Numerical Methods for Financial Derivatives

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Lecture 13: Boundary Conditions and Free Boundary Problems

- GBM:

$$dS = (\mu - \delta)Sdt + \sigma S dW$$

- Black-Scholes Equation for $V(S,t)$:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0, 0 < S < \infty, 0 < t < T$$

- Terminal Condition $(t = T)$:

$$\begin{aligned} V_c(S,T) &= (S-K)^+, \\ V_p(S,T) &= (K-S)^+ \end{aligned}$$

- Boundary Conditions $(S \to 0, S \to \infty)$:

$$\text{As } S \to 0 \begin{cases} V_c(S,t) & \to 0 \\ V_p(S,t) & \to Ke^{-r(T-t)} - Se^{-\delta(T-t)} \approx Ke^{-r(T-t)} \end{cases}$$

$$\text{As } S \to \infty \begin{cases} V_p(S,t) & \to 0 \\ V_c(S,t) & \to Se^{-\delta(T-t)} - Ke^{-r(T-t)} \approx Se^{-\delta(T-t)} \end{cases}$$

Put-Call Parity

- Recall Put-Call Parity for a dividend paying underlying S:

$$V_c + Ke^{-r(T-t)} = V_p + Se^{-\delta(T-t)}$$

- Consider $V(S,t)$ as S is at each end of the space domain $(0,\infty)$:
  - As $S \to 0$,

$$V_c \to 0$$
$$V_p \to Ke^{-r(T-t)} - Se^{-\delta(T-t)} \approx Ke^{-r(T-t)}$$

  - As $S \to \infty$,

$$V_p \to 0$$
$$V_c \to Se^{-\delta(T-t)} - Ke^{-r(T-t)} \approx Se^{-\delta(T-t)}$$

The Heat Equation, Once Again

- The above BS equation is equivalent to the heat equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}, \ -\infty < x < \infty, \ 0 < \tau < \frac{\sigma^2 T}{2}$$

- This equivalence is due to the following transformations:
  - $S = Ke^x \Rightarrow x = \ln(S/K)$
  - $t = T - \frac{2\tau}{\sigma^2} \Rightarrow \tau = \frac{\sigma^2}{2}(T - t)$, setting $t = 0$
  - $q = 2r/\sigma^2$
  - $q_\delta = 2(r - \delta)/\sigma^2$
  - $V(S, t) = V(Ke^x, T - \frac{2\tau}{\sigma^2}) \equiv \upsilon(x, \tau)$
  - $\upsilon(x, \tau) = K \exp\{-\frac{1}{2}(q_\delta - 1)x - (\frac{1}{4}(q_\delta - 1)^2 + q)\tau\} y(x, \tau)$

The BS-Heat Equation Problem

- The BS-Heat Equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial^2 x^2}, \quad -\infty < x < \infty, \ 0 < \tau < \frac{\sigma^2 T}{2}$$

- Initial Conditions ($\tau = 0$; i.e. $t = T$)

$$\text{call: } y(x,0) = \max\{e^{\frac{x}{2}(q_\delta+1)} - e^{\frac{x}{2}(q_\delta-1)}, 0\}$$

$$\text{put: } y(x,0) = \max\{e^{\frac{x}{2}(q_\delta-1)} - e^{\frac{x}{2}(q_\delta+1)}, 0\}$$

- Boundary Conditions for Standard European Options:
  - As $x \to -\infty$(or $x \to x_{\min}$, $S \to 0$), $y_c = 0$ and

$$\begin{aligned} y_p &= e^{\frac{1}{2}(q_\delta-1)x_{\mathbf{min}}+[\frac{1}{4}(q_\delta-1)^2]\tau} - e^{\frac{1}{2}(q_\delta+1)x_{\mathbf{min}}+[\frac{1}{4}(q_\delta+1)^2]\tau} \\ &\approx e^{\frac{1}{2}(q_\delta-1)x_{\mathbf{min}}+[\frac{1}{4}(q_\delta-1)^2]\tau} \end{aligned}$$

