Numerical Methods for Financial Derivatives

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Lecture 10: Accuracy, Consistency, Convergence, and Stability

Introduction

- **Accuracy:** The order of the truncation error arising from a numerical scheme.
- **Consistency:** A numerical scheme is said to be consistent if the finite difference representation converges to the PDE that we try to solve as the space and time steps tend to zero.
- **Convergence:** A numerical scheme is said to be convergent if the difference between the numerical solution and the exact solution at a fixed point in the domain of interest tends to zero uniformly as the space and time discretizations tend to zero.
- **Stability:** A numerical scheme is said to be stable if the difference between the numerical solution and the exact solution remains bounded as the number of time steps tends to infinity.

The Lax Equivalence Theorem

Theorem

Given a properly posed linear initial value problem and a consistent finite difference scheme, stability is the only requirement for convergence. For formal proof, see Richtmeyer & Morton 1967.

Problem

$$\frac{\partial y(x,\tau)}{\partial \tau} = \frac{\partial^2 y(x,\tau)}{\partial x^2}$$

$$y(x,0) = \sin \pi x, \;\; 0 < x < 1$$

$$y(0,\tau) = y(1,\tau) = 0, \;\; \tau > 0$$

Solution

$$y(x,\tau) = e^{-\pi^2 \tau} \sin \pi x$$

- The initial and boundary data are consistent at the two corners,

$$y(0,0) = y(1,0) = 0,$$

- Thus, the solution does not have a discontinuity at the corners of the domain.

Explicit Scheme for the Heat Equation

Explicit Scheme (Forward Difference), pages 146, 147

$$\frac{y_{j,i+1} - y_{j,i}}{\triangle\tau} = \frac{y_{j+1,i} - 2y_{j,i} + y_{j-1,i}}{\triangle x^2} + TE \tag{1}$$

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle t} = \frac{w_{j+1,i} - 2w_{j,i} + w_{j-1,i}}{\triangle x^2}$$

$$w_{j,i+1} = \lambda w_{j-1,i} + (1 - 2\lambda) w_{j,i} + \lambda w_{j+1,i} \quad \text{with} \quad \lambda = \frac{\triangle\tau}{\triangle x^2}$$

$$w_{j,i} \approx y_{j,i} = y(j\triangle x, i\triangle\tau)$$

Explicit Scheme for the Heat Equation (2)

$$w_{j,i+1} = \lambda w_{j-1,i} + (1-2\lambda)w_{j,i} + \lambda w_{j+1,i}$$

In matrix form,

$$w^{(i+1)} = A_R \cdot w^{(i)}$$

$$\begin{bmatrix} w_{\mathbf{1},\boldsymbol{i}+\mathbf{1}} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ w_{\boldsymbol{N}-\mathbf{1},\boldsymbol{i}+\mathbf{1}} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1-2\lambda & \lambda \\ 0 & \cdots & \cdots & 0 & \lambda & 1-2\lambda \end{bmatrix}}_{\boldsymbol{A_R},\ (\boldsymbol{N}-\mathbf{1})\times(\boldsymbol{N}-\mathbf{1})} \begin{bmatrix} w_{\mathbf{1},\boldsymbol{i}} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ w_{\boldsymbol{N}-\mathbf{1},i} \end{bmatrix}$$

where as we have shown on Topic 9,

$$A_R = I + \triangle\tau \cdot A_{\triangle x}$$

The Discretization Matrix of the Heat Equation

- Consider the explicit scheme:

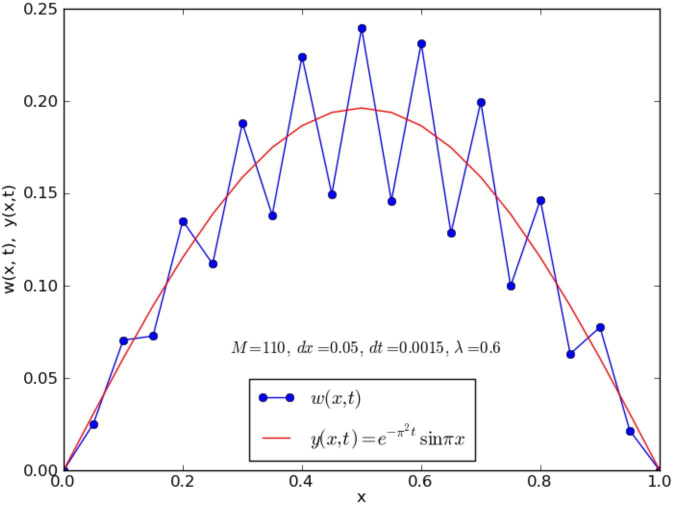
$$\frac{w_{j,i+1} - w_{j,i}}{\triangle\tau} = \frac{w_{j-1,i} - 2w_{j,i} + w_{j+1,i}}{\triangle x^2}$$

