5.4

Exercises: Find the general indefinite integrals:

a) $\int \left(\sqrt{x} - 3csc^2(x)\right)dx$ b) $\int \left(\frac{cos(x)}{sin^2(x)}\right)dx$

a) $\int \left(\sqrt{x} - 3csc^2(x)\right)dx = \frac{2}{3}x^{3/2} + 3cot(x) + C$

b) $\int \left(\frac{cos(x)}{sin^2(x)}\right)dx = \int cot(x)csc(x)dx = - csc(x) + C$

Exercises: Find the general definite integrals:

a) $\int_1^8 \frac{t^2 - \sqrt[3]{t} + 1}{t} dt$ b) $\int_0^1 \left(\frac{1}{\sqrt{1-x^2}} + \pi x\right)dx$

a)

$\int_1^8 \frac{t^2 - \sqrt[3]{t} + 1}{t} dt = \int_1^8 (t - t^{-2/3} + t^{-1})dt = \frac{1}{2}t^2 - 3t^{1/3} + ln(t)|_1^8 = 32 - 6 + ln(8) - \frac{1}{2} + 3 = \frac{57}{2} + ln$

b) $\int_0^1 \left(\frac{1}{\sqrt{1-x^2}} + \pi x\right)dx = sin^{-1}(x) + \frac{\pi}{2}x^2|_0^1 = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

Exercise: A particle moves along a line so that its velocity at time t is v(t) = $t^2 - 4t - 12$.

a) Find the displacement of the particle during the period $3 \le t \le 12$.

b) Find the distance traveled during this same time period.

a)

$\int_3^{12} (t^2 - 4t - 12)dt = \frac{t^3}{3} - 2t^2 - 12t|_3^{12} = 576 - 288 - 144 - (9 - 18 - 36) = 18$

b) $\int_3^{12} (t^2 - 4t - 12)dt = \left|\int_3^6 (t^2 - 4t - 12)dt\right| + \int_6^{12} (t^2 - 4t - 12)dt =$

$\left|\frac{t^3}{3} - 2t^2 - 12t|_3^6\right| + \frac{t^3}{3} - 2t^2 - 12t|_6^{12} =$

$= |72 - 72 - 72 - (9 - 18 - 36)| + (576 - 288 - 144) - (72 - 72 - 72) = 27 + 216 = 24$

5.5

Exercise: Find the following indefinite integrals:

a) $\int\left(sin(x^2 + 2x + 4)(x + 1)dx\right)$ b) $\int cot(x)dx$

c) $\int\left(\frac{9}{\sqrt{1-9x^2}}dx\right)$ d) $\int\left(e^{10x}dx\right)$

e) $\int\left(\frac{9x}{\sqrt{1-9x^2}}dx\right)$ f) $\int\left(\frac{6x}{2x^2+1}dx\right)$

a) $u = x^2 + 2x + 4,\quad du = (2x + 2)dx,\qquad \frac{du}{(2x+2)} = dx$

$\int\left(sin(x^2 + 2x + 4)(x + 1)dx\right) = \frac{1}{2}\int\left(sin(u)(x + 1)\frac{du}{(x+1)}\right) = \frac{1}{2}\int(sin(u)du)$

$= \ -\frac{1}{2}cos(u) + C = \ -\frac{1}{2}cos(x^2 + 2x + 4) + C$

b) $u = sin(x),\ du = \ cos(x)dx,$

$\int cot(x)dx = \int\frac{cos(x)dx}{sin(x)} = \int\frac{du}{u} = ln(|u|) + C = \ ln|sin(x)| + C$

c) $u = \ 3x,\qquad du = 3dx,\qquad \frac{du}{3} = dx$

$\int\left(\frac{9}{\sqrt{1-9x^2}}dx\right) = 9\int\left(\frac{1}{\sqrt{1-(3x^2)}}dx\right) = \ 9\int\left(\frac{1}{\sqrt{1-u^2}}\frac{du}{3}\right) = 3\int\left(\frac{du}{\sqrt{1-u^2}}\right) = \ 3sin^{-1}(u) + C = \ 3sin^{-1}(3x) + C$

d) $u = \ 10x,\quad du = 10dx,\qquad \frac{du}{10} = dx$

$\int\left(e^{10x}dx\right) = \frac{1}{10}\int\left(e^{u}du\right) = \frac{1}{10}\int\left(e^{u}du\right) = \frac{1}{10}e^{u} + C = \frac{1}{10}e^{10x} + C$

