5.4) Indefinite Integrals and the Net Change Theorem:

-Considering that the Fundamental Theorem of Calculus from last section showed us that integration and differentiation are inverse processes, it should be no surprise that the process of finding antiderivatives uses integral notation as well. The notation used for antiderivatives borrows from definite integral notation by stating that an antiderivative of a continuous function f is denoted by $\int f(x)dx$, and is called an **indefinite integral**.

$$\int f(x)dx = F(x) \qquad \text{where } F'(x) = f(x)$$

-When the integral symbol is shown without limits on top or bottom, you will not be getting back a number, because you are not being asked to evaluate anything, nor are you being asked to find area under a curve, nor plug anything into an antiderivative and subtract. Instead you are getting back an antiderivative of f(x), or even a **family of antiderivatives** of f(x) which is the entire collection of all functions that have a derivative of f(x).

-For example, we have seen that $\frac{x^3}{3}$ is an antiderivative of x^2, and so (as we will see) since every antiderivative of x^2 differs from one another by a constant term,

$$\int x^2 dx = \frac{x^3}{3} + C \qquad \text{where C is a constant number}$$

-C can be any constant number since no matter what C is, $\frac{d}{dx}\left[\frac{x^3}{3} + C\right] = x^2$, and in fact this means $\frac{x^3}{3} + C$ is the entire family of antiderivatives of x^2. By the Fund. The. of Calc. Part 2 in fact, given a continuous f on [a,b]:

$$\int_a^b f(x)dx = \int f(x)dx \,\Big]_a^b$$

-You can take a definite integral, or you can take an indefinite integral, then evaluate your family (or any single member of the family) of antiderivatives at b and a, and finally subtract the two evaluations. You would get the same result either way.

-Still, it is more important than ever that you remember how to find antiderivatives, which will involve some new strategies later, but depends heavily on your ability to remember famous antiderivatives of other functions from Calculus I. You may want to review them in the textbook in section 4.9 or section 5.4 if you are having trouble remembering.

-Antiderivatives and their families are also meant to be valid on an interval, even though unlike definite integral examples we may not know what that interval is. Usually it would be the interval of the implied domain of the original function (not the antiderivative, after all there is more than one possible F such that $F'(x) = f(x)$). For example:

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

-This integral is valid only on $(-\infty, 0)$ or $(0, \infty)$, even though a more general antiderivative of the function $f(x) = \frac{1}{x^2}$ is:

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & for\ x < 0 \\ -\frac{1}{x} + C_2 & for\ x > 0 \end{cases}$$

Example: Find the general indefinite integral: $\int (10x^4 - 2sec^2(x))dx$

Solution: Using the power rule in reverse and what we know about the derivative of tan(x), we have:

$$\int (10x^4 - 2sec^2(x))dx = 10\left(\frac{x^5}{5}\right) - 2(tan(x)) + C$$

$$\int (10x^4 - 2sec^2(x))dx = 2x^5 - 2tan(x) + C$$

If you are uncertain that you integrated correctly, you can take the derivative of your answer F(x) to see if you get the original function f(x).

Example: Find the general indefinite integral: $\int \frac{sin(x)}{cos^2(x)} dx$

Solution: Sometimes you need a little bit of rewriting before you can confidently determine the function in question has a famous antiderivative. You will want to remember your trigonometric identities from precalculus:

$$\int \frac{sin(x)}{cos^2(x)} dx = \int \left(\frac{1}{cos(x)} * \frac{sin(x)}{cos\,(x)}\right) dx$$

$$\int \frac{sin(x)}{cos^2(x)} dx = \int (sec(x) * tan(x))\, dx$$

$$\int \frac{sin(x)}{cos^2(x)} dx = sec(x) + C$$

Exercises: Find the general indefinite integrals:

a) $\int (\sqrt{x} - 3csc^2(x))\, dx$

b) $\int \left(\frac{cos(x)}{sin^2(x)}\right) dx$

Example: Find the definite integral and interpret the result in terms of areas.

