7.3) Trigonometric Substitutions:

-Sometimes trigonometric functions show up in integrals when you least expect them to, and not just before we have seen that $\frac{1}{x^2+1}$ is the derivative of arctangent and other derivative rules like this. Sometimes by using **trigonometric substitution**, that is, replacing an entire variable with a trigonometric function of another variable, we can integrate expressions we may have had trouble with earlier.

-Take for example, $\int_0^2 \sqrt{4-x^2}dx,$ which we solved earlier with geometry to be $\pi,$ but is there a way to integrate this definite integral to get that answer? As it turns out, we can do this with substitution, but specifically a special kind of substitution, where instead of replacing $4-x^2$ with u, we replace x with $2sin(\theta)$.

-Why do this? The logic involves some backwards thinking, where given any expression of the form $\sqrt{a^2-x^2}$, if x were replaced with a trigonometric function, like say $asin(\theta)$ the expression would become $\sqrt{a^2-a^2sin^2(\theta)},$ which by Pythagorean identity, becomes the expression $\sqrt{a^2cos^2(\theta)} = |acos(\theta)|$. By substituting x with an expression that allows us to get a perfect square, the problematic square root operator vanishes, leaving us (hopefully) with an expression we can more easily integrate.

-This procedure is sometimes called **inverse substitution**, since it involves substitution of what appears to be a simple expression with something more complicated. After all, we are replacing x, a single variable, with a composition of another variable, $x(\theta) = asin(\theta),$ which in turn means that for integration purposes, we would need the differential dx to be replaced with $dx = x'(\theta)d\theta$.

-This substitution adds factors more than gets rid of them, and is why this is sometimes called inverse substitution, making an integral that looks simple look more complicated. But remember, the important thing is whether you are capable of integrating it, even if it looks more complicated.

-Looking back at the example, letting $x = 2sin(\theta)$ then leads to the differential $dx = 2cos(\theta)d\theta$. Even the limits can be replaced in terms of $\theta,$ since $2sin(\theta) = 2$ at $\theta = \frac{\pi}{2},$ and equals 0 at $\theta = 0$. If we substitute all this into the integral, we get:

$$\int_0^{\pi/2} \sqrt{4-4sin^2\theta}(2cos(\theta))d\theta = \int_0^{\pi/2} 2\,|cos(\theta)|\,(2cos(\theta))d\theta = 4\int_0^{\pi/2} cos^2(\theta)d\theta$$

-Complicated? Sure. But it can be integrated by hand with substitution, which is not true for $\int_0^2 \sqrt{4-x^2}dx$.

-Where do these **trigonometric substitutions** come from? The Pythagorean identities of course. The following table is worth remembering for making proper substitutions to allow for easier integration when you see certain <u>expressions</u> in an integration. There are some limitations on what the resulting variable θ can equal as a result, so those limitations are mentioned in the table too, as well as the identity the substitution uses:

Expression	Substitution and Limitation:	Identity
$\sqrt{a^2 - x^2}$	$asin(\theta), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - sin^2\theta = cos^2\theta$
$\sqrt{x^2 - a^2}$	$asec(\theta), \quad 0 \leq \theta < \frac{\pi}{2}$ and $\pi \leq \theta < \frac{3\pi}{2}$	$sec^2\theta - 1 = tan^2\theta$
$\sqrt{a^2 + x^2}$	$atan(\theta), \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + tan^2\theta = sec^2\theta$

-Let the constant being combines with x^2 be a clue for what a should be in your substitution.

Example: Integrate $\int \frac{\sqrt{9-x^2}}{x^2} dx$

Solution: Let's substitute x with $3sin(\theta)$, and in turn substitute dx with $3cos(\theta)d\theta$. This makes the integral:

$$\int \frac{\sqrt{9-9sin^2(\theta)}}{9sin^2(\theta)} 3cos(\theta)d\theta$$

$$\int \frac{3cos(\theta)}{9sin^2(\theta)} 3cos(\theta)d\theta$$

$$\int \frac{cos^2(\theta)}{sin^2(\theta)} d\theta = \int cot^2(\theta)d\theta = \int (csc^2\theta - 1)d\theta = -cot(\theta) - \theta + C$$

Only one problem, $-cot(\theta) - \theta + C$ needs to be written in terms of x, so we will need the fact that $3sin(\theta) = x$ to write all this back in terms of x.

$sin(\theta) = \frac{x}{3}$ implies that $\theta = sin^{-1}\left(\frac{x}{3}\right)$. As for what $cot(\theta)$ is in terms of x, you can use a right triangle and some trigonometric ratios, or trigonometric identities:

$$cot(\theta) = \sqrt{csc^2(\theta) - 1} = \sqrt{\left(\frac{3}{x}\right)^2 - 1} = \sqrt{\frac{9-x^2}{x^2}} = \frac{\sqrt{9-x^2}}{x}$$

Therefore, we have: $-\frac{\sqrt{9-x^2}}{x} - sin^{-1}\left(\frac{x}{3}\right) + C$ as our answer in terms of x.

