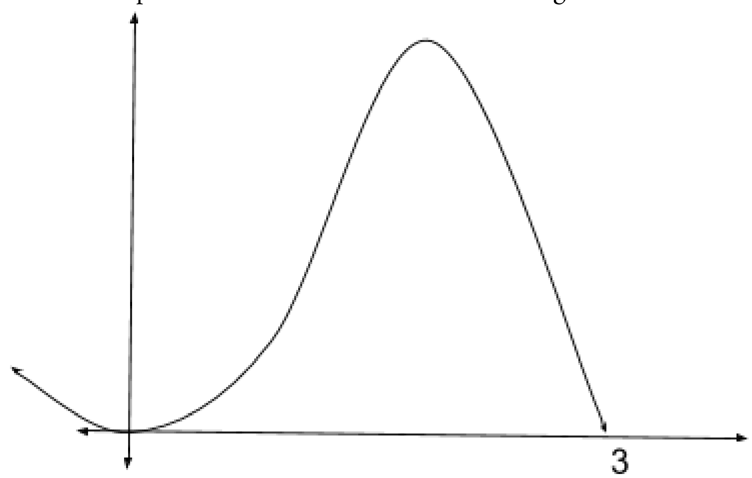
6.3) Volumes by Cylindrical Shells:

-What makes a volume-by-rotation problem difficult sometimes is that the cross-sectional area is difficult to express in terms of the proper variable. We have seen that when we rotate regions around the x-axis or other horizontal lines in the form of y = b, our cross-sectional areas are in terms of x, A(x). If we rotate regions around the y-axis or vertical lines in the form of x = a, our cross-sectional areas are in terms of y, A(y).

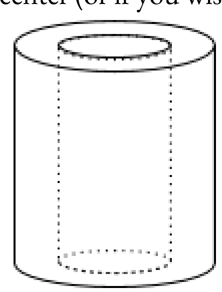
-But what if it is too difficult to express the cross section in terms of the given variable?



-This region for example is a rough sketch of the graph of $y = 3x^2 - x^3$ from x = 0 to x = 3. If you rotated the area enclosed by this curve and the x-axis around the **y-axis** instead of the x-axis, you would not be able to use the disk procedure of $V = \int_0^4 A(y)dy = V = \int_0^4 \pi((larger\ radius)^2 - (smaller\ radius)^2)dy$ unless you could solve $y = 3x^2 - x^3$ for x in terms of y. Which is far too difficult.

-One thing that was the case for the reasoning behind volume by rotation was that we were rotating rectangles that were perpendicular to the axis of rotation, which was what gave us our disks and washers that we would sum the volumes for to get our volume of the solid. But what if instead of creating lots of disks and washers that we could add the volumes of together to get our volume, we used slightly different shapes?

-These slightly different shapes would be **cylindrical shells**, which are essentially right circular cylinders with a smaller right circular cylinder cut out of their center (or if you wish, much taller washers).



-These cylindrical shells therefore are characterized has having an outer radius r_2 for the size of the larger cylinder and an inner radius r_1 for the size of the smaller cylinder cut out of the center. The two cylinders have the same height, so the volume of a shell like this would be the volume of the larger cylinder minus the volume of the space cut out of the center:

$$V = V_2 - V_1 = \pi(r_2)^2h - \pi(r_1)^2h$$

-With a little rewriting we get the following form instead:

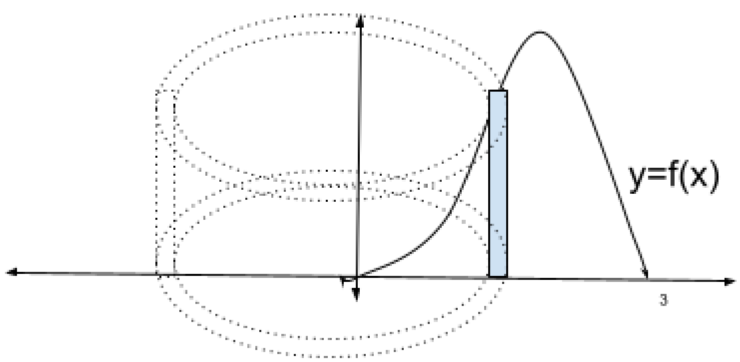
$$V = \left(r_2^2 - r_1^2\right)\pi h = (r_2 + r_1)(r_2 - r_1)\pi h$$
$$V = \left(r_2^2 - r_1^2\right)\pi h = \tfrac{(r_2+r_1)}{2}(r_2 - r_1)2\pi h$$

