11.1) Sequences:

-A **sequence** of numbers is essentially any list of numbers presented in a definite order, like:

-Integers greater than 0 in ascending order: 1, 2, 3,4 ,.....

-Prime numbers in ascending order: 2, 3, 5, 7,...

-Positive powers of $\frac{1}{2}$ written in descending order: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

-The **Fibonacci sequence**: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,...

-These in fact are all examples of an **infinite sequence** since there is no final value in the sequence. There is always a starting value though, and in a sequence, we denote this value as a_1, and the nth value in the sequence is a_n.

-Sequences can be thought of as functions with a domain of positive integers only, where $f(n) = a_n$ for any positive integer. Sometimes functions are written with the first few terms written to that we can be certain of the "rules" that take you from one term to the next, like above, or in general:

$$a_1, a_2, a_3 \ldots$$

-But if the "rules" of what the nth term in a sequence is equal to is algebraic, then the sequence can be defined as:

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

-The second version in particular tells you what the first value of n in the sequence rule should be. n can be any number we want, but if n is not given, we presume n=1 is the start (not n=0). The expression inside the brackets would be the rule for determining what the nth term in the sequence is, such as for the positive powers of ½ sequence above which can be expressed as:

$$\left\{\frac{1}{2^n}\right\}$$

-Many times, the rule is a function of n, like above, or for some of the following examples where the rule and the first few terms are both given:

$$\left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.\right\}$$

$$\left\{(-1)^n \frac{(n+1)}{3^n}\right\}_{n=0}^{\infty} = \left\{\frac{1}{1}, -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \ldots.\right\}$$

-The second sequence in particular is what is called an **alternating sequence**, since the terms in the sequence alternate from being positive to being negative. The power $(-1)^n$ is a factor that allows for the terms to alternate in sign since raising -1 to an even power gives +1 and raising -1 to an odd power gives -1. So if a sequences is alternating, some variation on $(-1)^n$ will be a factor in the sequence.

-And of course, depending on proper manipulation of the starting point and the rule, the same sequence can be expressed in multiple ways:

$$\left\{\sqrt{n+3}\right\}_{n=0}^{\infty} = \{\sqrt{n}\}_{n=3}^{\infty} = \left\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots.\right\}$$

-Notice that we don't have to simplify every term in the sequence if we don't want to. Sometimes it's actually easier to spot the rule of the sequence if the terms are not in simplified form.

Example: Find a formula for the general term a_n of the sequence:

$$\left\{-\frac{3}{5}, \frac{4}{25}, -\frac{5}{125}, \frac{6}{625}, \ldots.\right\}$$

given that the pattern of the first few terms continues.

Solution: There is more than one sequence rule that will work, so you can experiment a bit if you wish as long as the rule satisfies the given terms. We will presume for simplicity that the first term is where n = 1, the second term is where n = 2, and so on.

The terms alternate, so we need a factor of $(-1)^n$. If the first term were positive, then we could make the power of this factor n+1, so that the first power of -1 is even and the second is odd. However, since the first term is negative, we need the first power of -1 to be odd and the second even, so n is fine in this case.

We can also see that the terms are all fractions with a denominator that is always scaling up by a factor of 5, so a power of 5 will be needed as well, in the denominator. The numerator is increasing by one each time, so if we let the numerator equal n+2 that should make the first numerator 3, the second 4, and so on.

The following rule should satisfy the sequence:

$$\left\{-\frac{3}{5}, \frac{4}{25}, -\frac{5}{125}, \frac{6}{625}, \ldots.\right\} = \left\{(-1)^n \frac{n+2}{5^n}\right\}$$

You can also just set the rule equal to a_n : $a_n = (-1)^n \frac{n+2}{5^n}$

Exercise: Find a formula for the general term a_n of the sequence:

$$\left\{\frac{1}{1}, -\frac{4}{7}, \frac{9}{49}, -\frac{16}{343}, \ldots.\right\}$$

given that the pattern of the first few terms continues.

-Of course, some sequences don't have a simple defining equation. Like the Fibonacci sequence from the start of this section, where the "rule" is $a_1 = 1,\ a_2 = 1,$ and $a_n = a_{n-1} + a_{n-2}$. There is no way to determine what the nth term in this sequence is unless you know the two terms that came before it, and they require you know the terms before them.

