5.5) The Substitution Rule:

-Some integral rules we have seen so far as just derivative rules in reverse, but most of them have been very simple derivative rules in reverse. But what happens when you try to find an analogous integration rule to a more complicated derivative rule, like say the chain rule?

-How do you take the derivative of $f(x) = (5x^2+1)^{10}$? Most of you are probably aware of the more general version of the power rule, which is that we can take the derivative of this the way we take the derivative of $g(x) = x^{10}$, but tack on an extra factor of what is inside the parentheses to do so:

$$\frac{d}{dx}\left[(5x^2+1)^{10}\right] = 10(5x^2+1)^9 * 10x = 100x(5x^2+1)^9$$

-Sometimes students have trouble remembering the extra factor of 10x, or where it comes from. For that, don't forget the fact that the chain rule is used for compositions of functions. Let $u = 5x^2+1$ be a function of x, so the derivative of a function of u with respect to x needs to be a product of the derivative of the entire function with respect to u, multiplied by the derivative of u with respect to x:

$$\frac{d}{dx}\left[(5x^2+1)^{10}\right] = \frac{d}{dx}\left[u^{10}\right] = 10u^9 * \frac{du}{dx} = 10(5x^2+1)^9 * 10x = 100x(5x^2+1)^9$$

-If this is how you derive a composition of two functions, how would you integrate a composition of two functions? By doing the chain rule in reverse! You can even use a bit of differentials:

$$\text{If } u = 5x^2+1, \text{ then } du = 10x * dx$$

-Where else have we seen a literal factor of dx? Inside our integrals (be they indefinite or definite) we have always had a dx or dt factor that indicates what variable the integral is in terms of, and here it can act as a literal substitution factor. After all, derivatives using the chain rule often have a degree of substitution that occurs, so it's natural to have substitution happening when we integrate. Hence why this procedure of integration is called **the substitution rule** or **integration by substitution**.

$$\int 100x(5x^2+1)^9 dx$$

-How would you integrate the above expression? This is a composition of functions, so what we need is to substitute the expression $5x^2+1$ with another variable, like $u = 5x^2+1$ in the integral:

$$\int 100x(u)^9 dx$$

-This can be a problem though, since as you can see, we now have two variables: u and x (not to mention dx) all in the same integral. Is there a way to make sure the integral has nothing but u and du inside? If so, we can integrate more easily. That's where the differential comes in. Remember how $du = 10x * dx$? We can make another substitution, replacing dx with an expression in terms of x and du:

$$du = 10x * dx \qquad \Rightarrow \qquad \frac{du}{10x} = dx$$

-If we substitute this into the integral we get: $\int 100x(u)^9 dx = \int 100x(u)^9 \frac{du}{10x} = \int 10u^9 du$

-Now we have an integral we can work with. All in terms of u and du, and it quite obviously integrates to:

$$\int 10u^9 du = u^{10} + C = (5x^2 + 1)^{10} + C$$

-Once again this involves using the **Substitution Rule**:

If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then:

$$\int f(g(x)) * g'(x)dx = \int f(u)du$$

-There is a lot going on in an integration problem that involves compositions of functions:

-An expression in terms of x will have to be replaced with another expression in terms of a different variable. (u = f(x))

-Take the differential of your expression you substituted with u in the first step. ($u = f'(x)dx$)

-dx will in turn also have to be substituted using your expression from step 2.

-If your resulting substitutions result in an integral in terms of u and du only, integrate.

-If your integral still have x and u at the same time, start over with a different substitution.

-After integrating put your final answer back in terms of x by substituting f(x) back in for u.

-Knowing what expression needs to be replaced with u is not easy, nor is the procedure of finding the differential and substituting in the proper expression to rewrite the integral in terms of u and du. You have to be persistent; just like how the chain rule involves a factor seemingly appearing out of nowhere, integration by substitution will result in a factor seemingly disappearing out of nowhere. It takes practice to find the right substitution that will make everything clean for the integration.

