Numerical Methods for Financial Derivatives

Hwan C. Lin

Department of Economics
University of North Carolina at Charlotte

Lecture 6: Fourier Series and the Fourier Transform

- The Fourier series representation of periodic functions.
- The Fourier transform of non-periodic functions is the limiting case of the Fourier series representation of periodic functions.
- Applying the Fourier transform and the Convolution theorem to solve heat equations.

Definition

A function $f(t)$ is periodic of period T if there is a number $T > 0$ such that $f(t+T) = f(t)$ for all t. If there is such a T then the smallest one for which the equation holds is called the fundamental period of the function f.

- Examples:
  - $f(t) = \sin(t)$ is periodic of period 2π.
  - $f(t) = \cos(t)$ is periodic of period 2π.
  - $f(t) = \sin(2\pi t)$ is periodic of period 1.
  - $f(t) = \cos(2\pi t)$ is periodic of period 1.
  - $f(t) = \cos(4\pi t)$ is periodic of period $\frac{1}{2}$.
  - $f(t) = \cos(2\pi t) + \frac{1}{2}\cos(4\pi t)$ is periodic of period 1 (see graph below)

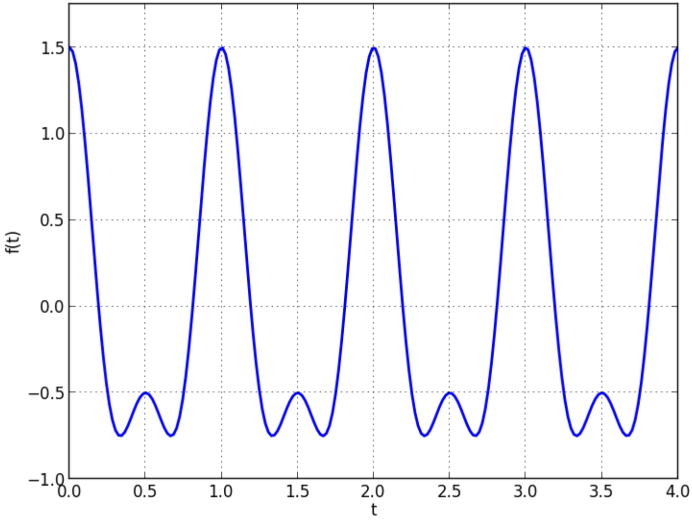

Figure 1: Periodic function $f(t) = \cos(2\pi t) + \frac{1}{2}\cos(4\pi t)$

- Suppose that $f(t)$ is a periodic function of period one and **square-integrable** in $t \in (t_0, t_0 + 1)$. Then this function can be approximated by

$$f(t) \approx f_N(t) = \sum_{n=1}^{N} A_n \sin(2\pi nt + \phi_n) \tag{1}$$

  where $f_N(t)$ converges to $f(t)$ at almost every point as $N \to \infty$.
- $f_N(t)$ is a sum of N sinusoids or harmonics.
- Def: $f(t)$ is square-integrable for $t \in (-\infty, \infty)$ if

$$\int_{-\infty}^{\infty} \mid f(t) \mid^2 dt < \infty$$

More Derivations

- Recall that $\sin(\alpha+\beta) = \sin(\beta)\cos(\alpha) + \cos(\beta)\sin(\alpha)$. So,

$$\begin{aligned}
f(t) \approx f_N(t) &= \sum_{n=1}^{N} A_n \sin(2\pi nt + \phi_n) \\
&= \sum_{n=1}^{N} A_n \left[\sin(\phi_n)\cos(2\pi nt) + \cos(\phi_n)\sin(2\pi nt)\right] \\
&= \sum_{n=1}^{N} \left[a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right] \qquad (2)
\end{aligned}$$

  where $a_n = A_n \sin(\phi_n)$ and $b_n = A_n \cos(\phi_n)$
- It is more common to write (2) as

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right] \qquad (3)$$

  where the harmonics have periods 1, 1/2, 1/3, . . . , 1/N , respectively, or frequencies 1, 2, 3, . . . , N.

We will use complex exponentials to represent the Fourier series of (3). To do so, we need the following:

$$e^{it} = \cos(t) + i\sin(t) \tag{4a}$$

$$e^{-it} = \cos(t) - i\sin(t) \tag{4b}$$

$$\cos(2\pi nt) = \frac{e^{2\pi int} + e^{-2\pi int}}{2} \tag{4c}$$

$$\sin(2\pi nt) = \frac{e^{2\pi int} - e^{-2\pi int}}{2} \tag{4d}$$

- Using the above trigonometric algebra, we can write

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi nt) + b_n \sin(2\pi nt)]$$

as

$$f(t) \approx \sum_{n=-N}^{N} c_n e^{2\pi int} \tag{5}$$

where c_n are called Fourier coefficients.

- In this final form, c_n are complex numbers and satisfy:
  - $c_0 = \frac{a_0}{2}$
  - $c_{-n} = \overline{c}_n$, i.e. c_{-n} is a complex conjugate of c_n.
  - $c_0 = \overline{c}_0$
  - The magnitudes of c_{-n} and c_n are equals; i.e., $\mid c_{-n} \mid = \mid c_n \mid$

Proof for $f(t) \approx \sum_{n=-N}^{N} c_n e^{2\pi i n t}$

$$
\begin{aligned}
f(t) &\approx \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi n t) + b_n \sin(2\pi n t)] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \left\{ \frac{e^{2\pi i n t} + e^{-2\pi i n t}}{2} \right\} + b_n \left\{ \frac{e^{2\pi i n t} - e^{-2\pi i n t}}{2i} \right\} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \left\{ \frac{e^{2\pi i n t} + e^{-2\pi i n t}}{2} \right\} + b_n \left\{ \frac{e^{2\pi i n t} - e^{-2\pi i n t}}{2i} \right\} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[(\frac{a_n}{2} - \frac{b_n}{2} i) e^{2\pi i n t} + (\frac{a_n}{2} + \frac{b_n}{2} i) e^{-2\pi i n t} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} c_n e^{2\pi i n t} + \sum_{n=-N}^{-1} \bar{c}_n e^{2\pi i n t} \\
&= \sum_{n=-N}^{N} c_n e^{2\pi i n t}, \quad \text{with} \quad c_0 \equiv \frac{a_0}{2}, \;\; c_{-n} = \bar{c}_n
\end{aligned}
$$

Attention to the Proof

- In fact, there two ways to write the halfway result of the proof:

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} \left[(\frac{a_n}{2} - \frac{b_n}{2} i) e^{2\pi i n t} + (\frac{a_n}{2} + \frac{b_n}{2} i) e^{-2\pi i n t} \right]$$

