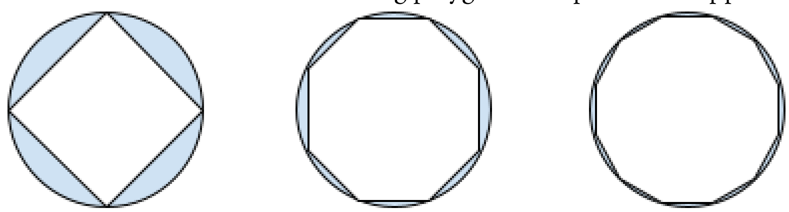
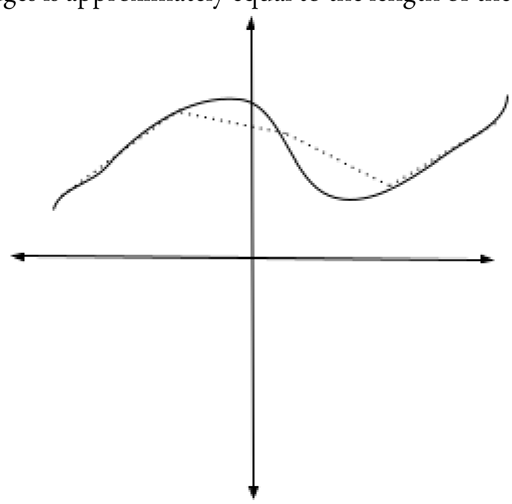
8.1) Arc Length:

-Can you approximate the circumference of a circle using polygons? It's possible to approximate:



-The circumference of a circle is not particularly close to the perimeter of a square inscribed within it. Though the circumference is closer to the perimeter of an octagon inscribed inside, and even closer to the perimeter of a dodecagon inscribed inside.

-The idea in general is that the length of a curve can be approximated by connecting one point on the curve to another point on the curve with a straight line, and then another straight line from there, and so on. The sum of the lengths of all these straight edges is approximately equal to the length of the curve:



-The length of the curve would be more accurately approximated if more straight edges were created and connected from one to another. So when approximating the length of a curve y=f(x) from one end at x = a to another at x = b, divide the interval into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. If we define $f(x_i) = P_i$ where x_i is the right endpoint of the nth subinterval, then let $|P_{i-1}P_i|$ be the length of the edge that connects the points P_{i-1} and P_i, making it the length of the ith edge used in the approximation.

-If L is the length of f(x) from x = a to x = b, then:

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

-If n approaches infinity, this approximation becomes the actual length of the curve:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \left|P_{i-1}P_i\right|$$

-Let $f(x_i) = y_i$, so $\left|P_{i-1}P_i\right|$ would be equal to $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. This can be rewritten even further using the Mean Value Theorem, which states that on the interval $[x_{i-1}, x_i]$ there is a value x_i^* at which $f'(x_i^*)(x_i - x_{i-1}) = f(x_i) - f(x_{i-1}) = y_i - y_{i-1}$.

-If we plug this expression $f'(x_i^*)(x_i - x_{i-1})$ in for $y_i - y_{i-1}$ in the distance formula $\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$, we would get

$$\sqrt{(x_i - x_{i-1})^2 + (f'(x_i^*)(x_i - x_{i-1}))^2} = (x_i - x_{i-1})\sqrt{1 + f'(x_i^*)^2}$$

-Recall however, that $(x_i - x_{i-1})$ is equal to Δx, so that makes this expression $\Delta x\sqrt{1 + f'(x_i^*)^2}$, and so the summation for the approximation of the length of the arc from a to b becomes:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x\sqrt{1 + f'(x_i^*)^2}$$

-As is always the case in this course when we take an infinite summation with a x_i^* and Δx, this is in fact a Riemann sum, and an integral:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x\sqrt{1 + f'(x_i^*)^2} = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

-This gives us a means of finding the length of an arc from point a to point b on the function y=f(x), as long as $g(x) = \sqrt{1 + f'(x)^2}$ is continuous.

The Arc Length Formula:
If $f'(x)$ is continuous on [a,b], then the length of the curve y = f(x) on [a,b] is:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

or, by Leibniz d Notation:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

-The Leibniz d Notation version can be very helpful for non-function equations.