Example: Integrate $\int x\sqrt{1-2x^2}dx$

Solution: The substitution that needs to be made involves the expression inside the radical (usually expressions inside parentheses, radicals, logarithms, and other operations, as they are usually a sign of being composed inside another function. So in this case, let $u = 1 - 2x^2$.

From here, take the differential of u: $du = -4xdx$.

Now try to solve for dx: $-\frac{du}{4x} = dx$.

Let's see if the integral can now be written in terms of u and du only:

$$\int x\sqrt{1-2x^2}dx \quad \Rightarrow \quad \int x\sqrt{u}\ \left(-\tfrac{du}{4x}\right)$$

$$-\tfrac{1}{4}\int \sqrt{u}\ du$$

Looks clean now, so let's integrate: $-\frac{1}{4}\int \sqrt{u}\ du$

$$-\frac{1}{4}\left(\frac{u^{1.5}}{1.5} + C\right)$$

$$-\frac{(1-2x^2)^{1.5}}{6}+C$$

You notice that when we distribute -1/4 through the parentheses we do not distribute into C. Why not? Because C is an arbitrary constant, so there's no need to have our final answer have coefficients on an arbitrary value, since the arbitrary value is arbitrary, with or without a coefficient. 2C, 5C, -0.25C, they are all still just an arbitrary constant, C.

-However, not every composition is going to solvable with the substitution rule (or at least, not this substitution rule).

Example: Can you integrate using the substitution rule here?

$$\int \sqrt[4]{x^2+1}\,dx$$

Solution: In short, no. Why not? If you try using the substitution rule here, you would run into a problem:

Let $u = x^2+1$, then the differential would be $du = 2xdx$, which means dx can be solved to get $dx = \frac{du}{2x}$. But what happens when we try to substitute this all in?

$$\int \sqrt[4]{x^2+1}\,dx$$

$$\int \sqrt[4]{u}\,\frac{du}{2x}$$

After substituting, we still have both u and x. That is a problem, as we cannot integrate when there is both a u and an x. It turns out, none of the substitution methods we have seen will work here. There is a way to do this, but we'll get to that another day.

-Is there an easy way to find the substitution expression for u? Nope.
Is there an easy way to get the differential cancellation for du? Nope.
Is there an easy way to tell if the substitution rule will even work? Nope.

-This is not easy, and it's common to get frustrated with the lengthy procedure that could lead to finding out you used the wrong substitution or that the expression can't be integrated with the substitution rule at all, but you have to keep trying. It takes practice with many different situations to be able to recognize the tells for what needs to be substituted, and whether the proper factors are around to "disappear" when we substitute for dx.

Example: Find $\int ((x^3+2x)cos(x^4+4x^2+4))\,dx$

Solution: First off, we need a substitution for u. Let's take the expression inside of cosine: $u = x^4+4x^2+4$. From here, let's take the differential of our expression: $du = (4x^3+8x)dx$, which means $\frac{du}{4x^3+8x} = dx$.

Now we substitute to see if the integral is any cleaner (by cleaner, we mean that we can integrate it by hand):

$$\int((x^3+2x)cos(x^4+4x^2+4))\,dx$$

$$\int((x^3+2x)cos(u))\,\frac{du}{4x^3+8x}$$

$$\int((x^3+2x)cos(u))\,\frac{du}{4(x^3+2x)}$$

$$\frac{1}{4}\int cos(u)du$$

Looks clean to me, so now let's integrate! $\frac{1}{4}(sin(u)+C)=\frac{1}{4}sin(x^4+4x^2+4)+C$

-You won't know until you have tried substituting with both the u expression and the differential if you have a proper substitution set. If you find you don't, don't get discouraged, just try again. Sometimes you just have to be creative with how you substitute.

Example: Find $\int\left(\sqrt{2x+1}\right)dx$

Solution: What can u be in this case? The obvious choice is to let u = 2x+1 and du = 2dx. That does work, but I will show you an alternate substitution to show that sometimes more than one substitution will give you the correct answer.

