5.2) The Definite Integral:

-We have seen that the area underneath a continuous positive curve on an interval [a,b] can be found by:

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(x_i^*))$$

-It turns out this infinite summation goes by another name and another notation. Suppose that f is a defined function on the interval [a, b] (not necessarily positive everywhere anymore), and the interval is divided into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, and we still let $x_0 = a,\ x_1,\ x_2, ..., x_n = b$ be the endpoints of these subintervals, with x_i^* a sample point on the ith subinterval $[x_{i-1},\ x_i]$. Then the **definite integral of f from a to b** is defined and denoted by:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(x_i^*))$$

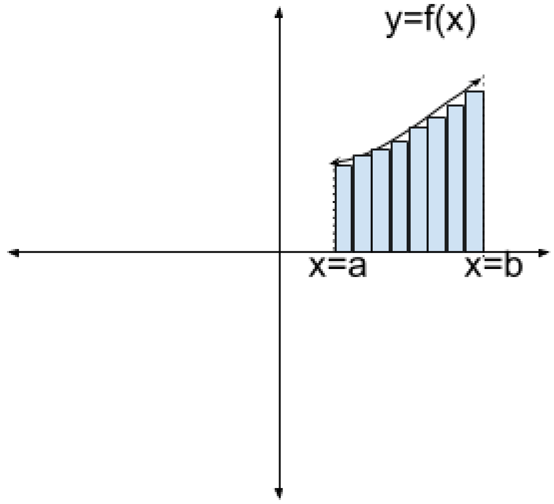
As long as the limit exists and gives the same value for all possible choices of sample points. If this limit exists, we say that f is **integrable** on [a, b].

-The symbol you see here, $\int$, is called an **integral sign**, and looks like a stretched out S since an integral is an infinite limit of **S**ums. As for the expression $\int_a^b f(x)dx$, f(x) is called the **integrand**, a is the **lower limit**, b is the **upper limit**, and while "dx" does not seem to have much meaning yet, it will eventually have to do with differences in x (Δx). For now, you just need to remember that dx is an indicator of what the independent variable of the expression is, even though by the time the calculation is over, x will be of little importance, to the point that x could be replaced with any other letter and the integral would compute to being the same number.

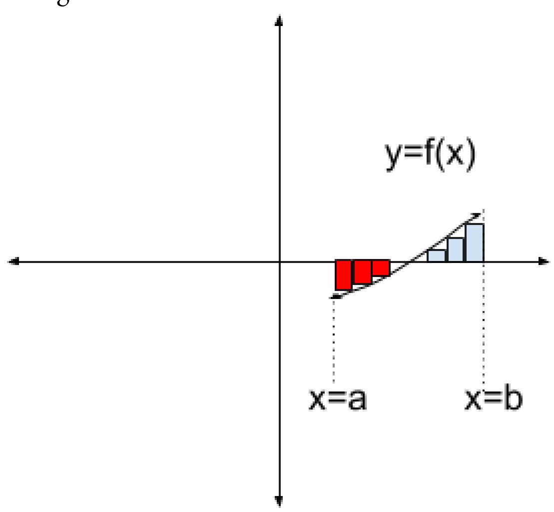
-The procedure of computing an integral is called **integration**, and $\int_a^b f(x)dx$ itself is a number, just like how if you computed $\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(x_i^*))$ over an interval [a, b] you would have a number. In fact, if $f(x) \geq 0$ everywhere on the interval [a, b], then $\int_a^b f(x)dx$ is the area under the curve y = f(x) from a to b.

-The summation $\sum_{i=1}^{n} (\Delta x * f(x_i^*))$ we have used and seen a few times will be coming back later on, and is called a **Riemann** sum, so do not forget the name. If you take the limit of that Riemann sum as n goes to infinity you get a definite integral, but a Riemann sum for a finite number n is an approximation of area under a curve. The next page reproduces one of the graphs from the previous section to show why.

-Reimann sums are summations of areas of rectangles found by multiplying a constant width of these rectangles (Δx) by the heights of these different rectangles $(f(x_i^*))$ then adding them all together.

