HW 4 Q 1,2 Matthew Vaughan

1) $X = \mu + \sqrt{W} A Z$ $W \sim \exp(\lambda)$ $F_W(w) = 1 - e^{-\lambda w}$ $\lambda > 0$

$\uparrow H_W(w)$

$Z \sim N(0, I_d)$

- This is of the form $X = \mu + AY$, $Y = \sqrt{W} Z$
and by the example on slide 52, Y is spherical so
X is elliptical.

- Dispersion matrix Σ can be seen from following slide 59
to get $\Sigma = AA^T$

$$\psi(t) = \int_0^\infty e^{-\frac{1}{2}wt} h(w)\, dw$$

$$= \int_0^\infty e^{-\frac{1}{2}wt} \left[\lambda e^{-\lambda w}\right] dw$$

$$= \int_0^\infty \lambda e^{(-\frac{1}{2}t - \lambda)w} dw$$

$$= \frac{\lambda}{(-\frac{1}{2}t - \lambda)} e^{(-\frac{1}{2}t - \lambda)w} \Big|_0^\infty$$

$$= -\frac{\lambda}{(-\frac{1}{2}t - \lambda)}$$

$$\boxed{\psi(t) = \frac{\lambda}{\frac{1}{2}t + \lambda}}$$

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2) $X = \mu + \sqrt{W} A Z \qquad W \sim N^{-1}(\lambda, \chi, \beta) \qquad Z \sim N(0, I_d)$

Like Q1, X is elliptical b/c $Y = \sqrt{W} Z$ is spherical

Like Q1, $\Sigma = AA^T$

$$h(w) = \frac{\chi^{-\lambda}\left[\sqrt{\chi\beta}\right]^{\lambda}}{2K_\lambda(\sqrt{\chi\beta})} w^{\lambda-1} e^{-(\chi w^{-1} + \beta w)/2}$$

$$\psi(t) = \frac{\chi^{-\lambda}\left[\sqrt{\chi\beta}\right]^{\lambda}}{2K_\lambda(\sqrt{\chi\beta})} \underbrace{\int_0^\infty w^{\lambda-1} e^{-\frac{1}{2}(\{\beta+t\}w + \chi w^{-1})} dw}$$

use $\int_0^\infty h(w)dw = 1$ and let $\lambda = \lambda$, $\beta = \beta + t$, $\chi = \chi$ then:

$$\int_0^\infty \left[\frac{\chi^{-\lambda}\left[\sqrt{\chi\{\beta+t\}}\right]^{\lambda}}{2K_\lambda(\sqrt{\chi(\beta+t)})}\right] w^{\lambda-1} e^{-(\chi w^{-1} + \{\beta+t\}w)/2} dw = 1$$

Doesn't depend on W, move to other side

$$\int_0^\infty w^{\lambda-1} e^{-(\chi w^{-1} + \{\beta+t\}w)/2} dw = \frac{2K_\lambda(\sqrt{\chi\{\beta+t\}})}{\chi^{-\lambda}\left[\sqrt{\chi\{\beta+t\}}\right]^{\lambda}}$$

This is what we wanted to solve for

$$\boxed{\psi(t) = \frac{\cancel{\chi^{-\lambda}}\left[\sqrt{\cancel{\chi}\beta}\right]^{\lambda}}{\cancel{2}K_\lambda\left[\sqrt{\chi\beta}\right]} \left[\frac{\cancel{2}K_\lambda\left[\sqrt{\chi\{\beta+t\}}\right]}{\cancel{\chi^{-\lambda}}\sqrt{\cancel{\chi}\{\beta+t\}}^{\lambda}}\right] = \frac{K_\lambda\left[\sqrt{\chi\{\beta+t\}}\right]}{K_\lambda\left[\sqrt{\chi\beta}\right]} \left\{\frac{\beta}{\beta+t}\right\}^{\lambda/2}}$$

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