  - As $x \to \infty$(or $x \to x_{\max}$, $S \to \infty$), $y_p = 0$ and

$$\begin{aligned} y_c &\to e^{\frac{1}{2}(q_\delta+1)x_{\mathbf{max}}+[\frac{1}{4}(q_\delta+1)^2]\tau} - e^{\frac{1}{2}(q_\delta-1)x_{\mathbf{max}}+[\frac{1}{4}(q_\delta-1)^2]\tau} \\ &\approx e^{\frac{1}{2}(q_\delta+1)x_{\mathbf{max}}+[\frac{1}{4}(q_\delta+1)^2]\tau} \end{aligned}$$

Step 1: Remove the S variable from the coefficients

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^S \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

Let

$$S = Ke^x$$

Then

$$\frac{\partial V}{\partial x} = S\frac{\partial V}{\partial S},\ \frac{\partial^2 V}{\partial x^2} = S\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2},\text{ and (1) reduces to}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}) + (r-\delta)\frac{\partial V}{\partial x} - rV = 0$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + (r - \delta - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0 \tag{2}$$

Rearranging (2),

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - rV + (r - \delta - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} = -\frac{\partial V}{\partial t}$$

Dividing each side by $\sigma^2/2$ yields

$$\frac{\partial^2 V}{\partial x^2} - (\frac{2r}{\sigma^2})V + (\frac{2(r-\delta)}{\sigma^2} - 1)\frac{\partial V}{\partial x} = -(\frac{2}{\sigma^2})\frac{\partial V}{\partial t}$$

or

$$\frac{\partial^2 V}{\partial x^2} - qV + (q_\delta - 1)\frac{\partial V}{\partial x} = -(\frac{2}{\sigma^2})\frac{\partial V}{\partial t} \tag{3}$$

where

$$q = \frac{2r}{\sigma^2},$$

$$q_\delta = \frac{2(r-\delta)}{\sigma^2}$$

Step 3: Re-scale the Time Variable

Let

$$\tau = \frac{1}{2}\sigma^2(T - t)$$

Then

$$\frac{\partial V}{\partial \tau} = \frac{\partial V}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{2}{\sigma^2}\frac{\partial V}{\partial t}$$

and (3) becomes

$$\frac{\partial^2 V}{\partial x^2} - qV + (q_\delta - 1)\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \tau} \tag{4}$$

Step 4: Transformation of V into y

Assuming α and β are constant and letting

$$V(S,t) = Ke^{\alpha x + \beta \tau} y(x,\tau)$$

Then

$$\frac{\partial V}{\partial x} = \alpha V + \frac{V}{y}\frac{\partial y}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2} = \alpha \frac{\partial V}{\partial x} + \alpha \frac{V}{y}\frac{\partial y}{\partial x} + \frac{V}{y}\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial V}{\partial \tau} = \beta V + \frac{V}{y}\frac{\partial y}{\partial \tau}$$

Substitutions from (4) yields

$$(\alpha \frac{\partial V}{\partial x} + \alpha \frac{V}{y}\frac{\partial y}{\partial x} + \frac{V}{y}\frac{\partial^2 y}{\partial x^2}) - qV + (q_\delta - 1)(\alpha V + \frac{V}{y}\frac{\partial y}{\partial x}) = \beta V + \frac{V}{y}\frac{\partial y}{\partial \tau}$$

$$(\alpha \frac{y}{V}\frac{\partial V}{\partial x} + \alpha \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}) - qy + (q_\delta - 1)(\alpha y + \frac{\partial y}{\partial x}) = \beta y + \frac{\partial y}{\partial \tau} \qquad (5)$$

Step 5: Find alpha and beta to finish up

Replacing $\frac{\partial V}{\partial x}$ with $\alpha V + \frac{V}{y}\frac{\partial y}{\partial x}$ in (5) and rearranging terms to obtain

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial \tau} = [-(q_\delta - 1) - 2\alpha]\frac{\partial y}{\partial x} + [q + \alpha(q_\delta - 1) - \alpha^2 + \beta]y$$

Thus, the heat equation obtains

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial \tau} = 0$$

if

$$-(q_\delta - 1) - 2\alpha = 0 \Rightarrow \alpha = -\frac{1}{2}(q_\delta - 1)$$

$$q + \alpha(q_\delta - 1) - \alpha^2 + \beta = 0 \Rightarrow \beta = -[\frac{1}{4}(q_\delta - 1)^2 + q]$$

