HW2 Problem 1 Examples

We use the formula

$$\psi(t) = \int_0^\infty e^{-\frac{1}{2}wt} h(w)\, dw \quad \text{for}$$

$h(w) =$ density of $h(w)$

Example 1 W is uniform on $[0,1]$

$$\psi(t) = \int_0^1 e^{-\frac{1}{2}wt}\, dw$$

$$= \frac{2}{t}\left(1 - e^{-t/2}\right)$$

Example 2 W is "half normal"

$$\Pr(0 \le W \le w) = 2\mathcal{N}(w) - 1$$

Then $h(w) = \frac{2}{\sqrt{2\pi}} e^{-w^2/2}$, and

$$\psi(t) = \int_0^\infty e^{-\frac{1}{2}wt} h(w)\, dw = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}wt} e^{-\frac{1}{2}w^2}\, dw$$

$$= \int_0^\infty \frac{2}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}(w^2 + wt)\right) dw$$

$$= \int_0^\infty \frac{2}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\left(w^2 + wt + \tfrac{1}{4}t^2 - \tfrac{1}{4}t^2\right)\right) dw$$

$$= \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{t^2/8} \exp\left(-\tfrac{1}{2}\left(w + \tfrac{t}{2}\right)^2\right) dw$$

$$= 2e^{t^2/8}\left(1 - \mathcal{N}(t/2)\right) = 2e^{t^2/8}\mathcal{N}\left(-\tfrac{t}{2}\right).$$