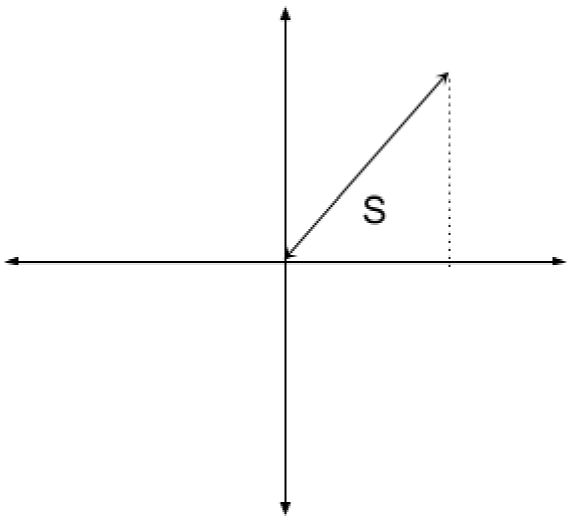
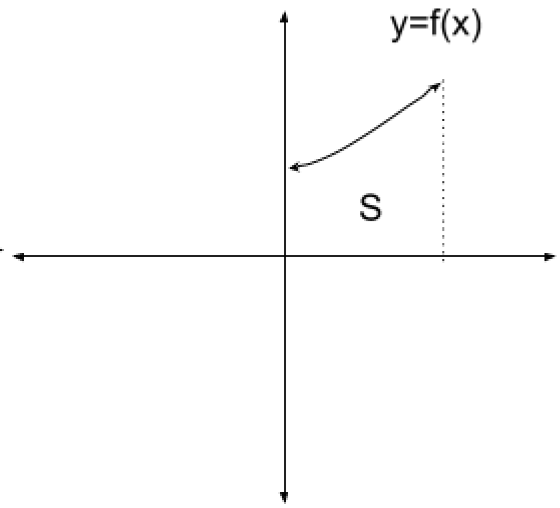
5.1)The Area and Distance Problem

-Given an area S underneath a curve y=f(x) in two-dimensional space but above the x-axis, can you find the area of S?



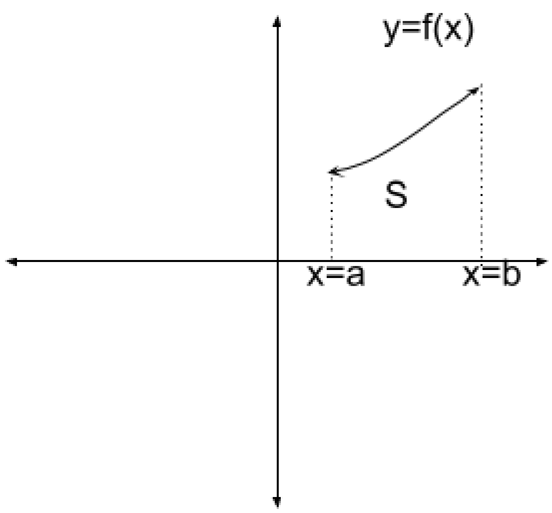


-If the area under the curve is a geometric shape you are familiar with, like above on the left, yes.
If the area under the curve is a non familiar shape, like above on the right, you still can, but it is trickier.

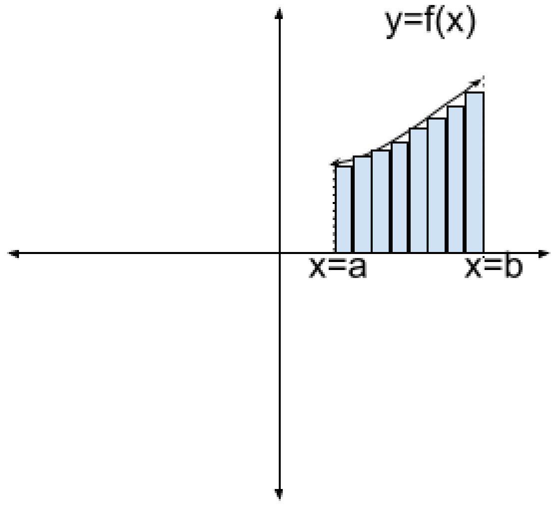
-First, we express the area S by the following:

$$S = \{(x, y) \mid a \le x \le b,\ 0 \le y \le f(x),\ 0 \le f(x)\}$$

Let the area S be bounded on top by the graph of the continuous function y = f(x) where $0 \le f(x)$ (f(x) is positive only for now). S is bounded on bottom by the x-axis, and is bounded on the sides by x = a and x = b.