Exercise: Integrate $\int \frac{\sqrt{4-x^2}}{x^2} dx$

-Since definite integrals are intended to find area under a curve, this use of trigonometric substitution can also be used for finding areas of trickier shapes we may not have before:

Example: Find the area enclosed by the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution: If you solve the equation for y, you have the following:

$$\frac{b^2}{a^2}x^2 + y^2 = b^2$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2$$

Rewrite slightly with some factoring to get:

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

If you can find the area under the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ from x = -a to x = a, you will have half of the area enclosed by an ellipse, so let's try integrating and using some trigonometric substitution to do so.

To substitute, we will need to substitute $x = asin(\theta)$ and $dx = acos(\theta)d\theta$. However, the limits have to change too. One of the benefits of the limitations section of the above table is that it helps to figure out what values of θ will make $asin(\theta) = a$ and $asin(\theta) = -a$. Since the limitations on x are $a \leq x \leq -a$, we have $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and can replace the limits with those numbers.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2 - (asin(\theta))^2}\,(acos(\theta))d\theta$$

$$2b\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2sin^2(\theta)}(cos(\theta))d\theta$$

$$2ab\int_0^{\frac{\pi}{2}} \sqrt{1 - sin^2(\theta)}(cos(\theta))d\theta$$

$$2ab\int_0^{\frac{\pi}{2}} cos(\theta)(cos(\theta))d\theta$$

$$2ab\int_0^{\frac{\pi}{2}} cos^2(\theta)d\theta$$

$$ab\int_0^{\frac{\pi}{2}}(1 + cos(2\theta))d\theta = ab(\theta + \tfrac{1}{2}sin(2\theta)\,|_0^{\pi/2} = ab(\tfrac{\pi}{2}) = \frac{ab\pi}{2}$$

If this is half of the area inside the ellipse, that means the total area inside the ellipse is $ab\pi$.

Example: Integrate $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

Solution: This time, the proper substitution will be $x = 2tan(\theta)$ since $x^2 + 4$ is in the radical. However, this also means that $dx = sec^2(\theta)$, and the integral becomes:

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{1}{4tan^2(\theta)\sqrt{4tan^2(\theta)+4}}2sec^2(\theta)d\theta$$

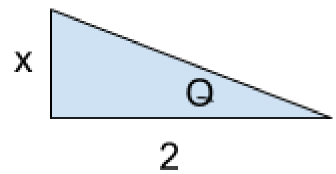
$$\int \frac{1}{4tan^2(\theta)2sec(\theta)}2sec^2(\theta)d\theta$$

$$\frac{1}{4}\int \frac{sec(\theta)}{tan^2(\theta)}d\theta$$

Write everything in terms of sine and cosine and you get:

$$\frac{1}{4}\int \frac{cos(\theta)}{sin^2(\theta)}d\theta = \frac{1}{4}\int cot(\theta)csc(\theta)d\theta \ = \frac{1}{4}(-\,csc(\theta)) + C$$

What does this equal in terms of x? We have that $x = 2tan(\theta)$, which means $\frac{x}{2} = tan(\theta)$. We will use a triangle ratio for this case, just to try something different:



We want to know what $csc(\theta)$ is, which is the ratio of the hypotenuse over the opposite side. The hypotenuse will be $\sqrt{x^2+4}$, so our answer is:

$$-\frac{1}{4}\left(\frac{\sqrt{4+x^2}}{x}\right) + C = \ -\frac{\sqrt{4+x^2}}{4x} \ + C$$

-How about an example involving a sec substitution?

Example: Integrate $\int \frac{dx}{\sqrt{x^2-16}}$

Solution: Let's allow x to be substituted with secant: $x = 4sec(\theta)$, which means $dx = 4tan(\theta)sec(\theta)d\theta$. Therefore, we have:

$$\int \frac{4tan(\theta)sec(\theta)d\theta}{\sqrt{16sec^2(\theta)-16}} = \int \frac{4tan(\theta)sec(\theta)d\theta}{4tan(\theta)} = \int sec(\theta)d\theta = ln\,|sec(\theta) + tan(\theta)| \ + \ C$$

As always though, we need our final answer to be in terms of x, not θ. We have that $x = 4sec(\theta)$, so we also have that $\frac{x}{4} = sec(\theta)$.

For tangent, we will use the identity: $tan^2\theta = sec^2(\theta) - 1 = \frac{x^2}{16} - 1 = \frac{x^2-16}{16}$. Therefore, we have:

$$ln\,|sec(\theta) + tan(\theta)| \ + \ C = ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C$$

Exercise: Integrate $\int \frac{dx}{\sqrt{x^2-4}}$

-However, sometimes you have to be a little more creative with your substitution of the variable.

