APPENDIX A

SIMULATION OF COPULAS

Copulas have primary and direct applications in the simulation of dependent variables. We now present general procedures to simulate bivariate, as well as multivariate, dependent variables. Other algorithms can be found in many of the exercises proposed by [207], as well as in Appendix C.

The mathematical kernel for simulating copulas is provided by the formulas in Eq. (3.26), where conditional probabilities are written in terms of the partial derivatives of copulas. Firstly we need to introduce the notion of a *quasi-inverse* of a distribution function.

DEFINITION A.1 (Quasi-inverse). *Let F be a univariate distribution function. A quasi-inverse of F is any function $F^{[-1]}\colon \mathbb{I} \to \overline{\mathbb{R}}$ such that:*

1. *if $t \in \mathrm{Ran}(F)$, then $F^{[-1]}(t)$ is any number $x \in \overline{\mathbb{R}}$ such that $F(x) = t$, i.e., for all $t \in \mathrm{Ran}(F)$,*

$$F\left(F^{[-1]}(t)\right) = t; \tag{A.1a}$$

2. *if $t \notin \mathrm{Ran}(F)$, then*

$$F^{[-1]}(t) = \inf\{x\colon F(x) \geq t\} = \sup\{x\colon F(x) \leq t\}. \tag{A.1b}$$

Clearly, if F is strictly increasing it has a single quasi-inverse, which equals the (ordinary) inverse function F^{-1} (or, sometimes, $F^{(-1)}$).

A.1. THE 2-DIMENSIONAL CASE

A general algorithm for generating observations (x, y) from a pair of r.v.'s (X, Y) with marginals F_X, F_Y, joint distribution F_{XY}, and 2-copula $\mathbf{C}$ is as follows. By virtue of Sklar's Theorem (see Theorem 3.1), we need only to generate a pair (u, v) of observations of r.v.'s (U, V), Uniform on $\mathbb{I}$ and having the 2-copula $\mathbf{C}$. Then, using the *Probability Integral Transform*, we transform (u, v) into (x, y), i.e.

$$\begin{cases} x = F_X^{[-1]}(u) \\ y = F_Y^{[-1]}(v) \end{cases}. \tag{A.2}$$

In order to generate the pair (u, v) we may consider the conditional distribution of V given the event $\{U = u\}$:

$$c_u(v) = \mathbb{P}\{V \le v \mid U = u\} = \frac{\partial}{\partial u}\mathbf{C}(u, v). \tag{A.3}$$

A possible algorithm is as follows.

1. Generate independent variates u, t Uniform on $\mathbb{I}$.
2. Set $v = c_u^{[-1]}(t)$.

The desired pair is then (u, v). For other algorithms see [71, 156].

ILLUSTRATION A.1. ►

The Frank and the Gumbel-Hougaard families of 2-copulas are widely used in applications (e.g., in hydrology — see [66, 253, 254, 90, 67, 255]). As clarified in Appendix C (see Section C.1 and Section C.2, respectively), Frank's copulas may model both negatively and positively associated r.v.'s, whereas Gumbel-Hougaard's copulas only describe positive forms of association. In addition, Frank's copulas do not feature tail dependence, as opposed to Gumbel-Hougaard's copulas.

We show shortly several comparisons between simulated (U, V) samples extracted from Frank's and Gumbel-Hougaard's 2-copulas, where both U and V are Uniform on $\mathbb{I}$. In all cases the sample size is $N = 200$, and the random generator is the same for both samples, as well as the value of Kendall's τ_K. This gives us the possibility of checking how differently these copulas affect the joint behavior of r.v.'s subjected to their action.

The first example is given in Figure A.1. Here $\tau_K \approx 0.001$, i.e. U and V are very weakly positively associated (practically, they are independent, that is $\mathbf{C} \approx \mathbf{\Pi}_2$ for both families). Indeed, the two plots are almost identical, and the points are uniformly sparse within the unit square.

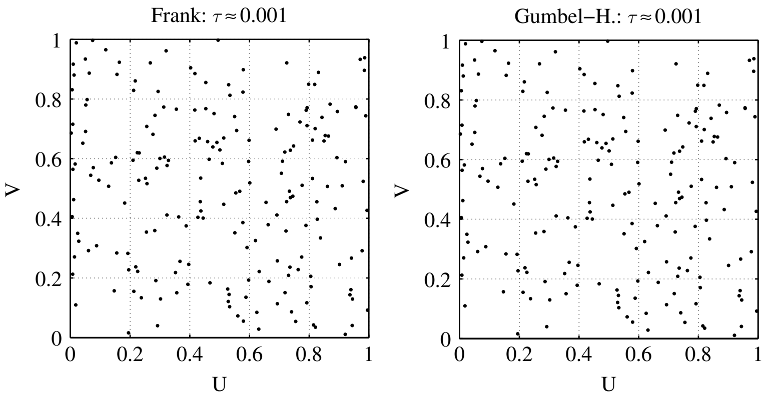

Figure A.1. Comparison between simulated samples extracted from Frank's and Gumbel-Hougaard's 2-copulas. Here $\tau_K \approx 0.001$

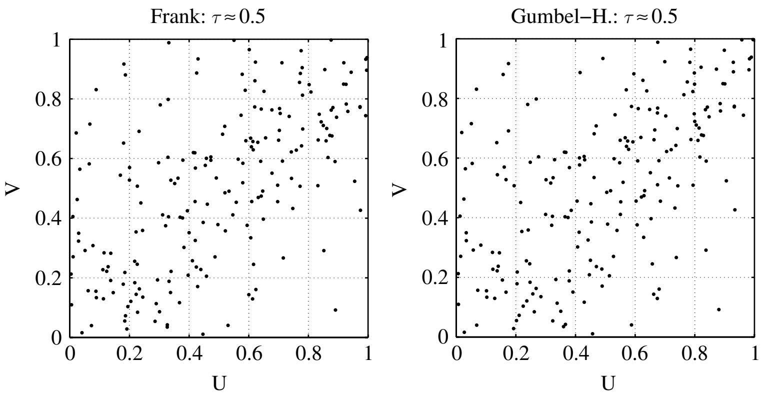

Figure A.2. Comparison between simulated samples extracted from Frank's and Gumbel-Hougaard's 2-copulas. Here $\tau_K \approx 0.5$

The second example is given in Figure A.2. Here $\tau_K \approx 0.5$, i.e. U and V are moderately positively associated. Actually, the points tend to dispose themselves along the main diagonal. The two plots are still quite similar. The main differences are evident only in the upper right corner, where the Gumbel-Hougaard copula tends to organize the points in a different way from that of the Frank copula.

The third example is given in Figure A.3. Here $\tau_K \approx 0.95$, i.e. U and V are strongly associated positively. Actually, the points clearly tend to dispose themselves along the main diagonal. The two plots are still quite similar. The main differences are evident only in the extreme upper right corner: the Gumbel-Hougaard copula tends to concentrate the points, whereas the Frank copula seems to make the points more sparse.

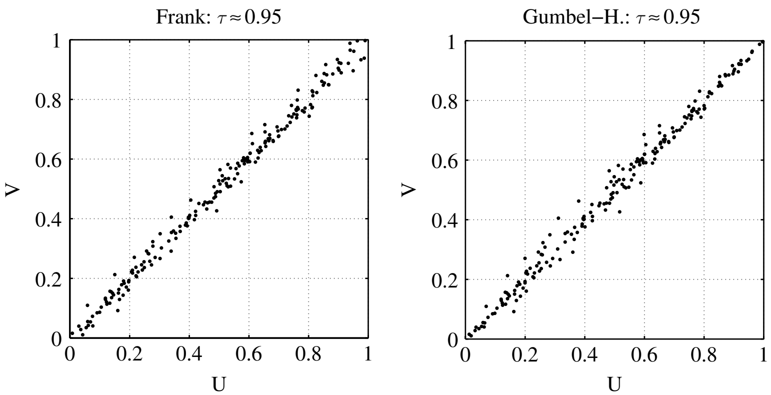

Figure A.3. Comparison between simulated samples extracted from Frank's and Gumbel-Hougaard's 2-copulas. Here $\tau_K \approx 0.95$

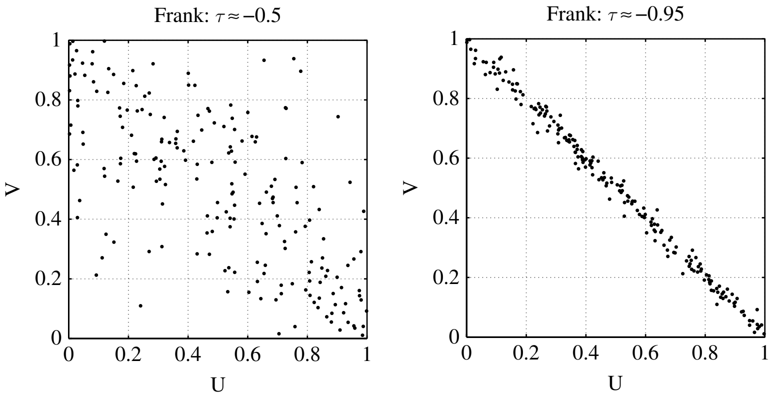

Figure A.4. Simulated samples extracted from Frank's 2-copulas: (*left*) $\tau_K \approx -0.5$; (*right*) $\tau_K \approx -0.95$

The last example concerns the negative association, and thus only samples extracted from the Frank family can be shown. In in Figure A.4 we show two simulations where $\tau_K \approx -0.5$ (corresponding to a moderate negative association), and $\tau_K \approx -0.95$ (corresponding to a strong negative association). In both cases the points clearly tend to disperse along the secondary diagonal of the unit square. As τ_K decreases, the observations get more concentrated along this line.

In Figures A.1–A.4 we show how the behavior of a bivariate sample may change when the degree of association between the variables considered (measured, e.g., by Kendall's τ_K) ranges over its domain, i.e. $[-1, +1]$. As an important conclusion of practical relevance, we must stress that visual comparisons are not sufficiente to decide which copula best describes the behavior of the available data. For instance, the plots for $\tau_K > 0$ show that Frank's and Gumbel-Hougaard's 2-copulas apparently behave in a similar way. Unfortunately, this is also a widespread practice as seen in the literature. On the contrary, only specific statistical tests (e.g., concerning tail dependence) may help in deciding whether or not a family of copulas should be considered for a given application. ◀

A.2. THE GENERAL CASE

Let F be a multivariate distribution with continuous marginals $F_1, \ldots, F_d$, and suppose that F can be expressed in a unique way via a d-copula $\mathbf{C}$ by virtue of Sklar's Theorem (see Theorem 4.2). In order to simulate a vector $(X_1, \ldots, X_d) \sim F$, it is sufficient to simulate a vector $(U_1, \ldots, U_d) \sim \mathbf{C}$, where the r.v.'s U_i's are Uniform on $\mathbb{I}$. By using Sklar's Theorem and the *Probability Integral Transform*

$$U_i = F_i(X_i) \Longleftrightarrow X_i = F_i^{[-1]}(U_i), \tag{A.4}$$

where $i = 1, \ldots, d$, the r.v.'s X_i's have marginal distributions F_i's, and joint distribution F. We now show how to simulate a sample extracted from $\mathbf{C}$. For the sake of simplicity, we assume that $\mathbf{C}$ is absolutely continuous.

1. To simulate the first variable U_1, it suffices to sample from a r.v. U_1' Uniform on $\mathbb{I}$. Let us call u_1 the simulated sample.
2. To obtain a sample u_2 from U_2, consistent with the previously sampled u_1, we need to know the distribution of U_2 conditional on the event $\{U_1 = u_1\}$. Let us denote this law by $G_2(\cdot \mid u_1)$, given by:

$$\begin{aligned} G_2(u_2 \mid u_1) &= \mathbf{P}(U_2 \le u_2 \mid U_1 = u_1) \\ &= \frac{\partial_{u_1} \mathbf{C}\,(u_1, u_2, 1, \ldots, 1)}{\partial_{u_1} \mathbf{C}\,(u_1, 1, \ldots, 1)} \\ &= \partial_{u_1} \mathbf{C}\,(u_1, u_2, 1, \ldots, 1)\,. \end{aligned} \tag{A.5}$$

   Then we take $u_2 = G_2^{-1}(u_2' \mid u_1)$, where u_2' is the realization of a r.v. U_2' Uniform on $\mathbb{I}$, that is independent of U_1'.
3. In general, to simulate a sample u_k from U_k, consistent with the previously sampled $u_1, \ldots, u_{k-1}$, we need to know the distribution of U_k conditional on the events $\{U_1 = u_1, \ldots, U_{k-1} = u_{k-1}\}$. Let us denote this law by $G_k(\cdot \mid u_1, \ldots, u_{k-1})$, given by:

$$\begin{aligned} &G_k(u_k \mid u_1, \ldots, u_{k-1}) \\ &\quad = \mathbb{P}\{U_k \le u_k \mid U_1 = u_1, \ldots, U_{k-1} = u_{k-1}\} \\ &\quad = \frac{\partial_{u_1,\ldots,u_{k-1}} \mathbf{C}\,(u_1, \ldots, u_k, 1, \ldots, 1)}{\partial_{u_1,\ldots,u_{k-1}} \mathbf{C}\,(u_1, \ldots, u_{k-1}, 1, \ldots, 1)}. \end{aligned} \tag{A.6}$$

   Then we take $u_k = G_k^{-1}(u_k' \mid u_1, \ldots, u_{k-1})$, where u_k' is the realization of a r.v. U_k' Uniform on $\mathbb{I}$, that is independent of $U_1', \ldots, U_{k-1}'$.

Using the *Probability Integral Transform*, it is easy to generate the sample $(x_1, \ldots, x_d)$ extracted from F:

$$(x_1, \ldots, x_d) = \left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right). \tag{A.7}$$

Below we show how the "conditional" approach to the construction of multivariate copulas, introduced in Section 4.3, is also well suited for the simulation of multivariate vectors, for this task can be made simply by calculating partial derivatives of the copulas of interest.

ILLUSTRATION A.2 (Sea storm simulation). ►

As shown in Illustration 4.9, in [68] there is a characterization of the sea storm dynamics involving four variables: the significant wave height H, the storm duration D, the waiting time I between two successive "active" storm phases, and the storm wave direction A.

In order to simulate the full sea-state dynamics, one needs the 4-copula $\mathbf{C}_{HDIA}$ associated with F_{HDIA}. The algorithm explained in this Section for simulating multivariate data using copulas has a "nested" structure. An initial variable (e.g., H) is first simulated. Then, by using the information provided by $\mathbf{C}_{HD}$, the variable D can be simulated consistently with the previous realization of H. As the next step, the variable I can be simulated by exploiting $\mathbf{C}_{HDI}$, and the pair (H, D) just simulated. Note that all the three 2-copulas $\mathbf{C}_{HD}, \mathbf{C}_{HI}, \mathbf{C}_{DI}$ are required to carry out this step. Finally, the variable A can be simulated by using $\mathbf{C}_{HDIA}$, and the triple (H, D, A) just simulated. Here, all the six 2-copulas linking the four variables H, D, I, A are needed.

If $\mathbf{C}_{HDIA}$ is constructed as explained in Subsections 4.3.1–4.3.2, then an important result follows. In fact, only the knowledge of the four one-dimensional distributions F_H, F_D, F_I, F_A, and of the six 2-copulas $\mathbf{C}_{HD}$, $\mathbf{C}_{HI}$, $\mathbf{C}_{HA}$, $\mathbf{C}_{DI}$, $\mathbf{C}_{DA}$, $\mathbf{C}_{IA}$, is required to carry out the simulation. Clearly, this may represent a great advantage with respect to the estimation of parameters. In addition, the calculations greatly simplify, and the integral representations disappear: eventually, only trivial composite functions of partial derivatives of 2-copulas need to be evaluated. In turn, the numerical simulation of sea-states is quite fast.

Below we outline a step-by-step procedure to simulate a sequence of sea-states, assuming that the construction of the underlying model follows the "conditional" approach outlined above. We simplify and clarify the presentation at the expense of some abuse of mathematical notation. Here D_1 and D_2 denote, respectively, the partial derivatives with respect to the first and the second component. An obvious point is as follows: if the variables were simulated in a different order, then only the corresponding copulas should be changed, while the algorithm remains the same.

1. **Simulation of H.**
   Let U_1 be Uniform on $\mathbb{I}$. In order to simulate H set

$$H = F_H^{-1}(U_1). \tag{A.8}$$

2. **Simulation of D.**
   Let U_2 be Uniform on $\mathbb{I}$ and independent of U_1. In order to simulate D consistently with H, the function

$$G_2(u_2 \mid u_1) = \partial_{u_1} \mathbf{C}_{HD}(u_1, u_2) \tag{A.9}$$

   must be calculated first. Then set

$$D = F_D^{-1}(G_2^{-1}(U_2 \mid U_1)). \tag{A.10}$$

3. **Simulation of I.**
Let U_3 be Uniform on $\mathbb{I}$ and independent of $\{U_1, U_2\}$. In order to simulate I consistently with (H, D), the function

$$G_3(u_3 \mid u_1, u_2) = \frac{\partial_{u_1,u_2} \mathbf{C}_{HDI}(u_1, u_2, u_3)}{\partial_{u_1,u_2} \mathbf{C}_{HD}(u_1, u_2)} \tag{A.11}$$

must be calculated first, where the numerator ϕ equals

$$\phi = \partial_{u_1} \mathbf{C}_{HI}(\mathrm{D}_2\, \mathbf{C}_{HD}(u_1, u_2), \mathrm{D}_1\, \mathbf{C}_{DI}(u_2, u_3)). \tag{A.12}$$

In fact, according to Eq. (4.22), ϕ can be written as

$$\begin{aligned} \partial_{u_1,u_2} \mathbf{C}_{HDI}(u_1, u_2, u_3) &= \partial_{u_1,u_2} \int_0^{u_2} \mathbf{C}_{HI}(\ldots, \ldots)\, \mathrm{d}x \\ &= \partial_{u_1} \left(\partial_{u_2} \int_0^{u_2} \mathbf{C}_{HI}(\ldots, \ldots)\, \mathrm{d}x \right), \end{aligned} \tag{A.13}$$

leading to the expression of ϕ shown in Eq. (A.12). Moreover, by taking the partial derivative with respect to u_1, we have the following simplification:

$$\begin{aligned} \phi = \mathrm{D}_1\, \mathbf{C}_{HI}(\mathrm{D}_2\, \mathbf{C}_{HD}(u_1, u_2), \mathrm{D}_1\, \mathbf{C}_{DI}(u_2, u_3)) \cdot \\ \cdot \mathrm{D}_{12}\, \mathbf{C}_{HD}(u_1, u_2), \end{aligned} \tag{A.14}$$

and the right-most term equals the denominator in Eq. (A.11), which then cancels out. Then set

$$I = F_I^{-1}(G_3^{-1}(U_3 \mid U_1, U_2)). \tag{A.15}$$

4. **Simulation of A.**
Let U_4 be Uniform on $\mathbb{I}$ and independent of $\{U_1, U_2, U_3\}$. In order to simulate A consistently with (H, D, I), the function

$$G_4(u_4 \mid u_1, u_2, u_3) = \frac{\partial_{u_1,u_2,u_3} \mathbf{C}_{HDIA}(u_1, u_2, u_3, u_4)}{\partial_{u_1,u_2,u_3} \mathbf{C}_{HDI}(u_1, u_2, u_3)} \tag{A.16}$$

must be calculated first. Here the denominator equals $\partial_{u_3}\phi$, whereas the numerator ψ can be calculated as in Eq. (A.13), and is given by

$$\psi = \mathrm{D}_1\, \mathbf{C}_{HA}(\psi_1, \psi_2) \cdot \partial_{u_3}\phi, \tag{A.17}$$

where

$$\psi_1 = \mathrm{D}_2\, \mathbf{C}_{HI}(\mathrm{D}_2\, \mathbf{C}_{HD}(u_1, u_2), \mathrm{D}_1\, \mathbf{C}_{DI}(u_2, u_3)), \tag{A.18a}$$

$$\psi_2 = \mathrm{D}_2\, \mathbf{C}_{AI}\left(\mathrm{D}_2\, \mathbf{C}_{AD}\left(u_4, u_2\right), \mathrm{D}_1\, \mathbf{C}_{DI}\left(u_2, u_3\right)\right). \tag{A.18b}$$

As in the simulation of I, cancellations occur, and only a simplified version of the numerator remains in the expression of G_4. Then set

$$A = F_A^{-1}(G_4^{-1}(U_4 \mid U_1, U_2, U_3)). \tag{A.19}$$

Note how the whole simulation procedure simply reduces to the calculation of partial derivatives of 2-copulas. Also, in some cases, the inverse functions used above can be calculated analytically; otherwise, a simple numerical search can be performed.

As an illustration, in Figure A.5 we show the same comparisons as those presented in Figure 4.2, but using now a data set of about 20,000 simulated sea storms. As in Illustration 4.9, the three variables H, D, and A are considered, and the 2-copula used for the pair (H, D) belongs to the Ali-Mikhail-Haq family (see Section C.4), while those used for (H, A) and (D, A) belong to the Frank family (see Section C.1). As expected, the agreement is good in all cases.

As a further illustration, the same comparison as just given is shown in Figure A.6, where the variables D, I, and A are considered. The 2-copula used for the pairs (D, I) and (I, A) belong to the Gumbel-Hougaard family (see Section C.2), while that used for (D, A) belongs to the Frank family (see Section C.1). Again, the fit is good in all cases. These plots should be compared to those in Figure 4.3, where a much smaller data set is used.