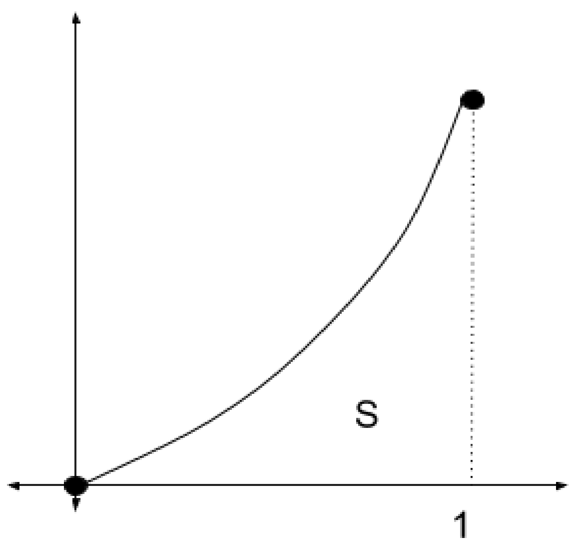


-Since S is not necessarily going to be a shape we can find the exact area of using geometric formulas, the way to approximate the area of S is to fill S up with geometric shapes we can find the area of easily, then add up the areas of these geometric shapes. The sum of these geometric shape areas should be approximately equal to the area of S, if not exactly equal:

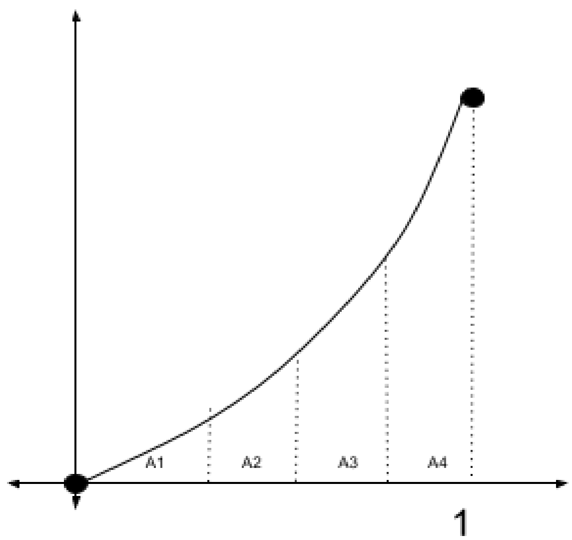


-If draw eight rectangles underneath the curve y=f(x) like we see above and then find the areas of these 8 rectangles, then add those 8 areas up, that sum of areas will approximately be the same as the area of S. We use rectangles due to the ease in finding the area of a rectangle.

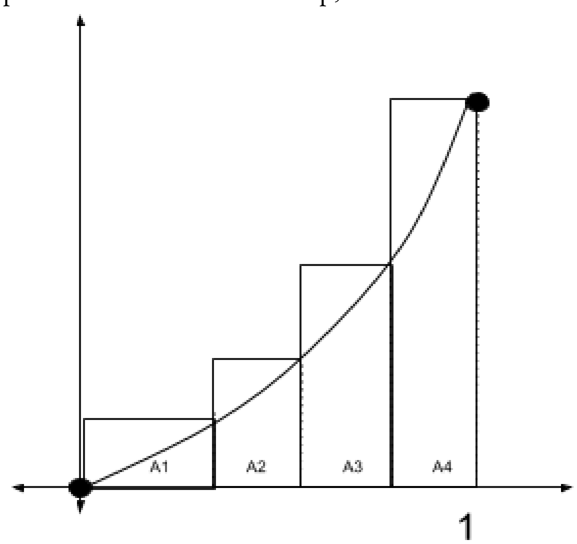
Example: Estimate the area underneath the parabola $f(x) = x^2$ from 0 to 1:

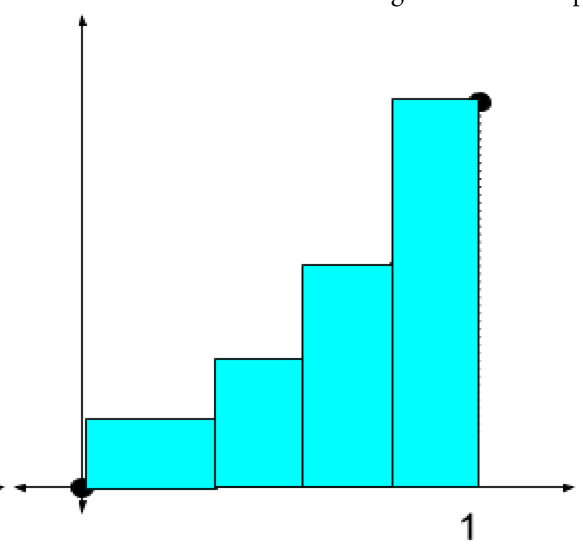


Solution: We start by trying to envision the area S as being four sections or "strips" of area under the curve, each of the same width for simplicity. We will call these strips $A_1,\ A_2,\ A_3,\ and\ A_4$, and as you can imagine, since they all have the same width and they take up all the space between x = 0 and x = 1, the strips are all of width $\frac{1}{4}$:



Can we find the exact areas of the strips? No, but we can fit a rectangle in-between each strip that has an area comparable to the area of the strip, and it will be much easier to find the areas of the rectangles than the strips:





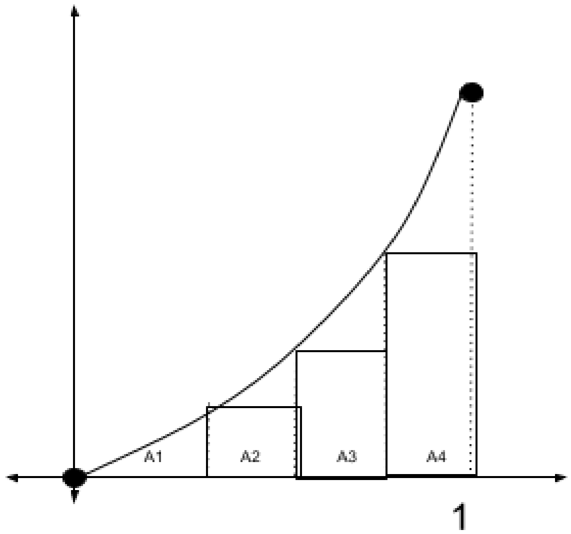
The area of these four rectangles will add up to equal approximately the area under the curve $f(x) = x^2$, but how do you find the area of these rectangles? We know the area of a rectangle is length times width, and that the width of these rectangles is $\frac{1}{4}$, but what about the heights?

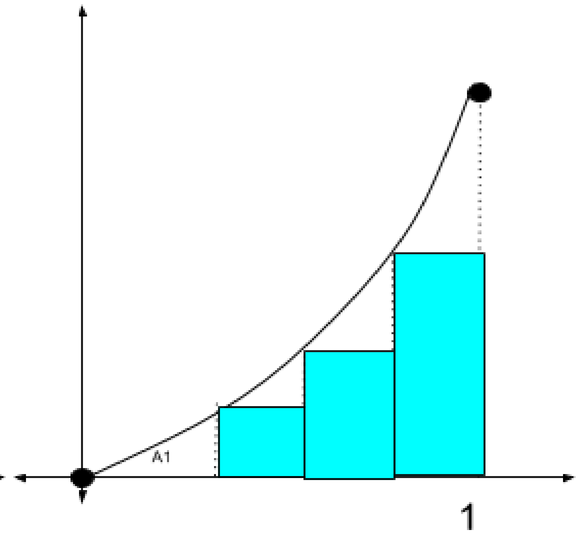
The rectangles all touch the curve $f(x) = x^2$, at $\left(\frac{1}{4}, f(\frac{1}{4})\right)$, $\left(\frac{2}{4}, f(\frac{2}{4})\right)$, $\left(\frac{3}{4}, f(\frac{3}{4})\right)$, and $\left(\frac{4}{4}, f(\frac{4}{4})\right)$. Therefore the heights of these rectangles are $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$, $\left(\frac{2}{4}\right)^2 = \frac{1}{4}$, $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$, and $\left(\frac{4}{4}\right)^2 = 1$.

