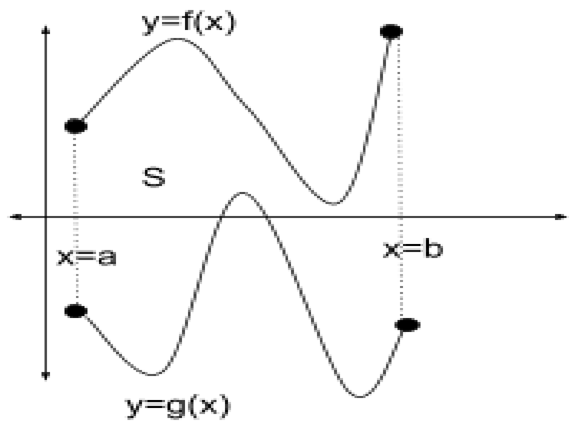
6.1) Area Between Curves:

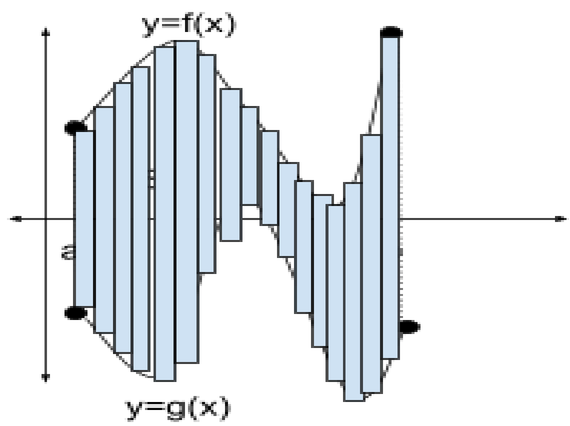
-Don't forget that definite integrals measure the net area between the x-axis and a curve, but what if you wanted to find the area between two different curves? The procedure is actually not as difficult as you may think.



-As pictured in the graph above, let S be the area from x = a to x = b in-between two continuous curves y=f(x) and y=g(x) such that $f(x) \geq g(x)$ for all x in [a, b]. In set notation, the area can be expressed as:

$$S = \{(x, y) |\ a \leq x \leq b,\ \ g(x) \leq y \leq f(x)\}$$

-Like in chapter 5 when we needed to find the area of areas under a curve, one way to do so is to approximate the area between f(x)and g(x) by drawing many rectangles between the curves and adding up the area of all these rectangles. This will approximately equal to the area between the curves:



-Once again, like chapter 5, the approximation is better when there are more rectangles, which means that the sum of the product of heights of the rectangles times a constant width Δx should approximate the area between the two curves. What are the heights of these rectangles? Considering that these rectangles make up the distance between two curves, the value of one curve minus the value of the other should give the height of the rectangles.

-As a result, taking a hint from the last chapter gives us that the area between the curves would be equal to:

$$Area\ between\ f\ and\ g = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\, \Delta x$$

-However, using some algebraic manipulation of limits and summation, we can rewrite this as:

$$Area\ between\ f\ and\ g = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*))\, \Delta x - \lim_{n\to\infty} \sum_{i=1}^{n} (g(x_i^*))\, \Delta x$$

-From what we know about Riemann sums however, we know:

$$Area\ between\ f\ and\ g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx$$

-From there, we have our method for finding area between two continuous curves:

Area Between Two Curves:
Given two continuous functions f and g where $f \geq g$ on [a, b], the area between the two curves from x=a to x=b equals:

$$\int_a^b (f(x) - g(x))dx$$

-You notice there are no restrictions besides that f is greater than g. f and g themselves could be positive, negative, or zero as long as f is always bigger than g. You also notice that this is **area** not **net area** between curves, implying that this integral always will be positive. This is logical given what we know about comparing integrals of one function to another.

-The most logic that goes into a problem like this is making sure that one function is always bigger than the other on an interval, which we will get to later, but otherwise the computation is very straightforward given what we were doing in the last section.

