Numerical Methods for Financial Derivatives

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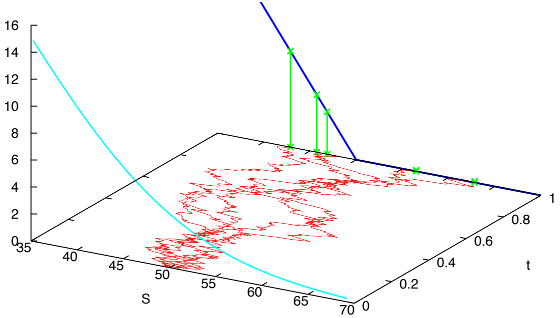
Lecture 4: Algorithms for Simulating Sample Paths and Computing European Options (Ch. 3)

Introduction

- We present three methods for generating sample paths via Monte Carlo simulations.
- We demonstrate the use of Ito-Taylor expansions to derive these numerical methods.
- We discuss how to apply Monte Carlo simulations for computing European options.
  - In Chapter 1 we introduced the formula of risk-neutral valuation of options,
    $$V(S_0, 0) = e^{-rT} E_Q[\Psi(S_T)|S_0],$$
    where $\Psi(S_T)$ denotes the payoff.
  - In the Black–Scholes model, specifically, this is
    $$V(S_0, 0) = e^{-rT} \int_0^\infty \Psi(S_T) \cdot f_{GBM(\boldsymbol{S_T}, T; \boldsymbol{S_0}, r, \sigma)} dS_T.$$
- There are two approaches to calculate the above integral:
  1. The integral (Int) is approximated using numerical quadrature.
  2. One applies Monte Carlo simulation.

Monte Carlo Simulation for European Options

- One draws random numbers that match the underlying risk-neutral probability.
- Calculates many sample paths of asset prices S_t. This is the bulk of the work.
- To complete, compute the mean of the payoff values, and discount.



Five simulated asset paths with payoff.

Numerical Methods for Sample Paths

- For instance, given the process

$$dX(t) = a(X(t), t)dX(t) + b(X(t), t)dW(t),$$

how to general the trajectory of S in the time domain?

- In general, there are to directions:
  - Analytical solution
  - Some time-discretized numerical integration schemes
- We present two numerical integration schemes to simulate sample paths for the dynamic trajectory of a stochastic process. These schemes are:
  - the explicit Euler method

$$y_{t_{i+1}} = y_{t_i} + a(y_{t_i})\triangle t + b(y_{t_i})\triangle W_i$$

  - the Milstein method

$$\begin{aligned} y_{t_{i+1}} = y_{t_i} &+ a(y_{t_i})\triangle t + b(y_{t_i})\triangle W_i \\ &+ \frac{1}{2} b(y_{t_i}) b'(y_{t_i})[(\triangle W_i)^2 - \triangle t] \end{aligned}$$

- where y_{t_i} is an approximation of $X(t_i)$.

Taylor Expansion for Deterministic Function Equations

- Given a real-valued function,

$$f(X)$$

- Taylor expansion of $f(X)$ at X_0:

$$f(X) = f(X_0) + (X - X_0)f'(X_0) + \frac{1}{2}(X - X_0)^2 f''(X_0) \\ + \ldots + \frac{1}{n!}(X - X_0)^2 f^{(n)}(X_0) + \ldots$$

  - Linear approximation:

$$f(X) \approx f(X_0) + (X - X_0)f'(X_0)$$

  - Quadratic approximation:

$$f(X) \approx f(X_0) + (X - X_0)f'(X_0) + \frac{1}{2}(X - X_0)^2 f''(X_0)$$

- The truncation error

- Consider the autonomous ODE

$$\frac{dX(t)}{dt} = a(X(t))$$

- Let f be a function of $X(t)$, then the evolution of $f(X(t))$ is governed by

$$\frac{df(X(t))}{dt} = \frac{dX((t))}{dt} \cdot \frac{\partial}{\partial X} f(X(t)), \quad \text{or}$$

$$\frac{df(X((t))}{dt} = a(X(t)) \cdot \frac{\partial}{\partial X} f(X(t)) \tag{1}$$

- **Integral equation**: By defining a linear operator

$$\mathcal{L} \equiv a(X) \cdot \frac{\partial}{\partial X},$$

we can rewrite eq.(1) in terms of the integral equation:

$$f(X(t)) = f(X(t_0)) + \int_{t_0}^{t} \mathcal{L} f(X(\tau_1)) d\tau_1 \tag{2}$$

where t_0 is the initial time.

