9.1) Modeling with Differential Equations:

-One of the first things you learned when you first heard of exponential derivatives was that the rate of change of exponential functions is proportional to the value of the function itself.

-The rate of change of the function f(t) is the derivative of the function with respect to time. This exponential function has a growth rate constant of k, which states the rate (usually a percentage) at which the function is growing. However, this is another way of saying that the rate of change of the function is just what the function is equal to multiplied by the growth rate k.

$$f'(t) = k * f(t)$$

-This is logical given what we know about the derivative of exponential functions (this one in particular). Given a function variable f that grows exponentially, and the variable of time t, it is in fact true that:

$$\frac{df}{dt} = k * f$$

-k could also be considered a proportionality constant, since the rate of change of f and f itself are thereby proportional to each other. However, this last function is also considered a **differential equation** because it is an equation that combines and compares a variable and its derivative with each other. Can you determine what function f(t) would need to be to satisfy this differential equation?

-Considering that we knew what f was in terms of t already, it's not too hard to see that any exponential function of the form $f(t) = Ce^{kt}$ satisfies the differential equation $\frac{df}{dt} = k * f$. Even with the constant coefficient of C attached, it is true that:

$$f'(t) = C(ke^{kt}) = k * Ce^{kt} = k * f$$

-So the entire family of functions $f(t) = Ce^{kt}$ are solutions to $\frac{df}{dt} = k * f$. C would distinguish one function from another, and can be considered the initial value of any member of the family since $f(0) = Ce^{k(0)} = C$.

-A population function (P(t)) that belongs to this family would in theory be a function that is always increasing without bound at a faster and faster rate, with an initial value of C. However, logically speaking, populations should really only behave this way when the population is small, since the larger the population gets, the more other factors like limited resources will cause growth rate to be different from this idea that $\frac{dP}{dt} = kP$.

-Let the **carrying capacity** (not so much a maximum but more a breaking point at which growth needs to start going in the other direction in order to stabilize) be denoted by M, such that the population should be approaching this carrying capacity M.

-If the population is below M, the population growth will be positive, causing the population to grow and get closer to M. If the population is above M, the population growth will be negative, causing the population shrink and get closer to M. So in general, for a population model to be logical, it should satisfy the following:

-If the population is small, the growth rate is proportional to P, implying that $\frac{dP}{dt} = kP$.

-If the population exceeds the carrying capacity M, then P starts to decrease, implying that $\frac{dP}{dt} < 0$.

-Is there a way to modify the differential equation for P and $\frac{dP}{dt}$ so that when P is small that $\frac{dP}{dt} \approx kP$, and when P is greater than M, that $\frac{dP}{dt} < 0$? It turns out we can if we let the rate of population growth be proportional to not only the population but also the difference in population and carrying capacity. In other words, create a joint proportional equation with constant of proportionality c>0:

$$\frac{dP}{dt} = cP(M - P)$$

-You can also rewrite this by letting k = cM which makes the equation:

$$\frac{dP}{dt} = kP(1 - \frac{P}{M})$$

-If P is small, then $1 - \frac{P}{M}$ is approximately 1, which means $\frac{dP}{dt} \approx kP$, and if P > M, then $1 - \frac{P}{M} < 0$ so $\frac{dP}{dt} < 0$.

-This more logical differential equation is called a **logistic differential equation**. Can you determine a function P(t) that satisfies $\frac{dP}{dt} = kP(1 - \frac{P}{M})$? We can think of two without even trying: $P(t) = 0$ and $P(t) = M$, since these two values are constant so they make $\frac{dP}{dt}$ equal 0, and they are both roots for $kP(1 - \frac{P}{M})$. These are both constant functions, and constant functions that satisfy differential equations are called **equilibrium solutions**.