Example: Find the area between the curves $f(x) = e^{2x}$ and $g(x) = 1 - x$ on [0, 2].

Solution: These two curves intersect at x = 0, and f(x) is increasing everywhere while g(x) is decreasing everywhere. Therefore we know that f(x) is equal to or greater than g(x) everywhere on this interval. Thus we just have to evaluate the integral of f(x) - g(x) on [0, 2].

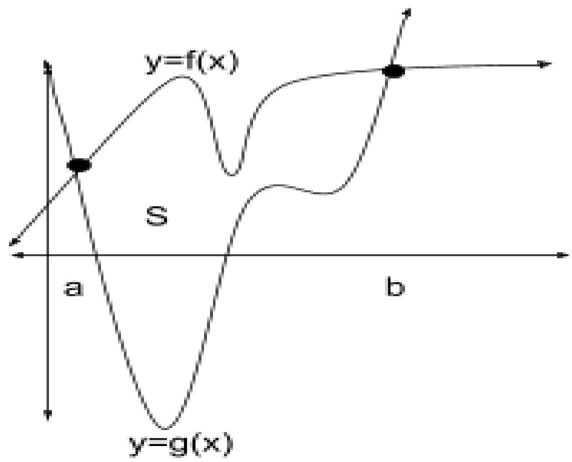
$$\int_0^2 (e^{2x} - (1 - x))dx$$

$$\int_0^2 (e^{2x} - 1 + x)dx$$

$$\frac{e^{2x}}{2} - x + \frac{x^2}{2} \,]^2{}_0 = \left(\frac{e^4}{2} - 2 + 2\right) - \left(\frac{1}{2} - 0 + 0\right) = \frac{e^4}{2} - \frac{1}{2}$$

-The integration is not difficult, it's making sure you have the right function subtracted by the right function. Sometimes it's obvious, but if it isn't, you will want to take the time to make sure you know which function is the top function (sometimes called the **upper bound**) and the bottom function (sometimes called the **lower bound**).

-Sometimes you won't even be given the interval for free. In which case the phrasing will be that you must "find the area of the region enclosed by the functions." In a situation like this, you must find the interval to integrate over yourself, because it is implied that the functions cross over each other multiple times, thus created a compact region (or multiple regions) in which one curve is on top and the other is on bottom at all times.



-Unlike the previous visual, the functions literally create a region between them that is implied to start where the functions cross paths the first time and end where the functions cross paths the second time. As a result, you will have to find the points of intersection between the curves, as those points of intersection are the upper and lower limits of the integral.

Example: Find the enclosed area between the curves $f(x) = x^2$ and $g(x) = 2x - x^2$.

Solution: First, we need to find where the functions intersect:

$$x^2 = 2x - x^2$$

$$0 = 2x - 2x^2 = 2x(1 - x)$$

The functions intersect at x = 0 and x = 1 and nowhere else, so the integral must start at x = 0 and end at x = 1. However, which one is the larger one? Plug any number between 0 and 1 (not 0 or 1 themselves, they are equal there) into both functions to see which is bigger. At x = 0.5, f(0.5)=0.25 and g(0.5)=0.75.

So that must mean that g(x) is the upper bound and f(x) is the lower bound. Make sure you subtract g(x) by f(x), not the other way around:

$$\int_0^1 ((2x - x^2) - x^2)dx = \int_0^1 (2x - 2x^2)dx = x^2 - \tfrac{2}{3}x^3 \,|^1_0 = 1 - \tfrac{2}{3} - (0 - 0) = \tfrac{1}{3}$$

Exercise: a) Find the area between the curves $f(x) = x^2 + 4$ and $g(x) = -3 - 2x$ on [-1, 3].
b) Find the enclosed area between the curves $f(x) = 3x + 6$ and $g(x) = x^2 + 2x$.

