Numerical Methods for Financial Derivatives

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Lecture 2: Introduction to Options (Ch. 1)

Options

What can the holder of an option choose to do?

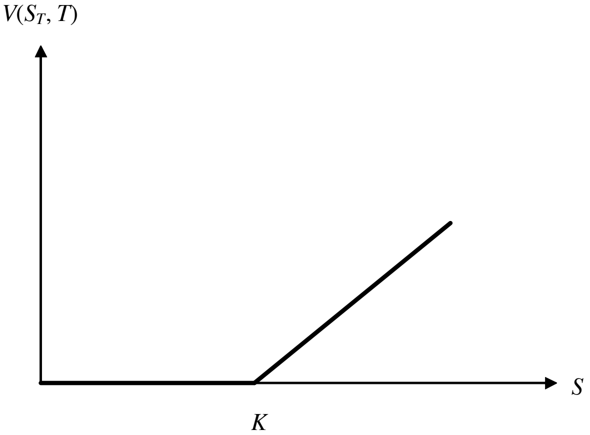
- Sell the option at the current market price.
- Retain the option and do nothing.
- exercise the option ($t \leq T$), or
- let the option expire worthless.

Value of Options

- What factors determine the value of an option?
- $V(S_t, t; T, K, r, \sigma)$
  - S_t, price of the underlying
  - $T - t$, remaining time to expire
  - K, exercise or strike price,
  - r, risk-free rate of interest, continuously compunded
  - σ, volatility, or standard deviation
- It is easy to determine the terminal value of an option, $V(S_T, T)$.
- But it is not easy to determine the value of an option before expiry, $V(S_t, t)$ for $t < T$.

- The Payoff Function of an European call

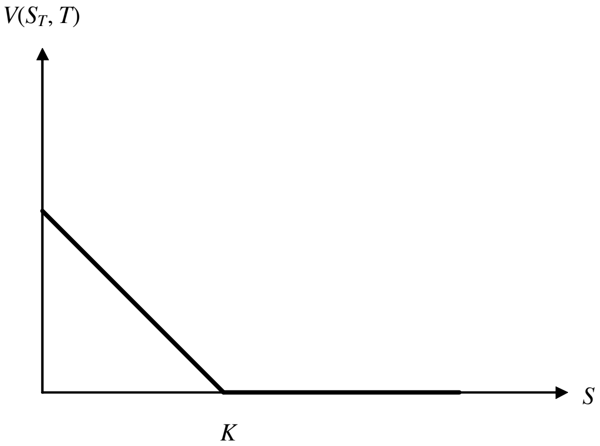
$$V_C^{eur}(S_T, T) = \max\{S_T - K, 0\} \text{ or } (S_T - K)^+$$



Payoff Functions (cont.)

- The Payoff Function of an European put

$$V_P^{eur}(S_T, T) = \max\{K - S_T, 0\} \text{ or } (K - S_T)^+$$



- The Payoff function of an American call

$$V_C^{am}(S_t, t) = \max\{S_t - K, 0\} \text{ or } (S_t - K)^+, \; t \leq T$$

- The payoff function of an American put

$$V_P^{am}(S_t, t) = \max\{K - S_t, 0\} \text{ or } (K - S_t)^+ \; t \leq T$$

Some Boundary Conditions

- The value of an American option can never fall below the payoff:

$$V_C^{am}(S_t, t) \geq (S_t - K)^+$$
$$V_P^{am}(S_t, t) \geq (K - S_t)^+$$

- The value of an American option should never be smaller than that of an European option:

$$V^{am}(S_t, t) = V^{eur}(S_t, t), \ t \leq T$$

- For European options, the values of put and call must satisfy the put-call parity,

