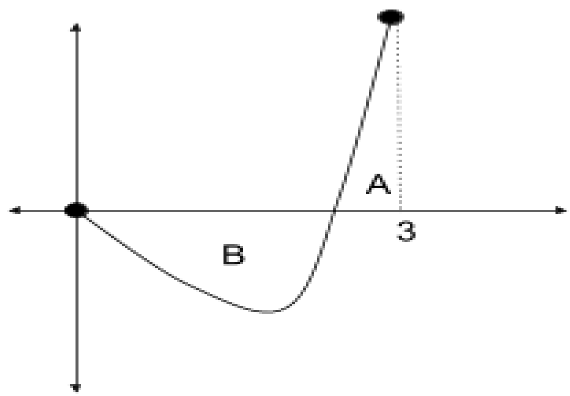
5.3) The Fundamental Theorem of Calculus:

-If there is one theorem that students remember from Calculus (while being unable to recite it verbatim....) it is the **Fundamental Theorem of Calculus**. Before we define it (and redefine it) suppose we have a continuous function f(x) on [a,b] where x is a variable on [a,b], then let g be the function of x seen below:

$$g(x) = \int_a^x f(t)dt$$

-This <u>is</u> a function of x, since no matter what value of x we choose on [a,b], g(x) will return one value back: the net area of f from a to x. Even better, if f(x) is a positive function, then g(x) will return the area under f and above the x-axis from a to x, and g(x) will be a constantly increasing function, making g(x) invertible.

-For example, for the function $f(x) = x^3 - 6x$ on [0,3] which we saw last section (which you can also solve quickly to find has a root at $x = \sqrt{6}$), the graph was given to be:



-If you evaluated $g(x) = \int_0^x (t^3 - 6t)dt$ at x = 0, you would have g(0)=0, if you evaluated g(x) at 3, you would have g(3)= the area of A minus the area of B, and if you evaluated g(x) at $\sqrt{6}$, you would have $g(\sqrt{6})$ = the negative of the area of B.

-First off, remember these areas under the curve evaluated by g <u>must</u> start at the start of the interval. You cannot use a single output of g to find an area starting somewhere else than the starting point. Now if you had two outputs, like say $g(\sqrt{6})$ and g(3), you can use the properties from last section to find that the area of A is equal to $g(3) - g(\sqrt{6})$.

-Second off, do not try using x both <u>inside</u> the integral as well as a limit. The reason we use "t" inside and "x" as a limit is to avoid confusion of what the integral is in terms of, and the variable the function g is in terms of.

-What would happen if you actually tried to evaluate the definite integral used for g(x)? The result depends on the function f(t) inside the integral, but we will attempt it here for a simple example:

Example: Evaluate $\int_0^x (t)dt$

Solution: $$\int_0^x (t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta t * f(t_i))$$

Remember, the variable inside the integral is t, not x. x plays the role for what Δt equals and t_i we use. We will use right endpoints here arbitrarily to find the width and inputs:

$$\Delta t = \frac{x-0}{n} = \frac{x}{n} \qquad \text{and} \qquad t_i = 0 + \frac{i*x}{n} = \frac{ix}{n}$$

We can now evaluate the Riemann sum limit since our function $f(t_i)$ is simply t_i.

$$\int_0^x (t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta t * f(t_i))$$

$$\int_0^x (t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{x}{n} * \frac{xi}{n}\right)$$

$$\int_0^x (t)dt = \lim_{n\to\infty} \frac{x^2}{n^2} \sum_{i=1}^{n} (i)$$

$$\int_0^x (t)dt = x^2 \left(\lim_{n\to\infty} \frac{1}{n^2} \frac{(n(n+1))}{2}\right)$$

$$\int_0^x (t)dt = x^2 \left(\lim_{n\to\infty} \frac{(n(n+1))}{2n^2}\right) = x^2 \left(\frac{1}{2}\right) = \frac{x^2}{2}$$

-It would seem that g(x) is not only an area function, but it's also an **antiderivative** of f(x). That means that $g'(x) = f$ (you can verify it yourself if you want), at least when f(x) = x. Can we prove it in general?

-Turns out, we can, but we'll need help from the definition of a derivative (no, not $f'(x) = nx^{n-1}$, we're talking about the limit definition!):

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

-We also work with how the definition can be adjusted for approximation purposes. Given h>0, g(x+h)-g(x) is a difference in areas under the graph of f. Remember earlier when we said the difference of two outputs of g give you the area starting at one point and ending at another under the curve of f? This is exactly that, starting at x and ending at x + h.