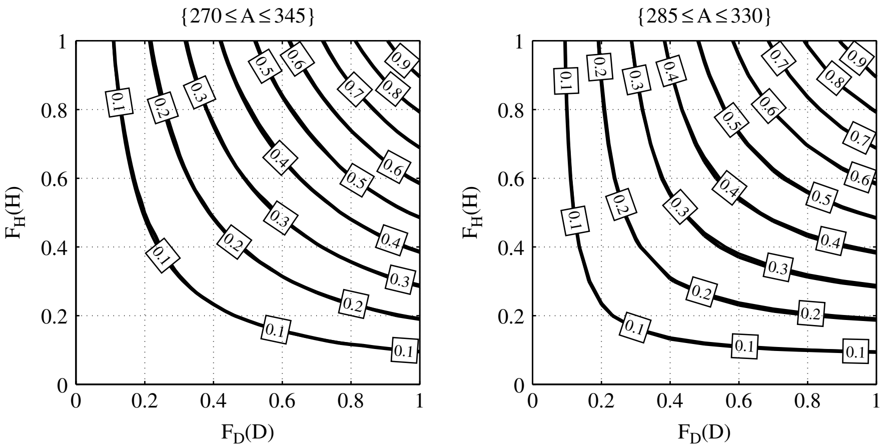

Figure A.5. Comparison between the level curves of the theoretical copulas fitted to the simulated (D, A, H) observations (*thin* lines), and those of the empirical copulas constructed using the same data (*thick* lines). The probability levels are as indicated (*labels*), as well as the conditioning events $\{\bullet \leq A \leq \bullet\}$

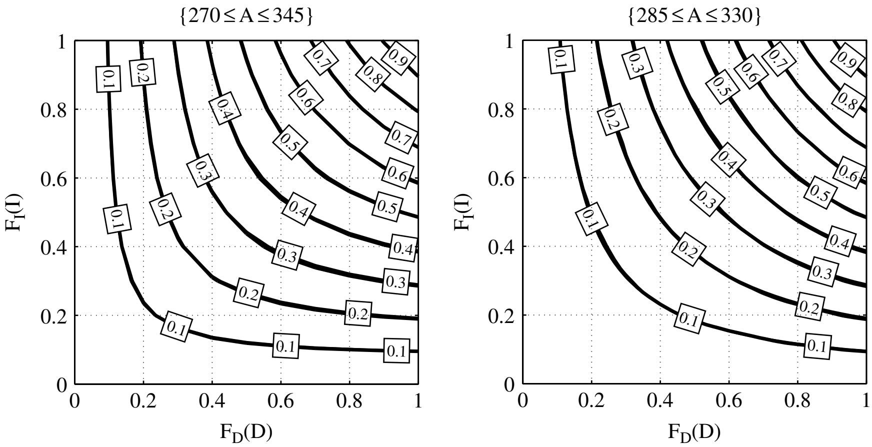

Figure A.6. Comparison between the level curves of the theoretical copulas fitted to the simulated (D, A, I) observations (*thin* lines), and those of the empirical copulas constructed using the same data (*thick* lines). The probability levels are as indicated (*labels*), as well as the conditioning events $\{\bullet \leq A \leq \bullet\}$

As a difference with the results shown in Illustration 4.9, the figures presented illustrate clearly how the situation may improve if sufficient data are made available when working with multivariate distributions. Indeed, in our examples the graphs of the theoretical copulas cannot be distinguished from those of the empirical ones. For these reasons, the possibility offered by copulas to simulate multidimensional vectors easily is invaluable in many practical applications. ◀

APPENDIX B

DEPENDENCE

(Written by Fabrizio Durante — Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz (Austria))

Dependence relations between random variables is one of the most studied subjects in Probability and Statistics. Unless specific assumptions are made about the dependence, no meaningful statistical model can be constructed. There are several ways to discuss and measure the dependence between random variables: valuable sources of information are [207, 155, 75], and references therein.

The aim of this Appendix is to collect and clarify the essential ideas about dependence, by emphasizing the role that copulas play in this context. In Section B.1 we present the basic concepts of dependence, and introduce the measures of dependence. In Section B.2 we present the measures of association for a pair of r.v.'s, and, in particular, we restrict ourselves to consider those measures (such as Kendall's τ_K and Spearman ρ_S) which provide information about a special form of dependence known as *concordance*.

B.1. BIVARIATE CONCEPTS OF DEPENDENCE

In this Section we review some bivariate dependence concepts for a pair of continuous r.v.'s X and Y. Wherever possible, we characterize these properties by using their copula $\mathbf{C}$. Analogous properties can be given in the d-variate case, $d \geq 3$. We refer the reader to [155, 75, 207].

B.1.1 Quadrant Dependence

The notion of *quadrant dependence* is important in applications, for risk assessment and reliability analysis.

DEFINITION B.1 (Quadrant dependence). *Let X and Y be a pair of continuous r.v.'s with joint distribution function F_{XY} and marginals F_X, F_Y.*

1. *X and Y are positively quadrant dependent (briefly, PQD) if*

$$\forall (x, y) \in \mathbb{R}^2 \quad \mathbb{P}\{X \leq x, Y \leq y\} \geq \mathbb{P}\{X \leq x\}\, \mathbb{P}\{Y \leq y\}. \tag{B.1}$$

2. *X and Y are negatively quadrant dependent (briefly, NQD) if*

$$\forall(x, y) \in \mathbb{R}^2 \quad \mathbb{P}\{X \leq x, Y \leq y\} \leq \mathbb{P}\{X \leq x\}\,\mathbb{P}\{Y \leq y\}. \tag{B.2}$$

Intuitively, X and Y are PQD if the probability that they are simultaneously large or simultaneously small is at least as great as it would be if they were independent.

In terms of distribution functions, the notion of quadrant dependence can be expressed as follows:

1. X and Y are positively quadrant dependent if

$$\forall(x, y) \in \mathbb{R}^2 \quad F_{XY}(x, y) \geq F_X(x)F_Y(y).$$

2. X and Y are negatively quadrant dependent if

$$\forall(x, y) \in \mathbb{R}^2 \quad F_{XY}(x, y) \leq F_X(x)F_Y(y).$$

Note that the quadrant dependence properties are invariant under strictly increasing transformations and, hence, they can be easily expressed in terms of the copula $\mathbf{C}$ of X and Y:

1. X and Y are PQD if, and only if,

$$\mathbf{C}(u, v) \geq \mathbf{\Pi}_2(u, v)$$

for all $(u, v) \in \mathbb{I}^2$.

2. X and Y are NQD if, and only if

$$\mathbf{C}(u, v) \leq \mathbf{\Pi}_2(u, v)$$

for all $(u, v) \in \mathbb{I}^2$.

Intuitively, X and Y are PQD if the graph of their copula $\mathbf{C}$ lies on or above the graph of the independence copula $\mathbf{\Pi}_2$. By using Proposition 3.2 and Proposition 3.3, we can easily prove that, if (X, Y) is PQD, then $(-X, -Y)$ is PQD, and $(-X, Y)$ and $(X, -Y)$ are NQD. The PQD property of X and Y can also be characterized in terms of covariance between X and Y (if it exists), as shown in [176] — see also Definition 2.5, where the stronger notion of (positively) associated r.v.'s was introduced, and Theorem 5.5.

PROPOSITION B.1. *Two r.v.'s X and Y are PQD if, and only if, the covariance* $\mathbb{C}(f(X), g(Y)) \geq 0$ *for all increasing functions f and g for which the expectations* $\mathbb{E}(f(X))$, $\mathbb{E}(g(Y))$, *and* $\mathbb{E}(f(X)g(Y))$ *exist.*

The notion of *orthant dependence* was introduced in Definition 2.4 as a multi-dimensional generalization of that of quadrant dependence. In terms of d-variate copulas we may rewrite Eqs. (2.27)–(2.28) as, respectively,

$$\mathbf{C}(u_1, \ldots, u_d) \geq \mathbf{\Pi}_d(u_1, \ldots, u_d) = u_1 \cdots u_d \tag{B.3}$$

and

$$\widetilde{\mathbf{C}}(u_1, \ldots, u_d) \geq (1 - u_1) \cdots (1 - u_d) \tag{B.4}$$

for all $\boldsymbol{u} \in \mathbb{I}^d$, where $\widetilde{\mathbf{C}}$ denotes the d-dimensional joint survival function corresponding to $\mathbf{C}$ (see also Theorem 5.6).

B.1.2 Tail Monotonicity

The notion of *tail monotonicity* is important in applications, for risk assessment and reliability analysis.

DEFINITION B.2 (Tail monotonicity). *Let X and Y be a pair of continuous r.v.'s.*

1. *Y is left tail decreasing in X (briefly, LTD$(Y \mid X)$) if*

$$\mathbb{P}\{Y \leq y \mid X \leq x\} \quad \text{is decreasing in } x \tag{B.5}$$

for all y.

2. *X is left tail decreasing in Y (briefly, LTD$(X \mid Y)$) if*

$$\mathbb{P}\{X \leq x \mid Y \leq y\} \quad \text{is decreasing in } y \tag{B.6}$$

for all x.

3. *Y is right tail increasing in X (briefly, RTI$(Y \mid X)$) if*

$$\mathbb{P}\{Y > y \mid X > x\} \quad \text{is increasing in } x \tag{B.7}$$

for all y.

4. *X is right tail increasing in Y (briefly, RTI$(X \mid Y)$) if*

$$\mathbb{P}\{X > x \mid Y > y\} \quad \text{is increasing in } y \tag{B.8}$$

for all x.

Intuitively, LTD$(Y \mid X)$ means that Y is more likely to take on smaller values when X decreases. Analogously, RTI$(Y \mid X)$ means that Y is more likely to take on larger values when X increases. Note that, if X and Y satisfy each of the four properties of tail monotonicity, then X and Y are PQD.

In terms of the copula $\mathbf{C}$ of X and Y, the above properties have the following characterization:

1. LTD($Y \mid X$) if, and only if, for every $v \in \mathbb{I}$,

$$u \mapsto \frac{\mathbf{C}(u, v)}{u} \quad \text{is decreasing.}$$

2. LTD($X \mid Y$) if, and only if, for every $u \in \mathbb{I}$,

$$v \mapsto \frac{\mathbf{C}(u, v)}{v} \quad \text{is decreasing.}$$

3. RTI($Y \mid X$) if, and only if, for every $v \in \mathbb{I}$,

$$u \mapsto \frac{v - \mathbf{C}(u, v)}{1 - u} \quad \text{is decreasing.}$$

4. RTI($Y \mid X$) if, and only if, for every $u \in \mathbb{I}$,

$$v \mapsto \frac{u - \mathbf{C}(u, v)}{1 - v} \quad \text{is decreasing.}$$

In terms of partial derivatives of $\mathbf{C}$, the above conditions can be also expressed in the following forms:

1. LTD($Y \mid X$) if, and only if, for every $v \in \mathbb{I}$,

$$\frac{\partial \mathbf{C}(u, v)}{\partial u} \leq \frac{\mathbf{C}(u, v)}{u}$$

for almost all $u \in \mathbb{I}$.

2. LTD($X \mid Y$) if, and only if, for every $u \in \mathbb{I}$,

$$\frac{\partial \mathbf{C}(u, v)}{\partial v} \leq \frac{\mathbf{C}(u, v)}{v}$$

for almost all $v \in \mathbb{I}$.

3. RTI($Y \mid X$) if, and only if, for every $v \in \mathbb{I}$,

$$\frac{\partial \mathbf{C}(u, v)}{\partial u} \geq \frac{v - \mathbf{C}(u, v)}{1 - u}$$

for almost all $u \in \mathbb{I}$.

4. RTI($X \mid Y$) if, and only if, for every $u \in \mathbb{I}$,

$$\frac{\partial \mathbf{C}(u, v)}{\partial v} \geq \frac{u - \mathbf{C}(u, v)}{1 - v}$$

for almost all $v \in \mathbb{I}$.

B.1.3 Stochastic Monotonicity

The notion of *stochastic monotonicity* is important in applications, for risk assessment and reliability analysis.

DEFINITION B.3 (Stochastic monotonicity). *Let X and Y be a pair of continuous r.v.'s.*

1. *Y is stochastically increasing in X (briefly, SI$(Y \mid X)$) if, and only if,*

$$x \mapsto \mathbb{P}\{Y > y \mid X = x\} \tag{B.9}$$

   is increasing for all y.
2. *X is stochastically increasing in Y (briefly, SI$(X \mid Y)$) if, and only if,*

$$y \mapsto \mathbb{P}\{X > x \mid Y = y\} \tag{B.10}$$

   is increasing for all x.
3. *Y is stochastically decreasing in X (briefly, SD$(Y \mid X)$) if, and only if,*

$$x \mapsto \mathbb{P}\{Y > y \mid X = x\} \tag{B.11}$$

   is decreasing for all y.
4. *X is stochastically decreasing in Y (briefly, SD$(X \mid Y)$) if, and only if,*

$$y \mapsto \mathbb{P}\{X > x \mid Y = y\} \tag{B.12}$$

   is decreasing for all x.

Intuitively, SI$(Y \mid X)$ means that Y is more likely to take on larger values as X increases. In terms of the copula $\mathbf{C}$ of X and Y, the above properties have the following characterization:

1. SI$(Y \mid X)$ if, and only if, $u \mapsto \mathbf{C}(u, v)$ is concave for every $v \in \mathbb{I}$.
2. SI$(X \mid Y)$ if, and only if, $v \mapsto \mathbf{C}(u, v)$ is concave for every $u \in \mathbb{I}$.
3. SD(Y|X) if, and only if, $u \mapsto \mathbf{C}(u, v)$ is convex for every $v \in \mathbb{I}$.
4. SD(X|Y) if, and only if, $v \mapsto \mathbf{C}(u, v)$ is convex for every $u \in \mathbb{I}$.

Note that, if X and Y are r.v.'s such that SI$(Y \mid X)$, then LTD$(Y \mid X)$ and RTI$(Y \mid X)$ follow. Analogously, SI$(X \mid Y)$ implies LTD$(X \mid Y)$ and RTI$(X \mid Y)$. In particular, if X and Y are r.v.'s such that their joint distribution function H is a bivariate EV distribution, then SI$(Y \mid X)$ and SI$(X \mid Y)$ follow [106].

B.1.4 Corner Set Monotonicity

The notion of *corner set monotonicity* is important in applications, for risk assessment and reliability analysis.

DEFINITION B.4 (Corner set monotonicity). *Let X and Y be a pair of continuous r.v.'s.*

1. *X and Y are left corner set decreasing (briefly, LCSD(X, Y)) if, and only if, for all x and y,*

$$\mathbb{P}\{X \leq x, Y \leq y \mid X \leqslant x', Y \leq y'\} \tag{B.13}$$

is decreasing in x' and in y'.

2. *X and Y are left corner set increasing (briefly, LCSI(X, Y)) if, and only if, for all x and y,*

$$\mathbb{P}\{X \leq x, Y \leq y \mid X \leq x', Y \leq y'\} \tag{B.14}$$

is increasing in x' and in y'.

3. *X and Y are right corner set increasing (briefly, RCSI(X, Y)) if, and only if, for all x and y,*

$$\mathbb{P}\{X > x, Y > y \mid X > x', Y > y'\} \tag{B.15}$$

is increasing in x' and in y'.

4. *X and Y are right corner set decreasing (briefly, RCSD(X, Y)) if, and only if, for all x and y,*

$$\mathbb{P}\{X > x, Y > y \mid X > x', Y > y'\} \tag{B.16}$$

is decreasing in x' and in y'.

Note that, if LCSD(X, Y), then LTD$(Y \mid X)$ and LTD$(X \mid Y)$ follow. Analogously, if RCSI(X, Y), then RTI$(Y \mid X)$ and RTI$(X \mid Y)$ follow. In terms of the copula $\mathbf{C}$ of X and Y, and of the corresponding survival copula $\overline{\mathbf{C}}$, these properties have the following characterization:

1. LCSD(X, Y) if, and only if, $\mathbf{C}$ is TP_2, i.e. for every u, u', v, v' in $\mathbb{I}$, $u \leq u'$, $v \leq v'$,

$$\mathbf{C}(u, v)\,\mathbf{C}(u', v') \geq \mathbf{C}(u, v')\,\mathbf{C}(u', v)\,;$$

2. RCSI(X, Y) if, and only if, $\overline{\mathbf{C}}$ is TP_2, i.e. for every u, u', v, v' in $\mathbb{I}$, $u \leq u'$, $v \leq v'$,

$$\overline{\mathbf{C}}(u, v)\,\overline{\mathbf{C}}(u', v') \geq \overline{\mathbf{C}}(u, v')\,\overline{\mathbf{C}}(u', v)\,.$$

In Table B.1 we summarize the relationships between the positive dependence concepts illustrated above [207].

Table B.1. Relationships between positive dependence concepts

$\mathrm{SI}(Y \mid X)$	$\Longrightarrow$	$\mathrm{RTI}(Y \mid X)$	$\Longleftarrow$	$\mathrm{RCSI}(X, Y)$
$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathrm{LTD}(Y \mid X)$	$\Longrightarrow$	$\mathrm{PQD}(X, Y)$	$\Longleftarrow$	$\mathrm{RTI}(X \mid Y)$
$\Uparrow$		$\Uparrow$		$\Uparrow$
$\mathrm{LCSD}(X, Y)$	$\Longrightarrow$	$\mathrm{LTD}(X \mid Y)$	$\Longleftarrow$	$\mathrm{SI}(X \mid Y)$

B.1.5 Dependence Orderings

After the introduction of some dependence concepts, it is natural to ask whether one bivariate distribution function is more dependent than another, according to some prescribed dependence concept. Comparisons of this type are made by introducing a partial ordering in the set of all bivariate distribution functions having the same marginals (and, hence, involving the concept of copula). The most common dependence ordering is the *concordance ordering* (also called *PQD ordering*).

DEFINITION B.5 (Concordance ordering). *Let H and H' be continuous bivariate distribution functions with copulas $\mathbf{C}$ and $\mathbf{C}'$, respectively, and the same marginals F and G. H' is said to be more concordant (or more PQD) than H if*

$$\forall (x, y) \in \mathbb{R}^2 \qquad H(x, y) \le H'(x, y), \tag{B.17a}$$

or, equivalently,

$$\forall (u, v) \in \mathbb{I}^2 \qquad \mathbf{C}(u, v) \le \mathbf{C}'(u, v). \tag{B.17b}$$

If H' is more concordant than H, we write $H \prec_{\mathbf{C}} H'$ (or simply $H \prec H'$).

If we consider a family of copulas $\{\mathbf{C}_\theta\}$, indexed by a parameter θ belonging to an interval of $\mathbb{R}$, we say that $\mathbf{C}_\theta$ is *positively ordered* if $\theta_1 \le \theta_2$ implies $\mathbf{C}_{\theta_1} \prec \mathbf{C}_{\theta_2}$ in the concordance ordering.

A dependence ordering related to LTD and RTI concepts is given in [7]. An ordering based on SI concepts is presented in [155], where an ordering based on the TP_2 notion is also discussed.

B.1.6 Measure of Dependence

There are several ways to discuss and measure the dependence between random variables. Intuitively, a measure of dependence indicates how closely two r.v.'s X and Y are related, with extremes at mutual independence and (monotone) dependence. Most importantly, some of these measures are *scale-invariant*, i.e. they remain unchanged under strictly increasing transformations of the variables of interest. Thus, from Proposition 3.2, they are expressible in terms of the copula linking these variables [265].

Practitioners should primarily consider dependence measures that depend only upon the copula of the underlying random vector. Unfortunately, this is not true for the often used Pearson's linear correlation coefficient ρ_P, that strongly depends upon the marginal laws (especially outside the framework of elliptically contoured distributions — see Section C.11; for a discussion about ρ_P see [85]).

In 1959, A. Rényi [231] proposed the following set of axioms for a *measure of dependence*. Here we outline a slightly modified version of them.

DEFINITION B.6 (Measure of dependence). *A numerical measure δ between two continuous r.v.'s X and Y with copula $\mathbf{C}$ is a measure of dependence if it satisfies the following properties:*

1. *δ is defined for every pair (X, Y) of continuous r.v.'s;*
2. $0 \leq \delta_{X,Y} \leq 1$*;*
3. $\delta_{X,Y} = \delta_{Y,X}$*;*
4. $\delta_{X,Y} = 0$ *if, and only if, X and Y are independent;*
5. $\delta_{X,Y} = 1$ *if, and only if, each of X and Y is almost surely a strictly monotone function of the other;*
6. *if α and β are almost surely strictly monotone functions on* $\operatorname{Ran} X$ *and* $\operatorname{Ran} Y$, *respectively, then* $\delta_{\alpha(X),\beta(Y)} = \delta_{X,Y}$*;*
7. *if $\{(X_n, Y_n)\}$, $n \in \mathbb{N}$, is a sequence of continuous r.v.'s with copula $\mathbf{C}_n$, and if $\{\mathbf{C}_n\}$ converges pointwise to a copula $\mathbf{C}$, then*

$$\lim_{n\to\infty} \delta_{X_n,Y_n} = \delta_{X,Y}.$$

Note that, in the above sense, Pearson's linear correlation coefficient ρ_P is not a measure of dependence: in fact, $\rho_P(X, Y) = 0$ does not imply that X and Y are independent. A measure of dependence is given, instead, by the *maximal correlation coefficient* ρ_P^* defined by:

$$\rho_P^* = \sup_{f,g} \rho_P(f(X), g(Y)), \tag{B.18}$$

where the supremum is taken over all Borel functions f and g for which the correlation $\rho_P(f(X), g(Y))$ is well defined. However, this measure is too often equal to one, and cannot be effectively calculated.

Another example is given by the *Schweizer-Wolff measure of dependence* [265] defined, for continuous r.v.'s X and Y with copula $\mathbf{C}$, by

$$\delta_{\mathbf{C}} = 12 \iint_{\mathbb{I}^2} \left| \mathbf{C}(u, v) - uv \right| \mathrm{d}u \, \mathrm{d}v. \tag{B.19}$$

We anticipate here that, if X and Y are PQD, then $\delta_{\mathbf{C}} = \rho_S$, and if X and Y are NQD, then $\delta_{\mathbf{C}} = -\rho_S$, where ρ_S denotes the Spearman's coefficient (see Subsection B.2.3). More details can be found in [207].