-Just because there is a rule, does not mean it's an equation either, like the sequence of numbers where the nth term is the nth number after the decimal place in pi:

$$\{1,\ 4,\ 1,\ 5,\ 9,\ 2,\ 6, \ldots\}$$

-However, if we did have a defining equation for a sequence, you could associate each term in the sequence with its place on the list. For the sequence $\left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.\right\}$ for example, you could define this sequence as a function with the discrete set of ordered pairs:

$$\left\{(1, \tfrac{n}{n+1})\right\} = \left\{(1, \tfrac{1}{2}), (2, \tfrac{2}{3}), (3, \tfrac{3}{4}), (4, \tfrac{4}{5}), \ldots.\right\}$$

-Sequences can be graphed, but only as a series of dots, not lines. Since the domain of sequences are sets of discrete numbers only, you can plot points in the xy-plane for a sequence like $\left\{(1, \frac{n}{n+1})\right\}$ but you cannot connect the dots in-between since the decimals in-between the whole numbers in the domain are not in the domain.

-If you did take plot the ordered pairs of $\left\{(1, \frac{n}{n+1})\right\}$ however, you would see that the ordered pairs approach the line y = 1, and that's no surprise, because like a function, you can take the limit of the rule of a defined sequence like $\left\{1, \frac{n}{n+1}\right\}$ to see what the rule is approaching as n goes to infinity.

-If we took the limit of $\frac{n}{n+1}$ as n goes to infinity (keeping in mind that n is still supposed to be whole numbers only while going to infinity), we would get:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

We can take this limit the way we'd take the limit of any function where the input is growing without bound. If the rule is approaching a finite number as n goes to infinity, then we say the sequence has a **limit** L. The following is an intuitive definition of a limit of a sequence that borrows from the definition of a limit we are familiar with:

A sequence $\{a_n\}$ has the **limit** L and we write:

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L, \text{ as } n \to \infty$$

if we can make the terms a_n as lose to L as we like by making n sufficiently large.

-If $\lim_{n\to\infty} a_n$ exists (is finite), then we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

-However, the <u>actual</u> definition of a limit of a sequence is as follows:

A sequence $\{a_n\}$ has the **limit** L and we write:

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L, \text{ as } n \to \infty$$

if for any $\varepsilon > 0$ there exists an integer N such that if n>N, then $|a_n - L| < \varepsilon$.

-In other words, a limit of a sequence exists if given any distance from the supposed limit L you choose, there is an index N of the sequence such that every term in the sequence after the Nth term is no more than that arbitrary distance from L.

-And as we have seen in the past, there are such things as functions with infinite limits: as n increases without bound, so does the function (in one direction or the other). There are sequences that have this property too (Fibonacci for one), and so we can also define infinite limits for sequences too:

If $\lim_{n\to\infty} a_n = \infty$, then that means for every positive number M, there is an integer N such that:

$$\text{if } n > N, \text{ then } a_n > M.$$

-If a sequence has an infinite limit, it is certainly a divergent sequence, but let's talk a bit about convergent sequences: sequences with a finite limit as n goes to infinity. The following is fairly logical:

$$\text{If } \lim_{n\to\infty} f(x) = L \text{ and } f(x) = a_n, \text{ then } \lim_{n\to\infty} a_n = L$$

-This is why it is so useful to be able to write sequences as a function of n. If you can do so, the limit of the sequence as n goes to infinity can be found the same way that a function limit as n goes to infinity is found.

-And of course, that means the many limit rules you learned from Calculus I can be used to find limits of sequences as long as the rule of the sequence is written as a function:

1) $\lim_{n\to\infty} \frac{1}{n^r} = 0$ if r > 0.

(the following rules suppose that $\{a_n\}$ *and* $\{b_n\}$ are convergent sequences and c is a constant)

2) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
3) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
4) $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right) * \left(\lim_{n\to\infty} b_n\right)$
5) $\lim_{n\to\infty}(c a_n) = c\left(\lim_{n\to\infty} a_n\right)$
6) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
7) $\lim_{n\to\infty}(a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p$ if p > 0 and $a_n > 0$
8) $\lim_{n\to\infty} c = c$

-You may also remember the **Squeeze Theorem** for function limits as well. It applies to sequence limits too:

If $a_n \le b_n \le c_n$ for $n \ge N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

-Another useful theorem that can be used for alternating sequences is the following:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Example: Find $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^3+1}$

Solution: You can start by finding the limit of $\frac{n^2}{n^3+1}$, without the power of -1. This is an alternating sequence, so if the limit of $\frac{n^2}{n^3+1}$ is 0, then so is the limit of the original sequence. If it is not, then by the fact that this is an alternating sequence the limit would not exist (at best it would alternate between L and -L).

