9.5) Linear Equations:

-For all the work we've done so far on differential equations, there is one basic differential equation we have not tackled yet: **linear differential equations**. These are first-order linear differential equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P(x) and Q(x) are continuous functions on a given interval.

-This may look simple, but a differential equation like this is not separable, so the strategies we have learned won't work here. Even something as simple looking as:

$$xy' + y = 2x$$

is not separable and thus cannot be solved through separating, integrating, and evaluating.

-It can be rewritten to look more like the expression we have above though, just divide by x first:

$$y' + \frac{1}{x}y = 2$$

-It becomes clear why this is not separable when you see that if y' is isolated, you cannot write $2 - \frac{1}{x}y$ as a product of a function of x times a function of y (or a quotient either). So if we can't separate, what can we do? You may have to squint a little bit first, but you may notice that $xy' + y = 2x$ seems to borrow a bit from the product rule. Look at the left side of the equation in particular:

$$xy' + y = \frac{d}{dx}(xy)$$

-This means that we can just integrate with respect to x to begin with.

$$\int (xy' + y)dx = \int 2xdx$$

$$(xy) = x^2 + C$$

$$y = x + \frac{C}{x}$$

-We have a family of solutions to the differential equation, but admittedly we got a little lucky. We could tell by looking at $xy' + y = 2x$ that the left side was the result of the product rule. However, would we have been as likely to see it if we had the other version to begin with?

$$y' + \frac{1}{x}y = 2$$

-Not as easy to look at this version of the differential equation and tell that we needed to integrate. However, if we multiplied both sides of this version by x we would have the other version where it is easier to spot the product rule at work.

-In fact, any first order linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ can be integrated by doing the product rule in reverse as long as the expression is written properly. What is required is something called an

integrating factor, denoted by I(x), which is a function both sides of the equation are multiplied by to make one side of the equation look like the derivative of the product of the integrating factor and y.

$$I(x)(\frac{dy}{dx} + P(x)y) = \frac{d}{dx}(I(x) * y)$$

-Multiply by the right integrating factor, and the entire left side of the equation $I(x)\left(\frac{dy}{dx} + P(x)y\right) = I(x)Q(x)$becomes $\frac{d}{dx}(I(x) * y)$, which simplifies the equation to:

$$\frac{d}{dx}(I(x) * y) = I(x)Q(x)$$

-Integrating both sides gives us:

$$I(x) * y = \int I(x)Q(x)dx + C$$

so solving for y gives us:

$$y = \frac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right)$$

-Yes, remember to include the arbitrary integration term C in the quotient, and unlike dividing C by a constant, dividing C by I(x) does not simplify to another C.

-So if we can find the right integrating factor, I(x), we can integrate $\frac{dy}{dx} + P(x)y = Q(x)$easily. But how do we find I(x)?

-Remember, supposedly $I(x)(\frac{dy}{dx} + P(x)y) = \frac{d}{dx}(I(x) * y)$, so let's expand both sides to see what we really have:

$$I(x)(\frac{dy}{dx} + P(x)y) = \frac{d}{dx}(I(x) * y)$$

$$\frac{dy}{dx}I(x) + P(x)yI(x) = I'(x) * y + I(x) * \frac{dy}{dx}$$

-Let's solve for I(x) to see what we get, and don't worry, $\frac{dy}{dx}I(x)$ will cancel out:

$$\frac{dy}{dx}I(x) + P(x)yI(x) = I'(x) * y + I(x) * \frac{dy}{dx}$$

$$P(x)yI(x) = I'(x) * y$$

$$P(x)I(x) = I'(x)$$

-<u>This</u> differential equation <u>is</u> separable, so we can solve for I(x) now:

$$\frac{I'(x)}{I(x)} = P(x)$$

$$\int \frac{I'(x)}{I(x)}dx = \int P(x)dx$$

$$ln|I(x)| = \int P(x)dx + C$$

$$I(x) = e^{\int P(x)dx + C}$$

$$I(x) = Ae^{\int P(x)dx}$$

-Let $A = \pm e^{C}$ for an arbitrary constant C. Therefore, if we want a particular integrating factor, not a general one or a family, we can simply let A=1 and use the simplest form of the integrating factor:

$$I(x) = e^{\int P(x)dx}$$

-Rather than worrying about memorizing this formula, most people just remember the general procedure:

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)dx}$ and then integrate both sides.

