7.1) Integration by Parts:

-The chain rule of differentiation had the analogous integration substitution rule. Is there an analogue to the product rule of differentiation in integration? It turns out there is. What was the product rule again?

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g'(x)f(x)$$

-So what would happen if we took the indefinite integral of each half of the product rule formula?

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int (f(x)g'(x) + g'(x)f(x))dx$$

-We can rewrite this a little further to get the following:

$$f(x)g(x) = \int f(x)g'(x)dx + \int g'(x)f(x))dx$$

-After rearranging we get: $\int f(x)g'(x)dx = f(x)g(x) - \int g'(x)f(x))dx$

-The formula seen above is called the **integration by parts** formula, which is a means of being able to take the integral of a product of two functions, those two functions being $f(x)$ and $g'(x)$. The integral is in fact a combination of antiderivatives and integrals of the functions and their antiderivatives.

Integration by Parts:

Let u=f(x) and v=g(x) be two differentiable functions of x. The differentials of u and v would therefore be $du = f'(x)dx$ and $dv = g'(x)dx$. The integral therefore of udv would be:

$$\int udv = uv - \int vdu$$

-The differential version of the integration by parts formula is the more popular version and the one that is easier to remember. However, the hard part is not remembering the formula, the hard part is knowing what all the "parts" of the integration by parts formula are. What is u, what is dv, and so on.

-After all, when all is said and done, there are 4 parts to the integration by parts formula, u, v and their differentials. Being able to find them all can be tricky, not to mention the procedure of actually integrating the parts.

Example: Integrate $\int xcos(x)dx$

Solution: To integrate this expression by parts, we have to decide on the parts. We need to find what is u and what is dv, and then we have to find the corresponding du and v. Remember, if you are using the differential notation like we are here, feel free to include dx in your parts.

Should u be x, or should dv be xdx? This is the branching point that will decide the problem, because unlike a lot of other methods we have seen so far, one branch will solve the problem, and the other will not. **Choose carefully**, because assigning parts improperly will cause the problem to loop indefinitely.

How do you choose? In this case consider that differentials need to be integrated, and so if dv=xdx, that would make the corresponding part $v = \frac{1}{2}x^2$. The original integral requires integration by parts because it is a product of a trigonometric function times a power of x. If you make $dv = xdx$, then the other part will have a larger power of x, which is not an improvement on the previous integral.

So let u = x, and thus by default, dy = cos(x)dx. Therefore:

$$u = x, \qquad dv = cos(x)dx,$$
$$du = dx, \qquad v = sin(x)$$

Therefore, if these are the pieces, that would make the integration by parts procedure look like this:

$$\int udv = uv - \int vdu$$

$$\int xcos(x)dx = xsin(x) - \int sin(x)dx = xsin(x) + cos(x) + C$$

Notice how we do not add the extra constant term of C until the very end.

-When deciding what function is u and what function is dv, ask yourself, would vdu be easier to integrate by hand than udv is? If not, you may want to consider reassigning the parts. Of course, not every integration by parts situation is obvious for what the parts are.

Example: Integrate $\int ln(x)dx$

Solution: Ever wonder why we never bothered to show the antiderivative of ln(x) earlier? It's because this is harder to do than you think. It doesn't even look like an integration by parts problem, since there appears to be only one part, but don't forget, no coefficient is the same as a coefficient of 1:

$$\int (1)ln(x)dx$$

So let u = ln(x) and let dv = (1)dx, which means:

$$u = ln(x), \qquad du = \frac{1}{x} * dx, \qquad dv = 1dx, \qquad v = x$$

Put the parts together and you get:

$$\int (1)ln(x)dx = xln(x) - \int x(\frac{1}{x})dx = xln(x) - \int dx = xln(x) - x + C$$

Exercises: Find the integrals of the following:

a) $\int x^4 ln(x^3)dx$ b) $\int xe^{4x}dx$

-Integration by parts is similar to other rules like L'Hospital's Rule or the Chain Rule in that if need be, the rule can be repeated as many times as necessary to complete the problem, or at least until another method presents itself:

Example: Integrate $\int x^3 e^{-3x} dx$

Solution: As always, it's worth it to consider which assignment of parts will lead to an easier integral and which will not. If $dv = x^3 dv$, that would make $v = \frac{1}{4}x^4$. This will make the other integral in the procedure more difficult to integrate, so perhaps this is not a great assignment of parts. We should use the other combination:

$$u = x^3, \qquad du = 3x^2 dx, \qquad dv = e^{-3x} dx, \qquad v = -\frac{1}{3}e^{-3x}$$

The integral therefore becomes: $\int x^3 e^{-3x} dx = -\frac{1}{3}x^3 e^{-3x} - \int(-\frac{1}{3}e^{-3x} * 3x^2 dx) = -\frac{1}{3}x^3 e^{-3x} + \int x^2 e^{-3x} dx.$

Now we have to integrate the remaining integral using more integration by parts. This is fine, we just have to reassign parts and repeat the process. Usually the assignment of parts will be similar to what they were the first time around (what was dv before will likely still be dv later):

$$u = x^2, \qquad du = 2xdx, \qquad dv = e^{-3x} dx, \qquad v = -\frac{1}{3}e^{-3x}$$

The integral becomes: $-\frac{1}{3}x^3 e^{-3x} + \int x^2 e^{-3x} dx = -\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} - \int(-\frac{1}{3}e^{-3x} * 2x) dx$

$$-\frac{1}{3}x^3 e^{-3x} + \int x^2 e^{-3x} dx = -\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int(e^{-3x}x) dx$$

One more integration by parts should do the trick for that last integral. The important thing is to make progress:

$$u = x, \qquad du = dx, \qquad dv = e^{-3x} dx, \qquad v = -\frac{1}{3}e^{-3x}$$

The integral becomes:

$$-\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int(e^{-3x}x) dx = -\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}x e^{-3x} - \int -\frac{1}{3}e^{-3x} dx\right)$$

$$= -\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x}\right) + C$$

$$= -\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} - \frac{2}{27}e^{-3x} + C$$

-It's a bit long-winded and there are many steps, but it is possible. Sometimes the integration-by-parts loop will not end, in which case you may need a little algebra:

Example: Integrate $\int sin(x) e^{-x} dx$

Solution: Seemingly this seems like a never-ending integration by parts loop, since neither part will completely disappear or lose power whether you differentiate or integrate. We will arbitrarily assign the parts:

$$u = sin(x), \qquad du = cos(x)dx, \qquad dv = e^{-x}dx, \qquad v = -e^{-x}$$

Put the parts together and you get: $\int sin(x)e^{-x}dx = -sin(x)e^{-x} + \int cos(x)e^{-x}dx$

We need integration by parts again for the new integral:

$$u = cos(x), \qquad du = -sin(x)dx, \qquad dv = e^{-x}dx, \qquad v = -e^{-x}$$

So you get:

$$\int sin(x)e^{-x}dx = -sin(x)e^{-x} + \int cos(x)e^{-x}dx = -sin(x)e^{-x} - cos(x)e^{-x} - \int(-sin(x)(-e^{-x})dx$$

$$\int sin(x)e^{-x}dx = -sin(x)e^{-x} - cos(x)e^{-x} - \int sin(x)e^{-x}dx$$

If we attempt to use integration by parts again, we are using the same parts. However, if you take another look at the equation, you see you can actually solve for the original integral:

$$\int sin(x)e^{-x}dx = -sin(x)e^{-x} - cos(x)e^{-x} - \int sin(x)e^{-x}dx$$

$$2 * \int sin(x)e^{-x}dx = -sin(x)e^{-x} - cos(x)e^{-x}$$

$$\int sin(x)e^{-x}dx = \tfrac{1}{2}\left(-sin(x)e^{-x} - cos(x)e^{-x}\right)$$

$$\int sin(x)e^{-x}dx = -\tfrac{1}{2}\left(sin(x)e^{-x} + cos(x)e^{-x}\right) + C$$

Sometimes you need to keep your eyes open and not fall into a continuous loop of integrating by parts again, and again, and again, and again....

Exercise: Integrate the following:

a) $\int xe^{2x}dx$ b) $\int sin(x)e^{2x}dx$

-What about definite integrals? It turns out that much is different for definite integrals when it comes to integration by parts, you just need to make sure you evaluate before you are finished. Whether you choose to evaluate along the way or evaluate at the very end (be careful if you use any integration by substitution along the way) is up to you:

Integration by Parts:

Let u=f(x) and v=g(x) be two differentiable functions of x from x=a to x=b. The differentials of u and v would therefore be $du = f'(x)dx$ and $dv = g'(x)dx$. The definite integral therefore would be:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g'(x)f(x))dx \qquad \text{or} \qquad \int_a^b udv = uv|_a^b - \int_a^b vdu$$

-The u/v notation can be a little trickier to read, but the procedure is the same in practice.

