1) $x_t = a x_{t-1} - \frac{1}{2} x_{t-2} + \varepsilon_t$ $\quad \varepsilon_t \sim N(0, b^2)$ $\quad x_t \sim N(0,1)$

ε_t indep. of all x_s $s<t$

$$\boxed{\begin{array}{ll} p = 2 & \phi_1 = a \\ q = 0 & \phi_2 = -\frac{1}{2} \end{array}}$$

$$\gamma(h) = E[x_t x_{t-h}]$$

$$= E\left[\left\{a x_{t-1} - \tfrac{1}{2} x_{t-2} + \varepsilon_t\right\} x_{t-h}\right]$$

$$= E\left[a x_{t-1} x_{t-h} - \tfrac{1}{2} x_{t-2} x_{t-h} + \varepsilon_t x_{t-h}\right]$$

$$\gamma(h) = \begin{cases} a\gamma(h-1) - \frac{1}{2}\gamma(h-2) & h \neq 0 \\ a\gamma(h-1) - \frac{1}{2}\gamma(h-2) + b^2 & h = 0 \end{cases} \qquad \gamma(h) = \gamma(-h)$$

<u>3 equations</u>

$$\gamma(0) = a\gamma(1) - \tfrac{1}{2}\gamma(2) + b^2$$
$$\gamma(1) = a\gamma(0) - \tfrac{1}{2}\gamma(1)$$
$$\gamma(2) = a\gamma(1) - \tfrac{1}{2}\gamma(0)$$

$$\gamma(1) = \frac{a}{1-(-\frac{1}{2})}\gamma(0) = \tfrac{2}{3}a\,\gamma(0)$$

$$\gamma(0) = a\tfrac{2}{3}a\,\gamma(0) - \tfrac{1}{2}\left[a\tfrac{2}{3}a\,\gamma(0) - \tfrac{1}{2}\gamma(0)\right] + b^2$$

$$\gamma(0) = \tfrac{1}{3}a^2\gamma(0) + \tfrac{1}{4}\gamma(0) + b^2$$

$$\boxed{\gamma(0) = \frac{b^2}{(1 - \frac{1}{3}a^2 - \frac{1}{4})}}$$

$|a| < 3/2$ to be real / positive

Autocorrelation - Take $\rho(h) = \gamma(h)/\gamma(0)$

$$\boxed{\rho(h) = a\,\rho(h-1) - \tfrac{1}{2}\rho(h-2)} \qquad h \geq 1$$

$h=1$) $\rho(1) = a\,\rho(0) - \frac{1}{2}\rho(1) \Rightarrow \rho(1) = \frac{2}{3}a$

$\boxed{|a| < 3/2}$ for this to work

For ψ_i) $\quad X_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} \qquad \sum_{i=0}^{\infty} \psi_i < \infty$

in the most generic form, with

$$\Theta(L) = 1 + \sum_{j=1}^{q} \theta_j L^j \quad , \quad \Phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$$

$$\Psi(L) = \sum_{i=0}^{\infty} \psi_i L^i$$

we have) $\quad \Phi(L) X_t = \Theta(L) \varepsilon_t$

To be causal) $\quad X_t = \Psi(L) \varepsilon_t \qquad$ subst. X_t in above...

So) $\quad \Theta(L) = \Phi(L)\Psi(L)$

With $q=0$ and $p=2$ this gives

$$1 = (1 - \phi_1 L - \phi_2 L^2)(\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots)$$

$$1 = \psi_0 + (\psi_1 - \phi_1\psi_0)L + (\psi_2 - \phi_1\psi_1 - \phi_2\psi_0)L^2 + (\psi_3 - \phi_1\psi_2 - \phi_2\psi_1)L^3 + \cdots$$

There must be an L on the LHS matching the RHS:

$$1 = \psi_0$$
$$0 = \psi_1 - \phi_1 \psi_0$$
$$0 = \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0$$

any $i \geq 2$ is

$$0 = \psi_i - \phi_1 \psi_{i-1} - \phi_2 \psi_{i-2}$$

$$\boxed{\text{Giving) } \begin{array}{l} \psi_0 = 1 \\ \psi_1 = a \\ \psi_i = a\psi_{i-1} + \frac{1}{2}\psi_{i-2} \quad i \geq 2 \end{array}}$$

$|a| < 3/2$