e) $u = \ 1 - 9x^2,\qquad du = \ -18xdx,\qquad -\frac{du}{18x} = dx$

$\int\left(\frac{9x}{\sqrt{1-9x^2}}dx\right) = \ -\int\left(\frac{9x}{\sqrt{u}} * \frac{du}{18x}\right) = \ -\frac{1}{2}\int\left(\frac{du}{\sqrt{u}}\right) = \ -\frac{1}{2}(2\sqrt{u}) + C = \ -\sqrt{1 - 9x^2} + C$

f) $u = \ 2x^2 + 1,\qquad du = \ 4xdx,\qquad \frac{du}{4x} = dx$

$\int\left(\frac{6x}{2x^2+1}dx\right) = \ 6\int\left(\frac{x}{2x^2+1}dx\right) = 6\int\left(\frac{x}{u}\frac{du}{4x}\right) = \ \frac{6}{4}\int\left(\frac{du}{u}\right) = \ \frac{3}{2}ln|u| + C = \frac{3}{2}ln(2x^2 + 1) + C$

Exercises: Evaluate the following definite integrals:

a) $\int_{0}^{\frac{3\pi}{4}} cos(x)sin^3(x)\,dx$ b) $\int_{-3}^{4} (x)\sqrt[3]{x\,+\,4}\,dx$ c) $\int_{0}^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$

a) $u\,=\,sin(x)$, $du\,=\,cos(x)dx$, $\frac{du}{cos(x)}\,=\,dx$, $u\,=\,sin\frac{5\pi}{4}\,=\,-\frac{\sqrt{2}}{2}$,
$u\,=\,sin(0)\,=\,0$

$$\int_{0}^{\frac{3\pi}{4}} cos(x)sin^3(x)\,dx\,=\,\int_{0}^{-\sqrt{2}/2} cos(x)u^3\,\frac{du}{cos(x)}\,=\,\int_{0}^{-\sqrt{2}/2} u^3du\,=\,\frac{1}{4}u^4\Big|_{0}^{-\sqrt{2}/2}\,=\frac{1}{4}(-\sqrt{2}/2)^4\,-\,0\,=\,\frac{1}{16}$$

b) $u\,=\,x\,+\,4$, $du\,=\,dx$, $u\,-\,4\,=\,x$, $u\,=\,4\,+\,4\,=\,8$,
$u\,=\,-\,3\,+\,4\,=\,1$

$$\int_{-3}^{4} (x)\sqrt[3]{x\,+\,4}\,dx\,=\int_{1}^{8} (u\,-\,4)u^{1/3}\,dx\,=\int_{1}^{8} (u^{4/3}\,-\,4u^{1/3})dx\,=\frac{3}{7}u^{7/3}\,-\,3u^{4/3}\Big|_{1}^{8}\,=\,(\frac{384}{7}\,-\,48)\,-\,(\frac{3}{7}\,-\,3)\,=\,-$$

c) $u\,=\,2x$, $du\,=\,2dx$, $\frac{du}{2}\,=\,dx$, $u\,=\,2\,*\,\frac{1}{4}\,=\frac{1}{2}$, $u\,=\,2(0)\,=\,0$,

$$\int_{0}^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx\,=\,\int_{0}^{1/2}\frac{1}{\sqrt{1-u^2}}\frac{du}{2}\,=\,\frac{1}{2}\int_{0}^{1/2}\frac{du}{\sqrt{1-u^2}}\,=\frac{1}{2}sin^{-1}(u)\Big|_{0}^{1/2}\,=\,\frac{1}{2}sin^{-1}(1/2)\,-\frac{1}{2}sin^{-1}(0)\,=\frac{1}{2}\frac{\pi}{6}\,=\frac{\pi}{12}$$

Exercises: Evaluate the following definite integrals:

a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{x}{1+x^2}\,+\,\frac{1}{1+x^2}\right)dx$ b) $\int_{-1}^{1} tan(x)(cos(x)\,\,+\,\,sin^2(x))dx$

a)
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{x}{1+x^2}\,+\,\frac{1}{1+x^2}\right)dx\,=\,\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{x}{1+x^2}\right)dx\,+\,\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{1+x^2}\right)dx\,=\,2\int_{0}^{\frac{\pi}{4}} \left(\frac{1}{1+x^2}\right)dx\,=\,2tan^{-1}(x)\Big|_{0}^{\pi/4}\,=\,2tan^{-1}(\pi/4)$$

b) $\int_{-1}^{1} tan(x)(cos(x)\,\,+\,\,sin^2(x))dx\,=\,0,$ because the function is odd.

