Numerical Methods for Financial Derivatives

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Lecture 3: The Black-Scholes Equation and Monte Carlo Simulation
(Ch. 1 & 2)

- The BS Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty,\ 0 < t < T$$

- Terminal Condition for $t = T$:

$$V(S, T) = \text{payoff}$$

- Boundary Conditions: How will $V(S,t)$ behave at $S = 0$ and $S \longrightarrow \infty$? For instance,

$$V_C^{eur}(0,t) = 0,\ \ V_P^{eur}(\infty,t) \to 0$$

$$V_C^{eur}(S,t) \to Se^{-\delta(T-t)} - Ke^{-r(T-t)} \text{ for } S \to \infty.$$

$$V_P^{eur}(S_t,t) \to Ke^{-r(T-t)} - S_t e^{-\delta(T-t)} \text{ for } S \to 0.$$

Assumptions for the BS Equation

- A geometric Brownian motion of the underlying, S,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- A riskless bond, B,

$$dB_t = rB_t dt$$

- A replicating portfolio Π to hedge the derivative V:

$$\Pi_t = \alpha_t S_t + \beta_t B_t$$

- Self-financing property (or the portfolio is "closed" for $0 < t < T$):

$$d\Pi_t = \alpha_t dS_t + \beta_t dB_t$$

- No-arbitrage:

$$\Pi_t = V(S_t, t), \text{ for } t \in 0, T]$$

Derivation of the BS Equation

- Using Ito's lemma,

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V(S,t)}{\partial S} dW$$

- Using the above-assumed self-financing and GBM,

$$d\Pi = (\alpha \mu S + \beta r B) dt + \alpha \sigma S dW$$

- The hedging strategy (delta hedge):

$$\alpha = \frac{\partial V(S,t)}{\partial S}$$

- Matching the dt coefficients of dV and $d\Pi$ and replacing βB with $\Pi - \alpha S = V - \alpha S$ yield the BS equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Greeks

- BS equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Greeks:

$$\text{Theta} = \frac{\partial V}{\partial t}$$

$$\text{Gamma} = \frac{\partial^2 V}{\partial S^2}$$

$$\text{Delta} = \frac{\partial V}{\partial S}$$

$$\text{Rho} = \frac{\partial V}{\partial r}$$

The BS Equation as a parabolic PDE

- General formulation for PDE:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+f=0,\ \text{where } f=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$$

- The PDE is all *elliptic* if $B^2-4AC<0$.
- The PDE is all *parabolic* if $B^2-4AC=0$.
- The PDE is all *hyperbolic* if $B^2-4AC>0$.
- What about the BS equation?

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

Risk-Neutral Valuation

- GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The BS Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty,\ 0 < t < T$$

- Why doesn't the drift μ of dS/S appear in the BS equation?
- What are the implications of the absence of μ in the BS equation?

Risk-Neutral Valuation (2)

- GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- Change of Measure: (Radon-Nikodym derivative and Cameron-Martin-Girsanov theorem)

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t [\gamma dt + dW_t], \quad \gamma = \frac{\mu - r}{\sigma} \\ &= rS_t dt + \sigma S_t d\widetilde{W}_t \end{aligned}$$

- Risk neutral valuation principle:

$$\mu \to r, \quad \mathbb{P} \to \mathbb{Q}, \quad W_t \to \widetilde{W}_t$$

$$\widetilde{W}_t = \gamma t + W_t$$

- Measures of $\mathbb{P}$ and $\mathbb{Q}$:
- W_t is a $\mathbb{P}$-Wiener process with $W_t \backsim N(0, t)$ under $\mathbb{P}$.
- $\widetilde{W}_t$ is a $\mathbb{P}$-Wiener process with $\widetilde{W}_t \backsim N(\gamma t, t)$ under $\mathbb{P}$?
- W_t is a $\mathbb{Q}$-Wiener process with $W_t \backsim N(-\gamma t, t)$ under $\mathbb{Q}$.
- $\widetilde{W}_t$ is a $\mathbb{Q}$-Wiener process with $\widetilde{W}_t \backsim N(0, t)$ under $\mathbb{Q}$.

