(Tarea 1) Estadística Aplicada III

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1) Sea $Y_i = \beta_0 + \beta_1 X_i + u_i$

a) Expresa matricialmente la ecuación anterior.

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \iff \underline{Y} = X \underline{\beta} + \underline{u}$$

b) Utilizando la fórmula $\hat{\beta} = (X^T X)^{-1} X^T Y$ demuestra que

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n X_i Y_i - n\bar{X}\bar{Y}}{\sum_{i=1}^n X_i^2 - n\bar{X}^2}, \quad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

Para el modelo $\underline{Y} = X\underline{\beta} + \underline{u}$ tenemos al estimador $\hat{\beta} = (X^T X)^{-1} X^T \underline{Y}$, donde $X = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}$, $\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$

Analizamos los productos matriciales

$$X^T X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{pmatrix} \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n X_i \\ \sum_{i=1}^n X_i & \sum_{i=1}^n X_i^2 \end{pmatrix}$$

$$X^T \underline{Y} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n X_i Y_i \end{pmatrix}$$

De este modo, tenemos que $\hat{\beta} = (X^T X)^{-1} X^T Y$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n X_i \\ \sum_{i=1}^n X_i & \sum_{i=1}^n X_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n X_i Y_i \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{n}{n}\sum_{i=1}^{n} x_i\right)} \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i^2 & -\frac{1}{n}\sum_{i=1}^{n} x_i \\ -\frac{1}{n}\sum_{i=1}^{n} x_i & 1 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \begin{pmatrix} \bar{y}\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i y_i \\ -\bar{x}\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} x_i y_i \end{pmatrix} \quad \ldots (*)$$

Verificamos,

$$\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \frac{2n\bar{x}}{n}\sum_{i=1}^{n} x_i + n\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i\bar{x} + \sum_{i=1}^{n} \bar{x}^2 = \sum_{i=1}^{n} \left\{x_i^2 - 2x_i\bar{x} + \bar{x}^2\right\}$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2$$

De este modo, $\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$$\Leftrightarrow \sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 \quad //$$

De (*) resulta que

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \frac{1}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \begin{pmatrix} \bar{y}\left\{\sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{x}^2\right\} - \bar{x}\sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{y}\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x})^2\bar{y} - \bar{x}\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \\ \dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{y}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) + \left\{n\bar{x}\bar{y} - \sum_{i=1}^{n} x_i y_i\right\}\bar{x}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \\ \dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \bar{y} - \left(\dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}\right)\bar{x} \\ \left(\dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}\right) \end{pmatrix}$$

Así pues,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

2) Sea $Y_i = \beta_0 + u_i$

a) Encuentra el estimador de la anterior regresión lineal, utilizando la fórmula $\hat{\beta} = (X^T X)^{-1} X^T Y$.

La regresión se escribe como

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \beta_0 + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \iff \underline{Y} = \underline{X}\beta_0 + \underline{u}$$

Usamos $\hat{\beta} = (X^T X)^{-1} X^T Y$

Vamos los productos matriciales

$$X^T X = (1\ 1\ \ldots\ 1)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n, \quad X^T Y = (1\ 1 \ldots 1)\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \sum_{i=1}^{n} Y_i$$

De este modo, $\hat{\beta} = (X^T X)^{-1} X^T Y$

$$\iff \hat{\beta}_0 = n^{-1} \sum_{i=1}^{n} Y_i \iff \hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} Y_i \iff \hat{\beta}_0 = \overline{Y}$$

b) Encuentra las matrices H y M, demuestra que son simétricas, idempotentes y encuentra sus rangos.

i) La matriz $H = X(X^T X)^{-1} X^T$

$$H = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} n^{-1} (1\ 1 \ldots 1) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \left(\tfrac{1}{n}\ \tfrac{1}{n} \cdots \tfrac{1}{n}\right) = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

1.- H es simétrica porque

$$H^T = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}^T = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} = H \quad \therefore H = H^T$$

2:- H es idempotente porque

$$H^2=\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}=\begin{pmatrix} \frac{n}{n^2} & \frac{n}{n^2} & \cdots & \frac{n}{n^2} \\ \frac{n}{n^2} & \frac{n}{n^2} & \cdots & \frac{n}{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n}{n^2} & \frac{n}{n^2} & \cdots & \frac{n}{n^2} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}=H$$

H es simétrica e idempotente

3:- $\text{rango}(H)=1$ pues $\text{rango}(H)=\text{tr}(H)=\sum_{i=1}^{n}\frac{1}{n}=\frac{n}{n}=1$

ii) La matriz $M=I-H$

$$M=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}-\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}=\begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}$$