- The discretization matrix $A_{\triangle x}$ of $y_\tau = y_{xx}$ is given by

$$A_{\triangle x} = \frac{1}{\triangle x^2} \underbrace{\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}}_{(N-1)\times(N-1)}$$

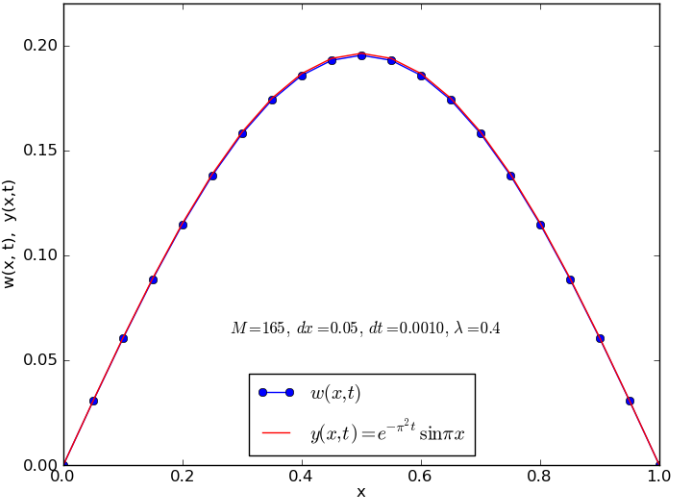
Scenario One: Explicit Scheme is Unstable

Explicit finite-difference approximation w(x, t=0.165), M=110, dt=0.0015



Scenario Two: Explicit Scheme is Stable

Explicit finite-difference approximation w(x, t=0.165), M=165, dt=0.0010



Explicit Scheme: Accuracy and Consistency

- Explicit Scheme:

$$
\begin{aligned}
TE(x_j, \tau_i) &= \frac{y_{j,i+1} - y_{j,i}}{\triangle\tau} - \frac{y_{j-1,i} - 2y_{j,i} + y_{j+1,i}}{\triangle x^2} \\
&= (y_\tau - y_{xx}) + (\frac{1}{2} y_{\tau\tau} \triangle\tau - \frac{1}{12} y_{xxxx} \triangle x^2) + \dots \\
&= \frac{1}{2} y_{\tau\tau} \triangle\tau - \frac{1}{12} y_{xxxx} \triangle x^2) + \dots \\
&= \frac{1}{2} y_{\tau\tau}(x,\eta) \triangle\tau - \frac{1}{12} y_{xxxx}(\xi,\tau) \triangle x^2
\end{aligned}
$$

  where $\eta \in (\tau, \tau + \triangle\tau)$, $\xi \in (x - \triangle x, x + \triangle x)$

- Assumptions:
  1. The initial and boundary data are consistent for $y(x,\tau)$.
  2. The initial data are sufficiently smooth.
  3. Due to assumptions (1) & (2), the upper bounds $M_{\tau\tau}$ and M_{xxxx} respectively for $|y_{\tau\tau}|$ and $|y_{xxxx}|$ hold uniformly over the closed domain $[0,1] \times [0, \tau_F]$.