Step 6: Summary

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial \tau} = 0$$

where

$$V(S,t) = Ke^{-\frac{1}{2}(q_\delta - 1)x - [\frac{1}{4}(q_\delta - 1)^2 + q]\tau} y(x,\tau)$$

$$S = Ke^x, \quad \tau = \frac{1}{2}\sigma^2(T-t)$$

$$q = \frac{2r}{\sigma^2},$$

$$q_\delta = \frac{2(r-\delta)}{\sigma^2}$$

Boundary Conditions for European Call and Put

- Put-Call parity in (S,t): For $0 < S < \infty$, $0 \le t \le T$,
$$V_c + Ke^{-r(T-t)} = V_p + Se^{-\delta(T-t)}$$
- Put-Call Parity in (x,τ): For $-\infty < x < \infty$, $0 \le \tau \le \frac{\sigma^2 T}{2}$,
$$y_c(x,\tau) + e^{\frac{1}{2}(q_\delta - 1)x + [\frac{1}{4}(q_\delta - 1)^2]\tau} = y_p(x,\tau) + e^{\frac{1}{2}(q_\delta + 1)x + [\frac{1}{4}(q_\delta + 1)^2]\tau}$$
- Consider boundary conditions for V and y:
  - As $S \to 0$ ($x \to -\infty$ or $x \to x_{\min}$), $V_c \to 0$, $y_c \to 0$, and
$$V_p \to Ke^{-r(T-t)} - Se^{-\delta(T-t)} \approx Ke^{-r(T-t)}$$
$$\begin{aligned} y_p &\to e^{\frac{1}{2}(q_\delta - 1)x_{\mathbf{min}} + [\frac{1}{4}(q_\delta - 1)^2]\tau} - e^{\frac{1}{2}(q_\delta + 1)x_{\mathbf{min}} + [\frac{1}{4}(q_\delta + 1)^2]\tau} \\ &\approx e^{\frac{1}{2}(q_\delta - 1)x_{\mathbf{min}} + [\frac{1}{4}(q_\delta - 1)^2]\tau} \end{aligned}$$
  - As $S \to \infty$ ($x \to \infty$ or $x \to x_{\max}$), $V_p \to 0$, $y_p \to 0$, and
$$V_c \to Se^{-\delta(T-t)} - Ke^{-r(T-t)}$$
$$\begin{aligned} y_c &\to e^{\frac{1}{2}(q_\delta + 1)x_{\mathbf{max}} + [\frac{1}{4}(q_\delta + 1)^2]\tau} - e^{\frac{1}{2}(q_\delta - 1)x_{\mathbf{max}} + [\frac{1}{4}(q_\delta - 1)^2]\tau} \\ &\approx e^{\frac{1}{2}(q_\delta + 1)x_{\mathbf{max}} + [\frac{1}{4}(q_\delta + 1)^2]\tau} \end{aligned}$$

Incorporating Boundary Conditions into FDM's

- The BS-Heat Equation:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

- Explicit finite difference equations system:

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle \tau} = \frac{w_{j+1,i} - 2w_{j,i} + w_{j-1,i}}{\triangle x^2}$$

$$\Rightarrow w_{j,i+1} = \lambda w_{j-1,i} + (1-2\lambda) w_{j,i} + \lambda w_{j+1,i}$$

- Implicit finite difference equations system:

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle \tau} = \frac{w_{j+1,i+1} - 2w_{j,i+1} + w_{j-1,i+1}}{\triangle x^2}$$

$$\Rightarrow -\lambda w_{j-1,i+1} + (1+2\lambda) w_{j,i+1} - \lambda w_{j+1,i+1} = w_{j,i}$$

- theta finite difference equations system:

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle \tau} = \theta[\frac{w_{j+1,i+1} - 2w_{j,i+1} + w_{j-1,i+1}}{\triangle x^2}] + (1-\theta)[\frac{w_{j+1,i} - 2w_{j,i} + w_{j-1,i}}{\triangle x^2}]$$

$$\Rightarrow w_{j,i+1} - \theta\lambda[w_{j+1,i+1} - 2w_{j,i+1} + w_{j-1,i+1}] = w_{j,i} + (1-\theta)\lambda[w_{j+1,i} - 2w_{j,i} + w_{j-1,i}]$$