-They are called equilibrium solutions because equations approach "equilibrium" if as time goes on (in one direction, in another direction, or both), their behavior imitates that of a constant function (they become more constant, more flat, have smaller rates of change). You can even imagine that a population function that satisfies $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ should in theory approach equilibrium over time, since a growing population should stabilize eventually around the carrying capacity: not getting much bigger or smaller than M, thus being more constant. A shrinking population would eventually stabilize around carrying capacity or (heaven forbid) 0.

-Population models and exponentials are not the other functions that satisfy differential equations. What about the motion of an object of mass m at the end of a vertical spring? Remember Hooke's Law from chapter 6.4? A spring that is stretched x units past its natural resting length (equilibrium if you will) will exert a force that is proportional to x:

$$restoring\ force \ = \ -kx$$

-Let k be a positive constant (call it the **spring constant**), so if all other factors like friction or air resistance are ignored, the restoring force should be (by Newton's 2nd Law):

$$restoring\ force \ = \ ma \ = \ m\frac{d^2x}{dt^2} \ = \ -kx$$

-This equation, $m\frac{d^2x}{dt^2} = -kx$, is a **second-order differential equation**. An equation that compares the second derivative of a variable with respect to time with that same variable. In this case, the second derivative of x is proportional to x. More specifically:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

-Is there a function that is proportional to its second derivative in this way? A function that when differentiated twice gives you the original function times a negative constant? I can think of two functions that do this, sine and cosine:

$\frac{d^2}{dx^2}[sin(x)] = \frac{d}{dx}[cos(x)] = -sin(x)$ and $\frac{d^2}{dx^2}[cos(x)] = \frac{d}{dx}[-sin(x)] = -cos(x)$

-Of course, we will need some modification for the second derivative to be the original function times $-\frac{k}{m}$. So what about:

$x = sin\left(\sqrt{\frac{k}{m}}t\right)$ and $x = cos\left(\sqrt{\frac{k}{m}}t\right)$

-Not only that, but linear combinations of these two functions would all technically satisfy $\frac{d^2x}{dt^2} = -\frac{k}{m}x$.

-These were all very specific examples though. What about general differential equations? By definition, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives (of any order). Though the **order** of the differential equation is the order of the highest derivative used in the equation (order 2 if $f''(t)$ is the highest derivative in the equation, order 5 if $f^{(5)}(t)$ is the highest derivative...)

-The functions don't have to be in terms of time either, for example, $y' = xy$ is a differential equation in which it is implied that y and its derivatives are functions of x.

-A function f is a solution to a differential equation if the equation is satisfied when y = f(x) and its derivatives are substituted into the equation.

-The truth is, we have solved many very simple differential equations already. Namely, any equation of the form:

$$y' = f(x)$$

is a differential equation. Integrate both sides to find an antiderivative F(x) where $F'(x) = f(x)$ and you have that the family of equations $y = F(x) + C$ for a constant C are solutions to the differential equation $y' = f(x)$.

-This is essentially just integration and we have plenty of practice with that already. But differential equations that are more complicated, like $xy' + y = 2x$ for example, are not so easy to find a solution to (or worse, find <u>all</u> the solutions to). We will be going over how to find solutions to differential equations with graphs, approximation, integration, and more, even though there is no systematic technique that allows us to solve for all differential equations.

-For starters though, we will begin by practicing when a given function is a solution to a differential equation, and when it is not:

Example: Determine whether $y = \frac{1}{x} + x$ is a solution to the following differential equations:

a) $xy' + y = 2x$ b) $xy'' + 2y' = 0$

Solution: To determine if a given equation is a solution to a differential equation, first find all the derivatives used in the equations so you can then plug those derivatives in:

$y = \frac{1}{x} + x, \qquad y' = -\frac{1}{x^2} + 1, \qquad y'' = \frac{2}{x^3}$

Now you can plug these expressions in to see if the differential equations are satisfied.