Explicit Scheme: Accuracy and Consistency (2)

- Absolute Value of the TE:

$$
\begin{aligned}
|TE| &= \left|\frac{1}{2}y_{\tau\tau}(x,\eta)\triangle\tau - \frac{1}{12}y_{xxxx}(\xi,\tau)\triangle x^2\right| \\
&\leq \frac{1}{2}|y_{\tau\tau}(x,\eta)|\triangle\tau + \frac{1}{12}|y_{xxxx}(\xi,\tau)|\triangle x^2 \\
&\leq \frac{1}{2}M_{tt}\triangle\tau + \frac{1}{12}M_{xxxx}\triangle x^2 \\
&= \frac{1}{2}\triangle\tau\left[M_{tt} + \frac{1}{6\lambda}M_{xxxx}\right], \qquad \lambda = \frac{\triangle\tau}{\triangle x^2}
\end{aligned}
$$

- Unconditionally Consistent: $TE \longrightarrow 0$ as $\triangle x$, $\triangle\tau \longrightarrow 0$ $\forall(x,\tau) \in (0,1)\times(\tau,\tau_F)$, independent of any relation between the two mesh sizes.
- First-order Accuracy: Given λ, $|TE|$ behaves asymptotically like $O(\triangle\tau)$ as $\triangle\tau \longrightarrow 0$.

Explicit Scheme: Eigenvalue-based Stability Analysis

- Note the difference between $w^{(i)}$ and $\overline{w}^{(i)}$:

$$\text{Theoretically Defined: } w^{(i)} = (w_{j0}, ..., w_{ji}, ..., w_{jM})^T, \; i = 0, ..., M$$

$$\text{Computer Computed: } \overline{w}^{(i)} = (\overline{w}_{j0}, ..., \overline{w}_{ji}, ..., \overline{w}_{jM})^T, \; i = 0, ..., M$$

$$\text{Propagated rounding error: } e^{(i)} = \overline{w}^{(i)} - w^{(i)}$$

- Rounding error $r^{(i+1)}$ refers to one that occurs during the computation of $\overline{w}^{(i+1)}$:

$$\overline{w}^{(i+1)} = A_R \cdot \overline{w}^{(i)} + r^{(i+1)}, \quad A_R = (I + \triangle\tau \cdot A_{\triangle x})$$

- Assume $r^{(i)} = 0$ for $i \geq 1$. Then:

$$A_R \cdot e^{(i)} = A_R \cdot \overline{w}^{(i)} - A_R \cdot w^{(i)} = \overline{w}^{(i+1)} - w^{(i+1)} = e^{(i+1)}$$

implying

$$e^{(i)} = (A_R)^i \cdot e^{(0)}, \quad \text{for } i \geq 1$$

Explicit Scheme: Eigenvalue-based Stability Analysis (2)

- Recall

$$X^{-1} A_R X = \Lambda \equiv \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_{N-1} \end{bmatrix}$$

- Therefore, $e^{(i)} = (A_R)^i \cdot e^{(0)}$ implies

$$X^{-1} e^{(i)} = (X^{-1} A_R^i X) X^{-1} e^{(0)}$$
$$\Rightarrow X^{-1} e^{(i)} = (X^{-1} A_R X)(X^{-1} A_R X) \cdots (X^{-1} A_R X) X^{-1} e^{(0)}$$
$$\Rightarrow X^{-1} e^{(i)} = (\Lambda)(\Lambda) \cdots (\Lambda) X^{-1} e^{(0)}$$
$$\widetilde{e}^{(i)} = \Lambda^i \widetilde{e}^{(0)} \text{ with } \widetilde{e}^{(i)} = X^{-1} e^{(i)},\ i \geq 0$$

where

$$\Lambda^i \equiv \begin{bmatrix} \Lambda_1^i & & 0 \\ & \ddots & \\ 0 & & \Lambda_{N-1}^i \end{bmatrix}$$

- Stability requires

$$e^{(i)} \to 0 \text{ (or } \widetilde{e}^{(i)} \to 0) \quad \text{as } i \to \infty$$

Lemma (page 149)

$$\begin{aligned} \rho(A_R) &< 1 \Leftrightarrow \Lambda_j^i \to 0 \text{ as } i \to \infty \\ &\Leftrightarrow \lim_{i\to\infty}[(A_R)^i]_{j,k} = 0, \quad j,k = 1,\dots N-1. \end{aligned}$$

where $\rho(A_R) = \max_j |\Lambda_j|$, $j = 1,\dots,N-1$, *is the spectral radius of* A_R, *and*

$$A_R = I - \lambda \cdot \underbrace{\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}}_{G}$$

Explicit Scheme: Eigenvalue-based Stability Analysis (4)

Lemma (page 150)

Let G be a $K \times K$ tridiagonal matrix:

$$G = \begin{bmatrix} \alpha & \beta & & 0 \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & \beta \\ 0 & & \gamma & \alpha \end{bmatrix}_{K \times K} \tag{2}$$