- Explicit:

$$\underbrace{\begin{bmatrix} w_{1,i+1} \\ w_{2,i+1} \\ \vdots \\ w_{N-2,i+1} \\ w_{N-1,i+1} \end{bmatrix}}_{w^{(i+1)}} = \underbrace{\begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \lambda & 1-2\lambda & \lambda \\ 0 & \cdots & 0 & \lambda & 1-2\lambda \end{bmatrix}}_{A_R} \underbrace{\begin{bmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ w_{N-2,i} \\ w_{N-1,i} \end{bmatrix}}_{w^{(i)}} + \underbrace{\begin{bmatrix} f_{1,i} \\ 0 \\ \vdots \\ 0 \\ f_{N-1,i} \end{bmatrix}}_{f^{(i)}}$$

- Boundary Conditions for European call:
  - $f_{1,i} = \lambda w_{0,i} \approx 0,$
  - $f_{N-1,i} = \lambda w_{N,i} \approx \lambda e^{\frac{1}{2}(q_\delta+1)(x_{\mathbf{max}})+\frac{1}{4}(q_\delta+1)^2(i\triangle\tau)}$
- Boundary Conditions for European put:
  - $f_{1,i} = \lambda w_{0,i} \approx \lambda e^{\frac{1}{2}(q_\delta-1)(x_{\mathbf{min}})+[\frac{1}{4}(q_\delta-1)^2](i\triangle\tau)}$
  - $f_{N-1,i} = \lambda w_{N,i} \approx 0$

- Implicit:

$$\underbrace{\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\lambda & 1+2\lambda & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}}_{A_L} \underbrace{\begin{bmatrix} w_{1,i+1} \\ w_{2,i+1} \\ \vdots \\ w_{N-2,i+1} \\ w_{N-1,i+1} \end{bmatrix}}_{w^{(i+1)}} + \underbrace{\begin{bmatrix} f_{1,i+1} \\ 0 \\ \vdots \\ 0 \\ f_{N-1,i+1} \end{bmatrix}}_{f^{(i+1)}} = \underbrace{\begin{bmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ w_{N-2,i} \\ w_{N-1,i} \end{bmatrix}}_{w^{(i)}}$$

- Boundary Conditions for European call:
  - $f_{1,i+1} = -\lambda w_{0,i+1} \approx 0$
  - $f_{N-1,i+1} = -\lambda w_{N,i+1} \approx -\lambda e^{\frac{1}{2}(q_\delta+1)(x_{\max})+\frac{1}{4}(q_\delta+1)^2(i+1)\triangle\tau}$
- Boundary Conditions for European put:
  - $f_{1,i+1} = -\lambda w_{0,i+1} \approx -\lambda e^{\frac{1}{2}(q_\delta-1)(x_{\min})+[\frac{1}{4}(q_\delta-1)^2](i+1)\triangle\tau}$
  - $f_{N-1,i+1} = -\lambda w_{N,i+1} \approx 0$

Incorporating the Boundary Conditions into FDM's (4)

- theta:

$$A_L \cdot w^{(i+1)} + f^{(i+1)} = A_R \cdot w^{(i)} + f^{(i)}$$

$$A_{L=} \begin{bmatrix} 1+2\theta\lambda & -\theta\lambda & 0 & \cdots & 0 \\ -\theta\lambda & 1+2\theta\lambda & -\theta\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\theta\lambda & 1+2\theta\lambda & -\theta\lambda \\ 0 & \cdots & 0 & -\theta\lambda & 1+2\theta\lambda \end{bmatrix}$$

$$A_{R=} \begin{bmatrix} 1-2(1-\theta)\lambda & (1-\theta)\lambda & 0 & \cdots & 0 \\ (1-\theta)\lambda & 1-2(1-\theta)\lambda & (1-\theta)\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & (1-\theta)\lambda & 1-2(1-\theta)\lambda & (1-\theta)\lambda \\ 0 & \cdots & 0 & (1-\theta)\lambda & 1-2(1-\theta)\lambda \end{bmatrix}$$

Incorporating the Boundary Conditions into FDM's (5)

