11.2) Series:

-So you have an infinite sequence of numbers. What can you do with this infinite sequence of numbers? The natural idea would be to combine them together, and the most natural of combinations would be to add them all together.

-Given an infinite sequence, $\{a_n\}$, the **infinite series**, or just a **series**, would be defined as:

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ... + a_n + ...$$

-Can one truly add together an infinite number of terms in the sequence together? Practically no, but in theory it would be possible to. For starters, you can begin by simply adding the first k terms in an infinite series together, giving us what is called a **partial sum** of a series.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n = s_n$$

-For example, you can have a sequence defined by $\left\{\frac{1}{2^n}\right\}$, which means that a partial sum of just the first 5 terms of the series defined by this sequence is denoted by s_5 and equal to:

$$\sum_{i=1}^{5} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$$

If you took a few more partial terms, you may see a pattern:

The first 6 terms sum up to $s_6 = \frac{63}{64}$

The first 7 terms sum up to $s_7 = \frac{127}{128}$

...

The first n terms sum up to $s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

-So if it can be proven that the partial sum of the first n terms <u>is</u> equal to $1 - \frac{1}{2^n}$, then the infinite sum should approach 1 as n goes to infinity, implying that the series "equals" 1.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} + ... = 1 - \frac{1}{2^n} = 1$$

-We'll get back to this example later, but for now let's talk a little bit about what it means when an infinite series sums up to a finite number.

-In general, the partial sums, $s_1, s_2, s_3, ...$ make up a sequence of their own, a sequence defined by:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

-If we take the limit of this sequence of partial sums as n goes to infinity, and this limit exists (is finite), then this limit $\lim_{n\to\infty} s_n$ is equal to the sum of the infinite series, and we can say the series is **convergent**.

Convergent and Divergent Series:

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + ... + a_n + ...$, let $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$. If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a finite real number, then $\sum_{i=1}^{\infty} a_i$ is called a **convergent** series, and we can write:

$$a_1 + a_2 + a_3 + ... + a_n + ... = s, \qquad \text{or} \qquad \sum_{i=1}^{\infty} a_i = s$$

The finite number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

-Not only can sequences be convergent or divergent, but series can be convergent or divergent too. The limit of a series, s, is technically the same value as the limit $\lim_{n\to\infty} s_n$ of the series. So if you find the limit of the partial sums as n goes to infinity, and find that that limit is finite, that means the series sum is finite too, and therefore the series is convergent.

-Before we continue though, take a moment to notice that an infinite sum has two denotations:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

-Remember the ways we used to define improper integrals of type I?

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^{t} f(x)dx$$

-To find an improper integral we integrate from 1 to t and then let t approach infinity. For a series, we sum from 1 to n and then let n approach infinity. We may just see more integral analogs later on when it comes to series.

-Of course, before you can worry about the sum of an infinite series, you need a reliable expression (either given or found) for the partial sum of a series. <u>Then</u> you can take the limit of the partial sum to find the infinite sum.

Example: Show that the series $\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$ is convergent, and find its sum.

Solution: It is an infinite series, so first we need a partial sum for the first n terms in the series. We could find what the first few terms in the partial sum sequence was and then predict what the nth partial sum would be, but that's hardly an exact method.

Instead, let's turn our attention to the expression in the series: $\frac{1}{i(i+1)}$. Can you rewrite this expression as a partial fraction decomposition?

$$\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1}$$

$$1 = A(i+1) + B(i)$$

$$1 = Ai + Bi + A$$

It appears that A = 1 and B = -1, so the series can be rewritten as:

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

So that means that the summation of the first n terms will have many terms cancelling out when added together.

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots \frac{1}{n} - \frac{1}{n+1}$$

In fact, when you add the first n terms in the series together, the only terms that will not cancel out are $\frac{1}{1}$ and $\frac{1}{n+1}$. So that means:

$$s_n = 1 - \frac{1}{n+1}$$

We have an expression for the nth partial sum, so that means that the series sum will be the limit of this partial sum expression as n goes to infinity:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Therefore, the series is convergent and converges to 1: $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1$

-When terms cancel out in a sum in pairs like this, it is called a **telescoping sum**. A telescoping sum is quite handy in infinite sums like this since it will turn infinitely many terms added together into a finite number of terms added together.