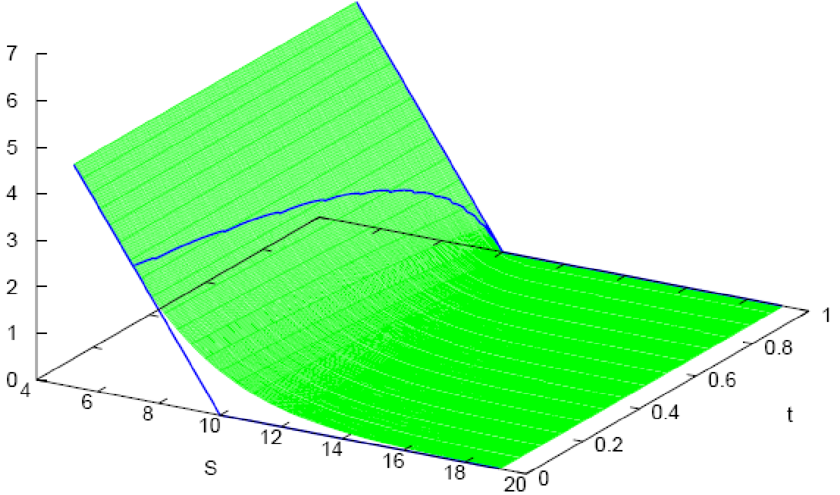
$$V_P^{eur}(S_t, t) + S_t - V_C^{eur}(S_t, t) = Ke^{-r(T-t)}$$
$$V_P^{eur}(S_t, t) + S_t e^{-\delta(T-t)} - V_C^{eur}(S_t, t) = Ke^{-r(T-t)}$$

- Lower bounds for European put:

$$V_P^{eur}(S_t, t) \geq Ke^{-r(T-t)} - S_t$$
$$V_P^{eur}(S_t, t) \geq Ke^{-r(T-t)} - S_t e^{-\delta(T-t)}$$

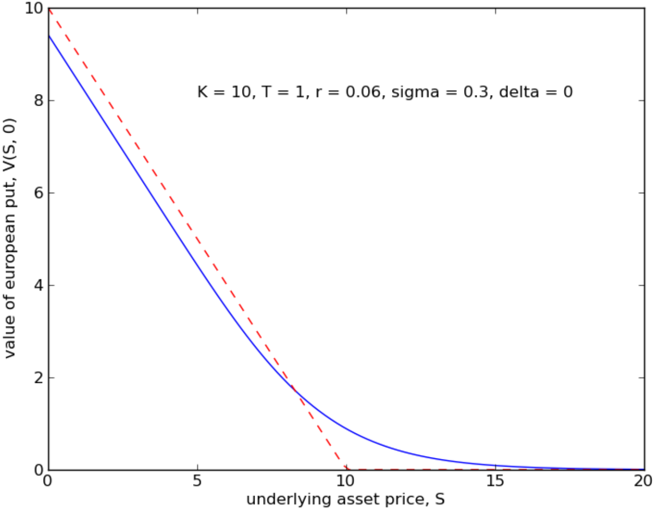
- For other bounds, see Appendix D1.

The Geometry of Options: Option Surface, Payoff Surface, Early Exercise Curve



American Put: $r = 0.06$, $\sigma = 0.03$

The Geometry of Options: Value of an European put at $t = 0$, $V_P^{eur}(S_0, 0)$



Analytical Solutions for European Call and Put

- The closed-form solution for European call:

$$V_C(S,t) = Se^{-\delta(T-t)} \cdot F(d_1) - Ke^{-r(T-t)} \cdot F(d_2)$$

- The closed-form solution for European put:

$$V_P(S,t) = -Se^{-\delta(T-t)} \cdot F(-d_1) + Ke^{-r(T-t)} \cdot F(-d_2)$$

- where

$$F(x) = \int_{-\infty}^{x} f(t)dt; \;\; f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Approximation Formula for Standard Normal Cumulation Distribution (see Appendix D2)

- Let us define

$$z = \frac{1}{1 + 0.2316419x}$$

- Let us also define

$$\begin{aligned} a_1 &= 0.319381530 \\ a_2 &= -0.356563782 \\ a_3 &= 1.781477937 \\ a_4 &= -1.821255978 \\ a_5 &= 1.330274429 \end{aligned}$$

- Then, we can approximate $F(x)$ using the formula:

$$\begin{aligned} F(x) &\approx 1 - f(x)(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5) \\ &= 1 - f(x)z((((a_5 z + a_4)z + a_3)z + a_2)z + a_1) \end{aligned}$$

  for $0 \leq x < \infty$.
- For $x < 0$, we can apply

$$F(x) = 1 - F(-x)$$