- Convention 1: define $c_n \equiv \frac{a_n}{2} - \frac{b_n}{2} i$ and $\bar{c}_n \equiv \frac{a_n}{2} + \frac{b_n}{2} i$, so that

$$\boxed{f(t) \approx c_0 + \sum_{n=1}^{N} c_n e^{2\pi i n t} + \sum_{n=-N}^{-1} \bar{c}_{-n} e^{2\pi i n t} = \sum_{n=-N}^{N} c_n e^{2\pi i n t}}$$

- Convention 2: define $c_n \equiv \frac{a_n}{2} + \frac{b_n}{2} i$ and $\bar{c}_n \equiv \frac{a_n}{2} - \frac{b_n}{2} i$, so that

$$\boxed{f(t) \approx \frac{a_0}{2} + \sum_{n=-N}^{-1} \bar{c}_{-n} e^{2\pi i n t} + \sum_{n=1}^{N} c_n e^{-2\pi i n t} = \sum_{n=-N}^{N} c_n e^{-2\pi i n t}}$$

- We adopted the popular convention 1.

Fourier Coefficients

- How to derive Fourier coefficients c_n? Here is a direct approach: Let's take the coefficient c_k for some fixed k. We can isolate it by multiplying both sides by $e^{-2\pi ikt}$:

$$\begin{aligned} e^{-2\pi ikt} f(t) &= e^{-2\pi ikt} \sum_{n=-N}^{N} c_n e^{2\pi int} \\ &= \cdots + e^{-2\pi ikt} c_k e^{2\pi ikt} + \cdots = \cdots + c_k + \cdots \end{aligned}$$

- Thus,

$$c_k = e^{-2\pi ikt} f(t) - \sum_{n=-N, n\neq k}^{N} c_n e^{2\pi i(n-k)t}$$

- Integrate from $t = 0$ to 1 (any period of one will do): since $\int_0^1 e^{2\pi i(n-k)t} dt = 0$, we obtain

$$c_k = \int_0^1 e^{-2\pi ikt} f(t) dt \qquad (6)$$

Summary: Fourier Series and Fourier Coefficients

Fourier series and coefficients

If we can write a period function $f(t)$ of period 1 as a (finite) Fourier series,

$$f(t) \approx \sum_{n=-N}^{N} c_n e^{2\pi int}, \quad \text{or} \quad f(t) \approx \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int}$$

then the Fourier coefficients are given by

$$c_n = \int_0^1 e^{-2\pi int} f(t)dt, \quad \text{or} \quad \hat{f}(n) = \int_0^1 e^{-2\pi int} f(t)dt, \quad n = 0, \pm 1, \ldots$$

Some Remarks

- The preceding results show the Fourier series representation and its corresponding Fourier coefficients for a periodic signal of period 1. In fact, **any interval of length 1** will do to calculate $\hat{f}(t)$:

$$\hat{f}(n) = \int_{a}^{a+1} e^{-2\pi int} f(t)dt$$

- The 0-th Fourier coefficient is the average value of the signal:

$$\hat{f}(0) = \int_{0}^{1} f(t)dt$$

Period, Frequency, and Spectrum

- We will look at some examples and applications in a moment. First, we want to make a few more general observations about the Fourier series representation of $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

  - The period is 1, due to $f(t) = f(t+1)$.
  - The **set of frequencies** present in a given periodic signal is the ***spectrum* of the signal**. It's the frequencies, like ± 1, ± 2, $\ldots$, $\pm\infty$, that make up the spectrum of $f(t)$, not the values of $\hat{f}(\pm 1)$, $\hat{f}(\pm 2)$, $\ldots$, $\hat{f}(\pm\infty)$.
  - Because of the symmetry relationship $\hat{f}(-n) = \overline{\hat{f}(n)}$.
  - If the coefficients are all zero from some point on, say $f(n) = 0$ for $\mid n \mid > N$, then it's common to say that the signal has no spectrum from that point, or that the spectrum of the signal is limited to the points between $-N$ and N. One also says in this case that the ***bandwidth*** is N (or maybe 2N depending to whom you're speaking) and that the signal is ***bandlimited***.

Period, Frequency, and Spectrum (cont.)

- Some other terminologies:
  - The squared magnitudes of the coefficients $|\hat{f}(n)|^2$ can be identified as the **energy** of the (positive and negative) harmonics $e^{\pm 2\pi int}$.
  - The sequence of squared magnitudes $|\hat{f}(n)|^2$ is called the ***power spectrum*** or the ***energy spectrum***.

Rayleigh's Identity

Rayleigh's Identity

The energy of $f(t)$ can be calculated from its Fourier coefficients:

$$\int_0^1 |\, f(t) \,|^2 \, dt = \sum_{n=-\infty}^{\infty} |\, \hat{f}(n) \,|^2 < \infty$$

This is known as **Rayleigh's identity**, or **Parseval's theorem**.

- Loosely speaking, the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform.

Complex Exponentials as an Orthonormal Basis

- We will present a proof for **Rayleigh's Identity** using the following relationships:
  - Define a complex exponential
$$e_n(t) \equiv e^{2\pi int}$$
  - Then its complex conjugate
$$\bar{e}_n(t) \equiv e^{-2\pi int}$$
  - The complex exponentials are orthonormal:

$$\langle e_n, e_m \rangle = \int_0^1 e_n \overline{e}_m dt = \int_0^1 e^{2\pi i(n-m)t} dt = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

- What is the component of a function f(t) "in the direction" en(t)? It is given by the inner product

$$\langle f, e_n \rangle = \int_0^1 f(t) \overline{e}_n dt = \hat{f}(n)$$

Proof for Rayleigh's Identity

- pf: Given $f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi int} = \sum_{n=-\infty}^{\infty} \langle f(t), e_n \rangle e_n$,

$$
\begin{aligned}
\int_0^1 \mid f(t) \mid^2 dt &= \langle f, f \rangle \\
&= \langle \sum_{n=-\infty}^{\infty} \langle f(t), e_n \rangle e_n, \sum_{m=-\infty}^{\infty} \langle f(t), e_m \rangle e_m \rangle \\
&= \sum_{n,m=-\infty}^{\infty} \langle f(t), e_n \rangle \overline{\langle f(t), e_m \rangle} \langle e_n, e_m \rangle \\
&= \sum_{n,m=-\infty}^{\infty} \langle f(t), e_n \rangle \overline{\langle f(t), e_m \rangle} \delta_{nm} \\
&= \sum_{n=-\infty}^{\infty} \langle f(t), e_n \rangle \overline{\langle f(t), e_n \rangle} = \sum_{n=-\infty}^{\infty} \mid \langle f(t), e_n \rangle \mid^2 \\
&= \sum_{n=-\infty}^{\infty} \mid \hat{f}(n) \mid^2
\end{aligned}
$$

How Big are the Fourier Coefficients?