The eigenvalues Λ_k^G and eigenvectors $\upsilon^{(k)}$ of G are:

$$\Lambda_k^G = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}}\cos\frac{k\pi}{K+1}, \quad k = 1, \dots, K \tag{3}$$

$$\upsilon^{(k)} = \left(\sqrt{\frac{\gamma}{\beta}}\sin\frac{k\pi}{K+1}, \left(\sqrt{\frac{\gamma}{\beta}}\right)^2 \sin\frac{2k\pi}{K+1}, \dots, \left(\sqrt{\frac{\gamma}{\beta}}\right)^K \sin\frac{Kk\pi}{K+1}\right)^T$$

Explicit Scheme: Eigenvalue-based Stability Analysis (5)

- Consider $K \to N-1, \alpha \to 2, \beta = \gamma \to -1$. Then,

$$
\begin{aligned}
\Lambda_k^G &= 2 - 2\cos\frac{k\pi}{N}, \quad k = 1, \ldots, N-1 \\
&= 2 - 2[1 - 2\sin^2\frac{k\pi}{2N}] \\
&= 4\sin^2\frac{k\pi}{2N}
\end{aligned}
$$

Review of Trigonometry

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Explicit Scheme: Eigenvalue-based Stability Analysis (6)

- The eigenvalues Λ_k of $A_R (= I - \lambda G)$ is therefore given by:

$$\Lambda_k = 1 - 4\lambda \sin^2 \frac{k\pi}{2N}, \quad k = 1, ..., N-1$$

- Stability requires $|\Lambda_k| < 1$. So,

$$-1 < 1 - 4\lambda \sin^2 \frac{k\pi}{2N} < 1$$

$$\Rightarrow \lambda \sin^2 \frac{k\pi}{2N} < \frac{1}{2}$$

- But

$$\sin^2 \frac{k\pi}{2N} \leq \sin^2 \frac{(N-1)\pi}{2N} < \sin^2 \frac{\pi}{2} = 1, \text{ for } k \leq N-1$$

- Conclusion: the explicit method $w^{(i+1)} = A_R \cdot w^{(i)}$ is stable if

$$\lambda \leq \frac{1}{2} \text{ (i.e., } \triangle\tau \leq \frac{\triangle x^2}{2}\text{)}$$

Implicit Scheme for the Heat Equation

Implicit Scheme (Backward Difference)

$$\frac{y_{j,i+1} - y_{j,i}}{\triangle t} = \frac{y_{j-1,i+1} - 2y_{j,i+1} + y_{j+1,i+1}}{\triangle x^2} + TE,$$

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle t} = \frac{w_{j-1,i+1} - 2w_{j,i+1} + w_{j+1,i+1}}{\triangle x^2},$$

$$A_L \cdot w^{(i+1)} = w^{(i)},$$

$$A_L = (I - \triangle\tau \cdot A_{\triangle x}),$$

$$A_{\triangle x} = \frac{1}{\triangle x^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}$$

- Solve $A_L \cdot w^{(i+1)} = w^{(i)}$:

$$\underbrace{\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda & 1+2\lambda & -\lambda \\ 0 & \cdots & \cdots & 0 & -\lambda & 1+2\lambda \end{bmatrix}}_{A_L} \underbrace{\begin{bmatrix} w_{1,i+1} \\ w_{2,i+1} \\ \vdots \\ \vdots \\ w_{N-2,i+1} \\ w_{N-1,i+1} \end{bmatrix}}_{w^{(i+1)}} = \underbrace{\begin{bmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ \vdots \\ w_{N-2,i} \\ w_{N-1,i} \end{bmatrix}}_{w^{(i)}}$$

- Since A_L is a tridiagonal matrix, we can use the Thomas algorithm to solve the $(N-1)\times(N-1)$ system.

Implicit Scheme: Accuracy, Consistency and Stability

- Like the explicit scheme, the implicit scheme has first-order accuracy.
- Like the explicit scheme, the implicit scheme is unconditionally consistent.
- The explicit scheme is *conditionally* stable, but the implicit scheme is *unconditionally* stable.

