Numerical Methods for Financial Derivatives

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Lecture 14: Computation of American Options

Black-Scholes Inequality for American Options

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV\leq 0$$

- American put:
  - For $S \leq S_f$: $\quad V_p^{am}(S(t),t)=K-S(t)$
  - For $S > S_f$: $\quad V_p^{am} > K-S(t)$ and

$$V_p^{am} \text{ solves } \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0.$$

- American call:
  - For $S \geq S_f$: $\quad V_c^{am}(S(t),t)=S(t)-K$
  - For $S < S_f$: $\quad V_c^{am} > S(t)-K$ and

$$V_c^{am} \text{ solves } \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0.$$

- We must look for methods to compute V and S_f simultaneously.

Black-Scholes Inequality for American Options (2)

- Why does the BS inequality holds?

$$\frac{\partial V}{\partial t} + \underbrace{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV}_{\mathscr{L}_{BS}(V)} \le 0$$

- American put:
  - For $S \le S_f$, $V_p^{am} = K - S, \frac{\partial V}{\partial t} = 0, \frac{\partial V}{\partial S} = -1, \frac{\partial^2 V}{\partial S^2} = 0.$
  - Hence,

$$\frac{\partial V}{\partial t} + \mathscr{L}_{BS}(V) = -(r-\delta)S - r(K-S) = \delta S - rK.$$

  - But S_f is non-decreasing in t and $\lim_{t\to T} S_f(t) = \min(K, \frac{r}{\delta}K)$.
  - Therefore, whether $r > \delta$ or not,

$$S(t) \le S_f(t) < \lim_{\substack{t\to T \\ t<T}} S_f(t) = \min(K, \frac{r}{\delta}K) \Rightarrow \delta S(t) - rK < 0$$

  - That is, the BS inequality must hold:

$$\frac{\partial V}{\partial t} + \mathscr{L}_{BS}(V) \le 0, \text{ for } t \ge 0$$

Black-Scholes Inequality for American Options (3)

- The same BS inequality holds for both American put and call.
- Can you provide a proof for American call?

- BS Inequality:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV \le 0$$

$$\Rightarrow \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \ge 0$$

$$\text{where} \quad V(S,t) = Ke^{-\frac{1}{2}(q_\delta-1)x-(\frac{1}{4}(q_\delta-1)^2+q)\tau}y(x,\tau)$$

- American put: $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} = 0$ if $y(x,\tau) > g(x,\tau)$ and $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} > 0$ if $y(x,\tau) = g(x,\tau)$. Note:

$$V_p^{am} \ge (K-S)^+ = \max(K-S,0)$$

$$\Rightarrow y(x,\tau) \ge g(x,\tau) \equiv e^{(\frac{1}{4}(q_\delta-1)^2+q)\tau}\max[e^{\frac{1}{2}(q_\delta-1)x} - e^{\frac{1}{2}(q_\delta+1)x}, 0].$$

- American call: $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} = 0$ if $y(x,\tau) > g(x,\tau)$ and $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} > 0$ if $y(x,\tau) = g(x,\tau)$. Note:

$$V_c^{am} \ge (S-K)^+ = \max(S-K,0)$$

$$\Rightarrow y(x,\tau) \ge g(x,\tau) \equiv e^{(\frac{1}{4}(q_\delta-1)^2+q)\tau}\max[e^{\frac{1}{2}(q_\delta+1)x} - e^{\frac{1}{2}(q_\delta-1)x}), 0].$$

American Options as a Linear Complementarity Problem

Problem

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$$

$$y - g \geq 0$$

$$\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y - g) = 0$$

$$y(x, 0) = g(x, 0), \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T$$

$$\lim_{x \to \pm\infty} y(x, \tau) = \lim_{x \to \pm\infty} g(x, \tau)$$

Note that as outlined above, the free boundary condition does not show up explicitly.

Discretization with the Theta Method

- The heat equation $y_t = y_{xx}$ is discretized as:

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle\tau} = \theta\left[\frac{w_{j+1,i+1} - 2w_{j,i+1} + w_{j-1,i+1}}{\triangle x^2}\right] + (1-\theta)\left[\frac{w_{j+1,i} - 2w_{j,i} + w_{j-1,i}}{\triangle x^2}\right]$$

- The differential inequality $y_t - y_{xx} \geq 0$ becomes the discrete version:

$$w_{j,i+1} - \theta\lambda[w_{j+1,i+1} - 2w_{j,i+1} + w_{j-1,i+1}] - w_{j,i} - (1-\theta)\lambda[w_{j+1,i} - 2w_{j,i} + w_{j-1,i}] \geq 0$$

or

$$w_{j,i+1} - \theta\lambda[w_{j-1,i+1} - 2w_{j,i+1} + w_{j+1,i+1}] \geq w_{j,i} + (1-\theta)\lambda[w_{j-1,i} - 2w_{j,i} + w_{j+1,i}]$$

