11.5) Alternating Series and Absolute Convergence:

-Recall that an **alternating series** is any series in which the terms alternate from positive to negative:

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{1}{2}\right)^{n-1}$ $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$

-All of these above are examples of alternating series, and they can be identified by their rule, a_n containing some variation on the factor $(-1)^n$. This means alternating series rules in general can be written as:

$$a_n = (-1)^{n-1}b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

- b_n is intended to be a positive number, and in fact, $|a_n| = b_n$. If the first term is positive the left definition is usually the rule of the series, and if the first term is negative the right definition is the rule of the series.

-We will see later that the idea that $|a_n| = b_n$ helps us in the concept of absolute convergence, but we'll get to that later. For now, we have an important property that certain alternating series have:

The Alternating Series Test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots. \qquad (b_n > 0)$$

has the properties:

1) $b_{n+1} \le b_n$ for all n
2) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

-Remember the idea that in general, $\sum_{n=1}^{\infty} a_n$ does not have to be convergent if $\lim_{n\to\infty} a_n = 0$? Well, it turns out that if the series is alternating from positive terms to negative terms and is decreasing (eventually) then it actually <u>is</u> true that if $\lim_{n\to\infty} a_n = 0$, then the series is convergent. The alternating sign makes all the difference. Why is that?

-The proof for the alternating series test works in pairs. Consider partial sums that take terms in pairs:

$$s_2 = b_1 - b_2$$
$$s_4 = s_2 + b_4 - b_3$$
$$s_6 = s_4 + b_6 - b_5$$

We are considering the absolute values of the terms, $b_n = |a_n|$ to avoid worry of signs, though the proof can be done with signs without loss of generality. Also, since $b_{n+1} \le b_n$ for all n, all of these partial sums are positive:

$$s_2 = b_1 - b_2 \ge 0 \qquad \text{since} \qquad b_1 \ge b_2$$

$s_4 = s_2 + b_4 - b_3$ since $b_4 \geq b_3$ and $s_2 \geq 0$ ..... And so on.

This means that in general:

$$s_{2n} = s_{2n-2} + b_{2n-1} - b_{2n} \geq s_{2n-2}$$

So the even partial sums are increasing:

$$0 \leq s_2 \leq s_4 \leq s_6 \leq \ldots \leq s_{2n} \leq \ldots$$

However, the partial sum s_{2n} can be rewritten like so:

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots. - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Since $b_{2n-2} - b_{2n-1} \geq 0$ for all n, this means that $s_{2n} \leq b_1$, the first term in the alternating series. This implies that $\{s_{2n}\}$, the sequence of even partial sums, is bounded above, and increasing. This implies the sequence $\{s_{2n}\}$ is convergent. We will call the sum that $\{s_{2n}\}$ converges to s.

$$\lim_{n\to\infty} s_{2n} = s$$

Okay, but that's just even partial sums. What about the odd ones? Remember for any partial sum of 2n+1 terms, $s_{2n+1} = s_{2n} + b_{2n+1}$. So what would the limit $\lim_{n\to\infty} s_{2n+1}$ approach?

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} (s_{2n} + b_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1}$$

Part of the definition of an alternating series is that $\lim_{n\to\infty} b_n = 0$, so we have:

$$\lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} = \lim_{n\to\infty} s_{2n} + 0 = s + 0 = s$$

Both the even partial sums and the odd partial sums converge to the same finite limit, so that means the alternating series is convergent in general.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Solution: If we ignore the alternating factor, we have $b_n = \frac{1}{n}$.

- $b_{n+1} < b_n$ since $\frac{1}{n+1} < \frac{1}{n}$ for all n.
- $\lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, this series, the alternating harmonic series, is convergent.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n+1}$$

Solution: We have $b_n = \frac{3n}{4n+1}$. However, if we take the limit of $\frac{3n}{4n+1}$ as n goes to infinity, we have:

$$\lim_{n\to\infty} \frac{3n}{4n+1} = \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}} = \frac{3}{4}$$

Since the limit of the absolute value of the expression does not go to zero, this series is divergent.

Example: Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+1}$$

Solution: We have $b_n = \frac{n^2}{n^3+1}$, but is this a decreasing sequence? Let $f(x) = \frac{x^2}{x^3+1}$ and we will test it:

$$f'(x) = \frac{2x(x^3+1)-3x^2(x^2)}{(x^3+1)^2} = \frac{2x^4+2x-3x^4}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x)$ is negative for all positive x greater than $\sqrt[3]{2}$. While that is not an interval for all positive x, the point is that eventually $b_n \geq b_{n+1}$. That is good enough for the first condition of the alternating series test to be satisfied.

As for the second condition, it is not difficult to see that $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$. So this is a convergent series.

Exercise: Determine whether the following series converges or diverges:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} \frac{cos(\pi n)}{n^2}$ c) $\sum_{n=1}^{\infty} \frac{n^3}{n^4-2}$

-While we have seen how to show that an alternating series is convergent or divergent, knowing that an alternating series converges does not help us to determine the sum of the alternating series. The best we can do is estimate the sum of an alternating series.

-We use the usual notation for error and remainders of partial sums for alternating series:

$$R_n = s - s_n$$

-As it turns out though, it can be proven that the remainder for an alternating aries partial sum is simply smaller than b_{n+1}, the absolute value of the <u>next</u> term after the first n terms that have all been added together.

If the sum of a series is $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of the alternating series that satisfies:

1) $b_{n+1} \leq b_n$ for all n
2) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$

-Why is this so? Remember how in the proof of the alternating series test we stated that $\lim_{n\to\infty} s_{2n} = s$, which implies s is larger than all the even partial sums of the series. However, since all the odd partial sums would therefore have a term afterwards that would be positive to bring the partial summation closer to s, that means s is smaller than all the partial sums.