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$$

Solution: We are still going to be working with definite integrals, so the procedures from last section still hold:

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = 2\int_0^2 x^3 dx - 6\int_0^2 x dx \ + 3\int_0^2 \tfrac{1}{x^2+1} dx$$

Don't forget your famous derivative of arctan! Also, don't worry about using the definite integral evaluation notation for all the terms individually, you can put everything inside one bracket:

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = 2\int_0^2 x^3 dx - 6\int_0^2 x dx \ + 3\int_0^2 \tfrac{1}{x^2+1} dx$$

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = 2\tfrac{x^4}{4} - 6\tfrac{x^2}{2} \ + 3tan^{-1}(x) \ ]^2{}_0$$

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = \tfrac{x^4}{2} - 3x^2 \ + 3tan^{-1}(x) \ ]^2{}_0$$

Evaluate the entire expression at 2 first, then evaluate at 0 second, then subtract:

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = (8 - 12 + 3tan^{-1}(2)) \ - \ (0 - 0 + 3tan^{-1}(0))$$

$$\int_0^2 \left(2x^3 - 6x + \tfrac{3}{x^2+1}\right) dx = \ -4 + 3tan^{-1}(2) \ \approx \ -0.67855$$

This is the net area between $y = 2x^3 - 6x + \frac{3}{x^2+1}$ and the x-axis between 0 and 2, and since the net area is negative, we have that the areas this expression is below the x-axis are larger than the areas it is above the x-axis.

Example: Find the definite integral.

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt$$

Solution: It never hurts to rewrite if you are trying to make the expression easier to integrate, especially if you plan to integrate term-by-term. Just make sure you cancel carefully!

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt = \int_1^9 (2 + t^{0.5} \ - t^{-2})dt$$

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt = \ 2t \ + \ \tfrac{t^{1.5}}{1.5} + t^{-1}]^9{}_1$$

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt = \ 2t \ + \ \tfrac{t^{1.5}}{1.5} + t^{-1}]^9{}_1$$

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt = \left(18 \ + \ 18 \ + \tfrac{1}{9}\right) - \left(2 \ + \ \tfrac{2}{3} + 1\right)$$

$$\int_1^9 \left(\tfrac{2t^2+t^2\sqrt{t}-1}{t^2}\right) dt \ = \ 33 \ - \ \tfrac{5}{9} \ = \tfrac{292}{9}$$

Exercises: Find the general definite integrals:

a) $\int_1^8 \frac{t^2-\sqrt[3]{t}+1}{t}$ b) $\int_0^1 \left(\frac{1}{\sqrt{1-x^2}} + \pi x \right) dx$

-Our chosen notation for antiderivatives is $F'(x) = f(x)$, and so if we wanted we could rewrite the integral:

From This $\int_a^b f(x)dx = F(b) - F(a)$ To This $\int_a^b F'(x)dx = F(b) - F(a)$

-However, since derivatives in general are the rate of change of a function, $F'(x)$ is the rate of change of F(x), while F(b) - F(a) is the literal net change from one input, a, to another input, b.

-This talk of comparing rate of change to net change gives us an alternate version of the Fund. The. of Calc. part 2 sometimes called the **net change theorem**: The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

-As far as application is concerned, this allows us to recognize how certain variables (like velocity) that are the rate of change of another can be integrated over an integral to get the net change of another variable.

Velocity is the derivative of displacement with respect to time, so $\int_a^b v(t)dt = s(b) - s(a)$

The integral of velocity from time a to time b is the net displacement over that time.

Density is the derivative of mass with respect to position, so $\int_a^b \rho(x)dx = m(b) - m(a)$

The integral of density from point a to point b is the mass between one point and another.

Marginal cost is the derivative of cost with respect to units, so $\int_a^b C'(x)dx = C(b) - C(a)$

The integral of marginal cost from a units made to b units is the increase in cost from one number of units to another.

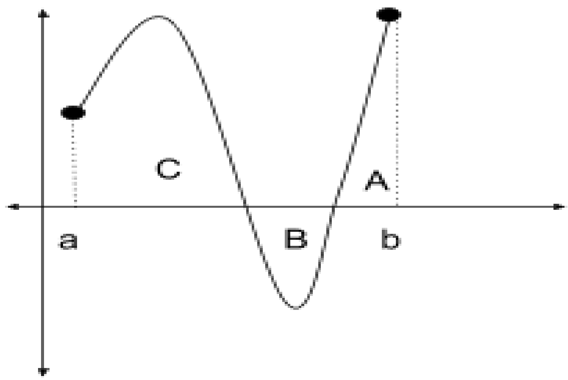
-Talking a bit more about displacement/velocity and their often mistook cousins, distance/speed, you may recall that the difference between velocity and speed is that velocity can be negative, but speed cannot, so most people remember that $speed = |velocity|$. So we have already that the integral of velocity over a time period is the net displacement:

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1) = \textit{total displacement}$$

-However, if we replace velocity with speed, we would have:

$$\int_{t_1}^{t_2} |v(t)|\,dt = \textit{total distance traveled}$$

-Notice we don't have the subtraction of start to end any longer, and that's because absolute value functions are not as readily differentiable as other functions, which can lead to problems with the computation of integral for absolute value functions.



