11.9) Representations of Functions as Power Series:

-Ever wonder how your calculator knows what the value of the square root of any input you want is? Or what sine of any input you want is? It's not because it has every single one of those values stored inside it, no computer in the world could possibly have the memory space for that.

-It's because these functions and other functions can be written as a geometric series. We have seen this at work already. Consider the power series below:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

-For all x where $|x| < 1$, this power series is a convergent geometric series. We have also seen that for convergent geometric series, the sum of the entire infinite series is where the first term is 1 and the common ratio is x equals:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

-Therefore the function $f(x) = \frac{1}{1-x}$ is the sum of a power series as long as x belongs to the interval of convergence (-1, 1). Likewise, we can say that $\sum_{n=0}^{\infty} x^n$ is a **power series representation** of $f(x) = \frac{1}{1-x}$ on the interval (-1, 1).

-We can also characterize the sum of a series as the limit of the sequence of partial sums $\{s_n(x)\}$, where

$$s_n = 1 + x + x^2 + \ldots + x^n$$

-As n increases, s_n becomes a better approximation for $f(x) = \frac{1}{1-x}$ on the interval (-1, 1). That of course means:

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n$$

-This also means a number of other functions can be expressed as the sum of a power series. However, these functions can only be the sum of a power series over a certain interval of x-values, namely the interval of convergence of the power series they represent.

Example: Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution: In this case, you can simply take the expression $\frac{1}{1-x}$ and replacing x with $-x^2$ and we get $\frac{1}{1+x^2}$:

$$\frac{1}{1-x} = \frac{1}{1-(-x^2)}$$

That means you can also replace x with $-x^2$ in the power series representation for $\frac{1}{1-x}$ to get the power series representation for $\frac{1}{1+x^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x)^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$$

This means that the power series representation of $\frac{1}{1+x^2}$ is $\sum_{n=0}^{\infty}(-1)^n(x)^{2n}$. So what is the interval of convergence?

You could use the Ratio or Root test to determine this, but that is more work than is necessary here. We got the power series by simply replacing x with $-x^2$ in $\sum_{n=0}^{\infty} x^n$, so we can find the interval of convergence with substitution too.

The interval of convergence of $\sum_{n=0}^{\infty} x^n$ is $|x| < 1$, so the interval of convergence of $\sum_{n=0}^{\infty}(-x^2)^n$ is where $\left|-x^2\right| < 1$. This appears to also be the interval (-1, 1). You don't even have to test the endpoints here, but if you did you would see that neither -1 or 1 is in the interval of convergence.

Example: Express $\frac{1}{x+2}$ as the sum of a power series and find the interval of convergence.

Solution: This is another variation on the sum of a power series, but to spot it this time you have to use a little factoring. To make $\frac{1}{2+x}$ look like $\frac{1}{1-x}$ in any way, that 2 has to be turned into a 1. The best way to do that is by factoring a 2 off everything (I repeat, everything) in the denominator:

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2} * \frac{1}{1-(-\frac{x}{2})}$$

So not only do we have to replace x with $-\frac{x}{2}$ in the power series, but we have an extra factor of $\frac{1}{2}$ too:

$$\frac{1}{2} * \frac{1}{1-(-\frac{x}{2})} = \frac{1}{2} * \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{1}{2} * \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n = \frac{1}{2} * \left(1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \ldots\right)$$

The series can be rewritten as $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} * x^n$. While the extra factor of $\frac{1}{2}$ changes the series look, it does not affect the process of finding the interval of convergence, which is why the form that is factored:

$$\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n$$

is usually used for the purposes of finding the interval of convergence.

Instead of $|x| < 1$ being the interval of convergence, we use $\left|\frac{x}{2}\right| < 1$, which means $|x| < 2$ is the interval of convergence. Once again, 2 and -2 don't have to be tested here, but neither is in the interval of convergence if you did test them.

Example: Express $\frac{x^3}{x+2}$ as the sum of a power series and find the interval of convergence.