Theorem

Theorem

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures. Given a path ω, for every ordered time mesh $\{t_1, \ldots, t_n\}$ with $(t_n = T)$, we define x_i to be $W_{t_i}(\omega)$, and then the derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ up to time T is defined to be the limit of likelihood ratios

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \lim_{n\to\infty} \frac{f^n_{\mathbb{Q}}(x_1, \ldots, x_n)}{f^n_{\mathbb{P}}(x_1, \ldots, x_n)}$$

as the mesh becomes dense in the interval $[0, T]$. The continuous-time derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ satisfies the results that

$$E_{\mathbb{Q}}(X_T) = E_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} X_T),$$

$$E_{\mathbb{Q}}(X_T|\mathscr{F}_s) = \zeta_s^{-1} E_{\mathbb{P}}(\zeta_t X_T|\mathscr{F}_s), \quad s \leq t \leq T,$$

where ζ_t is the process $E_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} X_T|\mathscr{F}_t)$, and X_t is any process adapted to the history $\mathscr{F}_t$.

Cameron-Martin-Girsanov theorem

Theorem

If W_t is a $\mathbb{P}$-Brownian notion and γ_t is an $\mathscr{F}$-previsible process satisfying the boundedness condition $E_{\mathbb{P}} \exp(\frac{1}{2}\int_0^T \gamma_t^2 dt) < \infty$, then there exists a measure $\mathbb{Q}$ such that

$$(i) \qquad \mathbb{Q} \textit{ is equivalent to } \mathbb{P}$$

$$(ii) \qquad \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$$

$$(iii) \qquad \widetilde{W}_t = W_t + \int_0^t \gamma_s ds \textit{ is a } \mathbb{Q}\textit{-Brownian motion.}$$

Ito's Lemma

Theorem

Let $F : R_+ \times R \to R$ be once continuously differentiable in its first argument and twice continuously differentiable in its second, and let X be the diffusion

$$X(t,\omega) = X(0,\omega) + \int_0^t \mu(s,X(s,\omega))ds + \int_0^t \sigma(s,X(s,\omega))dW(s,\omega),\ \forall t,\omega$$

$$\text{or}\ \ dX(t) = \mu(t,X(t))dt + \sigma(t,X(t))dW(t)$$

Then:

$$F(t,X(t,\omega)) = F(0,X(0,\omega)) + \int_0^t F_t(s,X)ds + \int_0^t F_x(s,X)\mu(s,X)ds$$

$$+ \int_0^t F_x(s,X)\sigma(s,X)dW(s,\omega) + \frac{1}{2}\int_0^t F_{xx}(s,X)\sigma^2(s,X)ds,\ \forall t,\omega$$

$$\text{or}\ \ dF = F_t dt + \mu F_x dt + \sigma F_x dW + \frac{1}{2}\sigma^2 F_{xx}(dW)^2$$

More on the Geometric Brownian Motion

- GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- Analytic solution to the GBM:

$$d\log S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\begin{aligned}\log S_t &= \log S_{t_0} + (\mu - \frac{1}{2}\sigma^2)(t - t_0) + \sigma(W_t - W_{t_0}) \\ &\Rightarrow E_{\mathbb{P}}[\log S_t | \mathscr{F}_{t_0}] = \log S_{t_0} + (\mu - \frac{1}{2}\sigma^2)(t - t_0) \\ &\Rightarrow Var_{\mathbb{P}}[\log S_t | \mathscr{F}_{t_0}] = \sigma^2(t - t_0)\end{aligned}$$

$$S_t = S_{t_0} e^{(\mu - \frac{1}{2}\sigma^2)(t - t_0) + \sigma(W_t - W_{t_0})}$$

Derivation

Proof.

Given $dS_t = \mu S_t dt + \sigma S_t dW_t$, define $F(S_t, t) = \log S_t$. Then

$$\begin{aligned} dF(S_t, t) &= \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 \\ &= \frac{1}{S}(\mu S dt + \sigma S dW_t) + 0 - \frac{1}{2}\frac{1}{S^2}(\mu S dt + \sigma S dW_t)^2 \\ &= (\mu dt + \sigma dW_t) - \frac{1}{2}\frac{1}{S^2}(\sigma^2 S^2 (dW)^2) \\ &= (\mu - \frac{1}{2}\sigma^2) dt + \sigma dW_t \end{aligned}$$

$$\Longrightarrow \log S_t = \log S_{t_0} + (\mu - \frac{1}{2}\sigma^2)t + \sigma(W_t - W_{t_0})$$

$$\Longrightarrow S_t = S_{t_0} e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma(W_t - W_{t_0})}$$

□

Lognormal Distribution

- Density function of $s_t \equiv \ln(S_t)$:

$$q(s_t) = \frac{1}{\sqrt{2\pi\sigma^2((t-t_0)}} e^{-\frac{\left[s_t - s_0 - (\mu - \sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}}$$

- Density function of S_t:

$$f(S_t) = \frac{1}{S_t} \cdot q(s_t) = \frac{1}{S_t\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{\left[\ln(S_t/S_0) - (\mu - \sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}}$$

Proof.

