7.7) Approximating Integrals and Estimations

Exercise: Use the midpoint rule and the trapezoidal rule to approximate $\int_1^2 e^x dx$ with n = 5 subintervals, and find the error of each compared to the actual area of $e^2 - e^1 \approx 4.67077427.$

Midpoint: $\Delta x = \frac{2-1}{5} = 0.2$

$\bar{x}_1 = 1 + \frac{(1)}{2} * 0.2 = 1.1, \quad \bar{x}_2 = 1.3, \quad \bar{x}_3 = 1.5, \quad \bar{x}_4 = 1.7, \quad \bar{x}_5 = 1.9$

$$\int_1^2 e^x dx \approx (0.2)\left(e^{1.1} + e^{1.3} + e^{1.5} + e^{1.7} + e^{1.9}\right) \approx 4.6629987,$$ error = 0.0077755513

Trapezoidal: $x_0 = 1, \quad x_1 = 1.2, \quad x_2 = 1.4, \quad x_3 = 1.6, \quad x_4 = 1.8, \quad x_5 = 2$

$$\int_1^2 e^x dx \approx \frac{0.2}{2}\left(e^1 + 2e^{1.2} + 2e^{1.4} + 2e^{1.6} + 2e^{1.8} + e^2\right) \approx 4.686333148,$$ error= -0.0155588779

Exercise: How large should n be in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 e^x dx$ are accurate to within 0.0001?

K = e^2

$$|E_T| \le \frac{K(b-a)^3}{12n^2} < 0.0001$$
$$\frac{e^2(2-1)^3}{12n^2} < 0.0001$$
$$\frac{1}{n^2} < \frac{0.0012}{e^2}$$
$$n^2 > 6157.546$$
$$n > 78.5$$

Trapezoidal Rule: n = 79 or more

$$|E_M| \le \frac{K(b-a)^3}{24n^2} < 0.0001$$
$$\frac{e^2(2-1)^3}{24n^2} < 0.0001$$
$$\frac{1}{n^2} < \frac{0.0024}{e^2}$$
$$n^2 > 3078.77$$
$$n > 55.5$$

Midpoint Rule: n = 56 or more. Pick 79 or more to be certain they both are.

Exercise: a) Use the Midpoint Rule with n = 10 to approximate the integral $\int_0^1 sin(x^2)dx$.

b) Give an upper bound for the error involved in this approximation.

a) $\frac{b-a}{n} = \frac{1-0}{10} = 0.1,$

$\bar{x}_1 = 0 + 0.05 = 0.05, \bar{x}_2 = 0.15, \bar{x}_3 = 0.25, \bar{x}_4 = 0.35, \bar{x}_5 = 0.45, \bar{x}_6 = 0.55, \bar{x}_7 = 0.65,$

$\bar{x}_8 = 0.75, \bar{x}_9 = 0.85, \bar{x}_{10} = 0.95$

$\int_0^1 sin(x^2)dx \approx 0.1(sin(0.0025) + sin(0.0225) + sin(0.0625) + sin(0.1225) + sin(0.2025) +$
$sin(0.3025) + sin(0.4225) + sin(0.5625) + sin(0.7225) + sin(0.9025)) \approx 0.3098162946$

b) $\frac{d^2}{dx^2}\left[sin(x^2)\right] = \frac{d}{dx}\left[2xcos(x^2)\right] = \left[2cos(x^2) - 4x^2sin(x^2)\right]$ Let K = 4sin(1)-2cos(1)

$$\left|E_M\right| \le \frac{K(b-a)^3}{24n^2}$$

$$\left|E_M\right| \le \frac{2.285279327(1-0)^3}{24(10)^2} = \frac{1}{1200} = 0.0009522$$

No more than 0.0009522.

Exercise: Use Simpson's Rule with n = 6 to approximate $\int_1^2 e^x dx$.