Solution: This function is quite simply the last example multiplied by an extra factor of x^3, so we can actually just take our power series expansion from the last example and multiply it by x^3.

$$\frac{x^3}{x+2} = x^3\left(\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n\right) = x^3\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}} * x^n\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}} * x^{n+3}$$

The higher power of x does not change anything as far as the interval of convergence is concerned. The interval is still $|x| < 2$. If you want to rewrite the expression so that x is raised to n instead of n+3, you can start the index at n=3 instead. However, all the powers have to adjust as well:

$$\sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{2^{n-2}} * x^n = 4 * \sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{2^n} * x^n$$

Now it becomes more obvious why the interval of convergence is still $|x| < 2$.

Exercise: Express the following as the sum of a power series and find the interval of convergence.

a) $\frac{1}{1+5x}$ b) $\frac{1}{3+2x}$ c) $\frac{1}{1-\sqrt{x}}$

-The benefit of writing a function as a sum of a power series is that it can be much easier to differentiate or integrate a series of power functions than it is the original function. Not always, some of the functions we will see we have known the derivative and integrals for a while, but power functions are generally easier to integrate and differentiate.

-If you can integrate or differentiate a function that is written in power series form, then you are integrating or differentiating term-by-term, which you can imagine is called **term-by-term integration and differentiation**. This will turn the power series into another power series. Even better, the integral/derivative will have the same radius of convergence as the original function does.

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence R>0, then the function f defined by:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and thus continuous) on the interval $(a-R,\ a+R)$ and:

- The derivative equals $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

and

- The integral equals $\int f(x) = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + ... = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radius of convergence for both of these other power series are R.

-One word of warning: The radius of the derivative and integral of a power series function will be the same as the original, but not necessarily the interval of convergence. The endpoints of one interval of convergence may or may not all be a part of the derivative or integral too. You will want to test the endpoints yourself to see if they are in the interval of convergence or not.

-Aside from being an alternative way of finding derivatives and integrals, writing a function as a power series can be a powerful tool in solving a differential equation, and a function expressed as a power series can be a solution to a differential equation.

Example: Express $\left(\frac{1}{1-x}\right)^2$ as a power series. What is its radius of convergence?

Solution: Do not take the expression from earlier and square it. It will be far too difficult to express in that way. So instead, let's recognize that it can be written as $\left(\frac{1}{1-x}\right)^2 = \frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$

We we can start with: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{d}{dx}\left[\frac{1}{(1-x)}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Note the index starts at 1 now, not 0. Therefore, the radius of convergence of the differentiated series is the same as the radius of convergence of the original, $R = 1$. You don't need the interval of convergence here, but if you did, you would find the endpoints do not work, so the interval is (-1, 1).

Example: Find a power series representing the function $ln(1 + x)$ and its radius of convergence:

Solution: You want to make sure you know how the function in question relates to the power series function you want to compareit it with. The derivative of $ln(1 + x)$ is $\frac{1}{1+x}$, and the power series for $\frac{1}{1+x}$ is:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \ldots$$

Therefore the integral would be:

$$\int \frac{1}{1+x}dx = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = \sum_{n=0}^{\infty}\int(-1)^n x^n dx$$

$$ln(1 + x) = \left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}\right) + C$$

It can be reindexed to be:

$$ln(1 + x) = \left(\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}\right) + C$$

It's tempting to say that C can stay any arbitrary constant we want, but remember, we have that this integral is equal to a known function, so we should solve for what C is.

Plug x = 0 into both sides and you have:

$$ln(1+0) = \left(\sum_{n=1}^{\infty} \frac{(-1)^n(0)^n}{n}\right) + C$$

$$ln(1) = C$$

C is 0. Therefore:

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

The radius of convergence is therefore R = 1, just like the original.

Example: Find a power series representing the function $tan^{-1}(x)$ and its radius of convergence:

Solution: We have that the derivative of $tan^{-1}(x)$ is $\frac{1}{1+x^2}$, and we know that this derivative has a power series representation of:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n(x)^{2n}$$

Therefore, the integral of this power series will be the power series representation of $tan^{-1}(x)$:

$$\int \frac{1}{1+x^2}dx = \int \sum_{n=0}^{\infty}(-1)^n(x)^{2n}dx = \sum_{n=0}^{\infty}\int(-1)^n(x)^{2n}dx$$

$$tan^{-1}(x) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n(x)^{2n+1}}{2n+1}\right) + C$$

What is C equal to? Plug x = 0 into each side:

$$tan^{-1}(0) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n(0)^{2n+1}}{2n+1}\right) + C$$

$$0 = \left(\sum_{n=0}^{\infty} 0\right) + C$$

So C = 0. Therefore:

$$tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x)^{2n+1}}{2n+1}$$

The radius of convergence must therefore be 1 since the radius of convergence for $\frac{1}{1+x^2}$ was also 1.