Discrete Version of Linear Complementarity Problem

- $A_L \cdot w^{(i+1)} \geq b^{(i)}$: (componentwise)

$$\underbrace{\begin{bmatrix} 1+2\theta\lambda & -\theta\lambda & & & 0 \\ -\theta\lambda & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix}}_{A_L} \underbrace{\begin{bmatrix} w_{1,i+1} \\ w_{2,i+1} \\ \vdots \\ \vdots \\ w_{N-1,i+1} \end{bmatrix}}_{w^{(i+1)}} \geq \underbrace{\begin{bmatrix} b_{1,i} \\ b_{2,i} \\ \vdots \\ \vdots \\ b_{N-1,i} \end{bmatrix}}_{b^{(i)}}$$

- $w^{(i)} \geq g^{(i)}$; $\left(A_L \cdot w^{(i+1)} - b^{(i)}\right)^{tr} \left(w^{(i+1)} - g^{(i+1)}\right) \geq 0$
- $w^{(i)} = [w_{1,i}, \ldots, w_{N-1,i}]^{tr}$, $g^{(i)} = [g_{1,i}, \ldots, g_{N-1,i}]^{tr}$, $b^{(i)} = [b_{1,i}, \ldots, b_{N-1,i}]^{tr}$
- Initial condition: $w_{j,0} = g_{j,0}$, $j = 1, \ldots, N-1$ (i.e. $w^{(0)} = g^{(0)}$)
- Boundary condition: $w_{0,i} = g_{0,i}$, $w_{N,i} = g_{N,i}$, $i \geq 1$

Discrete Version of Linear Complementarity Problem (2)

- Definitions of $b_{j,i}$:
  - $b_{j,,i} = w_{j,i} + (1-\theta)\lambda[w_{j-1,i} - 2w_{j,i} + w_{j+1,i}], \ \ j = 2, ..., N-2$
  - $b_{1,i} = w_{1,i} + (1-\theta)\lambda[w_{0,i} - 2w_{1,i} + w_{2,i}] + \theta\lambda w_{0,i+1}$
  - $b_{N-1,i} = w_{N-1,i} + (1-\theta)\lambda[w_{N-2,i} - 2w_{N-1,i} + w_{N,i}] + \theta\lambda w_{N,i+1}$
- Definitions of $g_{j,i} = g(x_j, \tau_i)$:
  - American put:

$$g_{j,i} = e^{(\frac{1}{4}(q_\delta-1)^2+q)\tau_i}\max[e^{\frac{1}{2}(q_\delta-1)x_j} - e^{\frac{1}{2}(q_\delta+1)x_j}, 0]$$

  - American call:

$$g_{j,i} = e^{(\frac{1}{4}(q_\delta-1)^2+q)\tau_i}\max[e^{\frac{1}{2}(q_\delta+1)x_j} - e^{\frac{1}{2}(q_\delta-1)x_j}, 0]$$

where

$$\begin{aligned} x_j &= x_{\min} + j\triangle x, \quad j = 0, 1, ..., N \\ &= x_{\min}, \ \text{for } j = 0 \\ &= x_{\max}, \ \text{for } j = N \end{aligned}$$

$$\tau_i = i\triangle\tau, \quad i = 0, 1, ..., M$$

The Projected SOR Method

- Variable Transformation for the Linear Complementarity Problem
  - $Aw - b \geq 0 \Rightarrow y \geq 0$, where $y = Aw - b$.
  - $w \geq g \Rightarrow x \geq 0$, where $x = w - g$.
  - $(Aw - b)^{tr}(w - g) = 0 \Rightarrow x^{tr} y = 0$
- Projected SOR [Cryer (1971)]:
  - Find vectors x and y such that for $\widehat{b} = b - Ag$,

$$Ax - y = \widehat{b}, \quad x \geq 0, \quad y \geq 0, \quad x^{tr} y = 0$$

  - Iteration: starting from $x^{(0)} \geq 0, \;\; k = 1, 2, \ldots.$

$$x_i^{(k)} = \max\{0, \; x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[\widehat{b} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k-1)} \right] \}$$

$$y_i^{(k)} = - \left[\widehat{b} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k-1)} \right] + a_{ii} \left(x_i^{(k)} - x_i^{(k-1)} \right)$$

The Projected SOR Method (2)

- Projected SOR
  - Problem: $x \geq 0, \;\; y \geq 0, \;\; x^{tr} y = 0, \;\; Ax - y = \widehat{b}, \;\; \widehat{b} = b - Ag$
  - Iteration:

$$x_i^{(k)} = \max\{0,\; x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[\widehat{b} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k-1)} \right] \}$$

- Projected SOR
  - Problem: $w \geq g, \;\; Aw - b \geq 0, \;\; (Aw - b)^{tr} (w - g) = 0$
  - Iteration:

$$w_i^{(k)} = \max\{g_i,\; w_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[b - \sum_{j=1}^{i-1} a_{ij} w_j^{(k)} - a_{ii} w_i^{(k-1)} - \sum_{j=i+1}^{N} a_{ij} w_j^{(k-1)} \right] \}$$

