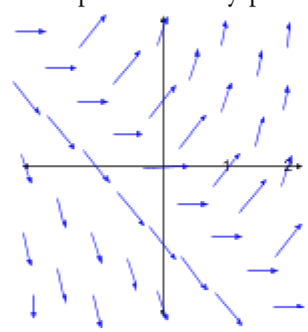
9.2) Directional Fields and Euler's Method:

-There is not an explicit way to solve differential equations in general, but one tool you have is graphing. If you were given the differential equation below:

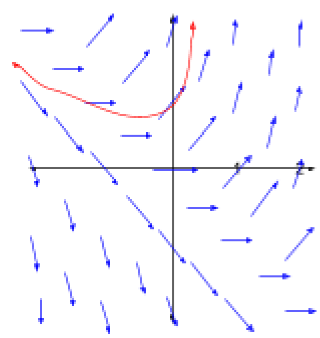
$$y' = x + y \qquad \text{for } y(0)=1$$

we could use the equation to determine that the equation in question has the property that the rate of change of the curve has a slope (y') equal to the sum of the coordinates. If (0,1) satisfied the equation, then the graph of the equation would have a slope of 1 at (0,1). If it then passed from there to about (1, 2), it would have a slope of 3. If it then passed from there to about (2, 5), it would have a slope of 7, and so on.

-Of course, this is very wishful thinking. This is a curve, not a connect-the-dots exercise, and just because the curve has a slope of 1 at (0,1) does not mean it continues to have a slope of 1 immediately before or after (0,1). So one thing that can be done is a **direction field** can be drawn that draws many imaginary lines that have the slope that the curve would in theory have at that point in the xy-plane.



-The smaller line segments point to where the curve would be heading if it were in that vicinity of the plane, so you can draw a solution curve through the point (0,1) by following the arrows of the direction field to determine where the curve is going and where it came from.

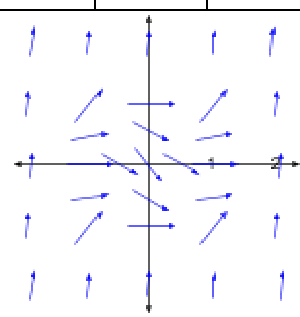


-Suppose that we have a first order differential equation $y' = F(x, y)$ where $F(x, y)$ is some expression in terms of x and y, where F(x,y) states that the slope of the curve passing through (x,y) is F(x,y). If we draw line segments with slope F(x,y) at several points (x,y) the result is a **direction field** (or **slope field**) that helps visualize the shape of the curve, provided that we have a point that the curve passes through (otherwise there's a whole family of curves that could fit the directions in a direction field).

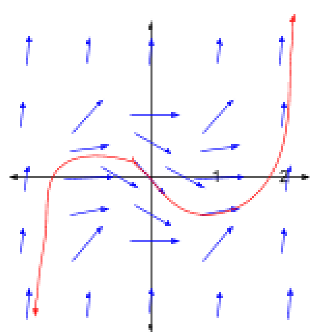
Example: a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
b) Use your direction field from part a) to sketch the solution curve that passes through the origin.

Solution: We should take the time to find the slope at some designated ordered pairs, and that will help get a sense of what the direction field will look like. Often the slopes of certain lines in the field can help determine the slopes of the lines around them.

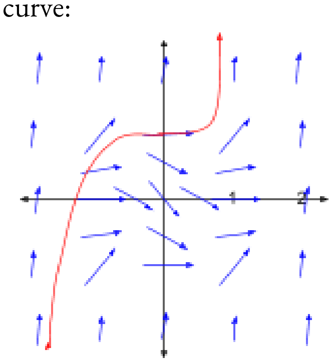
x	-2	-1	0	1	2	-2	-1	0	1	2
y	0	0	0	0	0	1	1	1	1	1
y'	3	0	-1	0	3	4	1	0	1	4

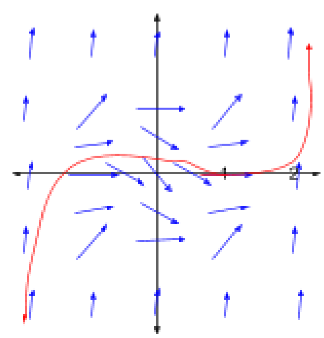


b) The curve passed through (0,0) would have to have a slope of -1 as it passes through the origin, so the direction of the slopes before and after this point can give us an idea of what the curve looks like in that vicinity of the xy-plane, and that in turn tells us what the rest of the curve would look like too:



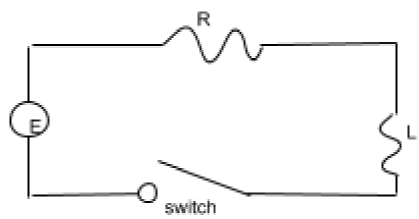
-Of course, if the given point was not (0,0) but was a different point, like say (0, 1), or (1,,0), that would lead to a different looking curve:





Exercise: a) Sketch the direction field for the differential equation $y' = x - y - 1$.
b) Use your direction field from part a) to sketch the solution curve that passes through the origin.