As for finding the limit of $\frac{n^2}{n^3+1}$, we can use all the usual strategies of limits, like dividing top and bottom by the biggest power of n in the denominator:

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{n^2}{n^3}}{\frac{n^3}{n^3}+\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = \lim_{n\to\infty} \frac{0}{1+0} = 0.$$

Therefore, since $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$, we have $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^3+1} = 0$ as well.

-Again, don't worry about being able to use all the limit tricks from Calculus I, they still work here:

Example: Is the sequence $a_n = \frac{n}{\sqrt{n+20}}$ convergent or divergent?

Solution: Let's find the limit of $\frac{n}{\sqrt{n+20}}$ as n goes to infinity. If it is finite, the sequence converges. If it is infinite, the sequence diverges:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n+20}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n}{n^2}+\frac{20}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n}+\frac{20}{n^2}}}$$

The numerator is approaching 1, and the denominator is approaching 0 as n goes to infinity, which suggests that the expression itself is going to infinity:

$$\lim_{n\to\infty} \frac{n}{\sqrt{n+20}} = \infty$$

This sequence is divergent.

Example: Is the sequence $a_n = \frac{ln(n)}{n}$ convergent or divergent?

Solution: If n goes to infinity, the numerator and denominator both go to infinity too. So can we use L'Hospital's Rule here? Technically, no. L'Hospital's Rule is for functions, not sequences. However, if we did use L'Hospital's Rule on the analogous function $f(x) = \frac{ln(x)}{x}$ as x goes to infinity, what would we get?

$$\lim_{x\to\infty} \frac{ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

We know in general therefore that if x gets arbitrarily large, $\frac{ln(x)}{x}$ goes to 0. Would that change if x was only restricted to whole numbers? No. Therefore, that means that $\lim_{n\to\infty} \frac{ln(n)}{n}$. So we can't use L'Hospital's rule on the sequence, but if the sequence has a rule that we can rewrite in terms of x, we could just use L'Hospital's rule on the x expression instead. The limit of the x expression will also be the limit of the n expression.

This sequence in convergent.

Exercise: Determine whether the following sequences are convergent or divergent:

a) $a_n = \frac{10}{5-e^{-n}}$ b) $a_n = \frac{sin(n)}{n}$ c) $a_n = \frac{(-1)^n n^2}{2^n}$

Example: Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution: First, take the absolute value of the expression and take the limit of that absolute value as n goes to infinity:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

Since the absolute value of the expression goes to 0 as n goes to infinity, so does the original expression:

$$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$$

-Some sequences don't need calculation to determine if they converge or divergent.

Example: Determine if the following sequence converges or diverges: $a_n = (-1)^n$

Solution: This sequence is quite simply: -1, 1, -1, 1, ...

It alternates between 1 and -1. If the sequence is merely alternating between two different non-zero values, then it cannot converge to a single value, and therefore cannot be convergent. Therefore this sequence is divergent. Remember, divergent sequences don't have to go to infinity, they just do not approach a finite value.

-One useful theorem of sequences is that compositions of sequences within functions allow us to take limits within the compositions:

If $\lim_{n\to\infty} a_n = L,$ and the function f(x) is continuous at L, then:

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example: Find $\lim_{n\to\infty} sin\left(\frac{\pi}{n}\right)$.

Solution: sin(x) is continuous everywhere, and it's not hard to see that $\lim_{n\to\infty}\frac{\pi}{n} = 0,$ so that means:

$$\lim_{n\to\infty} sin\left(\frac{\pi}{n}\right) = sin(\lim_{n\to\infty}\frac{\pi}{n}) = sin(0) = 0$$

Example: Find $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)$.