-A fun side note about that last example is that the interval of convergence does actually include both x = -1 and x =1. If you take the time to plug x = 1 into each side you get:

$$tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

However, we know that $tan^{-1}(1) = \frac{\pi}{4}$, so the infinite series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ equals $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$. Not only does this show that this series converges at x = 1 (and x=-1), but that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$ is equal to $tan^{-1}(1) = \frac{\pi}{4}$, and $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - ...$ is equal to $tan^{-1}(-1) = -\frac{\pi}{4}$.

Exercise: Express the following as the sum of a power series and find the interval of convergence.

a) $ln(1-4x)$ b) $tan^{-1}(3x)$

-Several non-geometric series that we have looked at so far can be computed using similar logic and manipulation, including a few that we have seen the sum for without explanation. So sit tight, we'll get back to them!

Example: a) Evaluate $\int \frac{1}{1+x^7} dx$ as a power series.

b) Approximate $\int_0^{0.5} \frac{1}{1+x^7} dx$ correct to within 10^{-7}.

Solution: a) Let's first expand $\frac{1}{1+x^7}$ into a power series:

$$\frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n dx = \sum_{n=0}^{\infty} (-1)^n (x^7)^n dx = \sum_{n=0}^{\infty} (-1)^n (x)^{7n} dx$$

Now we integrate term-by-term:

$$\int \frac{1}{1-(-x^7)} = \int \sum_{n=0}^{\infty} (-1)^n (x)^{7n} dx = \sum_{n=0}^{\infty} \int (-1)^n (x)^{7n} dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \right) + C$$

For once, we don't have to find C, since we are after a literal integral, so the expression we have is our answer.

$$\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \right) + C$$

Its radius of convergence is 1, and the interval of convergence is $-1 < x \leq 1$.

b) To approximate the value of this integral, we cannot simply integrate since we have no method for integrating a function of this type. However, we do know that $\int \frac{1}{1+x^7} dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \right) + C$. So if we use the Fundamental Theorem of Calculus here with any antiderivative/choice of C we want (use C = 0 for simplicity sake), we would have:

$$\int_0^{0.5} \frac{1}{1+x^7} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \Big|_0^{0.5} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{7n+1}}{7n+1}$$

We cannot find the exact value of this as it is not geometric, but we can approximate with a partial sum. How many terms should we use? They want the error to be less than 0.0000001, and since this is an alternating function the error for a partial sum using n terms will be less than whatever the (n+1)th term equals.

So what is the first term in the series that is less than $0.0000001 = 10^{-7}$ (in absolute value)?

$a_0 = \frac{1}{2},\ a_1 = -\frac{1}{8*2^8} = 4.9 * 10^{-4},\qquad a_2 = \frac{1}{15*2^{15}} = 2.0 * 10^{-6},\qquad a_3 = -\frac{1}{22*2^{22}} = 1.1 * 10^{-8},$

You can stop after a_2, but I would throw in one more term, just to be safe since $1.1 * 10^{-8}$ is not that much smaller than 10^{-7}, and the next term is a fair bit smaller:

$$a_4 = \frac{1}{29*2^{29}} = 7.1 * 10^{-11}$$

So our approximation of $\int_0^{0.5} \frac{1}{1+x^7}dx$ is: $a_0 + a_1 + a_2 + a_3 = \frac{1}{2} - \frac{1}{8*2^8} + \frac{1}{15*2^{15}} - \frac{1}{22*2^{22}} = 0.49951374$

Exercise: a) Evaluate $\int \frac{1}{1+x^3}dx$ as a power series.

b) Approximate $\int_0^{0.5} \frac{1}{1+x^3}dx$ correct to within 10^{-9}.