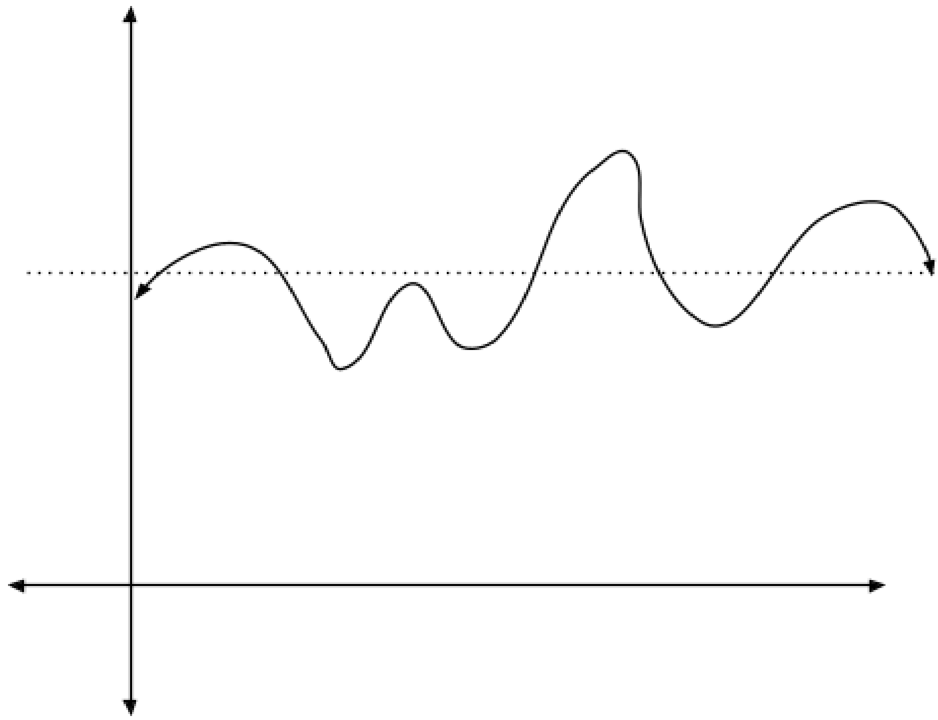
6.5) Average Value of a Function:

-Given a finite number of values, you most certainly know how to find the average value of those values:

$$y_{average} = \frac{y_1+y_2+...+y_n}{n}$$

-But what if you wanted to compute the average value of a variable that has not only infinitely many values, but uncountably many infinite values? Suppose you wanted to know what someone's average heart rate was from 8:00 a.m. to 10:00 a.m.. You cannot possibly take every heart rate from that period and add them up before dividing how many there are, since there are infinitely many points of time to consider.



-That's to say nothing of a function that has a graph like the one above, where the dotted line is about as close as we can get to approximating what the average output of the function is just by estimating what the average value in which the graph is evenly above and below the average value.

-In general, if you wanted to compute the average value of a function y = f(x), on $a \le x \le b$, you can start by dividing the interval [a,b] into n subintervals, each with length $\Delta x = \frac{b-a}{n}$. Your average value will be found by taking an output from each subinterval used, adding them up, and dividing by how many subintervals you used.

-Let x_i^* be the input from the ith subinterval, which means the average value of the outputs used will be:

$$\frac{f(x_1^*)+f(x_2^*)+...+f(x_n^*)}{n}$$

-Recall however that we said that $\Delta x = \frac{b-a}{n}$, so that means $n = \frac{b-a}{\Delta x}$. This can be plugged in for n in the average ratio:

$$\frac{f(x_1^*)+f(x_2^*)+...+f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{\Delta x(f(x_1^*)+f(x_2^*)+...+f(x_n^*))}{b-a} = \frac{1}{b-a}\Delta x\,(f(x_1^*)+f(x_2^*)+...+f(x_n^*))$$

Or simply: $$\frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

-As always, we can imagine that a more accurate average value will occur if n goes to infinity, so that means that the average value of our function will equal:

$$f_{average} = \lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

-The **average value** of a continuous function from x = a to x = b can be found by taking the definite integral of the function from x = a to x = b, and then dividing by the difference in limits.

-If f is a positive function, the outputs are essentially heights of the function, so the average value of f would also be the average height of the function, which means that since $\int_a^b f(x)dx$ is the area under the function from a to b, the average height ($y = f_{average}$, like the flat dotted line in the continuous graph seen earlier) is the area under the curve divided by b-a, the width of the area under the curve:

$$average\ height = \frac{area}{width}$$

Example: What is the average value of the function $f(x) = 1 + x^2$ on the interval $[-1,\ 2]$?

Solution: The function is continuous from x = -1 to x = 2, so the average value of f is:

$$f_{average} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)dx = \frac{1}{3}\left(x+\frac{x^3}{3}\,\Big|_{-1}^{2}\right) = \frac{1}{3}\left(\frac{14}{3}-\left(-\frac{4}{3}\right)\right) = \frac{1}{3}(6) = 2$$

Exercise: What is the average value of the function $f(x) = x + x^3$ on the interval $[-2,\ 6]$?

-Suppose you know the average value of a continuous function on an interval [a,b], $f_{average}$. Is it required that there is some value c in [a,b] that must make f(x) equal that average value? In fact, yes, it is a guarantee.

The Mean Value Theorem for Integrals:

Not to be confused with the mean value theorem for derivatives (which I 'm sure you all remember!), the mean value theorem for integrals states that if f is continuous on [a,b], then there exists a number c in [a,b] such that:

$$f(c) = f_{average} = \frac{1}{b-a}\int_a^b f(x)dx$$

In other words, $$f(c) * (b-a) = \int_a^b f(x)dx$$

-Remember, the Mean Value Theorem only says there is a c on the interval that makes f equal its average value, it does not tell you what that value is, or how many there are.

Example: Find all c that satisfies $\frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)dx = f(c)$ for $f(x) = 1 + x^2$ on [-1, 2].

Solution: We know that the average value of f(x) on [-1, 2] is 2 from a previous example, so where does $f(x) = 1 + x^2$ equal 2 on [-1, 2]? It may occur in one place, or multiple places:

$$f(c) = 2$$
$$1 + c^2 = 2$$
$$c^2 = 1$$
$$c = \pm 1$$

Both of these values are on [-1, 2], but if one of them was not, we can ignore that value. $f(x) = 1 + x^2$ equals its average value at x = 1 and x = -1.

Exercise: Find all c that satisfies the mean value theorem for $f(x) = 2x + x^2$ on the interval $[-3,\ 6]$

-Did you know you can prove that the average velocity of a car over a time interval [a, b] is the same as the average of its velocities over that same time period?

-What is the average velocity of a car over the time period [a,b]? That would be its difference in position divided by the difference in time from t = a to t = b. Let s(t) be the position of the car at time t:

$$average\ velocity = \frac{s(b)-s(a)}{b-a}$$

-How do you find the average of the velocities of this same car over the time period [a,b]? That would be the average value of the velocity function of the car, which if the position function is s(t), means the velocity function is $v(t) = s'(t)$.

-What is the average value of $s'(t)$ from t=a to t=b?

$$s'_{average} = \frac{1}{b-a}\int_{a}^{b}(s'(t))dt = \frac{1}{b-a}(s(t)\ |\ _a^b) = \frac{s(b)-s(a)}{b-a}$$

-There you have it, the average velocity and the average of the velocities are equal!