7.4) Integration of Rational Functions by Partial Fractions:

-Can you combine the following fractions? $\frac{1}{x+3} - \frac{1}{2x+1}$

$$\frac{1}{x+3} - \frac{1}{2x+1} = \frac{2x+1}{2x^2+7x+3} - \frac{x+3}{2x^2+7x+3} = \frac{x-2}{2x^2+7x+3}$$

-Simple, right? But how would you integrate the following function?

$$\int \frac{x-2}{2x^2+7x+3}\,dx$$

-However, if you could rewrite this fraction as two other fractions being added/subtracted together, you have:

$$\int \frac{x-2}{2x^2+7x+3}\,dx = \int \left(\frac{1}{x+3} - \frac{1}{2x+1}\right) dx$$

-Would you rather integrate the expression on the left, or the right? Naturally the one on the right, right? However, it's not nearly as easy to rewrite $\frac{x-2}{2x^2+7x+3}$ as two fractions as it is to rewrite $\frac{1}{x+3} - \frac{1}{2x+1}$ as one.

-**Partial fractions** is the term for a series of fractions like $\frac{1}{x+3} - \frac{1}{2x+1}$ that add together to another rational function. However, how do you find the partial fraction decomposition of a rational function? For starters, makes sure you are looking at a rational function, which is in the form:

$$f(x) = \frac{P(x)}{Q(x)} \text{ where P(x) and Q(x) are polynomials.}$$

-f(x) is considered an **improper** fraction if the top is bigger than the bottom, or in this case, if the degree of the top is bigger than the bottom. If f(x) is improper, you should divide P(x) by Q(x) using long division. This will give you a **proper** fraction, where the top (degree) is smaller than the bottom.

-If you do divide the top by the bottom, you will have a quotient polynomial (call it S(x)) and a remainder (R(x)) that will satisfy the following:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Example: Integrate $\int \frac{x^3-x^2-2x+1}{x^2-1}dx$

Solution: We have an improper fraction here, so before we try to integrate, let's see if we can write this in terms of $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ by using long division:

$$\begin{array}{r|l} & \quad\quad\quad x - 1 \\ \hline x^2 - 1 & x^3 - x^2 - 2x + 1 \\ & -\left(x^3 - x\right) \\ \cline{2-2} & -x^2 - x + 1 \\ & -x^2 \quad\;\; + 1 \\ \cline{2-2} & \quad\quad -x \end{array}$$

$$\int \frac{x^3-x^2-2x+1}{x^2-1}\,dx = \int \left(x - 1 + \frac{-x}{x^2-1}\right) dx$$

It's a much easier expression to integrate now than before we rewrote after division. Now we integrate more easily:

$$\int \left(x - 1 - \frac{x}{x^2-1}\right) dx = \frac{1}{2}x^2 - x - ln\left|x^2 - 1\right| + C$$

Exercise: Integrate $\int \frac{x^3-x^2-3x-1}{x^2-2x-3}\,dx$

-If you have already used long division, or the function is already a proper rational function, and you still need to break the function down further, consider factoring the denominator into a series of linear factors and irreducible quadratic factors $(ax^2 + bx + c,\ where\ b^2 - 4ac\ < 0)$. When you are aware of all the factors of the denominator, you should next try to find a way to express this proper rational function as a sum of partial fractions, all of which have a denominator that equals one or more of these factors of the original denominator.

<u>If the denominator factors into a product of distinct linear factors</u>:

-Suppose the fraction $f(x) = \frac{R(x)}{Q(x)}$ has a denominator $Q(x) = (a_1x - b_1)(a_2x - b_2)...(a_nx - b_n)$.

Then this means f(x) can be decomposed into: $f(x) = \frac{A_1}{a_1x-b_1} + \frac{A_2}{a_2x-b_2} + ... \frac{A_n}{a_nx-b_n}$

-Finding these constants in the numerators, $A_1,\ A_2,\ ...,\ A_n,$ requires some algebraic manipulation which is best explained through example:

Example: Integrate $\int \frac{2x^2-4x-1}{2x^3-x^2-x}\,dx$

Solution: First off, let's factor the denominator: $(2x^3 - x^2 - x) = x(2x^2 - x - 1) = x(2x + 1)(x - 1)$.

So the partial fraction deposition will have the following form:

$$\frac{2x^2-4x-1}{x(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x}$$

So to solve for A, B, and C, cancel out the denominators by multiplying by $x(2x + 1)(x - 1)$:

$$2x^2 - 4x - 1 = Ax(x - 1) + Bx(2x + 1) + C(x - 1)(2x + 1) = A(x^2 - x) + B(2x^2 + x) + C(2x^2 - x - 1)$$

$$2x^2 - 4x - 1 = Ax^2 - Ax + 2Bx^2 + Bx + 2Cx^2 - Cx - C$$

Consider refactoring in terms of the powers of x:

$$2x^2 - 4x - 1 = x^2(A + 2B + 2C) + x(-A + B - C) + 1(-C)$$

This means we can now solve the following system of linear equations to find out what A, B, and C are:

$$
\begin{aligned}
A + 2B + 2C &= 2\\
-A + B - C &= -4\\
-C &= -1
\end{aligned}
$$

Solve this system for A, B, and C and you have that C = 1, B = -1, and A = 2.