Solution: Unfortunately, we cannot use L'Hospital's rule here, despite both the numerator and denominator approaching infinity as n grows. Even if we did rewrite this in terms of x, x! Is not defined for non-integer values, so you can't take the derivative of x! However, we can use the Squeeze Theorem here:

Remember, for any integer n, $\frac{n!}{n^n} = \frac{1*2*3*...*n}{n*n*n*...*n} = \frac{1}{n} * \left(\frac{2*3*...*n}{n*n*...*n}\right)$

n is an integer greater than 0, so this is definitely a positive expression, and also the fraction $\frac{2*3*...*n}{n*n*...*n}$ is smaller than 1 since the numerator is smaller than the denominator. So therefore,

$$0 \leq \left(\frac{2*3*...*n}{n*n*...*n}\right) \leq 1$$

Which if we multiply by $\frac{1}{n}$ in each part of the compound inequality, becomes:

$$0 \leq \left(\frac{1*2*3*...*n}{n*n*n*...*n}\right) \leq \frac{1}{n}$$

So if we take the limit of all three components as n goes to infinity:

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \left(\frac{1*2*3*...*n}{n*n*n*...*n}\right) \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} \left(\frac{n!}{n^n}\right) \leq 0$$

Thus by the Squeeze Theorem, $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right) = 0$.

Example: For what values of r is $a_n = r^n$ a convergent sequence?

Solution: It is fairly well known that exponential functions $f(x) = r^x$ grows without bound if r is greater than 1, and goes to 0 if 0 < r < 1. If r=1, r^x is equal to 1 for all x. Therefore, if $0 \leq r \leq 1$, $a_n = r^n$ converges to either 0 or 1.

What about if r was negative however? If r was negative, then $r^x = |r|^x * (-1)^x$. Since the absolute value of $|r|^x * (-1)^x$ is equal to $|r|^x$, if r was between -1<r<0, $|r|^x$ converges to 0, if r<-1, $|r|^x$ goes to infinity, and if r= -1, $|r|^x$ equals 1.

Therefore, the functions $a_n = r^n$ converges for all r in $-1 \leq r \leq 1$. More specifically, $a_n = r^n$ converges to 1 if r = 1 or -1, and converges to 0 if $-1 < r < 1$.

-Some of the sequences we have seen so far (especially the alternating ones) get larger and smaller from term to term, but a sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$, and **decreasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$.

-A sequence is called **monotonic** if it is increasing or decreasing. Usually you can tell by the rule when a function is monotonic by basic arithmetic or algebraic logic. Larger denominators mean smaller numbers, so a sequence with a constant numerator and a denominator containing n getting larger is decreasing:

$\left\{\frac{1}{n+1}\right\}$, $\left\{\frac{1}{n^2+1}\right\}$, $\left\{\frac{1}{\sqrt{n+1}}\right\}$, and $\left\{\frac{1}{e^n}\right\}$ are all decreasing.

-Still, sometimes it's not obvious that a function is monotonic:

Example: Prove that the sequence $a_n = \frac{n}{n^2+1}$ is monotonic.

Solution: You can see that $a_1 = \frac{1}{2}$ and $a_2 = \frac{2}{5}$, so if this function is monotonic, it is decreasing. But can you prove that for any n, that $a_n \geq a_{n+1}$? First off, remember that n is a whole number greater than or equal to 1 for all terms. From here, we can use a little algebraic manipulation to see if it is true that $a_n \geq a_{n+1}$ for general n:

$$a_n \geq a_{n+1}$$

$$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1}$$

(try cross-multiplying here, after all, n is positive)

$$n((n+1)^2+1) \geq (n+1)(n^2+1)$$
$$n(n^2+2n+1+1) \geq n^3+n^2+n+1$$
$$n^3+2n^2+2n \geq n^3+n^2+n+1$$

If we move a few terms to the left side:

$$n^2+n \geq 1$$

If n is greater than or equal to 1, $n^2+n \geq 1$ is satisfied, as are all the other inequalities here. Therefore, we know that for any n, $a_n \geq a_{n+1}$, so this sequence is decreasing, and therefore monotonic.

-If that solution was a little too algebraic for you, you can also replace n with x and treat the sequence rule as a function.

-If you take the derivative of the function and show that the derivative is positive everywhere x>1, then the function/sequence is increasing.

-If you take the derivative of the function and show that the derivative is negative everywhere x>1, then the function/sequence is decreasing.

Exercise: Prove that the sequence $a_n = \frac{n}{\sqrt{n^2+1}}$ is increasing.

