Numerical Methods for Financial Derivatives

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Lecture 7: Characteristic Function, Fourier Transform and Pricing of Derivatives

Introduction

- In a large and growing family of financial models, explicit formulas exist for the characteristic functions of the state variables.
- Given any such characteristic function, we can compute the prices of a wide class of options on those underlying state variables using Fourier transform:
  - Fourier-analytic solutions to various forms of this problem have appeared in the finance literature.
  - They express option prices in terms of Fourier-inversion integrals, which are in practice evaluated numerically.
  - Carr, P. and D. Madan (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance* 3, 463–520.

Characteristic Function

Characteristic Function

If $f(.)$ is the probability density function of a random variable X, its *characteristic function* is

$$\varphi_X(\omega) \equiv E[e^{i\omega X}] = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx$$

Unlike the moment-generating function, the characteristic function always exists when treated as a function of a real-valued argument.

Conventions of the Continuous Fourier Transform

- We need to stay cautious when attempting to relating the characteristic function to the continuous Fourier transform. This is because there are different conventions for the definition of the Fourier transform of a probability density function $f(x)$, as shown in the following table:

Convention 1	Convention 2
$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi isx} dx$ $f(x) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi isx} ds$	$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{2\pi isx} dx$ $f(x) = \int_{-\infty}^{\infty} \hat{f}(n) e^{-2\pi isx} ds$
$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} dx$

- where $\omega \equiv 2\pi s$ is angular frequency in radians. There are more conventions.

Characteristic Function and Fourier Transform

- If convention 2 is chosen, the characteristic function for $f(x)$ is the Fourier transform of $f(x)$. That is,

$$\varphi_X(\omega) \equiv E[e^{i\omega X}] = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx = \hat{f}(\omega)$$

- If convention 1 is instead chosen, the characteristic function for $f(x)$ is the complex conjugate of the Fourier transform of $f(x)$. That is,

$$\begin{aligned} \varphi_X(\omega) \equiv E[e^{i\omega X}] &= \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx \\ &= \overline{\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx} = \overline{\hat{f}(\omega)} \\ &= \overline{\int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)dx} = \overline{\hat{f}(s)} \end{aligned}$$

- where $\varphi_X(\omega) = \overline{\hat{f}(\omega)} = \overline{\hat{f}(s)} \Leftrightarrow \overline{\varphi_X(\omega)} = \hat{f}(\omega) = \hat{f}(s)$.
- We choose the popular convention 1, which is consistent with Python's FFT module.

Moments of a Random Variable via Characteristic Function

- The characteristic function allows us to recover an arbitrary number of moments of that function.
- Suppose we have the characteristic function of random variable X

$$\varphi_X(\omega) = E[e^{i\omega X}]$$

  - The n^{th} derivative of $\varphi(\omega)$ is

$$\varphi^{(n)}(\omega) = E[(iX)^n e^{i\omega X}]$$

  - The n^{th} derivative of $\varphi(\omega)$ at $\omega = 0$ is

$$\varphi^{(n)}(0) = E[(iX)^n] = i^n E[X^n]$$

  - The n^{th}moment of X is

$$E[X^n] = i^{-n}\varphi^{(n)}(0)$$

The Characteristic Function of $Z \sim N(0,1)$

- If random variable $Z \sim N(0,1)$, its characteristic function is

$$\varphi_Z(\omega) = E[e^{i\omega Z}] = \int_{-\infty}^{\infty} \phi(z) e^{i\omega z} = e^{-\frac{1}{2}\omega^2}$$

where $\phi(z) = \frac{1}{\sqrt{(2\pi}} e^{-\frac{1}{2}z^2}$ is the standard normal density function.

- Recall: From Lecture 6, the Fourier transform of $\phi(z)$ is

$$\hat{\phi}(s) = e^{-2\pi^2 s^2} \Rightarrow \hat{\phi}(\omega) = e^{-\frac{1}{2}\omega^2}, \quad \omega = 2\pi s.$$

- Therefore, $\varphi_Z(\omega) = \bar{\hat{\phi}}(\omega) = \hat{\phi}(\omega) = e^{-\frac{\omega^2}{2}}$.

