Numerical Methods for Financial Derivatives

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Lecture 9: The Matrix Structure of Finite Difference Methods

Overview

- Finite Differences also referred to as a PDE method.
- The early example of applying finite differences to price financial derivatives is Brennan and Schwartz (1977): "Convertible Bonds: Valuation and Optimal Strategies for Call and Conversion." Journal of Finance 32: 1699-1715
- Various FDMs:
    - Explicit Finite Difference Method (or Explicit Euler Method)
    - Implicit Finite Difference Method (or Implicit Euler Method)
    - Crank-Nicholson Method
    - θ Method

FDM: Advantages vs Disadvantages

- Advantages:
  - A clearly understood scaling between computational effort and accuracy.
  - Finite differences can handle early exercise and complex boundaries & barriers.
  - The computation based on finite differences gives us the derivative's price for all the values of the initial spot price within the computational domain.
- Disadvantages:
  - The driving process must be Markovian.
  - The number of dimensions must be small.

Taylor Series Expansion and Truncation Error

- Forward Approximation:

$$f(x+\triangle x) = f(x) + f'(x)\triangle x + \underbrace{\frac{1}{2!}f''(x)\triangle x^2 + \frac{1}{3!}f'''(x)\triangle x^3 + \ldots}$$

$$f'(x) \equiv \frac{df}{dx} = \frac{f(x+\triangle x) - f(x)}{\triangle x} - TE$$

$$TE = \frac{1}{\triangle x}[\frac{1}{2!}f''(x)\triangle x^2 + \frac{1}{3!}f'''(x)\triangle x^3 + \ldots] \quad \sim \quad O(\triangle x)$$

- Backward Approximation:

$$f(x-\triangle x) = f(x) - f'(x)\triangle x + \underbrace{\frac{1}{2!}f''(x)\triangle x^2 - \frac{1}{3!}f'''(x)\triangle x^3 + \ldots}$$

$$f'(x) \equiv \frac{df}{dx} = \frac{f(x-\triangle x) - f(x)}{\triangle x} - TE$$

$$TE = \frac{1}{\triangle x}[\frac{1}{2!}f''(x)\triangle x^2 - \frac{1}{3!}f'''(x)\triangle x^3 + \ldots] \quad \sim \quad O(\triangle x)$$

Taylor Series Expansion and Truncation Error (2)

- First-Order Central Approximation:

$$f(x+\triangle x)-f(x-\triangle x)=2f'(x)\triangle x+\underbrace{2\times\frac{1}{3!}f'''(x)\triangle x^3+\ldots}$$

$$f'(x)\equiv\frac{df}{dx}=\frac{f(x+\triangle x)-f(x-\triangle x)}{2\triangle x}-TE$$

$$TE=\frac{1}{2\triangle x}[\frac{1}{3!}f'''(x)\triangle x^3+\ldots]\quad\sim\quad O(\triangle x^2)$$

- Second-Order Central Approximation:

$$f(x+\triangle x)+f(x-\triangle x)=2f(x)+f''(x)\triangle x^2+\underbrace{2\times\frac{1}{4!}f''''(x)\triangle x^4+\ldots}$$

$$f''(x)\equiv\frac{d^2f}{dx^2}=\frac{f(x+\triangle x)-2f(x)+f(x-\triangle x)}{\triangle x^2}-TE$$

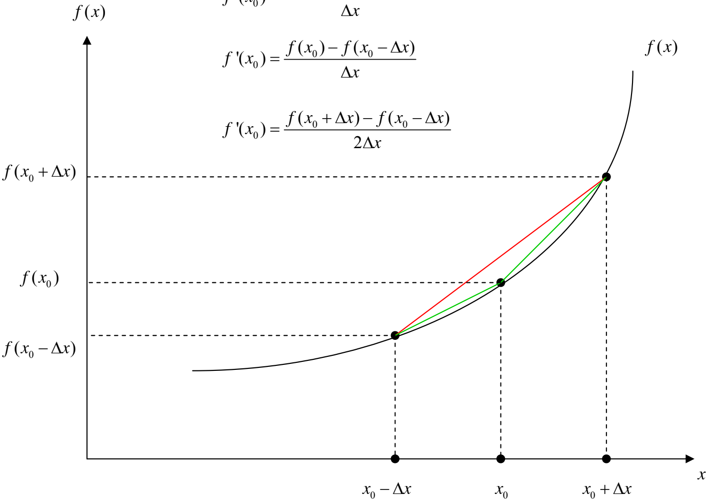
$$TE=\frac{1}{\triangle x^2}[2\times\frac{1}{4!}f''''(x)\triangle x^4+\ldots]\quad\sim\quad O(\triangle x^2)$$