-Suppose we let the difference in the two radii, the **thickness** of the shell if you will, is replaced with Δr, and we let the mean of the radii be replaced with r. That turns the volume of the shell into:

$$V = \tfrac{(r_2+r_1)}{2}(r_2 - r_1)2\pi h = r * \Delta r * 2\pi h = 2\pi r h \Delta r$$

-This means that the volume of a cylindrical shell has three components: $2\pi r$, which you probably recognize as the circumference of a circle with radius r, h, the height of the shell, and Δr, the thickness of the shell.

-Instead of rotating cross-sections that are perpendicular to the axis of rotation to get a series of disks to form the solid, what we will be doing is rotating cross-sections that are parallel to the axis of rotation to get a series of shells to form the solid.



-You notice that rotating the region around the y-axis using a series of rectangles that are parallel to the y-axis instead of perpendicular to the y-axis creates cylindrical shells. The rectangles that will form these shells are formed by subintervals $[x_{i-1}^*,\ x_i^*]$ all of equal width, which would make the width of the rectangle in the image above equal to $\Delta x = x_i^* - x_{i-1}^*$, the thickness of the cylindrical shell.

-If the point at which the rectangle intersects y=f(x) is the midpoint of the interval $[x_{i-1}^*,\ x_i^*]$, then that would make the area of this rectangle $f(\frac{x_i^*+x_{i-1}^*}{2}) * \Delta x = f(\overline{x}_i) * \Delta x$ and the resulting volume of the cylindrical shell would be $V_i = 2\pi\overline{x}_i * f(\overline{x}_i) * \Delta x$. Our three components of the volume of a cylindrical shell are all accounted for: $2\pi\overline{x}_i$ is the circumference, $f(\overline{x}_i)$ is the height, and Δx is the thickness.

-So the resulting volume of a solid approximated by shells instead of disks would be the sum of all n shells used:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi\overline{x}_i * f(\overline{x}_i) * \Delta x$$

-As always, the approximation of the volume is better when there are more shells, so let n approach infinity and you have:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} V_i = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x}_i * f(\overline{x}_i) * \Delta x$$

-From what we know about the definition of integrals and their relationship with Riemann sums, this becomes:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\overline{x}_i * f(\overline{x}_i) * \Delta x = \int_a^b 2\pi x f(x)dx$$

Volume of a Solid using Cylindrical Shells:

The volume of a solid obtained by rotating about the y-axis the region under the curve y = f(x) from a to b is:

$$V = \int_a^b 2\pi x f(x)dx \qquad \text{where} \quad 0 \le a < b$$

-The appeal of this function is that if rotating a function of x in terms of y around the y-axis proves too difficult (maybe because the definition of A(y) proves too difficult to find) you can instead integrate the above expression in terms of x.

-Most remember the disk rotation method of finding volume by the fact that the cross-sectional area function is associated with circle area, πr^2, but the cylindrical shell method (or shell method for short) is associated with circle circumference, $2\pi r$. Note the radius in both methods is meant to be the distance from the axis of rotation, but in the shell method r is associated with a variable, and in the disk method r is associated with an entire function of a variable (almost an "x" vs. "y" difference).

-So let's take a look at that example from the start of the section:

Example: Find the volume of the solid formed by rotating the region enclosed by $y = 3x^2 - x^3$ and y=0 around the y-axis.

Solution: We have already seen that this region starts at x = 0 and ends at x =3, and if we find the volume by using the shell method, we will need circumference, height, and thickness. The thickness will be Δx inside the integral, and almost always a given.

