11.4) The Comparison Tests:

-We have seen comparison of one integral to another can be a useful tool in determining whether an integral converges or diverges. Does a similar procedure exist for series? In some ways yes, but in other ways comparison is handled differently than before:

-We have seen that the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent and that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. However, what can we say about the similar series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$? Is it also convergent?

-In fact, it does converge, since for any n, $\frac{1}{2^n+1} < \frac{1}{2^n}$. Thus for every term in the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, the corresponding term in $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is smaller. So since every term is smaller, every partial sum s_n for $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ will be smaller than any partial sum for $\sum_{n=1}^{\infty} \frac{1}{2^n}$ too. Taking the limit as n goes to infinity of each series partial sum will imply that the sum s of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is smaller than the sum of $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < 1$$

-This is called **direct comparison**, in that we are not only comparing one series to another, but also the individual terms in one with the individual terms in the other. From this, we get the **direct comparison test**:

The Direct Comparison Test:

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both series with positive terms.

1) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
2) If $\sum_{n=1}^{\infty} a_n$ is divergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

-The proofs for both halves of the direct comparison test are in the textbook and in the ebook online, but neither proof is all that difficult, and the direct comparison test itself is fairly logical.

-Once again though, **do not get the rules mixed up**. Remember, if $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \leq b_n$ for all n, then we know nothing about $\sum_{n=1}^{\infty} a_n$. It could be convergent, it could be divergent. The same applies for if $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n \leq b_n$ for all n; then we know nothing about $\sum_{n=1}^{\infty} b_n$.

-Of course, comparison only is helpful if you have a series that you can compare with, specifically series that you remember (or can easily deduce) whether they are convergent or divergent. Here are some commonly used comparison examples:

-**p-series** of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are divergent when $p \leq 1$, and convergent when $p > 1$.

-**geometric ratio series** of the form $\sum_{n=1}^{\infty} ar^{n-1}$ are convergent when $-1 < r < 1$, and divergent when $r \geq 1$.

-Remember, a series is of no help to you for the direct comparison test if you do not know whether it is divergent or convergent.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{2n^2+7n+3}$$

Solution: This expression inside the series is continuous, increasing, and positive for all n >1, so let's compare this expression with another, simpler expression:

$$\frac{1}{2n^2+7n+3} < \frac{1}{2n^2}$$

Remove the 3, and the other term with n, and you have a smaller denominator, which means a larger fraction. What do we know about the following series?

$$\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This is a p-series with a power of 2, which is greater than 1, so therefore this other series, $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ is convergent.

However, since all the terms in $\sum_{n=1}^{\infty} \frac{1}{2n^2+7n+3}$ are smaller than the terms in this other series, we have:

$$\sum_{n=1}^{\infty} \frac{1}{2n^2+7n+3} < \sum_{n=1}^{\infty} \frac{1}{2n^2}$$

Therefore, since the larger series is convergent, this smaller series must also be convergent.

-Like the integral tests, it does not have to necessarily be the case that every term in one series is smaller than the analogous one in the other. It simply has to be the case that eventually there is an integer N such that for all n>N, $a_n \leq b_n$.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{ln(n)}{n}$$

Solution: We could compare this series to the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, however it is not true that $\frac{1}{n} \leq \frac{ln(n)}{n}$. When n=1 or n=2, ln(n) is smaller than 1. However, for $n \geq 3$, it is true that ln(n) > 1. Therefore we have that eventually, $\frac{1}{n} < \frac{ln(n)}{n}$, specifically for when $n \geq 3$.

So because $\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{ln(n)}{n}$, and the harmonic series $\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent, that means that the larger series, $\sum_{n=3}^{\infty} \frac{ln(n)}{n}$ is also divergent.

Exercises: Determine whether the following series converge or diverge:

a) $\sum_{n=1}^{\infty} \frac{sec^2(n)}{n}$ b) $\sum_{n=1}^{\infty} \frac{n}{n^3+n^2+2n+1}$

-Sometimes however, series don't have to be compared with each other term-by-term. In fact, always comparing the individual terms is a bit constricting, especially for cases like $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$, which has terms that are all larger than the terms in $\sum_{n=1}^{\infty} \frac{1}{2^n}$. That means $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is larger than $\sum_{n=1}^{\infty} \frac{1}{2^n}$, a convergent series. That tells us nothing.

-However, sometimes we are not concerned with just comparing the individual terms of the series to each other, but instead we compare the rules of the series to each other, and not in a term-by-term basis, but by taking the limits of the two rules together:

The Limit Comparison Test:

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c,$$

where c is a finite number greater than 0, then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, or they both diverge.

-The idea here is that if you compare the rules of the two series terms together and see that the behavior of each rule as n goes to infinity is similar enough that one is a multiple of sorts of the other, then that means their behavior is similar enough that they share the same relationship as n goes to infinity, and so therefore their series behave similarly enough that they are of the same type as well.

-Note, this does not mean take the limits of both a_n and b_n and see if you get the same or similar limit. It means that the ratio of these rules has to approach a <u>finite nonzero number</u> to suggest the series are similar in their convergence or divergence. If the limit of one rule divided by the other is 0 or infinite, nothing is certain.

-The proof of why this works is as follows. Suppose that for the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c,$$

Where c is finite and positive. Then that means there exists two positive numbers, call them d and D where

$$d < c < D$$

Since $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, there exists an integer, N, such that for all $n \geq N$,

$$d < \frac{a_n}{b_n} < D$$

That however, also means:

$$db_n < a_n < Db_n$$

So if $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$, which is greater than $\sum_{n=1}^{\infty} db_n$, is also divergent by comparison.