Exercise: Show that the series $\sum_{i=1}^{\infty} \frac{2}{i(i+2)}$ is convergent, and find its sum.

-One very famous and important kind of infinite series is the **geometric series**:

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{i=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

-Each new term in the series scales by a factor of r. We call this factor r a **common ratio**. The example from the start of this section is a geometric series with a common ratio $r = \frac{1}{2}$. Don't worry, we'll get back to that!

-If r = 1, then $s_n = a + a + \ldots + a = na$. If we take the limit of this partial sum expression as n goes to infinity, then the limit goes to infinity and so the geometric series is divergent. What if r did not equal 1?

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

Multiply both sides by r:

$$rs_n = r(a + ar + ar^2 + \ldots + ar^{n-1})$$

$$rs_n = ar + ar^2 + ar^3 + \ldots + ar^n$$

So that means rs_n minus s_n equals:

$$rs_n - s_n = (ar + ar^2 + ar^3 + \ldots + ar^n) - (a + ar + ar^2 + \ldots + ar^{n-1}) = ar^n - a$$

$$s_n(r-1) = a(r^n - 1)$$

Solve for s_n to get:

$$s_n = \frac{a(r^n-1)}{(r-1)}$$
$$s_n = a\frac{(r^n-1)}{(r-1)}$$

-So as long as r is not 1, the nth partial sum of a geometric series is:

$$s_n = a\frac{(r^n-1)}{(r-1)} = a\frac{(1-r^n)}{(1-r)}$$

-You may recall from the last section that if $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$, and it diverges to infinity for all other r. So what does that mean for the partial sum as n goes to infinity?

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a\frac{(r^n-1)}{(r-1)} = a\left(\lim_{n\to\infty}\frac{(1-r^n)}{(1-r)}\right) = a\left(\frac{(1-\lim_{n\to\infty} r^n)}{(1-r)}\right) = a\left(\frac{(1-0)}{(1-r)}\right) = \frac{a}{1-r}$$

-So now we have a means of being able to find the infinite sum of a geometric series, as well as when that series is divergent and when it is convergent.

Geometric Series:

$$\sum_{i=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

The geometric series above is convergent if $|r| < 1$, and it's sum is:

$$\sum_{i=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $r \geq 1$ or $r \leq -1$, then the geometric series $\sum_{i=1}^{\infty} ar^{n-1}$ diverges.

-This expression for the sum of a convergent geometric series can be a little easy to misread, so here's a verbal summary:

The sum of a convergent geometric series is the first term in the series divided by 1 minus the geometric ratio:

$$\frac{\textit{first term}}{1-\textit{common ratio}}$$

-Sometimes students overthink what r or a is equal to in the formula for the sum of a geometric series. It's easier to just remember it's the first term in the series divided by 1-r. Of course, you need to make sure to know what the first term and r are equal to.

Example: Find the sum of the following infinite series: $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots$

Solution: We can finally confirm that the sum of this series is 1 as we suspected. This is a geometric series since each term scales by $r = \frac{1}{2}$, and the first term in this series is $a = \frac{1}{2}$. Therefore, since 0 < r < 1, we know this is a convergent geometric series, and that the infinite sum of this convergent geometric series is:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{i-1} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

Example: Find the sum of the geometric series: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$

Solution: This is a geometric series, even though they didn't give us the expression for the nth term. We have that every term scales by a factor of $r = -\frac{2}{3}$, including the negative (include it, it is a part of the ratio), and this value of r is between -1 and 1, so this is a convergent geometric series.

When a series is stated term-by-term, finding the first term, a, is pretty straightforward. Therefore, we have a=5, and $r = -\frac{2}{3}$. The sum of the geometric series equals:

$$\sum_{i=1}^{\infty} 5\left(-\frac{2}{3}\right)^i = \frac{5}{1+2/3} = \frac{5}{\frac{5}{3}} = 3$$

-If you can verify that a series is geometric and find r, then it is not only straightforward to determine if the series converges, but also what the sum is if the series does converge. Thus, the hard part often is determining what r is.

