Factor Models

$$F = \frac{\sqrt{\rho}}{1+\rho(d-1)} \sum_{i=1}^{d} X_i + \sqrt{\frac{1-\rho}{1+\rho(d-1)}}\, Y$$

$\mathrm{Var}(X_i) = 1 \quad E(X_i) = 0 \quad Y, X_i$ independent

$\mathrm{Var}(Y) = 1$

$$B = \begin{pmatrix} \sqrt{\rho} \\ \vdots \\ \sqrt{\rho} \end{pmatrix} \qquad X = BF + \Sigma \qquad \mathrm{Cov}(X_i, X_j) = \rho$$

$$\Sigma_j = X_j - \sqrt{\rho}\, F$$

$$\mathrm{Cov}(F, \Sigma_j) = \mathrm{Cov}(F, X_j) - \sqrt{\rho}\,\mathrm{Var}(F)$$

$$= (d-1)\cdot\frac{\sqrt{\rho}}{1+\rho(d-1)}\cdot\rho + \frac{\sqrt{\rho}}{1+\rho(d-1)} - \sqrt{\rho}\,\mathrm{Var}(F)$$

$$= \sqrt{\rho} - \sqrt{\rho}\,\mathrm{Var}(F)$$

Now $$\mathrm{Var}(F) = \frac{d\rho}{(1+\rho(d-1))^2} + d(d-1)\frac{\rho^2}{(1+\rho(d-1))^2} + \frac{(1-\rho)}{1+\rho(d-1)}$$

$$= \frac{d\rho}{(1+\rho(d-1))} + \frac{1-\rho}{1+\rho(d-1)} = 1$$

$\Rightarrow \mathrm{Cov}(F, \Sigma_j) = 0.$

$$\mathrm{Cov}(\Sigma_i, \Sigma_j) = \mathrm{Cov}(X_i, X_j) - \sqrt{\rho}\,\mathrm{Cov}(X_j, F) - \sqrt{\rho}\,\mathrm{Cov}(X_i, F) + \rho\,\mathrm{Var}(F)$$

$$= \rho - \rho = 0.$$