-However, if you want the area under the curve of f(x) starting at x and ending at x + h, you can also approximate that by using a rectangle with a width of h and a height of f(x). The area of this rectangle is approximately the area under f(x) from x to x+h, and the area of the rectangle is found by multiplying h and f(x) together. As a result:

$$g(x+h) - g(x) \approx h * f(x)$$

-Of course, you can also solve this for f(x) to get:

$$\frac{g(x+h)-g(x)}{h} \approx f(x)$$

-Take the limit of each side as h goes to zero to get:

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} f(x) = f(x)$$

$$g'(x) = f(x)$$

-Thus we have the first part of the **Fundamental Theorem of Calculus**:

Fundamental Theorem of Calculus (Part 1):

If f is continuous on [a,b], then the function g(x) defined by $g(x) = \int_a^x f(t)dt$ for $a \le x \le b$. is continuous on [a,b], and differentiable on (a,b) such that $g'(x) = f(x)$.

-A thorough proof for this first half of the Fund. Thm. of Calc. can be found in the text, though you won't have to reproduce it. You will need to remember the theorem though, even though there is a part 2 that most remember more than this half. This half still has its uses, especially when written in its abbreviated form:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \qquad \text{if f is continuous}$$

-Again, this is a shorthand way of saying the area function g(x) is an antiderivative of f(x). Which makes finding the derivative of g(x) quite straightforward, most of the time anyway:

Example: Find the derivative of $\int_0^x \sqrt{1+t^2}\,dt$.

Solution: Don't forget to check that $f(x) = \sqrt{1+x^2}$ is a continuous function, since if it was not, the theorem does not hold anymore, and thus the derivative of the area function may not necessarily be what is inside the integral. In fact, there's a good chance the area function wouldn't be differentiable at all if what is inside is not continuous. In this case however, the solution is very easy:

$$\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}$$

-That said, don't get too comfortable with the theorem, since you still have to abide by other rules, like the chain rule of differentiation.

Example: Find the derivative of $\int_0^{2x^3} \sqrt{1+4t^3}\,dt$.

Solution: Remember, t is not the variable, x is. Therefore if the upper limit is anything besides x, do not forget to multiply by an extra factor of whatever the derivative of that expression is. If anything you can even try rewriting the expression using u notation:

$$\frac{d}{dx}\left[\int_0^{2x^3} \sqrt{1+4t^3}\,dt\right] = \frac{d}{dx}\left[\int_0^{u} \sqrt{1+4t^3}\,dt\right] = \sqrt{1+4u^3} \, * \frac{du}{dx} = \sqrt{1+4(2x^3)^3} \, * (6x^2) = \sqrt{1+32x^9} \, * (6x^2)$$

-You may also need to rewrite the expression first to ensure that your expression in terms of x is the upper limit, not the bottom limit, or break the integral into two pieces and take the derivative of each (use the properties of the integral from the previous section to do that).

Exercise: Find the derivative of the following with respect to x:

a) $\int_0^{8x} e^{t^2-2t-1}dt$ b) $\int_x^{2} cos(t^3-1)dt$ c) $\int_x^{x^2} \sqrt{t^4+t^2+1}dt$

-Now we address the other half of the **Fundamental Theorem of Calculus.** What if neither limit of the definite integral were a variable? What if they are both constants?

The Fundamental Theorem of Calculus (Part 2):

If f is a continuous function on $[a,b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f. In other words, F is any function where $F' = f$.

-The Fund. The. of Calc. part 2 also is a theorem of integrals, but this one is applied more often due to it not being restricted only to the area function $g(x) = \int_a^x f(t)dt$. Still, it is the second part of the theorem since it uses the first part for its proof.

-We know that $g(x) = \int_a^x f(t)dt$ is an antiderivative of f(x), so if F is our generic antiderivative of f, then let g(x) be F(x)+C where C is a constant number. After all, every antiderivative of a function differs by a constant term. This means that F(x) = g(x) - C. So, now let's take a look at what $F(b) - F(a)$ is:

$$F(b) - F(a) = (g(b) - C) - (g(a) - C)$$
$$F(b) - F(a) = g(b) - g(a)$$

However, what does g(a) equal? $g(a) = \int_a^a f(t)dt = 0$

And what does g(b) equal? $g(b) = \int_a^b f(t)dt = 0$

Replace t with x, and you have your proof for why $\int_a^b f(x)dx = F(b) - F(a)$.

-This means that definite integrals are much easier to find than they used to be (most of the time) and you will not need to use Riemann sums to find them (unless I ask you to, in which case, follow the directions!), as long as you can find an antiderivative of the expression inside the integral (which is not always the easiest thing to do).

Example: Evaluate the integral $\int_1^3 e^x dx$

Solution: Remember, you don't need an elaborate antiderivative to what is inside. Just pick the simplest function that you know has the derivative of the expression. e^x is it's own derivative, so an antiderivative of $f(x) = e^x$ is $F(x) = e^x$.

