11.11) Applications of Taylor Polynomials:

-A **Taylor polynomial** of a function f(x) is another name for a partial sum of a Taylor series centered at x=a. If:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

then the nth degree Taylor polynomial of f(x), $T_n(x)$ is the partial sum of the Taylor series up to n:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

-As a side note, the first-degree Taylor polynomial of a function f(x) is simply the linear approximation of f at x=a:

$$T_1(x) = f(a) + f'(a)(x - a) = L(x)$$

-If the first-degree Taylor polynomial can be used to approximate the value of a function near x = a, could the second-degree Taylor polynomial also be used? What about the third? The fourth? The answer to them all is yes. So which is the best approximation of f(x) at x = a?

-You probably see where this is going. We have seen that especially whenever the Taylor series represents the function, the Taylor series is practically equal to the function at x=a and near x=a. Thus, the nth degree Taylor polynomial is a partial sum of the series, it is an estimator at any order, with higher orders being more accurate than lower orders.

-In general, higher degree Taylor polynomials give more accurate approximations of f(x), but how accurate are they? You can guess that the remainder/error between the actual value of the function and the nth degree Taylor polynomial is merely the difference between f(x) and $T_n(x)$:

$$|R_n(x)| = |f(x) - T_n(x)|$$

-Estimating the error can be done in a few ways:

-Use a graph or calculator to compute $|R_n(x)| = |f(x) - T_n(x)|$ and estimate.

-Use the Alternating Series Estimation Theorem if the series is an alternating series.

-Use Taylor's Inequality which states that if $|R_n(x)| \leq M$, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

-The Alternating Series Estimation Theorem only works if the terms alternate in sign. The other two work all the time, though graphs and calculators are not recommended.

Example: a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at a=8.
b) How accurate is this approximation when $7 \leq x \leq 9$?

Solution: a) First we need the first and second derivative of f(x) for the function, and the third derivative for the Taylor Inequality:

$$f(x) = x^{1/3}, \qquad f(8) = 2$$
$$f'(x) = \frac{1}{3}x^{-2/3}, \qquad f'(8) = \frac{1}{12}$$
$$f''(x) = -\frac{2}{9}x^{-5/3} \qquad f''(8) = -\frac{1}{144}$$
$$f'''(x) = \frac{10}{27}x^{-8/3}$$

So we have a Taylor polynomial of

$T_2(x) = \sum_{i=0}^{2} \frac{f^{(i)}(8)}{i!}(x - 8)^i = f(8) + f'(8)(x - 8) + \frac{f''(8)}{2!}(x - 8)^2$, which is:

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2$$

b) This is not an alternating series, so we will use Taylor's Inequality to find a boundary for the error, $R_2(x)$.
First we need a boundary for the third derivative of f(x) on the interval [7, 9]:

$$f'''(x) = \frac{10}{27}x^{-8/3}$$

$x^{-8/3}$ gets larger when x gets smaller due to the negative power, so the biggest value of $f'''(x) = \frac{10}{27}x^{-8/3}$ using a value of x on [7, 9] is $f'''(7) = \frac{10}{27} * 7^{-8/3}$. We can round off $f'''(7)$ to 0.0021, so we have:

$$f'''(x) \leq 0.0021 = M \text{ on } [7, 9]$$

On top of that, the largest that $|x - 8|$ gets on the interval [7, 9] is 1, so $|x - 8|^{n+1} \leq 1$ for any n, including n=2. Therefore, the inequality becomes:

$$\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$
$$\left|R_2(x)\right| \leq \frac{0.0021}{3!}|x - 8|^3 \leq \frac{0.0021}{6} * 1^3 < 0.0004$$

If you plugged in a number between 7 and 9 into $\approx T_2(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2$, you would get a value that is less than 0.0004 away from $\sqrt[3]{x}$.

Exercise: a) Approximate the function $f(x) = \sqrt[4]{x}$ by a Taylor polynomial of degree 3 at a=16.
b) How accurate is this approximation when $15 \leq x \leq 17$?

Example: a) What is the maximum possible error in using the approximation below on the interval $-0.3 \leq x \leq 0.3$?

$$sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

b) Use this approximation to find $sin(12^{\circ})$ correct to six decimal places.
c) For what values of x is this approximation accurate to within 0.00005?

Solution: a) This is a partial sum of the Maclaurin series for sin(x), which you may recall was an alternating series. This means that no matter the value of x, the error of this approximation equation is no more than the whatever the next term in the series would be.

The next term in the Maclaurin series for sin(x), given that we have three in the approximation, would be the fourth term, given below:

$$-\frac{x^7}{7!}$$

What is the largest absolute value this term can have on the interval [-0.3, 0.3]? Plug in 0.3 to see:

$$\left|\frac{(0.3)^7}{7!}\right| = 4.3 * 10^{-8}$$

The farthest that any approximation by the equation will be from the actual value of sin(x) is approximately 0.000000043. That's pretty accurate!

b) To approximate $sin(12^{\circ})$we need to convert the degrees to radians:

$$12^{\circ} = \frac{12\pi}{180} = \frac{\pi}{15} \approx 0.21$$

The input is in the interval [-0.3, 0.3], so we are allowed to approximate using $x - \frac{x^3}{3!} + \frac{x^5}{5!}$. Plug in $\frac{\pi}{15}$ to get:

$$\frac{\pi}{15} - \frac{(\frac{\pi}{15})^3}{3!} + \frac{(\frac{\pi}{15})^5}{5!} \approx 0.20791169$$

Round that to six decimal places and you have that $sin(12^{\circ}) \approx 0.207912$

c) For what value of x will the error be less than 0.00005? We are still using the term $\frac{x^7}{7!}$ to determine the error of the Taylor polynomial, so we need to find what value of x makes $\frac{x^7}{7!}$ less than 0.00005:

$$\left|\frac{x^7}{5040}\right| < 0.00005$$

$$\left|x^7\right| < 0.252$$

$$|x| < \sqrt[7]{0.252} = 0.821$$

As long as x belongs to the interval [-0.821, 0.821], the approximation will be correct to less than 0.00005.

Exercise: a) What is the maximum possible error in using the approximation below on the interval $-0.2 \le x \le 0.2$?

$$cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

b) Use this approximation to find $cos(10^\circ)$ correct to eight decimal places.
c) For what values of x is this approximation accurate to within 0.0000005?

-Taylor's inequality could have been used too. You would have wanted to try using M=1 in the equality since trigonometric functions sin(x) and cos(x) are always less than or equal to 1.