Instead, I will get u be the entire expression: $u=\sqrt{2x+1}$.
You still need the differential for du : $du=\frac{1}{2\sqrt{2x+1}}*2dx=\frac{1}{\sqrt{2x+1}}dx$
This in turn means: $du\sqrt{2x+1}=dx$

While we are at it however, since we already have that $u=\sqrt{2x+1}$, so let's try plugging this into our dx expression too!

$$du*u=dx$$

Thus, our integral becomes: $\int\left(\sqrt{2x+1}\right)dx=\int(u)\,u*du=\int u^2du=\frac{1}{3}u^3+C=\frac{1}{3}\left(\sqrt{2x+1}\right)^3+C$.

Again, if you used u = 2x+1, you would have gotten the same answer. If you wanted to verify that this was the solution, you once again can take the derivative of your answer to make sure.

-Sometimes the simplest difference can cause the biggest change in answer. Check out the next two examples:

Example: Find $\int\left(\frac{x}{1+4x^2}\right)dx$

Solution: Let $u=1+4x^2$, then we have that $du=8xdx$, which means that $\frac{du}{8x}=dx.$
Now let's substitute:

$$\int\left(\frac{x}{1+4x^2}\right)dx$$

$$\int\left(\frac{x}{u}\right)\frac{du}{8x}$$

$$\frac{1}{8}\int\left(\frac{du}{u}\right)$$

$$\frac{1}{8}(ln(u)\ +\ C) = \frac{1}{8}ln(1+4x^2)\ +\ C$$

-Compare this example to:

Example: Find $\int\left(\frac{1}{1+4x^2}\right)dx$

Solution: The previous substitution will not work here, you will end up with an extra x. So what can you do instead? It's a bit trickier to spot (expect to grab substitutions out of thin air sometimes; they can be very hard to spot), but try letting u = 2x, and so du = 2dx. Since u = 2x however, that also means $u^2 = 4x^2$, therefore:

$$\int\left(\frac{1}{1+4x^2}\right)dx$$

$$\int\left(\frac{1}{1+u^2}\right)\frac{du}{2}$$

$$\frac{1}{2}\int\left(\frac{du}{1+u^2}\right)$$

This is the derivative of arctangent:

$$\frac{1}{2}(tan^{-1}(u)\ +\ C)$$

$$\frac{1}{2}tan^{-1}(2x)\ +\ C$$

-Just an extra factor of x turned the answer from a natural log to an inverse tangent. These substitutions can be very obtuse, and sometimes just a small change can alter the strategy or substitution completely. If anything, this last example showed that you can alter the substitution factors before you even substitute. If you have an expression for u, you can algebraically alter that expression however you need to to make the substitution clean.

Example: Find $\int x^5\sqrt{1+x^2}dx$

Solution: This is another instance of altering before substituting. You'd be correct to say that $u = 1+x^2$ is our substitution factor here, and that means $du = 2xdx$. However, we have more than just one factor of x outside the radical; we have five.

But consider that if $u = 1+x^2$, that also means that $u-1\ =\ x^2$. This may be useful for later. You may need multiple substitution factors. Remember, you can use algebra to make your u and du expression as simple or complicated as you want them to be.

$$\int x^5\sqrt{1+x^2}dx$$

$$\int (x^2)^2\sqrt{1+x^2} \ * xdx$$

$$\int (u-1)^2\sqrt{u} \ * x\frac{du}{2x}$$

Substituting in properly for dx, for $1+x^2$, and for x^4 turns this expression into a much more manageable integral. It's still not going to be easy, but it will be clean:

$$\frac{1}{2}\int (u^2-2u+1)\, u^{0.5} \ * du$$

$$\frac{1}{2}\int (u^{2.5}-2u^{1.5}+u^{0.5})\, du$$

$$\frac{1}{2}\left(\frac{u^{3.5}}{3.5}-2\frac{u^{2.5}}{2.5}+\frac{u^{1.5}}{1.5} \ + \ C\right)$$

$$\frac{u^{3.5}}{7}-\frac{2u^{2.5}}{5}+\frac{u^{1.5}}{3} \ + \ C$$

$$\frac{(1+x^2)^{3.5}}{7}-\frac{2(1+x^2)^{2.5}}{5}+\frac{(1+x^2)^{1.5}}{3} \ + \ C$$

-Sometimes the hardest substitution comes from the seemingly easiest looking integrals.