- Suppose that $f(t)$ is a square-integrable periodic function of period 1, and let

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi int}$$

be its Fourier series. Rayleigh's identity says

$$\int_0^1 |\, f(t)\,|^2 \, dt = \sum_{n=-\infty}^{\infty} |\, \hat{f}(n)\,|^2 < \infty$$

- In particular, the series $\sum_{n=-\infty}^{\infty} |\, \hat{f}(n)\,|^2$ converges, and it follows that

$$|\, \hat{f}(n)\,|^2 \to 0 \quad \text{as} \quad n \to \infty$$

- This is a general result on convergent series from good old calculus days — if the series converges the general term must tend to zero.

What if the period is not 1?

Fourier series and coefficients as period in not 1

If $f(t)$ is a square-integrable periodic function of period $T \neq 1$, then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

and the Fourier coefficients are given by

$$c_n = \frac{1}{T} \int_0^T e^{-2\pi int/T} f(t) dt$$

or

$$c_n = \frac{1}{T} \int_a^{T+a} e^{-2\pi int/T} f(t) dt$$

- We observe from this an important reciprocal relationship between
  - **the signal in the time (spatial) domain** (t is a generic variable) and
  - **the signal as displayed in the frequency domain** (i.e., in the spectrum).
- In the time domain the signal repeats after T seconds, while the points in the spectrum are 0, $\pm 1/T$, $\pm 2/T$, ..., which are spaced $1/T$ apart. (Of course for period $T = 1$ the spacing in the spectrum is also 1.)
- We will look such a reciprocal relationship based on the following examples.

Examples: Rectangle Functions

$$\Pi(t) = \begin{cases} 1, & \mid t \mid < 1/2 \\ 0, & \mid t \mid \geq 1/2 \end{cases} \qquad c_n = \frac{1}{T} \int_{-T/2}^{T/2} \Pi(t) e^{-2\pi i (\frac{n}{T}) t} dt, \;\; c_s = T c_n, \;\; s = \frac{n}{T}$$

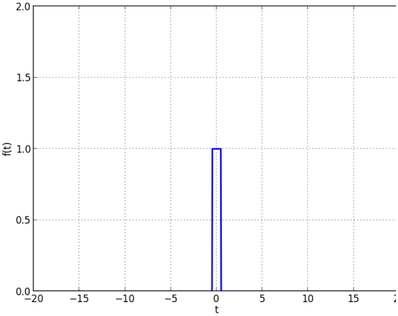

Figure 2: Rectangle Function (non-periodic)

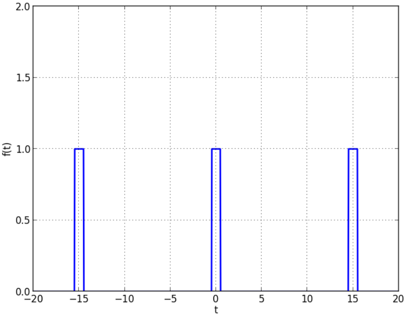

Figure 3: Rectangle Function of period $T = 15$

Example 1: Fourier Coefficients of Rectangle Function with Period 2

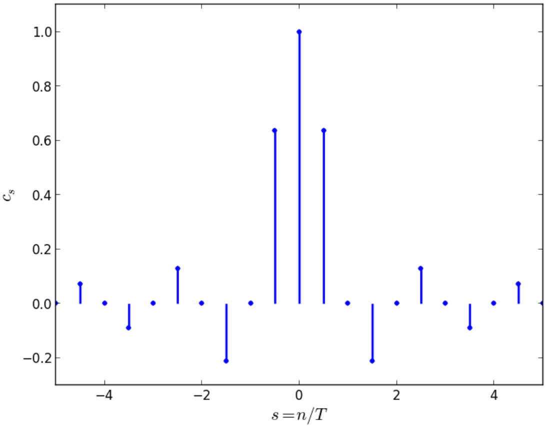

Figure 4: Fourier coefficients c_s of periodic rectangle function $\Pi(t)$ with period $T = 2$. Note: c_s is scaled up by T.

Example 2: Fourier Coefficients of Rectangle Function with Period 4

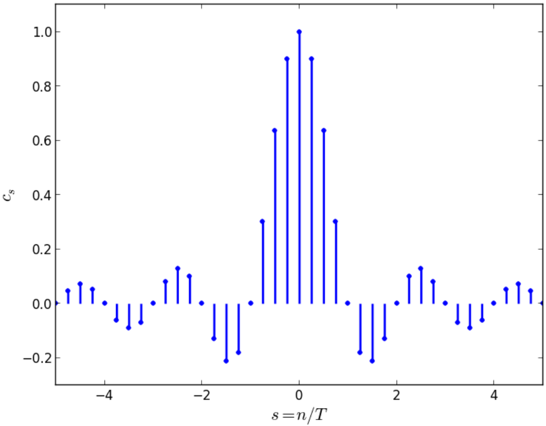

Figure 5: Fourier coefficients c_s of periodic rectangle function $\Pi(t)$ with period $T = 4$. Note: c_s is scaled up by T.

Example 3: Fourier Coefficients of Rectangle Function with Period 16

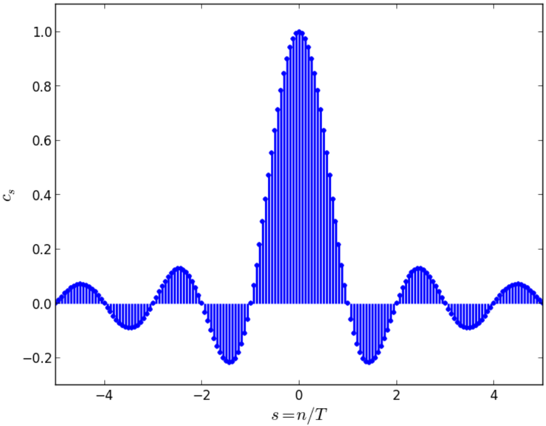

Figure 6: Fourier coefficients c_s of periodic rectangle function $\Pi(t)$ with period $T = 16$. Note: c_s is scaled up by T.

What We Observed from Examples?