If you wish, you could try to solve for A, B, and C using an alternative method described below:

$$2x^2 - 4x - 1 = Ax(x-1) + Bx(2x+1) + C(x-1)(2x+1)$$

At this step, you can also substitute values in for x that will cause some of the terms on the right side of the equation disappear, and that will allow you to solve for one of the variables.

For example, let x = 0, and the equation becomes:

$$-1 = C(-1)(1), \qquad C = 1$$

Let x = 1 and the equation becomes:

$$-3 = B(1)(3), \qquad B = -1$$

Let x= -0.5 and the equation becomes:

$$1.5 = A(-1.5)(-0.5), \quad A = 2$$

This method is not for everyone though, so feel free to stick to linear algebra to find A, B, and C if you please.

Which means the original function has a partial fraction decomposition of:

$$\frac{2x^2-4x-1}{x(2x+1)(x-1)} = \frac{2}{2x+1} - \frac{1}{x-1} + \frac{1}{x}$$

Therefore we can now more easily integrate:

$$\int \frac{2x^2-4x-1}{2x^3-x^2-x}\,dx = \int \left(\frac{2}{2x+1} - \frac{1}{x-1} + \frac{1}{x}\right) dx = ln\,|2x+1| - ln\,|x-1| + ln\,|x| + C$$

Exercise: Integrate $\int \frac{5x^2-40x-36}{x^3-4x^2-12x}\,dx$

-What if one of the linear factors repeats itself?

<u>If the denominator factors into a product of non distinct linear factors</u>:

-Suppose the fraction $f(x) = \frac{R(x)}{Q(x)}$ has a denominator that is divisible by a factor $(a_i x - b_i)^n$ for n>1.

Then instead of representing $(a_i x - b_i)^n$ with a single $\frac{A_i}{a_i x - b_i}$, use $\frac{A_1}{a_i x - b_i} + \frac{A_2}{(a_i x - b_i)^2} + \ldots + \frac{A_n}{(a_i x - b_i)^n}$.
The rest of the procedure is as before.

-Let's see an example:

Example: Integrate $\int \frac{2x^2+4x+1}{x^3-x^2-x+1} dx$

Solution: The denominator factors into $x^3 - x^2 - x + 1 = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$. So all the factors are linear, but one of them repeats. This means the partial fraction decomposition looks like this:

$$\frac{2x^2+4x+1}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Make sure every power of that repeated factor from 1 to n gets its own fraction.

The procedure from here is very similar to before, so multiply both sides by the $(x-1)^2(x+1)$:

$$2x^2 + 4x + 1 = A(x^2 - 2x + 1) + B(x^2 - 1) + C(x + 1)$$

$$2x^2 + 4x + 1 = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C$$

$$2x^2 + 4x + 1 = x^2(A + B) + x(-2A + C) + 1(A - B + C)$$

So what do A, B, and C equal? Solve the following system of linear equations:

$$2 = A + B$$
$$4 = -2A + C$$
$$1 = A - B + C$$

Solve and you will find $C = 3.5,\ B = 2.25,\ and\ A = -0.25$. This turns the integral into:

$$\int \frac{2x^2+4x+1}{x^3-x^2-x+1} dx = \int \left(\frac{-0.25}{x+1} + \frac{2.25}{x-1} + \frac{3.5}{(x-1)^2} \right) dx$$

Integrate and we get our solution: $\int \left(\frac{-0.25}{x+1} + \frac{2.25}{x-1} + \frac{3.5}{(x-1)^2} \right) dx = -0.25ln|x+1| + 2.25ln|x-1| - \frac{3.5}{x-1} + C$

-Polynomials can always be written as a product of linear factors and quadratic factors that cannot be factored, so what happens if the denominator contains quadratic factors? We'll start with the scenario that there are no repeated quadratic factors:

<u>If the denominator contains distinct quadratic factors</u>:

-Suppose $ax^2 + bx + c$ is one of the quadratic factors in the factorization of Q(x) in $f(x) = \frac{R(x)}{Q(x)}$ and cannot be factored/reduced. What does the partial fraction decomposition representative for this factor look like?

-The representative for $ax^2 + bx + c$ is: $\frac{Ax+B}{ax^2+bx+c}$. The algebra for solving for A and B in Ax+B is similar (if not more complicated) to the other scenarios, but the integration process will usually involve some kind of variation on the arctangent integration rule:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + C$$

-So let's see some examples for practice:

Example: Integrate $\int \frac{2x^2-x+4}{x^3+4x}dx$.

Solution: The bottom factors easily into: $x(x^2+4)$, which makes the representatives of our partial fraction decomposition as below:

$$\frac{A}{x} + \frac{Bx+C}{x^2+4}$$

The reason why the numerator is Bx+C and not just B is that the representatives are all meant to be improper, so if the denominator has a degree of 2, the numerator can (but does not have to) have a degree of 1. In case you are wondering, the reason why we don't do the same for the earlier examples that have larger powers of linear factors like $\frac{C}{(x-1)^2}$ is because having other factors like $\frac{B}{(x-1)}$ in the decomposition will account for higher degrees of x already.

