if we took any generic 2-dimensional region and rotated it around a straight line? We would have what is called a **solid of revolution**, and we will see that for these solids, cross-sections perpendicular to the line (also called the **axis of rotation**) are circular.

Example: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Solution:



The region rotated around the x-axis creates a series of cross-sections that are all circles with a radius of $y = \sqrt{x}$, which means the area of these cross-sections depend on x and are equal to $A(x) = \pi y^2 = \pi x$. The accompanying cylinder with this cross-section can be considered a disk with a radius of $\sqrt{x}$ and a thickness/height of Δx, and so the volume of this cylinder/disk is $A(x)\Delta x = \pi x \Delta x$.

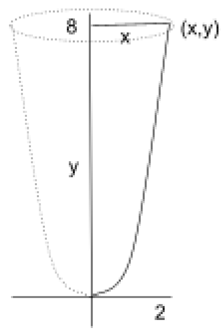
Therefore the solid formed by rotating $y = \sqrt{x}$ around the x-axis between x=0 and x=1 will have a volume of:

$$V = \int_0^1 A(x)dx = \int_0^1 \pi x dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

-You notice that revolving a shape around an axis will create circular cross-sections, which will have some variation of the area of a circle, meaning the only variable to worry about in the cross-section area is the radius of the circle created from rotating the region around the axis of rotation.

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, y = 8, and x=0 about the y-axis.

Solution: Let's visualize the region being rotated and the line it is being rotated around:



This time the solid is formed by rotating around the y-axis, which implies that the cross-sections should occur along the y-axis instead of the x-axis. However, that would require us to integrate with respect to y instead of x, which also means that instead of the cross-sectional area depending on x, they would have to depend on y.

The cross-sectional area should depend on the variable axis that the solid is rotating around, so we need an area function A(y) that depends on y. The cross-sections are still circles, so it is still true that A(y) is the area of a circle, but what is the radius of the cross-sectional circles?

From the picture we can see that the radius of the top cross section at x = 8 is x itself. But the area is still A(y), and must be written in terms of y. So since $y = x^3$, we solve for x to get $\sqrt[3]{y} = x$ is our radius for each A(y).

$$A(y) = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

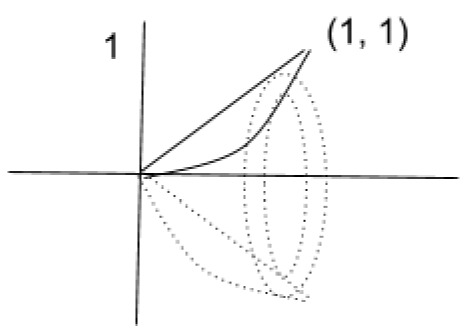
So if that's the area of the cross-section, the accompanying cylinder/disk volume would be $A(y)\Delta y = \pi y^{2/3}\Delta y$. Therefore, the volume would be the integral of this volume from y = 0 to y = 8:

$$V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3}dy = \pi \frac{y^{5/3}}{5/3}\Big|_0^8 = \frac{96\pi}{5}$$

-You might be seeing the pattern so far. Rotating around either axis will involve plugging an equation into a circle equation and integrating over an interval. In this next example however, we may need two.

Example: Find the volume of the solid formed by rotating the region enclosed by the curves $y = x$ and $y = x^2$ around the x-axis.

Solution: y=x and $y = x^2$ intersect at (0,0) and (1,1), but what does the enclosed region between these two curves look like when rotated around the x-axis?



The cross-sections in this case are not circles, but are actually hoops, or **washers**, which are a ring with an inner radius and an outer radius. The inner radius would be equal to the curve that is closer to the axis of rotation, which in this case is $y = x^2$, and the outer radius would be the curve farther from the axis of rotation, in this case $y = x$.