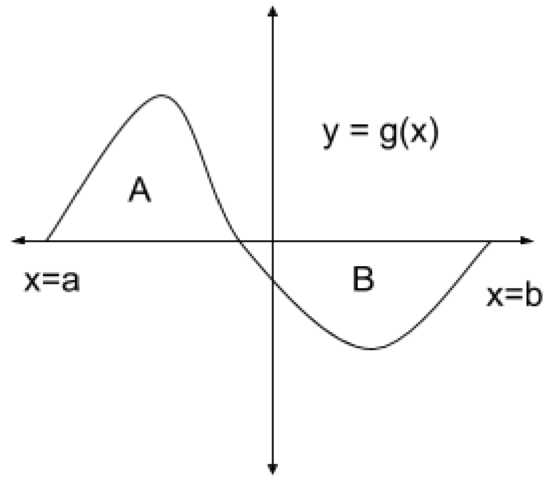


-But what happens if f is not positive everywhere on the interval x=a to x=b? That would mean that not every product in the Riemann sum is positive, and thus not every product is an area of a rectangle (areas have to be positive). When $f(x_i^*)$ is positive, then $\Delta x f(x_i^*)$ is an area of a rectangle, but when $f(x_i^*)$ is negative, $\Delta x f(x_i^*)$ is the negative of an area of a rectangle:



-The blue rectangles are on parts of the interval between x = a and x = b where f(x) is positive, and the red rectangles are on parts of the interval where f(x) is negative. If you took a Riemann sum on the interval of this function, you would get positive products that represent the area of the blue rectangles, and negative products that represent the negative of the area of the red rectangles.

-Since Riemann sums can be broken into a positive portion and a negative portion, and Riemann sums are analogous to area under a curve, that means definite integrals can also be broken up into two portions: a positive portion and a negative portion. As a result, definite integrals are more generally interpreted as a **net area**, or a difference of areas.



-The definite integral of the above function, y = g(x) from a to b is the area between the curve and the x-axis labeled A minus the area between the curve and the x-axis labeled B:

$$\int_a^b g(x)dx = A - B$$

If A is larger in area than B, then the net area/definite integral is positive, but if B is larger in area than A, then the net area is negative.

-You can use the strategies from last section to approximate net area considering that the only difference in the approximation process is that not every $f(x_i^*)$ output is positive anymore:

Example: Approximate the Riemann sum for $f(x) = x^3 - 6x$ on the interval [0, 3] with n = 6 subintervals and using right endpoints.

Solution: As always, we need the width of the intervals, $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$. We also need endpoints, and since we are using right endpoints, we have:

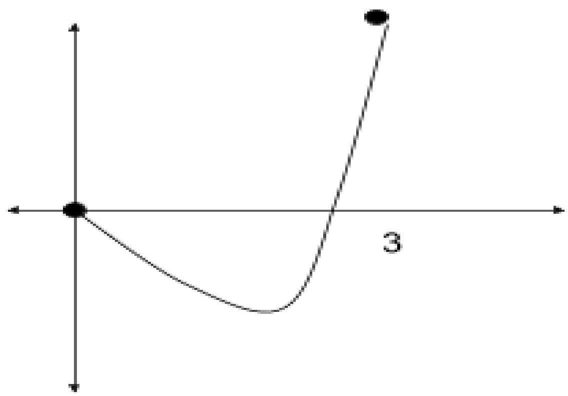
$$x_1 = a + \Delta x = 0 + 0.5 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2, \quad x_5 = 2.5, \quad x_6 = 3$$

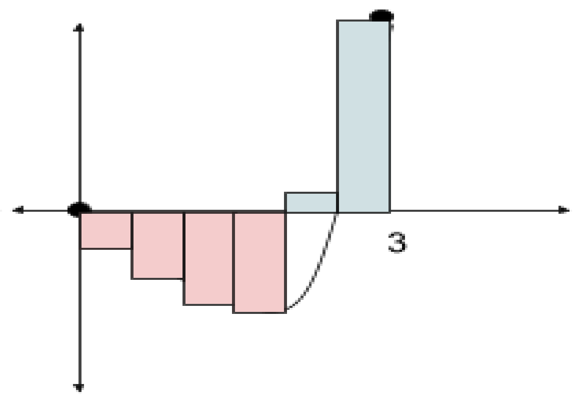
Therefore, our right endpoint Riemann sum is:

$$R_6 = \sum_{i=1}^{6} f(x_I)\Delta x = 0.5\,(f(.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3))$$

$$R_6 = 0.5\,(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) = -3.9375$$

If you graphed $f(x) = x^3 - 6x$ on the interval [0, 3] and drew some rectangles that all intersect on the right endpoint, the sum of the areas of the rectangles above the curve $(0.625 + 9)$ would be smaller than the sum of the areas of the rectangles beneath the curve $(-2.875 - 5 - 5.625 - 4)$.





