11.8) Power Series:

-Suppose we have a series that has more than just an index n, that increments in value by one unit for every term. Perhaps a series we are working with has a variable too, an x variable, which is not an index and does not have to be whole numbers only and does not have to increase in value from one term to the next.

-In the event of having both a variable x and an index n for a series, you have what is called a **power series**:

$$\sum_{n=0}^{\infty} c_n x^n$$

-The variable, x, can be any value it wants to be, and this value will be plugged into x for each term in the series. The index, n, is a different integer from 0 on up that is plugged into each term, with each term getting a different index value. That means c_n, the factors that change from one value of n to another, are **coefficients** of the series.

-Power series are essentially functions that resemble polynomial functions:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 +.... c_n x^n +...$$

but unlike polynomials they have infinite terms and powers of n so they don't tend to be labeled by degrees.

-Power series are series in which each term is a power function, but there are other types of series in terms of x as well, such as a **trigonometric series**, in which the terms are all trigonometric functions:

$$g(x) = \sum_{n=0}^{\infty} (a_n cos(nx) + b_n sin(nx))$$

-However, for now we are going to focus more on power series, such as the most basic of power series in which $c_n = 1$ for all n, and therefore the power series becomes a geometric series:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 ... + x^n +...$$

-Unlike some of the geometric series we have seen before however, this is a function of x, and x happens to be the common ratio from one term to the next in the series. Whether this is a geometric series that converges or diverges depends on x.

-If $-1 < x < 1$ then this is a convergent geometric series, like if $x = \frac{1}{2}$ we would have:

$$f(\tfrac{1}{2}) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} ... + \frac{1}{2^n} +... = 1$$

-If $x \geq 1$ or $x \leq -1$, then this is a divergent geometric series, like if x = 2 we would have:

$$f(2) = \sum_{n=0}^{\infty} (2)^n = 1 + 2 + 4 + 8 ... + 2^n +...$$

-However, a more general version of a power series may look more like this:

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 +.... c_n(x - a)^n +...$$

-Specifically, this is called a **power series in (x-a),** or **a power series centered at a**, or **a power series at a**. The a is there to indicate any horizontal shifting in the input that might be occurring.

-One thing worth mentioning also is that you may have noticed that the first term in a power series is found by using an index of n = 0, and the first term is usually either 1 *or* c_0 due to the first term being raised to the power of 0. Would the first term $c_0(x - a)^0$ still be c_0 if x = a? In other words, we would have the indeterminate $c_0(0)^0$ in this case, so would $0^0 = 1$? Without getting too much into it, yes, it would still be 1 and the first term is still designated to be c_0.

-What if you wanted to know in general if a power series converges or diverges? Or maybe for what values of x a power series converges or diverges? The test you would want to use to answer a question like this is almost always going to be the **Root Test** or the **Ratio Test** from two sections ago. Why? They have no prerequisites so you can use them whenever you want for any value of x you want, and because power series tend to be readily simplified by roots and ratios (they work well is what we are saying!).

-Remember, a series converges according to the Root/Ratio test if the limit as n goes to infinity of the ratio/root of the series rule is less than 1.

Example: For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution: If we apply the Ratio test to $\frac{(x-3)^n}{n}$ we would get:

$$\lim_{n \to \infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right|$$

$$\lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{n+1} \frac{n}{(x-3)^n} \right| = \lim_{n \to \infty} \left| (x - 3) \frac{n}{n+1} \right| = |(x - 3)| \lim_{n \to \infty} \frac{n}{n+1} = |(x - 3)|$$

Remember, x is not part of the limit, so this is why $|(x - 3)|$can be factored out of the limit before we evaluate. But we still have to ask ourselves, what values of x will make $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converge? If our result of the Ratio Test is smaller than 1, then the series converges. So what values of x will make $|(x - 3)|$ smaller than 1?

$$|(x - 3)| < 1$$
$$-1 < (x - 3) < 1$$
$$2 < x < 4$$

If x is between 2 and 4, the series converges, and if $x < 2$ or $x > 4$ then the series diverges. But what about when x = 2 or x = 4 themselves? It's tempting to lump them into the divergent category too, but you don't want to do that. If x =2 or x =4, then the Ratio Test result equals 1, which is inconclusive. So we have to take a closer look at each of these values.

If x = 2, then $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series. This is a convergent series.

If x = 4, then $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ becomes $\sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series. This is a divergent series.

One endpoint works, the other does not, so that means our range of values for x in which $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converges is:

$$2 \leq x < 4$$

-You want to get into the habit of testing the endpoints each time. There is no easy way to tell if both endpoints will be part of your range, or if neither, or if one or the other is.

Example: For what values of x does the series $\sum_{n=1}^{\infty} n!\,(x)^n$ converge?

Solution: Using the ratio test on $a_n = n!\,(x)^n$, we get:

$$\lim_{n \to \infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!(x)^{n+1}}{n!(x)^n} \right| = x \lim_{n \to \infty} |(n + 1)|$$

As n approaches infinity, n+1 goes to infinity as well. So is there any value x can be that will make $\lim_{n \to \infty} |(n + 1)|$ less than 1, or at the very least, not infinite? Only one, and that's when x = 0.

The only value of x that makes $\sum_{n=1}^{\infty} n!\,(x)^n$ convergent is x = 0.

-All power series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n$ have one value of x that makes the series converge, and that's x = a. It's a rather boring answer though, so it's always worth it to see if there is more.