How do you find the area of a washer? It is essentially a larger circle with a smaller circle cut out of the center, so take the area of the larger circle and subtract it by the area of the smaller circle. We have the radii of these two circles, so subtract one area from the other to get:

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

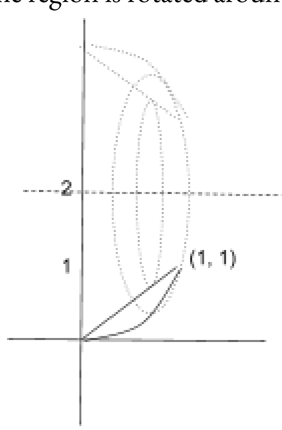
So now we can take the volume of this solid by integrating this cross-sectional area function from 0 to 1:

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^2 - x^4)dx = \pi(\tfrac{x^3}{3} - \tfrac{x^5}{5})|^1{}_0 = \tfrac{\pi}{3} - \tfrac{\pi}{5} = \tfrac{2\pi}{15}$$

-However, sometimes you will have to rotate your solid around a line other than an axis. When this happens, you will have to adjust what the radius of the circular cross-section circles will be, and the radius is essentially what the distance is between the axis of rotation and the line or lines rotating around it.

Example: Find the volume of the solid formed by rotating the region enclosed by the curves $y = x$ and $y = x^2$ around y=2.

Solution: Same region as before, but now the region is rotated around the horizontal line y = 2:



If you take a look at the cross-sections here, we again have washers, which means we can again subtract the area of a large circle by the area of a small circle to get our cross-sectional areas. Since the axis of rotation is no longer the x-axis however, the radii of the two circles has changed.

The smaller circles are now created from rotating y=x around the line y=2. The radius of the smaller circles would therefore be $2 - x$. The larger circles will be created from the other curve, $y = x^2$, and the radius will be $2 - x^2$. So the cross-sectional area would be:

$$A(x) = \pi(2 - x^2)^2 - \pi(2 - x)^2 = \pi(x^4 - 4x^2 + 4 \ - (x^2 - 4x + 4)) = \pi(x^4 - 5x^2 + 4x)$$

Now we can integrate this cross-section from x = 0 to x = 1 (since the enclosed area being rotated didn't change, neither does the limits of the integral):

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^4 - 5x^2 + 4x)dx = \pi(\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2)|^1{}_0 = \frac{\pi}{5} - \frac{5\pi}{3} + 2\pi = \frac{8\pi}{15}$$

-So in general, the volume of a solid of revolution can be calculated by one of the following:

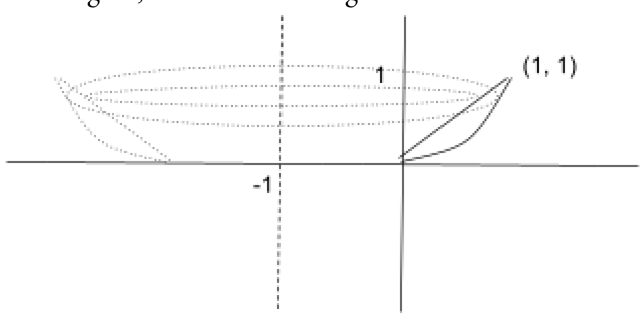
$V = \int_a^b A(x)dx$ if rotated about a horizontal line, or $V = \int_c^d A(y)dy$ if rotated about a vertical line

-We find the cross-sectional area if the cross-section is a disk/circle by finding the radius of the disk in terms of x or y and using $A = \pi(radius)^2$, or

-We find the cross-sectional area if the cross-section is a washer/hoop by finding the radius of the larger circle and the radius of the smaller circle and using $A = \pi(larger\ radius)^2 - \pi(smaller\ radius)^2$

Example: Find the volume of the solid formed by rotating the region enclosed by the curves $y = x$ and $y = x^2$ around x= -1.

Solution: Yet again, it's the same region, but it is now being rotated around the vertical axis x = -1:



The region is rotated around a vertical axis, and cross-sections are washers, so that means we need the integral $V = \int_c^d A(y)dy$ and the cross section area will be of the form $A(y) = \pi(larger\ radius)^2 - \pi(smaller\ radius)^2$.
So what are the radii?