Example: Is the series $\sum_{i=1}^{\infty} (2)^{2n}(3)^{1-n}$ convergent or divergent?

Solution: Is this a geometric series? It would be easier to determine if we rewrote this a little, perhaps by combining the exponentials.

$$\sum_{i=1}^{\infty} (2^2)^n(3)^{1-n} = \sum_{i=1}^{\infty} (4)^n(3)^{-(n-1)} = \sum_{i=1}^{\infty} 4(4)^{n-1}\frac{1}{3^{n-1}} = \sum_{i=1}^{\infty} 4\frac{4^{n-1}}{3^{n-1}} = \sum_{i=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

Now we can clearly see that this is a geometric series, and that the common ratio is $r = \frac{4}{3}$. This ratio is greater than 1, so this series is divergent.

You also could have written a few of the first terms and seen if the ratio from one to the other was always the same, but this is a bit more exact.

Exercise: Determine if the following series are convergent or divergent. If they are convergent, find the sum.

a) $3 - \frac{12}{3.5} + \frac{48}{12.25} - \frac{192}{42.875} + \ldots$ b) $\sum_{i=1}^{\infty} (15)^n(4)^{1-2n}$

Example: 0.2 mg/ml concentration of a drug is administered to a patient at the same time every day. Let C_n be the concentration of the drug in the bloodstream of the patient after the nth day. The drug dissipates when injected in the bloodstream to the point that by the next injection, only 30% of the drug will still be in the bloodstream. The concentrations for the first few days right after the administration are given:

$C_1 = 0.2$, $C_2 = 0.26$, $C_3 = 0.278$, ...

a) Find the concentration right after administration on day n.

b) What is the concentration right after administration on day 5?
c) What is the limiting concentration?

Solution: This may not look like a geometric series at first since the concentrations themselves do not share a common ratio from one to another. However, consider the following:

$$C_1 = 0.2$$
$$C_2 = 0.2 + 0.06 = 0.2 + 0.2(0.3)$$
$$C_3 = 0.2 + 0.06 + 0.018 = 0.2 + 0.2(0.3) + 0.2(0.3)^2$$

You can probably see it now. The concentrations are not the terms, they are the partial sums. Not only that, it appears we know what a and r are, a = 0.2, and r = 0.3.

a) Remember, the formula $\frac{a}{1-r}$ can tell us what the infinite series sum is, but not what any partial sums are. For that, we need the partial sum formula:

$$s_n = a\frac{(r^n-1)}{(r-1)} = (0.2)\frac{(1-(0.3)^n)}{(1-0.3)} = 0.2\frac{(1-0.3^n)}{0.7} = \frac{0.2}{0.7}(1-0.3^n) = \frac{2}{7}(1-0.3^n)$$

b) So what is the concentration on day 5?

$$s_5 = \frac{2}{7}(1-0.3^5) = 0.28502$$

c) The limiting concentration will be what the concentration approaches as n goes to infinity. Since r < 1, we know that the concentration is finite due to the partial sum sequence converging:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2}{7}(1-0.3^n) = \frac{2}{7}(1-0) = \frac{2}{7} = 0.\overline{285714}$$

-That decimal approximation has the bar signifying repeating digits, which brings up another application of infinite series.

Example: Write the number $2.3\overline{17}$ as a ratio of integers.

Solution: This expression is rational, as all decimals with repeating digits are. So let's find out what this rational number looks like when written as a ratio of integers:

$$2.3\overline{17} = 2.317171717... = 2.3 + \frac{17}{1000} + \frac{17}{100000} + \frac{17}{10000000} + ... = \frac{23}{10} + \frac{17}{10^3} + \frac{17}{10^5} + ...$$

This is in fact an infinite sum of fractions, all but one of which are part of a geometric series due to having a common ratio of $r = \frac{1}{100} = 0.01$. Rather than worry about including 2.3 in the infinite sum, let the infinite sum start at $\frac{17}{1000}$. Therefore we start with the first term being a = 0.017 and the ratio being r=0.01:

$$\frac{23}{10} + \sum_{i=1}^{\infty} \frac{17}{1000}\left(\frac{1}{100}\right)^{n-1} = \frac{23}{10} + \frac{\frac{17}{1000}}{1-\frac{1}{100}} = \frac{23}{10} + \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{23}{10} + \frac{17}{990} = \frac{2277}{990} + \frac{17}{990} = \frac{2294}{990} = \frac{1147}{495}$$

Exercise: Write the number $1.2\overline{34}$ as a ratio of integers.

