Conditional Log Likelihood Example

Suppose that $(X_t)_{t\in\mathbb{Z}}$ is ARCH(1)

That is
$$X_t = \sigma_t Z_t$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$$
$$\Rightarrow \quad X_t = \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}$$

Consider the special case where $Z_t \sim N(0,1)$, i.i.d.

Then $X_t \mid X_{t-1} \sim \mathcal{N}\left(0, \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}\right)$

Thus
$$\log\left[\prod_{t=1}^{n} f_{X_t|X_{t-1}}(X_t \mid X_{t-1})\right] =$$
$$\sum_{t=1}^{n} \log\left[\frac{1}{\sqrt{2\pi}\sqrt{\alpha_0+\alpha_1 X_{t-1}^2}}\, e^{-\frac{1}{2}\frac{X_t^2}{\alpha_0+\alpha_1 X_{t-1}^2}}\right]$$
$$= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\log(\alpha_0+\alpha_1 X_{t-1}^2) - \frac{1}{2}\sum_{t=1}^{n}\frac{X_t^2}{\alpha_0+\alpha_1 X_{t-1}^2}$$