B.2. MEASURES OF ASSOCIATION

A numerical *measure of association* is a statistical summary of the degree of relationship between variables. For the ease of comparison, coefficients of association are usually constructed to vary between -1 and $+1$. Their values increase as the strength of the relationship increases, with a $+1$ (or -1) value when there is perfect positive (or negative) association. Each coefficient of association measures a special type of relationship: for instance, Pearson's product-moment correlation coefficient ρ_P measures the amount of linear relationship.

The most widely known (and used), scale-invariant, measures of association are the *Kendall's* τ_K and the *Spearman's* ρ_S, both of which measure a form of dependence known as *concordance*. These two measures also play an important role in applications, since the practical fit of a copula to the available data is often carried out via the estimate of τ_K or ρ_S (see Chapter 3 and Appendix C). Note that both τ_K and ρ_S always exist (being based on the ranks), whereas the existence of other standard measures (such as ρ_P) may depend upon that of the second-order moments of the variables of interest — and is not guaranteed, e.g., for heavy tailed r.v.'s (see, e.g., the discussion in [67, 255]).

B.2.1 Measures of Concordance

Roughly speaking, two r.v.'s are *concordant* if small values of one are likely to be associated with small values of the other, and large values of one are likely to be associated with large values of the other.

More precisely, let (x_i, y_i) and (x_j, y_j) be two observations from a vector (X, Y) of continuous r.v.'s. Then, (x_i, y_i) and (x_j, y_j) are concordant if

$$(x_i - x_j)(y_i - y_j) > 0, \tag{B.20a}$$

and discordant if

$$(x_i - x_j)(y_i - y_j) < 0. \tag{B.20b}$$

A mathematical definition of a *measure of concordance* is as follows [258].

DEFINITION B.7 (Measure of concordance). *A numeric measure of association κ between two continuous r.v.'s X and Y with copula $\mathbf{C}$ is a measure of concordance if it satisfies the following properties:*

1. *κ is defined for every pair (X, Y) of continuous r.v.'s;*
2. *$-1 \le \kappa_{X,Y} \le 1$, $\kappa_{X,X} = 1$, and $\kappa_{X,-X} = -1$;*
3. *$\kappa_{X,Y} = \kappa_{Y,X}$;*
4. *if X and Y are independent then $\kappa_{X,Y} = 0$;*
5. *$\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$;*

6. *if* (X_1, Y_1) *and* (X_2, Y_2) *are random vectors with copulas* $\mathbf{C}_1$ *and* $\mathbf{C}_2$*, respectively, such that* $\mathbf{C}_1 \prec \mathbf{C}_2$*, then* $\kappa_{X_1,Y_1} \leq \kappa_{X_2,Y_2}$*;*
7. *if* $\{(X_n, Y_n)\}$*,* $n \in \mathbb{N}$*, is a sequence of continuous r.v.'s with copula* $\mathbf{C}_n$*, and if* $\{\mathbf{C}_n\}$ *converges pointwise to a copula* $\mathbf{C}$*, then*

$$\lim_{n\to\infty} \kappa_{\mathbf{C}_n} = \kappa_{\mathbf{C}}.$$

We anticipate here that, as a consequence of the above definition, both Kendall's τ_K and Spearman's ρ_S turn out to be measures of concordance. We need now to introduce the *concordance function*.

DEFINITION B.8 (Concordance function). *Let* $(X_1, Y_1), (X_2, Y_2)$ *be independent vectors of continuous r.v.'s with marginals* F_X, F_Y *and copulas* $\mathbf{C}_1, \mathbf{C}_2$*. The difference*

$$Q = \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \tag{B.21}$$

defines the concordance function Q*.*

The important point is that Q depends upon the distributions of (X_1, Y_1) and (X_2, Y_2) only through their copulas $\mathbf{C}_1$ and $\mathbf{C}_2$. In fact, it can be shown that

$$Q = Q(\mathbf{C}_1, \mathbf{C}_2) = 4 \iint_{\mathbb{I}^2} \mathbf{C}_2(u, v)\, \mathrm{d}\mathbf{C}_1(u, v) - 1. \tag{B.22}$$

Note that Q is symmetric in its arguments [207].

B.2.2 Kendall's τ_K

The population version of Kendall's τ_K [160, 170, 239, 207] is defined as the difference between the probability of concordance and the probability of discordance.

DEFINITION B.9 (Kendall's τ_K). *Let* (X_1, Y_1) *and* (X_2, Y_2) *be i.i.d. vectors of continuous r.v.'s. The difference*

$$\tau_K = \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \tag{B.23}$$

defines the population version of Kendall's τ_K*.*

Evidently, if $\mathbf{C}$ is the copula of X and Y, then

$$\tau_K^{X,Y} = \tau_K^{\mathbf{C}} = Q(\mathbf{C}, \mathbf{C}) = 4 \iint_{\mathbb{I}^2} \mathbf{C}(u, v)\, \mathrm{d}\mathbf{C}(u, v) - 1. \tag{B.24}$$

Note that the integral defining Q corresponds to the expected value of the r.v. $W = \mathbf{C}(U, V)$ introduced in Proposition 3.5, i.e.

$$\tau_K^{\mathbf{C}} = 4\,\mathbb{E}\left(\mathbf{C}(U, V)\right) - 1. \tag{B.25}$$

In particular, in the Archimedean case, Kendall's τ_K can be expressed [207] as a function of the generator γ of $\mathbf{C}$:

$$\tau_K^{\mathbf{C}} = 1 + 4\int_0^1 \frac{\gamma(t)}{\gamma'(t)}\,\mathrm{d}t. \tag{B.26}$$

Note that, for a copula $\mathbf{C}$ with a singular component, the expression in Eq. (B.24) for $\tau_K^{\mathbf{C}}$ may be difficult to calculate, and can be replaced by the following formula:

$$\tau_K^{\mathbf{C}} = 1 - 4\iint_{\mathbb{I}^2}\left(\frac{\partial \mathbf{C}(u, v)}{\partial u}\right)\cdot\left(\frac{\partial \mathbf{C}(u, v)}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v. \tag{B.27}$$

Note that, if X and Y are PQD, then $\tau_K^{X,Y} \geq 0$, and, analogously, if X and Y are NQD, then $\tau_K^{X,Y} \leq 0$. In particular, τ_K is increasing with respect to the concordance ordering introduced in Definition B.5. Possible multivariate extensions of τ_K are discussed in [152, 204].

The sample version t of Kendall's τ_K is easy to calculate:

$$t = \frac{c-d}{c+d}, \tag{B.28}$$

where c (d) represent the number of concordant (discordant) pairs in a sample of size n from a vector of continuous r.v.'s (X, Y). Unfortunately, in applications it often happens that continuous variables are "discretely" sampled, due to a finite instrumental resolution. For instance, the rainfall depth could be returned as an integer multiple of 0.1 mm, or the storm duration could be expressed in hours and rounded to an integer value (see, e.g., [66, 253]). Clearly, this procedure introduces repetitions in the observed values (called *ties* in statistics), which may adversely affect the estimation of τ_K. However, corrections to Eq. (B.28) are specified for solving the problem (see, e.g., the formulas in [223]).

ILLUSTRATION B.1 (Storm intensity–duration (cont.)). ►

In [255] a Frank's 2-copula is used to model the relationship between the (average) *storm intensity* I and the (wet) *storm duration* W — see also Illustration 4.4. The Authors carry out a seasonal analysis, and investigate the "strength" of the association between the r.v.'s I and W. In particular, they consider a sequence of increasing thresholds $\nu_1 < \cdots < \nu_n$ of the *storm volume* $V = IW$, and estimate the values of Kendall's τ_K for all those pairs (I, W) satisfying $IW > \nu_i$, $i = 1, \ldots, n$. In Figure B.1 we plot the results obtained; for the sake of comparison, the corresponding values of Pearson's linear correlation coefficient ρ_P are also shown.

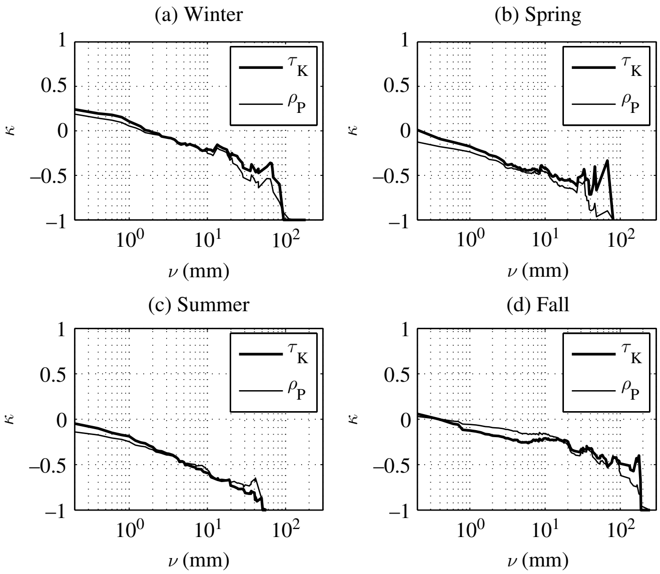

Figure B.1. Plot of the measures of association τ_K and ρ_P as a function of the volume threshold ν by considering the pairs (I, W) for which $V = IW > \nu$, for the data analysed by [255] in each season

Figure B.1 deserves several comments. The analysis of the behavior of τ_K is distribution-free, since there is no need to estimate the marginal laws of I and W. On the contrary, these marginals must be calculated before estimating ρ_P, since the existence of Pearson's linear correlation coefficient may depend upon such distributions, and must be proved in advance.

Except for very small values of the storm volume V (at most, $\nu = 2$ mm in Winter), I and W are always negatively associated, and such a link becomes stronger considering more and more extreme storms (i.e., for larger and larger values of V): apparently, the rate of "increase" of the association strength towards the limit value -1 is logarithmic. Note that the points in the upper tail are scattered simply because in that region ($\nu \gg 1$) very few storms are present, which affects the corresponding statistical analysis.

Overall, a negative association between I and W has to be expected (see, e.g., the discussion in [66]), for in the real world we usually observe that the strongest intensities are associated with the shortest durations, and the longest durations with the weakest intensities. ◄

B.2.3 Spearman's ρ_S

As with Kendall's τ_K, also the population version of Spearman's ρ_S [170, 239, 207] is based on concordance and discordance.

DEFINITION B.10 (Spearman's ρ_S). *Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent random vectors with a common joint continuous distribution. The difference*

$$\rho_S = 3\left(\mathbb{P}\{(X_1 - X_2)(Y_1 - Y_3) > 0\} - \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_3) < 0\}\right) \tag{B.29}$$

defines the population version of Spearman's ρ_S.

Evidently, if $\mathbf{C}$ is the copula of (X, Y), then $(X_1, Y_1) \sim \mathbf{C}$, but $(X_2, Y_3) \sim \mathbf{\Pi}_2$, since X_2 and Y_3 are independent. As a consequence,

$$\rho_S^{X,Y} = \rho_S^{\mathbf{C}} = 3\,Q(\mathbf{C}, \mathbf{\Pi}_2). \tag{B.30}$$

Also Spearman's ρ_S can be written in terms of a suitable expectation:

$$\rho_S^{\mathbf{C}} = 12 \iint_{\mathbb{I}^2} uv \,\mathrm{d}\mathbf{C}(u, v) - 3 = 12\,\mathbb{E}(UV) - 3. \tag{B.31}$$

A practical interpretation of Spearman's ρ_S arises from rewriting the above formula as

$$\rho_S^{\mathbf{C}} = 12 \iint_{\mathbb{I}^2} [\mathbf{C}(u, v) - uv]\,\mathrm{d}u\,\mathrm{d}v. \tag{B.32}$$

Thus, $\rho_S^{\mathbf{C}}$ is proportional to the signed volume between the graphs of $\mathbf{C}$ and the independence copula $\mathbf{\Pi}_2$. Roughly, $\rho_S^{\mathbf{C}}$ measures the "average distance" between the joint distribution of X and Y (as represented by $\mathbf{C}$), and independence (given by $\mathbf{\Pi}_2$). Note that, if X and Y are PQD, then $\rho_S^{X,Y} \geq 0$, and, analogously, if X and Y are NQD, then $\rho_S^{X,Y} \leq 0$. In particular, ρ_S is increasing with respect to the concordance ordering introduced in Definition B.5. Possible multivariate extensions of ρ_S are discussed in [152, 204].

The sample version r of Spearman's ρ_S is easy to calculate:

$$r = 1 - \frac{6\sum_{i=1}^{n}(R_i - S_i)^2}{n^3 - n}, \tag{B.33}$$

where $R_i = \mathrm{Rank}(x_i)$, $S_i = \mathrm{Rank}(y_i)$, and n is the sample size. As already mentioned before, instrumental limitations may adversely affect the estimation of ρ_S in practice due to the presence of ties. However, corrections to Eq. (B.33) are specified for solving the problem (see, e.g., the formulas in [223]).

ILLUSTRATION B.2 (Storm intensity–duration (cont.)). ►

As discussed in Illustration B.1, in [255] a Frank's 2-copula is used to model the relation between the (average) *storm intensity I* and the (wet) *storm duration*

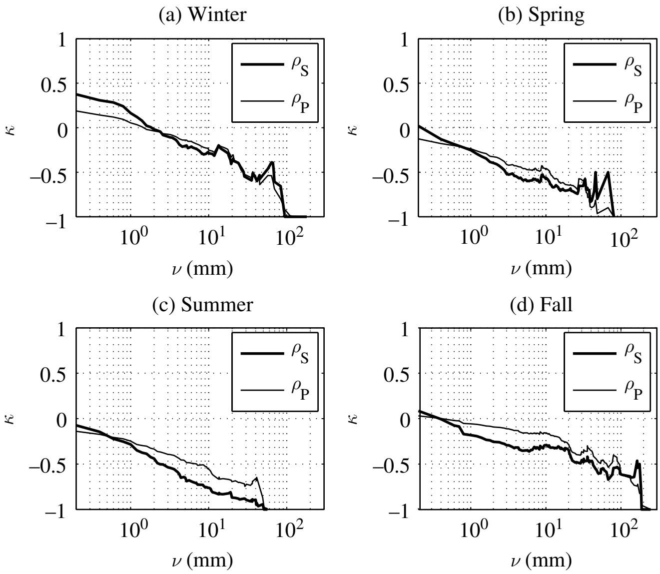

Figure B.2. Plot of the measures of association ρ_S and ρ_P as a function of the volume threshold ν by considering the pairs (I, W) for which $V = IW > \nu$, for the data analysed by [255] in each season

W. The Authors estimate the values of Spearman's ρ_S for all those pairs (I, W) satisfying $IW > \nu_i$, $i = 1, \ldots, n$, by considering a sequence of increasing thresholds $\nu_1 < \cdots < \nu_n$ of the *storm volume* $V = IW$. In Figure B.2 we plot the results obtained; for the sake of comparison, the corresponding values of Pearson's linear correlation coefficient ρ_P are also shown.

The same comments as in Illustration B.1 hold in the present case. ◀

APPENDIX C

FAMILIES OF COPULAS

(Written by Fabrizio Durante — Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz (Austria))

Research on copulas seems to generate new formulas endlessly. A full list of well known copulas includes several types. It is not feasible to include all of these in this book. Instead, we summarize a few families that find numerous applications in practice. For a more extensive list see [145, 155, 207], where additional mathematical properties are also found.

C.1. THE FRANK FAMILY

One of the possible equivalent expressions for members of this family is

$$\mathbf{C}_\theta(u, v) = \frac{1}{\ln\theta} \ln\left[1 + \frac{(\theta^u - 1)(\theta^v - 1)}{\theta - 1}\right], \tag{C.1}$$

where $u, v \in \mathbb{I}$, and $\theta \geq 0$ is a dependence parameter [99, 203, 108].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are PQD for $0 \leq \theta < 1$, and NQD for $\theta > 1$. The limiting case $\theta = 1$ occurs when U and V are independent r.v.'s, i.e. $\mathbf{C}_1 = \mathbf{\Pi}_2$. Moreover, $\mathbf{C}_0 = \mathbf{M}_2$ and $\mathbf{C}_\infty = \mathbf{W}_2$, and thus the Frank family is comprehensive. Also, since $\theta_1 \leq \theta_2$ implies $\mathbf{C}_{\theta_1} \prec \mathbf{C}_{\theta_2}$, this family is positively ordered. Further mathematical properties can be found in [207].

Every copula of the Frank family is absolutely continuous, and its density has a simple expression given by

$$\mathbf{c}_\theta(u, v) = -\frac{(\theta - 1)\theta^{u+v}\ln\theta}{\theta^{u+v} - \theta^u - \theta^v + \theta}. \tag{C.2}$$

Copulas belonging to the Frank family are strict Archimedean. The expression of the generator is, for $t \in \mathbb{I}$,

$$\gamma(t) = -\ln\left(\frac{\theta^t - 1}{\theta - 1}\right). \tag{C.3}$$

These 2-copulas are the only Archimedean ones that satisfy the functional equation $\mathbf{C} = \overline{\mathbf{C}}$ for radial symmetry.

Two useful relationships exist between θ and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\theta) = 1 - 4\,\frac{D_1(-\ln\theta) - 1}{\ln\theta}, \tag{C.4a}$$

$$\rho_S(\theta) = 1 - 12\,\frac{D_2(-\ln\theta) - D_1(-\ln\theta)}{\ln\theta}, \tag{C.4b}$$

where D_1 and D_2 are, respectively, the *Debye* functions of order 1 and 2 [181]. For a discussion on the estimate of θ using τ_K and ρ_S see [108].

The lower and upper tail dependence coefficients for the members of this family are equal to 0.

As an illustration, we plot the Frank 2-copula and the corresponding level curves in Figures C.1–C.3, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.1 $\mathbf{C}$ approximates $\mathbf{M}_2$, for $\theta \approx 0$, while in Figure C.2 $\mathbf{C}$ approximates $\mathbf{W}_2$, for $\theta \gg 1$. In Figure C.3 we show one of the Frank 2-copulas used in [66]: here a negative association is modeled.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Frank 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

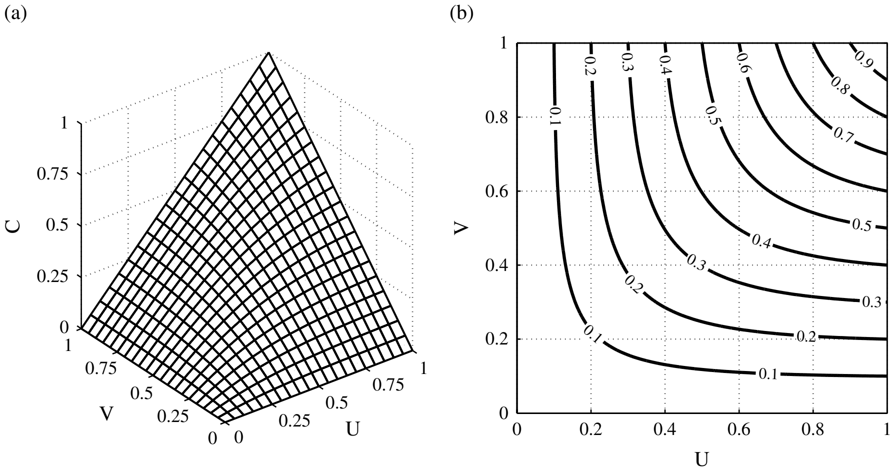

Figure C.1. The Frank 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.02$. The probability levels are as indicated

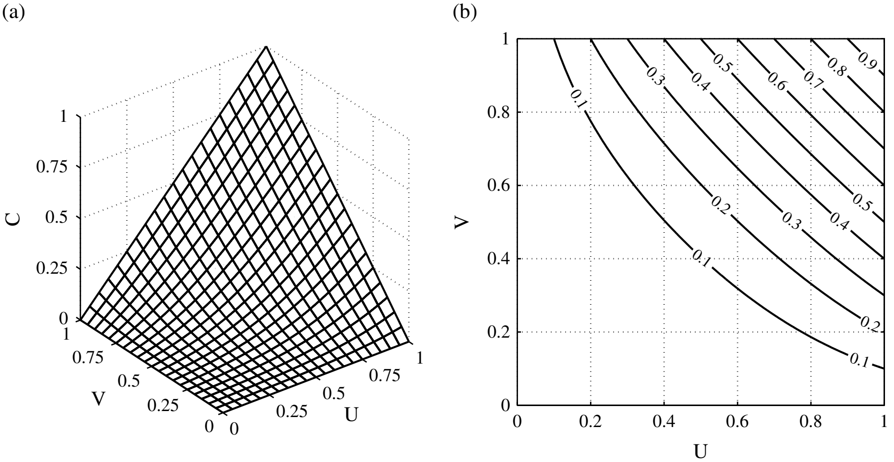

Figure C.2. The Frank 2-copula and the corresponding level curves. Here the parameter is $\theta = 50$. The probability levels are as indicated

Due to Proposition 4.1, the Frank family can be extended to the d-dimensional case, $d \geq 3$, if we restrict the range of the parameter θ to the interval $(0, 1)$, where γ^{-1} is completely monotonic. Its generalization is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = \frac{1}{\ln \theta} \ln \left[1 + \frac{(\theta^{u_1} - 1) \cdots (\theta^{u_d} - 1)}{(\theta - 1)^{d-1}} \right], \tag{C.5}$$

where $\theta \in (0, 1)$.

