7.2) Trigonometric Integrals

Exercise: Integrate $\int sin^4(x)cos^3(x)dx$.

$$\int sin^4(x)cos^3(x)dx \;=\int sin^4(x)(1-sin^2(x))cox(x)dx$$

$$\int sin^4(x)cos^3(x)dx \;=\int(sin^4(x)-sin^6(x))cox(x)dx$$

$$\int sin^4(x)cos^3(x)dx \;=\int(u^4-u^6)du$$

$$\int sin^4(x)cos^3(x)dx \;=\frac{u^5}{5}-\frac{u^7}{7}\;+C$$

$$\int sin^4(x)cos^3(x)dx \;=\frac{sin^5(x)}{5}-\frac{sin^7(x)}{7}\;+C$$

Exercise: Integrate the following:

a) $\int cos^4(x)sin^2(x)dx$ b) $\int sin^5(x)cos^3(x)dx$.

a)

$$\int cos^4(x)sin^2(x)dx \;=\int(cos^4(x)-cos^6(x))dx$$

$$\int cos^4(x)sin^2(x)dx \;=\int\left(\frac{(1+cos(2x))^2}{4}-\frac{(1+cos(2x))^3}{8}\right)dx$$

$$\int cos^4(x)sin^2(x)dx \;=\int\left(\frac{1}{4}+\frac{cos(2x)}{2}+\frac{cos^2(2x)}{4}-\frac{1}{8}-\frac{3cos(2x)}{8}-\frac{3cos^2(2x)}{8}-\frac{cos^3(2x)}{8}\right)dx$$

$$\int cos^4(x)sin^2(x)dx \;=\int\left(\frac{1}{8}+\frac{cos(2x)}{8}-\frac{cos^2(2x)}{8}-\frac{cos(2x)(1-sin^2(2x))}{8}\right)dx$$

$$\int cos^4(x)sin^2(x)dx \;=\int\left(\frac{1}{8}+\frac{cos(2x)}{8}-\frac{1+cos(4x)}{16}-\frac{cos(2x)}{8}+\frac{cos(2x)sin^2(2x)}{8}\right)dx$$

$$\int cos^4(x)sin^2(x)dx \;=\int\left(\frac{1}{16}-\frac{cos(4x)}{16}+\frac{cos(2x)sin^2(2x)}{8}\right)dx$$

$$\int cos^4(x)sin^2(x)dx \;=\frac{1}{16}x-\frac{sin(4x)}{64}+\frac{sin^3(2x)}{48}+C$$

b)

$$\int sin^5(x)cos^3(x)dx \;=\int sin^5(x)(1-sin^2(x))cox(x)dx$$

$$\int sin^5(x)cos^3(x)dx = \int (sin^5(x) - sin^7(x))cox(x)dx$$

$$\int sin^5(x)cos^3(x)dx = \int (u^5 - u^7)du$$

$$\int sin^5(x)cos^3(x)dx = \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$\int sin^5(x)cos^3(x)dx = \frac{sin^6(x)}{6} - \frac{sin^8(x)}{8} + C$$

Exercises: Integrate the following:

a) $\int csc^4(x)cot^3(x)dx$ b) $\int tan^5(x)dx$ c) $\int cos(5x)sin(3x)dx$

a)

$$\int csc^4(x)cot^3(x)dx = \int csc^2(x)(cot^3(x) + cot^5(x))dx$$

$$\int csc^4(x)cot^3(x)dx = -\int (u^3 + u^5)du$$

$$\int csc^4(x)cot^3(x)dx = -\frac{u^4}{4} - \frac{u^6}{6} + C$$

$$\int csc^4(x)cot^3(x)dx = -\frac{cot^4(x)}{4} - \frac{cot^6(x)}{6} + C$$

b)

$$\int tan^5(x)dx = \int tan(x)(sec^2(x) - 1)^2 dx$$

$$\int tan^5(x)dx = \int tan(x)(1 - 2sec^2(x) + sec^4(x))dx$$

$$\int tan^5(x)dx = \int tan(x)(1 - 2sec^2(x) + sec^2(x) + sec^2(x)tan^2(x))dx$$

$$\int tan^5(x)dx = \int tan(x)(1 - sec^2(x) + sec^2(x)tan^2(x))dx$$

$$\int tan^5(x)dx = \int (tan(x) - tan(x)sec^2(x) + sec^2(x)tan^3(x))dx$$

$$\int tan^5(x)dx = ln\,|sec(x)| - \tfrac{1}{2}tan^2(x) + \tfrac{1}{4}tan^4(x) + C$$

c)

$$\int cos(5x)sin(3x)dx = \int \tfrac{1}{2}(sin(8x) - sin(2x))dx$$

$$\int cos(5x)sin(3x)dx = -\tfrac{1}{16}cos(8x) + \tfrac{1}{4}cos(2x) + C$$

7.3)

