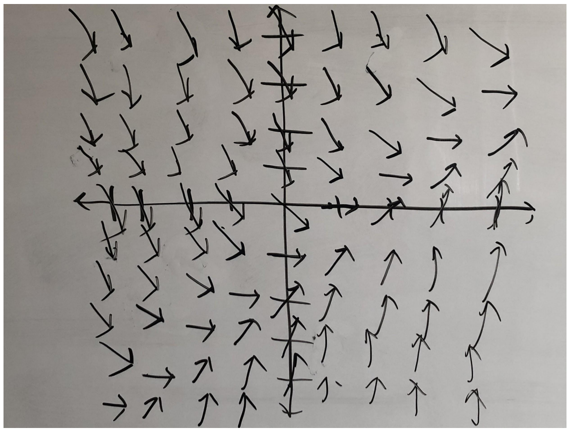
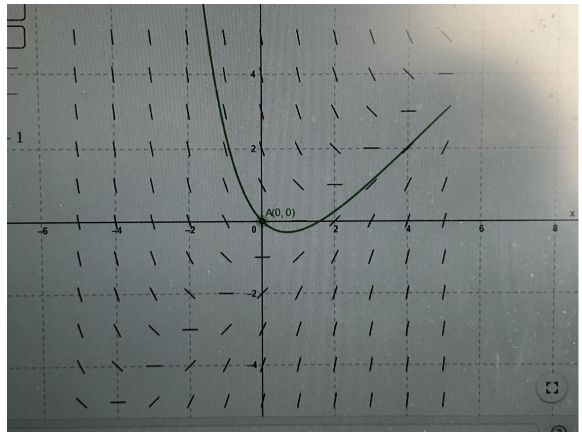
9.2) Directional Fields and Euler's Method:

Exercise: a) Sketch the direction field for the differential equation $y' = x - y - 1$.
b) Use your direction field from part a) to sketch the solution curve that passes through the origin.



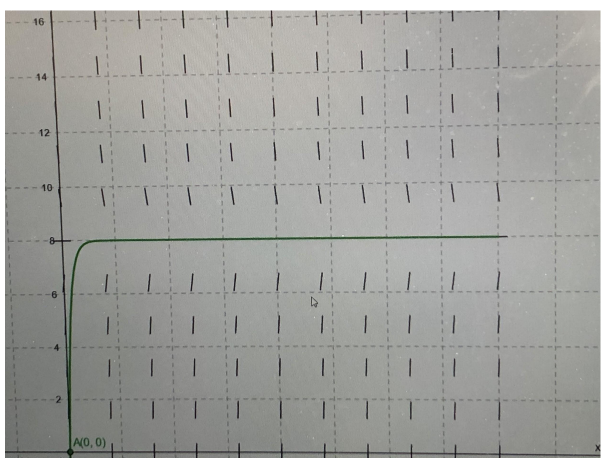


Exercise: Suppose that in the simple circuit the resistance is 10 ohms, the inductance is 2 henries, and a battery gives a voltage of 80 volts.

a) Draw a direction field for your differential equation.
b) What can you say about the limiting value of the current?
c) Identify any equilibrium solutions.
d) If the switch is closed when t = 0 so the current starts with I(0)=0, use the direction field to sketch a solution curve.

$$2 * \frac{dI}{dt} + 10I = 80 \qquad \frac{dI}{dt} = 40 - 5I$$

a) + d)



b) The limiting value of the current is I = 8. The current is approaching I=8 amperes as time goes on, regardless of where you start.

c)

$$0 = 40 - 5I$$
$$40 = 5I$$

I(t)=8, or (t, I(t)) = (t, 8) for any t.

Exercise: Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem. (use at least n = 10 values)

$$y' = F(x_{n-1}, y_{n-1}) = -2x + y, \quad y(0) = 1$$

$$y_n = y_{n-1} + 0.1 * F(x_{n-1}, y_{n-1})$$

n	x_n	y_n
1	0.1	1.1
2	0.2	1.19
3	0.3	1.269
4	0.4	1.3359
5	0.5	1.38949
6	0.6	1.428439
7	0.7	1.4512829
8	0.8	1.45641119
9	0.9	1.442052309
10	1.0	1.40625754

Exercise: We saw that a simple electric circuit with resistance 12 ohms, inductance 4 henries, and a battery with voltage 60 volts. If the switch is closed when t = 0 we modeled the current I at time t by the initial-value problem:

$$\frac{dI}{dt} = 15 - 3I, \qquad \text{I(0)=0}$$

Estimate the current in the circuit half a second after the switch is closed using h = 0.05.

$$I_n = I_{n-1} + 0.05(x_{n-1}, y_{n-1})$$

n	t_n	I_n
1	0.05	0.75
2	0.10	1.3875
3	0.15	1.929375
4	0.20	2.38996875
5	0.25	2.781473438
6	0.30	3.114252422
7	0.35	3.397114559
8	0.40	3.637547375
9	0.45	3.841915269
10	0.50	4.015627979

9.3) Separable Equations:

Exercises: Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{x\sin(x)}{(y-1)^2}$, for $y(0) = 1$ b) $\frac{dy}{dx} = \frac{xe^{x^2+1}}{y}$ for y > 0

a)

$$\frac{dy}{dx} = \frac{x\sin(x)}{(y-1)^2}$$

$$(y - 1)^2 dy = x\sin(x)dx$$

$$\frac{(y-1)^3}{3} = -x\cos(x) + \sin(x) + C$$

When x = 0, y = 1,

$$\frac{(1-1)^3}{3} = -0\cos(0) + \sin(0) + C$$

$$0 = C$$

$$\frac{(y-1)^3}{3} = -x\cos(x) + \sin(x)$$

or

$$y = \sqrt[3]{-3x\cos(x) + 3\sin(x)} + 1$$

b)

$$\frac{dy}{dx} = \frac{xe^{x^2+1}}{y}$$

$$dy * y = xe^{x^2+1}dx$$

$$\frac{y^2}{2} = \frac{1}{2}e^{x^2+1} + C$$

$$y = \sqrt{e^{x^2+1} + C}$$

Exercises: Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{e^{-2y}}{\sqrt{x}}$ b) $\frac{dy}{dx} = \sec(x)\cot(y)$ c) $\frac{dy}{dx} = 1 + y^2$ for y(0)

= 5

a)

$$\frac{dy}{dx} = \frac{e^{-2y}}{\sqrt{x}}$$

$$e^{2y}dy = \frac{dx}{\sqrt{x}}$$

$$\frac{1}{2}e^{2y} = 2\sqrt{x} + C$$

$$e^{2y} = 4\sqrt{x} + C$$

$$2y = \ln\left(4\sqrt{x} + C\right)$$

$$y = \ln\left(\sqrt{4\sqrt{x} + C}\right)$$

b)

$$\frac{dy}{dx} = sec(x)cot(y)$$

$$dy * tan(y) = sec(x)dx$$

$$dy * tan(y) = sec(x)dx$$

$$ln(sec(y)) = ln(sec(x) + tan(x)) + C$$

or

$$sec(y) = e^{ln(sec(x)+tan(x))+C} = Ae^{ln(sec(x)+tan(x))}$$

$$sec(y) = A(sec(x) + tan(x))$$

$$cos(y) = A(\frac{1}{sec(x)+tan(x)})$$

$$y = arccos\left(A(\frac{1}{sec(x)+tan(x)})\right)$$

c)

$$\frac{dy}{dx} = 1 + y^2$$

$$\frac{dy}{1+y^2} = dx$$

$$arctan(y) = x + C$$

$$arctan(5) = 0 + C$$

$$arctan(y) = x + arctan(5)$$

$$y = tan(x + arctan(5))$$

Exercise: Find the orthogonal trajectories of the family of curves $x = ky^4$, where k is an arbitrary constant.

$$1 = k(4y^3) * \frac{dy}{dx}$$

$$\frac{1}{4ky^3} = \frac{dy}{dx}$$

$$\frac{1}{4\left(\frac{x}{y^4}\right)y^3} = \frac{dy}{dx}$$

$$\frac{y}{4x} = \frac{dy}{dx}$$

Reciprocate

$$\frac{4x}{y} = \frac{dy}{dx}$$

$$4xdx = ydy$$

$$2x^2 + C = \frac{1}{2}y^2$$

$$y^2 + 4x^2 = C$$

Exercise: A lake contains 1 kg of chlorine dissolved in 10000 L of water. Run-off from a factory that contains 0.01 kg of chlorine per liter of water enters the lake at a rate of 50 liters per day. The water in the lake is also draining out into a river, and the mixture going in and the mixture in the lake and draining out are all thoroughly mixed. The solution going in and the mixture draining out are doing so at the same rate, so how much chlorine will be in the lake after 50 days?

$$\frac{dy}{dt} = rate\ of\ Cl\ going\ in\ -\ rate\ of\ Cl\ going\ out$$

$$\frac{dy}{dt} = 0.01\ *\ (50)\ -\ \frac{y(t)}{10000} * 50$$

$$\frac{dy}{dt} = 0.5\ -\ \frac{y(t)}{200}$$

$$\frac{dy}{0.5-\frac{y}{200}} = dt$$

$$\frac{dy}{100-y} = \frac{1}{200}dt$$

$$-\ ln(100\ -\ y) = \frac{1}{200}t\ +\ C$$

$$100\ -\ y\ = e^{\left(\frac{-1}{200}t+C\right)}$$

$$y\ =\ 100\ -\ Ae^{\frac{-1}{200}t}$$

$$y(t)\ =\ 100\ -\ 99e^{\frac{-1}{200}t}$$

$$y(50)\ =\ 100\ -\ 99e^{\frac{-1}{200}(50)}\ =\ 22.9\ kg$$

9.4) Population Growth:

Exercise: a) Write the solution of the initial-value problem $\frac{dP}{dt} = 0.05P\left(1\ -\ \frac{P}{5000}\right)$, where P(0)=200.
b) Use it to find the population sizes P(100) and P(150)
c) At what time does the population reach 4000?

a)

$$P(t)\ = \frac{M}{1+Ae^{-kt}}$$

$$P(t)\ = \frac{5000}{1+24e^{-0.05t}}$$

b)

$$P(100)\ = \frac{5000}{1+24e^{-0.05(100)}} = \frac{5000}{1+24e^{-5}} = 4304.0$$

$$P(150)\ = \frac{5000}{1+24e^{-0.05(150)}} = \frac{5000}{1+24e^{-7.5}} = 4934.5$$

c)

$$4000\ = \frac{5000}{1+24e^{-0.05t}}$$

$$4000(1\ +\ 24e^{-0.05t})\ =\ 5000$$

$$1\ +\ 24e^{-0.05t}\ =\ 1.25$$

$$e^{-0.05t} = \frac{0.25}{24}$$

$$-\ 0.05t\ =\ ln\left(\frac{0.25}{24}\right)$$

$$t\ = \frac{ln\left(\frac{0.25}{24}\right)}{-0.05} = 91.3$$