The circumference will be $2\pi x$ since we are rotating around the y-axis so the radius/distance from $y = 3x^2 - x^3$ and the y-axis is always just x itself. Finally, the height is merely $y = 3x^2 - x^3$ itself, in terms of x:

$$V = \int_0^3 2\pi x(3x^2 - x^3)dx = 2\pi\int_0^3 (3x^3 - x^4)dx = 2\pi\left(\frac{3x^4}{4} - \frac{x^5}{5}\Big|_0^3\right) = 2\pi\left(\frac{243}{20}\right) = \frac{243\pi}{10}$$

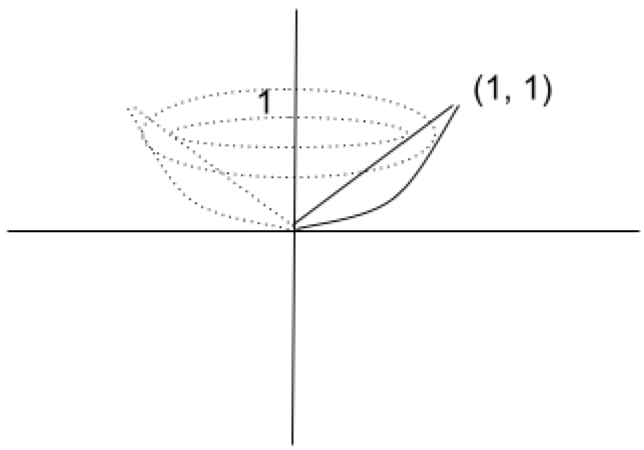
-Sometimes the shell method is easier, sometimes the disk method is easier. It depends on the region, the functions, the limits and other factors. If you are having trouble finding the limits of the region, or solving for a given variable in the function, or even integrating the resulting expression, you can always try using the other method. You may find it easier.

Exercise: Find the volume of the solid formed by rotating the region enclosed by $y = 4x - x^3$, y=0, and x = 2 around the y-axis.

-Some of the examples from the previous section (or examples similar to them) might even be easier now:

Example: Find the volume of the solid formed by rotating the region enclosed by the curves $y = x$ and $y = x^2$ around the y-axis

Solution: Same region as last section again, but it is now being rotated around the vertical axis x = 0, aka the y-axis:



But what if instead of solving for x in both functions and using the cross-section method, we used shells instead? One thing we have to account for is that the region has two curves this time, not one, so does that means we need two radii and two circumferences like we would have needed two radii and two cross-sectional areas? Not quite. If there are two curves, you need two heights, or at least the difference of two heights:

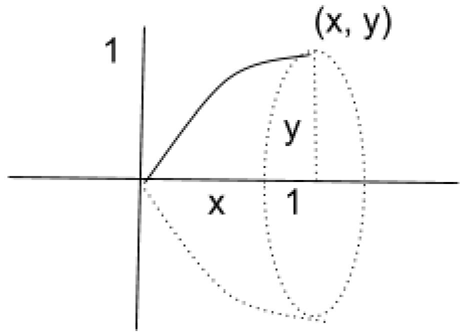
$$V = \int_a^b 2\pi x(f(x) - g(x))dx$$

The heights of the shells will start at one curve and go to the other when there are two curves for the enclosure. The first curve, f(x) should be the larger one when you plug x in, and the second curve, g(x) should be the smaller one. This is not the disk method, so "which is further from the axis of rotation" doesn't matter here.

y=x is larger than $y = x^2$ on the interval [0,1], so that means our volume will be:

$$V = \int_0^1 2\pi x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1 \right) = 2\pi\left(\frac{1}{12}\right) = \frac{\pi}{6}$$

