BANK OF ENGLAND

Advanced Analytical Tools

Value at Risk and Expected Shortfall

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Contents

A Volatility over several periods.

Main uses of Value at Risk methodology.

- **To measure the downside risk of an asset portfolio.**
- **To determine liquidity and capital buffers.**
  - **What size of liquid asset portfolio do we need in order to be 95% certain of not becoming illiquid in the next 30 days?**
  - **How much capital do we need in order to be 90% certain of not becoming insolvent in the next 3 months?**

Things *not* to focus on.

1. **Plus and minus signs i.e. whether losses are expressed as positive or negative numbers.**
2. **Whether the examples are about asset returns in currency or in percentage rates.**
3. **The difference between VaR(5%) and VaR(95%) - there is no difference, it's just notation.**

1. Risk and volatility.

- **Can we sum up the riskiness of an asset, or portfolio, in a single-value measure?**
  - **Obtaining a single-value measure is important if we want to rank assets and portfolios in terms of the risk that they emobody.**
- **Traditionally this was achieved using volatility i.e. standard deviation.**
- **While volatility is still used, as a single summary measure of risk it has some deficiencies:**
  - **It's symmetric i.e. upside and downside risks are measured in the same way.**

- **It does not deal well with heavy-tailed distributions.**
  * **‘Heavy’ being defined in relation to the Normal distribution.**

1.1. Illustrating a problem with volatility.

- **The following time series of asset returns all have mean zero, and volatility 1.**
- **But they come from three different distributions: N(0,1), t(3) (suitably scaled) and a 'jump distribution'.**

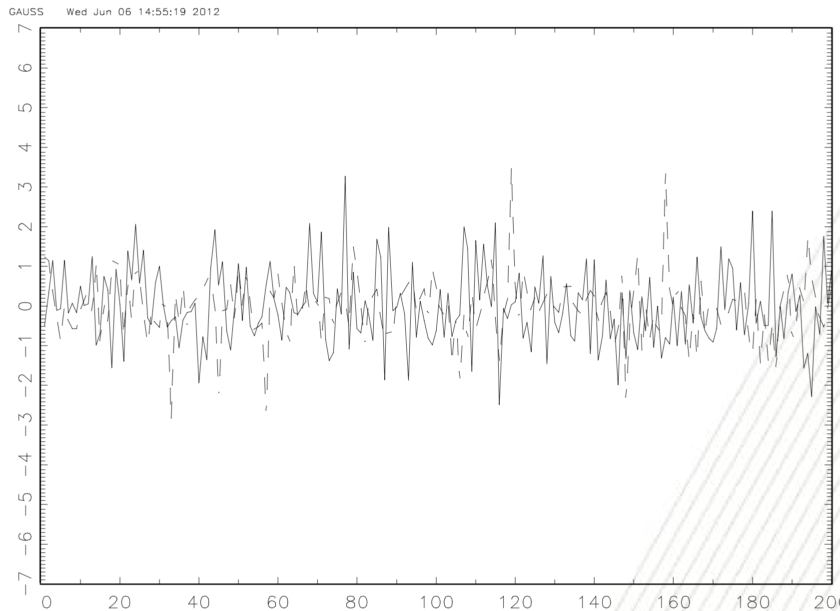

Figure 1: N(0,1) and t(3) variables.

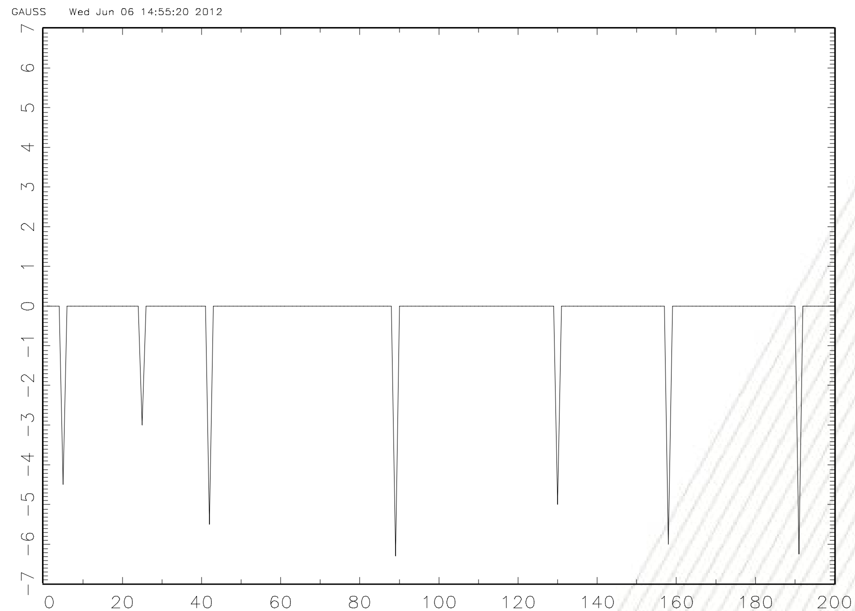

Figure 2: A jump variable.