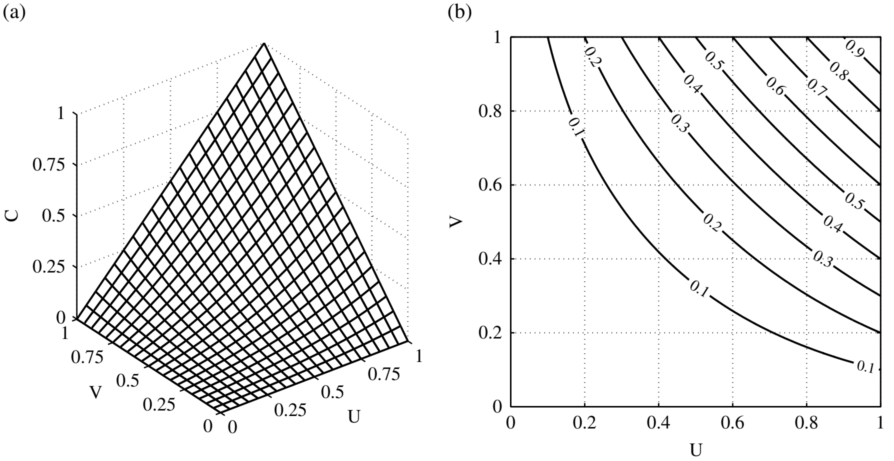

Figure C.3. The Frank 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 12.1825$, as used in [66]. The probability levels are as indicated

C.2. THE GUMBEL-HOUGAARD FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u,v) = e^{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}}, \tag{C.6}$$

where $u, v \in \mathbb{I}$, and $\theta \ge 1$ is a dependence parameter [207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 1$, i.e. $\mathbf{C}_1 = \mathbf{\Pi}_2$, and PQD for $\theta > 1$. In particular, this family is positively ordered, with $\mathbf{C}_\infty = \mathbf{M}_2$, and its members are absolutely continuous.

Copulas belonging to the Gumbel-Hougaard family are strict Archimedean. The expression of the generator is, for $t \in \mathbb{I}$,

$$\gamma(t) = (-\ln t)^\theta. \tag{C.7}$$

The following relationship exists between θ and Kendall's τ_K:

$$\tau_K(\theta) = \frac{\theta - 1}{\theta}, \tag{C.8}$$

which may provide a way to fit a Gumbel-Hougaard 2-copula to the available data.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 0$ and $\lambda_U = 2 - 2^{1/\theta}$.

As an illustration, we plot the Gumbel-Hougaard 2-copula and the corresponding level curves in Figures C.4–C.5, for different values of θ.

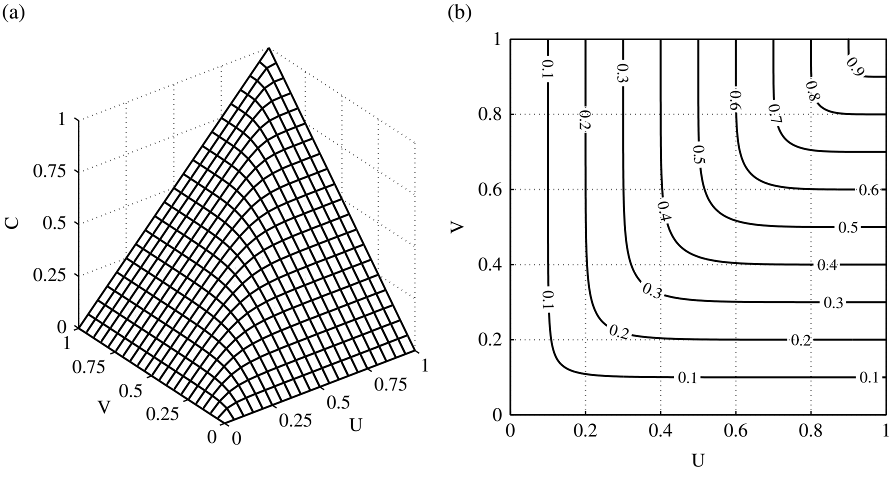

Figure C.4. The Gumbel-Hougaard 2-copula and the corresponding level curves. Here the parameter is $\theta = 5$. The probability levels are as indicated

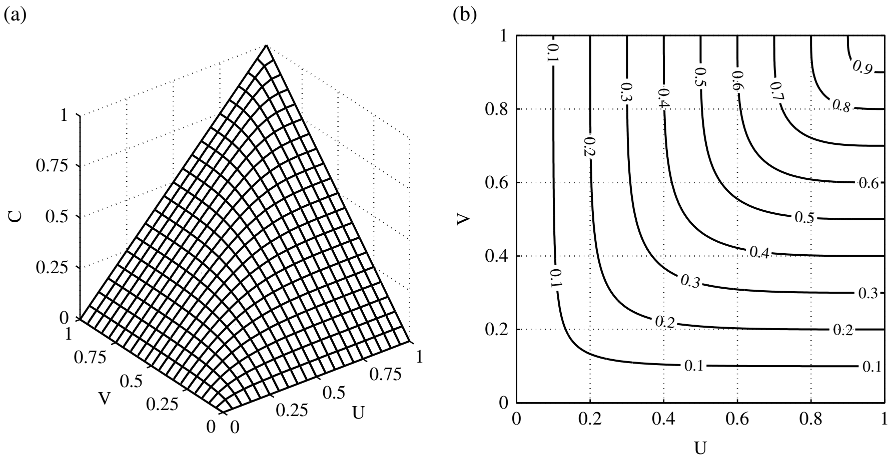

Figure C.5. The Gumbel-Hougaard 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 3.055$, as used in [67]. The probability levels are as indicated

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.4 $\mathbf{C}$ approximates $\mathbf{M}_2$, for sufficiently large θ. In Figure C.5 we show the Gumbel-Hougaard 2-copula used in [67]: here a weaker association is modeled.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Gumbel-Hougaard 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

Due to Proposition 4.1, the Gumbel-Hougaard family can be extended to the d-dimensional case, $d \geq 3$, and its expression is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = e^{-\left[(-\ln u_1)^\theta + \cdots + (-\ln u_d)^\theta\right]^{1/\theta}}. \tag{C.9}$$

Every member of this class is a MEV copula. As shown in [115], if H is a MEV distribution whose copula $\mathbf{C}$ is Archimedean, then $\mathbf{C}$ belongs to the Gumbel-Hougaard family (see Illustration 5.1).

C.3. THE CLAYTON FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = \left(\max\left\{(u^{-\theta} + v^{-\theta} - 1), 0\right\}\right)^{-1/\theta}, \tag{C.10}$$

where $u, v \in \mathbb{I}$, and $\theta \geq -1$ is a dependence parameter [207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are PQD for $\theta > 0$, and NQD for $-1 \leq \theta < 0$. The limiting case $\theta = 0$ corresponds to the independent case, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$. In particular, this family is positively ordered, and its members are absolutely continuous for $\theta > 0$.

The Clayton family is the only "truncation invariant family" [211], in the sense that, if U and V are r.v.'s with copula $\mathbf{C}$, then, given $u_0, v_0 \in (0, 1)$, the copula of the conditional r.v.'s U, given that $U \leq u_0$, and V, given that $V \leq v_0$, is again $\mathbf{C}$.

Copulas belonging to the Clayton family are Archimedean, and they are strict when $\theta > 0$. The expression of the generator is, for $t \in \mathbb{I}$,

$$\gamma(t) = \frac{1}{\theta}(t^{-\theta} - 1). \tag{C.11}$$

Using Eq. (B.26), it is possible to derive the following relationship between θ and Kendall's τ_K:

$$\tau_K(\theta) = \frac{\theta}{\theta + 2}, \tag{C.12}$$

which may provide a way to fit a Clayton 2-copula to the available data.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 2^{-1/\theta}$ and $\lambda_U = 0$, for $\theta \geq 0$.

As an illustration, we plot the Clayton 2-copula and the corresponding level curves in Figures C.6–C.8, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.6 $\mathbf{C}$ approximates $\mathbf{M}_2$, for sufficiently large θ, while in Figure C.7 $\mathbf{C}$ approximates $\mathbf{W}_2$, for sufficiently small θ. In Figure C.8 we show the Clayton 2-copula used in [109]: here a weak positive association is modeled.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Clayton 2-copula $\mathbf{C}_\theta$, is as follows:

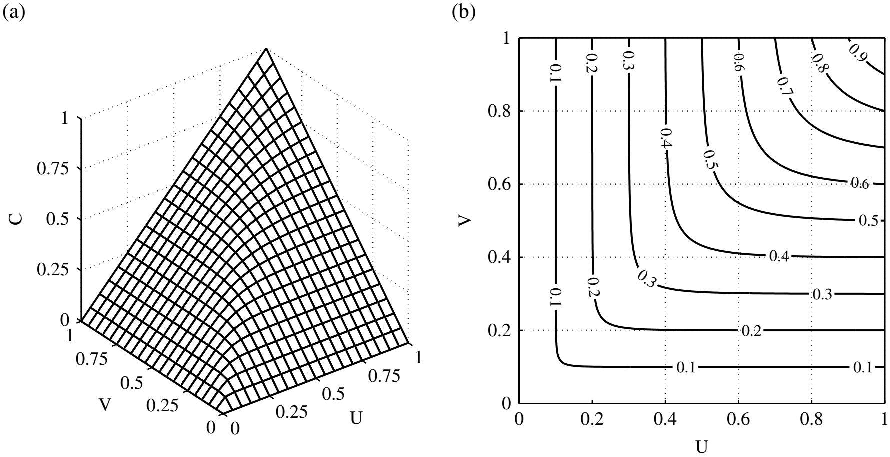

Figure C.6. The Clayton 2-copula and the corresponding level curves. Here the parameter is $\theta = 5$. The probability levels are as indicated

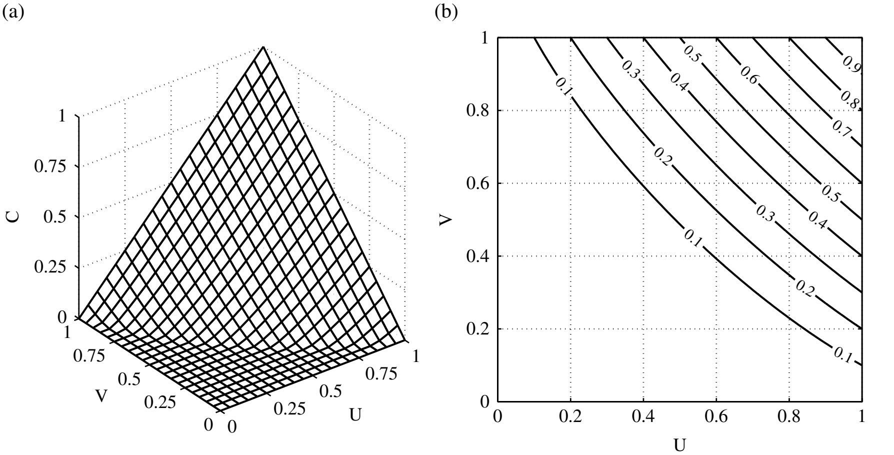

Figure C.7. The Clayton 2-copula and the corresponding level curves. Here the parameter is $\theta = -0.75$. The probability levels are as indicated

1. Generate independent variates x and y with a standard Exponential distribution.
2. Generate a variate z, independent of x and y, with Gamma distribution $\Gamma(\theta, 1)$.
3. Set $u = [1 + (x/z)]^{-\theta}$ and $v = [1 + (y/z)]^{-\theta}$.

The desired pair is then (u, v). For more details, see [71].

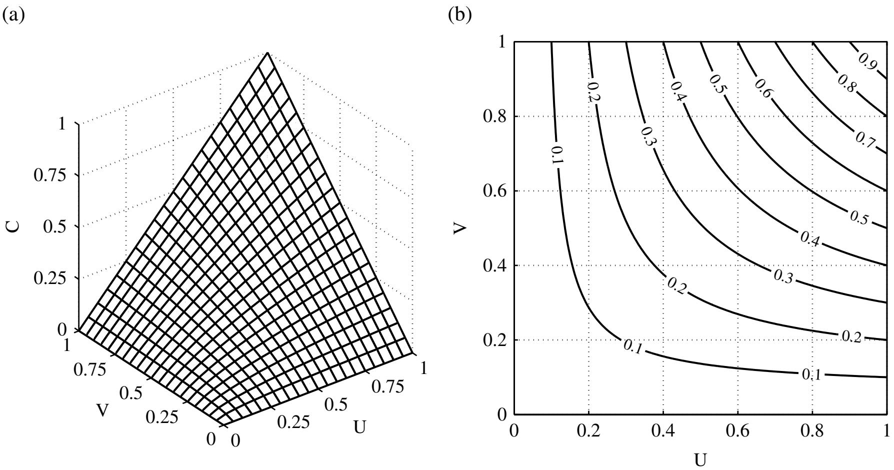

Figure C.8. The Clayton 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 0.449$, as used in [109]. The probability levels are as indicated

Due to Proposition 4.1, the Clayton family can be extended to the d-dimensional case, $d \geq 3$, if we consider the parameter range $\theta > 0$, where γ^{-1} is completely monotonic. Its generalization is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = (u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1)^{-1/\theta}, \tag{C.13}$$

where $\theta > 0$.

C.4. THE ALI-MIKHAIL-HAQ (AMH) FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}, \tag{C.14}$$

where $u, v \in \mathbb{I}$, and $-1 \leq \theta \leq 1$ is a dependence parameter [207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, PQD for $\theta > 0$, and NQD for $\theta < 0$. In particular, this family is positively ordered.

If $\theta = 1$, then $\mathbf{C}_1(u, v) = uv/(u + v - uv)$: this copula belongs to many families of Archimedean copulas, as noted in [206]. For instance, it belongs to the Clayton family by taking $\theta = 1$ in Eq. (C.10).

The harmonic mean of two Ali-Mikhail-Haq 2-copulas is again an Ali-Mikhail-Haq 2-copula, i.e. if $\mathbf{C}_{\theta_1}$ and $\mathbf{C}_{\theta_2}$ are given by Eq. (C.14), then their harmonic mean is $\mathbf{C}_{(\theta_1+\theta_2)/2}$. In addition, each Ali-Mikhail-Haq 2-copula can be written as a weighted harmonic mean of the two extreme members of the family, i.e.

$$\mathbf{C}_\theta(u, v) = \frac{1}{\frac{1-\theta}{2}\frac{1}{\mathbf{C}_{-1}(u,v)} + \frac{1+\theta}{2}\frac{1}{\mathbf{C}_1(u,v)}} \tag{C.15}$$

for all $\theta \in [-1, 1]$.

Copulas belonging to the Ali-Mikhail-Haq family are strict Archimedean. The expression of the generator is, for $t \in \mathbb{I}$,

$$\gamma(t) = \ln \frac{1 - \theta(1-t)}{t}. \tag{C.16}$$

A useful relationship exists between θ and Kendall's τ_K:

$$\tau_K(\theta) = 1 - \frac{2(\theta + (1-\theta)^2 \ln(1-\theta))}{3\theta^2}, \tag{C.17}$$

which may provide a way to fit a Ali-Mikhail-Haq 2-copula to the available data. Copulas of this family show a limited range of dependence, which restricts their use in applications: in fact, Kendall's τ_K only ranges from ≈ -0.1817 to $1/3$, as θ goes from -1 to 1.

The lower and upper tail dependence coefficients for the members of this family are equal to 0, for $\theta \in (-1, 1)$.

As an illustration, we plot the Ali-Mikhail-Haq 2-copula and the corresponding level curves in Figures C.9–C.11, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.9 $\mathbf{C}$ fails to approximate $\mathbf{M}_2$ sufficiently, although $\theta \approx 1$, and how in Figure C.10 $\mathbf{C}$ fails to approximate $\mathbf{W}_2$ sufficiently, although $\theta \approx -1$. In Figure C.11 we show the Ali-Mikhail-Haq 2-copula used in [68]: here a positive association is modeled.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having an Ali-Mikhail-Haq 2-copula $\mathbf{C}_\theta$, is as follows:

1. Generate independent variates u, t Uniform on $\mathbb{I}$.
2. Set $a = 1 - u$.
3. Set $b = -\theta(2at + 1) + 2\theta^2 a^2 t + 1$ and $c = \theta^2(4a^2t - 4at + 1) + \theta(4at - 4a + 2) + 1$.
4. Set $v = \frac{2t(a\theta-1)^2}{b+\sqrt{c}}$.

The desired pair is then (u, v). For more details, see [156].

Due to Proposition 4.1, this family can be extended for $d \geq 3$, if we consider the parameter range $\theta > 0$, where γ^{-1} is completely monotonic. Then, the generalization of the Ali-Mikhail-Haq family in d-dimensions is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = \frac{(1-\theta)\prod_{i=1}^d u_i}{\prod_{i=1}^d [1 - \theta(1-u_i)] - \theta \prod_{i=1}^d u_i}, \tag{C.18}$$

where $\theta > 0$.

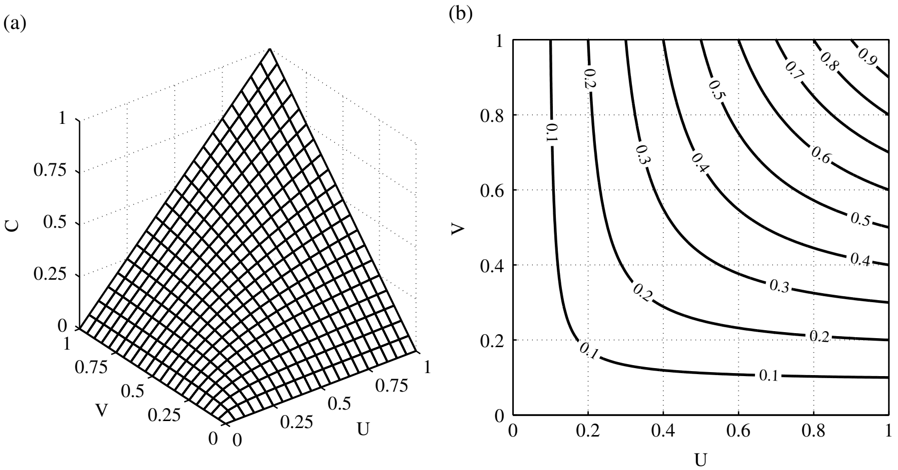

Figure C.9. The Ali-Mikhail-Haq 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.99$. The probability levels are as indicated

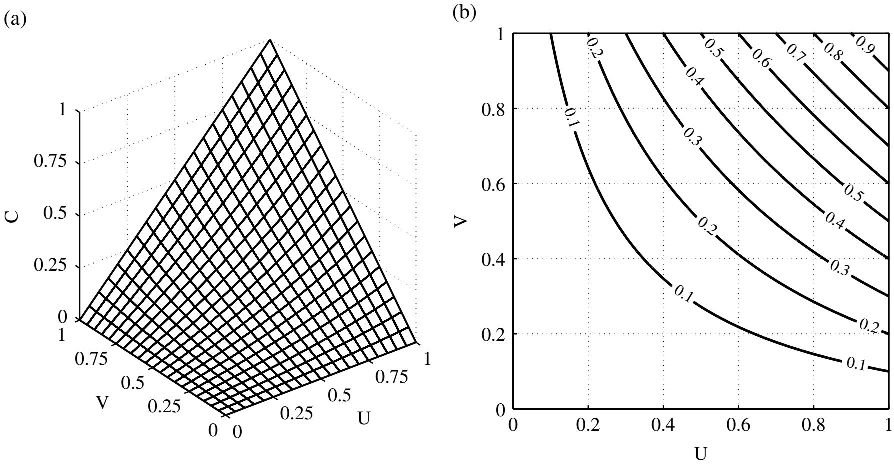

Figure C.10. The Ali-Mikhail-Haq 2-copula and the corresponding level curves. Here the parameter is $\theta = -0.99$. The probability levels are as indicated

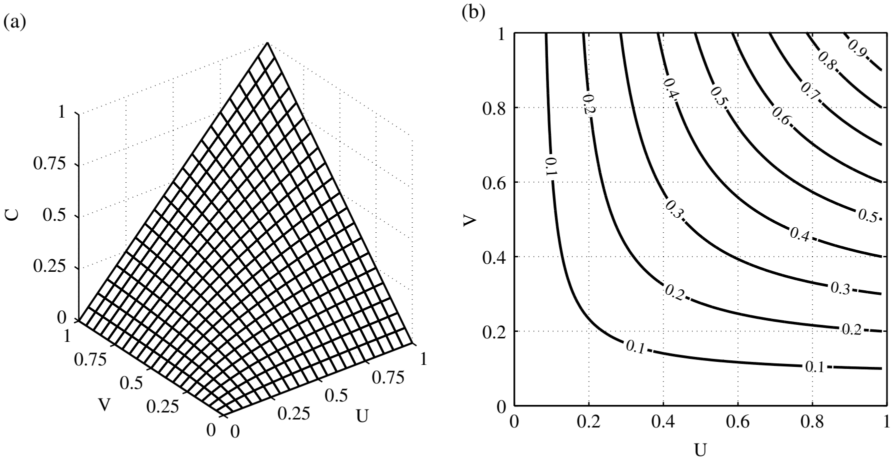

Figure C.11. The Ali-Mikhail-Haq 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 0.829$, as used in [68]. The probability levels are as indicated

C.5. THE JOE FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta\,(u, v) = 1 - \left[(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\right]^{1/\theta}, \tag{C.19}$$

where $u, v \in \mathbb{I}$, and $\theta \geq 1$ is a dependence parameter [153, 207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 1$, i.e. $\mathbf{C}_1 = \mathbf{\Pi}_2$, and PQD for $\theta > 1$. In particular, this family is positively ordered, with $\mathbf{C}_\infty = \mathbf{M}_2$, and its members are absolutely continuous.

Copulas belonging to the Joe family are strict Archimedean. The expression of the generator is, for $t \in \mathbb{I}$,

$$\gamma(t) = -\ln\left[1 - (1-t)^\theta\right]. \tag{C.20}$$

Using Eq. (B.26), it is possible to calculate the expression of Kendall's τ_K numerically.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 0$ and $\lambda_U = 2 - 2^{1/\theta}$.

