6.2)

Exercise: Find the volume of the region enclosed by y = $\sqrt{x}$ and $y = \frac{1}{2}x$ rotated around:

a) The x-axis
b) The y-axis
c) The line x = 4

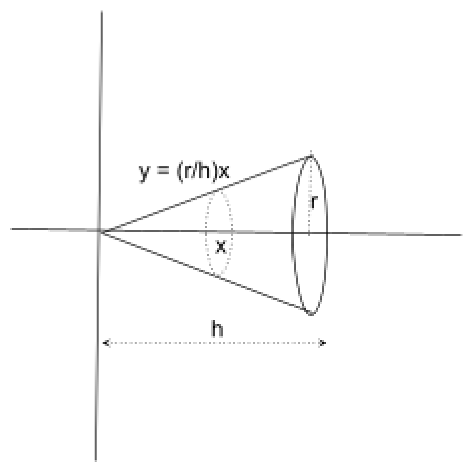
a) $V = \int_0^4 A(x)dx = \int_0^4 \pi((\sqrt{x})^2 - \left(\frac{1}{2}x\right)^2)dx = \int_0^4 \pi(x - \frac{1}{4}x^2)dx = \pi(\frac{x^2}{2} - \frac{x^3}{12})\Big|_0^4 = \pi(8 - \frac{16}{3}) = \frac{8\pi}{3}$

b) $V = \int_0^2 A(y)dy = \int_0^2 \pi((2y)^2 - (y^2)^2)dy = \int_0^2 \pi(4y^2 - y^4)dy = \pi(\frac{4y^3}{3} - \frac{y^5}{5})\Big|_0^2 = \pi(\frac{32}{3} - \frac{32}{5}) = \frac{64\pi}{15}$

c) $V = \int_0^2 A(y)dy = \int_0^2 \pi((4 - y^2)^2 - (4 - 2y)^2)dy = \int_0^2 \pi(16 - 8y^2 + y^4 - 16 + 16y - 4y^2)dy =$

$\int_0^2 \pi(-12y^2 + y^4 + 16y)dy = \pi(-4y^3 + \frac{y^5}{5} + 8y^2)\Big|_0^2 = \pi(-32 + \frac{32}{5} + 32) = \frac{32\pi}{5}$

Exercise: Show that the volume of a cone is $V = \frac{1}{3}r^2\pi h$. Start with the following picture to get your cross-section area:



$V = \int_0^h A(x)dx = \int_0^h \pi(\frac{r}{h}x)^2 dx = \int_0^h \pi\frac{r^2x^2}{h^2}dx = \pi\frac{r^2}{h^2}(\frac{x^3}{3})\Big|_0^h = \pi\frac{r^2}{h^2}(\frac{h^3}{3}) = \pi\frac{r^2}{h^2}(\frac{h^3}{3}) = \frac{1}{3}\pi r^2 h$

6.3)

Exercise: Find the volume of the solid formed by rotating the region enclosed by $y = 4x - x^3$, y=0, and x = 2 around the y-axis.

$$V = \int_0^2 2\pi x(4x - x^3)dx = 2\pi \int_0^2 (4x^2 - x^4)dx = 2\pi \left(\frac{4x^3}{3} - \frac{x^5}{5} \right) \Big|_0^2 = 2\pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{128\pi}{15}$$

Exercise: Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the curve $y = \sqrt{x}$, the y-axis, and y = 1, from y=0 to y=1.

$$V = \int_0^1 2\pi y(y^2)dy = 2\pi \int_0^1 y^3 dy = 2\pi \left(\frac{y^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{4} \right) = \frac{\pi}{2}$$

$$V = \int_0^1 \pi(1 - x)dx = \pi \left(x - \frac{x^2}{2} \right) \Big|_0^1 = \pi \left(\frac{1}{2} \right) = \frac{\pi}{2}$$

Exercise: Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and y = 0 about the line x = -1

$$V = \int_0^4 2\pi(1 + x)(4x - x^2)dx = 2\pi \int_0^4 (4x + 3x^2 - x^3)dx = 2\pi \left(2x^2 + x^3 - \frac{x^4}{4} \right) \Big|_0^4 = 2\pi\,(32 + 64 - 64) = 64\pi$$

6.4)

Exercise: A spring needs a force of 60 N to hold a spring at a length of 8 cm when its natural length is 12cm. How much work is needed to push that same spring from 10 cm to 4 cm?