Geometric Illustration of Finite Differences Approximation

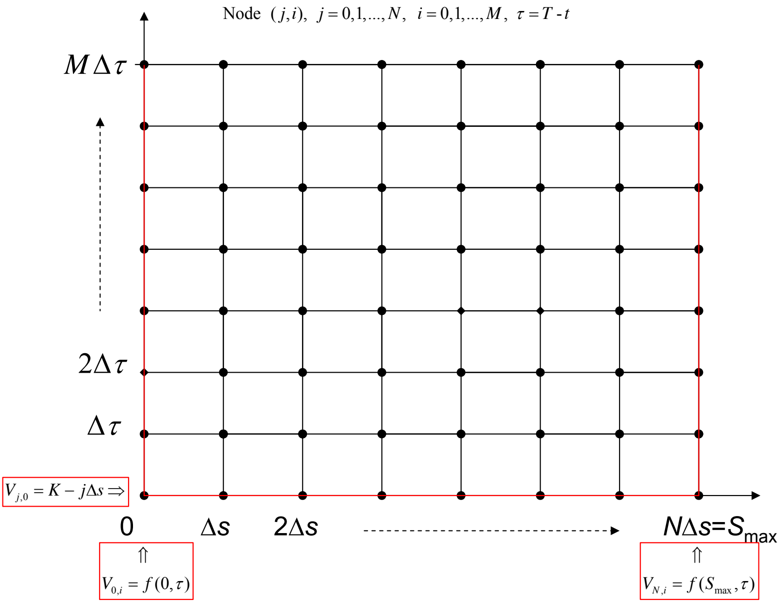
$$f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$f'(x_0) = \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x}$$

$$f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x}$$



Finite Difference Grid in Reverse Time



The BS PDE in Reverse Time

- The BS pricing equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- The BS pricing equation with $x = \log S$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0$$

- In Reverse Time, $\tau = T - t$ (time to maturity): $\frac{\partial V}{\partial \tau} = \frac{\partial V}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{\partial V}{\partial t}$
- Therefore,

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial x^2} + rS\frac{\partial V}{\partial x} - rV$$

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV$$

- Applying central differences to approximate $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial V}{\partial x}$ for time level i $(= 0, 1, ..., M)$:

$$\frac{\partial^2 V(j\triangle x, i\triangle \tau)}{\partial x^2} = \frac{V_{j+1,i} - 2V_{j,i} + V_{j-1,i}}{\triangle x^2}, \quad j = 1, 2, ...N-1$$

$$\frac{\partial V(j\triangle x, i\triangle \tau)}{\partial x} = \frac{V_{j+1,i} - V_{j-1,i}}{2\triangle x}, \; j = 1, 2, ...N-1$$

where

$$V_{j,i} = V(j\triangle x, i\triangle \tau)$$

- Space discretization of the BS equation thus leads to:

$$\frac{\partial V_{j,i}}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{V_{j+1,i} - 2V_{j,i} + V_{j-1,i}}{\triangle x^2} + (r - \frac{1}{2}\sigma^2)\frac{V_{j+1,i} - V_{j-1,i}}{2\triangle x} - rV_{j,i}$$

Implementation of Space Discretization (2)

- In fact, implementing space discretization generates a linear system of ordinary differential equations (ignoring subscript i):

$$\frac{dV_j}{d\tau} = aV_{j-1} + bV_j + cV_{j+1}, \quad j = 1, 2, ..., N-1$$

where

$$a = \frac{\sigma^2}{2\triangle x^2} - \frac{r}{2\triangle x} + \frac{\sigma^2}{4\triangle x}$$

$$b = -\frac{\sigma^2}{\triangle x^2} - r$$

$$c = \frac{\sigma^2}{2\triangle x^2} + \frac{r}{2\triangle x} - \frac{\sigma^2}{4\triangle x}$$

- Note that these coefficients (a, b, c) are independent of space step.

Implementation of Space Discretization (3)

- In matrix form, the linear system of ODEs is:

$$\underbrace{\begin{bmatrix} \frac{dV_1}{d\tau} \\ \frac{dV_2}{d\tau} \\ \vdots \\ \vdots \\ \frac{dV_{N-1}}{d\tau} \end{bmatrix}}_{(N-1)\times 1} = \underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \cdots & \cdots & 0 & a_{N-2} & b_{N-2} & c_{N-2} & 0 \\ \cdots & \cdots & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \end{bmatrix}}_{(N-1)\times(N+1)} \underbrace{\begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ \vdots \\ V_N \end{bmatrix}}_{(N+1)\times 1}$$

- This is a tridiagonal *rectangular* system:
  - $(N-1)$ ordinary differential equations,
  - $(N+1)$ unknowns,
  - $a_j = a, b_j = b, c_j = c,\ j = 1, \ldots, N-1$in this specific case of space discretization.