- Given a period T function of $\Pi(t)$, the frequencies are 0, $\pm\frac{1}{T}$, $\pm\frac{2}{T}$, ... that is, points in **the spectrum are spaced by $\frac{1}{T}$ apart**.
- As the signal in the time domain repeats with a longer delay (T increases), the spectrum is getting **more tightly packed.**
- That is, if T goes infinity, frequency $s = \frac{n}{T}$ ($n = 0, \pm 1, \pm 2, \ldots$) becomes **a continuous variable**. But if T is infinitely large, then $\Pi(t)$ is **a non-periodic function**. From this perspective, we can derive the Fourier transform of a non-periodic function. In what follows we explain the nice perspective of letting a periodic function transition to a non-periodic function.

- Given $\Pi(t) = \begin{cases} 1, & \mid t \mid < 1/2 \\ 0, & \mid t \mid \geq 1/2 \end{cases}$ with period T, the Fourier coefficients are

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T}\Pi(t)dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T}\cdot 1dt \\
&= \frac{1}{T}\left[-\frac{1}{2\pi in/T}e^{-2\pi int/T}\Big|_{-1/2}^{1/2}\right] = \frac{1}{2\pi in}\left[e^{\pi in/T} - e^{-\pi in/T}\right] \\
&= \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) \to 0, \text{ as } T \to \infty
\end{aligned}
$$

From Periodic to Non-Periodic (cont.)

- Scaled Fourier coefficients:

$$c_s = Tc_n = \frac{\sin(\pi n/T)}{\pi n/T} \to 1, \text{ as } T \to \infty$$

or using a scaled frequency $s = n/T$,

$$c_s = \frac{\sin(\pi s)}{\pi s} \equiv \text{sinc}(\pi s), \quad \text{with sinc}(0) = 1$$

- So, for a non-periodic function ($T \to \infty$), we define $c_s = \hat{\Pi}(s)$ and obtain

$$\hat{\Pi}(s) = T \times \frac{1}{T} \int_{-T/2}^{T/2} e^{-2\pi int/T} \Pi(t) dt$$

or

$$\hat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} \Pi(t) dt, \quad s \in (-\infty, \infty)$$

- A Fourier transform is born!

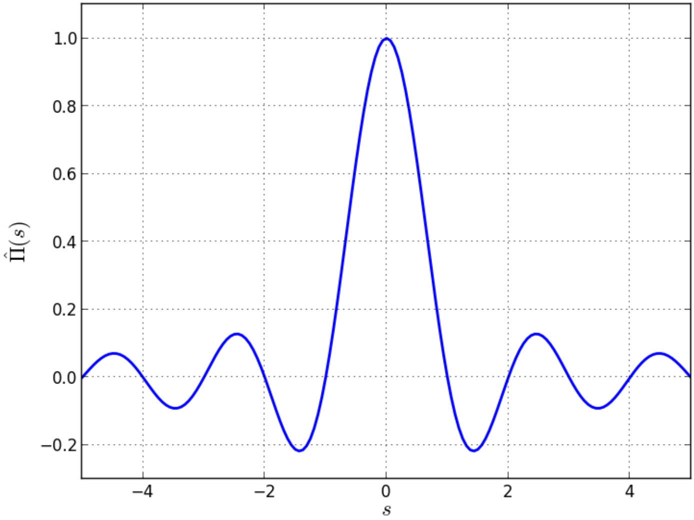

Figure 7: The Fourier transform of a non-periodic function $\Pi(t)$: $\hat{\Pi}(s) = \text{sinc}(s)$

Fourier Transform and Inverse Fourier Transform

The Fourier transform pair

Given a square-integrable function $f(t)$, $t \in \mathbb{R}$,

- the Fourier transform of f is

$$\hat{f}(s) = \mathcal{F}\{f(t)\}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)dt$$

- the inverse Fourier transform is

$$f(t) = \mathcal{F}^{-1}\{\hat{f}(s)\}(t) = \int_{-\infty}^{\infty} e^{2\pi ist} \hat{f}(s)ds$$

where i is the imaginative unit that solves $i^2 = -1$.

Some Remarks on the Fourier Transform Pair

- $\hat{f}(s)$, the Fourier transform of $f(t)$, is a complex-valued function of $s \in \mathbb{R}$.
- If $f(t)$ is a real-valued function, as it most often is, $\hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt$ must be a real number, even though other values of the Fourier transform may be complex. Also, $f(0) = \int_{-\infty}^{\infty} \hat{f}(s)ds$ is real.
- The spectrum of a periodic function is a discrete set of frequencies. By contrast, the Fourier transform of a non-periodic signal produces a continuous spectrum, or a continuum of frequencies.
- **Rayleigh's identy**:

$$\int_{-\infty}^{\infty} \mid f(t) \mid^2 dt = \int_{-\infty}^{\infty} \mid \hat{f}(s) \mid^2 ds$$

  where its says the energy in the time domain is equal to the energy in the frequency domain.
- Alternative notations:
  - $f(t) \rightleftharpoons \hat{f}(s); \quad f(t) \rightleftharpoons F(s); \quad f(t) \rightleftharpoons \mathcal{F}\{f(t)\}(s)$
  - $\mathcal{F}^{-1}\{\hat{f}(s)\}(t) = f(t)$

Alternative Expressions of Fourier Transforms

- Our definition of the Fourier transform is a standard one, but it's not the only one. The question is where to put the 2π.
- In general, the Fourier transform has alternative expressions:

$$\hat{f}(s) = \frac{1}{A} \int_{-\infty}^{\infty} e^{iBst} f(t) dt$$

where the choices of A and B are summarized below,

- $A = \sqrt{2\pi}, \quad B = \pm 1$
  - $A = 1, \quad B = \pm 2\pi$
  - $A = 1, \quad B = \pm 1$
- The definition we have chosen has $A = 1$ and $B = -2\pi$.

Basic Properties of the Fourier Transform

Assume $f(t)$, $g(t)$ are square-integrable functions on the real line.