-In case you were wondering, yes, this would work for the previous example, $y' + \frac{1}{x}y = 2$. Letting $P(x) = \frac{1}{x}$, we have $I(x) = e^{\int \frac{1}{x}dx} = e^{ln(x)} = x$. So multiply by x on each side to get:

$$xy' + y = 2x$$

Where the left side is: $$xy' + y = \frac{d}{dx}(xy) = \frac{d}{dx}(I(x)y)$$

-Integrate each side to get: $$\int (xy' + y)dx = \int (2x)dx$$

$$xy = x^2 + C$$

$$y = x + \frac{C}{x}$$

-Let's try a different example now:

Example: Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$

Solution: We have that $P(x) = 3x^2$, and $Q(x) = 6x^2$.We need an integrating factor I(x). Let's find out what $I(x)$ is:

$$I(x) = e^{\int 3x^2dx} = e^{x^3}$$

Multiply both sides of the differential equation by e^{x^3} and we have:

$$\frac{dy}{dx}e^{x^3} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

The left side is the derivative of the product of I(x) and y:

$$\frac{d}{dx}\left(e^{x^3} y\right) = 6x^2 e^{x^3}$$

Now we can integrate both sides:

$$e^{x^3} y = \int 6x^2 e^{x^3} dx$$

Use integration by substitution to integrate the right side:

$$e^{x^3} y = 2\int 3x^2 e^{x^3} dx$$

$$e^{x^3} y = 2e^{x^3} + C$$

Finally, divide by the original $I(x) = e^{x^3}$:

$$y = 2 + Ce^{-x^3}$$

-You don't have to worry about solving for C unless it's an initial value problem.....

Example: Find the solution of the initial-value problem:

$x^2 y' + xy = 1$, where x >0, and y(1)=2

Solution: First off, we should divide the factor of x^2 off of y', then we can worry about an integrating factor:

$$x^2 y' + xy = 1$$
$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

You may recall that when the factor on y is $\frac{1}{x}$ that makes the integrating factor I(x) equal x. So let's multiply both sides by $I(x) = x$.

$$xy' + y = \frac{1}{x}$$

The left side is the derivative of xy,so let's integrate both sides to get:

$$\int (xy' + y)dx = \int \frac{1}{x}dx$$

$$xy = ln|x| + C$$

Divide by x to solve for y:

$$y = \frac{ln|x|}{x} + \frac{C}{x}$$

Since x is greater than 0, we can write $ln(x)$ instead of $ln|x|$, and using y(1)=2, we can solve for C:

$$2 = \frac{ln(1)}{1} + \frac{C}{1} = 0 + C$$
$$2 = C$$

Our solution to the initial value problem is: $y = \frac{ln|x|}{x} + \frac{2}{x}$

-Sometimes you will not be able to complete the integration process once you introduce your integrating factor. If this happens, don't worry, just write your final answer with the integral included.

Example: Solve $y' + 2xy = 1$

Solution: We have an integrating factor of: $I(x) = e^{\int 2xdx} = e^{x^2}$. So multiply both sides by this factor to get:

$$y' + 2xy = 1$$
$$y'e^{x^2} + 2xe^{x^2}y = e^{x^2}$$

Integrate both sides and you get:

$$ye^{x^2} = \int e^{x^2}dx + C$$

We could integrate the left side because we know it is the derivative of the product of y and the integrating factor. We cannot integrate the right side though, so we can just leave it as it looks and solve around it:

$$y = e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}$$

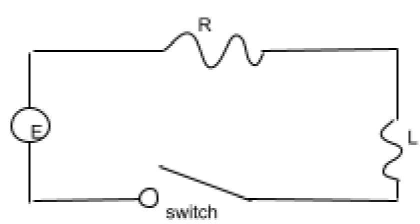
Exercise: Find the following solutions:

a) $x^2\frac{dy}{dx} + 3xy = 4$ for x>0 and y(2) = 4

b) $\frac{dy}{dx} + 4x^3y = 12x^3$

-There is one application this all should look familiar to, electric circuits (again!):

-A simple electric circuit contains an electromotive force that produces a voltage of E(t) volts and a current of I(t) amperes at time t. The circuit also contains a resistor with a resistance of R ohms and an inductor with an inductance of L henries. An image of such a circuit is below:



-Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L * (dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal tot he supplied voltage E(t). From this, we have a first-order differential equation that models the current I at time t:

$$L * \frac{dI}{dt} + RI = E(t)$$

-The differential equation used for electric circuits are in fact first-order linear equations. You can even rewrite the original equation a little to see it:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}E(t)$$

Example: Suppose that in the simple circuit the resistance is 12 ohms, the inductance is 4 henries, and a battery gives a constant voltage of 60 volts. If we plug these numbers into $L * \frac{dI}{dt} + RI = E(t)$ we get:

$$4 * \frac{dI}{dt} + 12I = 60$$

a) Find an expression for the current in a circuit when the switch is turned on at t = 0.
b) Find the current after 1 second.
c) Find the limiting value of the current.