The sum of the areas of the grey rectangles minus the sum of the area of the red rectangles gives us our Riemann sum. Notice that when a graph is increasing at some points and decreasing at others, not every rectangle will be intersecting above the curve or below the curve. Some will be above, some will be below.

Exercise: Approximate the Riemann sum for $g(x) = x^3 - 4x$ on the interval [1, 3] with n = 5 subintervals and using left endpoints.

-We mentioned last section that while we tend to make sure all the intervals are the same width to get the same Δx, not every interval must be the same to approximate, and as n gets arbitrarily large, the widths can differ in size and the selected points don't all have to be endpoints and the summation will still approach the net area between the x-axis and the curve.

-In fact, as long as a function is integrable, it doesn't matter what points are chosen on each interval, the limit of the Riemann sum will still approach the net area. In this course, just about every function we will be dealing with will be integrable, as the following theorem states (the proof for this theorem is too advanced for this course):

-If f is continuous on [a,b] or if f has only a finite number of jump discontinuities, then f is integrable on [a,b], and therefore $\int_a^b f(x)dx$ exists and is equal to the Riemann sum as n approaches infinity:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(x_i)) \quad \text{where } \Delta x = \tfrac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

-It is quite simple to rewrite an infinite Riemann sum in integral form as long as you know where each component fits in.

Example: What is the integral form of the following Riemann sum?

$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 - cos(x_i) + ln(x_i))\Delta x$ on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$

Solution: Since an integral has the form $\int_a^b f(x)dx$ you need the lower and upper limits (a and b) and a function f(x). The function f(x) should simply be the expression inside the Riemann sum with x_i replaced with x. As for the limits, since the expression in the Riemann sum is continuous on the entire interval in question, the limits of the integral are just the endpoints of the interval the Riemann sum takes place on.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x^3 - cos(x) + ln(x))dx$$

You can also think of the symbols $\lim_{n\to\infty} \sum_{i=1}^{n}$ being replaced with the integrand symbol, and Δx replaced with dx.

-You may be needing the following summation rules again in this section:

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1(2n+1))}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

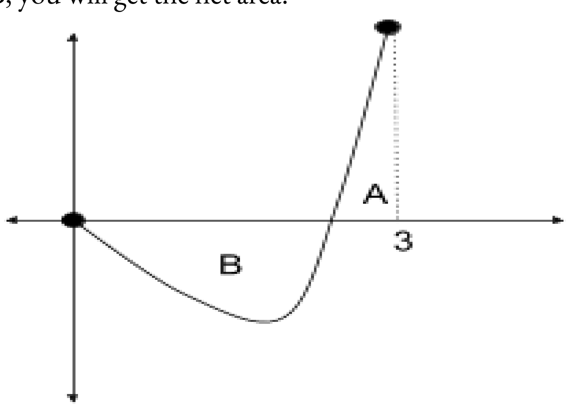
-But you may also need to add on top of the the following properties of summation:

$$\sum_{i=1}^{n} ca_1 = c\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

-These properties are all proven by the fact that a summation is essentially a sum of numbers, so these are essentially all commutative, distribution, and associative properties of addition all at work.

Example: Evaluate $\int_0^3 (x^3 - 6x)dx$.

Solution: We have approximated the net area of $f(x) = x^3 - 6x$ from a = 0 to b = 3, but now we find the exact net area. If you take the area above the x-axis and beneath the curve, A, and subtract that from the area above the curve and beneath the x-axis, B, you will get the net area:



This will require $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, and inputs x_i^*. The inputs can arbitrarily be the right endpoints (again, it shouldn't matter) where $x_1^* = a + \Delta x = 0 + \frac{3}{n} = \frac{3}{n}$, $x_2^* = a + 2\Delta x = \frac{6}{n}$, and in general, $x_i^* = \frac{3i}{n}$.