Basic Properties

- **Linearity**: For any complex numbers α and β,

$$\begin{aligned}\mathcal{F}\{\alpha f(t) + \beta g(t)\}(s) &= \mathcal{F}\{\alpha f(t)\}(s) + \mathcal{F}\{\beta g(t)\}(s) \\ &= \alpha \mathcal{F}\{f(t)\}(s) + \beta \mathcal{F}\{g(t)\}(s) \\ &= \alpha \hat{f}(s) + \beta \hat{g}(s)\end{aligned}$$

- **Shift**: For a shift in variable t (say, a delay in time) by a constant $b \in \mathbb{R}$,

$$\mathcal{F}\{f(t+b)\}(s) = e^{2\pi isb}\mathcal{F}\{f(t)\}(s) = e^{2\pi isb}\hat{f}(s)$$

- **Stretch**: For the scaling of t to at, $a > 0$ or $a < 0$,

$$\mathcal{F}\{f(at)\}(s) = \frac{1}{|a|}\mathcal{F}\{f(t)\}(\frac{s}{a}) = \frac{1}{|a|}\hat{f}(\frac{s}{a})$$

Proof for The Shift Theorem

Shift with $b > 0$ or $b < 0$

$$\begin{aligned}
\mathcal{F}\{f(t+b)\}(s) &= \int_{-\infty}^{\infty} e^{-2\pi ist} f(t+b)dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi is(u-b)} f(u)du, \quad u = t+b, \quad du = dt \\
&= e^{2\pi isb} \int_{-\infty}^{\infty} e^{-2\pi isu} f(u)du = e^{2\pi isb} \hat{f}(s)
\end{aligned}$$

Proof for the Stretch Theorem

Stretch with $a > 0$

$$\begin{aligned}
\mathcal{F}\{f(at)\}(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s t} f(at)dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi i s(\frac{u}{a})} f(u)(\frac{1}{a})du, \quad u = at, \quad du = adt \\
&= \frac{1}{a}\int_{-\infty}^{\infty} e^{-2\pi i(\frac{s}{a})u} f(u)du \\
&= \frac{1}{a}\mathcal{F}\{f(u)\}(\frac{s}{a}) = \frac{1}{a}\hat{f}(\frac{s}{a})
\end{aligned}$$

Stretch with $a < 0$

$$
\begin{aligned}
\mathcal{F}\{f(at)\}(s) &= \int_{-\infty}^{\infty} e^{-2\pi ist} f(at)dt \\
&= \int_{\infty}^{-\infty} e^{-2\pi is(\frac{u}{a})} f(u)(\frac{1}{a})du, \quad u = at, \quad du = adt \\
&= -\frac{1}{a} \int_{-\infty}^{\infty} e^{-2\pi i(\frac{s}{a})u} f(u)du \\
&= -\frac{1}{a}\mathcal{F}\{f(u)\}(\frac{s}{a}) = -\frac{1}{a}\hat{f}(\frac{s}{a}),
\end{aligned}
$$

where $-\frac{1}{a} = \frac{1}{|a|}$

The Fourier Transform of Standard Normal Density

- Standard normal density and its Fourier transform:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Longrightarrow \hat{\phi}(s) = e^{-2\pi^2 s^2}$$

- Derivation of the Fourier transform $\hat{\phi}(x)$:

$$\hat{\phi}(s) = \int_{-\infty}^{\infty} e^{-2\pi isx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

- First, taking the derivative of $\hat{\phi}$ w.r.t. s:

$$\begin{aligned} \frac{d\hat{\phi}}{ds} &= 2\pi \int_{-\infty}^{\infty} (-ix) e^{-2\pi isx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= 2\pi \int udv \end{aligned}$$

where $u = e^{-2\pi isx}$, $dv = (-\frac{ix}{\sqrt{2\pi}}) e^{-\frac{x^2}{2}} dx$, and $v = \frac{i}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

- Second, obtaining the ODE:

$$\begin{aligned}
\frac{d\hat{\phi}}{ds} &= 2\pi\left(uv\,|_{-\infty}^{\infty} - \int vdu\right)\\
&= 2\pi\left(0 - \int_{-\infty}^{\infty} \frac{i}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot (-2\pi is) e^{-2\pi isx} dx\right)\\
&= -4\pi^2 \int_{-\infty}^{\infty} e^{-2\pi isx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\\
&= -4\pi^2 s\hat{\phi}
\end{aligned}$$

- Solving the ODE $\frac{d\hat{\phi}}{ds} = -4\pi^2 s\hat{\phi}$ for:

$$\hat{\phi}(s) = \hat{\phi}(0)e^{-2\pi^2 s^2} = e^{-2\pi^2 s^2}$$

due to $\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x) = 1$.

The Fourier Transform of Normal Density

- Normal density and its Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Longrightarrow \hat{f}(s) = e^{-2\pi i s\mu - 2\pi^2\sigma^2 s^2}$$

- Derivation of the Fourier transform:

$$\begin{aligned}
\hat{f}(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{-2\pi i s(\sigma z+\mu)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2}} dz, \quad z = \frac{x-\mu}{\sigma}, \quad dz = \frac{dx}{\sigma} \\
&= e^{-2\pi i s\mu} \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i(\sigma s)z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{\hat{\phi}(\sigma s) = e^{-2\pi^2(\sigma s)^2}} \\
&= e^{-2\pi i s\mu - 2\pi^2\sigma^2 s^2}
\end{aligned}$$

Derivatives

Th Fourier transform of the 1^{th} and n^{th}derivatives of function $f(t)$ with respect to t is

$$\mathcal{F}\{f'(t)\}(s) = 2\pi is\hat{f}(s)$$

$$\mathcal{F}\{f^{(n)}(t)\}\quad(s) = \quad(2\pi is)^n\hat{f}(s)$$

- With $f(t) = \int_{-\infty}^{\infty}\hat{f}(s)e^{2\pi ist}ds$, we have

$$f'(t) = \int\limits_{-\infty}^{\infty} 2\pi is\hat{f}(s)e^{2\pi ist}ds$$

$$f^{(n)}(t) = \int\limits_{-\infty}^{\infty} (2\pi is)^n\hat{f}(s)e^{2\pi ist}ds$$

The Convolution Theorem

Definition

The convolution product of two functions $f(t)$ and $g(t)$ is defined as

$$h(t) \equiv (f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx$$
$$= \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

Theorem

The Fourier transform of $(f * g)(t)$ is given by

$$\mathcal{F}(f * g)(s) \equiv \hat{h}(s) = \hat{f}(s)\hat{g}(s)$$

- The product of the Fourier transforms of $f(t)$ and $g(t)$ is

$$
\begin{aligned}
\mathcal{F}f(s)\cdot\mathcal{F}g(s) &\equiv \hat{f}(s)\hat{g}(s) \\
&= \left(\int_{-\infty}^{\infty} e^{-2\pi ist} f(t)dt\right)\cdot\left(\int_{-\infty}^{\infty} e^{-2\pi isx} g(x)dx\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi is(t+x)} f(t)g(x)dtdx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi is(t+x)} f(t)dt\right) g(x)dx
\end{aligned}
$$

- Let $u = t + x$ in the inner integral. Then $t = u - x$, $dt = du$ and the limits are the same:

$$\mathcal{F}f(s)\cdot\mathcal{F}g(s) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2\pi isu} f(u-x)du\right) g(x)dx$$