- **The volatility measure suggests that all three are equally risky.**
- **But which is the most risky for a bank that will become insolvent if the loss hits -4%?**
- **Value at Risk (VaR) offers an alternative single-value measure of risk.**

2. Value at risk: Introduction.

- **VaR is a popular, albeit imperfect, alternative to volatility.**
- **Invented by JPMorgan and propagated in 'RiskMetrics',**
  - **VaR is not symmetric (it looks only at losses).**
  - **It treats heavy tails sensibly, up to a point.**
- **VaR analysis provides values for x and k in the following statement:**

> ***"We can be x% confident that we will not lose more than \$$k$ in the next n days."***

- **But, before we can make use of the VaR measure we need to decide on:**

  1. **The length of the time period over which the risk is of concern.**
  2. **The probability distribution of the risky variable over that period.**
     - **N.B. The VaR approach does not require Normality.**
     - **We will see later that we don't need the whole of the distribution either.**
  3. **A 'confidence level' i.e. x in the above statement. Or, occasionally, the loss level \$k instead.**

2.1. VaR for a simple discrete distribution.

- **Let a risky return have the following distribution:**

Return (%)	Probability
-12	0.02
-11	0.03
-10	0.06
$\geq$ -9	0.89

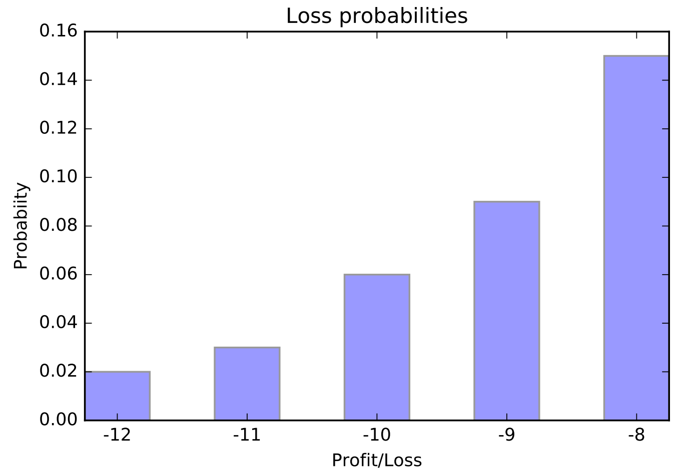

Figure 3: Left tail of probability function.

- **We can be 89% certain that we will not lose more than 9.**
- **We can be 95% certain that we will not lose more than 10, so...**
- **... the Value at Risk at 95% is 10.**

- **This can also be expressed as 'the Value at Risk at 5% is 10'.**
- **We will use the latter convention for the rest of these notes.**

- **What is the Value at Risk at 2%?**
- **What is the Value at Risk at 0%?**
- **The rest of the work involved in calculating VaR has to do with applying it to more realistic distributions.**

3. A complementary risk measure: Expected shortfall (ES).

3.1. Where VaR fails.

- **We alter the discrete example we had earlier,**

Return (%)	Probability
-50	0.02
-20	0.03
-10	0.06
$\geq$ -9	0.89

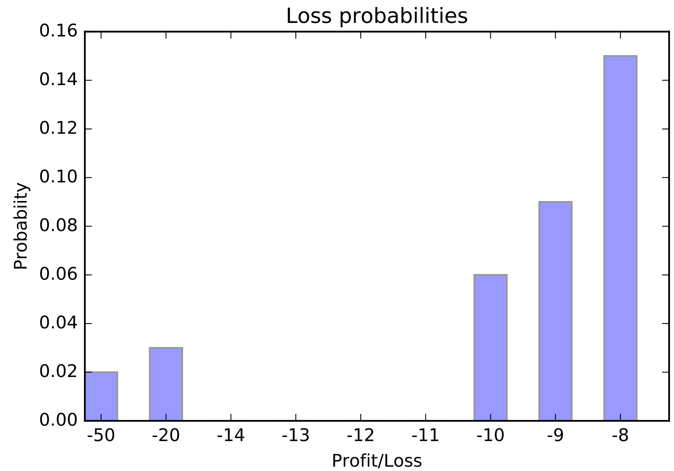

Figure 4: Left tail of probability function.