-A simple electric circuit contains an electromotive force that produces a voltage of E(t) volts and a current of I(t) amperes at time t. The circuit also contains a resistor with a resistance of R ohms and an inductor with an inductance of L henries. An image of such a circuit is below:



-Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L * (dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal tot he supplied voltage E(t). From this, we have a first-order differential equation that models the current I at time t:

$$L * \frac{dI}{dt} + RI = E(t)$$

Example: Suppose that in the simple circuit the resistance is 12 ohms, the inductance is 4 henries, and a battery gives a voltage of 60 volts. If we plug these numbers into $L * \frac{dI}{dt} + RI = E(t)$ we get:

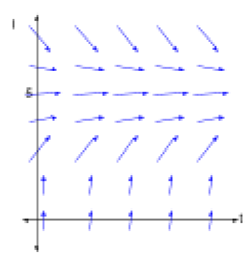
$$4 * \frac{dI}{dt} + 12I = 60$$

a) Draw a direction field for your differential equation.
b) What can you say about the limiting value of the current?
c) Identify any equilibrium solutions.
d) If the switch is closed when t = 0 so the current starts with I(0)=0, use the direction field to sketch a solution curve.

Solution: a) We can rewrite the equation to get the rate of change by itself, since we are after different slopes for given values for time t and current I:

$$\frac{dI}{dt} = 15 - 3I$$

t is the input variable and I is the output variable when you graph, but you also notice there is no t variable in the equation itself. This means that the slope depends entirely on values for I, and that the slopes will be the same for different times as long as the current remains the same:

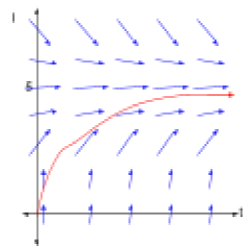


We can see that the slopes are flat when I = 5, and all other slopes point towards I = 5.

b) The limiting value of the current is what the current appears to be approaching as time goes on. We can see from the direction field that no matter where the curve begins, just about every curve approaches I = 5. Therefore, the limiting value of the current is I = 5 amperes.

c) The equilibrium solutions will be any constant functions I(t) = k that satisfy $\frac{dI}{dt} = 15 - 3I$. Since I(t) would be constant, we need all values of I that make $15 - 3I = 0$. Therefore we have an equilibrium solution at I = 5 only.

d) To draw a solution curve that passes through (0,0), start at (0,0) and use the slope directions to determine where to draw next:



-Though any solution curve (starting from below I=5 at least) would look pretty similar to this one, since every solution curve approaches the limiting value of I = 5. Part of the reason why is because $\frac{dI}{dt}$ was a function of L only with no t variables. A differential equation of the form:

$$y' = f(y)$$

in which the independent variable is missing entirely is called **autonomous**. In an autonomous differential equation, any two ordered pairs with the same y-coordinate will have the same slope.

Exercise: Suppose that in the simple circuit the resistance is 10 ohms, the inductance is 2 henries, and a battery gives a voltage of 80 volts.

a) Draw a direction field for your differential equation.
b) What can you say about the limiting value of the current?
c) Identify any equilibrium solutions.
d) If the switch is closed when t = 0 so the current starts with I(0)=0, use the direction field to sketch a solution curve.

-As you can see, you can use direction fields to find numerical solutions to differential equations, but numerical solutions can also be approximated using **Euler's Method**, which we can illustrate with a previous example:

$$y' = x + y \qquad \text{where} \qquad y(0) = 1$$

-We know from the initial condition that the curve y passes through (0,1) and has a slope of $y' = 0 + 1 = 1$. So using linear approximation methods, we know that points on the curve near (0,1) can be approximated by $L(x) = x + 1$. Remember, the linear approximation L(x) is a tangent line to the curve of y at (0,1), and tangent lines can be used to approximate the value of a curve at points near the intersection.

-You probably know however, that if you tried approximating the value of a curve at a point that was far from (0,1) using this linear approximation, it would not be very accurate. Euler decided to improve upon these straight line approximations by course-correcting the approximation shortly after the point of intersection.

-For example, at x = 0.5, L(0.5)=1.5, so what if we create another line segment that starts at this point, (0.5, 1.5). We will need another slope to go along with this new point to create a new straight line approximation, but we can get that slope by plugging (0.5, 1.5) into $y' = x + y$ again:

$$y' = 0.5 \ + 1.5 \ = 2$$

-The line with slope 2 passing through (0.5, 1.5) would be the equation $y = 2x + 0.5$. You can still approximate values for the curve for points near x=0 can still be approximated with y=x+1, but points near x=0.5 can be approximated with y=2x+0.5. This can be done again and again, approximation after approximation, as many times as we want.

-However, the sooner it is that we create the next line segment, the more accurate the resulting approximations will be. If we decided at x=0.25 to course correct, the line segments would approximate more accurately overall.