- Part I:
  - Set up $g(x_j, \tau_i)$.
  - Choose the value of θ.
  - Choose the value of overrelaxation parameter ω.
  - Fix the discretization by choosing $x_{\min}, x_{\max}, N$ (space steps), M (time steps).
  - Fix an error bound ε.
  - Calculate mesh sizes:
    - $\triangle x = (x_{\max} - x_{\min})/N$
    - $\triangle \tau = (1/2)\sigma^2 T/M$
    - $x_j = x_{\min} + j\triangle x,\ j = 1, ..., N$
  - Initialize the iteration vector $w^{(0)} = g^{(0)} = [g(x_1, 0), ..., g(x_{N-1}, 0)$
  - Calculate $\lambda = \triangle\tau/(\triangle x)^2$

- Part II:
  - τ-loop (for $i = 0, 1, ..., M-1$):
    - $\tau_i = i\triangle\tau$
    - $b_j = w_j + (1-\theta)\lambda[w_{j-1} - 2w_j + w_{j+1}]$, $j = 2, ..., N-2$
    - $b_1 = w_1 + (1-\theta)\lambda[g_{0,i} - 2w_1 + w_2] + \theta\lambda g_{0,i+1}$
    - $b_{N-1} = w_{N-1} + (1-\theta)\lambda[w_{N-2} - 2w_{N-1} + g_{N,i}] + \theta\lambda g_{N,i+1}$
    - Set $v = \max\{w, g^{(i+1)}\}$ componentwise (v is starting vector for iteration)
    - SOR-loop (for $k = 1, 2, ...$) as long as $\| v^{new} - v \|_2 > \varepsilon$
  - For $j = 1, 2, ... N-1$:

$$v_j^{new} = \max\{g_{j,i+1}, v_j + \frac{\omega}{1+2\theta\lambda}[b_j + \theta\lambda v_{j-1}^{new} - (1+2\theta\lambda)v_j + \theta\lambda v_{j+1}]\}$$

with $v_0^{new} = v_N = 0$

  - $v = v^{new}$(after testing for convergence)
  - $w^{(i+1)} = w = v$

- Inputs: $K, \quad T, \quad S_0, \quad r, \quad \delta, \quad \sigma$
- Perform Prototype Core Algorithm
- Revert to Actual Financial Variables at $t = 0$:
  - For $j = 1, ..., N-1$: w_j approximates $y(j\triangle x, \frac{1}{2}\sigma^2 T)$

$$S_j = Ke^{x_{\mathbf{min}} + j\triangle x},$$

$$V(S_j, 0) = K\exp\{-\frac{1}{2}(q_\delta - 1)j\triangle x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau_{\max}\}w_j$$

  - Test for Early Exercise : pick error bound $\varepsilon^* = K \cdot 10^{-5}$
- For American put:

$$\begin{aligned} j_f &= \max\{j : \ |V(S_j, 0) - (K - S_j)| < \varepsilon^*\} \\ &\Rightarrow S_0 < S_{j_f}: \ \text{stopping region!} \end{aligned}$$

- For American call

$$\begin{aligned} j_f &= \min\{j : \ |V(S_j, 0) - (S_j - K)| < \varepsilon^*\} \\ &\Rightarrow S_0 > S_{j_f}: \ \text{stopping region!} \end{aligned}$$

Remarks

- Can the Prototype core algorithm be adapted to compute European options?
  - Yes.
  - But the projected SOR must be recovered to standard SOR:

$$v_j^{new} = \max\{g_{j,i+1}, v_j + \frac{\omega}{1+2\theta\lambda}[b_j + \theta\lambda v_{j-1}^{new} - (1+2\theta\lambda)v_j + \theta\lambda v_{j+1}]\}$$

$\Rightarrow$

$$v_j^{new} = v_j + \frac{\omega}{1+2\theta\lambda}[b_j + \theta\lambda v_{j-1}^{new} - (1+2\theta\lambda)v_j + \theta\lambda v_{j+1}]$$

- Can the LU decomposition be incorporated into the Prototype core algorithm to compute American options?
  - Yes.
  - Use the Brennan-Schwartz algorithm.

Brennan-Schwartz algorithm

- Tridiagonal Matrix

$$\begin{bmatrix} \alpha_1 & \beta_1 & & & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{N-2} & \alpha_{N-2} & \beta_{N-2} \\ 0 & & & \gamma_{N-1} & \alpha_{N-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-2} \\ w_{N-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}$$

Solution

[Brennan-Schwartz Algorithm] $\widehat{\alpha}_1 = \alpha_1, \widehat{b}_1 = b_1$
(forward loop) for j = 2,...,N-1:

$$\widehat{\alpha}_j = \alpha_j - \beta_{j-1}\left(\frac{\gamma_j}{\widehat{\alpha}_{j-1}}\right), \widehat{b}_j = b_j - \widehat{b}_{j-1}\left(\frac{\gamma_j}{\widehat{\alpha}_{j-1}}\right)$$

(backward loop) for j = N-1,...,1:

$$w_{N-1} = \max\{g_{N-1}, \ \widehat{b}_{N-1}/\widehat{\alpha}_{N-1}\}$$
$$w_j = \max\{g_j, \ (\widehat{b}_j - \beta_j w_{j+1})/\widehat{\alpha}_j\}$$