- **It should be clear that we still get $VaR(5\%) = 10$.**
- **However, if $VaR(5\%) = 10$ is exceeded, the two possible losses are now -20 and -50.**
- **This new distribution is clearly more risky than the old one.**

3.2. Definition of expected shortfall.

- **ES is the expected size of the loss *given* that the VaR has been exceeded**
- **"If losses exceed the VaR how much should we expect to lose?"**
- **Also known as 'expected tail loss'.**
- **For the discrete example we had earlier,**

Return (%)	Probability
-12	0.02
-11	0.03
-10	0.06
$\geq$ -9	0.89

if the loss exceeds $VaR(5\%) = 10$, there are only two possible outcomes, -11 and -12.

- **To get the expected value of the tail loss we have to scale the probabilities to sum to 1, without changing their *relative* values.**

- **We do this as follows:**

$$\begin{aligned}
p'_{-12} &= \frac{p_{-12}}{p_{-11}+p_{-12}} && (1)\\
&= \frac{0.02}{0.02+0.03} && (2)\\
&= 0.4 && (3)
\end{aligned}$$

$$\begin{aligned}
p'_{-11} &= \frac{p_{-11}}{p_{-11}+p_{-12}} && (4)\\
&= \frac{0.03}{0.02+0.03} && (5)\\
&= 0.6 && (6)
\end{aligned}$$

- **Note that the denominator $0.02 + 0.03 = 0.05 = 5\%$ is the VaR confidence level that we are basing the ES on.**

- **We then get the expected tail loss as (using the standard convention that ES is expressed as a positive number):**

$$
\begin{aligned}
ES &= -[(0.4 \times -12) + (0.6 \times -11)] && (7)\\
&= 11.4 && (8)
\end{aligned}
$$

- **For the riskier distribution the expected tail loss is:**

$$
\begin{aligned}
ES &= -[(0.4 \times -50) + (0.6 \times -20)] && (9)\\
&= 32 && (10)
\end{aligned}
$$

i.e. ES reflects the greater risk; VaR does not.

4. VaR and the Normal distribution.

4.1. A numerical example.

- **We start by looking at the risk created by possible changes in the price of a financial asset, and we will measure this in terms of its rate of return.**
- **We assume that log returns are Normally distributed.**

- **Returns in natural units (r_n) are**

$$r_n \equiv \left(\frac{P_{t+1} - P_t}{P_t}\right) \tag{11}$$

- r_n **however, cannot be Normally distributed because it has a lower bound of -1, so we use log returns (r_l) instead, which are unbounded i.e.**

$$r_l = \log\left(1 + r_n\right) \tag{12}$$

where '$\log(\ldots)$**' denotes natural logs.**

- **For 'small' values of r_n, $r_l \approx r_n$.**
  - **'Small' includes the typical rates of return that we see e.g. if $r_n = 0.1$ then $r_l = 0.0953$.**
  - **If you are not familiar with the concept of log returns you will not lose anything here by treating them as if they were pretty much the same as 'ordinary' returns (i.e. r_n above) since the focus of these slide is on VaR and ES, not on how we define returns.**
- **Finally, just for convenience, we will express returns in percentage terms as**

$$r = 100 * r_l \tag{13}$$

- **We assume for now that**

$$r \sim N(\mu, \sigma) \tag{14}$$

- **We will be looking for the value of the loss that has only a 5% chance of being exceeded.**

- **The probability density function (pdf) for our distribution is:**

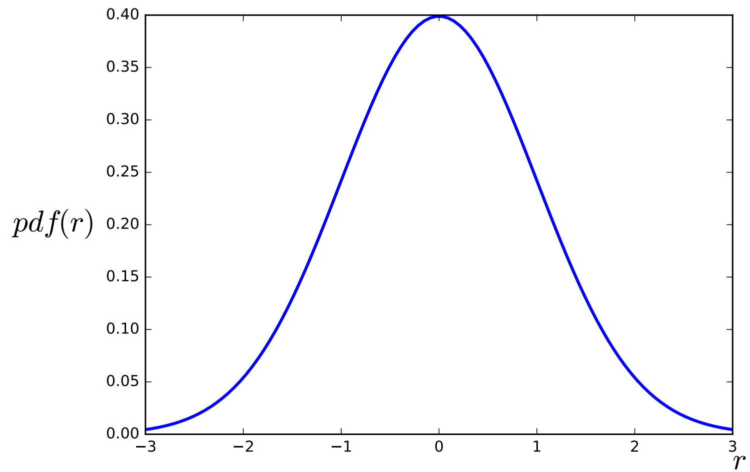

Figure 5: Probability density function for $r \sim N(0, 1)$.

