7.8) Improper Integrals:

-What is the area above the x-axis and under the curve $f(x) = \frac{1}{x^2}$ to the right of x = 1? This is not as straightforward to determine as you may think, considering that the area that we are talking about here is not enclosed; it goes on forever. So does that mean the area is infinite as well?

-The way to determine this is to consider a function that can determine the area under the curve f(x) starting at x=1 and ending at x = t, and then see what happens as t gets infinitely large. Maybe this area function will grow infinitely large as well, or maybe it will not.

-The region (call it S) under the curve $f(x) = \frac{1}{x^2}$ starting at x = 1 and ending at x = t can be defined as A(t), where:

$$A(t) = \int_1^t \frac{1}{x^2}dx$$

This can be evaluated easily enough: $A(t) = \int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1$

-From here, we can see that if we take the limit of A(t) as t goes to infinity, the limit is not in fact infinite:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} -\frac{1}{t} + 1 = 1$$

-Despite being a region that is not enclosed and goes on infinitely, the area of S is not infinite. This example shows that it is in fact possible to find definite integrals over infinite intervals. A definite integral over a region that is infinite in length (though not necessarily infinite in area) is called an **improper integral (of type I)**. It can be defined using ∞ as an upper or lower limit, but it is not recommended:

Improper Integrals of Type I:

-If $\int_a^t f(x)dx)$ exists for every number t $\geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx)$$

provided that this limit exists (as a finite number).

-If $\int_t^b f(x)dx)$ exists for every number t $\leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to -\infty}\int_t^b f(x)dx)$$

provided that this limit exists (as a finite number)

-These improper integrals, $\int_a^\infty f(x)dx$, and $\int_{-\infty}^b f(x)dx$, are called **convergent** if their limits exist and **divergent** if they do not. In other words, if these integrals are finite, they are convergent, if infinite, they are divergent.

-If $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are both convergent, then we can say that $\int_{-\infty}^{\infty} f(x)dx = \int_a^{\infty} f(x)dx + \int_{-\infty}^{a} f(x)dx$. In fact, you can use any a you want for the "separation" of integrals in this.

-These improper integrals can be interpreted as areas as long as f is a positive function, and as long as the integral is convergent, you can even let the area of S = $\{(x,y)|\ x \geq a,\ 0 \leq y \leq f(x)\}$ be defined as:

$$A(S) = \int_a^{\infty} f(x)dx$$

-Areas are finite positive numbers, so if the integral is divergent, I would not associate this with an area function.

Example: Is $\int_1^{\infty} \frac{1}{x}dx$ convergent or divergent? Show why or why not.

Solution: Let's find out by taking the limit of $\int_1^{t} \frac{1}{x}dx$ as t goes to infinity:

$$\lim_{t\to\infty}\int_1^{t} \frac{1}{x}dx = \lim_{t\to\infty} ln\,|x|\ \big|_1^t = \lim_{t\to\infty} ln(t)$$

This limit is infinite and does not exist, which means the integral is divergent.

-Don't be fooled by how similar the functions are. Just because they are both rational and they both approach 0 as the input goes to infinity, that doesn't mean their integrals behave the same way.

Exercise: Determine if the following integrals are convergent or divergent:

a) $\int_1^{\infty} \sqrt{\frac{1}{x}}dx$ b) $\int_{-\infty}^{-1} \frac{1}{x^3}dx$

Example: Is $\int_{-\infty}^{0} xe^x dx$ convergent or divergent? Show why or why not.

Solution: You may have guessed already, but the expression in the integral does in fact go to zero as x gets infinitely large in the negative direction. Don't believe it? See for yourself with help from L'Hospital's Rule:

$$\lim_{t\to -\infty} te^t = \lim_{t\to -\infty} \frac{t}{e^{-t}} = \lim_{t\to -\infty} \frac{1}{-e^{-t}} = 0$$

You may have also guessed that this does <u>not</u> necessarily mean that the integral is convergent. We just saw two examples where the integrand went to zero over the interval, but the integral was convergent for one and divergent for the other.