Example: Find $\int tan(x)dx$

Solution: You may think you know what this is from the chart in 5.4 in the textbook, but believe me, it's not there. It might help if you rewrote what tan(x) was in terms of sin(x) and cos(x) first (we will see more trigonometric identity use later, so don't forget them!):

$$\int tan(x)dx = \int \frac{sin(x)}{cos(x)}dx$$

So let's let u = cos(x) (I know it's tempting to use u = sin(x), but trust me, that won't work), and in turn, we have that du = -sin(x)dx. What happens when we substitute?

$$\int \frac{sin(x)}{cos(x)}dx$$

$$\int \frac{sin(x)}{u} * \frac{du}{-sin(x)}$$

$$-\int \frac{du}{u}$$

$$-(ln\,|u|+C)$$

$$-ln\,|cos(x)|+C$$

There is actually more than one way to write this, if you use the reciprocal rule for logarithms to get:

$$-\,ln\,|cos(x)| + C = \; ln\left|\frac{1}{cos(x)}\right| + C = \; ln\,|sec(x)| \;\; + C$$

Exercise: Find the following indefinite integrals:

a) $\int (sin(x^2 + 2x + 4)(x + 1)dx)$ b) $\int \; cot(x)dx$

c) $\int\left(\frac{9}{\sqrt{1-9x^2}}dx\right)$ d) $\int (e^{10x}dx)$

e) $\int\left(\frac{9x}{\sqrt{1-9x^2}}dx\right)$ f) $\int\left(\frac{6x}{2x^2+1}dx\right)$

-What about definite integrals? The integration process using the substitution rule is the same for definite integrals, but the evaluations can be different (if you choose). Remember, in any definite integral, the upper and lower limits are values to plugged into the original variable, not the substituted u variable.

-Some get around this by making sure everything is written back in terms of the original variable before plugging any numbers in for evaluation, which is not much more than just indefinite integration with an extra step at the end.

-However, a **substitution rule for definite integrals** procedure exists where the limits themselves are replaced with new values, allowing us to evaluate the integrals without substituting the original variables back in. It can save time and space not having to substitute everything back in before evaluating.

-If g' is continuous on [a,b] and f is continuous on the range of u = g(x), then:

$$\int_a^b f(g(x))g'(x)dx \;=\; \int_{g(a)}^{g(b)} f(u)du$$

-In other words, trade your old limit values for x for new limit values for u by plugging the limits a and b into your u = g(x) expression. What you get back are the new limits you can plug into your integrated f(u)du expression, which will save you the trouble of plugging extra variables or numbers back in.

Example: Evaluate: $\int_2^6 \sqrt{4x+1}\; dx$

Solution: If you use the substitution rule to integrate this expression, you would let u = 4x+1 and du=4dx. However, on top of that let's find out what u=4x+1 equals at the limits x=2 and x= 6:

$g(2) = 4(2) + 1 = 9,$ $g(6) = 4(6) + 1 = 25$

So now when we substitute, we substitute in not just u, not just du, but also the new limits:

$$\int_2^6 \sqrt{4x+1}\; dx = \int_9^{25} (\sqrt{u})\; \frac{du}{4}$$

$$\int_2^6 \sqrt{4x+1}\; dx = \frac{1}{4}\int_9^{25} (u^{0.5})\; du$$

$$\int_2^6 \sqrt{4x+1}\,dx = \tfrac{1}{4}\left(\tfrac{u^{1.5}}{1.5}\right)\Big]_9^{25}$$

Rather than plug the expressions for x back in and evaluate though, let's now just plug in 25 and 9 for u:

$$\int_2^6 \sqrt{4x+1}\,dx = \left(\tfrac{25^{1.5}}{6} - \tfrac{9^{1.5}}{6}\right) = \tfrac{125}{6} - \tfrac{27}{6} = \tfrac{49}{3}$$

Just a little extra rewriting at the start saves us space and evaluating later.