1:- M es simétrica porque

$$M^T=\begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}^T=\begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}=M$$

2:- M es idempotente porque

$$M^2=\begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}\begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}=$$

$$= \begin{pmatrix} \frac{(n-1)^2}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} , & \frac{1-n}{n^2} + \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} , & \ldots , & \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{1-n}{n^2} \\ \frac{1-n}{n^2} + \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} , & \frac{1}{n^2} + \frac{(n-1)^2}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} , & \ldots , & \frac{1}{n^2} + \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{1-n}{n^2} \\ \vdots & & & \\ \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{1-n}{n^2} , & \frac{1}{n^2} + \frac{1-n}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{1-n}{n^2} , & \ldots , & \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{(n-1)^2}{n^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n^2 - 2n + 1 + n - 1}{n^2} , & \frac{2 - 2n + n - 2}{n^2} , & \ldots , & \frac{2 - 2n + n - 2}{n^2} \\ \frac{2 - 2n + n - 2}{n^2} , & \frac{n^2 - 2n + 1 + n - 1}{n^2} , & \ldots , & \frac{2 - 2n + n - 2}{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2 - 2n + n - 2}{n^2} , & \frac{2 - 2n + n - 2}{n^2} , & \ldots , & \frac{n^2 - 2n + 1 + n - 1}{n^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n^2 - n}{n^2} & -\frac{n}{n^2} & \cdots & -\frac{n}{n^2} \\ -\frac{n}{n^2} & \frac{n^2 - n}{n^2} & \cdots & -\frac{n}{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{n}{n^2} & -\frac{n}{n^2} & \cdots & \frac{n^2 - n}{n^2} \end{pmatrix} = \begin{pmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix} = M$$

3.- Rango $(M) = n-1$ porque

$$\underbrace{\text{Rango}(M) = Tr(M)}_{M \text{ idempotente y simétrica}} = tr(I - H) = tr(I) - tr(H)$$

$$= n - 1$$

3) Sea $X \sim N(\mu, \sigma^2 I)$ con $\mu = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

a) Encuentra la distribución de $Y = \frac{x^T A x}{\sigma^2}$, con $A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

Usaremos el teorema siguiente

Teorema:

Sea $\underline{X} \sim N(\mu, \sigma^2 I)$, $\underline{\mu}, \underline{x} \in \mathbb{R}^n$, $I \in \mathbb{R}^{n \times n}$. Sea $A \in \mathbb{R}^{n \times n}$ idempotente, simétrica de rango $(A) = k \leq n$, $A\underline{\mu} = 0$, entonces

$$\frac{x^T A x}{\sigma^2} \sim \chi^2_{(k)}$$

Vemos que

1) $A^T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}^T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = A$, por lo que A es simétrica

2) $A^2 = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 3/9 & 3/9 & 3/9 \\ 3/9 & 3/9 & 3/9 \\ 3/9 & 3/9 & 3/9 \end{pmatrix}$

$= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = A$, por lo que A es idempotente

3) Entonces, rango $(A) = tr(A) = \sum_{i=1}^{3} \frac{1}{3} = \frac{3}{3} = 1 \leq 3$

4) Verifiquemos el producto

$A\underline{\mu} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} + \frac{1}{3} - \frac{2}{3} \\ \frac{1}{3} + \frac{1}{3} - \frac{2}{3} \\ \frac{1}{3} + \frac{1}{3} - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$

Como cumple con las hipótesis, $Y = \frac{x^T A x}{\sigma^2} \sim \chi^2_{(1)}$

b) Sea $w = B^T x$, donde B es un vector de tamaño 3×1, encuentra todos los B, tal que Y y W sean independientes. ⑧

Usaremos el teorema siguiente.

Teorema:

Sea $\underline{X} \sim N(\underline{\mu}, \sigma^2 I)$, $\underline{X}, \underline{\mu} \in \mathbb{R}^{n\times 1}$. Sea A simétrica e idempotente, rango $(A) = k$, $A\underline{\mu} = 0$, $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times 1}$ tal que $AB = \underline{0}$ entonces

$\frac{\underline{x}^T A \underline{x}}{\sigma^2}$ y $B^T \underline{X}$ son independientes.