Example: Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

Solution: While it is tempting to derive the equation $y^2 = x^3$ implicitly, it would be best to start by solving for y and then derivative with respect to x:

$$y = \pm x^{1.5}$$

We will toss the negative version since the interval is upon positive values of x and y only:

$$y = x^{1.5}$$
$$\frac{dy}{dx} = 1.5x^{0.5}$$

Now we can integrate the arc length formula from x = 1 to x = 4:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Some integration by substitution will help here. Let $u = 1 + \frac{9}{4}x, du = \frac{9}{4}dx$, and so the limits are now from u = $\frac{13}{4}$ to 10.

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx \int_{13/4}^{10} \sqrt{u}(\frac{4}{9})du = \frac{4}{9}(\frac{2}{3}u^{1.5})\Big|_{13/4}^{10} = \frac{8}{27}(10\sqrt{10} - \frac{13\sqrt{13}}{8})$$

-One of the reasons why we consider Leibniz d Notation in our arc length formula is because depending on the equation or curve being used, it might even be advantageous for x and y to interchange: integrate with respect to y, use the derivative of x with respect to y, and use limits from one value, y = c, to another value, y = d:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

-The procedures and logic are fairly easy to translate from one example to another:

Example: Find the length of the arc on the equation $y^2 = x$ from (0,0) to (1,1).

Solution: If you try solving for y you would have $y = \sqrt{x}$ which has a derivative of $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, and plugging this into the arc length formula will lead to a fairly difficult integral. So instead, let's try the arc length formula in terms of y instead.

We will need the derivative of x in terms of y instead, which will be $\frac{dx}{dy} = 2y$. Plug this value into the arc length function and integrate from y = 0 to y = 1 (not too hard when the x and y values are all equal, right?):

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + (2y)^2}\, dy$$

Some trigonometric substitution will help greatly. Let $2y = tan(\theta)$, so $dy = sec^2(\theta)$. The limits can change also. Plug y = 0 into $2y = tan(\theta)$ to get $\theta = 0$, and plug y =1 in to get $\theta = tan^{-1}(2)$. Let $tan^{-1}(2) = \delta$ for brevity:

$$L = \int_0^1 \sqrt{1 + 4y^2}\,dy = \int_0^{\delta} \sqrt{1 + tan^2\theta}\left(\frac{1}{2}sec^2(\theta)\right)d\theta = \frac{1}{2}\int_0^{\delta}\left(sec^3(\theta)\right)d\theta$$

To integrate, we will do so by parts where $u = sec(\theta)$ and $dv = sec^2(\theta)$. So $du = sec(\theta)tan(\theta)$ and $v = tan(\theta)$

$$\int_0^{\delta}\left(sec^3(\theta)\right)d\theta = sec(\theta)tan(\theta) - \int_0^{\delta} sec(\theta)tan^2(\theta)d\theta = sec(\theta)tan(\theta) - \int_0^{\delta} sec(\theta)d\theta - \int_0^{\delta} sec^3(\theta)d\theta$$

Combine the like integrals and divide by 2 to get:

$$\int_0^{\delta}\left(sec^3(\theta)\right)d\theta = \frac{1}{2}\left(sec(\theta)tan(\theta) - \int_0^{\delta} sec(\theta)d\theta\right) = \frac{1}{2}(sec(\theta)tan(\theta) - ln|sec(\theta) + tan(\theta)|)$$

So back to the original problem:

$$L = \frac{1}{2}\int_0^{\delta}\left(sec^3(\theta)\right)d\theta = \frac{1}{4}(sec(\theta)tan(\theta) - ln|sec(\theta)|)|\,_0^{\delta} = \frac{1}{4}(sec(\delta)tan(\delta) - ln|sec(\delta) + tan(\delta)|) - 0$$

Remember, $tan^{-1}(2) = \delta$, so $tan(\delta) = 2$, and by $tan^2(\delta) + 1 = sec^2(\delta)$, we have $sec(\delta) = \sqrt{5}$.

$$L = \frac{1}{4}(sec(\delta)tan(\delta) - ln|sec(\delta) + tan(\delta)|) = \frac{1}{4}((\sqrt{5})(2) - ln\left|\sqrt{5} + 2\right|) = \frac{\sqrt{5}}{2} - \frac{ln(\sqrt{5}+2)}{4}$$

-Sometimes though, you may still not be able to integrate no matter how you set the integral up:

Example: a) Set up an integral the calculates the length of the hyperbola xy=1 from (1, 1) to $(2, \frac{1}{2})$.
b) Use Simpson's Rule for n = 10 to approximate the length of the arc.

