11.6) The Ratio and Root Test

Exercise: Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\sum_{n=1}^{\infty} \frac{5^n}{(n+1)2^{2n+1}}$ b) $\sum_{n=1}^{\infty} \frac{(n)10^n}{(n-1)!}$ c) $\sum_{n=1}^{\infty} \frac{cos(\pi n)}{\sqrt{n}}$

a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{5^{n+1}}{(n+2)2^{2n+3}}}{\frac{5^n}{(n+1)2^{2n+1}}}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1}}{(n+2)2^{2n+3}} * \frac{(n+1)2^{2n+1}}{5^n}\right| = \frac{5}{4} \lim_{n\to\infty} \left|\frac{n+1}{(n+2)}\right| = \frac{5}{4}$

Divergent

b) $\lim_{n\to\infty} \left|\frac{\frac{(n+1)10^{n+1}}{(n)!}}{\frac{(n)10^n}{(n-1)!}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)10^{n+1}}{(n)!} * \frac{(n-1)!}{n*10^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)*10}{n^2}\right| = 0$ Absolutely Convergent

c) $\lim_{n\to\infty} \left|\frac{\frac{cos(\pi(n+1))}{\sqrt{n+1}}}{\frac{cos(\pi n)}{\sqrt{n}}}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}}{\sqrt{n}}\right| = 1$ Inconclusive

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, conditionally convergent only.

Exercise: Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(ln\, n)^n}$ b) $\sum_{n=1}^{\infty} (tan^{-1}(n))^n$ c) $\sum_{n=1}^{\infty} \frac{n3^{2n}}{6^{n+1}}$

a) $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(ln(n))^n}\right|} = \lim_{n\to\infty} \frac{1}{ln(n)} = 0$ Absolutely Convergent

b) $\lim_{n\to\infty} \sqrt[n]{(tan^{-1}(x))^n} = \lim_{n\to\infty} tan^{-1}(x) = \frac{\pi}{2}$ Divergent

c) $\lim_{n\to\infty} \sqrt[n]{\frac{n3^{2n}}{6^{n+1}}} = \frac{9}{6} * \lim_{n\to\infty} \sqrt[n]{\frac{n}{6}} = \frac{3}{2}$ Divergent

11.8) Power Series:

Exercise: For what values of x do the following series converge?

a) $\sum_{n=2}^{\infty} \frac{(5-2x)^n}{ln(n)}$ b) $\sum_{n=1}^{\infty} (x - 1)^n \sqrt{n^2 + 1}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$

a) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(5-2x)^{n+1}}{ln(n+1)}}{\frac{(5-2x)^n}{ln(n)}}\right| = |5 - 2x| \lim_{n\to\infty} \left|\frac{ln(n+1)}{ln(n)}\right| = |5 - 2x| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |5 - 2x|$$

$$|5 - 2x| < 1$$
$$2 < x < 3$$

When x = 2, $\sum_{n=2}^{\infty} \frac{(5-2x)^n}{ln(n)} = \sum_{n=2}^{\infty} \frac{1}{ln(n)} > \sum_{n=1}^{\infty} \frac{1}{n}$ divergent.

When x = 3, $\sum_{n=2}^{\infty} \frac{(5-2x)^n}{ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{ln(n)}, \frac{1}{ln(n)}$ decreasing and approaches 0 as x goes to infinity, so convergent.

Interval of convergence = (2, 3].

b) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}\sqrt{n^2+2n+2}}{(x-1)^n\sqrt{n^2+1}}\right| = |x - 1| \lim_{n\to\infty} \left|\frac{\sqrt{n^2+2n+2}}{\sqrt{n^2+1}}\right| = |x - 1|$$

$$|x - 1| < 1$$
$$0 < x < 2$$

When x = 0, $\sum_{n=1}^{\infty} (- 1)^n \sqrt{n^2 + 1}$, divergent. When x = 2, $\sum_{n=1}^{\infty} (1)^n \sqrt{n^2 + 1}$, divergent.

Interval of convergence = (0, 2).

c) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)5^{n+1}} x^{n+1}}{\frac{1}{n5^n} x^n}\right| = \left|\frac{x}{5}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \left|\frac{x}{5}\right|$$

$$\left|\frac{x}{5}\right| < 1$$
$$- 5 < x < 5$$

When x = -5, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(- 5)^n = \sum_{n=1}^{\infty} \frac{-1}{n}$ divergent.

When x = 5, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ decreasing and approaches 0 as x goes to infinity, so convergent.

Interval of convergence = (-5, 5].

Exercise: Find the radius of convergence and interval of convergence for the following series:

a) $\sum_{n=1}^{\infty} \frac{(7-3x)^n}{n^2}$ b) $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{\sqrt{2n-1}}$ c) $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n ln(n)}$

a) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(7-3x)^{n+1}}{(n+1)^2}}{\frac{(7-3x)^n}{n^2}}\right| = |7-3x| \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = |7-3x|$$

$$|7-3x| < 1$$
$$2 < x < \frac{8}{3}$$

When x = 2, $\sum_{n=1}^{\infty} \frac{(7-3x)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ divergent.