- **Iterations**: iterating eq.(2) leads to

$$\begin{aligned} f(X(t)) &= f(X(t_0)) + \int_{t_0}^{t} \mathcal{L} \left[f(X(t_0)) + \int_{t_0}^{\tau_1} \mathcal{L} f(X(\tau_2)) d\tau_2 \right] d\tau_1 \\ &= f(X(t_0)) + \mathcal{L} f(X(t_0)) \int_{t_0}^{t} d\tau_1 + \int_{t_0}^{t} \int_{t_0}^{\tau_1} \mathcal{L}^2 f(X(\tau_2)) d\tau_2 d\tau_1 \\ &= f(X(t_0)) + \mathcal{L} f(X(t_0)) \cdot (t - t_0) + \int_{t_0}^{t} \int_{t_0}^{\tau_1} \mathcal{L}^2 f(X(\tau_2)) d\tau_2 d\tau_1 \end{aligned} \tag{3}$$

- If we iterate once more using Eq. (2) for $f(X(\tau_2))$ in (3), we have

$$\begin{aligned}\mathcal{L}^2 f(X(\tau_2)) &= \mathcal{L}^2 \left[f(X(t_0)) + \int_{t_0}^{\tau_2} \mathcal{L} f(X(\tau_3)) d\tau_3 \right] \\ &= \mathcal{L}^2 f(X(t_0)) + \underbrace{\int_{t_0}^{t_2} \mathcal{L}^3 f(X(\tau_3)) d\tau_3}_{O(\mathcal{L}^3)}\end{aligned}$$

- Then the double integral term becomes

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathcal{L}^2 f(X(\tau_2)) d\tau_2 d\tau_1 &= \int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathcal{L}^2 f(X(t_0)) d\tau_2 d\tau_1 + O(\mathcal{L}^3) \\ &= \mathcal{L}^2 f(X(t_0)) \int_{t_0}^{t}\int_{t_0}^{\tau_1} d\tau_2 d\tau_1 + O(\mathcal{L}^3) \\ &= \mathcal{L}^2 f(X(t_0)) \cdot \frac{1}{2}(t-t_0)^2 + O(\mathcal{L}^3)\end{aligned}$$

- All these together, we can rewrite eq.(3) as

$$f(X(t)) = f(X(t_0)) + \mathcal{L}f(X(t_0)) \cdot (t - t_0) + \frac{1}{2}\mathcal{L}^2 f(X(t_0)) \cdot (t - t_0)^2 + O(\mathcal{L}^3) \tag{4}$$

- This is precisely the Taylor expansion we are familiar with.
- **Recap**: The Taylor expansion for the process

$$\frac{df(X((t))}{dt} = a(X(t)) \cdot \frac{\partial}{\partial X} f(X(t))$$

or

$$f(X(t)) = f(X(t_0)) + \int_{t_0}^{t} \mathcal{L}f(X(\tau_1))d\tau_1,$$

is given by eq.(4).