Example: For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ converge?

Solution: Using the ratio test on $a_n = \frac{x^n}{(2n)!}$, we get:

$$\lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{(2(n+1))!}}{\frac{x^n}{(2n)!}} \right| = \lim_{n \to \infty} \left| \frac{(x)^{n+1}}{(2n+2)*(2n+1)*(2n)!} * \frac{(2n)!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{(2n+2)*(2n+1)} \right|$$

$$x * \lim_{n \to \infty} \left| \frac{1}{(2n+2)*(2n+1)} \right|$$

The expression $\lim_{n \to \infty} \left| \frac{1}{(2n+2)*(2n+1)} \right| = 0$, and all before any value of x is used. Therefore, all values of x will make the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ converge, since no matter what x is used, the Ratio Test results in the limit being less than 1.

Exercise: For what values of x do the following series converge?

a) $\sum_{n=2}^{\infty} \frac{(5-2x)^n}{ln(n)}$ b) $\sum_{n=1}^{\infty} (x - 1)^n \sqrt{n^2 + 1}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$

-As you have seen from the examples and exercises so far, there are in fact only three possibilities for what values of x make a power series $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$ converge:

1) The series converges only when x = a.
2) The series converges for all x.
3) There exists some positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

-This positive number R mentioned in scenario #3 above is called the **radius of convergence** for a power series. For scenario #1 and #2, the radius of convergence is 0 and ∞ respectively. It serves as the length of half of the interval of values of x that will make a power series converge, which is why when only one value of x works, the radius is 0 due to there not being an interval, and when all values of x work, the interval is the infinitely large range of all real numbers.

-As you may have seen from previous exercises though, #3) is the trickiest scenario, not only because you will have to find out what R is, but you will have to find an interval, complete with endpoints that may or may not be a part of the solution. So remember, you may actually have one of four solution intervals for scenario #3:

$(a - R, a + R)$, $[a - R, a + R]$, $[a - R, a + R)$, $(a - R, a + R]$

-Test those endpoints independently. There is no easy way to be certain which is the endpoint you want.

Example: For what values of x does the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ converge?

Solution: Let $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$. By the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}}.}{\frac{(-3)^n x^n}{\sqrt{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{\sqrt{n+2}} * \frac{\sqrt{n+1}}{(3x)^n} \right| = |3x| \lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

First focus on the limit:

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{\sqrt{n+2}} \right| = \lim_{n\to\infty} \left| \sqrt{\frac{n+1}{n+2}} \right| = \left| \sqrt{\lim_{n\to\infty} \frac{1+1/n}{1+2/n}} \right| = \left| \sqrt{1} \right| = 1$$

Since the limit is 1, that means if $|3x| < 1$, the series converges:

$$|3x| < 1$$
$$-1 < 3x < 1$$
$$-\frac{1}{3} < x < \frac{1}{3}$$

So we know $\left(-\frac{1}{3}, \frac{1}{3}\right)$ is part of the solution, but is it the entire interval of convergence? We have to test the endpoints to find out:

If $x = -\frac{1}{3}$, then $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$.

This is the same series as: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series where p = 0.5 < 1, which means the series diverges.

If $x = \frac{1}{3}$, then $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$.

This is an alternating series in which the terms are always decreasing and the term $\frac{1}{\sqrt{n+1}}$ goes to zero when n approaches infinity. Therefore, this series converges.

When $x = -\frac{1}{3}$, the series diverges. When $x = \frac{1}{3}$,the series diverges. Therefore, our interval of convergence is:

$$(-\frac{1}{3}, \frac{1}{3}]$$

Example: Find the radius of convergence and interval of converge for $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Solution: Let $a_n = \frac{n(x+2)^n}{3^{n+1}}$, so by the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+1}}}{\frac{n(x+2)^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+1}} * \frac{3^n}{n(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+2)(n+1)}{3n} \right| = \left| \frac{(x+2)}{3} \right| \lim_{n\to\infty} \left| \frac{(n+1)}{n} \right.$$

Once again, the expression with x can leave the limit while we evaluate the limit as n goes to infinity.

$$\lim_{n\to\infty} \left| \frac{(n+1)}{n} \right| = \lim_{n\to\infty} \left| 1 + \frac{1}{n} \right| = 1$$

This means that if $\left| \frac{(x+2)}{3} \right| < 1$, the series converges. This can be rewritten as $\frac{|(x+2)|}{3} < 1$,or $|(x+2)| < 3$. This form lets us determine that the radius of convergence is 3. As for the interval of convergence, let's solve for x:

$$|(x+2)| < 3$$
$$-3 < x+2 < 3$$
$$-5 < x < 1$$

What about the endpoints? Let's test them one at a time:

If x = -5, then:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-5+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-1)^n(3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(-1)^n}{3}$$

This expression $\frac{n(-1)^n}{3}$ does not approach zero as n goes to infinity, so therefore the series diverges if x = -5.

If x = 1, then:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(1+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{3}$$

This expression $\frac{n}{3}$ does not approach zero as n goes to infinity, so therefore the series diverges if x = 1.

Therefore, neither endpoint makes the series converge, so our interval of convergence is $(-5, 1)$.

Exercise: Find the radius of convergence and interval of convergence for the following series:

a) $\sum_{n=1}^{\infty} \frac{(7-3x)^n}{n^2}$ b) $\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{\sqrt{2n-1}}$ c) $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n ln(n)}$