-When one function crosses over another, one function changes from being the larger function before intersection to being the smaller function after intersection. So what happens when you want to find the area between two curves when one curve is the upper bound over one region of the interval and the other curve is the upper bound over another region of the interval?

-Remember the distance vs. displacement example from last chapter, and how we used absolute values to express our expressions that we were integrating? We use a similar idea here. In fact, we can use an absolute value to express the area between two curves from x = a to x = b in general:

The area between two curves, y = f(x) and y = g(x) and between x = a and x = b is:

$$\int_a^b |f(x) - g(x)|\, dx$$

-However, how do you integrate or even define the absolute value of the difference between f(x) and g(x)? We should consider the region between x = a and x = b as a series of smaller regions, $S_1, S_2, S_3, \ldots.$ with areas $A_1, A_2, A_3, \ldots$ where for each subregion S_i, either f(x) is the upper bound, or g(x) is the upper bound. Therefore, we can think of the absolute value of f(x) minus g(x) as a piecewise-defined function depending on which function is the upper bound:

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } g(x) \geq f(x) \end{cases}$$

-As a result, finding the area between two functions in general involves finding all the subregions in which f(x) is greater than g(x) and integrating f(x) - g(x) over all these subregions, and then finding all the subregions in which g(x) is greater than f(x) and integrating g(x) - f(x) over those subregions. Usually this is a matter of finding where (if anywhere) that they intersect over a particular region of values, and then using a graph or algebra to determine which function is larger over each region.

Example: Find the area between the curves y = cos(x) and y = sin(x) on the interval $[0, \frac{\pi}{2}]$.

Solution: We know that areas will start at x = 0 and end at x = $\frac{\pi}{2}$, but in-between the functions may cross over. So let's find out where (if anywhere) that they cross over each other:

$$sin(x) = cos(x)$$

$$\frac{sin(x)}{cos(x)} = 1$$

$$tan(x) = 1$$

Between 0 and $\frac{\pi}{2}$, there is only one value at which tan(x) equals 1, and that is $\frac{\pi}{4}$. Therefore, if they intersect at $\frac{\pi}{4}$, they will likely (though not always) change positions of which is the upper bound and which is the lower bound.

Plug x = 0 into both functions and you have that $cos(0) = 1 > 0 = sin(0)$.
Plug x = $\frac{\pi}{2}$ into both functions and you have that $cos(\frac{\pi}{2}) = 0 < 1 = sin(\frac{\pi}{2})$.

We know that on the subregion $[0, \frac{\pi}{4}]$, $cos(x) \geq sin(x)$, and on the subregion $[\frac{\pi}{4}, \frac{\pi}{2}]$, $sin(x) \geq cos(x)$. Now we just have to integrate the proper difference of functions on each subregion separately:

$$\int_0^{\pi/4} (cos(x) - sin(x))\, dx + \int_{\pi/4}^{\pi/2} (sin(x) - cos(x))\, dx$$

$$\left(sin(x) + cos(x) \mid^{\pi/4}{}_0\right) + \left(-cos(x) - sin(x) \mid^{\pi/2}{}_{\pi/4}\right)$$

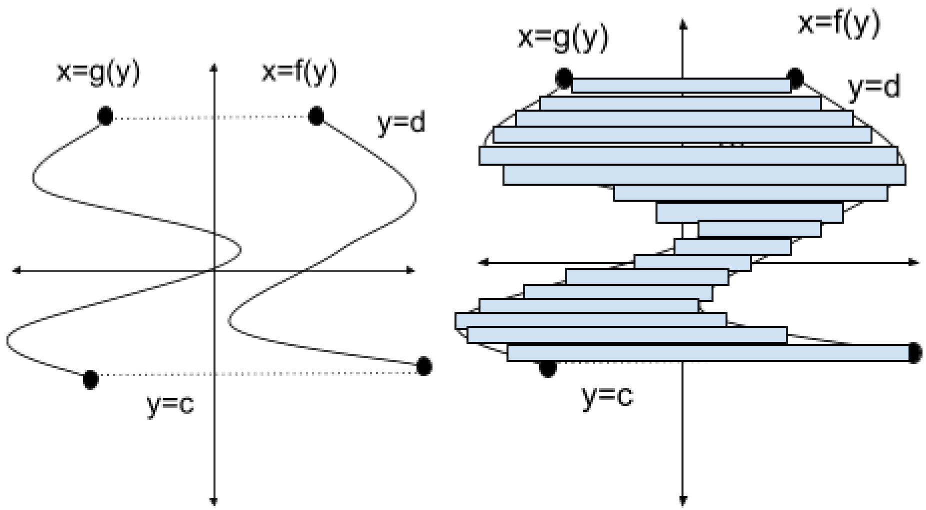
$$\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} - (0+1)\right) + \left((0-1) - (-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2})\right)$$