So the answer is the difference of F(3) - F(1): $\int_1^3 e^x dx = e^3 - e^1$. No need to simplify.

A common notation practice is that when you have found an antiderivative F(x) for the expression in the integral, but you have not plugged your values in yet, you can write $\int_a^b f(x)dx = F(x)]^b{}_a$. A square bracket (or sometimes a vertical line) with a superscript and a subscript on it. The superscript would be the upper limit, and the subscript would be the lower limit, both about to be plugged in and then subtracted by each other.

Example: Evaluate the integral $\int_0^1 (x^2)dx$

Solution: This has been long coming. What has a derivative of x^2 ? You have to do a little backwards thinking sometimes, as there are some antiderivatives that are easy to find if you apply a famous derivative rule in reverse. In this case, you need to undo the power rule so to speak.

When you differentiate a power function you multiply the function by its degree, <u>then</u> reduce that degree by one unit.

So to integrate a power function, you increase the degree of the function by one unit, <u>then</u> divide by that new degree.

Thus an antiderivative of x^2 would be $\frac{1}{3}x^3$, so $\int_0^1 (x^2)dx = \frac{1}{3}x^3]^1{}_0 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3} - 0 = \frac{1}{3}$.

Exercise: Evaluate the integral $\int_1^2 (x^3 - 6x^2 - 8x + 12 - \frac{4}{x^2})dx$

-Sometimes there is not trick to finding the derivative except to just remember that certain functions are famous derivatives of other functions:

Example: Evaluate the integrals: a) $\int_{1}^{e^2} \frac{1}{x}dx$ b) $\int_{0}^{\pi} sin(x)dx$

Solution: a) What function has a derivative of $\frac{1}{x}$?

If you cannot remember, this will be a difficult time. The derivative of $ln(x)$ is $\frac{1}{x}$, but another function that has $\frac{1}{x}$ as a derivative is $ln\,|x|$. Between the two though, the absolute value version is more encompassing for what can be plugged into it, so we will use that one for our evaluation:

$$\int_{1}^{e^2} \frac{1}{x}dx = ln\,|x|\,]^{e^2}{}_{1} = ln(e^2) - ln(1) = 2 - 0 = 2$$

b) What function has a derivative of sin(x)?

If you guessed cos(x), you are wrong. It's a common mistake to forget that the derivative of cos(x) is -sin(x), not sin(x). But you can add an extra factor of -1 (more on that when we get to integration by substitution) to make up for the fact that all is missing is a negative:

$$\int_{0}^{\pi} sin(x)dx = -cos(x)]^{\pi}{}_{0} = -cos(\pi) - (-cos(0)) = -(-1) + (1) = 2$$

Be careful with all those negatives floating around. It's quite easy to forget a negative, forget to distribute a negative, or incorrectly combine a negative. Also, don't forget that definite integrals are still returning net area, and sin(x) on this interval is nonnegative, meaning the final net area should be positive. If you get a negative number, you did something wrong.

-That logic should never be forgotten.

Example: What is wrong with this solution:

$$\int_{-1}^{3} \left(\frac{1}{x^2}\right) dx = \int_{-1}^{3} (x^{-2})\, dx = \frac{x^{-1}}{-1}]^{3}{}_{-1} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Correction: Nothing is incorrect here algebraically, as all the steps taken are the correct ones, but clearly something is wrong because the final answer is negative, but $\frac{1}{x^2}$ is a positive function on its domain, meaning the net area over any section of its domain should be positive.

What's the problem? Take another look at those limits: the interval they cover is [-1, 3], which includes 0. 0 is not in the domain of $\frac{1}{x^2}$, and so the fundamental theorem of calculus does not apply here due to $\frac{1}{x^2}$ not being continuous over the interval.

How do we evaluate an integral like this? That's a topic for another day.

Exercise: Evaluate the integrals: a) $\int_{3}^{48} \frac{1}{2x}dx$ b) $\int_{-\pi/4}^{\pi/4} sec^2(x)dx$

-We've danced around the topic all section, but for our purposes and taking both parts of the Fundamental Theorem of Calculus into consideration, you should come away from this section remembering the following:

The Fundamental Theorem of Calculus:

Given a continuous function f on [a,b]:

1) The function g(x) defined by $g(x) = \int_a^x f(t)dt$ for $a \le x \le b.$ is continuous on [a,b], and differentiable on (a,b) such that $g'(x) = f(x).$
2) $\int_a^b f(x)dx = F(b) - F(a),$ where F is any antiderivative of f. In other words, F is any function where $F' = f.$

-Taken together, you can think of integration and differentiation as inverse processes, just like multiplication and division, or addition and subtraction, or exponentials and logarithms.