Key Facts about Continuous Increasing Functions and Generalized Inverses – PROOFS

(1) $$\Pr(G(X) \le G(x)) = \Pr(X \le x) + \Pr(X > x \;\&\; G(X) = G(x)) + \Pr(X > x \;\&\; G(X) < G(x))$$

But G is increasing so the last term goes away.

(2) $$\Pr(G(X) \le G(x)) = \Pr(X \le x) + \Pr(X > x \;\&\; G(X) = G(x)) = \Pr(X \le x)$$

(3) Given G increasing.

(a) Suppose G is continuous

Now suppose that $u_2 > u_1$ and $G^{\leftarrow}(u_2) = G^{\leftarrow}(u_1)$

$$G^{\leftarrow}(u_2) = \inf\{x \mid G(x) \ge u_2\} = \bar{x}$$
$$G^{\leftarrow}(u_1) = \inf\{x \mid G(x) \ge u_1\} = \bar{x}$$

$\Rightarrow$ if $x < \bar{x}$ $\quad G(x) < u_1$

$x \ge \bar{x}$ $\quad G(x) \ge u_2$ $\quad$ no good

(b) $G^{\leftarrow}$ strictly increasing

if G is not continuous $\Rightarrow \lim_{x \uparrow \bar{x}} G(x) < G(\bar{x})$

$$\lim_{x \uparrow \bar{x}} G(x) = u_1 \qquad G(\bar{x}) = u_2 \qquad u_1 < u_2$$

Now $\inf\{x \mid G(x) \geq u_1\} = \bar{x}$

$\inf\{x \mid G(x) \geq u_2\} \leq \bar{x}$

$\Rightarrow G^{\leftarrow}$ not strictly increasing

4. G right continuous

Suppose $G(x) \geq y$ and $x < G^{\leftarrow}(y)$

$x < \inf\{x' \mid G(x') \geq y\}$

But $G(x) \geq y$ no good

Now if $x \geq G^{\leftarrow}(y)$

$\geq \inf\{\bar{x} \mid G(\bar{x}) \geq y\} = \tilde{x}$

Since G is right continuous, as

$x \downarrow \tilde{x}$ we always have $G(x) \geq y$

$\Rightarrow G(\tilde{x}) \geq y$

$G(x) \geq y.$

5. G strictly increasing

let $u = G(x)$

$G^{\leftarrow}(u) = \inf\{\bar{x} \mid G(\bar{x}) \geq u\}$

$= \inf\{\bar{x} \mid G(\bar{x}) \geq G(x)\}$

We cannot have $\bar{x} < x$ so $\bar{x} = x$.

(6) G is continuous

(3)

let $x = G^{\leftarrow}(u)$

$x = \inf \{ \bar{x} \mid G(\bar{x}) \geq u \}$

if $G(x) < u$ then as $\bar{x} \downarrow x$

$$G(x) = \lim_{\bar{x} \downarrow x} (G(\bar{x})) \geq u$$

$\Rightarrow G(x) \geq u.$

if $G(x) > u$ $\quad \bar{x} \uparrow x$

$G(\bar{x}) < u$ but $G(x) > u$

not possible if G is continuous