Solution: a) The red flags should be going up already since we are only asking you to set up the integral, not find the arc length. More red flags will arise as we find our expression of y in terms of x (no, setting up the integral for x in terms of y is no easier).

If xy=1, then $y = \frac{1}{x}$, so $\frac{dy}{dx} = -\frac{1}{x^2}$.The integral will therefore be:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\,dx$$

This is as far as we can go, since we have no method that will allow us to integrate this expression.

b) So let's not bother integrating, let's try estimating! We are using n = 10 subintervals, so $\Delta x = \frac{2-1}{10} = 0.1$.
This and the endpoints will be: $x_0 = 1,\ x_1 = 1.1,\ \ldots,\ x_9 = 1.9,\ x_{10} = 2$.

Remember, the first and last endpoints will be plugged into $f(x) = \sqrt{1 + \frac{1}{x^4}}$, the rest will alternate from 4 times the output, and 2 times the output:

$L \approx \frac{0.1}{3}(f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.$

Plug in each and compute, and you will get $L \approx 1.1321$

Exercise: a) Find the length of the arc of the parabola $y = 2x^2$ between the points (0,0) and (2,8).
b) Find the length of $36y^2 = (x^2 - 4)^3$ from x = 2 to x = 3, and where $y \geq 0$.

-Borrowing a bit of strategy from the Fundamental Theorem of Calculus, you could create an arc length function the same way we had an area function: starting at point t = a and ending at t = x, the length of the function curve for f(t) can be computed by:

$$s(x) = \int_a^x \sqrt{1 + f'(t)^2}\,dt$$

-By the Fundamental Theorem of Calculus, we can also differentiate to get:

$$s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

-Rewritten in differential form, this equation can be written in two different ways:

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{(dx)^2 + (dy)^2} \qquad \text{or} \qquad ds^2 = dx^2 + dy^2$$

-Had the integral been integrated in terms of y instead, you could also use $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$. These are all mnemonic devices used to remember the arc length formula. They borrow from the Pythagorean Theorem, but instead of $s^2 = x^2 + y^2$, it's the differentials. You will have to connect the relationship back to the integral from there, but many do use these formulas to help with their memory of arc length.

-We also still have the arc length function if we wish to use it: $s(x) = \int_a^x \sqrt{1 + f'(t)^2}\,dt$

Example: Find the arc length function for the curve $y = x^2 - \frac{1}{8}ln(x)$ by letting $P_0 = (1, 1)$ be the starting point.

Solution: We have that $f(x) = x^2 - \frac{1}{8}ln(x)$ is our function for the integral, so let a = 1. The function is:

$$s(x) = \int_1^x \sqrt{1 + (2t - \frac{1}{8t})^2}dt = \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}}dt = \int_1^x \sqrt{4t^2 + \frac{1}{2} + \frac{1}{64t^2}}dt = \int_1^x \sqrt{(2t + \frac{1}{8t})^2}dt = \int_1^x (2$$

This can actually be integrated:

$$s(x) = \int_1^x (2t + \frac{1}{8t})dt = t^2 + \frac{1}{8}ln(t) \Big|_1^x = x^2 + \frac{1}{8}ln(x) - 1$$

-Not only do we have an arc length integral, we have a simplified arc length function that we can plug any value of x greater than 1 into to find the length along the curve of $y = x^2 - \frac{1}{8}ln(x)$ starting at x = 1 and ending at the input of our choice. If you let x = e for example, you would have that the arc length would be: $e^2 - \frac{7}{8}$.

Exercise: a) Find the arc length function for the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$ by letting $P_0 = (1, \frac{5}{12})$ be the starting point.

b) Use the function to find the arc length from $(1, \frac{5}{12})$ to $(\frac{9}{4}, \frac{2251}{576})$