9.3) Separable Equations:

-What about algebraic procedures for solving differential equations? We don't have a reliable algebraic method for solving any differential equation, but we have methods for certain kinds of differential equations.

-A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored into a product or quotient of a function of only x and a function of only y:

$$\frac{dy}{dx} = f(x) * g(y) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

-The quotient version in differential form in particular can even be rewritten as:

$$dy * g(y) = dx * f(x)$$

-When written in this way, we can integrate both sides of the equation:

$$\int g(y)dy = \int f(x)dx$$

-This implies that y is a function of x. Sometimes we are even capable of solving for y in terms of x, but how would we do it? Considering that we just said y is a function of x, we would therefore be able to differentiate both sides of the integral equation in terms of x:

$$\frac{d}{dx}\int g(y)dy = \frac{d}{dx}\int f(x)dx$$

By the chain rule, this becomes:

$$\left(\frac{d}{dy}\int g(y)dy\right)\frac{dy}{dx} = f(x)$$

The resulting $\frac{ay}{dx}$ would therefore satisfy the original expression.

$$(g(y))\frac{dy}{dx} = f(x)$$
$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

-Whatever expression or function y in terms of x that results from this procedure would be our equation that satisfies the separable differential equation from earlier.

Example: a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$

b) Find the solution of this equation that satisfies the initial condition y(0)=2.

Solution: a) If we rewrite the differential equation in terms of differentials and separate, then we get:

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$
$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

Don't worry about putting an arbitrary constant C in for each integral; we only need one, as having two would result in the two combining together to give you a single arbitrary constant anyway. So therefore the function y in terms of x that satisfies:

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

Is a function that satisfies the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$, but can we go further and solve for what this function y is?

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$y^3 = x^3 + 3C = x^3 + K$$

$$y = \sqrt[3]{x^3 + K}$$

The function $y = \sqrt[3]{x^3 + K}$ where K is any arbitrary constant will satisfy $\frac{dy}{dx} = \frac{x^2}{y^2}$.You can test it out yourself if you aren't sure!

b) So if we want a particular solution that satisfies y(0)=2, then we should plug x = 0 and y = 2 into our family of solutions, from there we can solve for K:

$$y = \sqrt[3]{x^3 + K}$$

$$2 = \sqrt[3]{(0)^3 + K} = \sqrt[3]{K}$$

$$8 = K$$

The particular solution we want is: $y = \sqrt[3]{x^3 + 8}$

-Remember though, sometimes you can't explicitly write your answer as a function in terms of x, and if that happens, your answer will simply be the y that satisfies your expression after integration:

Example: Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos(y)}$

Solution: Let's rewrite this in terms of differentials with a different variable on each side:

$$dy(2y + \cos(y)) = dx(6x^2)$$

Now we integrate:

$$\int (2y + \cos(y))dy = \int (6x^2)dx$$

$$y^2 + sin(y) = 2x^3 + C$$

C is a constant term. Unlike the last example, we cannot solve for y here, so we do not have an explicit solution y in terms of x. We simply have that any y in terms of x that satisfies this equation is a solution to the differential equation. It's like stating how a family of functions is our solution.

-We don't have room here to expression what these families of functions look like when graphed. However I will say that if you did take the time to graph $y = \sqrt[3]{x^3 + K}$ for different K values, you'd see that the family is a series of cube root functions shifted to different units left or right. For the other example however, the curves graphed from $y^2 + sin(y) = 2x^3 + C$ will not necessarily look similar to each other.

Exercises: Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{xsin(x)}{(y-1)^2}$, for $y(0) = 1$

b) $\frac{dy}{dx} = \frac{xe^{x^2+1}}{y}$ for y > 0

Example: Solve the differential equation $y' = x^2 y$

Solution: Remember, a quotient is merely a product of a factor and a fraction, so this can be rewritten in quotient form, and then solved like the other examples:

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{dx} = \frac{x^2}{1/y}$$

$$dy * \frac{1}{y} = dx * x^2$$

We can now integrate the differentials:

$$\int \frac{1}{y} dy = \int x^2 dx$$

$$ln(y) = \frac{x^3}{3} + C$$

We can also solve for y:

$$y = e^{\frac{x^3}{3} + C}$$

$$y = Ae^{\frac{x^3}{3}}$$

Where A is an arbitrary constant coefficient.

-Even the example from last section with the current can be done with separable functions, though you may need to be creative with how the functions are separated:

Example: Suppose that in the simple circuit the resistance is 12 ohms, the inductance is 4 henries, and a battery gives a constant voltage of 60 volts. If we plug these numbers into $L * \frac{dI}{dt} + RI = E(t)$ we get:

$$4 * \frac{dI}{dt} + 12I = 60$$

Find an expression for the current in a circuit when the switch is turned on at t = 0. What is the limiting value of the current?

