11.10) Taylor Series and Maclaurin Series:

Exercise: For the function $f(x) = e^{2x}$, find the Maclaurin series and its radius of convergence.

$$f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)(x) + \frac{f''(0)}{2}(x)^2 + \frac{f'''(0)}{6}(x)^3 +... \qquad \text{where } |x| < 1$$

$$f(0) = \sum_{n=0}^{\infty} \frac{(2)^n}{n!}(x)^n = 1 + 2x + 4x^2 + \frac{4}{3}x^3 +...$$

$\lim_{n\to\infty} \left|\frac{2}{n+1}x\right| = 0$ for all x, radius of convergence is infinite.

Exercise: Find the Taylor series for f(x)=sin(x) centered at $x = \frac{\pi}{4}$

$$f(\tfrac{\pi}{4}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{4})}{n!}(x - \tfrac{\pi}{4})^n = f(\tfrac{\pi}{4}) + f'(\tfrac{\pi}{4})(x - \tfrac{\pi}{4}) + \frac{f''(\frac{\pi}{4})}{2}(x - \tfrac{\pi}{4})^2 + \frac{f'''(\frac{\pi}{4})}{6}(x - \tfrac{\pi}{4})^3 +...$$

$$f(\tfrac{\pi}{4}) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2}(-1)^n\left(\frac{(x-\frac{\pi}{4})^{2n}}{(2n)!} + \frac{(x-\frac{\pi}{4})^{2n+1}}{(2n+1)!}\right) = \frac{\sqrt{2}}{2}\left(1 + (x - \tfrac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{2} - \frac{(x-\frac{\pi}{4})^3}{6} + \frac{(x-\frac{\pi}{4})^4}{24}...\right)$$

Exercise: For the function $f(x) = \frac{1}{\sqrt{1-x}}$, find the Mclaurin series and its radius of convergence.

$$f(0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n)!n!}(x)^n = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{15}{48}x^3...$$

$\lim_{n\to\infty} \left|\frac{(2n+2)*(2n+1)}{2^2(n+1)(n+1)} * x\right| = \lim_{n\to\infty} \left|\frac{4n^2+6n+2}{4n^2+8n+4} * x\right| = |x| < 1$ Radius of convergence = 1.

Exercise: a) Evaluate $\int e^{-x^3} dx$ as an infinite series.

b) Evaluate $\int_0^1 e^{-x^3} dx$ correct to within an error of 0.00001.

a) $e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6}....$

$$\int e^{-x^3} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{n!(3n+1)} + C$$

b) $\int_0^1 e^{-x^3} dx = \sum_{n=0}^{6} \frac{(-1)^n}{n!(3n+1)} = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \frac{1}{1920} + \frac{1}{13680} = 0.80752$

11.11) Applications of Taylor Polynomials:

Exercise: a) Approximate the function $f(x) = \sqrt[4]{x}$ by a Taylor polynomial of degree 3 at a=16.
b) How accurate is this approximation when $15 \leq x \leq 17$?

$$f(16) = \sum_{n=0}^{3} \frac{f^{(n)}(16)}{n!}(x-16)^n = f(16) + f'(16)(x-16) + \frac{f''(16)}{2}(x-16)^2 + \frac{f'''(16)}{6}(x-16)^3$$

$$f(16) = 2 + \frac{1}{32}(x-16) - \frac{3}{4096}(x-16)^2 + \frac{21}{786432}(x-16)^3$$

b)

$$f^{(4)}(x) = \frac{231}{256}x^{-15/4}$$

$$f^{(4)}(x) < \frac{231}{256}(15)^{-15/4}$$

$$f^{(4)}(x) < 0.000035078 = M$$

$$\left|R_3(x)\right| < \frac{M}{4!}|x-16|^4 < \frac{0.000035078}{24} * (1)^4 = 0.000001462$$

The error will be no larger than 0.000001462

Exercise: a) What is the maximum possible error in using the approximation below on the interval $-0.2 \leq x \leq 0.2$?

$$cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

b) Use this approximation to find $cos(10^\circ)$ correct to eight decimal places.
c) For what values of x is this approximation accurate to within 0.0000005?

a) Maximum Error = $\frac{(0.2)^8}{8!} = 0.0000000000635$

b) $cos(\frac{\pi}{18}) \approx 1 - \frac{(\frac{\pi}{18})^2}{2!} + \frac{(\frac{\pi}{18})^4}{4!} - \frac{(\frac{\pi}{18})^6}{6!} = 0.98480775$

c)

$$\left|\frac{x^8}{8!}\right| < 0.0000005$$

$$\left|x^8\right| < 0.02016$$

$$-0.61384866 < x < 0.61384866$$