Still, we have to solve for A, B, and C:

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Now we remove the denominators by multiplying by $x(x^2+4)$:

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)(x)$$
$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

Regroup by x powers:

$$2x^2 - x + 4 = x^2(A+B) + x(C) + (1)(4A)$$

We can see from this example that A = 1, B = 1, and C = -1. So that makes the partial fraction decomposition:

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4} = \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}$$

Now we can integrate: $$\int \frac{2x^2-x+4}{x(x^2+4)}dx = \int(\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4})dx$$

$$\int \frac{2x^2-x+4}{x(x^2+4)}dx = ln\left|x\right| + \frac{1}{2}ln\left|x^2+4\right| - \frac{1}{2}tan^{-1}(\frac{x}{2}) + C$$

Remember, the C on the end is the constant term for the indefinite integral, not the C used in the partial fraction decomposition.

<u>If the denominator contains non distinct quadratic factors</u>:

-You can probably guess what to do if the denominator factorization includes quadratic factors that are repeated.

-For $(a_i x^2 + b_i x + c)^n$, use $\frac{A_1}{a_1 x^2+b_i x+c} + \frac{A_2}{(a_1 x^2+b_i x+c)^2} + \ldots + \frac{A_n}{(a_i x^2+b_i x+c)^n}$.

Example: Integrate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx$

Solution: The denominator is already fully factored, so let's go straight to the partial fraction decomposition:

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply by $x(x^2+1)^2$ on both sides to get:

$$1 - x + 2x^2 - x^3 = A(x^2+1)^2 + (Bx + C)(x(x^2+1)) + (Dx + E)(x)$$

If you attempt to solve this by plugging in values you'll have a hard time given how many repeated factors there are as well as how few numbers will make terms disappear, which is why most people use this method instead of linear algebra. So I would suggest you do this the traditional way:

$$1 - x + 2x^2 - x^3 = A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x) + (Dx + E)(x)$$
$$1 - x + 2x^2 - x^3 = Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

Regroup by powers of x:

$$1 - x + 2x^2 - x^3 = x^4(A + B) + x^3(C) + x^2(2A + B + D) + x(C + E) + (1)(A)$$

We have the following: 0 = A+B, -1 = C, 2 = 2A+B+D, -1 = C+E, 1 = A

Solving for the variables gives us: A = 1, B = -1, C = -1, D = 1, and E = 0, and so we have the following partial fraction decomposition:

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}$$

Therefore we can now integrate:

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx = ln\,|x| - \frac{1}{2} ln\,\left|x^2+1\right| - tan^{-1}(x) - \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$$

Exercises: Integrate the following:

a) $\int \frac{10x^2-7x+9}{x^3-x^2-x+1} dx$

b) $\int \frac{x^3+4x^2+5x+14}{x^4+4x^2-5} dx$

-Of course, not all functions that have a denominator are rational functions, in which case you must also remember that sometimes you just have to be a little creative with your substitutions:

Example: Integrate $\int \frac{\sqrt{x+4}}{x} dx$

Solution: This is not a partial fraction problem (yet), but you can integrate with some careful substitution:

Let $u = \sqrt{x+4}$, and you may be tempted to now find out what du is, but the differential would be rather messy to find. Instead, let's play with the substitution a bit further.

If $u = \sqrt{x+4}$, then $u^2 = x + 4$, and $u^2 - 4 = x$. Now we can find that $2udu = dx$.

This may seem trivial, but the substitution takes a much different turn now that we know all of this:

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2-4} 2udx = \int \frac{2u^2}{u^2-4} du$$

Before you try partial fraction decomposition, try dividing with long division to get the following:

$$2\int \frac{u^2}{u^2-4} du = 2\int (1 + \frac{4}{u^2-4}) du$$

Now we can turn our attention to $\frac{4}{u^2-4}$ which has a partial fraction decomposition of: $\frac{4}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$

What are A and B? Let's find out:

$$\frac{4}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$$
$$4 = A(u+2) + B(u-2)$$
$$4 = u(A+B) + (1)(2A - 2B)$$

Since A+B=0 and 2A-2B = 4, we have that A = 1 and B = -1, and so the partial fraction decomposition of $\frac{4}{u^2-4}$ becomes:

$$\frac{4}{u^2-4} = \frac{1}{u-2} - \frac{1}{u+2}$$

Therefore we can now integrate:

$$2\int (1 + \frac{4}{u^2-4}) du = 2\int (1 + \frac{1}{u-2} - \frac{1}{u+2}) du = 2(u + ln\,|u-2| - ln\,|u+2|) + C$$

When we plug $\sqrt{x+4}$ back in for u, we get:

$$2(\sqrt{x+4} + ln\left|\sqrt{x+4} - 2\right| - ln\left|\sqrt{x+4} + 2\right|) + C$$

-Sometimes partial fraction decomposition will appear when you don't expect it, so always be on the lookout!