-So in general, given a first-order initial-value problem $y' = F(x, y)$ with initial condition $y(x_0) = y_0$, we want to find approximate values for the solution at equally spaced numbers, $x_0,\ x_1 = x_0 + h, \ldots,\ x_i = x_0 + ih.$ h is our "step size" the distance it takes before we course correct each time.

-We will always get our slopes for these linear approximation course corrections from $F(x, y)$, so we can create a recursive rule to determine the slope needed to create the next linear approximation equation at each course correction:

$$y_1 = y_0 + hF(x_0, y_0), \qquad y_2 = y_1 + hF(x_1, y_1), \qquad \ldots \qquad y_i = y_{i-1} + hF(x_{i-1}, y_{i-1})$$

-This gives a series of ordered pairs that are approximately ordered pairs that the solution to the differential equation passes through. Again, the solution is highly graphical rather than algebraic, like with direction fields, but a visual solution is still better than no solution at all.

Euler's Method:

Approximate values for the solution of the initial-value problem $y' = F(x, y),\ y(x_0) = y_0,$ with step size h, at $x_n = x_{n-1} + h,$ are $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$ at n = 1, 2, 3, ...

-h will have to be given at the beginning of a problem like this, since unlike Riemann sums, there is no finite interval to use to determine what Δx is. As always, the smaller the steps, the better the approximations.

Example: Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem.

$$y' = x + y, \qquad y(0) = 1.$$

Solution: You will need time and space to do this, so set up a table for many steps and course-corrections. When n = 0, $x_n = 0$, and $y_n = 1$, for the remaining values, we use the previous x_{n-1} and y_{n-1} values.

h equals 0.1, F(x,y) = x+y, and the approximation will be $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$ starting at (0,1):

$$y_1 = y_0 + hF(x_0, y_0) \qquad y_1 = 1 + 0.1F(0,1) = 1 + 0.1(0+1) = 1.1$$

$$y_2 = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = 1.22 + 0.1F(0.2, 1.22) = 1.362$$

$$y_4 = 1.362 + 0.1F(0.3, 1.362) = 1.5282$$

This is how we compute each new value. This gives us a table of:

n	x_n	y_n
1	0.1	1.1
2	0.2	1.22
3	0.3	1.362
4	0.4	1.5282
5	0.5	1.72102
6	0.6	1.943122
7	0.7	2.197434
8	0.8	2.487178
9	0.9	2.815895
10	1.0	3.187485

Exercise: Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem. (use at least n = 10 values)

$$y' = -2x + y, \quad y(0) = 1$$

Example: We saw that a simple electric circuit with resistance 12 ohms, inductance 4 henries, and a battery with voltage 60 volts. If the switch is closed when t = 0 we modeled the current I at time t by the initial-value problem:

$$\frac{dI}{dt} = 15 - 3I, \qquad I(0)=0$$

Estimate the current in the circuit half a second after the switch is closed.

Solution: You could approximate the current after 0.5 seconds by using one linear approximation equation, which gives us $L(x) = I_0 + F(t_0, I_0)(t - t_0) = 0 + 15(t - 0) = 15t,$ and then plugging t = 0.5 in, but just one approximation equation with no course corrections is not very accurate, especially since t = 0.5 and t = 0 are not that close to one another all things considered. (this doesn't even get into how we have seen already that curves that start below I=5 approach I=5 without crossing it, and plugging 0.5 into 15t gives us 7.5)

Instead, we will use Euler's Formula. They didn't give us a value for h, so we'll arbitrarily use h = 0.1.

h equals 0.1, F(t,I) = 15-3I, and the approximation will be $I_n = I_{n-1} + hF(t_{n-1}, I_{n-1})$ starting at (0,0):

$$I_1 = I_0 + hF(t_0, I_0) \qquad y_1 = 0 + 0.1F(0,0) = 0 + 0.1(15 - 3(0)) = 1.5$$

$$y_2 = 1.5 + 0.1F(0.1, 1.5) = 1.5 + 0.1(15 - 3(1.5)) = 2.55$$

$$y_3 = 2.55 + 0.1F(0.2, 2.55) = 2.55 + 0.1(15 - 3(2.55)) = 3.285$$

$$y_4 = 3.285 + 0.1F(0.3, 3.285) = 3.285 + 0.1(15 - 3(3.285)) = 3.7995$$

$$y_5 = 3.7995 + 0.1F(0.4, 3.7995) = 3.7995 + 0.1(15 - 3(3.7995)) = 4.15965$$

We have that the solution for the differential equation going through (0,0) approximately goes through $(0.5,\ 4.15965)$.

Exercise: We saw that a simple electric circuit with resistance 12 ohms, inductance 4 henries, and a battery with voltage 60 volts. If the switch is closed when t = 0 we modeled the current I at time t by the initial-value problem:

$$\frac{dI}{dt} = 15 - 3I, \qquad \text{I(0)=0}$$

Estimate the current in the circuit half a second after the switch is closed using h = 0.05.