Example: Evaluate $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$

Solution: In this case, there is a coefficient on x, which means that we need to be a little more careful with our substitution. It might help to rewrite slightly before substituting:

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx$$

Now instead of setting x equal to a trigonometric expression, we will set 2x equal to a trigonometric expression. Specifically let $2x = 3tan(\theta)$. This in turn means $x = \frac{3}{2}tan(\theta)$ and $dx = \frac{3}{2}sec^2(\theta)d\theta$.

What about the limits? At x = 0, $0 = \frac{3}{2}tan(\theta)$, which means $\theta = 0$. At $x = \frac{3\sqrt{3}}{2}$, $\frac{3\sqrt{3}}{2} = \frac{3}{2}tan(\theta)$, or $\sqrt{3} = tan(\theta)$, which means $\theta = \frac{\pi}{3}$. Now we have everything we need to integrate:

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \int_0^{\pi/3} \frac{\frac{27}{8}tan^3(\theta)}{(9tan^2(\theta)+9)^{3/2}}\left(\frac{3}{2}sec^2(\theta)\right)d\theta$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \int_0^{\pi/3} \frac{\frac{27}{8}tan^3(\theta)}{27sec^3(\theta)}\left(\frac{3}{2}sec^2(\theta)\right)d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{tan^3(\theta)}{sec(\theta)}d\theta$$

Let's rewrite in terms of sine and cosine:

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \frac{3}{16}\int_0^{\pi/3} \frac{\frac{sin^3(\theta)}{cos^3(\theta)}}{\frac{1}{cos(\theta)}}d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{sin^3(\theta)}{cos^2(\theta)}d\theta$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \frac{3}{16}\int_0^{\pi/3} \frac{(1-cos^2(\theta))}{cos^2(\theta)}sin(\theta)d\theta$$

Now we have to substitute a second time, this time let $u = cos(\theta)$, which would make $du = -sin(\theta)d\theta$. The limits have to be changed too. At $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$, and at $\theta = 0$, u = 1. So we have:

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \frac{3}{16}\int_1^{1/2} \frac{(1-u^2)}{u^2}sin(\theta)\frac{du}{-sin(\theta)} = -\frac{3}{16}\int_1^{1/2}\left(\frac{1}{u^2}-1\right)du = -\frac{3}{16}\left(\frac{-1}{u}-u\,\Big|_1^{1/2}\right) = -\frac{3}{16}\left(-\frac{5}{2}-(-2)\right)$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+9)^{3/2}}dx = \frac{3}{32}$$

-What many take for granted is how helpful it is to change limits whenever you use integration by substitution. If you attempted the last example without changing the limits, you first would have had to substitute the θ expressions back in, and then after that you have to substitute the x expressions back into those, and <u>then</u> you can plug the original x limits into x. That's a lot of extra evaluation.

Example: Integrate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$

Solution: You have to be a little creative with your substitution. Don't just set x equal to a trigonometric expression, be flexible. For instance, the fact that the expression under the radical is a bit different from the usual substitution expression, so first let's complete the square so it's a bit more like what we are used to:

$$3-2x-x^2 = 4-1-2x-x^2 = 4-(1+2x+x^2) = 4-(x+1)^2$$

Since it's a binomial being squared instead of just x, why not set the binomial equal to a trigonometric expression? We will still use a = 2, but instead:

$$x + 1 = 2sin(\theta), \qquad x = 2sin(\theta) - 1, \qquad dx = 2cos(\theta)d\theta$$

Let's now substitute:

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{x}{\sqrt{4-(x+1)^2}}dx = \int \frac{2sin(\theta) - 1}{\sqrt{4-4sin^2(\theta)}}2cos(\theta)d\theta$$

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{2sin(\theta) - 1}{2cos(\theta)}2cos(\theta)d\theta$$

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int(2sin(\theta) - 1)d\theta = -2cos(\theta) - \theta + C$$

As always, we write in terms of x. We know $x + 1 = 2sin(\theta)$, so $sin(\theta) = \frac{x+1}{2}$, and $\theta = sin^{-1}\left(\frac{x+1}{2}\right)$.

Then there is $cos(\theta)$, where we know from trigonometric identity that $cos(\theta) = \sqrt{1 - sin^2(\theta)}$, so $cos(\theta) = \sqrt{1 - \left(\frac{x+1}{2}\right)^2} = \sqrt{\frac{3-2x-x^2}{4}} = \frac{\sqrt{3-2x-x^2}}{2}$.

Therefore, our final answer is $-\sqrt{3 - 2x - x^2} - sin^{-1}\left(\frac{x+1}{2}\right) + C$

Exercises: Integrate the following:

a) $\int \frac{x^3}{\sqrt{9x^2 + 4}}dx$ b) $\int \frac{x}{\sqrt{x^2+4x+5}}dx$