We have to determine this using a limit of an integral, just like the last few examples. Unlike the last few examples though, you will need to integrate by parts (I suggest letting x = u and $dv = e^x$):

$$\int_{-\infty}^{0} xe^x dx = \lim_{t \to -\infty} \int_{t}^{0} xe^x dx = \lim_{t \to -\infty} \left(xe^x \Big|_t^0 - \int_t^0 e^x dx \right) = \lim_{t \to -\infty} (xe^x - e^x) \Big|_t^0 = \lim_{t \to -\infty} -1 - te^t + e^t$$

As t goes to negative infinity, both te^t and e^t go to zero. So therefore the integral goes to -1, which is finite (though not positive) so this is a convergent integral.

Example: Is $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ convergent or divergent? Show why or why not.

Solution: It might seem like circular logic here, but what we can do is separate this integral into two improper integrals and evaluate the limits of each. If they are both finite, then this integral is convergent. If either is not, this integral is divergent. You can choose any value you wish to break the integral off into two, but we'll keep it simple and use a = 0:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx = \left(\lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} dx \right) + \left(\lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} dx \right)$$

$$= \left(\lim_{t \to -\infty} tan^{-1}(x) \Big|_t^0 \right) + \left(\lim_{t \to \infty} tan^{-1}(x) \Big|_0^t \right) = \left(\lim_{t \to -\infty} 0 - tan^{-1}(t) \right) + \left(\lim_{t \to \infty} tan^{-1}(t) - 0 \right) = \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)$$

The total integral value is π, which is finite, so this is a convergent integral.

-You've probably noticed that many of the examples we have seen are rational functions. This begs the question: What does p need to be in order for $\int_{1}^{\infty} \frac{1}{x^p} dx$ to be convergent? We have seen that when p = 1, the integral is divergent, and when p = 2 the integral is convergent. Can we find a definite cutoff point?

-Presuming that p is not 1, we can integrate $\int_{1}^{\infty} \frac{1}{x^p} dx$ using the power rule, so let's see what this integral becomes in general when $p \neq 1$:

$$\int_{1}^{\infty} \frac{1}{x^p} dx = \lim_{t \to \infty} \int_{1}^{t} x^{-p} dx = \lim_{t \to \infty} \left(\frac{x^{-p+1}}{-p+1}\right) \Big|_1^t = \lim_{t \to \infty} \left(\frac{t^{-p+1}}{1-p} - \frac{1}{1-p}\right)$$

-We did say that p was not 1, so no worries about the denominators being equal to zero, but let's first consider if p was greater than 1. If p is greater than 1, then 1-p is less than 0, so if $t \to \infty$, $t^{1-p} \to 0$.

-Therefore, if p is greater than 1, $\lim_{t \to \infty} \left(\frac{t^{-p+1}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}$, which is finite, meaning $\int_{1}^{\infty} \frac{1}{x^p} dx$ is convergent.

-If p is less than 1 however, then 1-p is greater than 0, so if $t \to \infty$, $t^{1-p} \to \infty$.

-Therefore, if p is less than 1, $\lim_{t \to \infty} \left(\frac{t^{-p+1}}{1-p} - \frac{1}{1-p}\right)$ is infinite, meaning $\int_{1}^{\infty} \frac{1}{x^p} dx$ is divergent.

-This gives us one general convergence/divergence rule:

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ is convergent if } p > 1\text{, and divergent if } p \leq 1.$$

-We have not yet talked about another type of improper integral that deals with another kind of infinity in its limits. What if instead of one or more of the limits approaching infinity, what if the function itself approached infinity in some way over its definite integral interval?

-Suppose f(x) is defined on [a, b), but has a vertical asymptote as x approaches b: $\lim_{x\to b^-} f(x) = \infty$. If we defined another area function A(t) on [a, b) such that:

$$A(t) = \int_a^t f(x)dx$$

-Can this integral be finite as t approaches b from the left? If it is, then you could say the area of the region S in this case is finite and you can write:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

-This is called an **improper integral (of type II)**. Unlike type I improper integrals, there is no restriction or worry about redefining if f is not a positive function, and no matter what discontinuity f has at b, we can call it an improper integral of type II.

Improper Integrals of Type II:
-If f is continuous on [a, b) and is discontinuous at b, then:

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

if this limit exists (is finite)

-If f is continuous on (a, b] and is discontinuous at a, then:

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

if this limit exists (is finite)

-Once again, if these improper integrals are finite, the integral is **convergent**, and if the limit of the improper integrals does not exist, the integral is **divergent**. So if f has a discontinuity at x = c, but is defined everywhere else on [a, b], then if $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are both convergent, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example: Is $\int_2^6 \frac{1}{\sqrt{x-2}}dx$ convergent or divergent? Show why or why not.