- **We are looking for the value of r for which the shaded area to its left is 5% of the total area under the curve.**

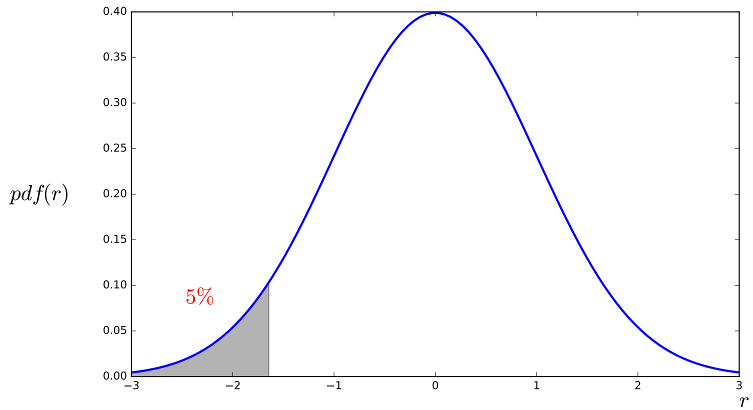

Figure 6: Probability density function for $r \sim N(0, 1)$.

- **We can find this from standard Normal tables...**
- **...but it is easier to use Excel or some other programme.**
- **E.g. in Python's SciPy library the code is: 'norm.ppf(0.05)'**

- **We get the probability that the asset price will fall by 1.645 or more as 5%.**

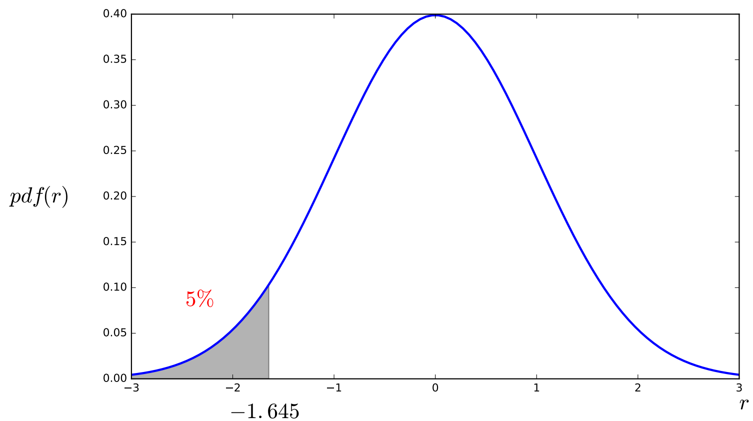

Figure 7: Probability density function for $r \sim N(0, 1)$.

- **This gives us the VaR at 5% as**

$$VaR(5\%) = 1.645\%$$

- **In terms of the typical VaR statement that we saw earlier...**
  - **We can be 95% certain that we will not lose more than than 1.645%.**
- **Similarly,**

$$VaR(1\%) = 2.33\%$$

- **Note that these numerical VaR results are only for returns that are N(0,1) distributed.**

4.1.1. A general formula for Normally distributed returns.

- **For the more general distribution $r \sim N(\mu, \sigma)$, we transform r into the standard normal variate z where**

$$z = \frac{r - \mu}{\sigma} \sim N(0, 1) \tag{15}$$

- **Introducing z just makes the presentation clearer, we will dispense with it later.**
- **For the lhs of the distribution $r \sim N(\mu, \sigma)$ we have,**

$$Pr\left(\frac{r - \mu}{\sigma} < -1.645\right) = 0.05 \tag{16}$$

- **We find the critical values, $z_{5\%}$ and $r_{5\%}$, as**

$$z_{5\%} = \frac{r_{5\%} - \mu}{\sigma} = -1.645 \tag{17}$$

$$\Rightarrow r_{5\%} = \mu - 1.645\sigma \tag{18}$$

- **Note that this will typically give us a negative number, but because the convention is to treat losses as positive numbers, the general $VaR(5\%)$ formula for a Normally distributed return is,**