- Switching the order of integration:

$$\begin{aligned}
\mathcal{F}f(s)\cdot\mathcal{F}g(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi isu} f(u-x)g(x)dudx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi isu} f(u-x)g(x)dxdu \\
&= \int_{-\infty}^{\infty} e^{-2\pi isu}\left(\int_{-\infty}^{\infty} f(u-x)g(x)dx\right) du \\
&= \int_{-\infty}^{\infty} e^{-2\pi isu} h(u)du = \hat{h}(s) \equiv \mathcal{F}(f*g)(s)
\end{aligned}$$

Heat Equation

The heat equation

$$\begin{aligned} \frac{\partial u(x,\tau)}{\partial \tau} &= \frac{\partial^2 u(x,\tau)}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < \tau < \tau^* \\ u(x,0) &= \phi(x) \end{aligned}$$

About Solving Heat Equation

- There are alternative ways to solve the heat equation:
    - Separation of variables:

$$u(x,\tau) = f(x)g(\tau)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow fg' = f''g \Rightarrow g'/g = f''/f$$

    - Fourier transform:

$$\mathcal{F}\left\{\frac{\partial u(x,\tau)}{\partial \tau}\right\}(s) = \mathcal{F}\left\{\frac{\partial^2 u(x,\tau)}{\partial x^2}\right\}(s)$$

        - Fourier transform of functional derivatives
        - Convolution theorem
        - Use of the fact:

$$\int_{-\infty}^{\infty} \exp[-x^2] dx = \sqrt{\pi}$$

- Given the time level τ, the Fourier transform of the heat equation is

$$\mathcal{F}\left\{\frac{\partial u(x,\tau)}{\partial \tau}\right\}(s) = \mathcal{F}\left\{\frac{\partial^2 u(x,\tau)}{\partial x^2}\right\}(s)$$

where

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial u(x,\tau)}{\partial \tau}\right\}(s) &= \int_{-\infty}^{\infty} e^{-2\pi isx}\frac{\partial u(x,\tau)}{\partial \tau}dx \\
&= \frac{\partial}{\partial \tau}\left[\int_{-\infty}^{\infty} e^{-2\pi isx}u(x,\tau)dx\right] \\
&= \frac{\partial}{\partial \tau}\mathcal{F}\{u(x,\tau\}(s) \\
&= \frac{\partial}{\partial \tau}\hat{u}(s,\tau)
\end{aligned}$$

$$\mathcal{F}\left\{\frac{\partial^2 u(x,\tau)}{\partial x^2}\right\}(s) = (2\pi is)^2\mathcal{F}\{u(x,\tau\}(s) = -4\pi^2 s^2\hat{u}(s,\tau)$$

Solving Heat Equation using Fourier Transform (2)

- Given frequency s, the Fourier transform of the heat equation is a first-order ordinary differential equation (ODE):

$$\frac{\partial}{\partial \tau}\hat{u}(s,\tau) + 4\pi^2 s^2 \hat{u}(s,\tau) = 0$$

- The solution to the ODE is

$$\hat{u}(s,\tau) = \hat{u}(s,0)e^{-4\pi^2 s^2 \tau}$$

where $\hat{u}(s,0)$ is the Fourier transform of the initial condition $u(x,0) = \phi(x)$; that is,

$$\hat{u}(s,0) = \mathcal{F}\{u(x,0)\}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s x} u(x,0) dx$$

Solving Heat Equation using Fourier Transform (3)

- We have obtained $\hat{u}(s,\tau)$ as shown above. To find $u(x,\tau)$, we can apply the inverse Fourier transform:

$$u(x,\tau) = \mathcal{F}^{-1}\{\hat{u}(s,\tau)\}$$

- To do so, first, define

$$\hat{u}_1(s,\tau) = e^{-4\pi^2 s^2 \tau} = e^{-2\pi^2(\sqrt{2\tau})^2 s^2}$$

$$\hat{u}_2(s,\tau) = \hat{u}(s,0)$$

such that

$$\hat{u}(s,\tau) = \hat{u}_1(s,\tau)\hat{u}_2(s,\tau)$$

- From the convolution theorem, $\mathcal{F}(u_1 * u_2) = \hat{u}_1\hat{u}_2 = \hat{u}$, implying $u = u_1 * u_2$.

- Next, derive u_1 and u_2 using the inverse Fourier transforms of $\hat{u}_1$ and $\hat{u}_2$, respectively:

$$u_1(x,\tau) = \mathcal{F}^{-1}\{\hat{u}_1(s,\tau)\} = \int_{-\infty}^{\infty} e^{2\pi isx} e^{-4\pi^2 s^2 \tau} ds$$

$$u_2(x,\tau) = \mathcal{F}^{-1}\{\hat{u}_2(s,\tau)\} = \int_{-\infty}^{\infty} e^{2\pi isx} \hat{u}(s,0) ds$$

- Now we can obtain u based on the convolution product of u_1 and u_2:

$$u(x,\tau) = (u_1 * u_2)(x,\tau) = \int_{-\infty}^{\infty} u_1(x-y,\tau) u_2(y,\tau) dy$$

- We recognize (can you?) that $\hat{u}_1(s,\tau) = e^{-2\pi^2(\sqrt{2\tau})^2 s^2}$ is the Fourier transform of the Gaussian,

$$u_1 = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \quad \text{with } \sigma = \sqrt{2\tau}$$

- Thus, using the Gaussian and $u_2(x,\tau) = u(x,0)$, we have the final solution

$$\begin{aligned} u(x,\tau) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} * u(x,0)\right)(x,\tau) \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}}\right] \cdot u(y,0)dy \end{aligned}$$

- It implies that the temperature of the rod at a point x at a time τ is some kind of averaged, smoothed version of the initial temperature $u(x,0)$. That's convolution at work.

- Solution: $u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4\tau}\right] \cdot u(y,0)dy$
- Verify initial condition: let $\tau \to 0$ (i.e., $\sigma \to 0$) to obtain

$$\begin{aligned}
\lim_{\tau\to 0} u(x,\tau) &= \int_{-\infty}^{\infty} \left\{ \lim_{\tau\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-x)^2}{2\sigma^2}\right]\right\} \cdot u(y,0)dy \\
&= \int_{-\infty}^{\infty} \left\{ \lim_{\sigma\to 0} f(y) \right\} \cdot u(y,0)dy, \quad y \sim N(x,\sigma) \\
&= \int_{-\infty}^{\infty} \delta(y-x) \cdot u(y,0)dy, \quad y \sim N(x,\sigma) \\
&= u(x,0)
\end{aligned}$$

where $\delta(.)$ is Dirac delta function.