- If random variable $X \sim N(\mu, \sigma^2)$, its characteristic function is

$$\varphi_X(\omega) = E[e^{i\omega X}] = e^{i\mu\omega - \frac{1}{2}\sigma^2\omega^2}$$

- Recall: from Lecture 6,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Longrightarrow \hat{f}(s) = e^{-2\pi i s\mu - 2\pi^2\sigma^2 s^2}$$

- Therefore, $\varphi_X(\omega) = \overline{\hat{f}}(s) = e^{2\pi i s\mu - 2\pi^2\sigma^2 s^2} = e^{i\mu\omega - \frac{1}{2}\sigma^2\omega^2}$
- Q: What is the characteristic function of $W_t \sim$ Brownian motion?

$$\varphi_W(\omega) = E[e^{i\omega W_t}] = e^{-\frac{1}{2}t\omega^2}$$

since

$$W_t \sim N(0, t)$$

Review of Geometric Brownian Motion

- GBM:

$$dS_t = (r-\delta)S_t dt + \sigma S_t dW_t$$

- Using Ito's lemma, the solution to the stochastic differential equation is

$$\begin{aligned} S_t &= S_0 e^{(r-\delta-\frac{\sigma^2}{2})t+\sigma W_t} \\ &= S_0 e^{(r-\delta-\frac{\sigma^2}{2})t+\sigma\sqrt{t}Z_t} \end{aligned}$$

- Probability density function of $X_t = \ln(S_t)$:

$$q(x_t) = \frac{1}{\sqrt{2\pi\sigma^2((t-t_0)}} e^{-\frac{\left[x_t - x_0 - (r-\delta-\frac{\sigma^2}{2})(t-t_0)\right]^2}{2\sigma^2(t-t_0)}}$$

- Probability density function of S_t:

$$f(S_t) = \frac{1}{S_t}\cdot q(x_t) = \frac{1}{S_t\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{\left[(x_t-x_0)-(\mu-\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}}$$

The Characteristic Function of $\ln(S_t)$

- The characteristic function of $X_t = \ln(S_t)$:

$$X_t \equiv \ln S_t \sim N(x_0 + (r - \delta - \frac{\sigma^2}{2})t, \sigma^2 t), \quad t_0 = 0$$

$$\begin{aligned} \varphi_X(\omega) = E[e^{i\omega X_t}] &= \int_{-\infty}^{\infty} e^{i\omega x} q(x) dx \\ &= e^{i(x_0 + (r - \delta - \frac{\sigma^2}{2})t)\omega - \frac{1}{2}\sigma^2 t \omega^2} \end{aligned}$$

- The first major development in the pricing of options using Fourier techniques was proposed by **Carr and Madan (1999)**
  - Peter Carr and Dilip B. Madan. 1999. Option valuation using the Fast Fourier transform (FFT). The *Journal of Computational Finance*, 2 (4): 61-73.
- Resources:
  - Ali Hirsa, *Computational Methods in Finance* (2013)
  - U. Cherubini, G. Lunga, S. Mulinacci, P.Rossi, *Fourier Transform Methods in Finance* (2010)

How to Use Fourier Transform Techniques?

Steps for pricing via Fourier transform techniques

1. **Derive** the Fourier transform of the expected value of the derivative under the risk-neutral distribution.
2. **Express** this transform in terms of a known characteristic function.
3. **Apply** the inverse Fourier transform to recover the derivative price.
4. **Implement** the FFT algorithm to compute the price.

How good is the Fourier Transform Method?