Exercise: Integrate $\int \frac{\sqrt{4-x^2}}{x^2}dx$

$$\text{Let } x = 2sin(\theta), \text{ and } dx = 2cos(\theta)d\theta, \text{ and } \theta = sin^{-1}\left(\frac{x}{2}\right)$$

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \int \frac{\sqrt{4-4sin^2(\theta)}}{(4sin^2(\theta))}(2cos(\theta))d\theta$$

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \int \frac{2\sqrt{1-sin^2(\theta)}}{(4sin^2(\theta))}(2cos(\theta))d\theta$$

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \int \frac{cos(\theta)}{sin^2(\theta)}(cos(\theta))d\theta$$

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \int \frac{cos^2(\theta)}{sin^2(\theta)}d\theta = \int cot^2(\theta)d\theta = \int(csc^2(\theta) - 1)d\theta$$

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \; -cot(\theta) - \theta \; + C = \; -\frac{\sqrt{4-x^2}}{x} - sin^{-1}\left(\frac{x}{2}\right) + C$$

Exercise: Integrate $\int \frac{dx}{\sqrt{x^2-4}}$

$$\text{Let } x = 2sec(\theta), \text{ and } dx = sec(\theta)tan(\theta)d\theta, \text{ and } \theta = sec^{-1}\left(\frac{x}{2}\right)$$

$$\int \frac{2tan(\theta)sec(\theta)d\theta}{\sqrt{4sec^2(\theta)-4}} = \int \frac{2tan(\theta)sec(\theta)d\theta}{2tan(\theta)} = \int sec(\theta)d\theta = ln\,|sec(\theta) + tan(\theta)| \; + \; C$$

$$ln\,|sec(\theta) + tan(\theta)| \; + \; C = ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Exercises: Integrate the following:

a) $\int \frac{x^3}{\sqrt{9x^2+4}}dx$ b) $\int \frac{x}{\sqrt{x^2+4x+5}}dx$

a)

$$\text{Let } x = \frac{2}{3}tan(\theta), \text{ and } dx = \frac{2}{3}sec^2(\theta)d\theta, \text{ and } \theta = tan^{-1}\left(\frac{3x}{2}\right)$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \int \frac{\frac{8}{27}tan^3(\theta)}{\sqrt{4tan^2(\theta)+4}}\left(\frac{2}{3}sec^2(\theta)\right)d\theta$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{16}{81}\int \frac{tan^3(\theta)}{2sec(\theta)}sec^2(\theta)d\theta$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{8}{81}\int tan^3(\theta)sec(\theta)d\theta = \frac{8}{81}\int tan(\theta)\,(sec^2(\theta)-1)\,sec(\theta)d\theta$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{8}{81}\int (sec^2(\theta)-1)\,tan(\theta)sec(\theta)d\theta$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{8}{81}\int (u^2-1)du$$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{8}{81}\left(\frac{u^3}{3}-u\right)+C = \frac{8sec^3(\theta)}{243} - \frac{8}{81}sec(\theta)+C$$

Let $\frac{3x}{2} = tan(\theta)$, so $sec(\theta) = \frac{\sqrt{9x^2+4}}{2}$

$$\int \frac{x^3}{\sqrt{9x^2+4}}dx = \frac{(9x^2+4)^{3/2}}{243} - \frac{4(9x^2+4)^{1/2}}{81}+C$$

b)

Let $x+2 = tan(\theta)$, and $dx = sec^2(\theta)d\theta$, and $\theta = tan^{-1}(x+2)$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \int \frac{x}{\sqrt{(x+2)^2+1}}dx = \int \frac{tan(\theta)-2}{\sqrt{tan^2(\theta)+1}}sec^2(\theta)d\theta$$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \int\left(\frac{tan(\theta)}{sec(\theta)} - \frac{2}{sec(\theta)}\right)sec^2(\theta)d\theta$$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \int\left(\frac{tan(\theta)}{sec(\theta)} - \frac{2}{sec(\theta)}\right)sec^2(\theta)d\theta$$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \int (tan(\theta)sec(\theta) - 2sec(\theta))d\theta$$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = sec(\theta) - 2ln\,|sec(\theta)+tan(\theta)|+C$$

$$\int \frac{x}{\sqrt{x^2+4x+5}}dx = \sqrt{x^2+4x+5} - 2ln\left|\sqrt{x^2+4x+5}+x+2\right|+C$$