-However, this also means that for any partial sum s_n, s is closer to s_n than s_{n+1} is. This implies:

$$|s - s_n| \leq \left|s_{n+1} - s_n\right| = b_{n+1}$$

Thus, the remainder is less than whatever the next term in the series would be: $R_n = |s - s_n| \leq b_{n+1}$

-Therefore, if you want to find an approximation of an alternating sums that has a remainder of R or less, simply find the first term that is smaller than R in the series, and add together all the terms that come before that term.

Example: Find the sum of the series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Solution: First off, this is convergent by the alternating series theorem.

- $b_{n+1} < \frac{1}{(n+1)!} = \frac{1}{n+1} * \frac{1}{n!} < \frac{1}{n!} = b_n$
- $\lim_{n\to\infty} \frac{1}{n!} = 0$

To have a series sum that is correct to three decimal places means we want the remainder to be less than 0.0005. What is the first term in the sequence $\left\{\frac{1}{n!}\right\}$ that is less than 0.0005? (it helps to know 0.0005 equals $\frac{1}{2000}$)

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \frac{1}{5040}, \cdots$$

The 7th term, $\frac{1}{5040}$ is the first term that is less than the remainder. Since the (n+1)th term in the sum of an alternating series is always the remainder of the partial sum up to n terms, that means the partial sum of the first six terms of this sum is accurate to the sum to three decimal places.

So what is the partial sum of the first six terms? Don't forget the alternating sign!

$$s = \sum_{n=1}^{6} \frac{(-1)^n}{n!} = -\frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = -0.6319\overline{4}$$

Exercise: Find the sum of the series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ correct to three decimal places.

-Given any series $\sum_{n=1}^{\infty} a_n$, we can make an analogous series from $\sum_{n=1}^{\infty} |a_n|$. All the same terms, but we take the absolute values of each term before we add them together. So if you took a convergent alternating series and removing the alternating factor, $(-1)^n$, would the series still be convergent?

-A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the analogous series $\sum_{n=1}^{\infty} |a_n|$ is also convergent.

-First off, if the series $\sum_{n=1}^{\infty} a_n$ is already positive for all its terms, then absolute convergence and regular convergence are one and the same. Meaning that absolute convergence is a topic saved for alternating series and other series that can have positive and negative terms.

-Some alternating series, like:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

are absolutely convergent, because if you took the absolute value of the terms, you would have $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a p-series with a power greater than 1, which makes this series convergent. So because the absolute value version is convergent, the alternating series is absolutely convergent. Also, in case you were wondering, a series that is absolutely convergent by default is also convergent.

-Some alternating series, like:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

are not absolutely convergent, because if you took the absolute value of the terms here, you would have $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series, a divergent series. Still, the original alternating harmonic series is convergent by the alternating series test, so while not absolutely convergent, this alternating series is what we call **conditionally convergent**.

-A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

-A series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} |a_n|$ is divergent, but $\sum_{n=1}^{\infty} a_n$ is convergent.

Example: Determine if the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{cos(n)}{n^2}$$

Solution: This series has positive and negative terms, but is not alternating. However, we can try to determine what is true about the absolute value version of this series:

$$\sum_{n=1}^{\infty} \left|\frac{cos(n)}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|cos(n)|}{n^2}$$

However, one thing we do know absolutely about cos(x) is that $-1 \leq cos(x) \leq 1$, which in turn means:

$$|cos(x)| \leq 1$$

Therefore, we have:

$$\sum_{n=1}^{\infty} \left|\frac{cos(n)}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|cos(n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We have that the absolute value version of this series is less than the p-series with p>1. By the comparison test, that means the absolute value series $\sum_{n=1}^{\infty} \left|\frac{cos(n)}{n^2}\right|$ is convergent, which means $\sum_{n=1}^{\infty} \frac{cos(n)}{n^2}$ is absolutely convergent, which means $\sum_{n=1}^{\infty} \frac{cos(n)}{n^2}$ by default is convergent.

Exercises: Determine if the following series are absolutely convergent, conditionally convergent, or divergent.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$ c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

-One strategy that can prove fruitful for determining if a series is convergent or divergent is **rearrangement** of the terms. It is exactly what it sounds like: rearranging the order the terms are added together. After all, addition is commutative, so it shouldn't matter what order the terms are written in, right?

-On top of that, if $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series with sum s, then any rearrangement of that same series will have the same sum of s. You may think that would hold even if the series was merely conditionally convergent, but that is not always true. In fact, any conditionally convergent series can be rearranged to give a different sum.

-Take the alternating harmonic series which is conditionally convergent and we will see (if there is time) has a sum of ln(2):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = ln(2)$$

-What if we divide everything in this summation by 2?

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + ... = \frac{1}{2}ln(2)$$

-With a little bit of rewriting, you can insert extra zeros into the summation in the previous equation to get something that looks like this:

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + ... = \frac{1}{2}ln(2)$$

-Now watch what happens when I combine this equation together with the original alternating harmonic series:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + ... = ln(2)$$
$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + ... = \frac{1}{2}ln(2)$$

$$\frac{1}{1} + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + ... = \frac{3}{2}ln(2)$$

-What do you see in this final equation? I see all the same alternating terms of the original alternating harmonic series, simply written in a different order. However, the other side of the equal sign is $\frac{3}{2}ln(2)$, not $ln(2)$ as the original equation had.

-Same numbers, different order, and somehow that managed to give us two different sums!

-In fact, you can do this with any conditionally convergent series you want to get any sum you want. Given a conditionally convergent series $\sum_{n=1}^{\infty} a_n$, and any real number r, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that has a sum equal to r.