Therefore, the areas of the rectangles, from left to right, added up is:

$$\frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{9}{16}\right) + \frac{1}{4}(1) = \left(\frac{1}{64}\right) + \left(\frac{1}{16}\right) + \left(\frac{9}{64}\right) + \frac{1}{4} = \frac{30}{64} = \frac{15}{32} = 0.46875$$

Of course, this is not the only way we could have approximated the area under the curve, since we could have drawn the rectangles differently. What would we have gotten if we drew the rectangles so the other corner of the rectangle intersected the curve each time?





The first rectangle has a height of 0, and thus is practically non existent for the picture or the sum of the areas. Now we have that the area is approximately:

$$\frac{1}{4}(0) + \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{9}{16}\right) = (0) + \left(\frac{1}{64}\right) + \left(\frac{1}{16}\right) + \left(\frac{9}{64}\right) = \frac{15}{64} = \frac{7}{32} = 0.21875$$

-The first summation we found is sometimes called a right-endpoint sum, denoted by R_n where n is the number of strips used to form the rectangles, and the second summation is called a left-endpoint sum, denoted by L_n. Both of these are summations of rectangles that approximate the area under a curve from one point to another.

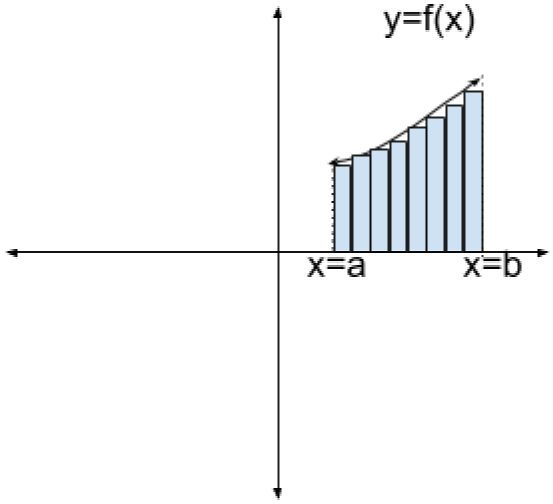
-Between L_n and R_n, will one always be larger? In general, no. However:

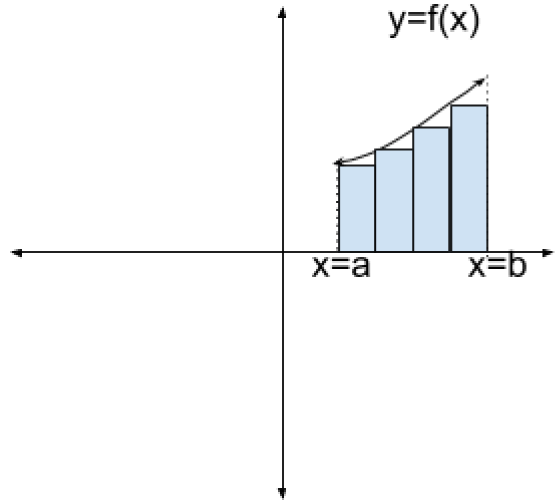
-If y = f(x) is increasing everywhere from x=a to x=b, then $L_n < R_n$.

-If y = f(x) is decreasing everywhere from x=a to x=b, then $L_n > R_n$

-Is either of these approximations actually the area under the curve? No. The R_n was a little too large, since you saw the rectangles were spilling over the top of the curve, and the L_n was a little too small, since you saw the rectangles were all under the curve. On top of that, the bigger problem is that there were not enough rectangles to truly give an accurate approximation. Will L_n and/or R_n approach the actual area under the curve if n gets large enough? Perhaps.