Example: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

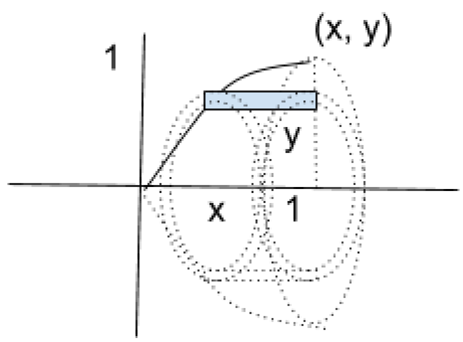


Solution: Using disks from the last section, we found the answer was $V = \int_0^1 A(x)dx = \int_0^1 \pi x dx = \pi \frac{x^2}{2}|^1{}_0 = \frac{\pi}{2}$. But what if we wanted to use shells? Using shells will require us to integrate with respect to the other variable. Most remember that with disks, you integrate with respect to the variable whose axis you are rotating around, but with shells you integrate with the opposite variable whose axis you are rotating around. So the shell method:

$$V = \int_c^d 2\pi y f(y) dy$$

should give us the correct answer, but you still want to proceed carefully.

$2\pi y$ is a given here since we are rotating around the x-axis, but what about f(y)? You may think that $f(y) = y^2$ here since it is what you get when you solve for x in $y = \sqrt{x}$, but is that actually the height of the shells you are creating? Imagine what one of these shells might look like, and what the rectangle that forms one of these shells looks like:



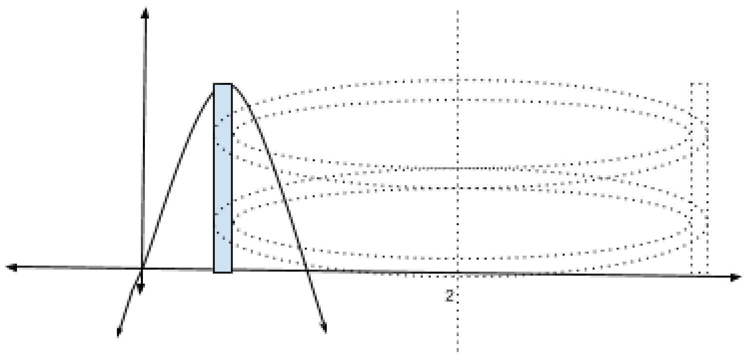
Remember, f(y) should return an x-coordinate for a given y-coordinate. From the look of this image, given any y-coordinate, the height of the bar is not the x-coordinate that is associated with $x = y^2$, but instead it appears to be the distance from that x-coordinate and the line x=1. Thus the height of the shells is: $1 - y^2$.

Now we can find the volume (which will be the same answer as last section!):

$$V = \int_0^1 2\pi y(1-y^2)dy = 2\pi \int_0^1 (y - y^3)dy = 2\pi \left(\frac{y^2}{2} - \frac{y^4}{4}|^1{}_0\right) = 2\pi(\tfrac{1}{4}) = \frac{\pi}{2}$$

Exercise: Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the curve $y = \sqrt{x}$, the y-axis, and y = 1, from y=0 to y=1.

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and y = 0 about the line x = 2.



Solution: The height in this case is most certainly just the function $f(x) = x - x^2$, but the radius of the circumference is another matter. Given the rectangle in the picture above, it appears the distance from the curve and the axis of rotation is the distance from x and x = 2, so we have a radius of $r = 2 - x$. This means we can find the volume by integrating from x = 0 to x = 1:

$$V = \int_0^1 2\pi(2-x)(x-x^2)dx = 2\pi\int_0^1(2x - 3x^2 + x^3)dx = 2\pi\left(x^2 - x^3 + \tfrac{x^4}{4}\Big|_0^1\right) = 2\pi(\tfrac{1}{4}) = \tfrac{\pi}{2}$$

Exercise: Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and y = 0 about the line x = -1

Disk Method or Shell Method?

-Which is the better method to use to find the volume of a solid found by rotating a region around an axis?

-Is it easier to define the cross-section from one value to another along the axis of rotation, or is it easier to define the heights from one end to the other of the region on the axis parallel to the axis of rotation?

-Can you find the limits more easily in terms of one variable or the other?

-Will you be able to evaluate the integral(s) you set up to find the volume?

-These questions and others are all worth asking, and sometimes you won't know that one method is easier than the other until you try a method and find it's just a little too difficult to continue, in which case you may want to try the other.