Let $q(s_t)$ be the density function and $Q(s_t)$ its cumulative distribution. Then

$$\begin{aligned} dQ(s_t) &= q(s_t) \cdot ds_t = q(s_t) \cdot \frac{ds_t}{dS_t} \cdot dS_t \\ &= q(s_t) \cdot \frac{1}{S_t} \cdot dS_t = f(S_t) \cdot dS_t \equiv dF(S_t) \end{aligned}$$

□

1st & 2nd Moments of Lognormal

- Lognormal distribution: S_t is lognormally distributed with
- first moment: $E_{\mathbb{P}}(S_t) = S_0 e^{\mu(t-t_0)} \Rightarrow E_{\mathbb{Q}}(S_t) = S_0 e^{r(t-t_0)}$
- second moment:
  $E_{\mathbb{P}}(S_t^2) = S_0^2 e^{(2\mu+\sigma^2)(t-t_0)} \Rightarrow E_{\mathbb{Q}}(S_t^2) = S_0^2 e^{(2r+\sigma^2)(t-t_0)}$

Derivation of 1st Moment

Using properties of probability density functions

Setting $Z_t = \log(S_t/S_0)$, we have $dZ_t = dS_t/S_t$ and $S_t = S_0 e^{Z_t}$. Then,

$$
\begin{aligned}
E_{\mathbb{P}}(S_t) &= \int_0^{\infty} S_t f(S_t) dS_t \\
&= \int_0^{\infty} S_t \frac{1}{S_t\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{\left[\log(S_t/S_0)-(\mu-\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}} dS_t \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} S_0 e^{Z_t} e^{-\frac{\left[Z_t-(\mu-\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}} dZ_t \\
&= S_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{2\sigma^2((t-t_0)Z_t-\left[Z_t-(\mu-\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}} dZ_t \\
&= S_0 e^{\mu(t-t_0)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{\left[Z_t-(\mu+\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}} dZ_t \\
&= S_0 e^{\mu(t-t_0)}
\end{aligned}
$$

Given $dS_t = \mu S_t dt + \sigma S_t dW_t$, it holds that

$$S_t = S_0 + \mu \int_0^t S_\theta d\theta + \sigma \int_0^t S_\theta dW$$

$$\Rightarrow E_{\mathbb{P}}[S_t] = S_0 + \mu \int_0^t E_{\mathbb{P}}[S_\theta] d\theta, \ \forall t$$

Define $h[t] \equiv E_{\mathbb{P}}[S_t]$. Then

$$h[t] = S_0 + \mu \int_0^t h[\theta] d\theta$$

$$\Rightarrow h'[t] = \mu h[t] \text{ with } h[0] = S_0$$

$$\Rightarrow E_{\mathbb{P}}[S_t] \equiv h[t] = S_0 e^{\mu t}$$

Derivation of 2nd Moment

- By definition,

$$
\begin{aligned}
E_{\mathbb{P}}(S_t^2) &= \int_0^\infty S_t^2 f(S_t) dS_t \\
&= \int_0^\infty S_t \frac{1}{S_t\sqrt{2\pi\sigma^2(t-t_0)}} e^{-\frac{\left[\log(S_t/S_0)-(\mu-\sigma^2/2)(t-t_0)\right]^2}{2\sigma^2((t-t_0)}} dS_t
\end{aligned}
$$

- How to proceed using properties of probability density functions?
- How to derive $E_{\mathbb{P}}(S_t^2)$ using Ito's lemma?

Moment-Generating Function

- Moment-generating Function: Another approach to proving $E_{\mathbb{P}}(S_t) = S_0 e^{\mu(t-t_0)}$.
- Consider
$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)(t-t_0) + \sigma(W_t - Wt_0)}$$
- Then
$$E_{\mathbb{P}}(S_t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)(t-t_0)} E_{\mathbb{P}}(e^{\sigma(W_t - W_{t_0})})$$
- Recall: A random variable X is a normal $N(\mu, \sigma^2)$ under measure $\mathbb{P}$ iff
$$E_{\mathbb{P}}(e^{\theta X}) = e^{\theta\mu + \theta^2\sigma^2/2} \text{ for all real } \theta$$
- Therefore, due to $(W_t - W_{t_0}) \sim N(0, (t - t_0))$, it hold that (take σ as θ)
$$E_{\mathbb{P}}(e^{\sigma(W_t - W_{t_0})}) = e^{\sigma\cdot 0 + \sigma^2\cdot(t-t_0)/2} = e^{\sigma^2\cdot(t-t_0)/2}$$
$$E_{\mathbb{P}}(S_t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)(t-t_0)} e^{\sigma^2\cdot(t-t_0)/2} = S_0 e^{\mu(t-t_0)}$$
- What about $E_{\mathbb{P}}(S_t^2) = S_0^2 e^{(2\mu+\sigma^2)(t-t_0)}$?