Stochastic Taylor Expansion

- Consider the stochastic differential equation,

$$dX(t) = a(X(t))dt + b(X(t))dW(t) \tag{5}$$

  Again for simplicity, we consider the autonomous case.
- **Question**: what is the stochastic Taylor (i.e., Ito-Taylor) expansion for the process (5)?
- First of all, let us apply Ito's lemma to an arbitrary function, $f(X(t))$,

$$\begin{aligned} df(X(t)) &= \frac{\partial f(X(t))}{\partial X} dX + \frac{\partial f(X(t))}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f(X(t))}{\partial X^2} (dX)^2 \\ &= \left\{ a(X(t) \frac{\partial}{\partial X} f(X(t)) + \frac{1}{2} [b(X(t))]^2 \frac{\partial^2}{\partial X^2} f(X(t)) \right\} dt \\ &\quad + b(X(t) \frac{\partial}{\partial X} f(X(t)) dW(t) \end{aligned} \tag{6}$$

Stochastic Taylor Expansion (2)

- Define these two linear operators:

$$\mathcal{L}^0 = a(X(t))\frac{\partial}{\partial X} + \frac{1}{2}[b(X(t))]^2\frac{\partial^2}{\partial X^2}$$

$$\mathcal{L}^1 = b(X(t))\frac{\partial}{\partial X}$$

- Then, we can rewrite eq.(6) as

$$df(X(t)) = \mathcal{L}^0 f(X(t))dt + \mathcal{L}^1 f(X(t))dW(t) \tag{7}$$

or

$$f(X(t)) = f(X(t_0)) + \int_{t_0}^{t} \mathcal{L}^0 f(X(s))ds + \int_{t_0}^{t} \mathcal{L}^1 f(X(s))dW(s) \tag{8}$$

Stochastic Taylor Expansion (2)

Next, we can choose $f(X(t))$ in eq.(8):

- If we choose $f(x) = x$, then eq.(8) becomes

$$X(t) = X(t_0) + \int_{t_0}^{t} a(X(s))ds + \int_{t_0}^{t} b(X(s))dW(s) \tag{9}$$

- If we choose $f(x) = a(x)$, then eq.(8) becomes

$$a(X(t)) = a(X(t_0)) + \int_{t_0}^{t} \mathcal{L}^0 a(X(s))ds + \int_{t_0}^{t} \mathcal{L}^1 a(X(s))dW(s) \tag{10}$$

- If we choose $f(x) = b(x)$, then eq.(8) becomes

$$b(X(t)) = b(X(t_0)) + \int_{t_0}^{t} \mathcal{L}^0 b(X(s))ds + \int_{t_0}^{t} \mathcal{L}^1 b(X(s))dW(s) \tag{11}$$

- Substituting eqs.(10) and (11) into eq.(9), we have

$$
\begin{aligned}
X(t) = X(t_0) & \\
&+ \int_{t_0}^{t} \left[a(X(t_0)) + \int_{t_0}^{s_1} \mathcal{L}^0 a(X(s_2)) ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1 a(X(s_2)) dW(s_2) \right] ds_1 \\
&+ \int_{t_0}^{t} \left[b(X(t_0)) + \int_{t_0}^{s_1} \mathcal{L}^0 b(X(s_2)) ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1 b(X(s_2)) dW(s_2) \right] dW(s_1)
\end{aligned} \tag{12}
$$

- where in terms of the linear operators (page 11),
  - $\mathcal{L}^0 a = a \frac{\partial}{\partial X} a(X) + \frac{1}{2} b^2 \frac{\partial^2}{\partial X^2} a(X) \equiv aa' + \frac{1}{2} b^2 a''$
  - $\mathcal{L}^0 b = ab' + \frac{1}{2} b^2 b''$
  - $\mathcal{L}^1 a = b \frac{\partial}{\partial X} a(X) = ba'$
  - $\mathcal{L}^1 b = bb'$

Stochastic Taylor Expansion (4)

- Moving those constant terms out of the integrands in eq.(12), we obtain

$$X(t) = X(t_0) + a(X(t_0)) \int_{t_0}^{t} ds_1 + b(X(t_0)) \int_{t_0}^{t} dW(s_1) + R \quad (13)$$

- where R is given by

$$\begin{aligned} R = & \int_{t_0}^{t} \int_{t_0}^{s_1} \mathcal{L}^0 a(X(s_2)) ds_2 ds_1 + \int_{t_0}^{t} \int_{t_0}^{s_1} \mathcal{L}^1 a(X(s_2)) dW(s_2) ds_1 \\ & + \int_{t_0}^{t} \int_{t_0}^{s_1} \mathcal{L}^0 b(X(s_2)) ds_2 dW(s_1) + \int_{t_0}^{t} \int_{t_0}^{s_1} \mathcal{L}\mathcal{L}^1 b(X(s_2)) dW(s_2) dW(s_1) \end{aligned} \quad (14)$$

- Note that the essence of the method is to use the substitution repeatedly to obtain constant integrands in higher and higher order terms. For instance, we can apply the substitution to the last term in eq.(14).