As an illustration, in Figures C.12–C.13 we plot the Joe 2-copula and the corresponding level curves, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.12 $\mathbf{C}$ approximates $\mathbf{\Pi}_2$, for $\theta \approx 1$, while in Figure C.13 $\mathbf{C}$ approximates $\mathbf{M}_2$, for sufficiently large θ.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Joe 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

Due to Proposition 4.1, this family can be extended for $d \geq 3$, if we consider the parameter range $\theta \geq 1$, where γ^{-1} is completely monotonic. Then, the generalization of the Joe family in d-dimensions is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = 1 - \left[1 - \prod_{i=1}^{d}\left(1 - (1-u_i)^\theta\right)\right]^{1/\theta}, \tag{C.21}$$

where $\theta \geq 1$.

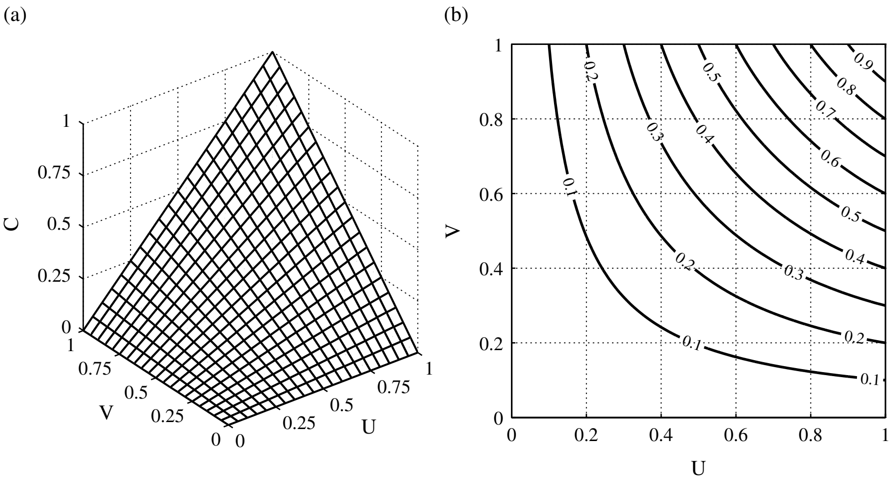

Figure C.12. The Joe 2-copula and the corresponding level curves. Here the parameter is $\theta = 1.05$. The probability levels are as indicated

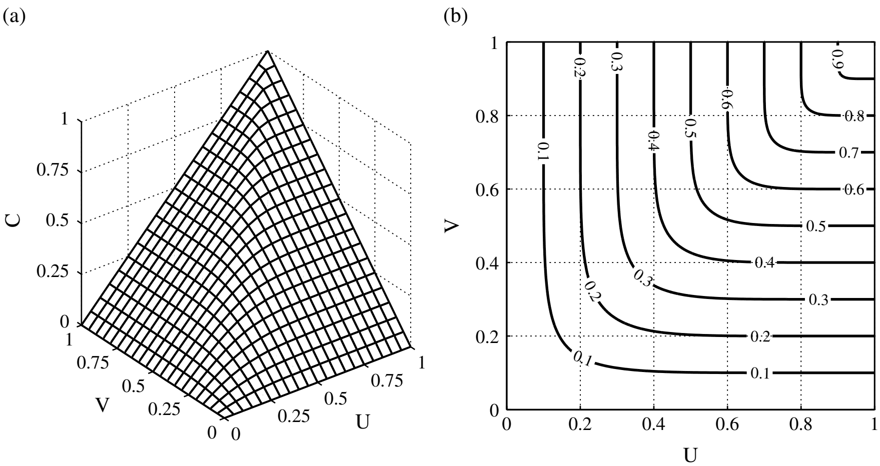

Figure C.13. The Joe 2-copula and the corresponding level curves. Here the parameter is $\theta = 7$. The probability levels are as indicated

C.6. THE FARLIE-GUMBEL-MORGENSTERN (FGM) FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = uv + \theta uv(1-u)(1-v). \tag{C.22}$$

where $u, v \in \mathbb{I}$, and $\theta \in [-1, 1]$ is a dependence parameter [207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, PQD for $\theta > 0$ and NQD for $\theta < 0$. In particular, this family is positively ordered.

Every member of the FGM family is absolutely continuous, and its density has a simple expression given by

$$\mathbf{c}_\theta(u, v) = 1 + \theta(1-2u)(1-2v). \tag{C.23}$$

The arithmetic mean of two FGM 2-copulas is again an FGM 2-copula, i.e. if $\mathbf{C}_{\theta_1}$ and $\mathbf{C}_{\theta_2}$ are given by Eq. (C.22), then their arithmetic mean is $\mathbf{C}_{(\theta_1+\theta_2)/2}$. In addition, each FGM 2-copula can be written as the arithmetic mean of the two extreme members of the family, i.e.

$$\mathbf{C}_\theta(u, v) = \frac{1-\theta}{2}\mathbf{C}_{-1}(u, v) + \frac{1+\theta}{2}\mathbf{C}_1(u, v) \tag{C.24}$$

for all $\theta \in [-1, 1]$.

The FGM 2-copulas satisfy the functional equation $\mathbf{C} = \overline{\mathbf{C}}$ for radial symmetry.

Two useful relationships exist between θ and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\theta) = \frac{2\theta}{9}, \tag{C.25a}$$

$$\rho_S(\theta) = \frac{\theta}{3}. \tag{C.25b}$$

Therefore, the values of $\tau_K(\theta)$ have a range of $[-2/9, 2/9]$, and the values of $\rho_S(\theta)$ have a range of $[-1/3, 1/3]$. These limited intervals restrict the usefulness of this family for modeling.

The lower and upper tail dependence coefficients for the members of this family are equal to 0.

As an illustration, in Figures C.14–C.15 we plot the Farlie-Gumbel-Morgenstern 2-copula and the corresponding level curves, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.14 $\mathbf{C}$ fails to approximate $\mathbf{M}_2$ sufficiently, although $\theta \approx 1$, and how in Figure C.15 $\mathbf{C}$ fails to approximate $\mathbf{W}_2$ sufficiently, although $\theta \approx -1$.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Farlie-Gumbel-Morgenstern 2-copula $\mathbf{C}_\theta$, is as follows:

1. Generate independent variates u, t Uniform on $\mathbb{I}$.
2. Set $a = \theta(1-2u) - 1$ and $b = \sqrt{a^2 - 4(a+1)t}$.
3. Set $v = 2t/(b-a)$.

The desired pair is then (u, v). For more details, see [156].

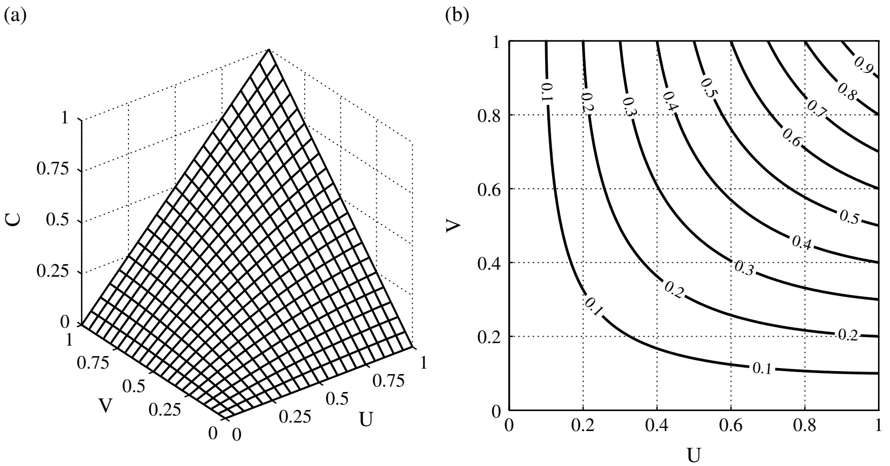

Figure C.14. The Farlie-Gumbel-Morgenstern 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.99$. The probability levels are as indicated

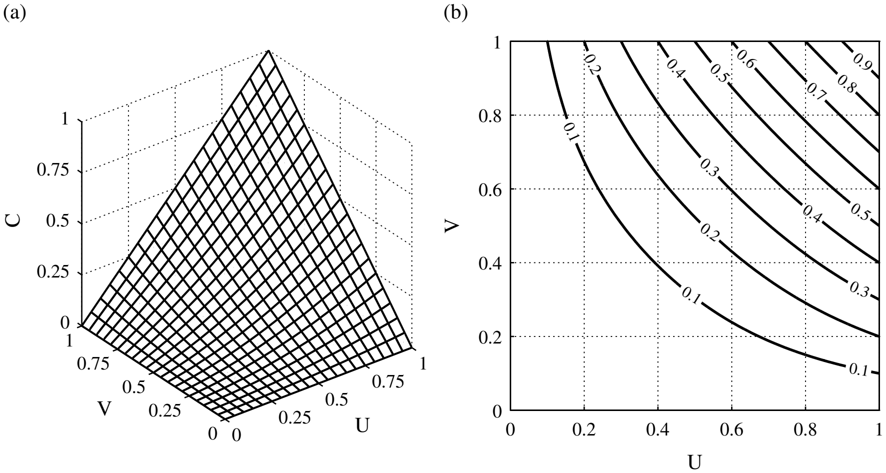

Figure C.15. The Farlie-Gumbel-Morgenstern 2-copula and the corresponding level curves. Here the parameter is $\theta = -0.99$. The probability levels are as indicated

In order to extend the range of dependence of the FGM family, several generalizations are proposed [75]. In particular, the following family of copulas is introduced in [238]:

$$\mathbf{C}_{f,g}(u, v) = uv + f(u)g(v), \tag{C.26}$$

where f and g are two real functions defined on $\mathbb{I}$, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 0$, which satisfy the conditions:

$$|f(u_1) - f(u_2)| \le M|u_1 - u_2|, \tag{C.27a}$$

$$|g(u_1) - g(u_2)| \le \frac{1}{M}|u_1 - u_2|, \tag{C.27b}$$

for all $u_1, u_2 \in \mathbb{I}$, with $M > 0$. Note that, by taking $f(u) = \theta u(1-u)$ and $g(v) = v(1-v)$, Eq. (C.26) describes the FGM family. Instead, by taking $f(u) = \theta u^\alpha (1-u)^\beta$ and $g(v) = v^\alpha (1-v)^\beta$, Eq. (C.26) describes the family given in [174].

The FGM family can be extended to the d-dimensional case, $d \ge 3$ — see [75, 207] for more details. Such an extension depends upon $2^d - d - 1$ parameters, and has the following form:

$$\begin{aligned}\mathbf{C}_\theta(\boldsymbol{u}) = {} & \left(\prod_{i=1}^{d} u_i\right) \\ & \cdot \left[1 + \sum_{k=2}^{d} \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant d} \theta_{j_1 \cdots j_k} (1 - u_{j_1}) \cdots (1 - u_{j_k})\right],\end{aligned} \tag{C.28}$$

for suitable values of the parameters. In particular, for $d = 3$, four parameters are required, yielding the expression

$$\begin{aligned} \mathbf{C}_\theta(\boldsymbol{u}) = u_1u_2u_3[1 + \theta_{12}(1 - u_1)(1 - u_2) + \theta_{13}(1 - u_1)(1 - u_3) \\ + \theta_{23}(1 - u_2)(1 - u_3) + \theta_{123}(1 - u_1)(1 - u_2)(1 - u_3)], \end{aligned} \tag{C.29}$$

for suitable values of the θ's.

C.7. THE PLACKETT FAMILY

The standard expression for members of this family is

$$\begin{aligned} \mathbf{C}_\theta(u, v) = \frac{1 + (\theta - 1)(u + v)}{2(\theta - 1)} \\ - \frac{\sqrt{[1 + (\theta - 1)(u + v)]^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)}, \end{aligned} \tag{C.30}$$

where $u, v \in \mathbb{I}$, and $\theta \geq 0$ is a dependence parameter [219, 145, 207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are PQD for $\theta > 1$, and NQD for $\theta < 1$. The limiting case $\theta = 1$ correspondes to the independent case, i.e. $\mathbf{C}_1 = \mathbf{\Pi}_2$. Moreover, $\mathbf{C}_0 = \mathbf{W}_2$ and $\mathbf{C}_\infty = \mathbf{M}_2$, and thus the Plackett family is comprehensive. In particular, this family is positively ordered. Every Plackett copula is absolutely continuous, and satisfies the functional equation $\mathbf{C} = \overline{\mathbf{C}}$ for radial symmetry.

No closed form exists for the value of Kendall's τ_K. Instead, Spearman's ρ_S is given by:

$$\rho_S(\theta) = \frac{\theta + 1}{\theta - 1} - \frac{2\theta \ln \theta}{(\theta - 1)^2} \tag{C.31}$$

for $\theta \neq 1$.

The lower and upper tail dependence coefficients for the members of this family are equal to 0.

As an illustration, we plot the Plackett 2-copula and the corresponding level curves in Figures C.16–C.17, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.16 $\mathbf{C}$ approximates $\mathbf{M}_2$, for $\theta \gg 1$, while in Figure C.17 $\mathbf{C}$ approximates $\mathbf{W}_2$, for $\theta \approx 0$.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V), Uniform on $\mathbb{I}$ and having a Plackett 2-copula $\mathbf{C}_\theta$ is as follows:

1. Generate independent variates u, t Uniform on $\mathbb{I}$.
2. Set $a = t(1 - t)$.
3. Set $b = \theta + a(\theta - 1)^2$, $c = 2a(u\theta^2 + 1 - u) + \theta(1 - 2a)$, and $d = \sqrt{\theta \cdot [\theta + 4au(1 - u)(1 - \theta)^2]}$.
4. Set $v = \frac{c-(1-2t)d}{2b}$.

The desired pair is then (u, v). For more details, see [156].

A multivariate extension of this family is presented in [196].

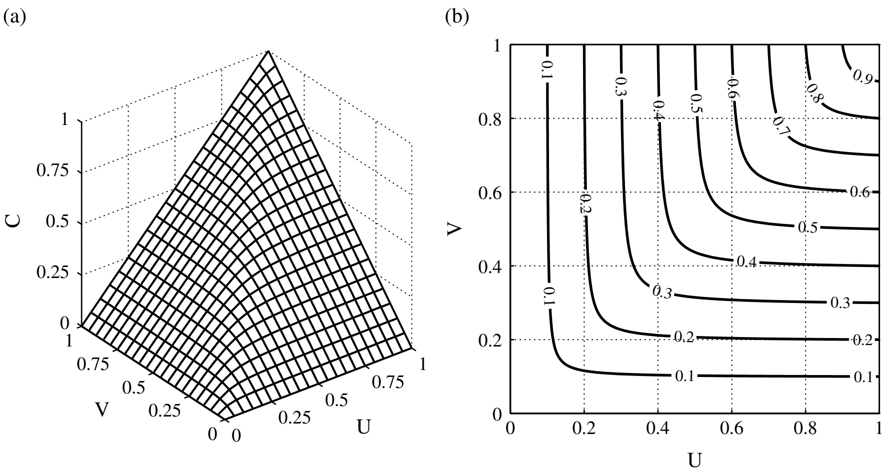

Figure C.16. The Plackett 2-copula and the corresponding level curves. Here the parameter is $\theta = 50$. The probability levels are as indicated

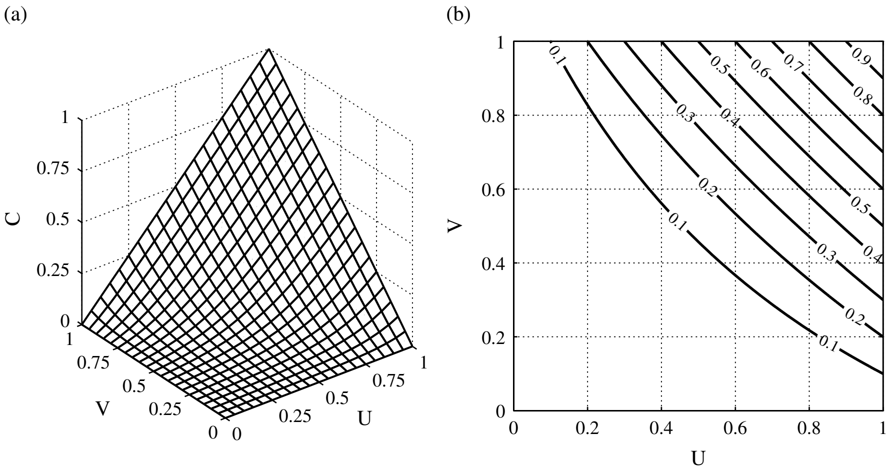

Figure C.17. The Plackett 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.1$. The probability levels are as indicated

C.8. THE RAFTERY FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = \mathbf{M}_2(u, v) + \frac{1-\theta}{1+\theta}(uv)^{1/(1-\theta)}\left[1 - (\max\{u, v\})^{-(1+\theta)/(1-\theta)}\right], \tag{C.32}$$

where $u, v \in \mathbb{I}$, and $\theta \in \mathbb{I}$ is a dependence parameter [227, 207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, and PQD for $\theta > 0$, with $\mathbf{C}_1 = \mathbf{M}_2$. Moreover, this family is positively ordered, and its members are absolutely continuous.

Two useful relationships exist between θ and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\theta) = \frac{2\theta}{3-\theta}, \tag{C.33a}$$

$$\rho_S(\theta) = \frac{\theta(4-3\theta)}{(2-\theta)^2}, \tag{C.33b}$$

which may provide a way to fit a Raftery 2-copula to a sample.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 2\theta/(\theta+1)$ and $\lambda_U = 0$.

As an illustration, we plot the Raftery 2-copula and the corresponding level curves in Figures C.18–C.19, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.18 $\mathbf{C}$ approximates $\mathbf{M}_2$, for $\theta \approx 1$, while in Figure C.19 $\mathbf{C}$ approximates $\mathbf{\Pi}_2$, for $\theta \approx 0$.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Raftery 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

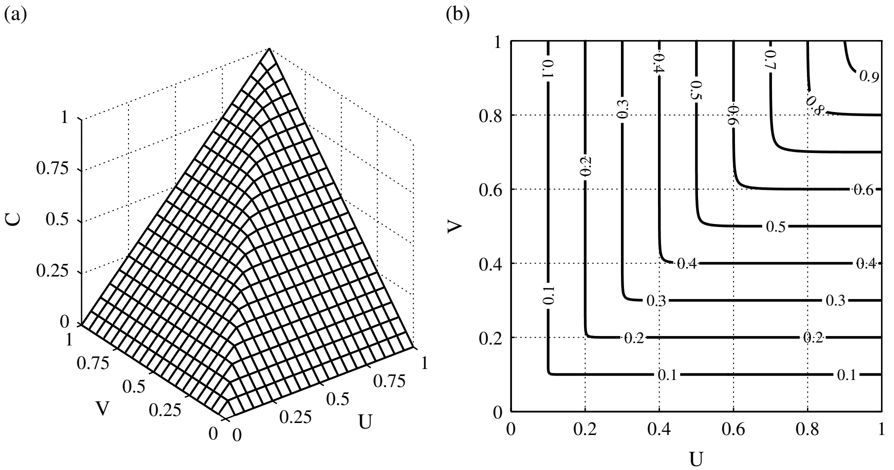

Figure C.18. The Raftery 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.95$. The probability levels are as indicated.

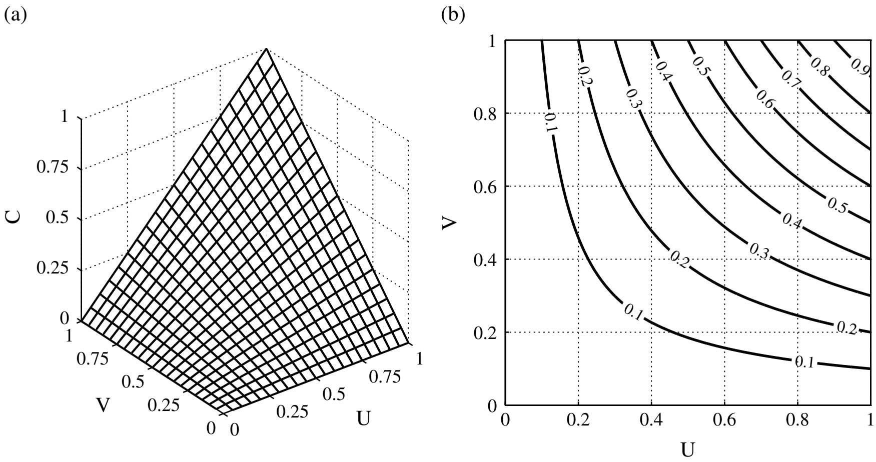

Figure C.19. The Raftery 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.05$. The probability levels are as indicated.

C.9. THE GALAMBOS FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = uv \exp\left(\left[(-\ln u)^{-\theta} + (-\ln v)^{-\theta}\right]^{-1/\theta}\right), \tag{C.34}$$

where $u, v \in \mathbb{I}$, and $\theta \geq 0$ is a dependence parameter [104, 153].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, and PQD for $\theta > 0$. In particular, this family is positively ordered, with $\mathbf{C}_\infty = \mathbf{M}_2$, and its members are absolutely continuous. Most importantly, copulas belonging to the Galambos family are EV copulas.

Using Eq. (B.26) it is possible to calculate the expression of Kendall's τ_K numerically.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 0$ and $\lambda_U = 2^{-1/\theta}$.

As an illustration, we plot the Galambos 2-copula and the corresponding level curves in Figures C.20–C.22, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.20 $\mathbf{C}$ approximates $\mathbf{M}_2$, for θ large enough, while in Figure C.21 $\mathbf{C}$ approximates $\mathbf{\Pi}_2$, for $\theta \approx 0$. In Figure C.22 we show the Galambos 2-copula used in [109]: here a low positive association is found to a large extent.

