11.1) Sequences:

Exercise: Find a formula for the general term a_n of the sequence:

$$\left\{\frac{1}{1}, -\frac{4}{7}, \frac{9}{49}, -\frac{16}{343}, \ldots\right\} \qquad a_n = \frac{n^2}{(-7)^{(n-1)}}$$

given that the pattern of the first few terms continues.

Exercise: Determine whether the following sequences are convergent or divergent:

a) $a_n = \frac{10}{5-e^{-n}}$ b) $a_n = \frac{sin(n)}{n}$ c) $a_n = \frac{(-1)^n n^2}{2^n}$

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{10}{5-e^{-n}} = \frac{10}{5-0} = 2$ Convergent

b)

$$\lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} a_n \leq 0$$

$$\lim_{n\to\infty} \frac{sin(n)}{n} = 0 \text{ Convergent}$$

c) $\lim_{n\to\infty} \left|a_n\right| = \lim_{n\to\infty} \left|\frac{(-1)^n n^2}{2^n}\right| = \lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n}{2^n(ln(2))} = \lim_{n\to\infty} \frac{2}{2^n(ln(2))^2} = 0$ Convergent

Exercise: Prove that the sequence $a_n = \frac{n}{\sqrt{n^2+1}}$ is increasing.

$f(x) = \frac{x}{\sqrt{x^2+1}}$, $f'(x) = \frac{\sqrt{x^2+1} - x*\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0 \; for \; all \; x, \; so \; a_n = \frac{n}{\sqrt{n^2+1}}$ increases for all n.

11.2) Series:

Exercise: Show that the series $\sum\limits_{i=1}^{\infty} \frac{2}{i(i+2)}$ is convergent, and find its sum.

$$\sum_{i=1}^{\infty} \frac{2}{i(i+2)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2}\right) = \lim_{n \to \infty} \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) +\ldots+ \left(\frac{1}{n} - \frac{1}{n-2}\right) =$$

$$\lim_{n \to \infty} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} +\ldots.+ \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} +\ldots+ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}\right) = \lim_{n \to \infty} \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{1} + \frac{1}{2} =$$

Exercise: Determine if the following series are convergent or divergent. If they are convergent, find the sum.

a) $3 - \frac{12}{3.5} + \frac{48}{12.25} - \frac{192}{42.875} +\ldots$

b) $\sum\limits_{i=1}^{\infty} (15)^{n}(4)^{1-2n}$

a) $\sum\limits_{i=1}^{\infty} \frac{3*(-4)^{i-1}}{(3.5)^{i-1}} = 3\sum\limits_{i=1}^{\infty} (-1)^{i-1} * \left(\frac{4}{3.5}\right)^{i-1}$ r > 1, so this series diverges.

b) $\sum\limits_{n=1}^{\infty} \frac{4*(15)^{n}}{(4)^{2n}} = \sum\limits_{n=1}^{\infty} 4 * \left(\frac{15}{16}\right)^{n} = \sum\limits_{n=1}^{\infty} 4 * \frac{15}{16}\left(\frac{15}{16}\right)^{n-1} = \sum\limits_{n=1}^{\infty} \frac{15}{4}\left(\frac{15}{16}\right)^{n-1}$ r < 1, so this series converges.

$$\sum_{n=1}^{\infty} \frac{15}{4}\left(\frac{15}{16}\right)^{n-1} = \frac{15/4}{1-15/16} = \frac{15/4}{1/16} = \frac{15}{4} * 16 = 60$$

Exercise: Write the number $1.2\overline{34}$ as a ratio of integers.

$$1.2\overline{34} = 1.2 + \sum_{n=1}^{\infty} 0.034\left(\frac{1}{100}\right)^{n-1} = 1.2 + \frac{0.034}{1-1/100} = \frac{12}{10} + \frac{34/1000}{99/100} = \frac{1188}{990} + \frac{34}{990} = \frac{1222}{990} = \frac{611}{495}$$

11.3) The Integral Test and Estimates of Series:

Exercise: a) Test $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ for convergence or divergence.

b) Test $\sum_{n=1}^{\infty} \frac{1}{n^3}$ for convergence or divergence.

a) $\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{n \to \infty} \int_1^{n} \frac{x}{x^2+1} dx = \lim_{n \to \infty} \frac{1}{2} ln(x^2 + 1)|_1^n = \lim_{n \to \infty} \frac{1}{2} ln(n^2 + 1) - \frac{1}{2} ln(2)$ is infinite, so the series diverges.

b) $\int_1^{\infty} \frac{1}{x^3} dx = \lim_{n \to \infty} \int_1^{n} \frac{1}{x^3} dx = \lim_{n \to \infty} - \frac{1}{2x^2}|_1^n = \lim_{n \to \infty} \frac{1}{2} - \frac{1}{2n^2} = \frac{1}{2}$, so the series converges.

Exercise: a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 10 terms. Estimate the error involved in this approximation.

b) Approximate the sum of the series using your error and your approximation.

c) How many terms are needed to ensure the sum is accurate to 5 decimal places?

a) $\sum_{n=1}^{10} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} = 1.54977$

$\int_n^{\infty} \frac{1}{x^2} dx = \lim_{t \to \infty} \int_n^{t} \frac{1}{x^2} dx = \lim_{t \to \infty} (- \frac{1}{t} + \frac{1}{n}) = \frac{1}{n}$

$$\frac{1}{11} \leq R_{10} \leq \frac{1}{10}$$

b)

$$\sum_{n=1}^{10} \frac{1}{n^2} + \frac{1}{11} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{10} \frac{1}{n^2} + \frac{1}{10}$$

$$1.54977 + 0.0909090... \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.54977 + 0.1$$

$$1.640677 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.64977$$

c)

$$R_n \leq \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$$

$$\frac{1}{n} < 0.000005$$

$$200000 < n$$