- Case 1: For instance, for an European call, the lower boundary may be zero ($V_{0,i} = V(0, \tau_i) = 0$) and the upper boundary may be given by the present value of the intrinsic value at maturity; i.e.

$$V_{N,i} = V(N\triangle x, \tau_i) = e^{-\tau_i}(e^{N\triangle x} - K)^+, \ \tau_i = i\triangle\tau$$

- We can drop the 1st and 2nd columns to obtain a square system:

$$\underbrace{\begin{bmatrix} \frac{dV_1}{d\tau} \\ \frac{dV_2}{d\tau} \\ \vdots \\ \vdots \\ \frac{dV_{N-1}}{d\tau} \end{bmatrix}}_{(N-1)\times 1} = \underbrace{\begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots \\ a_2 & b_2 & c_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & a_{N-2} & b_{N-2} & c_{N-2} \\ \cdots & \cdots & 0 & a_{N-1} & b_{N-1} \end{bmatrix}}_{(N-1)\times(N-1)} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_{N-1} \end{bmatrix}}_{(N-1)\times 1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ F_N \end{bmatrix}}_{(N-1)\times 1}$$

where we ignore subscript i, and $F_N = c_{N-1}[e^{-\tau_i}(e^{N\triangle x} - K)^+]$. That is,

$$\frac{dV}{d\tau} = A_{\triangle x} \cdot V + F$$

How to Fix the Rectangular System? (2)

- Case 2: We may be able to linearly extrapolate V_0 and V_1:

$$V_{0,i} = 2V_{1,i} - V_{2,i}, \quad V_{N,i} = 2V_{N-1,i} - V_{N-2,i}$$

- Then we enlarge the system by adding two rows and make it square:

$$\underbrace{\begin{bmatrix} \frac{dV_0}{d\tau} \\ \frac{dV_1}{d\tau} \\ \vdots \\ \vdots \\ \frac{dV_N}{d\tau} \end{bmatrix}}_{(N+1)\times 1} = \underbrace{\begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_{N-2} & b_{N-2} & c_{N-2} & 0 \\ 0 & 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}}_{(N+1)\times(N+1)} \underbrace{\begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ \vdots \\ V_N \end{bmatrix}}_{(N+1)\times 1}$$

- As indicated above (ignoring i),

$$\frac{dV_0}{d\tau} = \frac{dV_N}{d\tau} = 0$$

Summary of the Space Discretization

- We have discretized

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0$$

  through the space domain into

$$\frac{dV^{(i)}}{d\tau} = A_{\triangle x} \cdot V^{(i)} \quad \text{or} \quad \frac{dV^{(i)}}{d\tau} = A_{\triangle x} \cdot V^{(i)} + F^{(i)}, \quad \text{with}$$

$$V^{(i)} = [V_{0,i}, V_{1,i}, \ldots, V_{N,i}]', \text{ or } V^{(i)} = [V_{1,i}, V_{1,i}, \ldots, V_{N-1,i}]'$$

- Other ways to deal boundary conditions:
  - One-sided difference for both $V_{0,i}$ and $V_{N,i}$.
  - The discrete linear complementary problem.
- The ODEs system with initial conditions, $V_{j,0}$ (payoffs).
- Corresponding to each specific time level.
- Solving the system must use a time-discretization scheme.

Time Discretization: Explicit, Implicit, Weighted Averages

- Explicit Method

$$\frac{dV^{(i)}}{d\tau} = \frac{V^{(i+1)} - V^{(i)}}{\triangle\tau} = A_{\triangle x} \cdot V^{(i)}$$

- Implicit Method

$$\frac{dV^{(i+1)}}{d\tau} = \frac{V^{(i+1)} - V^{(i)}}{\triangle\tau} = A_{\triangle x} \cdot V^{(i+1)}$$

- Crank-Nicolson Method

$$\frac{1}{2}(\frac{dV^{(i+1)}}{d\tau} + \frac{dV^{(i)}}{d\tau}) = \frac{V^{(i+1)} - V^{(i)}}{\triangle\tau} = \frac{1}{2}A_{\triangle x}(V^{(i+1)} + V^{(i)})$$

- θ Method

$$\frac{V^{(i+1)} - V^{(i)}}{\triangle\tau} = A_{\triangle x}(\theta V^{(i+1)} + (1-\theta)V^{(i)})$$

- $\theta = 0 \Rightarrow$ Explicit; $\theta = 1 \Rightarrow$ Implicit; $\theta = \frac{1}{2} \Rightarrow$ Crank-Nicolson