Solution: There is a vertical asymptote at x = 2, so we have a discontinuity at x = 2, so let's integrate with a one-sided limit as t approaches 2 from the right:

$$\int_2^6 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+}\int_t^6 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+} (2\sqrt{x-2})|_2^6 = \lim_{t\to 2^+} (2\sqrt{6-2} - 2\sqrt{t-2}) = 2(2) - 0 = 4$$

This is a convergent integral.

Exercise: Is $\int_3^5 \frac{x}{\sqrt{x^2-9}}dx$ convergent or divergent? Show why or why not.

Example: Is $\int_0^{\pi/2} sec(x)dx$ convergent or divergent? Show why or why not.

Solution: You may recall there is a vertical asymptote on the graph of sec(x) at $\pi/2$, and even better, when x approaches $\pi/2$ from the left, sec(x) goes to positive infinity. So we will need to use a limit:

$$\int_0^{\pi/2} sec(x)dx = \lim_{t\to\pi/2^-}\int_0^t sec(x)dx = \lim_{t\to\pi/2^-} ln|sec(x) + tan(x)|\,|_0^t = \lim_{t\to\pi/2^-} ln|sec(t) + tan(t)| - ln(1).$$

As t approaches $\pi/2$, from the left, both sec(t) and tan(t) approach positive infinity. Therefore, $\lim_{t\to\pi/2^-} ln|sec(t) + tan(t)|$ goes to infinity. This integral is divergent.

Example: Is $\int_0^3 \frac{1}{x-1}dx$ convergent or divergent? Show why or why not.

Solution: There is a vertical asymptote at x = 1. As x approaches one from the left, $\frac{1}{x-1}$ goes to negative infinity, but as x approaches one from the right, $\frac{1}{x-1}$ goes to positive infinity. We may want to remember <u>both</u> of those for later.

In this case, we will attempt to break this integral in two pieces and take the limit of both as x approaches 1:

$$\int_0^3 \frac{1}{x-1}dx = \int_0^1 \frac{1}{x-1}dx + \int_1^3 \frac{1}{x-1}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}dx + \lim_{t\to 1^+}\int_t^3 \frac{1}{x-1}dx = \lim_{t\to 1^-} ln|t-1|\,|_0^t + \lim_{t\to 1^+} ln|t-1|\,|_t^3$$

The first limit as t approaches one from the left goes to $-\infty$, so that alone is enough to say the integral will not be a definite finite number overall, so you don't have to take the other limit, this integral is divergent.

-As a side note to that last example, many will argue that the other limit also approaches negative infinity which means one negative infinity limit minus another negative infinity limit should equal zero. This is incorrect logic.

-Remember, ∞ is not a number. You cannot arithmetically combine ∞ with another ∞ the way you combine two numbers together. At the least, this expression would be indeterminate, which is still not enough to say the integral is convergent. Remember, the only way the integral can be convergent is if **both** pieces are convergent. If one half is divergent, so is the whole, no matter what is happening to the other half.

Exercise: Is $\int_0^{\pi/2} \frac{sin(x)}{\sqrt{cos(x)}}dx$ convergent or divergent? Show why or why not.

Example: Is $\int_0^1 ln(x)dx$ convergent or divergent? Show why or why not.

Solution: ln(x) has a vertical asymptote at 0, and as x approaches 0 from the right, ln(x) goes to negative infinity. Let's integrate:

$$\int_0^1 ln(x)dx = \lim_{t\to 0^+}\int_t^1 ln(x)dx = \lim_{t\to 0^+}(xln(x) - x)\Big|_t^1 = \lim_{t\to 0^+}(1ln(1) - 1 \; - (tln(t) - t)) = \lim_{t\to 0^+}(t - 1 - tln(t))$$

What is the limit of t(ln(t)) as t approaches 0 from the right? This is indeterminate, but can be found using L'Hospital's Rule:

$$\lim_{t\to 0^+} tln(t) = \lim_{t\to 0^+}\frac{ln(t)}{\frac{1}{t}} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to 0^+} \; -t = 0$$

Since t(ln(t)) goes to zero, (t - 1 - t(ln(t))) goes to 0 - 1 - 0 = -1. This is a convergent integral.

