Rank Correlations are Attainable

PROOF

KENDALL'S TAU:

We use the formula

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 [\lambda W(u_1,u_2) + (1-\lambda) M(u_1,u_2)] \times$$
$$d[\lambda W(u_1,u_2) + (1-\lambda) M(u_1,u_2)] \quad -1$$

$$= 4\lambda^2 \int_0^1\int_0^1 W\,dW + 4(1-\lambda)^2 \int_0^1\int_0^1 M\,dM$$
$$+ 4(\lambda-\lambda^2)\int_0^1\int_0^1 M\,dW + 4(\lambda-\lambda^2)\int_0^1\int_0^1 W\,dM$$

Now $W(u_1,u_2) = P_r(U < u_1,\ 1-U < u_2) = \text{Max}(1-u_1-u_2, 0)$

But to compute $\int_0^1\int_0^1 W\,dW$ we can assume that a.s. $U_2 = 1-U_1$

so that $\int_0^1\int_0^1 W\,dW = \int_0^1 \text{Max}(1-u_1-(1-u_1))\,du_1 = 0$

Similarly, $\int_0^1\int_0^1 M\,dM = \int_0^1\int_0^1 \min(u_1,u_2)\,dM$

$$= \int_0^1 u\,du = \frac{1}{2}$$

Next, $dW = 0$ unless $u_2 = 1-u_1$, so

$$\int_0^1\int_0^1 M\,dW = \int_0^1 \min(u, 1-u)\,du = \frac{1}{4}$$

Also, $dM = 0$ unless $u_2 = u_1$, so

$$\int_0^1\int_0^1 W\,dM = \int_0^1 (1-2u)^+\,du$$

$$= \int_0^{\frac{1}{2}} (1-2u)\,du$$

$$= \frac{1}{4}$$

Thus $\rho_\tau(X_1, X_2) = 4(1-\lambda)^2 \cdot \frac{1}{2} + 4(\lambda-\lambda^2)\cdot\frac{1}{4} + 4(\lambda-\lambda^2)\cdot\frac{1}{4} - 1$

$$= 1-2\lambda.$$

SPEARMAN'S RHO:

In this case,

$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 \{[\lambda W(u_1,u_2) + (1-\lambda)M(u_1,u_2)] - u_1u_2\}\,du_1\,du_2$$

$$= 12\lambda\int_0^1\int_0^1 W(u_1,u_2)\,du_1\,du_2 + 12(1-\lambda)\int_0^1\int_0^1 M(u_1,u_2)\,du_1\,du_2 - 3$$

$$= 12\lambda\int_0^1\int_0^1 (1-u_1-u_2)^+\,du_1\,du_2 + 12(1-\lambda)\int_0^1\int_0^1 \min(u_1,u_2)\,du_1\,du_2 - 3$$

$$= 12\lambda\int_0^1\int_0^{1-u_2} (1-u_1-u_2)\,du_1\,du_2 + 12(1-\lambda)\cdot 2\int_0^1\int_0^{u_2} u_1\,du_1\,du_2 - 3$$

$$= 12\lambda \int_0^1 \left(1-u_2 - \frac{1}{2}(1-u_2)^2 - u_2(1-u_2)\right) du_2$$

$$+ 24(1-\lambda) \int_0^1 \frac{1}{2} u_2^2 \, du_2 \; - 3$$

$$= 12\lambda \int_0^1 \frac{1}{2} u^2 du + 12(1-\lambda) \cdot \frac{1}{3} - 3$$

$$= 2\lambda + 4 - 4\lambda - 3 = 1 - 2\lambda$$