Consider an ARMA (1,1) process

$$X_t - \phi X_{t-1} = \varepsilon_t + \theta \varepsilon_{t-1}$$

Assume that

$$\varepsilon = SWN(0, \sigma^2)$$

Now suppose we have samples

$X_0, X_1, \ldots, X_N$.

Conditional log likelihood

$$X_1 = \phi X_0 + \varepsilon_1 + \theta \varepsilon_0$$

Assume that $\varepsilon_t = \mathcal{N}(0, \sigma^2)$

$$\Rightarrow X_1 = \mathcal{N}(\phi X_0, \sigma^2(1+\theta^2))$$

hence $f(X_1 | X_0)$

$$= \frac{1}{\sigma\sqrt{2\pi(1+\theta^2)}} \exp\left(-\frac{1}{2}\left(\frac{(X_1 - \phi X_0)^2}{\sigma^2(1+\theta^2)}\right)\right)$$

More generally

$$X_n = \mathcal{N}(\phi X_{n-1}, \sigma^2(1+\theta^2))$$

ARMA(1,1) is causal if

$1 - \phi z$ has no roots in $|z| \le 1$

$\Rightarrow |\phi| < 1$

$z = \frac{1}{\phi} \quad |z| > 1.$

also $\frac{1}{\phi} \neq -\frac{1}{\theta}$

$\theta \neq -\phi$

$$\rho(h) = \frac{\phi^{|h|-1}(\phi+\theta)(1+\phi\theta)}{1+\theta^2+2\phi\theta}$$

if $\phi = 0$ then

$X_t = \varepsilon_t + \theta\varepsilon_{t-1}$

$Var(X_t) = \sigma^2 + \sigma^2\theta^2 = \sigma^2(1+\theta^2)$

but the formula in the slides is

$$\sigma^2\left[\frac{(1-\phi^2) + (\phi+\theta)^2\phi^2}{(1-\phi^2)}\right] \Rightarrow \sigma^2$$

should be $$\sigma^2\left[\frac{(1-\phi^2) + (\phi+\theta)^2}{1-\phi^2}\right] \Rightarrow \sigma^2(\theta^2+1)$$

$$X_t = \sigma_t Z_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$X_t^2 = \alpha_0 Z_t^2 + \alpha_1 X_{t-1}^2 Z_t^2 + \beta \sigma_{t-1}^2 Z_t^2$$

$$E(X_t^2) = \alpha_0 + \alpha_1 E(X_{t-1}^2) + \beta E(\sigma_{t-1}^2)^*$$

fitting a VARMA process

$$X_t = \Phi X_{t-1} + \varepsilon_t + \Theta \varepsilon_{t-1}$$

$$\Phi = \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \theta_{11} & 0 \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

$$\begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \theta_{11} & 0 \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{pmatrix}$$

$$X_{1,t} = \phi_{11} X_{1,t-1} + \theta_{11} \varepsilon_{1,t-1}$$

$$Var(X_{1,t}) = \theta_{11}^2$$

$$E(X_{1,t} \mid X_{1,t-1}) = \phi_{11} X_{1,t-1}$$

$$X_{2,t} = \phi_{21} X_{1,t-1} + \phi X_{2,t-1} + \theta_{21} \varepsilon_{1,t-1} + \theta_{22} \varepsilon_{2,t-1}$$

$$E(X_{2,t} \mid X_{1,t-1}, X_{2,t-1}) = \phi_{21} X_{1,t-1} + \phi X_{2,t-1}$$

Simple case

$$\varepsilon \sim \mathcal{N}(0, I_2)$$

$$\mathrm{Var}(X_{2,t} \mid X_{1,t-1}, X_{2,t-1}) = \theta_{21}^2 + \theta_{22}^2$$

$$\mathrm{Cov}(X_{1,t}, X_{2,t} \mid X_{1,t-1}, X_{2,t-1}) = \theta_{11}\theta_{21}$$

$$\rho = \frac{\theta_{11}\theta_{21}}{\theta_{11}\sqrt{\theta_{21}^2 + \theta_{22}^2}}$$

$$= \frac{\theta_{21}}{\sqrt{\theta_{21}^2 + \theta_{22}^2}}$$

$$f(X_{1,t}, X_{2,t} \mid X_{1,t-1}, X_{2,t-1})$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right]$$

where $\Sigma = \begin{pmatrix} \theta_{11}^2 & \theta_{11}\theta_{21} \\ \theta_{11}\theta_{21} & \theta_{21}^2 + \theta_{22}^2 \end{pmatrix}$

$$\sigma_1^2\sigma_2^2 - \sigma_1^2\sigma_2^2\rho^2 = \theta_{11}^2(\theta_{21}^2 + \theta_{22}^2) - \theta_{11}^2\theta_{21}^2$$

$$= \theta_{11}^2\theta_{22}^2$$

so we get

$$\frac{1}{2\pi\,\theta_{11}\theta_{22}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

$$\mu = (\phi_{11} x_{1,t-1},\ \phi_{21} x_{1,t-1} + \phi\, x_{2,t-1})$$