Ya sabemos que $Y = \frac{\underline{x}^T A \underline{x}}{\sigma^2}$ cumple con $A \in \mathbb{R}^{n\times m}$ simétrica e idempotente, con $A\mu = \underline{0}$, rango$(A) = 1$, $\frac{\underline{x}^T A \underline{x}}{\sigma^2} \sim \chi^2_{(1)}$.

Falta verificar, y hallar, $B \in \mathbb{R}^{n\times 1}$ tal que $AB = \underline{0}$

$$AB = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} \frac{1}{3} b_1 + \frac{1}{3} b_2 + \frac{1}{3} b_3 = 0 \\ \frac{1}{3} b_1 + \frac{1}{3} b_2 + \frac{1}{3} b_3 = 0 \\ \frac{1}{3} b_1 + \frac{1}{3} b_2 + \frac{1}{3} b_3 = 0 \end{cases}$$

$$\Leftrightarrow \frac{1}{3} b_1 + \frac{1}{3} b_2 + \frac{1}{3} b_3 = 0 \Leftrightarrow b_1 + b_2 + b_3 = 0$$

Así, $Y = \frac{\underline{x}^T A \underline{x}}{\sigma^2}$ y $W = B^T \underline{X}$ son independientes si

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ tal que } b_1 + b_2 + b_3 = 0.$$

Escaneado con CamScanner

c) Sea $q = \frac{\underline{x}^T C \underline{x}}{\sigma^2}$ con $C = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$. Encuentra la distribución de q y demuestre que es independiente a Y.

Usaremos el teorema

Teorema:

Sea $\underline{X} \sim N(\mu, \sigma^2 I)$, $\underline{X}, \underline{\mu} \in \mathbb{R}^{n \times 1}$, $I \in \mathbb{R}^{n \times n}$. Sean $A, B \in \mathbb{R}^{n \times n}$ simétricas, idempotentes, $AB = 0$, $A = P D_1 P^{-1}$, $B = P D_2 P^{-1}$, con rango $(A) = k_1$, rango $(B) = k_2 \leq n$, $A\underline{\mu} = 0$, $B\underline{\mu} = 0$

$$\frac{\underline{x}^T A \underline{x}}{\sigma^2} \sim \chi^2_{(k_1)}, \quad \frac{\underline{x}^T B \underline{x}}{\sigma^2} \sim \chi^2_{(k_2)}$$

entonces

$$Y = \frac{\underline{x}^T A \underline{x}}{\sigma^2}, \quad W = \frac{\underline{x}^T B \underline{x}}{\sigma^2}$$ son independientes

Por los incisos anteriores, sabemos que se cumple $A \in \mathbb{R}^{3 \times 3}$ simétrica, idempotente, rango $(A) = 1 \leq 3$, $A\underline{\mu} = 0$ y $\frac{\underline{x}^T A \underline{x}}{\sigma^2} \sim \chi^2_{(1)}$

Estudiamos a $C \in \mathbb{R}^{3 \times 3}$

1) C es simétrica porque

$$C^T = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}^T = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = C$$

2) C es idempotente ya que

$$C^2 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9}+\frac{1}{9}+\frac{1}{9} & -\frac{2}{9}-\frac{2}{9}+\frac{1}{9} & -\frac{2}{9}+\frac{1}{9}-\frac{2}{9} \\ -\frac{2}{9}-\frac{2}{9}+\frac{1}{9} & \frac{1}{9}+\frac{4}{9}+\frac{1}{9} & \frac{1}{9}-\frac{2}{9}-\frac{2}{9} \\ -\frac{2}{9}+\frac{1}{9}-\frac{2}{9} & \frac{1}{9}-\frac{2}{9}-\frac{2}{9} & \frac{1}{9}+\frac{1}{9}+\frac{4}{9} \end{pmatrix} = \begin{pmatrix} 6/9 & -3/9 & -3/9 \\ -3/9 & 6/9 & -3/9 \\ -3/9 & -3/9 & 6/9 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = A$$

3) Al ser C simétrica e idempotente,

$$\text{rango}(C) = tr(C) = \sum_{i=1}^{3} \frac{2}{3} = \frac{6}{3} = 2 \leq 3$$

4) Veamos al producto

$$AC = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{9}-\frac{1}{9}-\frac{1}{9} & -\frac{1}{9}+\frac{2}{9}-\frac{1}{9} & -\frac{1}{9}-\frac{1}{9}+\frac{2}{9} \\ \frac{2}{9}-\frac{1}{9}-\frac{1}{9} & -\frac{1}{9}+\frac{2}{9}-\frac{1}{9} & -\frac{1}{9}-\frac{1}{9}+\frac{2}{9} \\ \frac{2}{9}-\frac{1}{9}-\frac{1}{9} & -\frac{1}{9}+\frac{2}{9}-\frac{1}{9} & -\frac{1}{9}-\frac{1}{9}+\frac{2}{9} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