Example: Evaluate: $\int_1^{e^2} \frac{ln(x)}{x}\,dx$

Solution: The substitution here is a little tricky. Let u = ln(x), which means $du = \frac{dx}{x}$. The only other option was to let u = x, which is never a good substitution choice. So if u = ln(x), what about the limits for u?

$g(e^2) = ln(e^2) = 2,$ $\qquad g(1) = ln(1) = 0$

So now we can substitute and integrate:

$$\int_1^{e^2} \frac{ln(x)}{x}\,dx = \int_0^2 \frac{u}{x}\,(xdu)$$

$$\int_1^{e^2} \frac{ln(x)}{x}\,dx = \int_0^2 udu$$

$$\int_1^{e^2} \frac{ln(x)}{x}\,dx = \frac{u^2}{2}\Big]_0^2 = 2 - 0 = 2$$

-Again, if you don't want to substitute in for the limits, you can always just plug your expressions back in for x and then evaluate with the original limits, but that can be overly time-consuming, so it is not recommended.

Exercises: Evaluate the following definite integrals:

a) $\int_0^{\frac{3\pi}{4}} cos(x)sin^3(x)\,dx$ $\qquad$ b) $\int_{-3}^{4} (x)\sqrt[3]{x+4}\,dx$ $\qquad$ c) $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$

-The last topic involves symmetry of functions, which hopefully you remember. If not, here's a recap:

-Given a function continuous on [-a, a], f is even if f(x)=f(-x) for all x in [-a, a]

-Given a function continuous on [-a, a], f is odd if f(-x) = -f(x) for all x in [-a, a]

-Functions with symmetry of either type on an interval [-a, a] have the following integral properties:

a) If f is even on [-a, a], then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

b) If f is odd on [-a, a], then $\int_{-a}^{a} f(x)dx = 0$

-This means that if a function has symmetry of one type or the other, you can cut your work (at least) in half by recognizing that the symmetry allows you to evaluate your definite integrals much quicker.

Example: Evaluate the following $\int_{-5}^{5} xln(x^4 + x^2 + 5)dx$

Solution: Before you worry about substitution, integrating, evaluating, or anything else that would make this problem any longer, take a look at the function inside the integral. It is continuous on [-5,5], it is an odd function (replace x with -x if you are unsure), and so the integral of the function over [-5,5] is 0.

$$\int_{-5}^{5} xln(x^4 + x^2 + 5)dx = 0$$

Done.

Example: Evaluate the following $\int_{-1}^{1} (x^5 + x^4 + x^3 + x^2 + x + 1)dx$

Solution: This is a slightly harder situation to spot since the function itself is not even nor is it odd. However, you can separate the terms into their own integrals:

$$\int_{-1}^{1} (x^5 + x^4 + x^3 + x^2 + x + 1)dx = \int_{-1}^{1} (x^5 + x^3 + x)dx + \int_{-1}^{1} (x^4 + x^2 + 1)dx$$

The first integral is of an odd function since all the powers are odd, and the second integral is of an even function since all the powers are even. This means if we want to integrate over [-1, 1], the first integral can be ignored entirely since it will evaluate to 0, and the second can be doubled and integrated over [0, 1] instead:

$$\int_{-1}^{1} (x^5 + x^3 + x)dx + \int_{-1}^{1} (x^4 + x^2 + 1)dx = 2\int_{0}^{1} (x^4 + x^2 + 1)dx$$

$$2\left(\frac{x^5}{5} + \frac{x^3}{3} + x\right)\Big|_0^1 = 2\left(\frac{1}{5} + \frac{1}{3} + 1 - (0)\right) = \frac{46}{15}$$

Exercises: Evaluate the following definite integrals:

a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{x}{1+x^2} + \frac{1}{1+x^2}\right) dx$

b) $\int_{-1}^{1} tan(x)(cos(x) + sin^2(x))dx$