$$VaR(5\%) = -(\mu - 1.645\sigma) \tag{19}$$

$$\Rightarrow VaR(5\%) = 1.645\sigma - \mu \tag{20}$$

- **Similarly,**

$$VaR(1\%) = 2.33\sigma - \mu \tag{21}$$

- **Finally, for an initial portfolio value of $\$V(0)$ we get**

$$VaR(5\%)_{\$} = V(0)(1.645\sigma - \mu) \tag{22}$$

4.2. VaR again, in words this time.

- **Given a distribution we can we find numbers x and k such that we can state that:**
  - **'We can be x% confident that we will not lose more than k.'**
  - **Or, 'The probability that we will not lose more than k is $x\%$.'**
- **VaR allows us to rewrite this as:**
  - **'We can be 95% certain that we will not lose more than $VaR(5\%)_{\$}$.'**
- **And,**

– **‘We can be 99% certain that we will not lose more than** $VaR(1\%)$**.’**

5. Value at Risk: An example.

5.1. SP500: Daily returns.

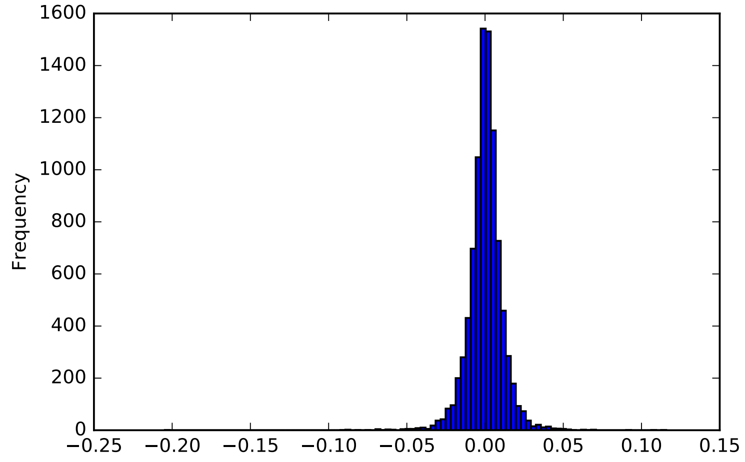

Figure 8: Histogram of SP500 daily returns (% per day).

- **The estimated s.d. and mean for SP500 daily returns from 1 Feb 1980 to 18 May 2016 are 0.0377% and 1.1192%.**
- **The distribution appears to be approximately Normal, so we assume that**

$$r \sim N(0.0377, 1.1192)$$

- **Find the 5% VaR for a \$1m investment in the index.**
  - **The 5% 'critical value' for r, is found from:**

$$\begin{aligned} r_{5\%} &= 1.645\sigma - \mu \\ &= 1.645 \times 1.1192 - .0377 \\ &= 1.8034 \end{aligned}$$

  - **We can be 95% certain that losses will not be greater than 1.8034% on any given day.**

- **So, with a $1m portfolio, we can be 95% certain that we will not lose more than $18034 in any one day:** $VaR_{\$1m}(5\%) = \$18034$
- **The time period is important. Here the data are for *daily* returns, and are used to construct *daily* moments for the probability distribution.**
- **As a result, we get only a 1-day VaR.**

5.2. Expected shortfall and the SP500 example.

- **The SP500 estimated parameters were** $\hat{\mu} = 0.0377\%$, $\hat{\sigma} = 1.1192\%$
- **For the expected shortfall we get, using a standard formula,**

$$
\begin{aligned}
ES(0.05) &= \sigma \left(\frac{\phi(\Phi^{-1}(0.05))}{0.05} \right) - \mu && (23)\\
&= 1.1192 \left(\frac{\phi(-1.645)}{0.05} \right) - 0.0377 && (24)\\
&= 1.1192 \left(\frac{0.1031}{0.05} \right) - 0.0377 && (25)\\
&= 1.1192 \times 2.063 - 0.0377 && (26)\\
&= 2.31 - 0.0377 && (27)\\
&= 2.27 && (28)
\end{aligned}
$$

5.3. Combining the VaR and ES results.

- $VaR(5\%)_{\$}$ = **\$18034 implies that the probability of losing \$18034 or more on any single day is 5%.**
- **ES tells us that on these days our expected loss is** $2.27\% \times \$1000000 =$ **\$22712.**

6. Some problems with common applications of VaR.

Richard Berner sums these up rather nicely as follows:

> ***"First, leverage and volatility risk are procyclical. That's because risk is often managed by looking to metrics like value at risk, or VAR. In other words, a risk manager gives a portfolio a risk budget by imposing a VAR constraint. When VAR rises, the manager must sell securities to reduce risk. But the security sales further depress prices, which amplifies incentives to sell. This phenomenon is true not only in banks, but in asset managers and other portfolio managers.***

- **What this means is that if a portfolio manager is told to keep their VaR(5%) to \$100m say, then when the volatility of the portfolio increases, and the $VaR(5\%) = 1.645\sigma - \mu$ goes with it, the manager will have an incentive to reduce the portfolio's size in order to keep to the \$100m.**

- **The resulting sales may push the prices of the portfolio's assets down which may reduce the estimate μ, and raise the estimated σ of the portfolio.**

- **The result of this is to increase the VaR again, with further encouragement to sell.**

- **While a single institution's sales may not impact the market, it is likely that many will be using the same values for σ and μ which can lead to multiple sales.**

VAR has a variety of well-known shortcomings:

- *It depends on contemporaneous volatility.*
- *It underestimates worst-case loss because it looks at historical not forward-looking correlations.*
- *It may not capture correlations across a portfolio.*
- *Low VAR, like low volatility, creates incentives for more leverage. Even stress tests may look deceptively good if the scenarios are selected from a low volatility regime.*

Nonetheless, VAR is widely used."

Richard Berner, Director (2013–2017), Office of Financial Research. Bard College conference, April 2016.

7. Extensions to the basic VaR/ES presented here.

1. Dealing with more-realistic 1-period distributions of returns:
   (a) ‘Extreme value’ distributions.
   (b) Empirical distributions.
2. Multi-period distributions.
3. Time-varying distributions.
4. Stressed VaR
5. Numerical solution methods, mainly Monte Carlo simulations.
6. Copulas.
7. VaR-related measures of systemic risk.

8. Further reading.

The first paper below is suitable for general reading, the others are more specialised.

1. ***Fundamental review of the trading book: A revised market risk framework.*** **BCBS, 2013.**
2. ***Why is risk so hard to measure?*** **Danielsson and Zhou, De Nederlandischebank, 2016.**
3. ***Financial Risk Forecasting*, Danielsson, 2011.**
4. ***Comparative Analyses of Expected Shortfall and Value-at-Risk under Market Stress*, Yamai and Yoshiba, Bank of Japan.**

5. *Value-at-risk versus expected shortfall: A practical perspective.* Yamai and Yoshiba, Journal of Banking and Finance 29, 2005 pp 997–1015

A. Volatility over several periods.

- **To convert a 1-day volatility (σ_1) to an n-day volatility we often see the following formula being used:**

$$\sigma_n = \sqrt{n}\sigma_1 \tag{29}$$

with this value being used for the calculation of an n-day VaR.

- **This may appear rather odd if you are used to seeing the the following equation for the variance of a multiple of a random variable x:**

$$\begin{aligned} V(nx) &= n^2 V(x) && (30) \\ \Rightarrow \sigma_n &= n\sigma_1 && (31) \end{aligned}$$

- **So why do we have**

$$\sigma_n = \sqrt{n}\sigma_1 \tag{32}$$

  in equation (29)?

- **This can be explained as follows, using a 2-day example :**
  - **Let the random returns for the 2 days be x_1, and x_2, then**

$$V(x_1 + x_2) = V(x_1) + V(x_2) + 2Cov(x_1, x_2) \tag{33}$$

  - **Equation (29) is based on 2 implicit assumptions:**
    1. **The returns on the 2 days are independently distributed, i.e. that $Cov(x_1, x_2) = 0$.**
    2. **They have the same variance, call it $V(x)$.**

– **Hence**

$$V(x_1 + x_2) = V(x) + V(x) + 0 = 2V(x) \tag{34}$$

and

$$\sigma_{x_1+x_2} = \sqrt{2V(x)} = \sqrt{2}\sigma_x \tag{35}$$

- **For equation (30) we are multiplying a single realisation by** n**, but for (29)we are adding two different realisations from the same distribution, which is what we require for an n-day measure.**

- **The covariance of a variable with itself is, of course, the**

same as its variance, so

$$
\begin{aligned}
V(2x) &= V(x+x) && (36)\\
&= V(x)+V(x)+2Cov(x,x) && (37)\\
&= V(x)+V(x)+2V(x) && (38)\\
&= 4V(x) && (39)\\
\Rightarrow \sigma_{2x} &= 2\sigma_x && (40)
\end{aligned}
$$

- **So it all revolves around the Cov term, and whether or not it is equal to zero.**