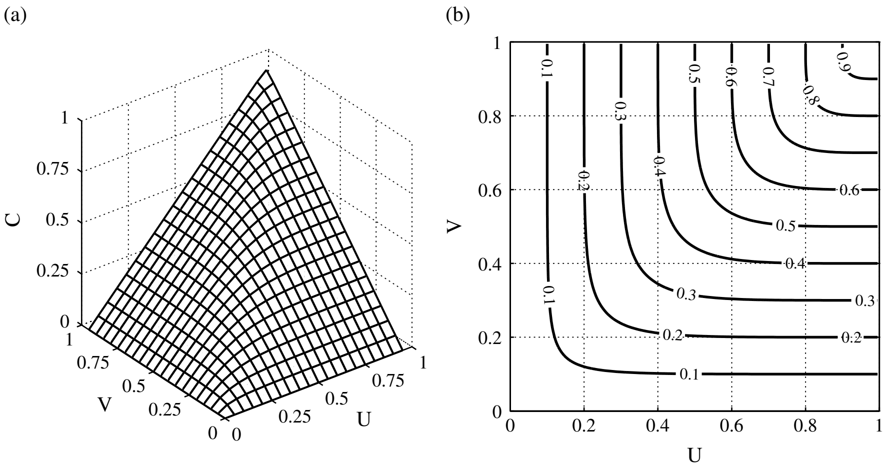

Figure C.20. The Galambos 2-copula and the corresponding level curves. Here the parameter is $\theta = 3$. The probability levels are as indicated

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Galambos 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

A multivariate (partial) extension of this family to the d-dimensional case, $d \geq 3$, is presented in [155].

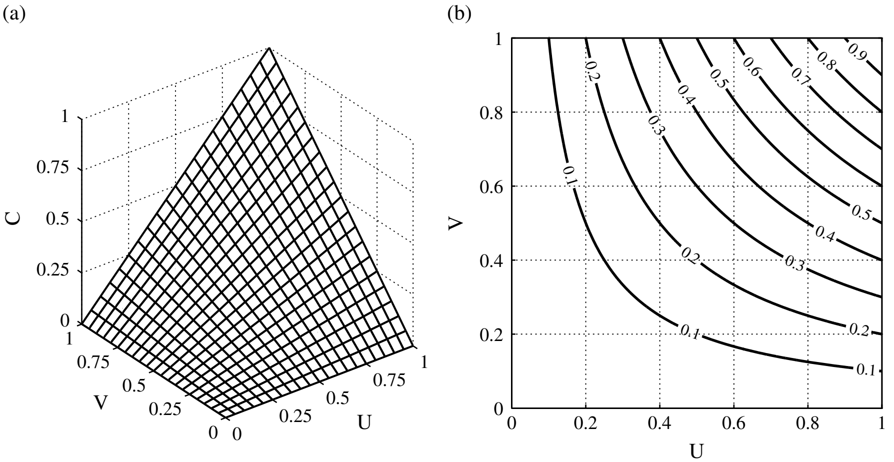

Figure C.21. The Galambos 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.01$. The probability levels are as indicated

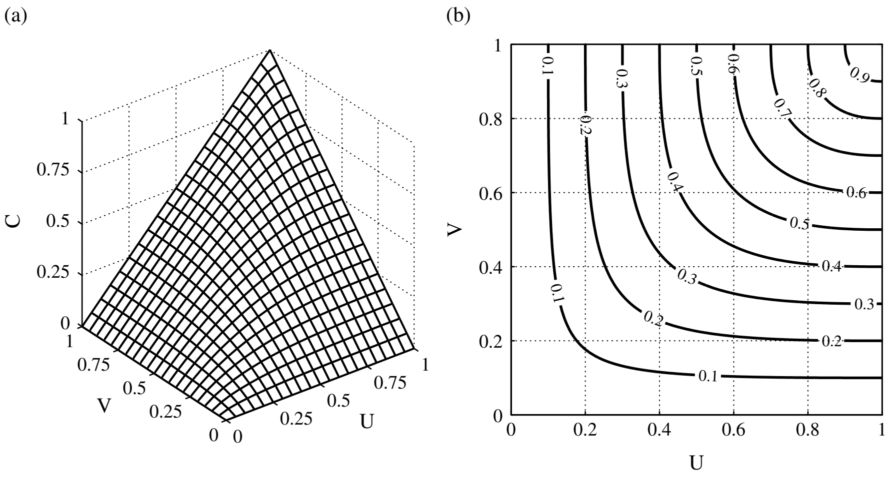

Figure C.22. The Galambos 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 1.464$, as used in [109]. The probability levels are as indicated.

C.10. THE HÜSLER-REISS FAMILY

The standard expression for members of this family is

$$\mathbf{C}_\theta(u, v) = \exp\left[(\ln u)\,\Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\left(\frac{\ln u}{\ln v}\right)\right) + (\ln v)\,\Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\left(\frac{\ln v}{\ln u}\right)\right)\right], \tag{C.35}$$

where $u, v \in \mathbb{I}$, $\theta \geq 0$ is a dependence parameter, and Φ is the univariate standard Normal distribution [144, 153].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, and PQD for $\theta > 0$. In particular, this family is positively ordered, with $\mathbf{C}_\infty = \mathbf{M}_2$, and its members are absolutely continuous. Most importantly, copulas belonging to the Hüsler-Reiss family are EV copulas.

Using Eq. (B.26) it is possible to calculate the expression of Kendall's τ_K numerically.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 0$ and $\lambda_U = 2 - 2\Phi(1/\theta)$.

As an illustration, we plot the Hüsler-Reiss 2-copula and the corresponding level curves in Figures C.23–C.25, for different values of θ.

As a comparison with Figure 3.1 and Figure 3.4, note how in Figure C.23 $\mathbf{C}$ approximates $\mathbf{M}_2$, for sufficiently large θ, while in Figure C.24 $\mathbf{C}$ approximates $\mathbf{\Pi}_2$, for sufficiently small θ. In Figure C.25 we show the Hüsler-Reiss 2-copula used in [109]: here a low positive association is seen to a large extent.

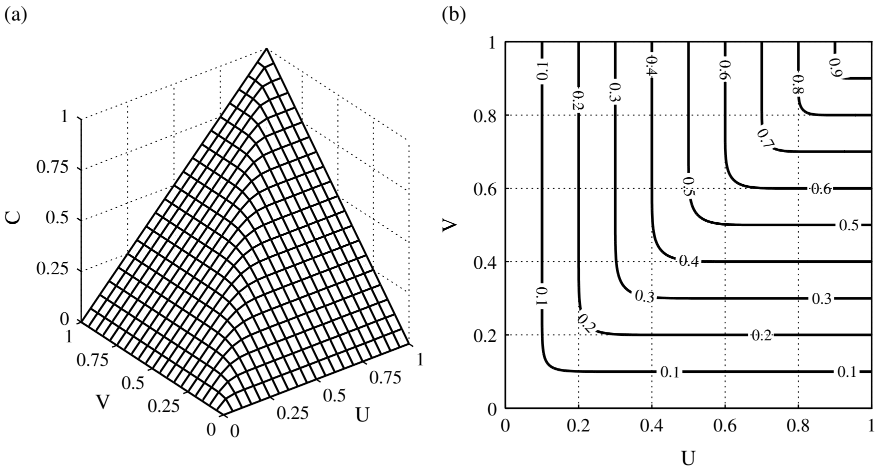

Figure C.23. The Hüsler-Reiss 2-copula and the corresponding level curves. Here the parameter is $\theta = 10$. The probability levels are as indicated.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Hüsler-Reiss 2-copula $\mathbf{C}_\theta$, can be constructed by using the method outlined in Section A.1.

A multivariate (and partial) extension of this family to the d-dimensional case, $d \geq 3$, is presented in [155].

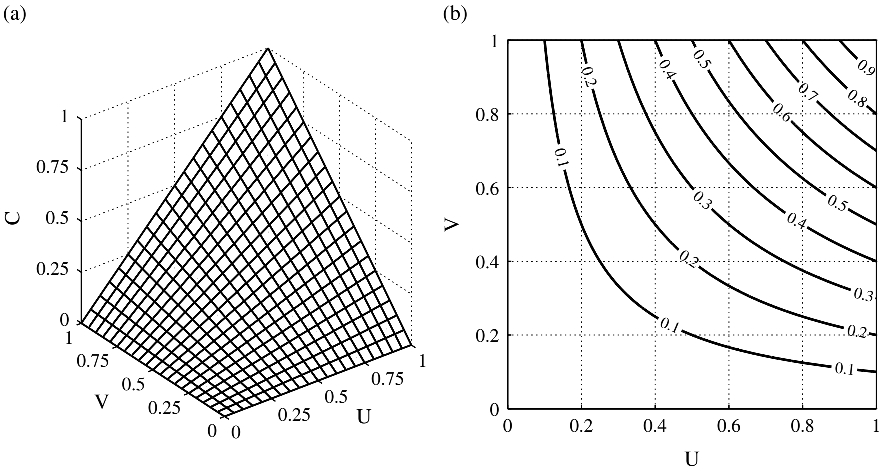

Figure C.24. The Hüsler-Reiss 2-copula and the corresponding level curves. Here the parameter is $\theta = 0.05$. The probability levels are as indicated.

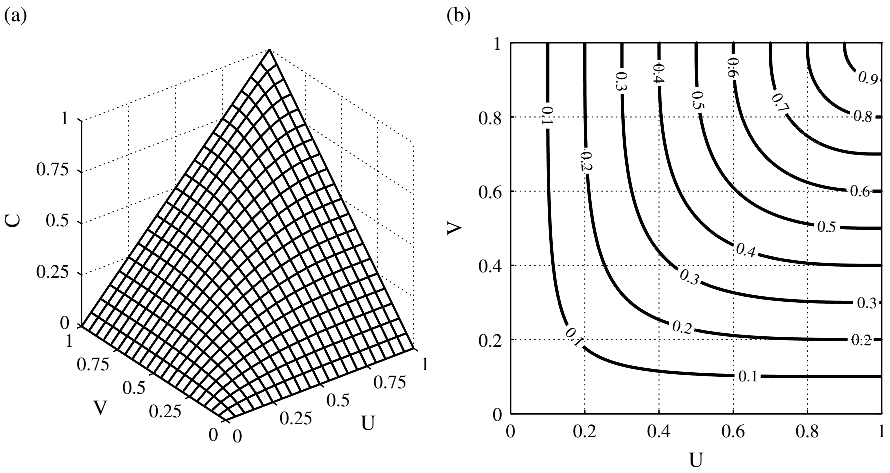

Figure C.25. The Hüsler-Reiss 2-copula and the corresponding level curves. Here the parameter is $\theta \approx 2.027$, as used in [109]. The probability levels are as indicated.

C.11. THE ELLIPTICAL FAMILY

A random vector $\boldsymbol{X} \in \mathbb{R}^d$ is said to have an *elliptical distribution* if it admits the stochastic representation

$$\boldsymbol{X} = \boldsymbol{\mu} + R\boldsymbol{A}\boldsymbol{U}, \tag{C.36}$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$, R is a positive r.v. independent of $\boldsymbol{U}$, $\boldsymbol{U}$ is a random vector Uniform on the unit sphere in $\mathbb{R}^d$, and $\boldsymbol{A}$ is a fixed $d \times d$ matrix such that $\Sigma = \boldsymbol{A}\boldsymbol{A}^T$ is non-singular.

The density function of an elliptical distribution (if it exists) is given by, for $\boldsymbol{x} \in \mathbb{R}^d$,

$$f(\boldsymbol{x}) = |\Sigma|^{-1/2} g((\boldsymbol{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})) \tag{C.37}$$

for some function $g : \mathbb{R} \to \mathbb{R}_+$, called a *density generator*, uniquely determined by the distribution of R (for more details, see [89, 88, 84]). For instance, the function

$$g(t) = C \exp(-t/2) \tag{C.38}$$

generates the multivariate Normal distribution, with a suitable normalizing constant C. Similarly,

$$g(t) = C\,(1 + t/m)^{-(d+m)/2} \tag{C.39}$$

generates the multivariate t-Student distribution, with a suitable normalizing constant C and $m \in \mathbb{N}$.

All the marginals of an elliptical distribution are the same, and are elliptically distributed. The unique copula associated to an elliptical distribution is called *elliptical copula*, and can be obtained by means of the inversion method (see, e.g., Corollary 3.1 or Corollary 4.1). If H is a d-variate elliptical distribution, with univariate marginal distribution functions F, then the corresponding elliptical d-copula is given by

$$\mathbf{C}(\boldsymbol{u}) = H(F^{-1}(u_1), \ldots, F^{-1}(u_d)). \tag{C.40}$$

The copula of the bivariate Normal distribution, also called *Gaussian* copula, is given by

$$\mathbf{C}_\theta(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \exp\left(-\frac{s^2 - 2\theta st + t^2}{2(1-\theta^2)}\right) \mathrm{d}s\,\mathrm{d}t, \tag{C.41}$$

where $\theta \in [-1, 1]$, and Φ^{-1} denotes the inverse of the univariate Normal distribution.

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are PQD for $\theta > 0$, and NQD for $\theta < 0$. The limiting case $\theta = 0$ corresponds to the independent case, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$. This family is positively ordered, and also comprehensive, since $\mathbf{C}_{-1} = \mathbf{W}_2$ and $\mathbf{C}_1 = \mathbf{M}_2$. Every Gaussian 2-copula satisfies the functional equation $\mathbf{C} = \overline{\mathbf{C}}$ for radial symmetry.

Two useful relationships exist between θ and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\theta) = \frac{2}{\pi} \arcsin\theta, \tag{C.42a}$$

$$\rho_S(\theta) = \frac{6}{\pi} \arcsin\frac{\theta}{2}, \tag{C.42b}$$

which may provide a way to fit a Gaussian 2-copula to the available data.

Gaussian copulas have lower and upper tail dependence coefficients equal to 0.

The copula of the bivariate t-Student distribution with $\nu > 2$ degrees of freedom, also called the *t-Student* copula, is given by

$$\mathbf{C}_\theta(u, v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\theta^2}} \left(1 + \frac{s^2 - 2\theta st + t^2}{\nu(1-\theta^2)}\right)^{-(\nu+2)/2} \mathrm{d}s\ \mathrm{d}t, \tag{C.43}$$

where $\theta \in [-1, 1]$, and t_ν^{-1} denotes the inverse of the univariate t-distribution.

The expressions of Kendall's τ_K and Spearman's ρ_S of a t-copula are the same as of those in Eqs. (C.42).

The lower and upper tail dependence coefficients are equal, and are given by

$$\lambda_U = 2t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\,\sqrt{1-\theta}}{\sqrt{1+\theta}}\right). \tag{C.44}$$

Therefore, the tail dependence coefficient is increasing in θ, and decreasing in ν, and tends to zero as $\nu \to \infty$ if $\theta < 1$ (see also [261]).

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a 2-copula either of Gaussian or t-Student type, is outlined in [84].

The d-variate extensions, $d \geq 3$, of the Gaussian and the t-Student copulas can be constructed in an obvious way by using, respectively, the multivariate version of Normal and t-Student distributions.

In applications, multivariate elliptical copulas are sometimes used together with different types of univariate marginals (not necessarily elliptically distributed), in order to obtain new classes of multivariate distributions functions called *meta-elliptical* (see [88, 1] and also [110]).

C.12. THE FRÉCHET FAMILY

The standard expression for members of this family is

$$\mathbf{C}_{\alpha,\beta}(u, v) = \alpha\mathbf{M}_2(u, v) + (1-\alpha-\beta)\mathbf{\Pi}_2(u, v) + \beta\mathbf{W}_2(u, v), \tag{C.45}$$

where $u, v \in \mathbb{I}$, and α, β are dependence parameters, with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$ [207]. Therefore, the copulas of this family are *convex combinations* of $\mathbf{\Pi}_2, \mathbf{W}_2, \mathbf{M}_2$. In fact, $\mathbf{C}_{1,0} = \mathbf{M}_2$, $\mathbf{C}_{0,0} = \mathbf{\Pi}_2$ and $\mathbf{C}_{0,1} = \mathbf{W}_2$. In turn, this family is comprehensive. When $\alpha, \beta \neq 0$, $\mathbf{C}_{\alpha,\beta}$ has a singular component, and hence it is not absolutely continuous.

Two useful relationships exist between α and β and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\alpha, \beta) = \frac{(\alpha-\beta)(\alpha+\beta+2)}{3}, \tag{C.46a}$$

$$\rho_S(\alpha, \beta) = \alpha - \beta, \tag{C.46b}$$

which may provide a method of fitting a Fréchet 2-copula to the available data.

The lower and upper tail dependence coefficients for the members of this family are equal to α.

As an illustration, in Figures C.26–C.27 we plot the Fréchet 2-copula and the corresponding level curves, for different values of the parameters α, β.

Figures C.26–C.27 should be compared to Figure 3.1 and Figure 3.4.

Although the Fréchet family does not seem to have desirable properties, it was recently rediscovered in applications. In fact, as shown in [306], every 2-copula can be approximated in a unique way by a member of the Fréchet family, and the error bound can be estimated. Accordingly, practitioners can deal with a complicated copula by focusing on its approximation by means of a suitable Fréchet copula.

A slight modification of this family is the so-called *Linear Spearman* copula (see [155, family B11] and [142]), given by, for $\theta \in \mathbb{I}$,

$$\mathbf{C}_\theta(u, v) = (1-\theta)\mathbf{\Pi}_2(u, v) + \theta\mathbf{M}_2(u, v). \tag{C.47}$$

This family is positively ordered, and the dependence parameter θ is equal to the value of Spearman's ρ_S, i.e. $\rho_S(\theta) = \theta$.

The Linear Spearman copula can be extended to the d-dimensional case, $d \geq 3$, in an obvious way. The corresponding expression is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = (1-\theta)\mathbf{\Pi}_d(\boldsymbol{u}) + \theta\mathbf{M}_d(\boldsymbol{u}), \tag{C.48}$$

for $\theta \in \mathbb{I}$. Note that, instead, the Fréchet family cannot be extended in the same manner: in fact, $\mathbf{W}_d$ is not a copula for $d \geq 3$ (see Illustration 4.1).

C.13. THE MARSHALL-OLKIN FAMILY

The standard expression for members of this family is

$$\mathbf{C}_{\alpha,\beta}(u, v) = \min\left\{u^{1-\alpha}v, uv^{1-\beta}\right\}, \tag{C.49}$$

where $u, v \in \mathbb{I}$, and α, β are dependence parameters, with $0 \leq \alpha, \beta \leq 1$ [185, 186, 207]. Note that $\mathbf{C}_{\alpha,0} = \mathbf{C}_{0,\beta} = \mathbf{\Pi}_2$ and $\mathbf{C}_{1,1} = \mathbf{M}_2$.

Copulas of this family have an absolutely continuous component, and a singular component given by, respectively,

$$A_{\mathbf{C}} = uv^{1-\beta} - \frac{\alpha\beta}{\alpha+\beta-\alpha\beta}\left(u^\alpha\right)^{(\alpha+\beta-\alpha\beta)/\alpha\beta} \tag{C.50a}$$

for $u^\alpha < v^\beta$, and

$$S_{\mathbf{C}} = \frac{\alpha\beta}{\alpha+\beta-\alpha\beta}\left(\min\left\{u^\alpha, v^\beta\right\}\right)^{(\alpha+\beta-\alpha\beta)/\alpha\beta}. \tag{C.50b}$$

Two useful relationships exist between the parameters α, β and, respectively, Kendall's τ_K and Spearman's ρ_S:

$$\tau_K(\alpha, \beta) = \frac{\alpha\beta}{\alpha-\alpha\beta+\beta}, \tag{C.51a}$$

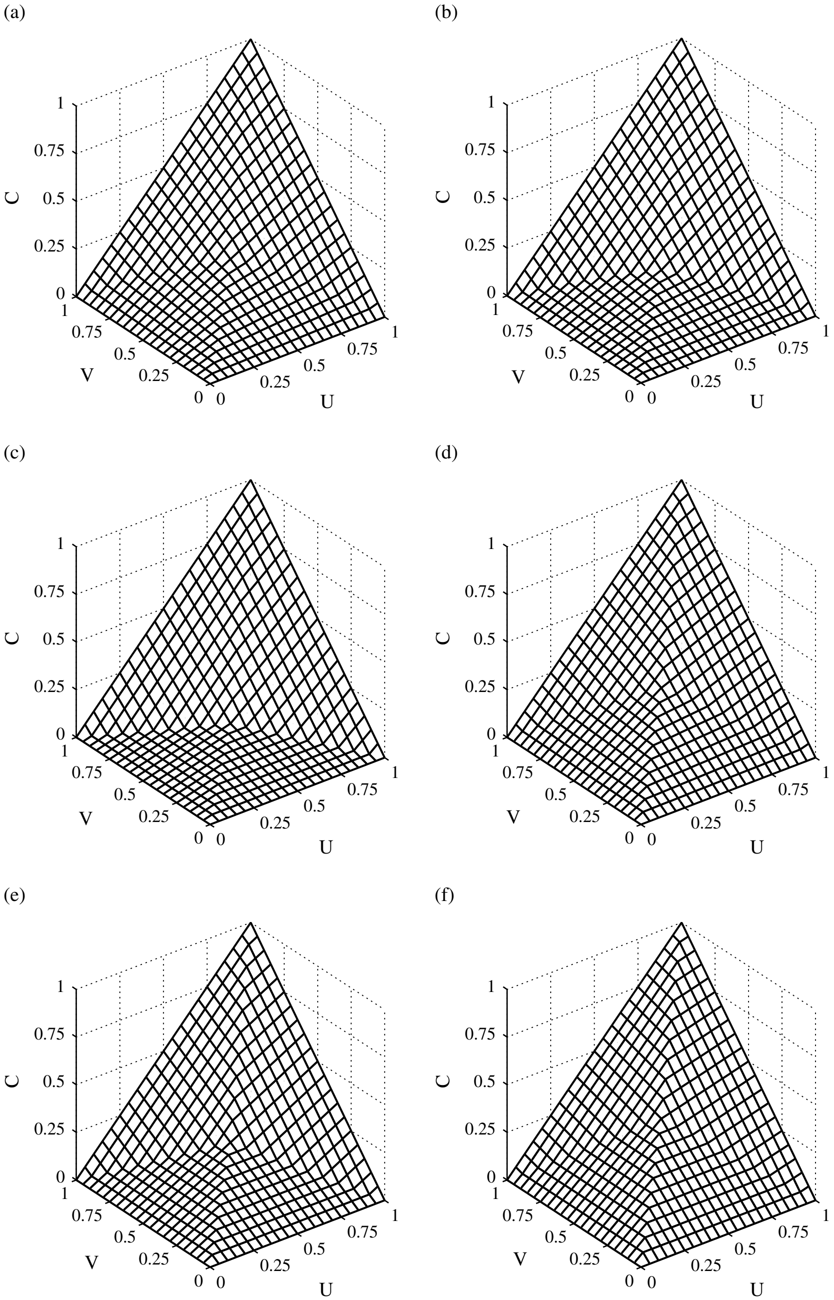

Figure C.26. The Fréchet 2-copula for several values of the parameters α, β: (*a*) $\alpha = 0.25, \beta = 0.25$, (*b*) $\alpha = 0.25, \beta = 0.5$, (*c*) $\alpha = 0.25, \beta = 0.75$, (*d*) $\alpha = 0.5, \beta = 0.25$, (*e*) $\alpha = 0.5, \beta = 0.5$, (*f*) $\alpha = 0.75, \beta = 0.25$.