-The more rectangles there are, the more the area of the sum of those rectangles will more closely match the actual area under the curve. Compare the areas of the rectangles under the curve in these two pictures:





-Which group of rectangles takes up more area under the curve in these two pictures, the one with 4 or the one with 8 rectangles? Naturally the one with more rectangles is more accurate. Going back to the $y = x^2$ example, we can actually take more summations L_n and R_n to see what they approach as n gets bigger and bigger:

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

-It appears that L_n and R_n are approaching $\frac{1}{3}$ as n gets bigger. However, can we be sure of that? The only way to be sure is to compute the limit of what either L_n or R_n approaches as n goes to infinity:

Example: Prove that for $f(x) = x^2$ on $[0, 1]$ $\lim\limits_{n\to\infty} R_n = \frac{1}{3}$.

Solution: Do we have a formula for what R_n equals for any general n over this interval? Remember, the sum of the right-endpoint rectangle areas is a sum of rectangle areas that all have a width of $\frac{1}{n}$, and their heights are all computed by plugging a different multiple of $\frac{1}{n}$ into f(x):

$$R_n = \frac{1}{n} * f\left(\frac{1}{n}\right) + \frac{1}{n} * f\left(\frac{2}{n}\right) + \frac{1}{n} * f\left(\frac{3}{n}\right) + \ldots \frac{1}{n} * f\left(\frac{n}{n}\right)$$

Since $f(x) = x^2$, this becomes: $R_n = \frac{1}{n} * \left(\frac{1}{n}\right)^2 + \frac{1}{n} * \left(\frac{2}{n}\right)^2 + \frac{1}{n} * \left(\frac{3}{n}\right)^2 + \ldots \frac{1}{n} * \left(\frac{n}{n}\right)^2$

$$R_n = \frac{1}{n} * \left(\frac{1}{n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2)$$
$$R_n = \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

What is the sum of the first n square numbers? You may recall the formula is:

$$(1^2 + 2^2 + 3^2 + \ldots + n^2) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Therefore we have that: $$R_n = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{(n+1)(2n+1)}{6n^2}$$

What is the limit of this expression as n goes to infinity?

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{2}{6} = \frac{1}{3}$$

-We could have found $\lim_{n\to\infty} L_n$ and we would have gotten a similar result. Finding either of these limits will in fact give you the area under the curve of y = f(x) and above the x-axis if y = f(x) is positive from x = a to x = b.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

-These limits can be tricky depending on how complicated a function f(x) is, but we can come up with a generalized function for what R_n or L_n is.

-Remember, the interval for the area starts at x = a and ends at x = b, so if we divide this interval into n strips, the strips all have a width of $\frac{b-a}{n}$. We call this width $\Delta x = \frac{b-a}{n}$.

-As for the strips themselves, they can be considered smaller intervals on their own:

$$A_1 = [x_0, x_1],\ A_2 = [x_1, x_2],\ \ldots,\ A_n = [x_{n-1}, x_n]$$

-We can presume $x_0 = a$ and $x_n = b$. In fact, we can write all the endpoints in terms of a and Δx since every strip has a width of Δx, so you can just keep adding by Δx to get from one interval to the next. Therefore we can presume that

$$x_1 = x_0 + \Delta x = a + \Delta x$$
$$x_2 = x_1 + \Delta x = a + 2\Delta x$$
$$x_3 = x_2 + \Delta x = a + 3\Delta x$$
$$\ldots$$
$$x_i = x_{i-1} + \Delta x = a + i(\Delta x)$$
$$\ldots$$
$$x_n = x_{n-1} + \Delta x = a + n\Delta x = a + n\frac{b-a}{n} = a + b - a = b$$

-We also still have that the area of each rectangle depends on the input, which for the right-endpoint sums gives us areas of:

$$A_i = f(x_i)\Delta x = f(a + i(\Delta x))\Delta x$$

-Thus the right-endpoint sum itself is: $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \Delta x\left(\sum_{i=1}^{n} f(x_i)\right)$

-In case you were interested in finding a left-endpoint sum instead, the only difference is which endpoints go into f(x) to find the heights of the rectangles. Instead of starting with x_1, we start with $x_0 = a$. This makes this endpoint sum look like this: $L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x = \Delta x\left(\sum_{i=1}^{n} f(x_{i-1})\right)$