-A common misconception about series is the idea that as long as the expression inside the summation converges goes to 0, that implies that the series converges too. This is not true, as we have a very famous counterexample to that, the **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

-What do the partial sums for the harmonic series look like? Let's take a look at a few:

$$s_1 = \sum_{n=1}^{1} \frac{1}{n} = 1$$

$$s_2 = \sum_{n=1}^{2} \frac{1}{n} = 1 + \frac{1}{2} = 1.5$$

$$s_4 = \sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$$

$$s_8 = \sum_{n=1}^{8} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 2.5$$

-As you can imagine, $s_{16} > 3,\ s_{32} > 3.5,\ s_{64} > 4,$ and so on. In general, for any integer n, $s_{2^n} > 1 + \frac{n}{2}$. If we take the limit of this as n goes to infinity, we see that the partial sum of the harmonic series is greater than $\lim_{n\to\infty} 1 + \frac{n}{2}$, which is infinite. So therefore the partial sum does not approach a finite number as n goes to infinity, and is thus divergent.

-It is not be true that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ is also convergent. However, the converse is true:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The proof is fairly simple. If $\sum_{n=1}^{\infty} a_n$ is convergent, then it converges to a finite number s. It is also true that $\lim_{n\to\infty} s_n = s$. In general however, given a_n, it is true that $a_n = s_n - s_{n-1}$. However, if you take the limit of each side as n goes to infinity, you get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

-If you know anything about logic, you probably also remember that since:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

is true, the contrapositive of this statement is also true:

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Is the series $\sum_{i=1}^{\infty} \frac{n^2}{5n^2+4}$ convergent or divergent?

Solution: To begin, you can take the limit of the expression in the sum to see if it equals 0 as n goes to infinity. Because if it does not, the series by default is divergent.

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5+0} = \frac{1}{5}$$

Since the expression in the series does not approach 0 as n goes to infinity, the series is divergent.

-Beware though. If the expression in the series <u>does</u> approach 0 as n goes to infinity, that tells us ***<u>NOTHING</u>***. The series may converge, it may diverge, but you can't be sure just from the expression going to zero.

-Fortunately, if you are certain that a series or two is convergent, certain combinations of those series tend to be convergent too:

Suppose that $\sum_{i=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ are both convergent series, then so are the following series:

- $\sum_{i=1}^{\infty} ca_n$ where c is a constant number.
- $\sum_{i=1}^{\infty} (a_n + b_n)$
- $\sum_{i=1}^{\infty} (a_n - b_n)$

Example: Find the sum of the following series: $\sum_{i=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

Solution: This series is actually a linear combination of series we have already seen in this section:

$$\sum_{i=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = \sum_{i=1}^{\infty} \frac{3}{n(n+1)} + \sum_{i=1}^{\infty} \frac{1}{2^n} = 3\left(\sum_{i=1}^{\infty} \frac{1}{n(n+1)}\right) + \sum_{i=1}^{\infty} \frac{1}{2^n}$$

In an earlier example, we saw that $\sum_{i=1}^{\infty} \frac{1}{2^n}$ was convergent and equals 1.

In another earlier example, we saw $\sum_{i=1}^{\infty} \frac{1}{n(n+1)}$ was convergent and also equals 1. So therefore, this series is also convergent, and it's sum is:

$$\sum_{i=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\left(\sum_{i=1}^{\infty} \frac{1}{n(n+1)}\right) + \sum_{i=1}^{\infty} \frac{1}{2^n} = 3(1) + 1 = 3 + 1 = 4$$