- Boundary Conditions for European call:

$$f_{1,i} = (1-\theta)\lambda w_{0,i} \approx 0,$$

$$\begin{aligned} f_{N-1,i} &= (1-\theta)\lambda w_{N,i} \\ &\approx (1-\theta)\lambda e^{\frac{1}{2}(q_\delta+1)(x_{\max})+\frac{1}{4}(q_\delta+1)^2(i\triangle\tau)} \end{aligned}$$

$$f_{1,i+1} = -\theta\lambda w_{0,i+1} \approx 0,$$

$$\begin{aligned} f_{N-1,i+1} &= -\theta\lambda w_{N,i+1} \\ &\approx -\theta\lambda e^{\frac{1}{2}(q_\delta+1)(x_{\max})+\frac{1}{4}(q_\delta+1)^2(i+1)\triangle\tau} \end{aligned}$$

- Boundary Conditions for European put:

$$\begin{aligned} f_{1,i} &= (1-\theta)\lambda w_{0,i} \\ &\approx (1-\theta)\lambda e^{\frac{1}{2}(q_\delta-1)(x_{\min})+[\frac{1}{4}(q_\delta-1)^2](i\triangle\tau)} \end{aligned}$$

$$f_{N-1,i} = (1-\theta)\lambda w_{N,i} \approx 0,$$

$$\begin{aligned} f_{1,i+1} &= -\theta\lambda w_{0,i+1} \\ &\approx -\theta\lambda e^{\frac{1}{2}(q_\delta-1)(x_{\min})+[\frac{1}{4}(q_\delta-1)^2](i+1)\triangle\tau} \end{aligned}$$

$$f_{N-1,i+1} = -\theta\lambda w_{N,i+1} \approx 0$$

- Early Exercise
- Elementary Lower boundaries for American options

$$V^{am}(S,t) \geq V^{eur}(S,t),\ \forall (S,t)$$

$$V_p^{am}(S,t) \geq (K-S)^+,\ \forall (S,t)$$

$$V_c^{am}(S,t) \geq (S-K)^+,\ \forall (S,t)$$

- Put-Call Parity implies bounds for European options, but not American options:

$$V_c + Ke^{-r(T-t)} = V_p + Se^{-\delta(T-t)}$$

As $S \to \infty :$ (OK) $V_p^{am} = 0$, but (???) $V_c^{am} \to Se^{-\delta(T-t)} - Ke^{-r(T-t)}$

As $S \to 0 :$ (OK) $V_c^{am} = 0$, but (???) $V_p^{am} \to Ke^{-r(T-t)} - Se^{-\delta(T-t)}$

Diagram of Values of American/European Puts (Fig. 4.5)

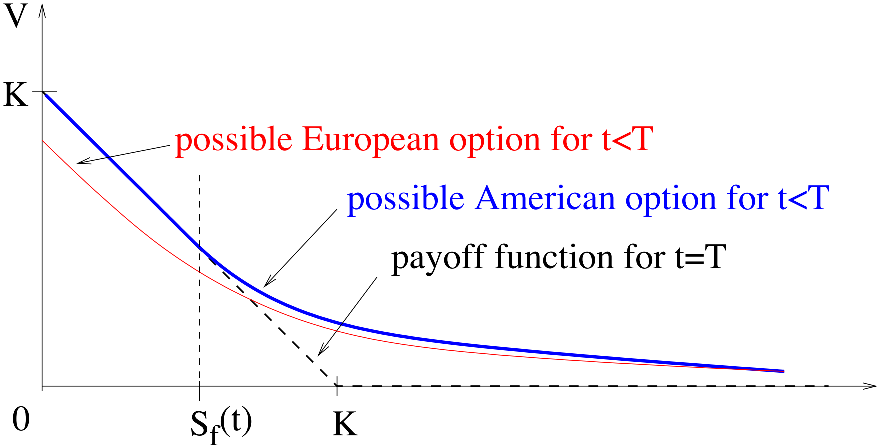
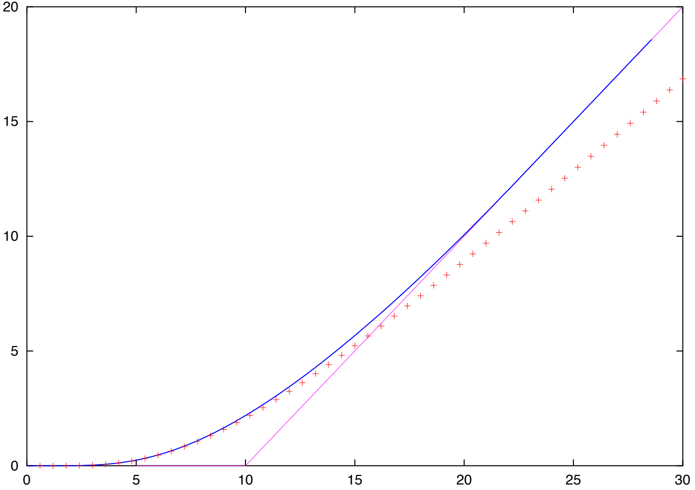