-These two sums approach the same value as n goes to infinity, since in general the only differences between these two sums is that L_n uses $\Delta x f(a)$ as a rectangle area and R_n does not, while R_n uses $\Delta x f(b)$ and L_n does not. So you given one, you can find the other sum by the following:

$$R_n = L_n + \Delta x * f(a) - \Delta x * f(b)$$

-As n goes to infinity however, Δx goes to 0, thus $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \textit{Area under } f(x) \textit{ from } a \textit{ to } b.$

-Thus we don't need both, we can just say that the **area** A of the region S under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x) = \lim_{n\to\infty} \left(\left(\sum_{i=1}^{n} f(x_i)\right)\Delta x\right)$$

-Other variations of area under the curve limits exist where the widths of the strips and the x_i inputs are not necessarily all uniform (maybe certain strips are wider than others, maybe not every input is an endpoint), but the idea for why $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$ still applies. If there are infinitely many strips/rectangles/intervals being used, whether the strips all have the same width or all the inputs are endpoints are negligible to the point that you don't need uniform inputs or widths for the limit to still approach the area under the curve:

$$A = \lim_{n\to\infty} \left(\Delta x\left(\sum_{i=1}^{n} f(x_i{}^*)\right)\right)$$

where $x_i{}^*$ is any point on the interval $[x_{i-1}, x_i]$.

-That said, still need to be able to find the limit, and depending on the function, $\sum_{i=1}^{n} f(x_i{}^*)$ can be difficult to find or evaluate as n goes to infinity. For starters, you might want to remember some of the more famous summation rules:

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1(2n+1))}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Exercise: a) Prove that for $f(x) = x^3$, on $[0, 1]$, $\lim_{n\to\infty} R_n = \frac{1}{4}$.

b) Prove that for $f(x) = x^2$, on $[0, 3]$, $\lim_{n\to\infty} R_n = 9$.

-Some functions are a bit more complicated to work with summations however, like exponentials. Sometimes you will just have to do the computations the long way by hand. In these non-limit situations, you will want to pay close attention to what you are being asked to do:

-How many strips are they asking you to use? This will tell you what n is.

-What is the starting point a and what is the endpoint b of the interval? You need these to find Δx.

-What kind of inputs should you use for f(x)? They could be:

-Left endpoints. In which case your first input is $x_0 = a,$ and the ith input is

$x_{i-1} = a + \Delta x\,(i-1)$

-Right endpoints. In which case your last input is $x_n = b,$ and the ith input is

$x_i = a + \Delta x\,(i)$

-Midpoints. In which case your first input is $x_1{}^* = a + \frac{\Delta x}{2},$ and the ith input is

$x_i{}^* = a + \Delta x\left(\frac{2i-1}{2}\right)$

Example: a) Estimate the area under the curve $f(x) = e^{-x}$ between x = 0 and x = 2 by using midpoints and using four subintervals.
b) Repeat part a) but use ten subintervals instead.

Solution: a) They are not asking for the exact area under the curve of $f(x) = e^{-x}$ from [0, 2], just an estimate.

If we wanted to find the exact area under the curve we could use the fact that the interval is [0,2] to find that $\Delta x = \frac{2}{n}$ and arbitrarily use right endpoints to let each x_i input be $\frac{2i}{n}$. From there we evaluate:

$$A = \lim_{n\to\infty} (\Delta x\left(\sum_{i=1}^{n} f(x_i{}^*)\right)) = \lim_{n\to\infty} \frac{2}{n}\left(\sum_{i=1}^{n} e^{-\frac{2i}{n}}\right)$$

We won't be doing that since $\left(\sum_{i=1}^{n} e^{-\frac{2i}{n}}\right)$ can't be found for infinitely large n without a computer. Instead we will deal with a couple finite cases only.