Solution: First we should solve for $\frac{dI}{dt}$:

$$\frac{dI}{dt} = 15 - 3I$$

Can we rewrite the expression on the right so that it is a function of t (f(t)) divided by a function of I (g(I))? At first it seems the answer is no since there are no t's, but remember. The function can be constant, so let f(t)=1:

$$\frac{dI}{dt} = (1)(15 - 3I)$$

$$\frac{dI}{dt} = \frac{1}{1/(15-3I)}$$

So f(t) = 1, and g(I) = $\frac{1}{(15-3I)}$. Now we can separate, then integrate, then evaluate!

$$\frac{dI}{dt} = \frac{1}{1/(15-3I)}$$

$$dI * \frac{1}{(15-3I)} = dt$$

$$\int dI * \frac{1}{(15-3I)} = \int dt$$

$$-\frac{1}{3} ln|15 - 3I| = t + C$$

$$ln|15 - 3I| = -3t + C$$

$$15 - 3I = e^{-3t+C}$$

$$-3I = Ae^{-3t} - 15$$

$$I = Ae^{-3t} + 5$$

Now we need to solve for A since they did say that the switch is turned on at t = 0, that means I(0)=0. From here we have:

$$I = Ae^{-3t} + 5$$

$$0 = Ae^{-3(0)} + 5$$

$$0 = A + 5$$

The equation therefore i: $I = -5e^{-3t} + 5$. The limiting value is what I approaches as t approaches infinity, so take the limit as t goes to infinity to get:

$$\lim_{n \to \infty} I = \lim_{n \to \infty} -5e^{-3t} + 5 = 0 + 5 = 5$$

As time goes on indefinitely, the current in the circuit will approach 5 amperes.

Exercises: Solve the following differential equations:

a) $\frac{dy}{dx} = \frac{e^{-2y}}{\sqrt{x}}$ b) $\frac{dy}{dx} = sec(x)cot(y)$ c) $\frac{dy}{dx} = 1 + y^2$ for y(0) = 5

-You may have heard the term orthogonal used to describe when two curves meet each other at a 90° angle (don't use the term "perpendicular," since that is only used when two straight lines meet at a 90° angle). So an **orthogonal trajectory** for a family of curves is a curve that intersects all the members of a family of curves orthogonally.

-For the family of curves $x^2 + y^2 = r^2$ for example, concentric circles centered at the origin, any member of the family of straight lines $y = mx$ through the origin is an orthogonal trajectory of the circle family. In fact, since every member of one family is orthogonal to every member of the other, we say the families are orthogonal trajectories of each other.

-So how do you find the orthogonal trajectories of a family of curves?

Example: Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Solution: This family is the set of all parabolas with their vertex at the origin and their axis at the symmetry on the x-axis. The first step should be to find a single differential equation that is satisfied by all members of the family. The way to do this is to differentiate the family of curves equation with respect to x:

$$x = ky^2$$
$$1 = 2ky\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2ky}$$

k is a differentiating factor from one member of the family to another, but in this equation it is a bit of a problem. We need an equation that will be orthogonal to all members of the family, implying that k should be removed so that the value of k does not matter whether our equation is orthogonal or not. We do have that $x = ky^2$, so that also means that $k = \frac{x}{y^2}$, which can be plugged into our differential equation:

$$\frac{dy}{dx} = \frac{1}{2ky}$$
$$\frac{dy}{dx} = \frac{1}{2\left(\frac{x}{y^2}\right)y} = \frac{y}{2x}$$

Solving for $\frac{dy}{dx}$ with the original family of equations gives us a means to find the slope at (x,y) of any member of this family given that it passes through (x,y). However, we want a function that will be orthogonal to these curves all at once, which would imply that we want a curve that has a slope that is the negative reciprocal of this slope of $\frac{y}{2x}$.

Thus, an equation that has a slope of $\frac{dy}{dx} = -\frac{2x}{y}$ will be orthogonal to every member of the family of $x = ky^2$. However, we still have to solve this reciprocal differential equation:

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$ydy = -2xdx$$

$$\int ydy = \int -2xdx$$

$$\frac{1}{2}y^2 = -x^2 + C$$

$$y^2 = -2x^2 + C$$

$$y^2 + 2x^2 = C$$

Any ellipse of the form $y^2 + 2x^2 = C$ will be orthogonal to the family of parabolas.

Exercise: Find the orthogonal trajectories of the family of curves $x = ky^4$, where k is an arbitrary constant.

