7.5) Integration Strategies:

-This section is more or less a review section, since up to this point we have covered many integration rules and methods.

-A big table of these rules are summarized on the first section of the 7.5 textbook online, so feel free to recap the material on that table. But rules are only part of the process. Here's a list of things you may want to consider doing whenever you have to integrate an expression:

1) **Try to simplify the integrand first:** Rewriting an integral is key to spotting what you need to do to integrate it (it's not like derivatives where you can brute force your way through if you feel like it, you need to strategize), and the first and easiest writing technique to use is always simplifying.

Can you collect like terms? Can you distribute? Can you rewrite your trigonometric equations in terms of sine and cosine only, or use trigonometric identities? All of these are worth asking if you are trying to simplify.

2) **Look for an obvious substitution**: It might seem intuitive to go into a problem ready to think outside the box and come up with a complicated or unlikely substitution, but remember, the most logical answer is most often the correct one. Look for substitutions that are obvious: inside parentheses, under radicals, in denominators, etc. If they don't work, then you can worry about being clever.

3) **Classify the Integrated According to its Form**: We have seen that some substitution tricks and methods are more likely to work on certain expressions:

-Trigonometric functions and algebraic combinations of said functions will almost always require substitution using pythagorean identities, formulas for reducing powers, and other trigonometric identities.

-Rational functions require the use of partial fraction decomposition often.

-Products of two functions that have no relation to one another will often integrate using integration by parts.

-Radical expressions will often be easier to integrate if x is replaced with a trigonometric functions, especially if the radicals contain $x^2 - a^2$ , $a^2 - x^2$, or $x^2 + a^2$. Otherwise, if $\sqrt[n]{ax+b}$ is used you may want to use substitution where u = $\sqrt[n]{ax+b}$.

4) **Try Again**: If you have tried the first three steps and still don't have an expression you can integrate, feel free to try again using substitution, parts, or manipulating the integral (which are essentially the three main methods of integrating we have learned).

-In general, once you have the expression in the proper form, you can integrate, so much of the work in this section is more practice on how to write the expressions in the proper format first:

**Example**: Rewrite the expression to more easily integrate: $\int \frac{tan^3(x)}{cos^3(x)}dx$

Solution: This looks complicated but remember, $\frac{1}{cos(x)} = sec(x)$, so:

$$\int \frac{tan^3(x)}{cos^3(x)}dx = \int tan^3(x)\frac{1}{cos^3(x)}dx = \int tan^3(x)sec^3(x)dx$$

Now we can use pythagorean identities to more easily integrate. Save a tan(x) and a sec(x), and write in terms of secant:

$$\int tan^2(x)sec^2(x)(tan(x)sec(x))dx = \int(1+sec^2(x))sec^2(x)(tan(x)sec(x))dx = \int(sec^2(x)+sec^4(x))(tan(x)sec(x))dx$$

This wasn't the only way to approach the problem of course. You could also have rewritten everything in terms of sine and cosine:

$$\int \frac{tan^3(x)}{cos^3(x)}dx = \int \frac{sin^3(x)}{cos^6(x)}dx$$

Let u = cos(x) and du = -sin(x)dx, so save one sin(x)dx and write everything else in terms of cos(x).

**Example**: Rewrite the expression to more easily integrate: $\int sin(\sqrt{x})dx$

Solution: This looks complicated, but if you go for the obvious substitution and work from there, things will present themselves:

Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}dx$, so we have: $\int sin(\sqrt{x})dx = \int sin(u)(2\sqrt{x}du)$

Since $u = \sqrt{x}$, substitute that in: $\int sin(\sqrt{x})dx = \int sin(u)(2\sqrt{x}du) = \int 2usin(u)du$

From here you can now integrate by parts, with U = u and dV = sin(u)du

**Example**: Rewrite the expression to more easily integrate: $\int \frac{dx}{x\sqrt{ln(x)}}$

Solution: It might be tempting to let $u = \frac{1}{x}$ or $u = \frac{1}{\sqrt{ln(x)}}$. But again, try the simple stuff before anything tough:
If $u = ln(x)$, then $du = \frac{1}{x}dx$, or $xdu = ln(x)$. So we have:

$$\int \frac{dx}{x\sqrt{ln(x)}} = \int \frac{xdu}{x\sqrt{u}} = \frac{du}{\sqrt{u}}$$

**Example**: Rewrite the expression to more easily integrate: $\int\sqrt{\frac{1-x}{1+x}}dx$

Solution: Using some integral manipulation will help immensely here:

$$\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}} = \frac{1-x}{\sqrt{1-x^2}}$$

You may think the denominator needs to be rationalized, but now we can separate the terms into their own fractions:

$$\int \frac{1-x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}dx - \int \frac{x}{\sqrt{1-x^2}}dx$$

From here we can integrate each term individually (sine inverse for the first, power rule for the second).

**Elementary Functions**:

An elementary function (which are the only functions we have worked with so far this semester) are polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric functions and their inverses, hyperbolic functions and their inverses, and any combination of these functions through addition, subtraction, multiplication, division, and composition.

-Why bring this up now? Because a question that always arises by now is: "Are there any functions that we cannot integrate using any of the methods we have seen?" As it turns out, yes, and here are some examples:

$f(x) = e^{x^2}$, $g(x) = sin(x^2)$, $h(x) = cos(e^x)$, $j(x) = \sqrt{x^3 + 1}dx$, $k(x) = \frac{sin(x)}{x}$

-These are all continuous functions, and so their integral exists, but we are unable to find an equation to express what the integrals of these functions are by hand.

-The reason why is that elementary functions are functions for which the derivative is also elementary. However, the integral of an elementary function does not have to be elementary, which is why we are unable to find the integral of the above functions by hand, or at the least, express the integrals of these functions in terms of other elementary functions.

-We will have a way to express the integrals of these functions later, but not using any of the functions we have seen. Some of this is covered in section 7.6, but we will be skipping that section due to how technology focused that section is.

**Note, there are no exercises in this section, but you can still prepare for section 7.5 online on WebAssign**