$$\left(\sqrt{2} - 1\right) + \left(\sqrt{2} - 1\right) = 2\sqrt{2} - 2$$

-A graph would have also helped determine which functions were above which in the region in question, or even just some foreknowledge of what the shape of the graphs of sin(x) and cos(x) are like. Sometimes it will be easy to find the intersections and regions necessary to create your integrals, and sometimes it will not.

Exercise: Find the area between the curves $f(x) = x^2 - 5x + 6$ and $g(x) = x^3 - 6x^2 - x + 18$ on the interval $[-1, 5]$.

-Most of the integration we have done has been done on functions of y in terms of x, but almost all the procedures and rules we have covered will work just as well if we interchange x and y. If the area we are trying to find lies between two functions of x in terms of y instead, the areas in-between can still be found using rectangles and the sum of the areas of those rectangles would still approach the area in-between the curves:



-Most of the differences are aesthetic only. Instead of starting at x=a and ending at x=b, this region starts at y=c and ends at y=d. Instead of the region lying between y=f(x) and y=g(x), these functions are x=f(y) and x=g(y). The analogue that is usually the hardest is that we presume in this case that $f(y) \geq g(y)$, in that f(y) is further to the right than g(y) is. Instead of upper bound and lower bound (though we often still use the same terms), the boundaries are more **right bound** for the larger x values, and **left bound** for the smaller s values.

-The rest of the procedure for finding area between two curves that are functions in terms of y however are identical to what we have already seen:

Given two continuous functions f and g where $f \geq g$ on [a, b], the area between the two curves from y=c to y=d equals:

$$\int_c^d (f(y) - g(y))dy$$

-Of course, you have to make sure the functions are functions in terms of y, which may require more work on your part. The reason for having this method of finding area between curves is that sometimes it is easier to envision the area being between functions of x instead of functions of y.

Example: Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

Solution: These are equations of y and x, but if I had the choice of solving these equations for y to get functions of x, or solving these equations for x to get functions of y, the algebra is easier solving for x:

$$x = y + 1 \qquad \text{and} \qquad x = \tfrac{1}{2}y^2 - 3$$

Either way, we do need to know where they intersect so we know where the enclosure begins and where it ends:

$$y + 1 = \tfrac{1}{2}y^2 - 3$$
$$0 = y^2 - 2y - 8 = (y - 4)(y + 2)$$

y=4 or -2. So which is the right bound and which is the left bound? Plug y = 0 in to see when x value is bigger, and you get that $x = y + 1 = 1$ and $x = \frac{1}{2}y^2 - 3 = -3$, meaning the right boundary is x=y+1 and $x = \frac{1}{2}y^2 - 3$ is the left boundary.