Solution: If we plug R=12, L=4, E(t)=60 into the differential equation, we get:

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}E(t)$$
$$\frac{dI}{dt} + 3I = 15$$

Our integrating factor will therefore be $e^{\int 3dt} = e^{3t}$, so we multiply both sides by e^{3t} to get:

$$\frac{dI}{dt}e^{3t} + 3e^{3t}I = 15e^{3t}$$

Integrate with respect to t to get:

$$\int \frac{d}{dt}\left[e^{3t}I\right]dt = \int 15e^{3t}dt$$

$$e^{3t}I = 5e^{3t} + C$$
$$I = 5 + Ce^{-3t}$$

We were also told that the switch is closed at t = 0, so we have I(0)=0, and so we can solve for C:

$$0 = 5 + Ce^{-3(0)}$$
$$0 = 5 + C$$
$$-5 = C$$

Our current equation is: $I = 5 - 5e^{-3t}$

b) What is the current after 1 second? Plug in t = 1 to get:

$$I = 5 - 5e^{-3(1)} = 5 - 5e^{-3} \approx 4.75\ amperes$$

c) The limiting value is what the current approaches as t goes to infinity:

$$\lim_{t\to\infty} I = \lim_{t\to\infty} 5 - 5e^{-3(1)} = 5 - 5(0) = 5\, amperes$$

-Let's try a different version of this circuit example; one that has a non-constant voltage.

Example: Suppose that in the simple circuit the resistance is 12 ohms, the inductance is 4 henries, and a battery gives a voltage of $E(t) = 60sin(30t)$ volts. If we plug these numbers into $L * \frac{dI}{dt} + RI = E(t)$ we get:

$$4 * \frac{dI}{dt} + 12I = 60sin(30t)$$

Find I(t).

Solution: We can still divide by 4 on both sides to get:

$$\frac{dI}{dt} + 3I = 15sin(30t)$$

We can even use the same integrating factor, e^{3t}. When we multiply both sides by e^{3t}, we have to integrate the right side carefully:

$$\frac{dI}{dt}e^{3t} + 3e^{3t}I = 15e^{3t}sin(30t)$$

$$\int \frac{d}{dt}(e^{3t}I)dt = 15\int e^{3t}sin(30t)dt$$

To integrate $\int e^{3t}sin(30t)dt$ we will need integration by parts:

$$\int e^{3t}sin(30t)dt = \frac{-e^{3t}cos(30t)}{30} + \frac{1}{10}\int e^{3t}cos(30t)dt$$

$$\int e^{3t}sin(30t)dt = \frac{-e^{3t}cos(30t)}{30} + \frac{1}{10}\left(\frac{e^{3t}sin(30t)}{30} - \frac{1}{10}\int e^{3t}sin(30t)dt\right)$$

$$\int e^{3t}sin(30t)dt = \frac{-e^{3t}cos(30t)}{30} + \frac{e^{3t}sin(30t)}{300} - \frac{1}{100}\int e^{3t}sin(30t)dt)$$

$$\frac{101}{100}\int e^{3t}sin(30t)dt = \frac{-e^{3t}cos(30t)}{30} + \frac{e^{3t}sin(30t)}{300}$$

$$\int e^{3t}sin(30t)dt = \frac{100}{101}\left(\frac{-e^{3t}cos(30t)}{30} + \frac{e^{3t}sin(30t)}{300}\right)$$

$$\int e^{3t}sin(30t)dt = \left(\frac{-10e^{3t}cos(30t)}{303} + \frac{e^{3t}sin(30t)}{303}\right)$$

Whew, so now we can get back to the problem:

$$\int \frac{d}{dt}(e^{3t}I)dt = 15\int e^{3t}sin(30t)dt$$

$$e^{3t}I = 15\left(\frac{-10e^{3t}cos(30t)}{303} + \frac{e^{3t}sin(30t)}{303}\right) + C$$

$$I = 15\left(\frac{-10cos(30t)}{303} + \frac{sin(30t)}{303}\right) + Ce^{-3t}$$

$$I = \frac{-50cos(30t)}{101} + \frac{5sin(30t)}{101} + Ce^{-3t}$$