Crank-Nicolson Scheme, page 153

$$\frac{y_{j,i+1} - y_{j,i}}{\triangle t} = \frac{1}{2}\left(\frac{y_{j-1,i+1} - 2y_{j,i+1} + y_{j+1,i+1}}{\triangle x^2} + \frac{y_{j-1,i} - 2y_{j,i} + y_{j+1,i}}{\triangle x^2}\right) + TE,$$

$$\frac{w_{j,i+1} - w_{j,i}}{\triangle t} = \frac{1}{2}\left(\frac{w_{j-1,i+1} - 2w_{j,i+1} + w_{j+1,i+1}}{\triangle x^2} + \frac{w_{j-1,i} - 2w_{j,i} + w_{j+1,i}}{\triangle x^2}\right),$$

$$A_L \cdot w^{(i+1)} = A_R \cdot w^{(i)},$$

$$A_L = I - \frac{1}{2}\triangle\tau \cdot A_{\triangle x}, \quad A_R = I + \frac{1}{2}\triangle\tau \cdot A_{\triangle x}$$

$$A_L = \begin{bmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots & \cdots & 0 \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda/2 & 1+\lambda & -\lambda/2 \\ 0 & \cdots & \cdots & 0 & -\lambda/2 & 1+\lambda \end{bmatrix}_{(N-1)\times(N-1)} \tag{4}$$

$$A_R = \begin{bmatrix} 1-\lambda & \lambda/2 & 0 & \cdots & \cdots & 0 \\ \lambda/2 & 1-\lambda & \lambda/2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda/2 & 1-\lambda & \lambda/2 \\ 0 & \cdots & \cdots & 0 & \lambda/2 & 1-\lambda \end{bmatrix}_{(N-1)\times(N-1)} \tag{5}$$

- Truncation Error:

$$
\begin{aligned}
TE &= \frac{y_{j,i+1} - y_{j,i}}{\triangle\tau} \\
&\quad -\frac{1}{2}\left(\frac{y_{j-1,i+1} - 2y_{j,i+1} + y_{j+1,i+1}}{\triangle x^2} + \frac{y_{j-1,i} - 2y_{j,i} + y_{j+1,i}}{\triangle x^2}\right) \\
&= O(\triangle\tau^2) + O(\triangle x^2)
\end{aligned}
$$

- Crank-Nicolson has second order accuracy in both $\triangle\tau$ and $\triangle x$.
- Crank-Nicolson is unconditionally consistent.

- Crank-Nicolson (p. 155): $A_L \cdot w^{(i+1)} = A_R \cdot w^{(i)}$. From (4) & (5),

$$A_L = I + \frac{\lambda}{2} G, \quad G = \underbrace{\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix}}_{(N-1)\times(N-1)}, \quad A_R = I - \frac{\lambda}{2} G$$

- Thus,

$$\begin{aligned} \underbrace{(2I + \lambda G)}_{C} \cdot w^{(i+1)} &= (2I - \lambda G) \cdot w^{(i)} \\ &= (4I - \underbrace{(2I + \lambda G)}) \cdot w^{(i)} \end{aligned}$$

$$w^{(i+1)} = (4C^{-1} - I) \cdot w^{(i)}$$

- Recall the k^{th} eigenvalue of matrix G:

$$\Lambda_k^G = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}}\cos\frac{k\pi}{K+1}, \quad k = 1, ..., K = N-1$$
$$= 4\sin^2\frac{k\pi}{2N}$$

- Given $C = 2I + \lambda G$, $eig(C) = eig(2I + \lambda G)$ –> k^{th} eigenvalue:

$$\Lambda_k^C = 2 \cdot eig(I) + \lambda \cdot eig(G)$$
$$= 2 + \lambda \cdot 4\sin^2\frac{k\pi}{2N}, \;\; k = 1, \cdots, N-1$$

- Similarly,

$$eig(4C^{-1} - I) = 4 \cdot eig(C^{-1}) - eig(I) = 4 \cdot [eig(C)]^{-1} - eig(I)$$
$$\Lambda_k^{4C^{-1}-I} = 4 \cdot (\Lambda_k^C)^{-1} - 1$$

- Stability requires

$$|\Lambda_k^{4C^{-1}-I}| < 1.$$

- Since $\Lambda_k^C > 2$, Crank-Nicolson is unconditionally stable for all $\lambda > 0$.

von Neumann Stability Analysis (Fourier approach)

- Two fundamental ways of analyzing the stability of finite difference methods:
    - The Eigenvalue-based stability analysis (also known as the matrix approach)
    - The von Neumann stability analysis (also known as the Fourier analysis approach)
- Comments on the two approaches:
    - The matrix approach is more comprehensive because it captures the effect of boundary conditions.
    - In contrast, the Fourier approach is much more straightforward and is very popular.