If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$, which is smaller than $\sum_{n=1}^{\infty} Db_n$, is also convergent by comparison.

-Of course, you will have to be certain you know whether one of the two original series converges or diverges, and you will have to be able to find $\lim_{n \to \infty} \frac{a_n}{b_n}$.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

Solution: For the limit comparison test we need another series that we know converges or diverges, so let's choose $\sum_{n=1}^{\infty} \frac{1}{2^n}$. This is crucial. Sometimes picking the right series to compare makes all the difference. It must be a series you know converges or diverges off the bat (this one converges), and it must be a series whose rule is comparable to the rule of $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$, otherwise the algebra of the limit may be too difficult to evaluate:

$$\lim_{n \to \infty} \frac{1/2^n}{1/(2^n - 1)} = \lim_{n \to \infty} \frac{2^n - 1}{2^n} = \lim_{n \to \infty} 1 - \frac{1}{2^n} = 1 - 0 = 1$$

For the record, it does not matter which series you denote with a_n and which you denote with b_n. You may get a slightly harder expression to take the limit of, and you may even get a different limit. However, in either case you should end up with the same conclusion.

In this case, the limit of the rules as n goes to infinity is 1, which is finite and greater than 0, so therefore either both series converge, or both series diverge. Since we already know $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, that means $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges too.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^3 + n^2 + n + 4}}$$

Solution: What would be a good comparison series for this example? Sometimes taking a look at the original series and comparing powers is a good way of determining what might make for a good comparison series.

Remember, you are going to have to take the limit of the ratio of the two series yourself eventually, so if you don't pick a good comparison, the limit may be difficult to find. In this case, consider the following:

$$\frac{x+3}{\sqrt{x^3 + x^2 + x + 4}} \Rightarrow \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$$

When you remove all but the largest powers of x from top and bottom of this ratio, we are left with an expression that reduces to $\frac{1}{\sqrt{x}}$. What do we know about the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$? It is a p-series with a power below 1, so it is divergent. So let's try taking the limit of the ratio of this series and the original $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^3 + n^2 + n + 4}}$ series:

$$\lim_{n \to \infty} \frac{\frac{n+3}{\sqrt{n^3 + n^2 + n + 4}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{\sqrt{n}(n+3)}{\sqrt{n^3 + n^2 + n + 4}} = \lim_{n \to \infty} \frac{(n^{3/2} + 3n^{1/2})}{\sqrt{n^3 + n^2 + n + 4}}$$

Divide top and bottom by $n^{3/2}$ and we get:

$$\lim_{n\to\infty} \frac{(1+3n^{-1})}{\sqrt{1+n^{-1/2}+n^{-1}+4n^{-3/2}}} = \frac{1}{\sqrt{1}} = 1$$

Success! We got a limit that is bigger than 0, but finite. Remember, success does not mean the series $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^3+n^2+n+4}}$ converges, it means the series $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^3+n^2+n+4}}$ has the same behavior as $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We said that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, so therefore $\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^3+n^2+n+4}}$ diverges too.

Exercises: Determine whether the following series converge or diverge:

a) $\sum_{n=1}^{\infty} \frac{3n^2+n}{n^3+2n^2-4n+8}$ b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n3^n}$

-Last section we saw how using remainders can estimate what the sum of an infinite series can equal. Here we will use similar logic to see how remainders can help us determine how the sum of one infinite series can be estimated using the sums of other infinite series.

-We use similar notation here, where R_n of a series $\sum_{n=1}^{\infty} a_n$ is the sum of the remaining terms in the infinite series $\sum_{n=1}^{\infty} a_n$ after the first n terms:

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + ...$$

-Suppose we have a second series, $\sum_{n=1}^{\infty} b_n$, where the partial sum up to the nth term is denoted by t_n, the sum of the entire series is t, and the sum of the remaining terms in this series after the first n terms is the remainder denoted by T_n:

$$T_n = t - t_n = b_{n+1} + b_{n+2} + b_{n+3} + ...$$

-If by the direct comparison test we have that $a_n \le b_n$ for all n and have shown that by direct comparison that $\sum_{n=1}^{\infty} a_n$ converges by comparison to $\sum_{n=1}^{\infty} b_n$, then we have $R_n \le T_n$. If we know enough about $\sum_{n=1}^{\infty} b_n$, like that $\sum_{n=1}^{\infty} b_n$ is a geometric series, we can find the exact sum of $\sum_{n=1}^{\infty} b_n$ and determine that $\sum_{n=1}^{\infty} a_n$ is less than that. If we know that $\sum_{n=1}^{\infty} b_n$ is a p-series we can estimate its sum with methods from last section to find an upper bound for the sum of $\sum_{n=1}^{\infty} a_n$.

Example: The sum of the first 100 terms in $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is approximately 0.6864538. Find the error involved in this approximation.

Solution: We know that $\frac{1}{n^3+1} < \frac{1}{n^3}$ and that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series, so by the direct comparison test we know $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is a convergent series too. Therefore, the sum of $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is finite. But what is the remainder?

We know that R_{100}, the remainder of $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is less than T_{100}, the remainder of $\sum_{n=1}^{\infty} \frac{1}{n^3}$. As for T_{100}, we know:

$$T_{100} \leq \int_{100}^{\infty} \frac{1}{x^3} dx = \lim_{t \to \infty} \int_{100}^{t} \frac{1}{x^3} dx = \lim_{t \to \infty} \left(-\frac{1}{2t^2} + \frac{1}{2(100)^2} \right) = (0 + 0.00005) = 0.00005$$

So since $R_{100} \leq T_{100}$ and $T_{100} \leq 0.00005$, we know the remainder for our sum of the first 100 terms in $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is below 0.00005.