11.6) The Ratio and Root Test

-We are still not quite done taking a look at all the different ways to determine if a series is convergent or divergent. After comparing series to other series as we have done a few times so far, and taking integrals, and other methods, now we turn our attention to comparing terms in a series to themselves.

-The following is known as **the Ratio Test**, since it involves taking a ratio of general consecutive terms in a series.

1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (not just convergent, but absolutely convergent!)
2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; the series may be absolutely convergent, conditionally convergent, or divergent, and you can only find out by doing a different test.

-The ratio test borrows from the idea of geometric series, which states that if a series has the property that the ratio from one term to another is common (a common ratio r) and this ratio is smaller than 1, then the series converges, but if this ratio is greater than 1, the series diverges.

-The same idea is at play here. If you take the limit of the ratio from one term to the next and find that the ratio of consecutive terms is approaching a finite number, then you are likening the series to a geometric series. So if this eventual ratio between terms is small enough, then the series converges, but if this ratio is too large, the series diverges. This idea is used in the proof of the Ratio Test which is shown in the ebook and textbook, but we won't worry about replicating the proof here.

Example: Test the following series for absolute convergence:

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$$

Solution: One of the nice things about the ratio test is how there are no prerequisites that need to be true about the series before you try conducting the test, so you can use it whenever you want. Still, it is one of the few tests we have seen so far with an "inconclusive" result, so don't be so surprised if you use it all the time and find out that your answer is inconclusive and you have to try something else anyway.

Our term is $a_n = (-1)^n\frac{n^3}{3^n}$, and you can forget about the alternating factor, it will always disappear in the absolute value anyway. So by the Ratio test, we get:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}\right|$$

$$\lim_{n\to\infty}\left|-\frac{(n+1)^3}{3n^3}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3n^3} = \lim_{n\to\infty}\frac{1}{3}(1+\frac{1}{n}) = \frac{1}{3}(1+0) = \frac{1}{3}$$

By the ratio test, the series is absolutely convergent.

Example: Test the following series for absolute convergence:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

Solution: Forget the absolute value signs, there's no alternating factor anyway. So let $a_n = \frac{n^n}{n!}$, and:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}$$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} * \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)(n^n)}$$

Be careful cancelling here:

$$\lim_{n\to\infty} \frac{(n+1)^n}{(n^n)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

This very famous limit becomes e, which is finite and greater than 1, so therefore, this series is divergent.

Example: Test the following series for absolute convergence:

a) $\sum_{n=1}^{\infty} \frac{1}{n}$ b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution: You may already know whether these series are convergent or not (and how), but let's see what happens when we use the Ratio Test on each:

a) Let $a_n = \frac{1}{n}$, so the Ratio Test gives us:

$$\lim_{n\to\infty} \frac{1/(n+1)}{1/(n)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

The ratio test resulted in a 1. This is **inconclusive**, and you have to do something else to find out if the series converges or diverges. Remember, the inconclusive result does not mean you get to stop and just say the answer is inconclusive, you have to determine if the series is convergent or not in a different way. Thankfully, we know the harmonic series is divergent from several proofs in past sections already.

b) Let $a_n = \frac{1}{n^2}$, so the Ratio Test gives us:

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/(n)^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Again, the ratio test resulted in a 1. This is inconclusive again, so you have to try something else. Since the original expression was a p-series with p = 2>1, that means this is a convergent series, which for a non-alternating series, makes it absolutely convergent too.

Exercise: Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\sum_{n=1}^{\infty} \frac{5^n}{(n+1)2^{2n+1}}$ b) $\sum_{n=1}^{\infty} \frac{(n)10^n}{(n-1)!}$ c) $\sum_{n=1}^{\infty} \frac{cos(\pi n)}{\sqrt{n}}$

-One operation that doesn't always work so well with the Ratio Test is powers of n. If a factor or term is being raised to the nth power, especially if the base contains n as well, cancellation doesn't always come easy from the Ratio Test. Thankfully, there is a another test for convergence that is convenient when an nth power is in the rule for the series:

The Root Test:

1) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> (and also convergent)

2) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

3) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

-Once again, "inconclusive" is not a conclusion, it is an indication that you have to do something else. Also, if the Ratio Test is inconclusive, the Root Test will be too, and vice-versa, so don't try both.

Example: Test the following series for absolute convergence:

a) $\sum\limits_{n=1}^{\infty} \left(\frac{3n+1}{5n+3}\right)^n$ b) $\sum\limits_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$

Solution: The Root Test can be very easy to spot given how if you see a power of n, that's usually a good indicator that taking the nth root of everything (and I do mean everything) in the rule is a good idea to simplify. Though it won't always produce a conclusion:

a) Once again, no prerequisite material needed to conduct the Root Test, just let $a_n = \left(\frac{3n+1}{5n+3}\right)^n$, so:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+1}{5n+3}\right)^n} = \lim_{n\to\infty} \frac{3n+1}{5n+3} = \frac{3}{5}$$

This series is absolutely convergent.

b) If $a_n = \left(\frac{n+1}{n}\right)^n$, then $\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n}\right)^n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1$.

This is inconclusive. However, you may remember that the expression in the series is the same as $\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$, and the limit of this expression in the series is $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

By the test of divergence, this series is divergent because the expression inside does not go to 0 as n goes to infinity.

Exercise: Test the following series for absolute convergence, conditional convergence, or divergence:

a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(ln\ n)^n}$ b) $\sum\limits_{n=1}^{\infty} (tan^{-1}(x))^n$ c) $\sum\limits_{n=1}^{\infty} \frac{n3^{2n}}{6^{n+1}}$