-For example, suppose the above graph is a graph of y = v(t), the velocity of an object in motion from time a to time b. Sometimes it is moving in a positive direction, sometimes it is moving in a negative direction. The points on the interval in which the curve is above the x-axis and the area between the curve and the x-axis is above the x-axis are the times in which the object is moving in a positive direction (areas A and C) while the rest are the times when the object is moving in a negative direction (area B).

-If you wanted to find the total displacement of the object in motion, you could take the following integral, and what you would get is the sum of the areas of A and C, minus the area of B:

$$\int_a^b v(t)dt = s(b) - s(a) = A + C - B$$

-If you wanted to find the total distance traveled by the object in motion, you would have to take the integral of the absolute value of the velocity function, and what you would get is the sum of the areas of A, C, and B (no subtracting of any of these areas, they are all positive and added together now):

$$\int_a^b |v(t)|\, dt = A + C + B$$

-Displacement is found in a manner straight-forward and similar to what we have done so far. But to find distance traveled, you would need to:

- find the area of C which would require to know where y = v(t) intersects the x-axis the first time (call it t_1), and then take an integral of v(t) from a to t_1.

-find the area of B which would require to know where y = v(t) intersects the x-axis the second time ((call it t_2) and then an integral of v(t) from t_1 to t_2.

-find the area of A which would be the integral of v(t) from t_2 to b.

-take the absolute values of all these areas (or at the least the ones we know will be negative, the B area) then add them all up .

$$\int_a^b |v(t)|\, dt = \left|\int_a^{t_1} v(t)dt\right| + \left|\int_{t_1}^{t_2} v(t)dt\right| + \left|\int_{t_2}^{b} v(t)dt\right|$$

-It's harder, but not impossible.

Example: A particle moves along a line so that its velocity at time t is v(t) = $t^2 - t - 6$.

a) Find the displacement of the particle during the period $1 \le t \le 4$.

b) Find the distance traveled during this same time period.

Solution: a) You can find the displacement with a single integral of v(t) from 1 to 4:

$$\int_1^4 (t^2 - t - 6)dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t\,\Big]_1^4 = \left(\frac{64}{3} - 8 - 24\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right) = -\frac{9}{2}$$

The particle moved 4.5 units to the left (the standard negative direction)

b) However, we need multiple integrals to account for when the velocity is positive, and when it is negative. $v(1) = -4$, and $v(4) = 6$, so we know the object changes direction at least once. When? Whenever v(t) equals 0:

$$v(t) = t^2 - t - 6$$
$$0 = (t-3)(t+2)$$

At t = 3 and t = -2 (-2 is not on the interval, so ignore it) the velocity is zero, so that will be where we divide up our integral:

$$\int_1^4 \left|t^2 - t - 6\right| dt = \left|\int_1^3 (t^2 - t - 6)dt\right| + \left|\int_3^4 (t^2 - t - 6)dt\right|$$
$$\int_1^4 \left|t^2 - t - 6\right| dt = \left|\frac{t^3}{3} - \frac{t^2}{2} - 6t\,\Big]_1^3\right| + \left|\frac{t^3}{3} - \frac{t^2}{2} - 6t\,\Big]_3^4\right|$$

$$\int_1^4 \left|t^2 - t - 6\right| dt = \left|\left(-\frac{27}{2}\right) - \left(-\frac{37}{6}\right)\right| + \left|\left(-\frac{32}{3}\right) - \left(-\frac{27}{2}\right)\right| = \frac{44}{6} + \frac{17}{6} = \frac{61}{6}$$

It may be only 4.5 units to the left from where it started, but it traveled about 10.17 units overall to get there.

Exercise: A particle moves along a line so that its velocity at time t is v(t) = $t^2 - 4t - 12$.

a) Find the displacement of the particle during the period $3 \le t \le 12$.

b) Find the distance traveled during this same time period.