$\Delta x = \frac{1}{6}$

$$\int_1^2 e^x dx \approx \frac{(1/6)}{3}(e^1 + 4e^{7/6} + 2e^{8/6} + 4e^{9/6} + 2e^{10/6} + 4e^{11/6} + e^2) = \frac{1}{18}(84.07429608) = 4.670794227$$

Exercise: a) Use Simpson's rule with n = 10 to approximate the integral $\int_0^1 e^{x^2} dx$.

b) Estimate the error involved in this approximation (find the error bound).

a) $\Delta x = \frac{1}{10}$

$$\int_0^1 e^x dx \approx \frac{1}{30}(e^0 + 4e^{.01} + 2e^{.04} + 4e^{.09} + 2e^{.16} + 4e^{.25} + 2e^{.36} + 4e^{.49} + 2e^{.64} + 4e^{.81} + e^1) = \frac{1}{30}(43.880442) = 1.4625$$

b)

$$\frac{d^4}{dx^4}\left[e^{x^2}\right] = \frac{d^3}{dx^3}\left[2xe^{x^2}\right] = \frac{d^2}{dx^2}\left[2e^{x^2} + 4x^2e^{x^2}\right] = \frac{d}{dx}\left[12xe^{x^2} + 8x^3e^{x^2}\right] = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$$

Let K = 76e.

$$\left|E_S\right| \le \frac{K(b-a)^5}{180n^4} = \frac{76e(1-0)^5}{180(10)^4} = 0.000114771$$

7.8) Improper Integrals

Exercise: Determine if the following integrals are convergent or divergent:

a) $\int_1^{\infty} \sqrt{\frac{1}{x}}dx$ b) $\int_{-\infty}^{-1} \frac{1}{x^3}dx$

a) $\int_1^{\infty} \sqrt{\frac{1}{x}}dx = \lim_{t\to\infty} \int_1^{t} x^{-1/2}dx = \lim_{t\to\infty} \left(2t^{1/2} - 2\right)$ Divergent.

b) $\int_{-\infty}^{-1} \frac{1}{x^3}dx = \lim_{t\to\infty} \int_{-t}^{-1} x^{-3}dx = \lim_{t\to\infty} -\frac{1}{2} + \frac{1}{2}t^{-2} = -\frac{1}{2}$ Convergent

Exercise: Is $\int_3^{5} \frac{x}{\sqrt{x^2-9}}dx$ convergent or divergent? Show why or why not.

$\int_3^{5} \frac{x}{\sqrt{x^2-9}}dx = \lim_{t\to 3^+} \int_t^{5} \frac{x}{\sqrt{x^2-9}}dx = \lim_{t\to 3^+} \sqrt{5^2 - 9} - \sqrt{t^2 - 9} = 4 - 0 = 4$ Convergent

Exercise: Is $\int_0^{\pi/2} \frac{sin(x)}{\sqrt{cos(x)}}dx$ convergent or divergent? Show why or why not.

$\int_0^{\pi/2} \frac{sin(x)}{\sqrt{cos(x)}}dx = \lim_{t\to \pi/2^-} \int_0^{t} \frac{sin(x)}{\sqrt{cos(x)}}dx = \lim_{t\to \pi/2^-} -2\sqrt{cos(t)} + 2\sqrt{cos(0)} = 0 + 2 = 2$
Convergent

Exercises: Determine if the following integrals are convergent or divergent. Show why or why not.

a) $\int_1^{\infty} \frac{\sqrt{x^2+1}}{x}dx$ b) $\int_1^{\infty} \frac{1-e^{-x^2}}{x^2}dx$

a) The expression is positive everywhere on the interval x>1, and $\int_1^{\infty} \frac{\sqrt{x^2+1}}{x}dx > \int_1^{\infty} \frac{1}{x}dx$ everywhere on the interval x > 1, so this integral is divergent too.

b) The expression is positive everywhere on the interval x>1, and $\int_1^{\infty} \frac{1-e^{-x^2}}{x^2}dx < \int_1^{\infty} \frac{1}{x^2}dx$ everywhere on the interval x > 1, so this integral is convergent too.