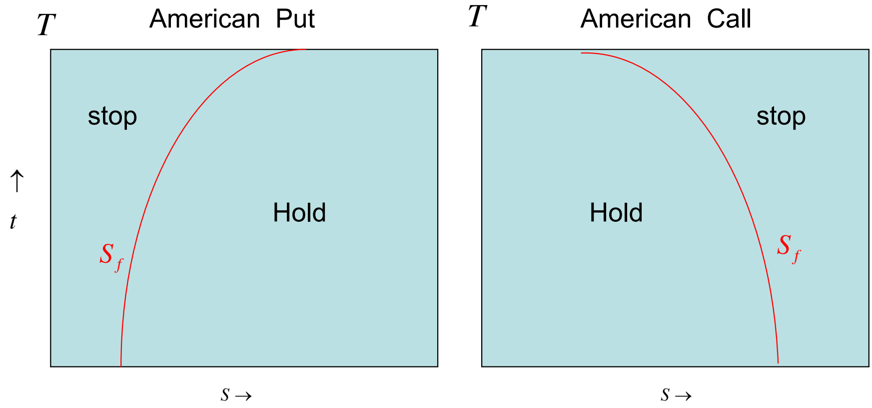
Diagram of Values of American/European Calls (Fig. 4.9)



Early Exercise and Free Boundary Problems

- When should an early exercise be launched?
- What is the optimal stopping time?
- What is the "hold" region? What is the "stop" region?
- What is the left-end boundary for American put to have an early exercise?
- What is the right-end boundary for American call to have an early exercise?
- The "free boundary (S_f)" refers to the left-end or right-end boundary
  - A priori, the boundary S_f is unknown;
  - S_f must be found along with the American option value.

Early Exercise Curve



Early Exercise Curve (2)

- The early exercise curve is the boundary between stopping and holding regions:
  - American put :

$$\begin{aligned} \text{Hold} &: \quad V_p^{am}(S,t) > (K-S)^+ \text{ for } S > S_f(t) \\ \text{Stop} &: \quad V_p^{am}(S,t) = (K-S)^+ \text{ for } S \leq S_f(t) \end{aligned}$$

  - American call (for dividend paying underlying assets, $\delta \neq 0$):

$$\begin{aligned} \text{Hold} &: \quad V_c^{am}(S,t) > (S-K)^+ \text{ for } S < S_f(t) \\ \text{Stop} &: \quad V_c^{am}(S,t) = (S-K)^+ \text{ for } S \geq S_f(t) \end{aligned}$$

Early Exercise Curve (3)

- $S_f = S_f(t)$, the contact point, varies with t.
- The early exercise curve S_f is continuously differentiable in t,
- non-decreasing (non-increasing) with t for American put (call),
  - $\lim_{t \to T} S_f(t) = \min(K, \frac{r}{\delta}K)$ for American put,
  - $\lim_{t \to T} S_f(t) = \max(K, \frac{r}{\delta}K)$ for American call.
- Does an early exercise curve exist for American call on non-dividend paying assets?
- Consider

$$V_c^{am}(S,t) \geq V_c^{eur}(S,t) \geq S - Ke^{-r(T-t)} > S - K.$$

- Does early exercise pay?