Matrix Forms After Both Space and Time Discretization

- General Form of A Discretized System

$$A_L \cdot V^{(i+1)} = A_R \cdot V^{(i)}$$

- Explicit Method

$$V^{(i+1)} = (I + \triangle\tau A_{\triangle x}) \cdot V^{(i)}$$

- Implicit Method

$$(I - \triangle\tau A_{\triangle x}) \cdot V^{(i+1)} = V^{(i)}$$

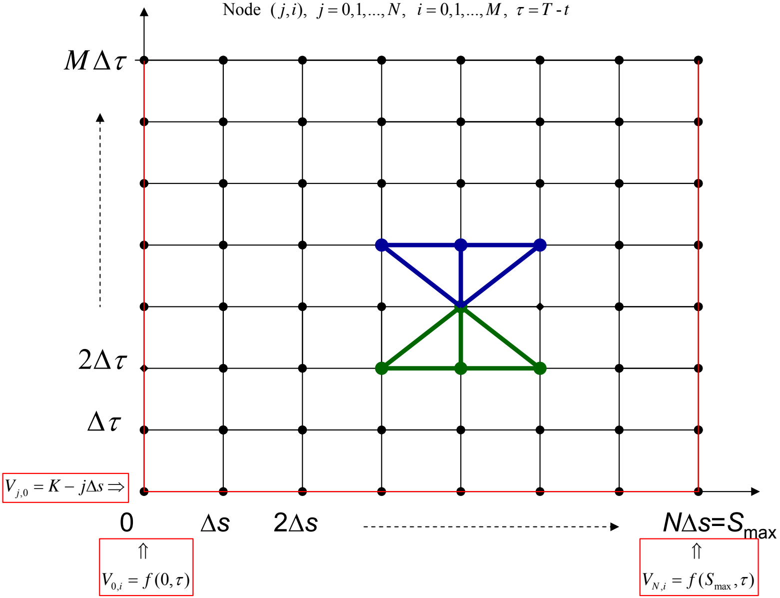
- Crank-Nicolson Method

$$(I - \frac{\triangle\tau}{2} A_{\triangle x}) \cdot V^{(i+1)} = (I + \frac{\triangle\tau}{2} A_{\triangle x}) V^{(i)}$$

- θ Method

$$(I - \theta\triangle\tau A_{\triangle x}) \cdot V^{(i+1)} = [I + (1-\theta)\triangle\tau A_{\triangle x}] V^{(i)}$$

Computational Views of Alternative FDMs



An Augmented Discretization Matrix under Explicit Method

$$V^{(i+1)} = \underbrace{(I + \triangle\tau A_{\triangle x})}_{A_R} \cdot V^{(i)}$$

$$A_R = \begin{bmatrix} 1+\triangle\tau & -2\triangle\tau & \triangle\tau & 0 & \cdots & \cdots & 0 \\ a_1\triangle\tau & 1+b_1\triangle\tau & c_1\triangle\tau & 0 & \cdots & \cdots & 0 \\ 0 & a_2\triangle\tau & 1+b_2\triangle\tau & c_2\triangle\tau & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & a_{N-2}\triangle\tau & 1+b_{N-2}\triangle\tau & c_{N-2}\triangle\tau & 0 \\ 0 & \cdots & \cdots & 0 & a_{N-1}\triangle\tau & 1+b_{N-1}\triangle\tau & c_{N-1}\triangle\tau \\ 0 & \cdots & \cdots & 0 & 1+\triangle\tau & -2\triangle\tau & 1+\triangle\tau \end{bmatrix}_{(N+1)\times(N+1)}$$

A Reduced Discretization Matrix under Explicit Method

$$V^{(i+1)} = \underbrace{(I + \triangle\tau A_{\triangle x})}_{A_R} \cdot V^{(i)} + \triangle\tau F$$

$$A_R = \begin{bmatrix} 1+b_1\triangle\tau & c_1\triangle\tau & 0 & \cdots & \cdots \\ a_2\triangle\tau & 1+b_2\triangle\tau & c_2\triangle\tau & 0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & a_{N-2}\triangle\tau & 1+b_{N-2}\triangle\tau & c_{N-2}\triangle\tau \\ \cdots & \cdots & 0 & a_{N-1}\triangle\tau & 1+b_{N-1}\triangle\tau \end{bmatrix}_{(N-1)\times(N-1)}$$

Primary Ways to Solve the Linear System

There are two ways to solve the linear system,

$$A_L \cdot V^{(i+1)} = A_R \cdot V^{(i)}$$

- Direct Solvers
  - Tridiagonal solver
- Iterative Solvers
  - The Jacobi Method
  - The Gauss-Seidel Method
  - The Successive Overrelaxation Method (SOR)