4) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}\right)$

Encuentra la distribución condicional de X_3 dado que $X_1=1$, $X_2=2$

Encuentra la distribución de $X_1+X_2-X_3$

Definimos a $\underline{X}_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ y $\underline{X}_2 = X_3$

Sabemos que

$$\underline{X}_2 \mid \underline{X}_1 \sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\underline{x}_1 - \underline{\mu}_1),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

entonces

Esperanza

$$\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\underline{x}_1 - \underline{\mu}_1) = 2 + (3,\ 5)\,\frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix}$$

$$= 2 + \frac{1}{7}(3,5)\begin{pmatrix} 4x_1 - 4 - x_2 + 1 \\ -x_1 + 1 + 2x_2 - 2 \end{pmatrix} = 2 + \frac{1}{7}(3,5)\begin{pmatrix} 4x_1 - x_2 - 3 \\ -x_1 + 2x_2 - 1 \end{pmatrix}$$

$$= 2 + \frac{1}{7}\left(12x_1 - 3x_2 - 9 - 5x_1 + 10x_2 + 5\right) = 2 + \frac{1}{7}\left(7x_1 + 7x_2 - 4\right)$$

$$= 2 + x_1 + x_2 - \frac{4}{7} = x_1 + x_2 + \frac{10}{7}$$

$X_1 = 1$, $X_2 = 2 \Rightarrow \mathbb{E}(X_3 \mid X_1 = 1, X_2 = 2) = 1 + 2 + \frac{10}{7} = 3 + \frac{10}{7} = \frac{31}{7}$

$\Sigma_{11} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, $\Sigma_{11}^{-1} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$

$\Sigma_{21} = (3\ \ 5)$ $\quad \Sigma_{12} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\Sigma_{22} = 8$

Varianza

$$\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} = 8 - (3,5)\,\frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= 8 - \frac{1}{7}(3,5)\begin{pmatrix} 12 - 5 \\ -3 + 10 \end{pmatrix} = 8 - \frac{1}{7}(3,5)\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$= 8 - (3,5)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 8 - 3 - 5 = 0$$

$$\therefore X_3 \mid X_1 = 1, X_2 = 2 \sim N\left(\frac{31}{7},\ 0\right)$$

4 b) $Y = X_1 + X_2 - X_3 = (x_1, x_2, x_3)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (1\ \ 1\ \ {-1})\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

Como $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}\right)$

$$\Rightarrow Y = (1,1,-1)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left((1,1,-1)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ (1,1,-1)\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right)$$

// $X \sim N_n(\mu, \Sigma) \Rightarrow AX \sim N(A\mu, A\Sigma A^T)$ //

$\mu_Y = (1,1,-1)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 + 1 - 2 = 0$

$\sigma^2_Y = (1,1,-1)\begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (1\ \ 1\ \ {-1})\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$

$\therefore Y \sim N(0, 0)$