Early Exercise Curve (4)

Proof: For American put, $S_f(t) < \lim_{t \to T} S_f(t) = \min(K, \frac{r}{\delta}K)$

- First, at $t = T$, $V_p^{am}(S,T) = K - S$ for $S < K$. Thus,

$$\begin{aligned}\frac{\partial V(S,T)}{\partial t} &= -\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - (r-\delta)S\frac{\partial V}{\partial S} + rV \\ &= -0 - (r-\delta)S \times (-1) + r(K-S) \\ &= rK - \delta S\end{aligned}$$

- But we observe $\frac{\partial V(S,T)}{\partial t} \leq 0$. Thus, for $t \to T$, it holds that

$$rK - \delta S \leq 0 \text{ or } \frac{r}{\delta}K \leq S < K$$

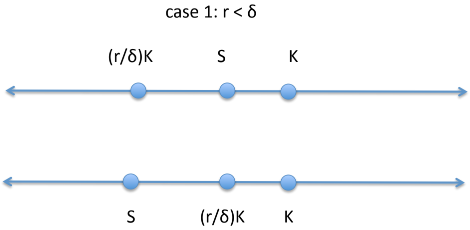
  which, however, makes sense only for $\delta > r$.

- Hence, given $\delta > r$, we conclude

$$0 < \frac{r}{\delta}K < S < K \text{ implies } \frac{\partial V(S,T)}{\partial t} < 0 \text{ and } V > K - S \text{ for } t \to T$$

$$0 < S < \frac{r}{\delta}K < K \text{ implies } \frac{\partial V(S,T)}{\partial t} = 0 \text{ and } V = K - S \text{ for } t \to T$$

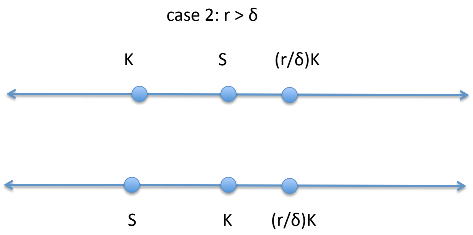
Early Exercise Curve (5)



Proof (cont.): For American put, $S_f(t) < \lim_{t \to T} S_f(t) = \min(K, \frac{r}{\delta}K)$

- Then given $\delta > r$, where is $S_f(T)$ located?
- Claim $S_f(T) = \frac{r}{\delta}K$ for $\delta > r$:
  - Suppose not. Then consider $S_f(T) > \frac{r}{\delta}K \Rightarrow$ a contradiction for $\frac{r}{\delta}K < S < S_f(T)$.
  - Suppose not. Then consider $S_f(T) < \frac{r}{\delta}K \Rightarrow$ a contradiction for $S_f(T) < S < \frac{r}{\delta}K$.

Early Exercise Curve (6)



Proof (cont.): For American put, $S_f(t) < \lim_{t \to T} S_f(t) = \min(K, \frac{r}{\delta}K)$

- Next, consider the case of $\delta \leq r$. Claim $S_f(T) = K$ for $\delta \leq r$:
  - Suppose not. Consider $S_f(T) > K \Rightarrow$ a contradiction by simple geometry.
  - Suppose not. Consider $S_f(T) < K \Rightarrow$ a contradiction for $S_f(T) < S < K$ and $t \to T$:

$$0 \geq \frac{\partial V}{\partial t} = rK - \delta S > 0 \Rightarrow 0 > 0$$

- The above two claims lead to $S_f(t) < \lim_{t \to T} S_f(t) = \min(K, \frac{r}{\delta}K)$.

Free Boundary Problems

- American put:
  - Left-end boundary ($S_f \leftarrow S$):

$$V_p^{am}(S_f(t), t) = K - S_f(t)$$

$$\frac{\partial V_p^{am}(S_f(t), t)}{\partial S} = -1$$

  - Right-end boundary ($S- > \infty$):

$$V_p^{am}(S(t), t) \to 0$$

- American call:
  - Left-end boundary ($0 \leftarrow S$)

$$V_c^{am}(S(t), t) \to 0$$

  - Right-end boundary ($S \to S_f$)

$$V_c^{am}(S_f(t), t) = S_f(t) - K$$

$$\frac{\partial V_c^{am}(S_f(t), t)}{\partial S} = 1$$