Pros and Cons: Fourier Transform

- **Models**: All models for which a characteristic function for the asset price distribution exists.
- **Pros**:
  1. Allows for pricing under any model with a characteristic function.
  2. Fast, n option prices in $O(n \ln(n))$ time.
  3. Generate $n$option prices in a single run.
- **Cons**:
  1. Restricted to path independent European options
  2. Restricted set of terminal payoffs, each needing to be rederived.
  3. Requires estimation of proper α, a damping parameter.
  4. Inaccurate for highly out-of-the-money options.

Definitions of variables

- S_t: price of the underlying security at time t
- $f(S_t) \equiv f(S_t \mid S_0)$: probability density function of S_t under some equivalent martingale measure
- $q(x_t) \equiv q(x_t \mid x_0)$: probability density function of $X_t \equiv \ln(S_t)$
- $k \equiv \ln(K)$: the log of the strike price K
- $V_C(S_0, t=0)$: price of European call option at present time $t = 0$
- $\varphi_s(\omega) = E[e^{i\omega X}]$: characteristic function of $X = \ln(S)$,

$$
\begin{aligned}
\varphi_X(\omega) &= \int_{-\infty}^{\infty} e^{i\omega x} q(x) dx \\
&= e^{i(x_0 + (r - \delta - \frac{\sigma^2}{2})t)\omega - \frac{\sigma^2 t}{2}\omega^2}
\end{aligned}
$$

The European call option V_C can be expressed as

$$
\begin{aligned}
V_C &= E_t[(S_T - K)^+] = e^{-rT}\int_{-\infty}^{\infty}(S_T - K)^+ f(S_T)dS_T \\
&= e^{-rT}\int_{K}^{\infty}(S_T - K)f(S_T)dS_T \\
&= e^{-rT}\int_{k}^{\infty}(e^{x_T} - e^k)q(x_T)dx_T \\
&= e^{-rT}\int_{k}^{\infty}(e^{x} - e^k)q(x)dx \equiv V_C(k) \qquad (1)
\end{aligned}
$$

where $x_T = \ln\, S_T$; $k = \ln\, K$; $f(S_T)$ is log-normally distributed; and $q(x_T)$ is normally distributed.

Next, from result (1), we derive the Fourier transform of $V_C(k)$:

$$\begin{aligned}
\hat{V}_C(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega k} V_C(k)dk, \quad \omega = 2\pi s \\
&= \int_{-\infty}^{\infty} e^{-i\omega k} \left(e^{-rT} \int_{-\infty}^{\infty} (e^x - e^k) q(x) dx \right) dk \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega k} (e^x - e^k) q(x) dx dk \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega k} (e^x - e^k) q(x) dk dx \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{-i\omega k} (e^x - e^k) q(x) dk dx \\
&= e^{-rT} \int_{-\infty}^{\infty} q(x) \left(\int_{-\infty}^{x} e^{-i\omega k} (e^x - e^k) dk \right) dx \qquad (2)
\end{aligned}$$

where we changed the order of integration (dx, dk) by Fubini's theorem and used the result $\int_x^{\infty} e^{-i\omega k}(e^x - e^k) q(x) dk = 0$.

- **Evaluate the inner integral of (2)**:

$$\begin{aligned}\int_{-\infty}^{x} e^{-i\omega k}(e^x - e^k)dk &= \int_{-\infty}^{x} e^{-i\omega k} e^x dk - \int_{-\infty}^{x} e^{-i\omega k} e^k dk \\ &= e^x \frac{e^{-i\omega k}}{-i\omega} \Big|_{-\infty}^{x} - \frac{e^{(1-i\omega)k}}{1-i\omega} \Big|_{-\infty}^{x}\end{aligned}$$

1. The 1st integral is undetermined due to $\lim_{k\to-\infty} e^{-i\omega k} \neq 0$.
2. The 2nd integral converges to zero, as $\lim_{k\to-\infty} e^{(1-i\omega)k} = 0$.

- To get around the "undetermined" problem, we introduce **a damping parameter** α to normalize the call option price

$$\nu_C(k) = e^{\alpha k} V_C(k)$$

- As shown in Step 2C, the damping parameter α can force **convergence,** thereby permitting a **computable** Fourier transform.