-A sequence is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$, and a sequence is **bounded below** if there is a number m such that $a_n \geq m$ for all $n \geq 1$.

-A sequence is a **bounded sequence** if it is bounded above and bounded below.

-Now intuitively, what can you say about a sequence that is increasing, but also bounded above? Since the sequence is increasing everywhere, but there is also a number M it never exceeds, there must be a finite limit less than or equal to M that it is converging to. Therefore, when a function can be shown to be bounded <u>and</u> monotonic in the proper combination, it is true that the function is convergent.

-Every increasing sequence that is bounded above is convergent.

-Every decreasing sequence that is bounded below is convergent.

-Every bounded sequence that is monotonic is convergent.

-The proof of these is in the textbook, but we will skip it here. All I will say is that you want to be careful of a few combinations:

-If a sequence is increasing and bounded below, it does not have to be convergent, and

-If a sequence is decreasing and bounded above, it does not have to be convergent.

-Mind the monotonic direction of the function and where it is bounded. If they do not match, you cannot say the function is convergent.

Example: Is the recursive sequence below convergent?

$$a_1 = 2,\ \ a_{n+1} = \tfrac{1}{2}(a_n + 6)\ \text{ for }\ n = 1, 2, 3...$$

Solution: We have no rule in terms of n we can use, but we can use induction. We can easily see what a_2 equals: $a_2 = \frac{1}{2}(a_1 + 6) = \frac{1}{2}(2 + 6) = \frac{1}{2}(8) = 4.$ So $a_2 > a_1$. However, can we now show that if $a_{k+1} > a_k$ if $a_k > a_{k-1}$? If we can, then by induction we have shown that every new term in the sequence after a_2 is greater than the one before it, and that in turn means this would be an increasing sequence.

Let's presume $a_k > a_{k-1}$, which means:

$$\begin{gathered} a_k > a_{k-1} \\ \tfrac{1}{2}(a_{k-1} + 6) > a_{k-1} \\ \tfrac{1}{2}a_{k-1} + 3 > a_{k-1} \\ a_{k-1} + 6 > 2a_{k-1} \\ 6 > a_{k-1} \end{gathered}$$

So does that also mean $a_{k+1} > a_k$?

$$a_{k+1} = \tfrac{1}{2}(a_k + 6) = \tfrac{1}{2}a_k + \tfrac{1}{2}(6) > \tfrac{1}{2}(a_k) + \tfrac{1}{2}(a_k) = \tfrac{1}{2}(2a_k) = a_k$$

Therefore, it is true that $a_{k+1} > a_k$. We have shown that if $a_k > a_{k-1}$, then $a_{k+1} > a_k$. We have also seen that a_2 is greater than a_1, so that means $a_3 > a_2$, and $a_4 > a_3$, and so on. Therefore, this sequence is increasing everywhere.

Is that all we know? Not quite. After all, we saw that if $a_k > a_{k-1}$, then $6 > a_{k-1}$ as well, and since every term is small than the next term in the sequence, that means 6 is greater than every term in the sequence. So this sequence is bounded above.

If the sequence is bounded above <u>and</u> also increasing, that means the sequence is convergent.

Is <u>that</u> all we know? Still not quite! Since we know that this sequence is convergent, that means it approaches a limit, call it L. Logically, we would presume that the limit, L, is 6. Can we prove it?

What would the limit of a_n be? We know it would be L. However, what would the limit of be in terms of the recursive rule?

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \tfrac{1}{2}(a_{n-1} + 6) = \lim_{n\to\infty} \tfrac{1}{2}(a_{n-1}) + \tfrac{1}{2}\lim_{n\to\infty} 6 = \tfrac{1}{2}\lim_{n\to\infty} a_{n-1} + 3$$

If n is going to infinity, so is n-1. And so if a_n approaches L when n goes to infinity, shouldn't a_{n-1} also approach L as n-1 goes to infinity? So that means $\frac{1}{2}\lim_{n\to\infty} a_{n-1} + 3 = \frac{1}{2}L + 3.$ That means:

$$L = \frac{1}{2}L + 3$$
$$\frac{1}{2}L = 3$$
$$L = 6$$

So we have shown not only that this sequence is bounded above, not only shown that it is increasing, not only shown that it is convergent, but we have also shown it converges to L=6.