- Dirac Delta function $\delta(x-a)$ has the properties:
  - $\delta(x-a) = \begin{cases} \infty, & x = a \in (-\infty, \infty) \\ 0, & x \neq a \end{cases}$
  - $\int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)dx = 1, \quad \varepsilon \in (0, \infty)$
  - $\int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)f(x)dx = f(a), \quad \varepsilon \in (0, \infty)$
- The Dirac delta function is not a function in the traditional sense.
- Loosely speaking, it is a function that is zero everywhere except at a point, where it is infinite, and the integral of any interval containing that one point has a value of 1.

- Fourier transform of $\delta(x)$:

$$\begin{aligned}\mathcal{F}\{\delta(x)\}(s) &\equiv \hat{\delta}(s) = \int_{-\infty}^{\infty} e^{-2\pi isx}\delta(x)dx \\ &= e^{-2\pi isx} \text{ at } x = 0 \\ &= 1\end{aligned}$$

- Inverse Fourier transform:

$$\mathcal{F}^{-1}\{\hat{\delta}(s)\}(x) = \delta(x) = \int_{-\infty}^{\infty} e^{2\pi isx} \cdot 1 \cdot ds$$

- Vanilla options $V(S_t, t)$:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r-\delta)S\frac{\partial V(S,t)}{\partial S} - rV \quad = \quad 0,$$

$$0 < S < \infty,\ 0 < t < T$$

- Reduction to heat equation:

$$\frac{\partial u(x,\tau)}{\partial \tau} \quad = \quad \frac{\partial^2 u(x,\tau)}{\partial x^2},$$

$$-\infty < x < \infty, \quad 0 < \tau < \frac{\sigma^2 T}{2}$$

- Variables:

$$S = Ke^x, \qquad \tau = \frac{1}{2}\sigma^2(T-t)$$

$$V(S,t) = Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} u(x,\tau)$$

$$q = \frac{2r}{\sigma^2}, \qquad q_\delta = \frac{2(r-\delta)}{\sigma^2}$$

- Terminal condition for BS:

$$V(S, t=T) = \varepsilon(S-K)^+ = \max\{\varepsilon[S-K], 0\}, \ \varepsilon \in \{1, -1\}$$

- Initial condition for heat equation:

$$u(x, \tau = 0) = \frac{\varepsilon(S-K)^+}{Ke^{-\frac{1}{2}(q_\delta - 1)x}} = \max\{\varepsilon[e^{\frac{x}{2}(q_\delta + 1)} - e^{\frac{x}{2}(q_\delta - 1)}], 0\}$$

Analytical solution to BS PDE

- Solution to heat equation:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4\tau}\right] \cdot u(y,0)dy$$

- Plug in initial condition:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4\tau}\right] \cdot \max\{\varepsilon[e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)}], 0\}dy$$

Analytical solution to BS PDE (2)

- We have $e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)} > 0$ iff $y > 0$ for all $q_\delta \in \mathbb{R}$. So, eur call and put are

$$
\begin{aligned}
u_C &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty \exp\left[-\frac{(x-y)^2}{4\tau}\right] \cdot [e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)}]dy \\
u_P &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^0 \exp\left[-\frac{(x-y)^2}{4\tau}\right] \cdot (-1)[e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)}]dy \\
&= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty \exp\left[-\frac{(x+y)^2}{4\tau}\right] \cdot (-1)[e^{\frac{-y}{2}(q_\delta+1)} - e^{\frac{-y}{2}(q_\delta-1)}]dy
\end{aligned}
$$

- Therefore, u_C and u_P can be presented by

$$
u(x,\tau) = \frac{\varepsilon}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(x-\varepsilon y)^2}{4\tau}} \cdot [e^{\frac{\varepsilon y}{2}(q_\delta+1)} - e^{\frac{\varepsilon y}{2}(q_\delta-1)}]dy
$$

- Changing variables with $z = \frac{y-x}{\sqrt{2\tau}}$, $dy = \sqrt{2\tau}dz$, $a_1 = \frac{1}{2}(q_\delta + 1)$ and $a_2 = \frac{1}{2}(q_\delta - 1)$, we can rewrite

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(x-y)^2}{4\tau}} [e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)}]dy$$

as

$$\begin{aligned} u(x,\tau) &= \frac{\sqrt{2\tau}}{\sqrt{4\pi\tau}} \int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{z^2}{2}} [e^{\frac{y}{2}(q_\delta+1)} - e^{\frac{y}{2}(q_\delta-1)}]dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{z^2}{2}} [e^{a_1(\sqrt{2\tau}z+x)} - e^{a_2(\sqrt{2\tau}z+x)}]dz \\ &= I_{a_1}(x,\tau) - I_{a_2}(x,\tau) \end{aligned}$$

where $I_{a_i}(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{z^2}{2}} e^{a_i(\sqrt{2\tau}z+x)} dz, \quad i = 1, 2$

- :Change $I_a(a, \tau)$ to include standard normal distribution:

$$
\begin{aligned}
I_a(x,\tau) &= \frac{e^{ax}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z^2 - 2a\sqrt{2\tau}z)} dz \\
&= e^{ax} e^{\frac{1}{2}(a\sqrt{2\tau})^2} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - a\sqrt{2\tau})^2} dz \\
&= e^{ax} e^{\frac{1}{2}(a\sqrt{2\tau})^2} \int_{-\frac{x}{\sqrt{2\tau}} - a\sqrt{2\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2} d\zeta \\
&= e^{ax} e^{\frac{1}{2}(a\sqrt{2\tau})^2} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}} + a\sqrt{2\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2} d\zeta \\
&= e^{ax} e^{\frac{1}{2}(a\sqrt{2\tau})^2} \Phi\left(\frac{x}{\sqrt{2\tau}} + a\sqrt{2\tau}\right)
\end{aligned}
$$

where $\Phi(.)$ is the standard normal distribution, $\zeta = z - a\sqrt{2\tau}$, $d\zeta = dz$, and $\underline{z} = -\frac{x}{\sqrt{2\tau}}$ implies $\underline{\zeta} = -\frac{x}{\sqrt{2\tau}} - a\sqrt{2\tau}$

- Solution for European call:

$$\begin{aligned} u_C(x,\tau) &= e^{a_1 x} e^{\frac{1}{2}(a_1\sqrt{2\tau})^2} \Phi\left(\frac{x}{\sqrt{2\tau}} + a_1\sqrt{2\tau}\right) \\ &\quad - e^{a_2 x} e^{\frac{1}{2}(a_2\sqrt{2\tau})^2} \Phi\left(\frac{x}{\sqrt{2\tau}} + a_2\sqrt{2\tau}\right) \end{aligned}$$

- But for the price of European call or put, V and u are related by

$$V(S,t) = Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} u(x,\tau)$$

- Using $a_1 = \frac{1}{2}(q_\delta + 1)$ and $a_2 = \frac{1}{2}(q_\delta - 1)$, collecting the exponential terms, and simply, we can obtain a simpler formula for $V_C(S,t)$, as shown next.