- Re-do:

$$
\begin{aligned}
\hat{\nu}_C(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega k} \nu_C(k) dk \\
&= \int_{-\infty}^{\infty} e^{-i\omega k} \left(e^{-r(T-t)} e^{\alpha k} \int_{k}^{\infty} (e^x - e^k) q(x) dx \right) dk \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{(\alpha - i\omega)k} (e^x - e^k) q(x) dk dx \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} q(x) \left(\int_{-\infty}^{x} e^{(\alpha - i\omega)k} (e^x - e^k) dk \right) dx \quad (3)
\end{aligned}
$$

- Compare (3) to (2): the term $\int_{-\infty}^{x} e^{-i\omega k}(e^x - e^k)dk$ has been changed $\int_{-\infty}^{x} e^{(\alpha - i\omega)k}(e^x - e^k)dk$.

- Evaluate the inner integral of (3):

$$\begin{aligned}\int_{-\infty}^{x} e^{(\alpha-i\omega)k}(e^x - e^k)dk &= \int_{-\infty}^{x} e^{(\alpha-i\omega)k} e^x dk - \int_{-\infty}^{x} e^{(\alpha-i\omega)k} e^k dk \\ &= e^x \cdot \frac{e^{(\alpha-i\omega)k}}{\alpha - i\omega} \Big|_{-\infty}^{x} - \frac{e^{(\alpha-i\omega+1)k}}{\alpha - i\omega + 1} \Big|_{-\infty}^{x} \\ &= e^x \cdot \frac{e^{(\alpha-i\omega)x}}{\alpha - i\omega} - \frac{e^{(\alpha-i\omega+1)x}}{\alpha - i\omega + 1} \\ &= \frac{e^{(\alpha-i\omega+1)x}}{(\alpha - i\omega)(\alpha - i\omega + 1)}\end{aligned}$$

- **Given** $\alpha > 0$, the exponential terms vanish for $k = -\infty$:

$$\lim_{k \to -\infty} e^{(\alpha-i\omega)k} = \lim_{k \to -\infty} e^{(\alpha-i\omega+1)k} = 0$$

- From steps 2c and 2d, we can write (3) as

$$
\begin{aligned}
\hat{\nu}_C(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega k}\nu_C(k)dk \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty} q(x)\left(\frac{e^{(\alpha-i\omega+1)x}}{(\alpha-i\omega)(\alpha-i\omega+1)}\right)dx \\
&= \frac{e^{-r(T-t)}}{(\alpha-i\omega)(\alpha-i\omega+1)}\int_{-\infty}^{\infty} e^{-i(\omega+(\alpha+1)i)x}q(x)dx \\
&= \frac{e^{-r(T-t)}}{(\alpha-i\omega)(\alpha-i\omega+1)}\cdot\hat{q}(\omega+(\alpha+1)i) \qquad (4)
\end{aligned}
$$

- Note: $\hat{q}(\omega+(\alpha+1)i)=\overline{\varphi}_X(\omega+(\alpha+1)i)$, a complex conjugate of the characteristic function.

- Recall:

$$\varphi_X(\omega) = e^{i(x_0+(r-\delta-\frac{\sigma^2}{2})t)\omega-\frac{\sigma^2 t}{2}\omega^2}$$

- Thus,

$$\overline{\varphi}_X(\omega) = \hat{q}(\omega) = e^{-i(x_0+(r-\delta-\frac{\sigma^2}{2})t)\omega-\frac{\sigma^2 t}{2}\omega^2}$$

and with $\omega' = \omega + (\alpha + 1)i$

$$\overline{\varphi}_X(\omega') = \hat{q}(\omega') = e^{-i(x_0+(r-\delta-\frac{\sigma^2}{2})t)\omega'-\frac{\sigma^2 t}{2}\omega'^2}$$