5) Verificamos si $A = P D P^{-1}$, $A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

Buscamos sus eigenvalores

$$|A-\lambda I| = \begin{vmatrix} 1/3-\lambda & 1/3 & 1/3 \\ 1/3 & 1/3-\lambda & 1/3 \\ 1/3 & 1/3 & 1/3-\lambda \end{vmatrix} = \left(\frac{1}{3}-\lambda\right)\begin{vmatrix} 1/3-\lambda & 1/3 \\ 1/3 & 1/3-\lambda \end{vmatrix} - \frac{1}{3}\begin{vmatrix} 1/3 & 1/3 \\ 1/3 & 1/3-\lambda \end{vmatrix}$$

$$+ \frac{1}{3}\begin{vmatrix} 1/3 & 1/3-\lambda \\ 1/3 & 1/3 \end{vmatrix}$$

$$= \left(\frac{1}{3} - \lambda\right)\left\{\left(\frac{1}{3} - \lambda\right)^2 - \frac{1}{9}\right\} - \frac{1}{3}\left\{\cancel{\frac{1}{9}} - \frac{1}{3}\lambda - \cancel{\frac{1}{9}}\right\} + \frac{1}{3}\left\{\cancel{\frac{1}{9}} - \cancel{\frac{1}{9}} + \frac{1}{3}\lambda\right\}$$

$$= \left(\frac{1}{3} - \lambda\right)\left\{\cancel{\frac{1}{9}} - \frac{2}{3}\lambda + \lambda^2 \cancel{- \frac{1}{9}}\right\} + \frac{1}{9}\lambda + \frac{1}{9}\lambda$$

$$= -\frac{2}{9}\lambda + \frac{1}{3}\lambda^2 + \frac{2}{3}\lambda^2 - \lambda^3 + \frac{2}{9}\lambda = -\lambda^3 + \lambda^2 = P_A(\lambda)$$

$P_A(\lambda) = 0 \iff -\lambda^3 + \lambda^2 = 0 \iff \lambda^2(-\lambda + 1) = 0 \iff \lambda_1 = 0$ {Raíz doble

$\lambda_2 = 1$ {Raíz simple

Buscamos los eigenvectores

Si $\lambda_1 = 0$

$$(A - \lambda_1 I \mid 0) = \left(\begin{array}{ccc|c} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{array}\right) \underset{\substack{-R_1+R_3 \\ 3R_1}}{\overset{-R_1+R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Vemos que $x_1 + x_2 + x_3 = 0 \iff x_1 = -x_2 - x_3 \iff \begin{cases} x_1 = -t - s \\ x_2 = t \\ x_3 = s \end{cases}, \ t, s \in \mathbb{R}$

$$\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \quad t, s \in \mathbb{R}$$

Así, $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ y $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ son los eigenvectores asociados

Si $\lambda_2 = 1$

$$(A - \lambda_2 I \mid 0) = \left(\begin{array}{ccc|c} -2/3 & 1/3 & 1/3 & 0 \\ 1/3 & -2/3 & 1/3 & 0 \\ 1/3 & 1/3 & -2/3 & 0 \end{array}\right) \underset{\substack{3R_2 \\ 3R_3}}{\overset{-\frac{3}{2}R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\underset{-R_1+R_3}{\overset{-R_1+R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \end{array}\right) \underset{-\frac{2}{3}R_2}{\overset{+R_2+R_3}{\sim}} \left(\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\frac{1}{2}R_2+R_1 \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Vemos que $\begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \Leftrightarrow \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix} \Leftrightarrow \begin{matrix} x_1 = t \\ x_2 = t \\ x_3 = t \end{matrix}, \ t \in \mathbb{R}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

por lo que v_3 es el eigenvector asociado

De esto modo, tenemos que

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P^{-1} = \overbrace{\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}}^{(*)}$$

Para P^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{-R_1+R_3}{\overset{-R_1+R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} -2R_3+R_2 \\ +R_3+R_1 \\ R_2 \leftrightarrow R_3 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 & 1 & -2 \end{array}\right) \underset{\begin{matrix} -2R_3+R_2 \\ -R_3+R_1 \end{matrix}}{\overset{-\frac{1}{3}R_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 0 & 1 & -1/3 & -1/3 & 2/3 \end{array}\right)$$

$$= P^{-1} \ (*)$$

Asimismo, vemos que

$$P D_1 P^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = A$$

Así, $A = P D_1 P^{-1}$

6) Vemos si $C = P D_1 P^{-1}$, $C = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$