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(x_i)) = \lim_{n\to\infty} \sum_{i=1}^{n} (\tfrac{3}{n} f(\tfrac{3i}{n}))$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{3}{n}((\tfrac{3i}{n})^3 - 6(\tfrac{3i}{n}))$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \tfrac{3}{n} \left[\sum_{i=1}^{n} (\tfrac{27i^3}{n^3} - \tfrac{18i}{n}) \right]$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \left(\tfrac{81}{n^4} \left[\sum_{i=1}^{n} i^3 \right] - \tfrac{54}{n^2} \left[\sum_{i=1}^{n} i \right] \right)$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \left(\tfrac{81}{n^4} \left(\tfrac{n^2(n+1)^2}{4} \right) - \tfrac{54}{n^2} \left(\tfrac{n(n+1)}{2} \right) \right)$$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \left(\tfrac{81n^2(n+1)^2}{4n^4} \right) - \lim_{n\to\infty} \left(\tfrac{27n(n+1)}{n^2} \right)$$

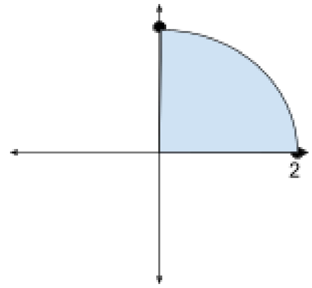
$$\int_0^3 (x^3 - 6x)dx = \tfrac{81}{4} - 27 = -\tfrac{27}{4}$$

Exercise: Evaluate $\int_0^2 (x^2 - 1)dx$.

-Of course, not every net area requires limits, Riemann sums, and algebra to find. Sometimes you can use a little geometric ingenuity to find net area.

Example: a) Evaluate $\int_0^2 \sqrt{4 - x^2}dx$. b) Evaluate $\int_0^3 (x - 2)dx$.

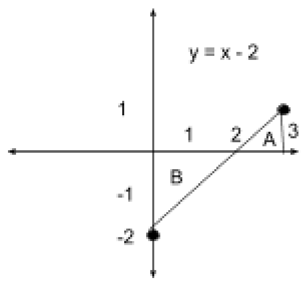
Solution: a)This expression would be rather difficult to set up for a Riemann sum, and even harder to evaluate since we haven't seen any summation tricks for square roots. However, when you remember that the definite integral is asking for net area, it becomes a bit more logical to think of this integral solely in terms of area, especially when you see the shape of the region that makes up the net area:



After all, if you set $y = \sqrt{4 - x^2}$, then you get $y^2 = 4 - x^2$, or $x^2 + y^2 = 4$, a circle with a radius of 2. The interval of values we are allowed to use for the x-axis is from 0 to 2 only, and its implied that y must be positive too. Thus the area under the curve $y = \sqrt{4 - x^2}$ and above the x-axis between 0 and 2 is one quarter of a circle of radius 2.

The area of a circle of radius 2 is $\pi r^2 = \pi(2)^2 = 4\pi$, so one quarter of this is π. The definite integral equals π, not found with a Riemann sum, but found with geometry.

b) This example can be done with a Riemann sum, but you can find it even faster if you graph the function and observe what the sections that make up the net area look like:



The net area is the area labeled A minus the area labeled B. A is a right triangle, and so is B. So therefore, the area of A is: $\frac{1}{2}(1 * 1) = \frac{1}{2}$, and the area of B is $\frac{1}{2}(2 * 2) = 2$. Thus, without Riemann sums, we know the net area is $-\frac{3}{2}$.

Exercise: a) Evaluate $\int_{-3}^{3} -\sqrt{9 - x^2}dx$. b) Evaluate $\int_{0}^{3}(x + 3)dx$.

-When using Riemann sums to find net area, it doesn't matter if the values used for x_i^* are left endpoints, right endpoints, or midpoints. However, when <u>approximating</u> net area, midpoints tend to give more accurate approximations than either kind of endpoint. In general:

$$\int_a^b f(x)dx \approx \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x * f(\overline{x}_i))$$

$\overline{x}_i$ is used for arithmetic means notation wise, but here $\overline{x}_i$ stands for a midpoint: $\overline{x}_i = \frac{x_i + x_{i-1}}{2}$

Example: Approximate the Riemann sum for $f(x) = x^3 - 6x$ on the interval [0, 3] with n = 6 subintervals and using midpoints.