- Fourier transform of ν_C:

$$\hat{\nu}_C(\omega) = \int_{-\infty}^{\infty} e^{-i\omega k}\nu_C(k)dk = \frac{e^{-r(T-t)} \cdot \hat{q}(\omega + (\alpha+1)i)}{(\alpha - i\omega)(\alpha - i\omega + 1)}, \ \ \alpha > 0$$

- Pricing integral via Inverse Fourier transform:

$$\begin{aligned} V_C(k) &= e^{-\alpha k} \cdot \nu_C(k), \quad \alpha > 0 \\ &= e^{-\alpha k} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega k}\hat{\nu}_C(\omega)d\omega \end{aligned} \tag{5}$$

- where $\hat{\nu}_C(\omega)$ is given by (4):

$$\hat{\nu}_C(\omega) = \frac{e^{-r(T-t)}}{(\alpha - i\omega)(\alpha - i\omega + 1)} \cdot \hat{q}(\omega + (\alpha+1)i)$$

Is the Pricing Integral (5) odd or even?

- From (5), it is clear that to compute V_C, we must compute the pricing integral. That is,

$$V_C(k) \;= e^{-\alpha k} \cdot \nu_C(k) = \; e^{-\alpha k} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega k} \hat{\nu}_C(\omega) d\omega, \quad \alpha > 0$$

- But note that ν_C is a real number. This implies the Fourier transform $\hat{\nu}_C$ is
  - odd in its imaginative part so that

$$\mathrm{Im}\{\hat{\nu}_C(\omega)\} = -\mathrm{im}\{\hat{\nu}_C(-\omega)\}$$

  - and even in its real part so that

$$\mathrm{Re}\{\hat{\nu}_C(\omega)\} = \mathrm{re}\{\hat{\nu}_C(-\omega)\}$$

- Thus, this allows to rewrite the pricing integral as

$$V_C(k) = \mathrm{Re}\left\{ \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{i\omega k} \hat{\nu}_C(\omega) d\omega \right\}$$

Put Options Pricing via Fourier Transform

Taking the same steps presented above, we can derive the pricing integral for European put options:

- Fourier transform:

$$\hat{\nu}_P(\omega) = \int_{-\infty}^{\infty} e^{-i\omega k}\nu_P(k)dk = \frac{e^{-r(T-t)} \cdot \hat{q}(\omega + (\alpha+1)i)}{(\alpha - i\omega)(\alpha - i\omega + 1)}, \ \alpha < 0$$

- Inverse Fourier transform:

$$\begin{aligned} V_P(k) &= e^{-\alpha k} \cdot \nu_P(k), \ \ \alpha < 0 \\ &= e^{-\alpha k} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega k}\hat{\nu}_P(\omega)d\omega \\ &= \mathrm{Re} \ \left\{ \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{i\omega k}\hat{\nu}_P(\omega)d\omega \right\} \end{aligned}$$

- Note: for European put options, $\alpha < 0$, but for European call options, $\alpha > 0$.

Computing the Fourier-Transform Pricing Integral

- The Fourier techniques discussed above allow us to compute European option prices for models with a known characteristic function is.
- However, we need algorithms for computing:

$$V(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega k} \hat{\nu}(\omega) d\omega \qquad (6)$$

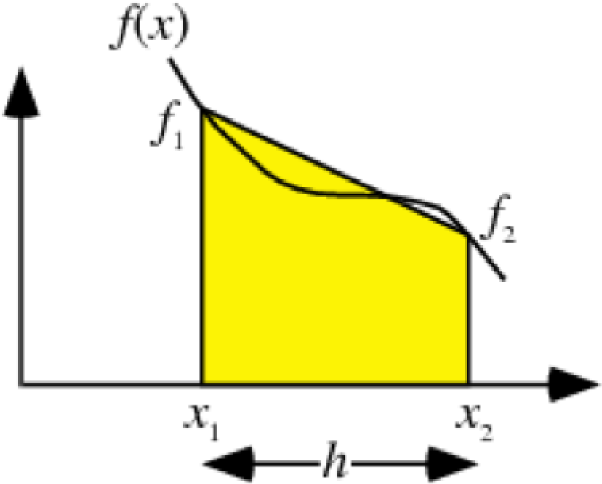
or

$$V(k) = \mathrm{Re}\left\{ \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{i\omega k} \hat{\nu}_P(\omega) d\omega \right\} \qquad (7)$$