Example: Evaluate $\int_0^1 tan^{-1}(x)dx$

Solution: The parts are there, even if you can't see them:

$$u = tan^{-1}(x), \quad du = \frac{1}{x^2+1}dx, \qquad dv = dx, \qquad v = x$$

Now let's integrate:

$$\int_0^1 tan^{-1}(x)dx = xtan^{-1}(x)|_0^1 - \int_0^1 x\left(\frac{1}{x^2+1}\right)dx$$

You can evaluate the first half now and then work on the second integral, or you can wait until after you have worked on the second integral before you plug anything in. We will wait till the end to plug anything in:

$$\int_0^1 tan^{-1}(x)dx = xtan^{-1}(x)|_0^1 - \int_0^1 x\left(\frac{1}{x^2+1}\right)dx = xtan^{-1}(x)|_0^1 - \frac{1}{2}ln(x^2+1)|_0^1 = (tan^{-1}(1) - 0) - (\frac{1}{2}ln(2) - 0)$$

$$\int_0^1 tan^{-1}(x)dx = \frac{\pi}{4} - \frac{1}{2}ln(2)$$

Exercise: Evaluate $\int_0^1 sin^{-1}(x)dx$

Reduction Formulas:

Taking another look at trigonometric function integrals (not for the last time!), what happens if you take an integral of a higher power of sin(x)? As it turns out, the procedure for taking the integral of a higher power of sin(x) involves integration by parts. Doing so will reduce the power that sin(x) is raised to, making the expression easier to integrate.

The procedure is popular enough however that there is a formula for reducing the power:

$$\int sin^n(x)dx = -\frac{1}{n}cos(x)sin^{n-1}(x) + \frac{n-1}{n}\int sin^{n-2}(x)dx \qquad \text{for } n \geq 2$$

-Why does this work the way it does? Let's take a look at the proof for why:

-To integrate $\int sin^n(x)dx$, start with separating $sin^n(x)$ into two parts: $sin(x)sin^{n-1}(x)$. Now we assign parts:

$$u = sin^{n-1}(x), \quad du = (n-1)sin^{n-2}(x)cos(x)dx, \qquad dv = sin(x)dx, \quad v = -cos(x)$$

-Now we integrate by parts:

$$\int sin^n(x)dx = sin^{n-1}(x)(-cos(x)) - \int((n-1)sin^{n-2}(x)cos(x))(-cos(x))dx$$

$$\int sin^n(x)dx = -sin^{n-1}(x)cos(x) + (n-1)\int(sin^{n-2}(x)cos^2(x))dx$$

-Use the trigonometric identity $cos^2(x) = 1 - sin^2(x)$ to get:

$$\int sin^n(x)dx = -sin^{n-1}(x)cos(x) + (n-1)\int(sin^{n-2}(x)(1 - sin^2(x))dx$$

$$\int sin^n(x)dx = -sin^{n-1}(x)cos(x) + (n-1)\int(sin^{n-2}(x) - sin^n(x))dx$$

$$\int sin^n(x)dx = -sin^{n-1}(x)cos(x) + (n-1)\int sin^{n-2}(x)dx - (n-1)\int sin^n(x))dx$$

-Now move the like integrals to the left side:

$$n\int sin^n(x)dx = -sin^{n-1}(x)cos(x) + (n-1)\int sin^{n-2}(x)dx$$

$$\int sin^n(x)dx = -\tfrac{1}{n}sin^{n-1}(x)cos(x) + \tfrac{(n-1)}{n}\int sin^{n-2}(x)dx$$

The proof is finished.

-However, remember this reduction formula only reduces the power; you still have to integrate the rest of the problem from here. Remember, you can keep reducing until $\int sin^{n-2}(x)dx$ is either $\int sin(x)dx$ or $\int sin^0(x)dx$, which can both be easily integrated.

Exercise: Integrate the following:

$a)\int sin^3(x)dx$ $\qquad\qquad$ $b)\int sin^4(x)dx$