Remember, the cross section area is A(y), so the radius have to be in terms of y, not x. Therefore we need $y = x$ and $y = x^2$ to be solved for x: $x = y$ and $x = \sqrt{y}$. As for the radii themselves, they will use these curves. The curve that is further from x = -1 is $x = \sqrt{y}$, and given any y from 0 to 1, the distance $x = \sqrt{y}$ is from x = -1 is $\sqrt{y} - (-1) = \sqrt{y} + 1$. That makes $\sqrt{y} + 1$ our larger radius. The smaller radius will be found by finding the distance from x = -1 to x = y: $y - (-1) = y + 1$.

We have our radii, so we can find our cross-sectional area function, A(y):

$$A(y) = \pi((\sqrt{y}+1)^2 - (y+1)^2) = \pi(y + 2\sqrt{y} + 1 - (y^2 + 2y + 1)) = \pi(2\sqrt{y} - y - y^2)$$

We can find the volume by integrating A(y) from 0 to 1:

$$V = \int_0^1 A(y)dy = \int_0^1 \pi(2\sqrt{y} - y - y^2)dx = \pi(\frac{2y^{3/2}}{3/2} - \frac{y^2}{2} - \frac{y^3}{3})|^1{}_0 = \frac{4\pi}{3} - \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{2}$$

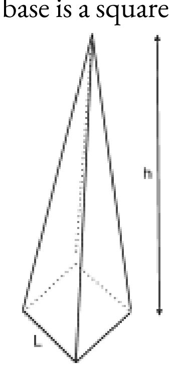
-As you can see, the exact same region rotated in a different way with a different axis resulting in a different distance to rotate around can create a solid with a different volume.

Exercise: Find the volume of the region enclosed by y = $\sqrt{x}$ and $y = \frac{1}{2}x$ rotated around:

a) The x-axis
b) The y-axis
c) The line x = 4

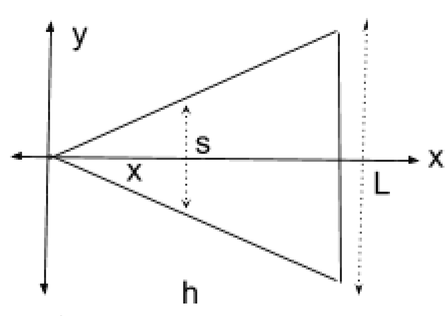
-Finally, we can work on some examples of finding volumes of objects that are not created by rotation, but do have cross-sections that we can find the functions of:

Example: Find the volume of a pyramid whose base is a square with side L and whose height is h.



Solution: Unlike cones, pyramids have a base that is a square, not a circle, and if you took cross-sections from the tip-top to the square base, each cross-section would be a square too. However, to help figure out the A function in this case, we will start by taking a look at a cross-section from the other direction.

Pretend the pyramid is on its side with the tip-top being at the origin in (x,y) space. Then if you took a look at the cross section on the xy-plane you would have:



From the tip-top of the pyramid to the base, every cross section will be perpendicular to the x-axis and will be a square of a different length. We will call the length of the cross-section after traveling x units down the x-axis "s."

However, this means we can use similar triangle ratios to find out what s is in terms of x, h, and :

$$\frac{s}{x} = \frac{L}{h} \qquad \Rightarrow \qquad s = x * \frac{L}{h}$$

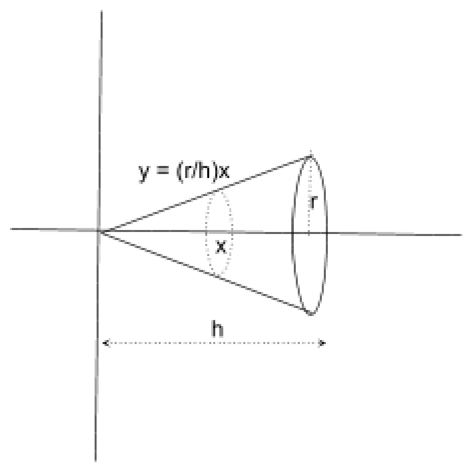
We know that the cross-sections are all squares, so the cross-section of the pyramid x units down the axis will be the square with a length of $s = x * \frac{L}{h}$, which means the area of the cross-section in terms of x is:

$$A(x) = s^2 = x^2 * \frac{L^2}{h^2}$$

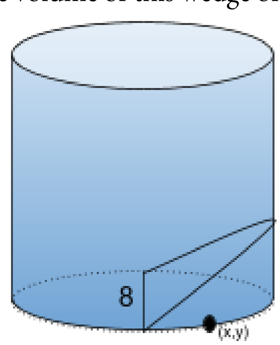
So what will be the volume of the pyramid? We now integrate the cross-section area from x = 0 to x = h:

$$V = \int_0^h A(x)dx = \int_0^h x^2 * \frac{L^2}{h^2}dx = \frac{L^2}{h^2}\int_0^h x^2 dx = \frac{L^2}{h^2}\left(\frac{x^3}{3} \Big|^h_0\right) = \frac{L^2}{h^2}\left(\frac{h^3}{3}\right) = \frac{1}{3}L^2 h$$

Exercise: Show that the volume of a cone is $V = \frac{1}{3}r^2\pi h$. Start with the following picture to get your cross-section area:

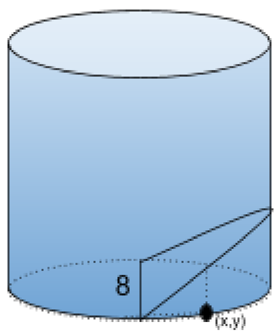


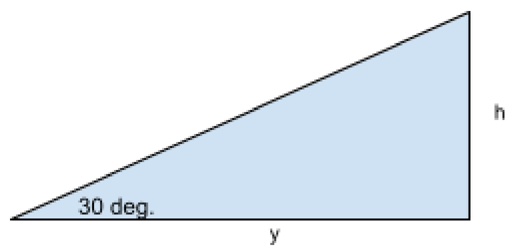
Example: A cylinder of radius 4 is sliced into by a plane that intersects the base along the diameter of the bottom base at an angle of 30°. What is the volume of this wedge of the cylinder that has been cut out?



Solution: Let the x-axis be the diameter where the plane reaches the base. Since the radius of the cylinder is 4, the diameter from one end where the plane meets to the other is 8 inches. However, we can let the origin of the x-axis be anywhere on this length of 8, so we will let x = 0 be the center of the base. This means we will eventually integrate the cross-section of the wedge from x = 4 to x = -4.

What is the shape of the cross-section? You may be able to see it from the earlier picture, but the cross-sections are right-triangles, specifically right triangles with an angle of 30° since that is the angle the plane intersected the base with:





So what are the dimensions of this right triangle? Take another look at the base of the cylinder. It is a circle with a radius of 4, so it satisfies $16 = x^2 + y^2$ where x is the dimension long the diameter and y is the dimension perpendicular to it. However, this means that y is essentially the base length of the right triangle that makes up the cross-section at x.

This is a 30-60-90 right triangle, so we know that if the larger length is y, then by tangent ratio, $tan(30^\circ) = \frac{h}{y}$. Solve for h to get the other side: $h = \frac{\sqrt{3}}{3}y$. Therefore, the area of this right triangle is:

$$A = \frac{1}{2}yh = \frac{1}{2}y * \frac{\sqrt{3}}{3}y = \frac{\sqrt{3}}{6}y^2$$

Remember however, that our cross section area must be in terms of x, not y. Since $16 = x^2 + y^2$, we can replace y^2 with $16 - x^2$. Therefore our cross-section area is $A(x) = \frac{\sqrt{3}}{6}(16 - x^2)$

Finally, we can now integrate our cross-sectional area function from -4 to 4:

$$V = \int_{-4}^{4} A(x)dx = \int_{-4}^{4} (16 - x^2)dx = \frac{\sqrt{3}}{6} * 2\int_{0}^{4}(16 - x^2)dx = \frac{\sqrt{3}}{3}\left(16x - \frac{x^3}{3} \Big|^4{}_0\right) = \frac{\sqrt{3}}{3}\left(\frac{128}{3}\right) = \frac{128\sqrt{3}}{9}$$