where

$$\hat{\nu}(\omega) = \frac{e^{-r(T-t_0)} \cdot \hat{q}(\omega + (\alpha+1)i)}{(\alpha - i\omega)(\alpha - i\omega + 1)}$$

- There are several algorithms available
    - Composite Trapezoidal Rule, Simpson's Rules, $\cdots$
    - **Faster Fourier Transform (FFT)**, Fractional FFT
    - COS method
    - Saddle Point method


$f(x)$
f_1
f_2
x_1
x_2
h

Trapezoidal Rule (2)

- To compute V_C in terms of (7):
  - Discretize the frequency domain $\omega \in [0, \infty)$:
  - Pick B to be the upper bound of the closed interval $[0, B] \approx [0, \infty)$
  - Let N be the number of equidistant sub-intervals so that $\triangle\omega = \frac{B}{N} \equiv h$.
  - Thus, $\omega_j = j\triangle\omega$, $j = 0, 1, \ldots, N$.
  - Trapezoidal rule:

$$
\begin{aligned}
V(k) &\approx \mathrm{Re} \left\{ \frac{e^{-\alpha k}}{\pi} \int_0^B e^{i\omega k} \hat{\nu}(\omega) d\omega \right\} \\
&\approx \mathrm{Re} \left\{ \frac{e^{-\alpha k}}{\pi} \sum_{m=0}^{N-1} \left[\frac{e^{i\omega_m k} \hat{\nu}(\omega_m) + e^{i\omega_{m+1} k} \hat{\nu}(\omega_{m+1})}{2} \right] \cdot h \right\} \\
&= \mathrm{Re} \left\{ \frac{e^{-\alpha k}}{\pi} \sum_{m=0}^{N} e^{i\omega_m k} \hat{\nu}(\omega_m) \triangle\omega_m \right\}, \quad \triangle\omega_m \begin{cases} = \frac{h}{2}, & m = 0 \text{ or } N \\ = h, & m \neq 0 \text{ or } N \end{cases}
\end{aligned}
$$

Trapezoidal Rule (3)

- To compute V_C in terms of (6):
  - Discretize the frequency domain $\omega \in (-\infty, \infty)$:
  - Pick B $(-B)$ to be the upper (lower) bound of the closed interval $[-B, B] \approx (-\infty, \infty)$
  - Let $2N$ be the number of equidistant sub-intervals so that $\triangle\omega = \frac{2B}{2N} = \frac{B}{N} \equiv h$.
  - Thus, $\omega_j = j\triangle\omega$, $j = -N, -(N-1), \ldots, 0, 1, \ldots, N$.
  - Trapezoidal rule:

$$
\begin{aligned}
V(k) &\approx \frac{e^{-\alpha k}}{2\pi} \int_{-B}^{B} e^{i\omega k} \hat{\nu}(\omega) d\omega \\
&\approx \frac{e^{-\alpha k}}{2\pi} \sum_{m=-N}^{N-1} \left[\frac{e^{i\omega_m k} \hat{\nu}(\omega_m) + e^{i\omega_{m+1} k} \hat{\nu}(\omega_{m+1})}{2} \right] \cdot h \\
&= \frac{e^{-\alpha k}}{2\pi} \sum_{m=-N}^{N} e^{i\omega_m k} \hat{\nu}(\omega_m) \triangle\omega_m, \quad \triangle\omega_m \begin{cases} = \frac{h}{2}, & m = \pm N \\ = h, & m \neq \pm N \end{cases}
\end{aligned}
$$