Simulation of a Standard Wiener Process

Algorithm 1.8 (page 29): Simulation of $W(t)$, $t = 0, 1, \ldots T$

Start: $t_0 = 0$, $W_0 = 0$, $\triangle t = T/M$
loop $j = 1, 2, \ldots M$:

$$t_j = t_{j-1} + \triangle t$$

draw $Z \sim \mathcal{N}(0,1)$

$$W_j = W_{j-1} + Z\sqrt{\triangle t}$$

Algorithm 2.12 (page 81): Box-Muller Method

1. generate $U_1 \sim \mathcal{U}[0,1]$ and $U_2 \sim \mathcal{U}[0,1]$.
2. $\theta = 2\pi U_2$; $\rho = \sqrt{-2\ln U_1}$
3. $Z_1 = \rho\cos\theta$ is a standard normal variate from $Z \sim \mathcal{N}(0,1)$; (as well as $Z_2 = \rho\sin\theta$).

Algorithm 2.13 (page 83): Marsaglia's Polar Method

1. Repeat: generate $U_1, U_2 \sim \mathscr{U}[0,1]$; $x_1 = 2U_1 - 1, x_2 = 2U_2 - 1$; until $s \equiv r^2 = x_1^2 + x_2^2 < 1$.
2. $Z_1 = x_1\sqrt{-2\ln(s)/s}$; $Z_2 = x_2\sqrt{-2\ln(s)/s}$; $Z_1, Z_2 \sim \mathscr{N}(0,1)$

Remarks:

- In the polar method, $x_1, x_2 \sim \mathscr{U}(-1,1)$, and the point (x_1, x_2) must be inside a unit circle whose radius is $r = \sqrt{s} < 1$.
- In the Box-muller method, $\cos\theta = x_1/r$, $\sin\theta = x_2/r$.
- Marsagalia's polar method is more efficient than Box-Miller, because the former does not apply trigonometric evaluations.

Simulation of Underlying Asset's Sample Paths

- Geometric Brownian Motion:

$$dS_t = a(S_t, t)dt + b(S_t, t)dW_t$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\Rightarrow S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

- Euler discretization of the GBM:

$$\triangle S_t = a(S_t, t)\triangle t + b(S_t, t)\triangle W_t$$

Algorithm 1.11 (page 34) Euler discretization of a SDE

Start: $t_0,\ y_0 = S_0,\ \triangle t,\ W_0 = 0$

loop: $j = 0, 1, 2, \ldots$

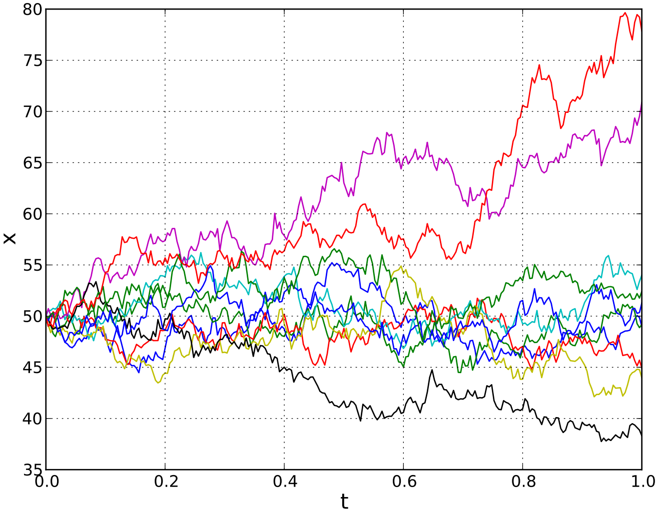
$$t_{j+1} = t_j + \triangle t$$

$$\triangle W = Z\sqrt{\triangle t} \text{ with } Z \sim \mathcal{N}(0,1)$$

$$y_{j+1} = y_j + a(y_j, t_j)\triangle t + b(y_j, t_j)\triangle W$$

Simulation of 10 Sample Paths

Geometric Brownian Motion: X(0)=50, mu=0.1, sigma=0.2, T=1, n=250, dt=T/n



Lognormal Density

f(x, t, x0=50, t0=0, mu=0.1, sigma=0.2)