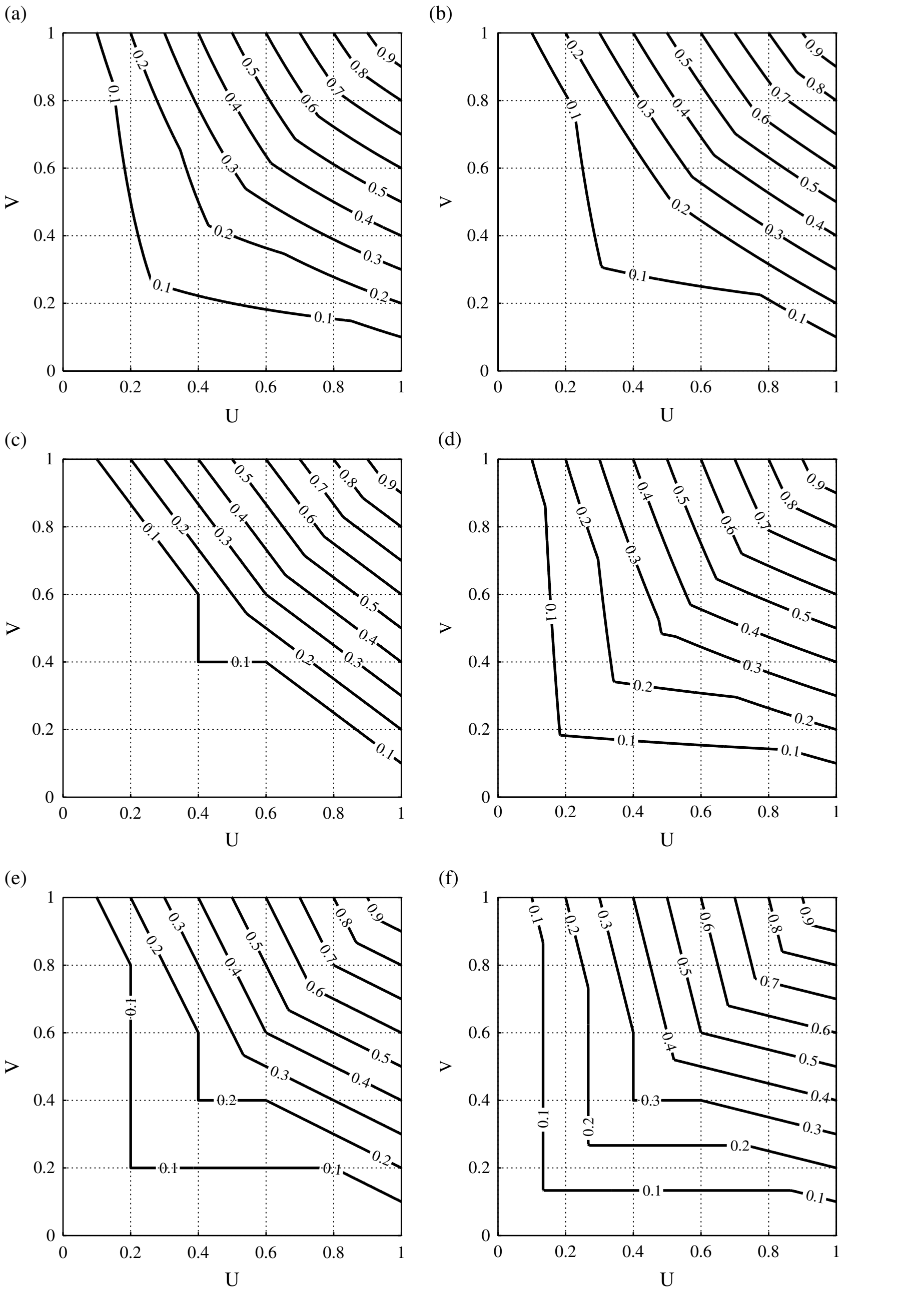

Figure C.27. The level curves of the Fréchet 2-copula for several values of the parameters α, β: (*a*) $\alpha = 0.25, \beta = 0.25$, (*b*) $\alpha = 0.25, \beta = 0.5$, (*c*) $\alpha = 0.25, \beta = 0.75$, (*d*) $\alpha = 0.5, \beta = 0.25$, (*e*) $\alpha = 0.5, \beta = 0.5$, (*f*) $\alpha = 0.75, \beta = 0.25$. The probability levels are as indicated.

$$\rho_S(\alpha, \beta) = \frac{3\alpha\beta}{2\alpha - \alpha\beta + 2\beta}, \tag{C.51b}$$

which may provide a way to fit a Marshall-Olkin 2-copula to the available data.

The lower and upper tail dependence coefficients for the members of this family are given by, respectively, $\lambda_L = 0$ and $\lambda_U = \min\{\alpha, \beta\}$.

As an illustration, we plot the Marshall-Olkin 2-copula and the corresponding level curves in Figures C.28–C.30, for different values of the parameters α, β.

Figures C.28–C.30 should be compared to Figure 3.1 and Figure 3.4.

A general algorithm for generating observations (u, v) from a pair of r.v.'s (U, V) Uniform on $\mathbb{I}$, and having a Marshall-Olkin 2-copula $\mathbf{C}_{\alpha,\beta}$, is as follows:

1. Generate independent variates r, s, and t Uniform on $\mathbb{I}$.
2. For any $\lambda_{12} > 0$, set $\lambda_1 = \lambda_{12}(1-\alpha)/\alpha$ and $\lambda_2 = \lambda_{12}(1-\beta)/\beta$
3. Set $x = \min\left\{\frac{-\ln r}{\lambda_1}, \frac{-\ln t}{\lambda_{12}}\right\}$ and $y = \min\left\{\frac{-\ln s}{\lambda_2}, \frac{-\ln t}{\lambda_{12}}\right\}$.
4. Set $u = \exp[-(\lambda_1 + \lambda_{12})x]$ and $v = \exp[-(\lambda_2 + \lambda_{12})y]$.

The desired pair is then (u, v). For more details, see [71].

The subfamily of Marshall-Olkin copulas obtained by taking $\alpha = \beta$ is of particular interest. This is known as the *Cuadras-Augé* family, and its members are given by

$$\mathbf{C}_\theta(u, v) = (\mathbf{\Pi}_2(u, v))^{1-\theta}(\mathbf{M}_2(u, v))^\theta, \tag{C.52}$$

where $u, v \in \mathbb{I}$, and $\theta \in \mathbb{I}$ is a dependence parameter [55, 207].

If U and V are r.v.'s with copula $\mathbf{C}_\theta$, then they are independent for $\theta = 0$, i.e. $\mathbf{C}_0 = \mathbf{\Pi}_2$, and PQD for $\theta > 0$, with $\mathbf{C}_1 = \mathbf{M}_2$. In particular, this family is positively ordered. These 2-copulas satisfy the functional equation $\mathbf{C} = \overline{\mathbf{C}}$ for radial symmetry.

As an illustration, see Figure C.28*ab*, Figure C.29*cd*, and Figure C.30*ef*, where the parameters α, β of the corresponding Marshall-Olkin 2-copulas are equal to one another.

The Cuadras-Augé copulas can be extended for $d \geq 3$ in an obvious way. The corresponding expression is given by

$$\mathbf{C}_\theta(\boldsymbol{u}) = (\mathbf{\Pi}_d(\boldsymbol{u}))^{1-\theta}(\mathbf{M}_d(\boldsymbol{u}))^\theta \tag{C.53}$$

for $\theta \in \mathbb{I}$. For an example see Illustration 5.5.

The multivariate extension of Marshall-Olkin 2-copulas has a complicated form, even if it can be easily simulated due to its probabilistic interpretation [84]. In the particular case $d = 3$, the Marshall-Olkin 3-copula depends on a multidimensional parameter $\boldsymbol{\theta}$, having nine components, and is given by

$$\begin{aligned}\mathbf{C}_{\boldsymbol{\theta}}(\boldsymbol{u}) = & u_1 u_2 u_3 \min\left\{u_1^{-\theta_{12}^1}, u_2^{-\theta_{12}^2}\right\} \min\left\{u_1^{-\theta_{13}^1}, u_3^{-\theta^2 13}\right\} \\ & \min\left\{u_2^{-\theta_{23}^1}, u_3^{-\theta_{23}^2}\right\} \max\left\{u_1^{-\theta_{123}^1}, u_2^{-\theta_{123}^2}, u_3^{-\theta_{123}^3}\right\}.\end{aligned}$$

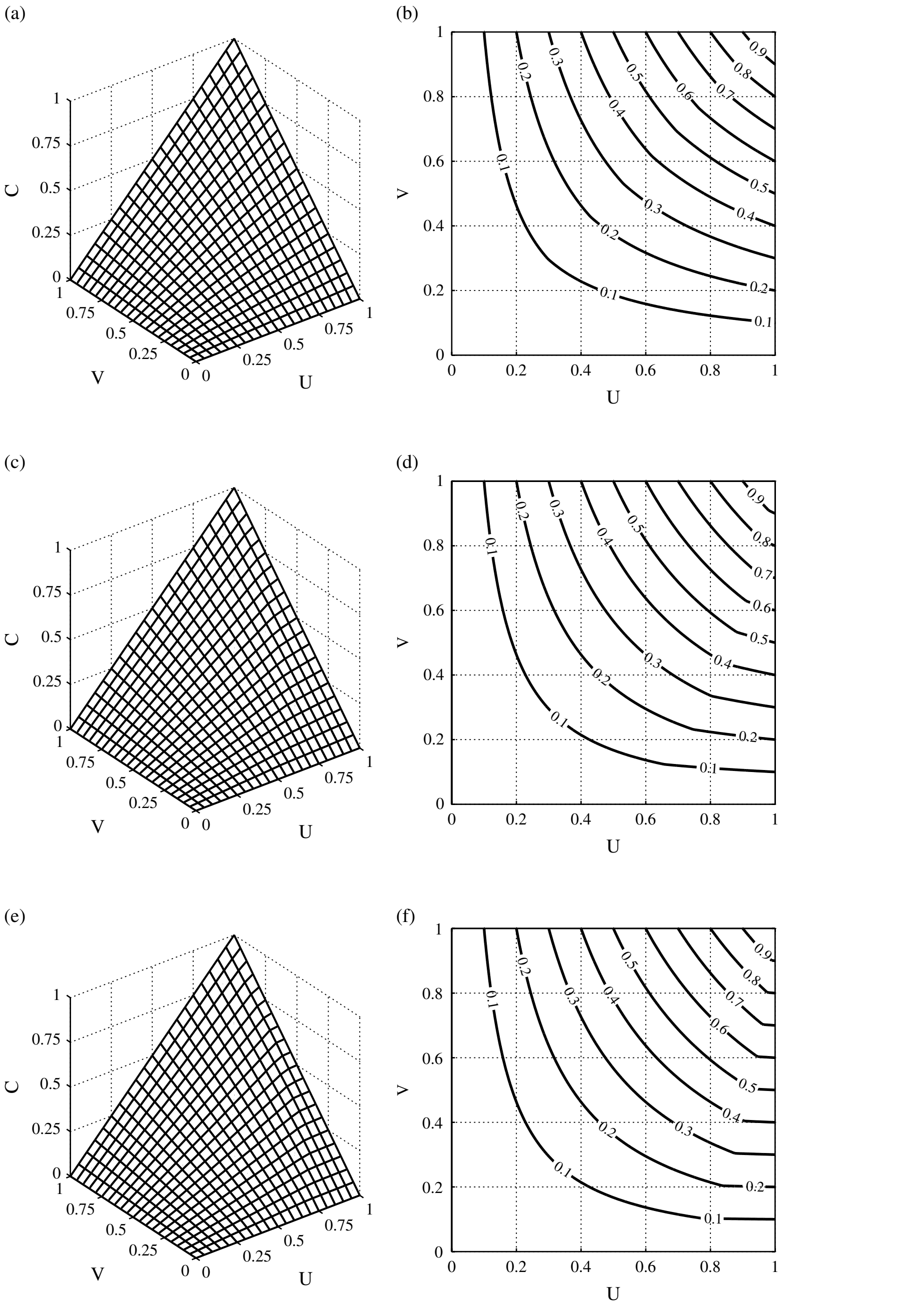

Figure C.28. The Marshall-Olkin 2-copula and the corresponding level curves for several values of the parameters α, β. Here $\alpha = 0.1$, and (*a,b*) $\beta = 0.1$, (*c,d*) $\beta = 0.5$, (*e,f*) $\beta = 0.9$. The probability levels are as indicated

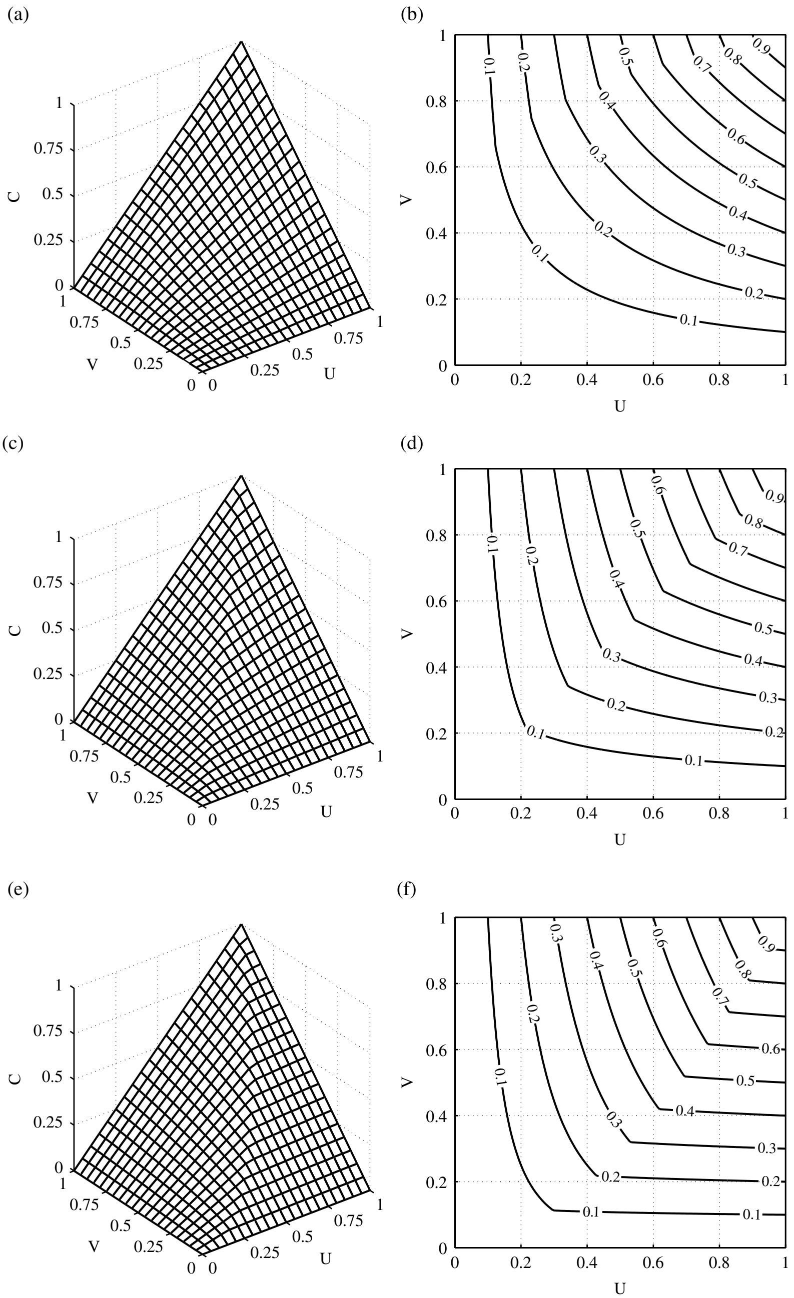

Figure C.29. The Marshall-Olkin 2-copula and the corresponding level curves for several values of the parameters α, β. Here $\alpha = 0.5$, and *(a,b)* $\beta = 0.1$, *(c,d)* $\beta = 0.5$, *(e,f)* $\beta = 0.9$. The probability levels are as indicated

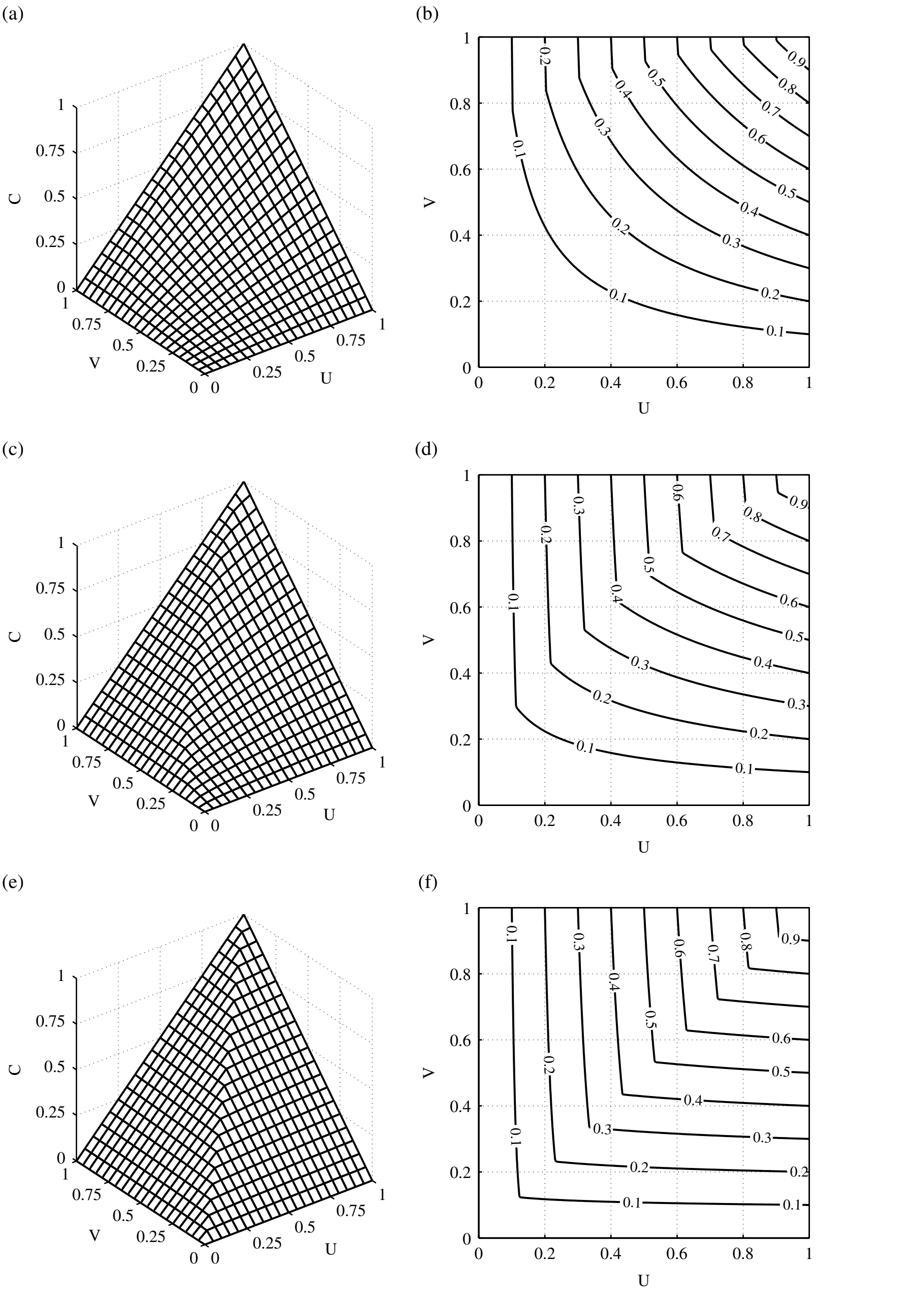

Figure C.30. The Marshall-Olkin 2-copula and the corresponding level curves for several values of the parameters α, β. Here $\alpha = 0.9$, and (*a,b*) $\beta = 0.1$, (*c,d*) $\beta = 0.5$, (*e,f*) $\beta = 0.9$. The probability levels are as indicated

C.14. THE ARCHIMAX FAMILY

The standard expression for members of this family is

$$\mathbf{C}_{\gamma,A}(u, v) = \gamma^{-1}\left[(\gamma(u)+\gamma(v))\, A\left(\frac{\gamma(u)}{\gamma(u)+\gamma(v)}\right)\right] \tag{C.54}$$

where $u, v \in \mathbb{I}$, γ is the generator of an Archimedean copula (see Definition 3.9), and A is a dependence function (see Definition 5.5).