$$600 = 0.04k$$
$$1500 = k$$

$$W = \int_{-0.02}^{-0.08} 1500x dx = -1500 \int_{-0.08}^{-0.02} x dx = -750x^2 \Big|_{-0.08}^{-0.02} = -750(0.0004 - 0.0064) = 4.5\ J$$

Exercise: A 30-lb rope is 20 feet long and hangs vertically down a well, fully extended. (there's no bucket at the end)

a) How much work is required to pull the rope up completely?
b) How much work is required to pull only half of the rope up?

a)

$$W = \int_{0}^{20} 1.5x dx = 0.75x^2\Big|_0^{20} = 300 \text{ ft - lbs}$$

b) Let W_1 be the work needed to get the first 10 feet up:

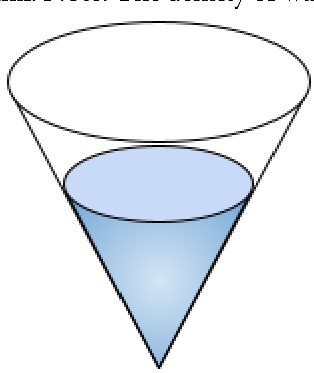
$$W_1 = \int_{0}^{10} 1.5x dx = 0.75x^2\Big|_0^{10} = 75 \text{ ft - lbs}$$

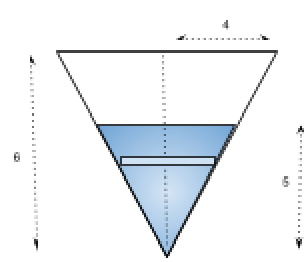
Let W_2 be the work needed to move the rest of the rope up 10 feet:

$$W_2 = \int_{10}^{20} (10) * 1.5 dx = 15x\Big|_{10}^{20} = 150 \text{ ft - lbs}$$

Total work is 225 ft - lbs.

Exercise: A tank has the shape of an inverted circular cone with height 6 meters and base radius 4 meters. It is filled with water to a height of 5 meters. Find the work required to empty the tank by pumping all the water out the top of the tank. Note: The density of water is $1000\ kg/m^3$





Solution:

$$V_i = (\tfrac{2}{3}(6 - x_i^*))^2 \pi * \Delta x = \tfrac{4\pi}{9}(6 - x_i^*)^2 \Delta x$$

$$m_i = V_i * \rho = \tfrac{4\pi}{9}(6 - x_i^*)^2 \Delta x * 1000 = \tfrac{4000\pi}{9}(6 - x_i^*)^2 \Delta x$$

$$F_i = m_i a = \tfrac{4000\pi}{9}(6 - x_i^*)^2 \Delta x \; * \; 9.8 = \tfrac{39200\pi}{9}(6 - x_i^*)^2 \Delta x$$

$$W = \int_1^6 \tfrac{39200\pi}{9} x(6 - x)^2 dx = \tfrac{39200\pi}{9} \int_1^6 (36x - 12x^2 + x^3) dx = \tfrac{39200\pi}{9}(18x^2 - 4x^3 + \tfrac{x^4}{4}) \Big|_1^6$$
$$= \tfrac{39200\pi}{9} * (108 - \tfrac{57}{4}) = \tfrac{1225000\pi}{3}\ J = 1282817\ J$$

6.5)

Exercise: What is the average value of the function $f(x) = x + x^3$ on the interval $[-2,\ 6]$?

$$\tfrac{1}{6-(-2)} \int_{-2}^{6} (x + x^3) dx = \tfrac{1}{8}(0.5x^2 + 0.25x^4) \Big|_{-2}^{6} = \tfrac{1}{8}(342 - 6) = 42$$

Exercise: Find all c that satisfies the mean value theorem for $f(x) = 2x + x^2$ on the interval $[-3,\ 6]$

$$\tfrac{1}{6-(-3)} \int_{-3}^{6} (2x + x^2) dx = \tfrac{1}{9}(x^2 + \tfrac{1}{3}x^3) \Big|_{-3}^{6} = \tfrac{1}{9}(108 - 0) = 12$$

$$12 = 2x + x^2$$

$$0 = x^2 + 2x - 12$$

$$x = \tfrac{-2+\sqrt{4+48}}{2} = 2.60555$$