-Now for a classic differential equations example: the mixing problem. Imagine a tank of fixed capacity contains a thoroughly mixed solution of some substance (no need to be stirred up to get uniform consistency, it already has it!) like salt. A solution of a given concentration enters the tank at a fixed rate and the mixture, still of uniform consistency, leaves at a fixed rate, while it may be of a different rate of how it is being added.

-Let y(t) be the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added to the tank, minus the rate at which the substance is being removed from the tank. (i.e. if more salt is being added to the tank than being removed, the salt water is getting more concentrated, and if less salt is being added to the tank than being removed, the salt water is getting diluted).

-This description is that of a first-order separable differential equation. This application can apply to a number of scenarios, like chemical discharge into the air of a lab, or removal of pollutants from a lake, or injection of drugs into a bloodstream. Any time a receiptical has a substance going in and coming out at the same time, this is technically a mixture problem solved with separable differential equations.

Example: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 liters per minute. The solution going in and the mixture in the tank and draining out are all thoroughly mixed. The solution going in and the mixture draining out are doing so at the same rate, so how much salt will be in the tank after half an hour?

Solution: Let $y(t)$ be the amount of salt (in kg) in the tank after t minutes. We were told that 20 kg of salt were in the tank to begin with, so y(0) = 20, but we want to know what is y(30). We need a differential equation satisfied by y(t), where $\frac{dy}{dt}$ is the rate at which salt is entering or leaving the tank:

$$\frac{dy}{dt} = rate\ of\ salt\ going\ in\ -\ rate\ of\ salt\ going\ out$$

Do we know the rate of the salt going in? We do, since we know the brine going into the tank contains 0.03 kg of salt per liter, and that the water is entering the tank at 25 liters per minute. If we multiply those rates together we get that the rate at which salt enters the tank is:

$$rate\ in\ = 25\ \frac{l}{min} * 0.03\ \frac{kg}{l} = 0.75\ \frac{kg}{min}$$

Do we know the rate of the salt going out? Not explicitly, since unlike the brine going in, the concentration of the mixture leaving the tank is a variable that depends on time. At t = 0, the tank has a concentration found by dividing the 20 kg of salt by the 5000 liters of water in the tank, or:

$$\frac{20\ kg}{5000\ l} = 0.004\ \frac{kg}{l}$$

However, the concentration will be different as time goes on. Still we find the concentration at any point of time t by dividing the amount of salt in the tank at time t, y(t), by the amount of water in the tank at time t. Since the rate at which brine is entering and mixture is leaving the tank are equal, we can be certain that the amount of liquid in the tank will always be 5000 liters. Therefore, the rate at which salt is going out of the tank is:

$$rate\ out\ = 25\ \frac{l}{min} * \frac{y(t)\ kg}{5000\ l} = \frac{1}{200}y(t)\ \frac{kg}{min}$$

Therefore, the rate of change for the salt in general is the difference in the rate going in and the rate going out:

$$\frac{dy}{dt} = 0.75\ \frac{kg}{min} - \frac{1}{200}y(t)\ \frac{kg}{min} = \frac{150-y(t)}{200}\ \frac{kg}{min}$$

$$\frac{dy}{dt} = \frac{150-y(t)}{200}$$

Now that we have a differential equation for y(t), we can now solve the differential equation to find out what y(t) is:

$$\frac{dy}{dt} = \frac{150-y(t)}{200}$$

$$\frac{1}{150-y(t)}dy = \frac{1}{200}dt$$

$$\int\frac{1}{150-y(t)}dy = \int\frac{1}{200}dt$$

$$-\ ln|150 - y(t)| = \frac{1}{200}t + C$$

$$ln|150 - y(t)| = -\frac{1}{200}t + C$$

$$150 - y(t) = e^{-\frac{1}{200}t+C} = Ae^{-\frac{1}{200}t}$$

$$y(t) = 150 - Ae^{-\frac{1}{200}t}$$

Now we have to find out what A is since we need to solve for the solution that satisfies the initial condition y(0)=20.

$$20 = 150 - Ae^{-\frac{1}{200}(0)}$$
$$130 = A$$

Now we know that the amount of salt in the tank at time t is:

$$y(t) = 150 - 130e^{-\frac{1}{200}t}$$

So how much salt is in the tank after 30 minutes? Plug in 30 for t to find out:

$$y(30) = 150 - 130e^{-\frac{1}{200}(30)} = 38.1\ kg$$

Exercise: A lake contains 1 kg of chlorine dissolved in 10000 L of water. Run-off from a factory that contains 0.01 kg of chlorine per liter of water enters the lake at a rate of 50 liters per day. The water in the lake is also draining out into a river, and the mixture going in and the mixture in the lake and draining out are all thoroughly mixed. The solution going in and the mixture draining out are doing so at the same rate, so how much chlorine will be in the lake after 50 days?