a)

$$xy' + y = 2x$$
$$x\left(-\frac{1}{x^2} + 1\right) + \left(\frac{1}{x} + x\right) = 2x$$
$$-\frac{1}{x} + x + \frac{1}{x} + x = 2x$$
$$2x = 2x$$

$y = \frac{1}{x} + x$ is a solution to the first differential equation we were given.

b)

$$xy'' + 2y' = 0$$
$$x\left(\frac{2}{x^3}\right) + 2\left(-\frac{1}{x^2} + 1\right) = 0$$
$$\frac{2}{x^2} - \frac{2}{x^2} + 2 = 0$$
$$2 = 0$$

$y = \frac{1}{x} + x$ is not a solution to the first differential equation we were given.

Exercise: a) Determine whether $y = \ sin(2x)$ is a solution to the following differential equation:

$$y^{(3)} - y'' - 4y' + 4y = 0$$

b) Determine whether $y = \ \sqrt{x}$ is a solution to the following differential equation:

$$4x^2y'' + 4xy' = y$$

-Still we can do better than test just a single function. How about an entire family of functions?

Example: Show that every member of the family of functions $y = \frac{1+ce^t}{1-ce^t}$ is a solution to: $y' = \frac{1}{2}(y^2 - 1)$.

Solution: It's probably easiest to tackle this one side at a time:

$$y' = \frac{d}{dt}\left[\frac{1+ce^t}{1-ce^t}\right] = \frac{ce^t(1-ce^t)+ce^t(1+ce^t)}{(1-ce^t)^2} = \frac{ce^t-c^2e^{2t}+ce^t+c^2e^{2t}}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

Now let's work on the other side:

$$\frac{1}{2}(y^2 - 1) = \frac{1}{2}\left(\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right) = \frac{1}{2}\left(\frac{1+2ce^t+c^2e^{2t}}{(1-ce^t)^2} - 1\right) = \frac{1}{2}\left(\frac{1+2ce^t+c^2e^{2t}-(1-2ce^t+c^2e^{2t})}{(1-ce^t)^2}\right) = \frac{1}{2}\left(\frac{4ce^t}{(1-ce^t)^2}\right) = \frac{2ce^t}{(1-ce^t)^2}$$

The left and right sides are equal to each other. There was no specification of what c was, so for any c, the entire family of equations of the form $\frac{1+ce^t}{1-ce^t}$ is a solution to this differential equation.

-As you have no doubt seen with antiderivative problems earlier, finding an entire family of solutions is one thing, but sometimes we are concerned with only one solution.

$$f'(x) = x^3 \qquad \text{has} \qquad f(x) = \tfrac{1}{4}x^4 + C \text{ as an entire family of solutions}$$

but if we want a specific solution f(x) where f(2) = 6,

then $f(x) = \frac{1}{4}x^4 + 2$ is the particular member of the family that satisfies this **initial condition**.

-A family of solutions is called a **general solution** to a differential equation, but if another condition, namely that $y(t_0) = y_0$, is also given, this is called an **initial condition**. Finding a solution of a differential equation that satisfies the initial condition is called an **initial-value problem**.

-Solving an initial-value problem is done by first finding the general solution, and then plugging the values of the initial condition in to solve for any missing coefficients, terms, or other factors that differentiate members of the family of solutions from each other.

Example: Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

Solution: We saw from the last example that the general solution to this differential equation is $y = \frac{1+ce^t}{1-ce^t}$. The initial condition however says that we want the member of the family where $2 = \frac{1+ce^0}{1-ce^0}$.

Let's solve for c:

$$2 = \frac{1+ce^0}{1-ce^0}$$

$$2 = \frac{1+c}{1-c}$$

$$2(1-c) = 1+c$$

$$2 - 2c = 1 + c$$

$$1 = 3c$$

$$c = \frac{1}{3}$$

The specific equation to satisfies this differential equation and the initial condition is:

$$y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t}$$

$$y = \frac{3+e^t}{3-e^t}$$

Exercise: Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = -3$.