When x = 8/3, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ decreasing and approaches 0 as x goes to infinity, so convergent.

Interval of convergence = $[2, \frac{8}{3}]$, Radius of Convergence = $\frac{1}{3}$.

b) $$\lim_{n\to\infty} \left|\frac{\frac{(x-3)^{2n+2}}{\sqrt{2n+1}}}{\frac{(x-3)^{2n}}{\sqrt{2n-1}}}\right| = (x-3)^2 \lim_{n\to\infty} \left|\sqrt{\frac{2n-1}{2n+1}}\right| = (x-3)^2.$$

$$|x-3| < 1$$
$$2 < x < 4$$

When x = 2, $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{2n-1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ divergent. When x = 4, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ divergent.

Interval of convergence = $(2, 4)$, Radius of Convergence = 1.

c) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x+2)^{n+1}}{2^{n+1} ln(n+1)}}{\frac{(x+2)^n}{2^n ln(n)}}\right| = \left|\frac{x+2}{2}\right| \lim_{n\to\infty} \left|\frac{ln(n)}{ln(n+1)}\right| = \left|\frac{x+2}{2}\right| \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = \left|\frac{x+2}{2}\right|$$

$$\left|\frac{x+2}{2}\right| < 1$$
$$-4 < x < 0$$

When x = -4, $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{ln(n)}$ convergent. When x = 0, $\sum_{n=2}^{\infty} \frac{(2)^n}{2^n ln(n)} = \sum_{n=2}^{\infty} \frac{1}{ln(n)}$ divergent.

Interval of convergence = $[-4, 0)$, Radius of Convergence = 2.

11.9) Representations of Functions as Power Series:

Exercise: Express the following as the sum of a power series and find the interval of convergence.

a) $\frac{1}{1+5x}$ b) $\frac{1}{3+2x}$ c) $\frac{1}{1-\sqrt{x}}$

a) $\frac{1}{1+5x} = \sum_{n=0}^{\infty} (-1)^n (5x)^n$ Interval of Convergence = $\left(-\frac{1}{5}, \frac{1}{5}\right)$

b) $\frac{1}{3+2x} = \frac{1/3}{1+\frac{2}{3}x} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n (\frac{2}{3}x)^n$ Interval of Convergence = $\left(-\frac{3}{2}, \frac{3}{2}\right)$

c) $\frac{1}{1-\sqrt{x}} = \sum_{n=0}^{\infty} (x)^{n/2}$ Interval of Convergence = (0, 1)

Exercise: Express the following as the sum of a power series and find the interval of convergence.

a) $ln(1-4x)$ b) $tan^{-1}(3x)$

a)

$$ln(1-4x) = \int \frac{-4dx}{1-4x} = \int \sum_{n=0}^{\infty} (-4) * (4x)^n dx = -\int \sum_{n=0}^{\infty} (4)^{n+1} x^n dx = C - \sum_{n=0}^{\infty} (4)^{n+1} \frac{x^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{(4x)^{n+1}}{n+1}$$

Interval of convergence: $[-\frac{1}{4}, \frac{1}{4})$

b) $$tan^{-1}(3x) = \int \frac{3dx}{1+9x^2} dx = \int \sum_{n=0}^{\infty} (3)^{2n+1} * (-1)^n * (x)^{2n} dx = \sum_{n=0}^{\infty} (3)^{2n+1} * (-1)^n * \int (x)^{2n} dx =$$

$$\sum_{n=0}^{\infty} (3)^{2n+1} * (-1)^n \frac{x^{2n+1}}{2n+1}$$

Interval of convergence: $[-\frac{1}{3}, \frac{1}{3}]$

Exercise: a) Evaluate $\int \frac{1}{1+x^3} dx$ as a power series.

b) Approximate $\int_0^{0.5} \frac{1}{1+x^3} dx$ correct to within 10^{-9}.

a)

$$\frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n dx = \sum_{n=0}^{\infty} (-1)^n (x^3)^n dx = \sum_{n=0}^{\infty} (-1)^n (x)^{3n} dx = \sum_{n=0}^{\infty} \int (-1)^n (x)^{3n} dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}\right) + C$$

b) $$\int_0^{0.5} \frac{1}{1+x^3} dx = \left(\sum_{n=0}^{8} (-1)^n \frac{1}{3n+1} * \frac{1}{2^{3n+1}}\right) =$$

$$\frac{1}{2} - \frac{1}{64} + \frac{1}{896} - \frac{1}{10240} + \frac{1}{106496} - \frac{1}{1048576} + \frac{1}{9961472} - \frac{1}{92274688} + \frac{1}{838860800} = 0.4854018521$$