7.4) **Exercise**: Integrate $\int \frac{x^3-x^2-3x-1}{x^2-2x-3}\,dx$

$$\frac{x^3-x^2-3x-1}{x^2-2x-3} = x + 1 + \frac{2x+2}{x^2-2x-3} = x + 1 + \frac{2}{x-3}$$

$$\int \frac{x^3-x^2-3x-1}{x^2-2x-3}\,dx = \int \left(x + 1 + \frac{2}{x-3}\right) dx = \frac{1}{2}x^2 + x + 2ln\,|x-3| + C$$

Exercise: Integrate $\int \frac{5x^2-40x-36}{x^3-4x^2-12x}\,dx$

$$x^3 - 4x^2 - 12x = x(x+2)(x-6)$$
$$\frac{5x^2-40x-36}{x^3-4x^2-12x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-6}$$

$$5x^2 - 40x - 36 = A(x+2)(x-6) + Bx(x-6) + Cx(x+2)$$

$$5x^2 - 40x - 36 = A(x^2-4x-12) + B(x^2-6x) + C(x^2+2x)$$

$$5x^2 - 40x - 36 = x^2(A+B+C) + x(-4A-6B+2C) + (-12A)$$

-12A = -36, so A = 3.

$$3 + B + C = 5$$
$$-12 - 6B + 2C = -40$$

$$B + C = 2$$
$$-6B + 2C = -28$$

$$2B + 2C = 4$$
$$6B - 2C = 28$$

8B = 32, so B = 4, and C = -2

$$\int \frac{5x^2-40x-36}{x^3-4x^2-12x}\,dx = \int\left(\frac{3}{x} + \frac{4}{x+2} - \frac{2}{x-6}\right)dx = 3ln\,|x| + 4ln\,|x+2| - 2ln\,|x-6| + C$$

Exercises: Integrate the following:

a) $\int \frac{10x^2-7x+9}{x^3-x^2-x+1}\,dx$ b) $\int \frac{x^3+4x^2+5x+14}{x^4+4x^2-5}\,dx$

a) $x^3 - x^2 - x + 1 = x^2(x-1) - 1(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2$

$$\frac{10x^2-7x+9}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$10x^2 - 7x + 9 = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$$10x^2 - 7x + 9 = A(x^2-2x+1) + B(x^2-1) + C(x+1)$$

$$10x^2 - 7x + 9 = x^2(A + B) + x(-2A + C) + 1(A - B + C)$$

$$10 = A + B$$
$$-7 = -2A + C$$
$$9 = A - B + C$$

Add them all up to get $\quad 12 = 2C$

C = 6. $\quad -7 = -2A + 6$

A = 6.5, and B = 3.5.

$$\int \frac{10x^2-7x+9}{x^3-x^2-x+1} dx = \int \left(\frac{6.5}{x+1} + \frac{3.5}{x-1} + \frac{6}{(x-1)^2} \right) dx = 6.5 ln\,|x+1| + 3.5 ln\,|x-1| - \frac{6}{x-1} + C$$

b) $x^4 + 4x^2 - 5 = (x^2 - 1)(x^2 + 5) = (x - 1)(x + 1)(x^2 + 5)$

$$\frac{x^3+4x^2+5x+14}{x^4+4x^2-5} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+5}$$

$$x^3 + 4x^2 + 5x + 14 = A(x - 1)(x^2 + 5) + B(x + 1)(x^2 + 5) + (Cx + D)(x^2 - 1)$$

$$x^3 + 4x^2 + 5x + 14 = A(x^3 - x^2 + 5x - 5) + B(x^3 + x^2 + 5x + 5) + C(x^3 - x) + D(x^2 - 1)$$

$$x^3 + 4x^2 + 5x + 14 = x^3(A + B + C) + x^2(-A + B + D) + x(5A + 5B - C) + 1(-5A + 5B - D)$$

$$1 = A + B + C$$
$$4 = -A + B + D$$
$$5 = 5A + 5B - C$$
$$14 = -5A + 5B - D$$

Add the first and third equations together and add the second and fourth equations together:

$$6 = 6A + 6B$$
$$18 = -6A + 6B$$

$$1 = A + B$$
$$3 = -A + B$$

B = 2, A = -1, C = 0, and D = 1.

$$\int \frac{x^3+4x^2+5x+14}{x^4+4x^2-5} dx = \int \left(\frac{-1}{x+1} + \frac{2}{x-1} + \frac{1}{x^2+5} \right) dx = -ln\,|x+1| + 2ln\,|x-1| + \frac{1}{\sqrt{5}} tan^{-1}\left(\frac{1}{\sqrt{5}}x\right) + C$$