We have that the interval is [0,2] and uses four subintervals, so n = 4. That means that $\Delta x = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}$. It is also the case that they want midpoints, so our four midpoints (which have to be found after we have a, b, and Δx) start at $\frac{1}{4}$ and differ from each other by $\frac{1}{2}$. Remember, use $x_i{}^* = a + \Delta x\left(\frac{2i-1}{2}\right)$:

$$x_1{}^* = 0 + \tfrac{1}{2}\left(\tfrac{1}{2}\right) = \tfrac{1}{4}$$
$$x_2{}^* = 0 + \tfrac{1}{2}\left(\tfrac{3}{2}\right) = \tfrac{3}{4}$$
$$x_3{}^* = 0 + \tfrac{1}{2}\left(\tfrac{5}{2}\right) = \tfrac{5}{4}$$
$$x_4{}^* = 0 + \tfrac{1}{2}\left(\tfrac{7}{2}\right) = \tfrac{7}{4}$$

Now we have that the area is approximately equal to the sum of $\frac{1}{2}f(x_i{}^*)$:

$$Area \approx \tfrac{1}{2}f(x_1{}^*) + \tfrac{1}{2}f(x_2{}^*) + \tfrac{1}{2}f(x_3{}^*) + \tfrac{1}{2}f(x_4{}^*)$$
$$Area \approx \tfrac{1}{2}(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}) = 0.8557$$

b) What if we did this again for 10 midpoints? The width would change, since $\frac{b-a}{n} = \frac{2}{10} = 0.2$, and the first midpoint would start at $a + \frac{1}{2}\Delta x = 0.1$. Thus we have a summation of:

$$Area \approx 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5} + e^{-0.7} + e^{-0.9} + e^{-1.1} + e^{-1.3} + e^{-1.5} + e^{-1.7} + e^{-1.9}) = 0.8632.$$

-Sometimes we have to buckle down and do the computations one at a time, either because the functions do not lend themselves to easy limit evaluation, or because the functions are expressed numerically instead of algebraically.

Exercise: Estimate the area under the curve $f(x) = e^{-x^2}$ between x = 0 and x = 2 by using midpoints and using five subintervals.

Example: A man riding a bicycle has his velocity measured in five second increments as seen below:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	5	12	16	13	15	17	14

Approximate the distance this man on the bicycle was able to travel in 30 seconds.

Solution: If we are going to approximate the distance this man traveled, we could multiply the 30 second he traveled by an average velocity over that 30 seconds, but that is rather sloppy and implies that he was traveling a constant velocity the whole time, which the table clearly contradicts. We can approximate better than that. Instead, we should use these different velocities over their own individual intervals of time to predict the distance.

For the first interval of time, [0, 5], we can use a velocity with this interval (5 ft/s or 12 ft/s) and multiply that velocity by the width of this interval to get a velocity times time, which gives us an approximate distance traveled over that interval. Then we can do the same for every other interval, always using an interval width of 5, and using the same kind of endpoint each time. Should we use left endpoints or right endpoints?

They gave us no instruction, so let's try both:

$$L_6 = 5 * (5) + 5 * (12) + 5 * (16) + 5 * (13) + 5 * (15) + 5 * (17) = 25 + 60 + 80 + 65 + 75 + 85 = 390\, ft.$$

$$R_6 = 5 * (12) + 5 * (16) + 5 * (13) + 5 * (15) + 5 * (17) + 5 * (14) = 60 + 80 + 65 + 75 + 85 + 70 = 435\, ft.$$

This man likely traveled somewhere between 390 and 435 feet over that 30 seconds.

-This last distance example is not far removed at all from the area functions we were just looking at, and not just because of the notation we used.

-The time intervals were practically widths of strips on the x-axis, and the given velocities were the heights of rectangles under a velocity curve, making the sum of these products a sum of rectangles under the velocity curve and an approximate area under the velocity curve as well. Of course, that means that distance traveled over a period of time is analogous to the area under a velocity curve over a period of time as well.

-As a result, you can find the distance traveled by an object in motion using the same formulas and procedures for finding area under a velocity curve:

$$distance = \lim_{n \to \infty} \left(\sum_{i=1}^{n} v(t_i) \Delta t \right)$$

-The widths of the rectangles are replaced with time interval lengths, and the function in question is a velocity function v in terms of time t.

Exercise: A man skiing down a slalom going faster and faster is measured having the following velocities at the following points of time:

Time (s)	0	4	8	12	16	20	24
Velocity (ft/s)	4	7	12	18	22	24	27

Find a lower limit and upper limit for the distance this man on the skis travels over this 24 second interval (remember, he is going faster and faster while going down the slalom).