8.1) Arc Length

**Exercise**: a) Find the length of the arc of the parabola $y = 2x^2$ between the points (0,0) and (2,8).
b) Find the length of $36y^2 = (x^2 - 4)^3$ from x = 2 to x = 3, and where $y \geq 0$.

$$a)\ L = \int_0^2 \sqrt{1 + (4x)^2}\, dx = \int_0^2 \sqrt{1 + 16x^2}\, dx = \int_0^{arctan(8)} \sqrt{1 + tan^2(u)}\left(\frac{1}{4}sec^2(u)\right)du =$$

$$\frac{1}{4} * \frac{1}{2}(sec(u)tan(u) + ln|sec(u) + tan(u)|)|_0^{arctan(8)} =$$

$$\frac{1}{8}\left(\frac{\sqrt{65}}{1}\left(\frac{8}{1}\right) + ln\left|\frac{\sqrt{65}}{1} + \frac{8}{1}\right|\right) = \sqrt{65} + \frac{1}{8}ln\left|\sqrt{65} + 8\right|$$

$$b)\ y = \frac{(x^2-4)^{3/2}}{6}, \qquad \frac{dy}{dx} = \frac{x(x^2-4)^{1/2}}{2}$$

$$L = \int_2^3 \sqrt{1 + \left(\frac{x(x^2-4)^{1/2}}{2}\wedge\right)^2}\, dx = \frac{1}{2}\int_2^3 \sqrt{4 + x^2(x^2 - 4)}\, dx = \frac{1}{2}\int_2^3 \sqrt{x^4 - 4x^2 + 4}\, dx = \frac{1}{2}\int_2^3 (x^2 - 2)dx$$

$$\frac{1}{2}\left(\frac{x^3}{3} - 2x\right)\Big|_2^3 = \frac{1}{2}(9 - 6) - \frac{1}{2}\left(\frac{8}{3} - 4\right) = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$$

**Exercise**: a) Find the arc length function for the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$ by letting $P_0 = (1, \frac{5}{6})$ be the starting point.
b) Use the function to find the arc length from $(1, \frac{5}{6})$ to $(\frac{9}{4}, \frac{2251}{576})$

a)

$$L = \int_1^x \sqrt{1 + (t^2 - \frac{1}{4t^2})^2}\, dt = \int_1^x \sqrt{1 + t^4 - \frac{1}{2} + \frac{1}{16t^4}}\, dt = \int_1^x \sqrt{t^4 + \frac{1}{2} + \frac{1}{16t^4}}\, dt = \int_1^x \sqrt{(t^2 + \frac{1}{4t^2})^2}dt =$$

$$\int_1^x (t^2 + \frac{1}{4t^2})dt = \frac{t^3}{3} - \frac{1}{4t}\Big|_1^x = \frac{x^3}{3} - \frac{1}{4x} - \frac{1}{12}$$

$$b)\ L|_{x=\frac{9}{4}} = \frac{\left(\frac{9}{4}\right)^3}{3} - \frac{1}{9} - \frac{1}{12} = \frac{729}{128} - \frac{1}{9} - \frac{1}{12} = \frac{2075}{576} = 3.602430556$$

9.1 Modeling with Differential Equations:

**Exercise**: a) Determine whether $y = sin(2x)$ is a solution to the following differential equation:

$$y^{(3)} - y'' - 4y' + 4y = 0$$

b) Determine whether $y = \sqrt{x}$ is a solution to the following differential equation:

$$4x^2y'' + 4xy' = y$$

a)

$\frac{d^3}{dx^3}[sin(2x)] - \frac{d^2}{dx^2}[sin(2x)] - 4 * \frac{d}{dx}[sin(2x)] + 4 * sin(2x) = - 8cos(2x) + 4sin(2x) - 4(2cos(2x)) + 4sin(2x) =$
$- 16cos(2x) + 8sin(2x)$

No, not a solution.

b) $4x^2\frac{d^2}{dx^2}[\sqrt{x}] + 4x\frac{d}{dx}[\sqrt{x}] = 4x^2 * (-\frac{1}{4x^{3/2}}) + 4x\frac{1}{2x^{1/2}} = -\sqrt{x} + 2\sqrt{x} = \sqrt{x} = y$

Yes, a solution.

**Exercise**: Find a solution of the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$ that satisfies the initial condition $y(0) = - 3$.

$$- 3 = \frac{1+ce^0}{1-ce^0}$$
$$- 3 + 3c = 1 + c$$
$$2c = 4$$
$$c = 2$$

$$y = \frac{1+2e^x}{1-2e^x}$$