- Simplify the exponential terms:

$$Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} \cdot e^{a_1 x} e^{\frac{1}{2}(a_1\sqrt{2\tau})^2} = Ke^{x + (q_\delta - q)\tau}$$

$$Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} \cdot e^{a_2 x} e^{\frac{1}{2}(a_2\sqrt{2\tau})^2} = Ke^{-q\tau}$$

- These simplifications makes $V_C(S,t)$ reduce to

$$\begin{aligned} V_C(S,t) &= Ke^{x+(q_\delta - q)\tau} \cdot \Phi(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(q_\delta + 1)\sqrt{2\tau}) \\ &\quad - Ke^{-q\tau} \cdot \Phi(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(q_\delta - 1)\sqrt{2\tau}) \end{aligned}$$

- Lastly, we need to return x to S and τ to t.

- Recall

$$x = \ln(\frac{S}{K}), \quad \tau = \frac{1}{2}\sigma^2(T-t), \quad q = \frac{2r}{\sigma^2}, \quad q_\delta = \frac{2(r-\delta)}{\sigma^2}$$

- Thus, in the formula of $V_C(S,t)$, we obtain the following substitutions:

$$e^{x+(q_\delta - q)\tau} = \frac{S}{K}e^{-\delta(T-t)}, \quad e^{-q\tau} = e^{-r(T-t)}$$

$$\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(q_\delta + 1)\sqrt{2\tau} = \frac{\ln(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \equiv d_1$$

$$\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(q_\delta - 1)\sqrt{2\tau} = \frac{\ln(\frac{S}{K}) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \equiv d_2$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- Using the above substitutions, we obtain the closed-form solution for European call:

$$V_C(S,t) == Se^{-\delta(T-t)} \cdot \Phi(d_1) - Ke^{-r(T-t)} \cdot \Phi(d_2)$$

- Using the same procedure, we can obtain the closed-form solution for European put:

$$V_P(S,t) == -Se^{-\delta(T-t)} \cdot \Phi(-d_1) + Ke^{-r(T-t)} \cdot \Phi(-d_2)$$

- Alternatively, we can obtain $V_P(S,t)$ by the put-call parity:

$$V_P + Se^{-\delta(T-t)} = V_C + Ke^{-r(T-t)}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^S \frac{\partial^2 V}{\partial S^2} + (r - \delta)S\frac{\partial V}{\partial S} - rV = 0 \tag{7}$$

Let

$$S = Ke^x$$

Then

$$\frac{\partial V}{\partial x} = S\frac{\partial V}{\partial S}, \ \frac{\partial^2 V}{\partial x^2} = S\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2}, \text{ and (7) reduces to}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}) + (r - \delta)\frac{\partial V}{\partial x} - rV = 0$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + (r - \delta - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0 \tag{8}$$

Rearranging (8),

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - rV + (r - \delta - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} = -\frac{\partial V}{\partial t}$$

Dividing each side by $\sigma^2/2$ yields

$$\frac{\partial^2 V}{\partial x^2} - (\frac{2r}{\sigma^2})V + (\frac{2(r-\delta)}{\sigma^2} - 1)\frac{\partial V}{\partial x} = -(\frac{2}{\sigma^2})\frac{\partial V}{\partial t}$$

or

$$\frac{\partial^2 V}{\partial x^2} - qV + (q_\delta - 1)\frac{\partial V}{\partial x} = -(\frac{2}{\sigma^2})\frac{\partial V}{\partial t} \tag{9}$$

where

$$q = \frac{2r}{\sigma^2},$$

$$q_\delta = \frac{2(r-\delta)}{\sigma^2}$$

Let

$$\tau = \frac{1}{2}\sigma^2(T - t)$$

Then

$$\frac{\partial V}{\partial \tau} = \frac{\partial V}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{2}{\sigma^2}\frac{\partial V}{\partial t}$$

and (9) becomes

$$\frac{\partial^2 V}{\partial x^2} - qV + (q_\delta - 1)\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \tau} \tag{10}$$

Assuming α and β are constant and letting

$$V(S,t) = Ke^{\alpha x+\beta\tau}u(x,\tau)$$

Then

$$\frac{\partial V}{\partial x} = \alpha V + \frac{V}{u}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2} = \alpha\frac{\partial V}{\partial x} + \alpha\frac{V}{u}\frac{\partial u}{\partial x} + \frac{V}{u}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial V}{\partial \tau} = \beta V + \frac{V}{u}\frac{\partial u}{\partial \tau}$$

Substitutions from (10) yields

$$(\alpha\frac{\partial V}{\partial x}+\alpha\frac{V}{u}\frac{\partial u}{\partial x}+\frac{V}{u}\frac{\partial^2 u}{\partial x^2})-qV+(q_\delta-1)(\alpha V+\frac{V}{u}\frac{\partial u}{\partial x}) = \beta V+\frac{V}{u}\frac{\partial u}{\partial \tau}$$

$$(\alpha\frac{y}{V}\frac{\partial V}{\partial x}+\alpha\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2})-qu+(q_\delta-1)(\alpha u+\frac{\partial u}{\partial x}) = \beta u+\frac{\partial u}{\partial \tau} \quad (11)$$

Replacing $\frac{\partial V}{\partial x}$ with $\alpha V + \frac{V}{u}\frac{\partial u}{\partial x}$ in (11) and rearranging terms to obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial \tau} = [-(q_\delta - 1) - 2\alpha]\frac{\partial u}{\partial x} + [q + \alpha(q_\delta - 1) - \alpha^2 + \beta]u$$

Thus, the heat equation obtains

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial \tau} = 0$$

if

$$-(q_\delta - 1) - 2\alpha = 0 \Rightarrow \alpha = -\frac{1}{2}(q_\delta - 1)$$

$$q + \alpha(q_\delta - 1) - \alpha^2 + \beta = 0 \Rightarrow \beta = -[\frac{1}{4}(q_\delta - 1)^2 + q]$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial \tau} = 0$$

where

$$V(S,t) = Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} u(x,\tau)$$

$$S = Ke^x, \;\; \tau = \frac{1}{2}\sigma^2 (T - t)$$

$$q = \frac{2r}{\sigma^2},$$

$$q_\delta = \frac{2(r - \delta)}{\sigma^2}$$