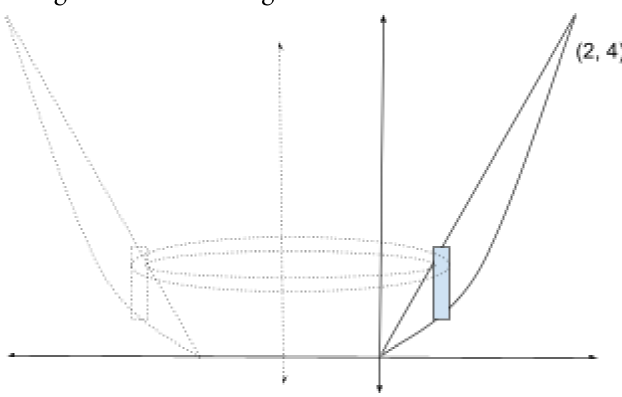
-Drawing a picture is never a bad idea either. Sometimes it will be a lot more obvious what your cross-section, rectangles, radius, whatever you may need is if you can visualize. Rectangles can be a good indicator for what method you should use too. Imagine the rectangle you draw rotating; if a rectangle is rotating around its base you should be using the disk method, but if it is rotating around its height you should be using the shell method.

-Of course, you could also always try both!

Example: Find the volume of the solid formed by rotating the area enclosed by $y = 2x$ and $y = x^2$ around the line x = -1.

a) Use x as the variable of integration
b) Use y as the variable of integration.

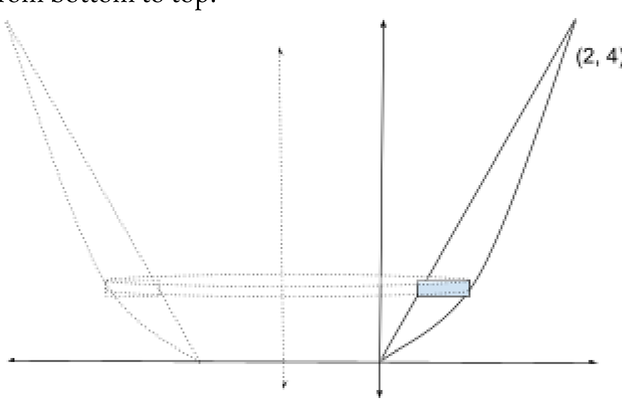
Solution: a) If we are using x as the variable of integration then that means the smaller solids within should be formed by traveling along the region from left to right:



However, since we are rotating around x = -1, we are <u>not</u> traveling along the axis of rotation. This implies we are using the shell method here. What is the height of each shell as a result? It appears that the heights (of the shells or rectangles) is y = 2x minus $y = x^2$.

$$V = \int_0^2 2\pi(x+1)(2x - x^2)dx = 2\pi \int_0^2 (2x + x^2 - x^3)dx = 2\pi \left(x^2 + \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^2\right) = 2\pi(4 + \frac{8}{3} - 4) = \frac{16\pi}{3}$$

b) If we are using y as the variable of integration then that means the smaller solids within should be formed by traveling along the region from bottom to top:



Since we are traveling along the axis of rotation this time, we are using the disk method here (or washer method since there are two curves here). We need our functions in terms of y, so instead of y = 2x, we have $x = \frac{1}{2}y$, and instead of $y = x^2$, we have $x = \sqrt{y}$. What is the larger and smaller radius? We are rotating around x = -1, and the curve closer to x = -1 is $x = \frac{1}{2}y$, and the one farther is $x = \sqrt{y}$.. The distance from -1 to $\frac{1}{2}y$ is $\frac{1}{2}y - (-1) = \frac{1}{2}y + 1$, and the distance from -1 to $x = \sqrt{y}$ is $\sqrt{y} + 1$.

Now we can integrate from y = 0 to y = 4 to find the volume (again):

$$V = \int_0^4 A(y)dy = \int_0^4 \pi((\sqrt{y} + 1)^2 - (\frac{1}{2}y + 1)^2)dy = \pi\int_0^4 (2\sqrt{y} - \frac{1}{4}y^2)dx = \pi((\frac{4y^{1.5}}{3} - \frac{y^3}{12})|^4{}_0) = \frac{16\pi}{3}$$