- Apply the substitution to the last term of eq.(14):

$$\int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1 b(X(s_2))dW(s_2)dW(s_1) \tag{15}$$

  - Select $f(x) \equiv \mathcal{L}^1 b(x)$;
  - Substitute for $L^1 b$ in terms of eq.(11):

$$\begin{aligned}\mathcal{L}^1 b(X(s_2)) = \mathcal{L}^1 b(X(t_0)) & \\ & + \int_{t_0}^{s_2}\mathcal{L}^0\mathcal{L}^1 b(X(s_3))ds_3 \\ & + \int_{t_0}^{s_2}\mathcal{L}^1\mathcal{L}^1 b(X(s_3))dW(s_3)\end{aligned}$$

  - where $\mathcal{L}^1 b(X(t_0)) = b(X(t_0))b'(X(t_0))$.

Stochastic Taylor Expansion (6)

- The Ito-Taylor expansion for the process (5) is

$$X(t) = X(t_0) + a(X(t_0)) \int_{t_0}^{t} ds_1 + b(X(t_0)) \int_{t_0}^{t} dW(s_1) + b(X(t_0))b'(X(t_0)) \int_{t_0}^{t} \int_{t_0}^{s_1} dW(s_2)dW(s_1) + \tilde{R} \quad (16)$$

- where $\tilde{R}$ is a new remainder and

$$\begin{aligned} \int_{t_0}^{t} \int_{t_0}^{s_1} dW(s_2)dW(s_1) &= \int_{t_0}^{t} W(s_1)dW(s_1) - \int_{t_0}^{t} W(t_0)dW(s_1) \\ &= \frac{1}{2} \int_{t_0}^{t} \left\{ d[W(s_1)]^2 - dt \right\} - W(t_0)[W(t) - W(t_0)] \\ &= \frac{1}{2} \left\{ [W(t)]^2 - [W(t_0)]^2 - (t - t_0) \right\} - W(t_0)[W(t) - W(t_0) \\ &= \frac{1}{2}[W(t) - W(t_0)]^2 - (t - t_0) \quad (17) \end{aligned}$$

- Note: $\int_{t_0}^{t} W(s_1)dW(s_1) = \frac{1}{2} \int_{t_0}^{t} \left\{ d[W(s_1)]^2 - dt \right\}$, (Ito's lemma)

- With the Ito-Taylor expansion (16), we can construct numerical integration schemes for the process (5). For the interval $[t_i, t_{i+1}]$, by choosing

$$t_0 = t_i, \quad t = t_{t+1}, \quad \triangle t = t_{i+1} - t_i,$$
$$\triangle W_i = W(t_{i+1}) - W(t_i),$$

combining (16) and (17) yields

$$X(t_{i+1}) = X(t_i) + a(X(t_i))\triangle t + b(X(t_i)\triangle W_i + \frac{1}{2}b(X(t_i))b'(X(t_i))[(\triangle W_i)^2 - \triangle t] + \tilde{R} \qquad (18)$$

- Keeping the first three terms, (18) gives the **explicit Euler method**:

$$y_{t_{i+1}} = y_{t_i} + a(y_{t_i})\triangle t + b(y_{t_i})\triangle W_i$$

- Keeping all terms but the remainder gives us the **Milstein method**:

$$y_{t_{i+1}} = y_{t_i} + a(y_{t_i})\triangle t + b(y_{t_i})\triangle W_i + \frac{1}{2}b(y_{t_i})b'(y_{t_i})[(\triangle W_i)^2 - \triangle t]$$

- where y_{t_i} is an approximation of $X(t_i)$.