Tail Dependence of Normal RVs and Gaussian Copulas

Suppose that $X_1, X_2 \sim \mathcal{N}(0,1)$, $\mathrm{Cov}(X_1, X_2) = \rho$.

The ~~density~~ d.f.

$$N_2(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\,\frac{u^2 - 2\rho u v + v^2}{1-\rho^2}\right] dv\,du$$

Hence density of $x \mid y$

$$f_{x|y} = \frac{\dfrac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\dfrac{1}{2}\,\dfrac{x^2 - 2\rho x y + y^2}{1-\rho^2}\right]}{\dfrac{1}{\sqrt{2\pi}}\, e^{-1/2\, y^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\,\frac{(x-\rho y)^2}{1-\rho^2}\right]$$

Thus $f_{x|y} \sim N(\rho y, 1-\rho^2)$

Therefore $\Pr(X \le y \mid y) =$

$$\Pr(\rho y + \sqrt{1-\rho^2}\, Z \le y) \qquad Z \sim \mathcal{N}(0,1)$$

$$= \mathcal{N}\left(\frac{y(1-\rho)}{\sqrt{1-\rho^2}}\right) = \mathcal{N}\left(y\sqrt{\frac{1-\rho}{1+\rho}}\right) \to 0 \text{ as}$$

$$y \to -\infty.$$

Thus $\lambda = 0$.