9.4) Population Growth:

-We have seen that exponential growth/decay functions of the form $P(t) = Ae^{kt}$ satisfy the differential equation $\frac{dP}{dt} = kP$. However, can we also prove that these are the only type of function that satisfy this differential equation? We can, but we would need to start with the differential equation and solve it for P(t):

$$\frac{dP}{dt} = kP.$$

-This equation is called the **law of natural growth**, that if P(t) is the value of a quantity y at time t, and that the rate of change of P with respect to t is proportional to its size P(t) at any time, then P(t) satisfies $\frac{dP}{dt} = kP$ for some constant number k. Remember, if k > 0, it's a growth function, and if k < 0, it's a decay function.

-Since the law of natural growth equation is a separable differential equation we can solve it for P(t):

$$\frac{dP}{dt} = kP$$
$$\frac{1}{P}dP = kdt$$
$$\int \frac{1}{P}dP = \int kdt$$
$$ln(P) = kt + C$$
$$P = e^{kt+C}$$
$$P = Ae^{kt}$$

where A is an arbitrary constant, but also the initial value of the function, since P(0)=A.

-Therefore, it is true that $P = Ae^{kt}$ is the only family of functions that are a solution to $\frac{dP}{dt} = kP$, where P(0)=A.

-Another way of phrasing the law of natural growth is that the ratio of the rate of change of P and P itself is a constant:

$$\frac{dP/dt}{P} = k$$

-If a function satisfies this function, it also belong to the family of $P = Ae^{kt}$.

-On the topic of populations, we can also account for populations that have the population growing or shrinking from population members immigrating in or out, in which case the rate of change has the natural growth rate added together with the number of members leaving or entering:

$\frac{dP}{dt} = kP - m$ for m>0 where m is the number of members leaving.

$\frac{dP}{dt} = kP + m$ for m>0 where m is the number of members entering.

-These differential equations can also be separable and solvable using methods similar to the mixture problems from last section.

-What about the more logical examples we saw from section 9.1? The models that suggested that in the beginning, population would grow with natural growth rate and so $\frac{dP}{dt} = kP$ when P was small, but would reach a **carrying capacity** eventually such that if P approached M, the growth would stop being natural and instead the population would approach and stay near M by having a positive growth rate if P<M and a negative growth rate if P>M?

-You may remember that we found a differential equation for this more logical case, and that it was:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

-This equation is sometimes called the **logistic differential equation**, and it does satisfy our conditions from earlier:

-If the population is small, the growth rate is proportional to P, implying that $\frac{dP}{dt} = kP$.

-If the population is smaller than the carrying capacity M, then P increases, implying that $\frac{dP}{dt} > 0$.

-If the population exceeds the carrying capacity M, then P starts to decrease, implying that $\frac{dP}{dt} < 0$.

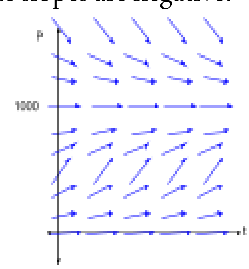
-So what kind of function could satisfy this differential equation? Let's take a look at what the graph of such an equation might look like?

**Example**: Draw a direction field for the logistic equation with k = 0.08 and carrying capacity M=1000. What can you deduce about the solutions?

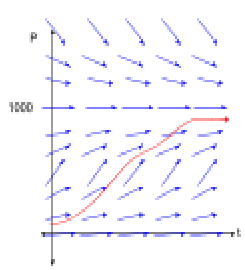
Solution: If we plug these numbers in we get:

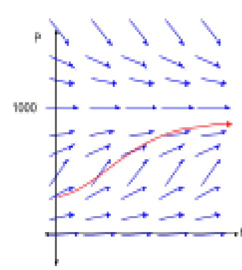
$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

If we draw a direction field (for t >0 and P>0 only, since the other quadrants are illogical), we have that this is an autonomous differential equation and the direction field slopes are flat at P=0 and P=1000. When 0<P<1000, the slopes are positive, and when P>1000, the slopes are negative:




We can even draw some solution curves starting at certain initial points, like say (0, 200) on the left, or (0,400) on the right.