Application of the Fourier Approach

- Explicit Scheme: page 251

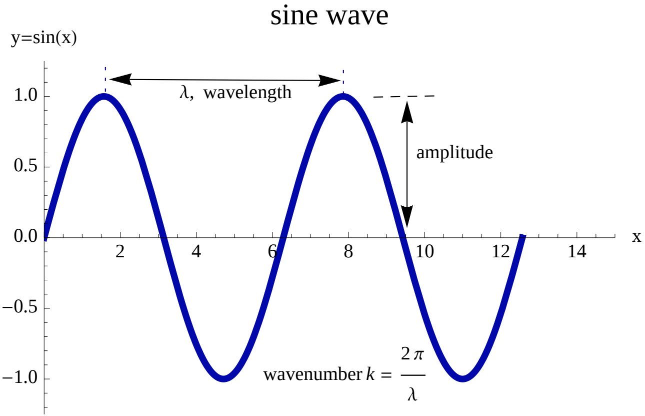
$$\frac{y_{j,i+1} - y_{j,i}}{\triangle\tau} = \frac{y_{j-1,i} - 2y_{j,i} + y_{j+1,i}}{\triangle x^2}$$

- On Topic 3, we have expressed the exact solution of the differential equation as a Fourier series. This expression is based on the observation that a particular set of Fourier modes are exact solutions.
- We can now easily show that an exact solution of the above difference equation derived from the explicit scheme is a similar Fourier mode given below:

$$y_{j,i} = \Lambda^i e^{\mathrm{i}k(j\triangle x)}$$

  - non-italic $\mathrm{i} = \sqrt{-1}$,
  - k is the wavenumber of a Fourier mode,
  - Λ is the amplification factor; related to the Λ in the matrix approach; but does not reflect boundary conditions
  - Λ^i is the amplitude.

Application of the Fourier Approach (2)



Application of the Fourier Approach (3)

- We substitute $y_{j,i} = \Lambda^i e^{\mathrm{i}k(j\triangle x)}$ into the difference equation:

$$(\Lambda - 1)\Lambda^i e^{\mathrm{i}k(j\triangle x)} = \lambda(\Lambda^i e^{\mathrm{i}k[(j-1)\triangle x]} - 2\Lambda^i e^{\mathrm{i}k(j\triangle x)} + \Lambda^i e^{\mathrm{i}k[(j+1)\triangle x]})$$

- Divide by $\Lambda^i e^{\mathrm{i}k(j\triangle x)}$:

$$\Lambda \equiv \Lambda(k) = 1 + \lambda(e^{\mathrm{i}k[-\triangle x]} - 2 + e^{\mathrm{i}k[\triangle x]})$$

- Noting the following relations:

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta, \quad e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

- The amplification factor can be rewritten as

$$\Lambda(k) = 1 - 4\lambda\sin^2\frac{1}{2}k\triangle x$$

- Taking $k = m\pi$, we can write our numerical approximation,

$$y_{j,i} = \sum_{-\infty}^{\infty} A_m e^{\mathrm{i}m\pi(j\triangle x)}[\Lambda(m\pi)]^i$$

Application of the Fourier Approach (4)

- Consider the amplification factor,

$$\Lambda(k) = 1 - 4\lambda \sin^2 \frac{1}{2} k \triangle x$$

- Stability requires

$$-1 \leq \Lambda(k) \leq 1$$

- But the most oscillatory mode is the one for which $k\triangle x = \pi \pm 2n\pi$, $n = 0, 1, \ldots$ such that

$$\sin^2 \frac{1}{2} k \triangle x = 1$$

- Thus:
    - The explicit scheme is unstable if

$$\Lambda < -1 \Longleftrightarrow \lambda > \frac{1}{2}$$

    - The explicit scheme is stable if

$$\Lambda \geq -1 \Longleftrightarrow \lambda \leq \frac{1}{2}$$