Solution: As always, we need the width of the intervals, $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$. The midpoints are:

$$\overline{x}_1 = a + \frac{\Delta x}{2} = 0 + \frac{0.5}{2} = 0.25, \quad x_2 = 0.75, \quad x_3 = 1.25, \quad x_4 = 1.75, \quad x_5 = 2.25, \quad x_6 = 2.75$$

Therefore, our midpoint Riemann sum is:

$$R_6 = \sum_{i=1}^{6} f(x_I)\Delta x = 0.5\,(f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75))$$

$$R_6 = 0.5\,(-1.4384375 - 4.078125 - 5.546875 - 5.140625 - 2.109375 + 4.296875) = -7.00828125$$

Still not the exact value of -6.75 we found earlier, but more accurate than our earlier approximation of -3.9375.

Properties of the Definite Integral:

-Be sure to commit the following properties to memory:

$$\int_b^a f(x)dx \;=\; -\int_a^b f(x)dx$$

-The above property is actually fairly obvious. If you swap the upper and lower limits of the integral, then the width $\Delta x = \frac{b-a}{n}$ in the Riemann sum becomes $\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$, and since the width is a distributive factor in the Riemann sum, an opposite sign for Δx is also an opposite sign for the entire Riemann sum. Therefore, if the limits swap places, the net area swaps signs.

$$\int_a^a f(x)dx \;=\; 0$$

-If the lower and upper limit are the same number, then the width $\Delta x = 0$, which makes the entire Riemann sum 0 as well.

$$\int_a^b c\,dx \;=\; c(b-a) \text{ where c is a constant number}$$

$$\int_a^b (f(x) \pm g(x))dx \;=\; \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \text{ where c is a constant number}$$

-These properties hold due to distributive and associative properties of the limit and summations in a definite integral. These properties have a fair bit of use given how they essentially allow definite integrals to be broken up into smaller pieces that can be evaluated separately:

Example: Evaluate $\int_0^3 (5 - 4x^3 + 24x)dx$.

Solution: With some rewriting, this definite integral is much easier to compute than as written now.

First off, we can separate the first term from the others by giving them their own definite integrals:

$$\int_0^3 (5 - 4x^3 + 24x)dx = \int_0^3 5dx + \int_0^3 (-4x^3 + 24x)dx \;=\; \int_0^3 5dx - 4\int_0^3 (x^3 - 6x)dx$$

A previous example told us that $\int_0^3 (x^3 - 6x)dx = -6.75,$ and we have a property that states that $\int_0^3 5dx$ is simply the constant number inside times the product of the difference in upper limit minus lower limit:

$$\int_0^3 5dx - 4\int_0^3 (x^3 - 6x)dx = 5(3-0) - 4(-6.75) = 15 + 27 = 42$$

Exercise: Evaluate the following definite integral without a Riemann sum. Try using previous examples and exercises to answer:

$$\int_0^3 (x^3 - 4x + 6)dx$$

-Some properties don't involve rewriting what is in the integral, but rather what is **on** the integral:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example: Given that $\int_4^{12} f(x)dx = -4$ *and* $\int_4^6 f(x)dx = 12,$ find $\int_6^{12} f(x)dx$

Solution: If we rewrite $\int_4^{12} f(x)dx = \int_4^6 f(x)dx + \int_6^{12} f(x)dx,$ and we know the values of two of these integrals, then we have the value of the third one:

$$\int_4^{12} f(x)dx = \int_4^6 f(x)dx + \int_6^{12} f(x)dx$$

$$-4 = 12 + \int_6^{12} f(x)dx$$

$$-16 = \int_6^{12} f(x)dx,$$

Exercise: Given $\int_4^{12} f(x)dx = 20,$ $\int_4^6 f(x)dx = 8,$ *and* $\int_8^{12} f(x)dx = -5,$ find $\int_6^{12} f(x)dx$ and $\int_6^8 f(x)dx.$

-Some properties are not about a single integral, but are about comparing one integral to another:

- If $f(x) \geq 0$ on $a \leq x \leq b,$ then $\int_a^b f(x)dx \geq 0.$
- If $f(x) \geq g(x)$ for $a \leq x \leq b,$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b,$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$