This family includes both the Archimedean copulas, by taking $A = 1$, and the EV copulas, by taking $\gamma(t) = \ln(1/t)$. The members of this family are absolutely continuous if $\gamma(t)/\gamma'(t) \to 0$ as $t \to 0^+$.

Archimax copulas that are neither Archimedean nor extreme can be constructed at will. For instance, consider the dependence function

$$A(t) = \theta t^2 - \theta t + 1,$$

with $\theta \in \mathbb{I}$ (this dependence function is used in *Tawn's mixed model*, see [280]), and as γ use the generator of the Clayton family (see Section C.3).

For an Archimax copula $\mathbf{C}_{\gamma,A}$, the value of Kendall's τ_K is given by

$$\tau_K(\gamma, A) = \tau_K(A) + (1 - \tau_K(A))\tau_K(\gamma), \tag{C.55}$$

where

$$\tau_K(A) = \int_0^1 \frac{t(1-t)}{A(t)}\, \mathrm{d}A'(t), \tag{C.56a}$$

and

$$\tau_K(\gamma) = 1 + 4\int_0^1 \frac{\gamma(t)}{\gamma'(t)}\, \mathrm{d}t, \tag{C.56b}$$

are, respectively, the values of Kendall's τ_K of an EV copula generated by A, and of an Archimedean copula generated by γ.

C.15. CONSTRUCTION METHODS FOR COPULAS

In this section we present three methods for constructing 2-copulas. These constructions are very important in applications, where families of copulas with more parameters may sometimes provide significantly better fits than any monoparametric subfamily.

The first method is discussed in [80, 163, 164, 198], and includes, as special cases, the family of copulas BB1–BB7 presented in [155]. The second method arises from the ideas presented in [111], and has recently been generalized in [76]. Based on the fact that several families of 2-copulas are symmetric, i.e. $\mathbf{C}(u, v) = \mathbf{C}(v, u)$, this method provides a simple way to generate copulas that may fail this property. Asymmetric families of copulas sometimes improve the fit of a model (see, for instance, [109, 121]). The third method presents some constructions of copulas based on their diagonal section. Other procedures (e.g., geometric and algebraic methods, shuffles, and ordinal sums) can be found in [207].

C.15.1 Transformation of Copulas

Let us denote by Θ the set of continuous and strictly increasing functions $h : \mathbb{I} \to \mathbb{I}$, with $h(0) = 0$ and $h(1) = 1$. Given a 2-copula $\mathbf{C}$ and a function $h \in \Theta$, let $\mathbf{C}_h$ be the *h-transformation* of $\mathbf{C}$ defined by

$$\mathbf{C}_h\left(u, v\right) = h^{-1}\left(\mathbf{C}\left(h(u), h(v)\right)\right). \tag{C.57}$$

Note that $\mathbf{C}_h$ need not be a copula. In fact, if $h(t) = t^2$ and $\mathbf{C} = \mathbf{W}_2$, then $(\mathbf{W}_2)_h$ is not 2-increasing. However, we have the following result.

PROPOSITION C.1. *For each $h \in \Theta$, the following statements are equivalent:*

1. *h is concave;*
2. *for every copula $\mathbf{C}$, the h-transformation of $\mathbf{C}$ given by Eq. (C.57) is a copula.*

We now give some examples.

ILLUSTRATION C.1. ►

Consider the strict generator $\gamma_\theta(t) = (t^{-\theta} - 1)/\theta$, $\theta > 0$, with inverse $\gamma_\theta^{-1}(t) = (1 + \theta t)^{-1/\theta}$, of an Archimedean copula belonging to the Clayton family $\mathbf{C}_\theta$ (see Section C.3). Set $h(t) = \exp\left(-\gamma(t)\right)$. Then $h(t)$ satisfies the assumptions of Proposition C.1, and it is easy to verify that $(\mathbf{\Pi}_2)_h = \mathbf{C}_\theta$. Therefore, every Clayton copula can be obtained as a transformation of the copula $\mathbf{\Pi}_2$ with respect to a suitable function h. The same procedure can be applied to obtain any family of strict Archimedean copulas. ◄

ILLUSTRATION C.2. ►

Consider the strict generator $\gamma_\theta(t) = (-\ln t)^\theta$ of the Gumbel-Hougaard family $\mathbf{C}_\theta$ (see Section C.2). Set $h_\delta(t) = \exp\left(-(t^{-\delta} - 1)/\delta\right)$, with $\delta > 0$. Then $h(t)$ satisfies the assumptions of Proposition C.1, and it is easy to verify that the h_δ-transformation of $\mathbf{C}_\theta$, $(\mathbf{C}_\theta)_{h_\delta}$, is a member of the family BB3 given in [155].

In general, let $\gamma_{1,\theta}$ and $\gamma_{2,\delta}$, respectively, be the strict generators of the two Archimedean families of copulas $\mathbf{C}_\theta$ and $\mathbf{C}_\delta$. Set $h_{1,\theta}(t) = \exp(-\gamma_{1,\theta}(t))$ and $h_{2,\delta}(t) = \exp(-\gamma_{2,\delta}(t))$. Then $(\mathbf{C}_\theta)_{h_{2,\delta}}$ is a two-parameter Archimedean copula, with additive generator $\gamma_{1,\theta} \circ h_{2,\delta}$. Similarly, $(\mathbf{C}_\delta)_{h_{1,\theta}}$ is a two-parameter Archimedean copula with additive generator $\gamma_{2,\delta} \circ h_{1,\theta}$. In this way, we can obtain the families of copulas BB1–BB7 given in [155]. ◄

Note that, if H is a bivariate distribution function with marginals F and G, and h satisfies the assumptions of Proposition C.1, then $\widetilde{H}(x, y) = h(H(x, y))$ is a bivariate distribution function with marginals $h(F)$ and $h(G)$, and the copula of $\widetilde{H}$ is $\mathbf{C}_{h^{-1}}$. A generalization of this transformation to the d-dimensional case, $d \geqslant 3$, is considered in [198].

C.15.2 Composition of Copulas

Let us denote by Θ the set of continuous and strictly increasing functions $h : \mathbb{I} \to \mathbb{I}$, with $h(0) = 0$ and $h(1) = 1$. Let $H : \mathbb{I}^2 \to \mathbb{I}$ be increasing in each variable, with $H(0, 0) = 0$ and $H(1, 1) = 1$. Given f_1, f_2, g_1 and g_2 in Θ, and $\mathbf{A}$ and $\mathbf{B}$ 2-copulas, the function

$$F_{\mathbf{A},\mathbf{B}}(u, v) = H\left(\mathbf{A}(f_1(u), g_1(v)), \mathbf{B}(f_2(u), g_2(v))\right) \tag{C.58}$$

is called the *composition* of $\mathbf{A}$ and $\mathbf{B}$. In general, $F_{\mathbf{A},\mathbf{B}}$ is not a copula. In [76] some conditions on H, f_1, f_2, g_1 and g_2 are given in order to ensure that $F_{\mathbf{A},\mathbf{B}}$ is a copula. Here, we only present two special cases of composition [111].

PROPOSITION C.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be 2-copulas. Then*

$$\mathbf{C}_{\alpha,\beta}(u, v) = \mathbf{A}(u^{\alpha}, v^{\beta})\,\mathbf{B}(u^{1-\alpha}, v^{1-\beta}). \tag{C.59}$$

defines a family of copulas $\mathbf{C}_{\alpha,\beta}$*, with parameters* $\alpha, \beta \in \mathbb{I}$.

In particular, if $\alpha = \beta = 1$, then $\mathbf{C}_{1,1} = \mathbf{A}$, and, if $\alpha = \beta = 0$, then $\mathbf{C}_{0,0} = \mathbf{B}$. For $\alpha \neq \beta$, the copula $\mathbf{C}$ in Eq. (C.59) is, in general, *asymmetric*, that is $\mathbf{C}(u, v) \neq \mathbf{C}(v, u)$ for some $(u, v) \in \mathbb{I}^2$.

An interesting statistical interpretation can be given for this family. Let U_1, V_1, U_2 and V_2 be r.v.'s Uniform on $\mathbb{I}$. If $\mathbf{A}$ is the copula of (U_1, U_2), and $\mathbf{B}$ is the copula of (V_1, V_2), and the pairs (U_1, U_2) and (V_1, V_2) are independent, then $\mathbf{C}_{\alpha,\beta}$ is the joint distribution function of

$$U = \max\left\{U_1^{1/\alpha}, V_1^{1/(1-\alpha)}\right\} \quad \text{and} \quad V = \max\left\{U_2^{1/\beta}, V_2^{1/(1-\beta)}\right\}. \tag{C.60}$$

In particular, this probabilistic interpretation yields an easy way to simulate a copula expressed by Eq. (C.59), provided that $\mathbf{A}$ and $\mathbf{B}$ can be easily simulated. A simpler way of constructing asymmetric copulas is based on the following result.

PROPOSITION C.3 (Khoudraji). *Let* $\mathbf{C}$ *be a symmetric copula,* $\mathbf{C} \neq \mathbf{\Pi}_2$*. A family of asymmetric copulas* $\mathbf{C}_{\alpha,\beta}$*, with parameters* $0 < \alpha, \beta < 1$, $\alpha \neq \beta$*, that includes* $\mathbf{C}$ *as a limiting case, is given by*

$$\mathbf{C}_{\alpha,\beta}(u, v) = u^{\alpha} v^{\beta} \cdot \mathbf{C}\left(u^{1-\alpha}, v^{1-\beta}\right). \tag{C.61}$$

ILLUSTRATION C.3. ►

Let **A** and **B** be Archimedean 2-copulas generated, respectively, by γ_1 and γ_2. In view of Proposition C.3, the following functions are copulas for all $\alpha, \beta \in \mathbb{I}$:

$$\mathbf{C}_{\alpha,\beta}(u,v) = \gamma_1^{-1}(\gamma_1(u^\alpha)+\gamma_1(v^\beta)) \cdot \gamma_2^{-1}(\gamma_2(u^{1-\alpha})+\gamma_2(v^{1-\beta})). \tag{C.62}$$

In particular, if $\gamma_1(t) = \gamma_2(t) = (-\ln t)^\delta$, with $\delta \geq 1$, then **A** and **B** belong to the Gumbel-Hougaard family of copulas (see Section C.2).

By considering Eq. (C.59), we obtain a three-parameter family of asymmetric copulas

$$\begin{aligned}\mathbf{C}_{\alpha,\beta,\delta}(u,v) = \exp\Big(&-\left[(-\alpha\ln u)^\delta + (-\beta\ln v)^\delta\right]^{1/\delta} \\ &-\left[(-\overline{\alpha}\ln u)^\delta + (-\overline{\beta}\ln v)^\delta\right]^{1/\delta}\Big),\end{aligned} \tag{C.63}$$

where $\overline{\alpha} = 1-\alpha$ and $\overline{\beta} = 1-\beta$, representing a generalization of the Gumbel-Hougaard family (also obtained by the method outlined in [111]). ◄

The construction in Eq. (C.59) has the following generalization to the d-dimensional case, $d \geqslant 3$.

PROPOSITION C.4. *Let* **A** *and* **B** *be d-copulas. Then*

$$\mathbf{C}_{\alpha_1,\dots,\alpha_d}(\mathbf{u}) = \mathbf{A}(u_1^{\alpha_1},\dots,u_d^{\alpha_d})\,\mathbf{B}(u_1^{1-\alpha_1},\dots,u_d^{1-\alpha_d}) \tag{C.64}$$

defines a family of d-copulas $\mathbf{C}_{\alpha_1,\dots,\alpha_d}$ *with parameters* $\alpha_1,\dots,\alpha_d \in \mathbb{I}$.

NOTE C.1 (EV copulas). If **A** and **B** are EV copulas, then the copula **C** given by Eq. (C.64) is also an EV copula: this is a direct consequence of the fact that every EV copula is max-stable — see Section 5.1. In particular, in the bivariate case, we can calculate explicitly the dependence function [111] associated with **C** — see Section 5.2.

ILLUSTRATION C.4. ►

Let $\mathbf{A}_\theta$ be a 3-dimensional Cuadras-Augé copula (see Section C.13), with $\theta \in \mathbb{I}$, and let $\mathbf{B} = \mathbf{\Pi}_3$. Given $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{I}$, let **C** be as in Eq. (C.64), i.e.

$$\mathbf{C}(\mathbf{u}) = u_1^{1-\theta\alpha_1}\,u_2^{1-\theta\alpha_2}\,u_3^{1-\theta\alpha_3}\,\min\left\{u_1^{\theta\alpha_1}, u_2^{\theta\alpha_2}, u_3^{\theta\alpha_3}\right\}. \tag{C.65}$$

Then, using Note C.1, **C** is an EV copula (see also Illustration 5.5). Moreover, the marginals $\mathbf{C}_{i,j}$ of **C**, with $i,j = 1,2,3$ and $i \neq j$, are Marshall-Olkin bivariate copulas, with parameters $(\theta\alpha_i, \theta\alpha_j)$. ◄

C.15.3 Copulas with Given Diagonal Section

The problem of finding a copula $\mathbf{C}$ with a given *diagonal section* (see Definition 3.3) has been discussed in several papers [101, 208, 102, 79]. Its relevance stems from the fact that, if $\mathbf{C}$ is a copula associated with two r.v.'s U and V Uniform on $\mathbb{I}$, then the diagonal section of $\mathbf{C}$ contains some information about the behavior of the r.v.'s max $\{U, V\}$ and min $\{U, V\}$ (see also Illustration 3.3).

The diagonal section of a copula $\mathbf{C}$ is the function $\boldsymbol{\delta}_{\mathbf{C}} : \mathbb{I} \to \mathbb{I}$ given by $\boldsymbol{\delta}_{\mathbf{C}}(t) = \mathbf{C}(t, t)$, and satisfies the following properties:

1. $\boldsymbol{\delta}_{\mathbf{C}}(0) = 0$ and $\boldsymbol{\delta}_{\mathbf{C}}(1) = 1$;
2. $\boldsymbol{\delta}_{\mathbf{C}}(t) \leq t$ for all $t \in \mathbb{I}$;
3. $\boldsymbol{\delta}_{\mathbf{C}}$ is increasing;
4. $\boldsymbol{\delta}_{\mathbf{C}}$ is 2-Lipschitz, i.e. $|\boldsymbol{\delta}_{\mathbf{C}}(t) - \boldsymbol{\delta}_{\mathbf{C}}(s)| \leq 2|t - s|$ for all $s, t \in \mathbb{I}$.

We denote by $\mathcal{D}$ the set of all functions with properties (1)–(4), and a function in $\mathcal{D}$ is called a *diagonal*. The question naturally arising is whether, for each diagonal $\boldsymbol{\delta}$, there exists a copula whose diagonal section equals $\boldsymbol{\delta}$. Here we write $\boldsymbol{\delta}_t$ to indicate $\boldsymbol{\delta}(t)$.

The first example is the *Bertino* copula $\boldsymbol{B}_{\delta} : \mathbb{I}^2 \to \mathbb{I}$ given by

$$\boldsymbol{B}_{\boldsymbol{\delta}}(u, v) = \min\{u, v\} - \min\left\{t - \boldsymbol{\delta}_t : t \in [\min\{u, v\}, \max\{u, v\}]\right\}. \tag{C.66}$$

The second example is the *Diagonal* copula $D_{\boldsymbol{\delta}} : \mathbb{I}^2 \to \mathbb{I}$ given by

$$\boldsymbol{D}_{\boldsymbol{\delta}}(u, v) = \min\left\{\frac{\boldsymbol{\delta}_u + \boldsymbol{\delta}_v}{2}, u, v\right\}. \tag{C.67}$$

If $\mathbf{C}$ is a copula whose diagonal section is $\boldsymbol{\delta}$, then $\mathbf{C} \succ \boldsymbol{B}_{\boldsymbol{\delta}}$. Moreover, if $\mathbf{C}$ is also symmetric, it follows that $\mathbf{C} \prec \boldsymbol{D}_{\boldsymbol{\delta}}$.

The values of Kendall's τ_K for, respectively, Bertino and Diagonal copulas are given by:

$$\tau_K(\boldsymbol{B}_{\boldsymbol{\delta}}) = 8 \cdot \int_0^1 \boldsymbol{\delta}_t \, \mathrm{d}t - 3, \tag{C.68a}$$

$$\tau_K(\boldsymbol{D}_{\boldsymbol{\delta}}) = 4 \cdot \int_0^1 \boldsymbol{\delta}_t \, \mathrm{d}t - 3, \tag{C.68b}$$

which may provide a method of fitting the copulas of interest to the available data. The interesting point is that Eqs. (C.68) yield bounds for the values of Kendall's τ_K of a copula $\mathbf{C}$, once the values at the quartiles $\mathbf{C}(i/4, i/4)$, $i = 1, 2, 3$, are known [209].

Constructions of (non-symmetric) copulas with given diagonal section are recently considered in [78], where a method for constructing a family of copulas with given lower and upper tail dependence coefficients is also presented. Specifically, given λ_L and λ_U (the prescribed tail dependence coefficients), consider the following piecewise linear function

$$\delta(t) = \begin{cases} \lambda_L t, & t \in \left[0, \frac{\lambda_U - 1}{2 - \lambda_U - \lambda_L}\right], \\ (2 - \lambda_U)t - 1 + \lambda_U, & \text{otherwise.} \end{cases}$$

Set $l_\delta = \max\{\lambda_L, 2 - \lambda_U\}$. Then, it can be shown that, for every $\lambda \in \left(1 - \frac{1}{l_\delta}, \frac{1}{l_\delta}\right)$, all the copulas in the family $\{\mathbf{C}_{\delta,\lambda}\}$ defined by

$$\mathbf{C}_{\delta,\lambda}(u, v) = \min\{\lambda\,\delta(u) + (1 - \lambda)\,\delta(v), u, v\}$$

have a upper tail dependence coefficient λ_U and a lower tail dependence coefficient λ_L.

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INDEX

GLOSSARY

- **Sets of numbers**

$\mathbb{N}$: naturals
$\mathbb{Z}$: integers
$\mathbb{R}$: reals
$\mathbb{I}$: unit interval [0, 1]

- **Copulas**

$\mathbf{C}$: copula
$\overline{\mathbf{C}}$: survival copula
$\mathbf{c}$: copula density
$\mathbf{W}_d$: d-dimensional Fréchet-Hoeffding lower bound
$\mathbf{M}_d$: d-dimensional Fréchet-Hoeffding upper bound
$\mathbf{\Pi}_d$: d-dimensional independence copula
$\boldsymbol{\delta}_{\mathbf{C}}$: diagonal section of $\mathbf{C}$
γ : Archimedean copula generator
$\gamma^{[-1]}$: pseudo-inverse of γ

- **Association measures**

τ_K : Kendall's τ
ρ_S : Spearman's ρ
ρ_P : Pearson's ρ

- **Probability**

X, Y, Z, W : random variables
U, V : random variables Uniform on $\mathbb{I}$
F_X : c.d.f. of X
f_X : p.d.f. of X
$\overline{F}_X$: survival distribution function of X
$F_{X_1 \cdots X_d}$: joint c.d.f. of $(X_1, \ldots, X_d)$
$f_{X_1 \cdots X_d}$: joint p.d.f. of $(X_1, \ldots, X_d)$
$\overline{F}_{X_1 \cdots X_d}$: joint survival distribution function of $(X_1, \ldots, X_d)$
$F^{-1}, F^{(-1)}$: ordinary inverse of F
$F^{[-1]}$: quasi-inverse of F
c.d.f.: cumulative distribution function

p.d.f. : probability density function
r.v. : random variable or vector
i.d. : identically distributed
i.i.d. : independent identically distributed

- **Statistical operators**

$\mathbb{P}\{\cdot\}$: probability
$\mathbb{E}(\cdot)$: expectation
$\mathbb{V}(\cdot)$: variance
$\mathbb{S}(\cdot)$: standard deviation
$\mathbb{C}(\cdot,\cdot)$: covariance

- **Order Statistics**

$X_{(i)}$: i-th order statistic
$F_{(i)}$: c.d.f. of i-th order statistic
$f_{(i)}$: p.d.f. of i-th order statistic
$F_{(i,j)}$: joint c.d.f. of $(X_{(i)}, X_{(j)})$
$f_{(i,j)}$: joint p.d.f. of $(X_{(i)}, X_{(j)})$

- **Miscellanea**

Dom : domain
Ran : range
Rank : rank
$\mathbf{R}(\cdot)$: risk function
$\zeta(\cdot)$: impact function
$\mathbf{1}_{\{\cdot\}}$: indicator function
$\#(\cdot)$: cardinality
$\Gamma(\cdot)$: Gamma function
$\Gamma_{\cdot}(\cdot)$: Incomplete Gamma function
$\mathrm{B}(\cdot,\cdot)$: Beta function
$\mathrm{B}_{\cdot}(\cdot,\cdot)$: Incomplete Beta function

- **Abbreviations**

EV : Extreme Value
EVC : Extreme Value Copula
MEV : Multivariate Extreme Value
GEV : Generalized Extreme Value
GP : Generalized Pareto
MDA : Maximum Domain of Attraction
CDA : Copula Domain of Attraction
POT : Peaks-Over-Threshold
AMC : Antecedent Moisture Condition