The slopes are steepest when they are far from P=0 or P=1000 (either being above 1000 or at P=500), and P=500 appears in general to be when the slopes are at their greatest. In fact, when the population begins at an initial point (0,a) where $0<a<500$, the slope goes from getting steeper to steepest at P=500 to flatter and flatter the closer they get to P=1000. This implies that for all functions that start below P=500, there is an inflection point when the function equals 500.

-So that might be what the curves of a logistic population growth function look like, but what does the equation equal? We need to solve the differential equation to find that out:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

$$\frac{1}{P(1-\frac{P}{M})}dP = kdt$$

-Before we try integrating, it may help to rewrite the left side using partial fractions:

$$\frac{1}{P(1-\frac{P}{M})} = \frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

-To solve for A and B, multiply by P(M-P):

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$M = A(M-P) + B(P)$$

$$M = AM - AP + BP$$

$$M = P(B-A) + 1(AM)$$

-So we have:

$$M = AM$$

$$0 = B - A$$

and so A = 1 and B = 1.

$$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$$

-Back to the integral, we have:

$$\frac{1}{P(1-\frac{P}{M})}dP = kdt$$

$$\int\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \int kdt$$

$$ln(P) - ln(M-P) = kt + C$$

-To simplify:

$$ln\left(\frac{P}{M-P}\right) = kt + C$$

$$\frac{P}{M-P} = e^{kt+C}$$

$$\frac{P}{M-P} = Ae^{kt}$$

-Now we need to solve for P:

$$P = (M-P)Ae^{kt}$$

$$P = MAe^{kt} - PAe^{kt}$$

$$P + PAe^{kt} = MAe^{kt}$$
$$P(1 + Ae^{kt}) = MAe^{kt}$$

$$P = \frac{MAe^{kt}}{(1+Ae^{kt})}$$

-You may recognize the function in its more commonly seen form by dividing top and bottom by $Ae^{kt}$ :

$$P = \frac{M}{(Ae^{-kt}+1)} = \frac{M}{1+Ae^{-kt}}$$

-Remember, A is an arbitrary constant, so $\frac{1}{Ae^{kt}} = \frac{1}{A}e^{-kt}$ is arbitrarily the same as writing $\frac{1}{Ae^{kt}} = Ae^{-kt}$. As for what A itself is equal to, it's tempting to say it's the initial population, but if you plug 0 into the P equation, you don't get that P(0) = A. Instead, if you know that the initial population is $P_0$, you get:

$$P_0 = \frac{M}{1+Ae^{-k(0)}}$$
$$P_0 = \frac{M}{1+A}$$
$$P_0 + P_0A = M$$
$$P_0A = M - P_0$$
$$A = \frac{M-P_0}{P_0}$$

-This is the family of solutions for our differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$. A population that has an initial value of $P_0$ and a carrying capacity of M will have a population at time t of:

$$P(t) = \frac{M}{1+Ae^{-kt}} \qquad \text{where } A = \frac{M-P_0}{P_0}$$

-We can also see that as time goes on indefinitely, $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{1+Ae^{-kt}} = \frac{M}{1} = M$, or the population approaches its carrying capacity. This is why the direction field for our example earlier had the property that no matter the initial value, the population would approach M eventually.

**Example**: a) Write the solution of the initial-value problem $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$, where P(0)=100.
b) Use it to find the population sizes P(40) and P(80)
c) At what time does the population reach 900?