6.1

Exercise: a) Find the area between the curves $f(x) = x^2 + 4$ and $g(x) = -3 - 2x$ on [-1, 3].
b) Find the enclosed area between the curves $f(x) = 3x + 6$ and $g(x) = x^2 + 2x$.

a)

$$\int_{-1}^{3} (x^2 + 4 - (-3 - 2x))dx = \int_{-1}^{3} (x^2 + 2x + 7)dx = \frac{1}{3}x^3 + x^2 + 7x|_{-1}^{3} = (9 + 9 + 21) - (-\frac{1}{3} + 1 -$$

b)

$$3x + 6 = x^2 + 2x$$
$$0 = x^2 - x - 6 = (x - 3)(x + 2)$$

g(x) is below f(x) everywhere on [-2, 3]

$$\int_{-2}^{3} (-x^2 + x + 6)dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x|_{-2}^{3} = (-9 + \frac{9}{2} + 18) - (\frac{8}{3} + 2 - 12) = \frac{27}{2} + \frac{22}{3} = \frac{125}{6}$$

Exercise: Find the area between the curves $f(x) = x^2 - 5x + 6$ and $g(x) = x^3 - 6x^2 - x + 18$ on the interval $[-1, 5]$.

$$x^3 - 6x^2 - x + 18 = x^2 - 5x + 6$$
$$x^3 - 7x^2 + 4x + 12 = (x + 1)(x^2 - 8x + 12) = (x + 1)(x - 2)(x - 6)$$

f(x) is below g(x) on [-1, 2], and g(x) is below f(x) on [2, 5]

$$\int_{-1}^{2} (x^3 - 7x^2 + 4x + 12)dx + \left|\int_{2}^{5} (x^3 - 7x^2 + 4x + 12)dx\right| = \frac{x^4}{4} - \frac{7x^3}{3} + 2x^2 + 12x|_{-1}^{2} + \left|\frac{x^4}{4} - \frac{7x^3}{3} + 2x^2\right.$$

$$4 - \frac{56}{3} + 8 + 24 - (\frac{1}{4} + \frac{7}{3} + 2 - 12) + \left|\frac{625}{4} - \frac{875}{3} + 50 + 60 - \left(4 - \frac{56}{3} + 8 + 24\right)\right| = \frac{99}{4} + \left| -\frac{171}{4}\right|$$

Exercise: Find the area enclosed by $y = x + 1$ and $y^2 = 3x + 1$

$$y^2 = 3x + 1$$
$$(x + 1)^2 = 3x + 1$$
$$x^2 + 2x + 1 = 3x + 1$$
$$x^2 - x = 0$$

Intersect at (0, 1) and (1, 2). On the interval [1,2] for y, y=x+1 is further to the right than $y^2 = 3x + 1$.

Rewrite to solve for x: $x = y - 1$ and $x = \frac{1}{3}y^2 - \frac{1}{3}$

$\int_1^2 (y - 1 - (\frac{1}{3}y^2 - \frac{1}{3}))dy = \int_1^2 (-\frac{1}{3}y^2 + y - \frac{2}{3})dy = -\frac{y^3}{9} + \frac{y^2}{2} - \frac{2y}{3}|_1^2 = (-\frac{8}{9} + 2 - \frac{4}{3}) - (-\frac{1}{9} +$

Exercise Find the area of the region enclosed by the following curves: $y = \frac{4}{x}$, y=x, and $y = \frac{1}{4}x$.

a) Do so using x as the variable of integration.
b) Do so using y as the variable of integration.

Intersections: y=x and $y = \frac{1}{4}x$ intersect at (0,0)

$y = \frac{4}{x}$ and $y = \frac{1}{4}x$ intersect at (4,1)

$y = \frac{4}{x}$ and y=x intersect at (2, 2)

a) On [0, 2], $y = \frac{1}{4}x$ is below y = x.

On [2, 4], $y = \frac{1}{4}x$ is below $y = \frac{4}{x}$.

$\int_0^2 (x - \frac{1}{4}x)dx + \int_2^4 (\frac{4}{x} - \frac{1}{4}x)dx = \int_0^2 (\frac{3}{4}x)dx + \int_2^4 (\frac{4}{x} - \frac{1}{4}x)dx = \frac{3x^2}{8}|_0^2 + (4ln(x) - \frac{x^2}{8})_2^4 = \frac{3}{2} + (4ln(4$

=4ln(2)=ln(16)

b) On [0, 1], $x = y$ is to the left of $x = 4y$

On [1, 2], $x = y$ is to the left of $x = \frac{4}{y}$.

$\int_0^1 (4y - y)dy + \int_1^2 (\frac{4}{y} - y)dy = \int_0^1 (3y)dy + \int_1^2 (\frac{4}{y} - y)dy = \frac{3y^2}{2}|_0^1 + (4ln(y) - \frac{y^2}{2})_1^2 = \frac{3}{2} + (4ln(2) -$

=4ln(2)=ln(16)