5.1

Exercise: a) Prove that for $f(x) = x^3$, on $[0, 1]$, $\lim\limits_{n\to\infty} R_n = \frac{1}{4}$.

b) Prove that for $f(x) = x^2$, on $[0, 3]$, $\lim\limits_{n\to\infty} R_n = 9$.

a) $R_n = \frac{1}{n} * f\left(\frac{1}{n}\right) + \frac{1}{n} * f\left(\frac{2}{n}\right) + \frac{1}{n} * f\left(\frac{3}{n}\right) + \ldots \frac{1}{n} * f\left(\frac{n}{n}\right) = \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \ldots f\left(\frac{n}{n}\right)\right)$

$R_n = \frac{1}{n}\left(\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^3\right) = \frac{1}{n}\left(\sum\limits_{i=1}^{n}\left(\frac{i^3}{n^3}\right)\right) = \frac{1}{n^4}\left(\sum\limits_{i=1}^{n} i^3\right) = \frac{1}{n^4}\left(\frac{(n(n+1))^2}{4}\right) = \frac{n^4+2n^3+n^2}{4n^4}$

$\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \frac{n^4+2n^3+n^2}{4n^4} = \frac{1}{4}$

b) $R_n = \frac{3}{n} * f\left(\frac{3}{n}\right) + \frac{3}{n} * f\left(\frac{6}{n}\right) + \frac{3}{n} * f\left(\frac{9}{n}\right) + \ldots \frac{3}{n} * f\left(\frac{3n}{n}\right) = \frac{3}{n}\left(f\left(\frac{3}{n}\right) + f\left(\frac{6}{n}\right) + f\left(\frac{9}{n}\right) + \ldots f\left(\frac{3n}{n}\right)\right)$

$R_n = \frac{3}{n}\left(\sum\limits_{i=1}^{n}\left(\frac{3i}{n}\right)^2\right) = \frac{3}{n}\left(\sum\limits_{i=1}^{n}\left(\frac{9i^2}{n^2}\right)\right) = \frac{27}{n^3}\left(\sum\limits_{i=1}^{n} i^2\right) = \frac{27}{n^3}\left(\frac{(n(n+1)(2n+1)}{6}\right) = \frac{27n(n+1)(2n+1)}{6n^3}$

$\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \frac{27n(n+1)(2n+1)}{6n^3} = \frac{27*2}{6} = 9$

Exercise: Estimate the area under the curve $f(x) = e^{-x^2}$ between x = 0 and x = 2 by using midpoints and using five subintervals.

$$\Delta x = \frac{2-0}{5} = 0.4,$$

$$x_1 = 0 + \tfrac{1}{2}(0.4) = 0.2, \quad x_2 = 0.6, \quad x_3 = 1.0, \quad x_4 = 1.4, \quad x_5 = 1.8$$

$$Area \approx 0.4\left(e^{-(0.04)} + e^{-(0.36)} + e^{-(1)} + e^{-(1.96)} + e^{-(3.24)}\right) = 0.88255$$

Exercise: A man skiing down a slalom going faster and faster is measured having the following velocities at the following points of time:

Time (s)	0	4	8	12	16	20	24
Velocity (ft/s)	4	7	12	18	22	24	27

Find a lower limit and upper limit for the distance this man on the skis travels over this 24 second interval (remember, he is going faster and faster while going down the slalom).

$Lower\ Bound \approx 4(4 + 7 + 12 + 18 + 22 + 24) = 348$

$Upper\ Bound \approx 4(7 + 12 + 18 + 22 + 24 + 27) = 440$

5.2

Exercise: Approximate the Riemann sum for $g(x) = x^3 - 4x$ on the interval [1, 3] with n = 5 subintervals and using left endpoints.

$$\Delta x = \frac{3-1}{5} = 0.4,$$

$$x_1 = 1.0, \qquad x_2 = 1.4, \qquad x_3 = 1.8, \qquad x_4 = 2.2, \qquad x_5 = 2.6$$

$$Area \approx 0.4\,(-3 - 2.856 - 1.368 + 1.848 + 7.176) = 0.72$$

Exercise: Evaluate $\int_0^2 (x^2 - 1)dx$.