Solution: a) The differential equation is logistic with k = 0.08, and M=1000. The initial condition is that $P_0 = 100$, so if we plug these values into $A = \frac{M-P_0}{P_0}$ we get $A = \frac{1000-100}{100} = 9$. As for the population equation itself, we get:

$$P(t) = \frac{M}{1+Ae^{-kt}}$$
$$P(t) = \frac{1000}{1+9e^{-0.08t}}$$

Not too bad when you don't have to solve the differential equation!

b) What does P(40) equal? Plug in t=40 to get:

$$P(40) = \frac{1000}{1+9e^{-3.2}} = 731.6$$

As for P(80), plug in t=80 to get:

$$P(80) = \frac{1000}{1+9e^{-6.4}} = 985.3$$

c) To solve for the time when P=900, you will need a little more algebra than the last two computations:

$$900 = \frac{1000}{1+9e^{-0.08t}}$$
$$900(1+9e^{-0.08t}) = 1000$$
$$(1+9e^{-0.08t}) = \frac{10}{9}$$
$$9e^{-0.08t} = \frac{1}{9}$$
$$e^{-0.08t} = \frac{1}{81}$$
$$-0.08t = -ln(81)$$
$$t = \frac{ln(81)}{0.08} = 54.9$$

After approximately 55 years the population will be 900.

**Exercise**: a) Write the solution of the initial-value problem $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{5000}\right)$, where P(0)=200.
b) Use it to find the population sizes P(100) and P(150)
c) At what time does the population reach 4000?

-So, what if you had numerical data for a population and you wanted to find an equation to model it? You would have to decide whether you think a natural growth model or a logistic growth model would be better.

**Example**: G.F. Gause conducted an experiment with Paramecium and used a logistic equation to model his data. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64. The table he used for his model is below:

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 2 | 3 | 22 | 16 | 39 | 52 | 54 | 46 | 50 | 76 | 69 | 51 | 57 | 70 | 53 | 59 | 57 |

a) Find exponential and logistic growth models for the data.
b) Compare the results with the data on the table.

Solution: a) To form an exponential growth model we need an initial population and an initial growth rate. P(0)=2 according to the table, and the relative growth rate was given to be k=0.7944. So we have a model of:

$$P(t) = 2e^{0.7944t}$$

As for the logistic growth model, we have an initial population of 2, a growth rate of 0.7944, and a carrying capacity of 64. This means we have $A = \frac{M-P_0}{P_0} = \frac{64-2}{2} = 31$, and as for the population equation itself, we get:

$$P(t) = \frac{64}{1+31e^{-0.7944t}}$$

b) So which is the better model? Was Gause right to think the population growth was logistic? Let $P_E$ be our exponential growth model population, and let $P_L$ be our logistic growth model population. To compare our models to what we have on the table, we have to plug the values t from 0 to 16 into each and see how close we are to the observed values. We will start with the exponential growth model:

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 2 | 3 | 22 | 16 | 39 | 52 | 54 | 46 | 50 | 76 | 69 | 51 | 57 | 70 | 53 | 59 | 57 |
| $P_E$ | 2 | 4 | 10 | 22 | 48 | 106 | ... | | | | | | | | | | |

We can stop at t = 5. Exponential growth models grow without bound, and we are already at a population larger than any of the values on the table. That probably means that our exponential growth model is not the ideal model to use here. It's likely the paramecium were <u>not</u> growing without bound. They probably had some kind of carrying capacity.

So is the logistic model we found any better?

| t | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 2 | 3 | 22 | 16 | 39 | 52 | 54 | 46 | 50 | 76 | 69 | 51 | 57 | 70 | 53 | 59 | 57 |
| $P_L$ | 2 | 4 | 9 | 17 | 28 | 40 | 51 | 58 | 61 | 62 | 63 | 64 | 64 | 64 | 64 | 64 | 64 |

It's not perfect, but it's definitely better. Also note that whenever the observed population went above the carrying capacity of 64, it immediately changed direction to go down. It's likely that the population was in fact logistic. Was Gause correct that the carrying capacity was 64? Probably not, but he did appear to be on the right track thinking the growth was logistic, not natural.

-There are other models that can be used for population growth, such as logistic growth with immigration taken into account, giving us a differential equation of:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

-Also, some populations run the risk of extinction if they dip below a minimum population m, and these populations have a differential equation of:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$