So what is the area enclosed by these two functions?

$$\int_{-2}^{4} (y + 1 - (\tfrac{1}{2}y^2 - 3))dy$$

$$\int_{-2}^{4} (y + 4 - \tfrac{1}{2}y^2))dy = \tfrac{y^2}{2} + 4y - \tfrac{1}{6}y^3 \,|^4_{-2} = \left(8 + 16 - \tfrac{64}{6}\right) - \left(2 - 8 + \tfrac{8}{6}\right) = 30 - \tfrac{72}{6} = 18$$

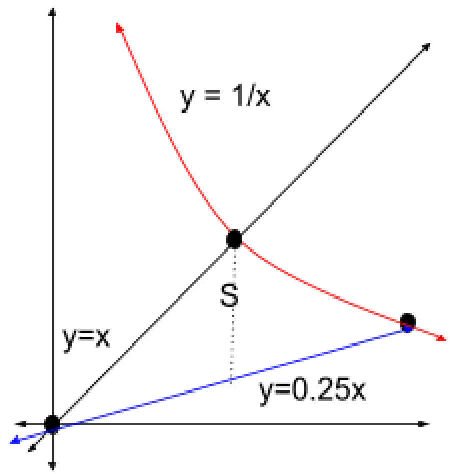
-This problem could have been done integrating with respect to x, but solving for y in each equation and then integrating would have been longer and more difficult. You can sometimes arrive to an answer much faster and easier if you are willing to integrate in terms of one variable as opposed to the other, so feel free to experiment.

Exercise: Find the area enclosed by $y = x + 1$ and $y^2 = 3x + 1$

Example: Find the area of the region enclosed by the following curves: $y = \frac{1}{x}$, y=x, and $y = \frac{1}{4}x$.

a) Do so using x as the variable of integration.
b) Do so using y as the variable of integration.

Solution: a) Once again, sometimes integrating with respect to one variable as opposed to the other is easier. So here we will attempt to do both so we can compare. Let's take a look at the graph of these three functions together so we can get an idea of the region and which functions are "larger" than others:



The black line is y = x, the red line is $y = \frac{1}{x}$, and the blue line is $y = \frac{1}{4}x$. S is the region we are trying to find the area of. Where do the intersections between these lines take place? We will need them to set up our integrals.

One intersection is obvious: (0,0) is where the two straight lines intersect. As for the intersection of y=x and $y = \frac{1}{x}$, we set the two equal to find they intersect at x = 1, or (1,1). Finally we need the intersection of $y = \frac{1}{x}$, and $y = \frac{1}{4}x$.

$$\frac{1}{4}x = \frac{1}{x}$$
$$x^2 = 4$$
$$x = 2$$

Ignore the intersection at x = -2, that is not in the image. The final intersection is at $(2, \frac{1}{2})$. Part of why the intersections are in ordered pair form when all we need is the x-coordinate is that we will use the y-coordinates for the second part of the problem.

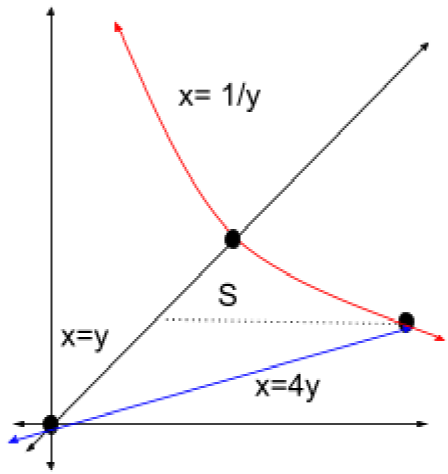
The dotted line in the problem is the "changing point" in which the upper bound changes from being one function to being another. From x = 0 to x=1, the upper bound is the black line. From x=1 to x=2 however, the upper bound is the red line. The lower bound appears to be the blue line for the entire region.

So now we can finally set up the integrals, one for each side of the dotted line, and find the area of S:

$$\int_0^1 (x - \frac{1}{4}x))dx + \int_1^2 (\frac{1}{x} - \frac{1}{4}x))dx$$
$$(\frac{1}{2}x^2 - \frac{1}{8}x^2 \,|^1{}_0) + (ln(x) - \frac{1}{8}x^2|^2{}_1)$$
$$(\frac{3}{8} - 0) + (ln(2) - \frac{1}{2} - (-\frac{1}{8})) = \frac{3}{8} + ln(2) - \frac{3}{8} = ln(2)$$

b) If they want us to redo this problem by integrating with respect to y, then we can keep the points of intersection (but be sure to use the y-coordinates, not the x-coordinates) (0,0), (2, $\frac{1}{2}$), and (1,1). Notice how they are written in ascending order of the y-coordinates this time, not the x-coordinates.