$$\int_0^2 (x^2-1)dx = \lim_{n\to\infty}\sum_{i=1}^{n}(\Delta x * f(x_i)) = \lim_{n\to\infty}\sum_{i=1}^{n}(\tfrac{2}{n}f(\tfrac{2i}{n})) = \lim_{n\to\infty}\tfrac{2}{n}\sum_{i=1}^{n}f(\tfrac{2i}{n}) =$$

$$\lim_{n\to\infty}\tfrac{2}{n}\sum_{i=1}^{n}(\tfrac{4i^2}{n^2}-1) = \lim_{n\to\infty}\tfrac{2}{n}\sum_{i=1}^{n}(\tfrac{4i^2}{n^2}) - \lim_{n\to\infty}\tfrac{2}{n}\sum_{i=1}^{n}1 = \lim_{n\to\infty}\tfrac{8}{n^3}\sum_{i=1}^{n}i^2 - \lim_{n\to\infty}\tfrac{2}{n}\sum_{i=1}^{n}1 = \lim_{n\to\infty}\tfrac{8}{n^3}\frac{n(n+1)(2n+1)}{6} - \lim_{n\to\infty}\tfrac{2}{n}(n) =$$

$$\lim_{n\to\infty}\frac{8n(n+1)(2n+1)}{6n^3} - \lim_{n\to\infty}2 = \tfrac{16}{6} - 2 = \tfrac{2}{3}$$

Exercise: a) Evaluate $\int_{-3}^{3} -\sqrt{9-x^2}dx$. b) Evaluate $\int_0^3 (x+3)dx$.

a) $\int_{-3}^{3} -\sqrt{9-x^2}dx = -\left(\tfrac{1}{2}\pi(3)^2\right) = -\tfrac{9\pi}{2}$

b) $\int_0^3 (x+3)dx = \tfrac{1}{2}(3)(3) + (3)(3) = \tfrac{9}{2} + 9 = \tfrac{27}{2}$

Exercise: Evaluate the following definite integral without a Riemann sum. Try using previous examples and exercises to answer:

$$\int_0^3 (x^3 - 4x + 6)dx$$

$$\int_0^3 (x^3 - 4x + 6)dx = \int_0^3 (x^3 - 6x + 2x + 6)dx = \int_0^3 (x^3 - 6x)dx + 2\int_0^3 (x+3)dx = (-6.75) + 2(13.5) = 20.25$$

Exercise: Given $\int_4^{12} f(x)dx = 20$, $\int_4^6 f(x)dx = 8$, and $\int_8^{12} f(x)dx = -5$, find $\int_6^{12} f(x)dx$ and $\int_6^8 f(x)dx$.

$$\int_6^{12} f(x)dx = \int_4^{12} f(x)dx - \int_4^6 f(x)dx = 20 - 8 = 12$$

$$\int_6^8 f(x)dx = \int_6^{12} f(x)dx - \int_8^{12} f(x)dx = 12 - (-5) = 17$$

5.3

Exercise: Find the derivative of the following with respect to x:

$$\text{a) } \int_0^{8x} e^{t^2-2t-1}dt \qquad \text{b) } \int_x^2 cos(t^3-1)dt \qquad \text{c) } \int_x^{x^2} \sqrt{t^4+t^2+1}dt$$

$$\text{a) } \frac{d}{dx}\left[\int_0^{8x} e^{t^2-2t-1}dt\right] = e^{64x^2-16x-1} * 8 = 8e^{64x^2-16x-1}$$

$$\text{b) } \frac{d}{dx}\left[\int_x^2 cos(t^3-1)dt\right] = \frac{d}{dx}\left[-\int_2^x cos(t^3-1)dt\right] = -cos(x^3-1)$$

$$\text{c) } \frac{d}{dx}\left[\int_x^{x^2} \sqrt{t^4+t^2+1}dt\right] = \frac{d}{dx}\left[\int_0^{x^2} \sqrt{t^4+t^2+1}dt - \int_0^{x} \sqrt{t^4+t^2+1}dt\right] = 2x\sqrt{x^8+x^4+1} - \sqrt{x^4+x^2+1}$$

Exercise: Evaluate the integral $\int_1^2 (x^3-6x^2-8x+12 - \frac{4}{x^2})dx$

$$\int_1^2 (x^3-6x^2-8x+12 - \frac{4}{x^2})dx = \frac{x^4}{4} - 2x^3 - 4x^2 + 12x + \frac{4}{x}\Big|_1^2 = (4-16-16+24+2) - (\frac{1}{4} - 2 - 4 + 12 + 4) =$$

$$\int_1^2 (x^3-6x^2-8x+12 - \frac{4}{x^2})dx = (-2) - (\frac{41}{4}) = -\frac{49}{4}$$

Exercise: Evaluate the integrals: $\quad$ a) $\int_3^{48} \frac{1}{2x}dx$ $\quad$ b) $\int_{-\pi/4}^{\pi/4} sec^2(x)dx$

$$\text{a) } \int_3^{48} \frac{1}{2x}dx = \frac{1}{2}\int_3^{48} \frac{1}{x}dx = \frac{1}{2}ln(48)| = \frac{1}{2}(ln(48) - ln(3)) = \frac{1}{2}(ln(16) = ln(4)$$

$$\text{b) } \int_{-\pi/4}^{\pi/4} sec^2(x)dx = tan(x)\,\Big|_{-\pi/4}^{\pi/4} = tan(\pi/4) - tan(-\pi/4) = 1 - (-1) = 2$$