-Of course, sometimes you don't need to go fully through the limit procedure to determine if an improper integral (of either type) is convergent or divergent. Sometimes some keen comparison skills can tell you everything.

-We saw a little bit of this already with the $\frac{1}{x^p}$ being convergent if p > 1 and divergent when p $\leq$ 1. We saw already that $\int_1^\infty \frac{1}{x}dx$ is divergent, which in this case means the integral is infinite. However compare that with $\int_1^\infty \frac{1}{\sqrt[3]{x}}dx$, which is a larger function for all x greater than or equal to 1.

-This would imply that the area under the curve from starting at x = 1 and going to the right is greater than the area under $\frac{1}{x}$ starting at x = 1 and going to the right. That would imply that $\int_1^\infty \frac{1}{\sqrt[3]{x}}dx$ is bigger than $\int_1^\infty \frac{1}{x}dx$, which was divergent. Without any computation, I can safely say that $\int_1^\infty \frac{1}{\sqrt[3]{x}}dx$ is divergent too.

-The definition uses type I improper integrals, but the rule holds for type II improper integrals too:

The Comparison Theorem:
Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then:

1) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent.

2) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty f(x)dx$ is also convergent.

-Remember the rules carefully, and do not use any combination not listed. All the comparison theorem states is that if one function is larger than the other function on the interval in question, then if the **larger** function has a convergent integral, then the smaller function integral is also convergent, and if the smaller function has a divergent integral, then the larger function integral is also divergent.

-If f(x) is larger and has a divergent integral, then we know **NOTHING** about the integral of g(x). It could be convergent or divergent, we don't know. The same applies if g(x) is smaller and has a convergent integral; we do not know what is true about f(x).

Example: Is $\int_0^\infty e^{-x^2}dx$ convergent or divergent? Show why or why not.

Solution: The limit is infinite so this definitely calls for an improper integral. Though you could take the time to break this down into two pieces first:

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

Let's focus on the second integral. For all x greater than or equal to 1, e^{-x^2} is smaller than e^{-x}, so $\int_1^\infty e^{-x^2}dx$ is smaller than $\int_1^\infty e^{-x}dx$. Let's try finding out whether $\int_1^\infty e^{-x}dx$ is convergent or not:

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}(-e^{-t}+e^{-1}) = e^{-1}$$

Therefore, we know that $\int_1^\infty e^{-x}dx$ is convergent. However, since this integral is convergent and e^{-x^2} is smaller than e^{-x}, we also know that $\int_1^\infty e^{-x^2}dx$ is convergent. (Remember, if $\int_1^\infty e^{-x}dx$ turned out to be divergent, you would know nothing about $\int_1^\infty e^{-x^2}dx$, and would have to try something different).

So if $\int_1^\infty e^{-x^2}dx$ is convergent, is $\int_0^\infty e^{-x^2}dx$ convergent too? Well, think of it this way. We know $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$, where $\int_0^1 e^{-x^2}dx$ is convergent since $\int_0^1 e^{-x^2}dx$ is a continuous function on a finite interval, and we just saw that $\int_1^\infty e^{-x^2}dx$ is also convergent.

When an improper integral is convergent, it is finite, and vice-versa, so if these two smaller integrals are finite, the sum of these two must also be finite, so $\int_0^\infty e^{-x^2}dx$ must be finite, which means $\int_0^\infty e^{-x^2}dx$ must converge.

-Notice that we never even found out what the integral of $\int_0^\infty e^{-x^2}dx$ was. You don't always need to actually find the integral value to prove the integral converges or diverges.

Example: Is $\int_1^\infty \frac{1+e^{-x}}{x}dx$ convergent or divergent? Show why or why not.

Solution: We have already seen that $\int_1^\infty \frac{1}{x}dx$ is divergent, but since $\frac{e^{-x}}{x}$ is positive everywhere on this interval, that must mean $\int_1^\infty \frac{1}{x}dx \leq \int_1^\infty \frac{1+e^{-x}}{x}dx$, so $\int_1^\infty \frac{1+e^{-x}}{x}dx$ must be divergent too.

Exercises: Determine if the following integrals are convergent or divergent. Show why or why not.

a) $\int_1^\infty \frac{\sqrt{x^2+1}}{x}dx$

b) $\int_1^\infty \frac{1-e^{-x^2}}{x^2}dx$