One thing that does need to change is that the functions need to be solved for x now, not y. y=x can stay the same as it was, but for the other two, $y = \frac{1}{x}$ becomes $x = \frac{1}{y}$ and $y = \frac{1}{4}x$ becomes $x = 4y$.



The dotted line has been readjusted so that they separate the region based upon what function is further to the right, as opposed to which function is on top. On the interval y = 0 to $y = \frac{1}{2}$ the blue line is further to the right than the black line, and on the interval $y = \frac{1}{2}$ to y = 1, the red line is further to the right than the black line.

Now we can set up the integrals:

$$\int_0^{1/2}(4y - y))dy + \int_{1/2}^{1}(\tfrac{1}{y} - y))dy$$

$$(2y^2 - \tfrac{1}{2}y^2\,|^{1/2}{}_0) + (ln(y) - \tfrac{1}{2}y^2|^1{}_{1/2})$$

$$(\tfrac{3}{8} - 0) + (-\tfrac{1}{2}\ -\ (ln(\tfrac{1}{2})\ -\tfrac{1}{8}))\ = \tfrac{3}{8}\ - \tfrac{3}{8} - ln(\tfrac{1}{2}) =\ - ln(\tfrac{1}{2})$$

Remember, $ln(\frac{1}{2})\ =\ - ln(2)$, so that means our answer is once again ln(2). Two different methods, same answer.

-Which one did you think was easier? It's opinion, some will say one, some will say the other, but again the option is there if you want to try it.

Exercise Find the area of the region enclosed by the following curves: $y = \frac{4}{x}$, y=x, and $y = \frac{1}{4}x$.

a) Do so using x as the variable of integration.
b) Do so using y as the variable of integration.

Example: Two people riding bicycles together have their velocity measured in five second increments as seen below:

Time (s)	0	5	10	15	20	25	30	35	40	45	50	55	60
Velocity #1 (ft/s)	5	12	16	13	15	17	14	15	19	14	13	15	12
Velocity #2 (ft/s)	8	13	18	14	17	18	17	17	20	18	20	21	22

Approximate the distance between the two bicycles after 60 seconds. Use 6 subintervals and use midpoints.

Solution: If we need to approximate the distance between these two bikers (biker #2 appears to have been riding faster at all times), then what we need are the difference in velocity between the two bikers at these points of time, which are found by subtracting the second biker velocity by the first biker velocity:

Time (s)	0	5	10	15	20	25	30	35	40	45	50	55	60
Velocity #1 (ft/s)	5	12	16	13	15	17	14	15	19	14	13	15	12
Velocity #2 (ft/s)	8	13	18	14	17	18	17	17	20	18	20	21	22
#2 - #1	3	1	2	1	2	1	3	2	1	4	7	6	10

We are approximating using six subintervals of time of equal width. The widths would be equal to $\Delta x = \frac{60-0}{6} = 10$. We can then multiply this width by corresponding differences in velocity for each subinterval. Which values do we use? We were asked to use midpoints, so the six midpoints will be:

$$x_1 = 0 + \frac{10}{2} = 5,\ \ x_2 = 15,\ \ x_3 = 25,\ \ x_4 = 35,\ \ x_5 = 45,\ \ x_6 = 55.$$

So the distance between the two will be the width, 10, multiplied by the difference in velocity at the points in time seen above (5, 15, 25, 35, 45, and 55).

$$Distance\ =\ 10(\ 1\ +\ 1\ +\ 1\ +\ 2\ +\ 4\ +\ 6)\ =\ 10(15)\ =\ 150\,feet$$