Buscamos sus eigenvalores

$$|C - \lambda I| = \begin{vmatrix} \frac{2}{3}-\lambda & -1/3 & -1/3 \\ -1/3 & \frac{2}{3}-\lambda & -1/3 \\ -1/3 & -1/3 & \frac{2}{3}-\lambda \end{vmatrix} = \left(\frac{2}{3}-\lambda\right)\begin{vmatrix} \frac{2}{3}-\lambda & -1/3 \\ -1/3 & \frac{2}{3}-\lambda \end{vmatrix} - \left(-\frac{1}{3}\right)\begin{vmatrix} -\frac{1}{3} & -1/3 \\ -1/3 & \frac{2}{3}-\lambda \end{vmatrix} + \left(-\frac{1}{3}\right)\begin{vmatrix} -1/3 & \frac{2}{3}-\lambda \\ -1/3 & -1/3 \end{vmatrix}$$

$$= \left(\frac{2}{3}-\lambda\right)\left\{\left(\frac{2}{3}-\lambda\right)^2 - \frac{1}{9}\right\} + \frac{1}{3}\left\{-\frac{2}{9}+\frac{\lambda}{3}-\frac{1}{9}\right\} - \frac{1}{3}\left\{\frac{1}{9}+\frac{2}{9}-\frac{\lambda}{3}\right\}$$

$$= \left(\frac{2}{3}-\lambda\right)\left\{\frac{4}{9}-\frac{4}{3}\lambda+\lambda^2-\frac{1}{9}\right\} + \frac{1}{3}\left\{\frac{\lambda}{3}-\frac{1}{3}\right\} - \frac{1}{3}\left\{\frac{1}{3}-\frac{\lambda}{3}\right\}$$

$$= \left(\frac{2}{3}-\lambda\right)\left\{\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}\right\} + \frac{\lambda}{9}-\frac{1}{9}-\frac{1}{9}+\frac{\lambda}{9}$$

$$= \frac{2}{3}\lambda^2 - \frac{8}{9}\lambda + \cancel{\frac{2}{9}} - \lambda^3 + \frac{4}{3}\lambda^2 - \frac{\lambda}{3} + \frac{2}{9}\lambda - \cancel{\frac{2}{9}}$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = P_C(\lambda)$$

$P_C(\lambda) = 0 \Leftrightarrow -\lambda^3 + 2\lambda^2 - \lambda = 0 \Leftrightarrow \lambda(\lambda^2 - 2\lambda + 1) = 0$

$\Leftrightarrow \lambda(\lambda-1)^2 = 0 \Leftrightarrow$ $\lambda_1 = 0$ { Raíz simple

$\lambda_2 = 1$ { Raíz doble

Buscamos los eigenvectores

Si $\lambda_1 = 0$ — análogo con $\lambda_2 = 1$ en A

$$(C - \lambda_1 I | 0) = \left(\begin{array}{ccc|c} 2/3 & -1/3 & -1/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Leftrightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \Leftrightarrow \begin{matrix} x_1 = t \\ x_2 = t \\ x_3 = t \end{matrix}$, $t \in \mathbb{R} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

Así, $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ es el eigenvector asociado

Si $\lambda_2 = 1$ análogo a $\lambda_1 = 0$ en A (14)

$$(C_2 - \lambda_2 I | 0) = \left(\begin{array}{ccc|c} -1/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & -1/3 & 0 \\ -1/3 & -1/3 & -1/3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Leftrightarrow x_1 + x_2 + x_3 = 0 \Leftrightarrow \begin{array}{l} x_1 = -t-s \\ x_2 = t \\ x_3 = s \end{array} ; t,s \in \mathbb{R} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, t,s \in \mathbb{R}$$

Así, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ son los eigenvectores asociados

Entonces,

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P^{-1} = \overbrace{\begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}}^{(**)}$$

Para P^{-1}

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{-R_1}{\overset{R_1+R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{R_3 \leftrightarrow R_2 \\ -R_3+R_1}}{\overset{R_3+R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \underset{\substack{-R_3+R_2 \\ 2R_3+R_1}}{\overset{\frac{1}{3}R_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right)$$

$$\underbrace{\qquad}_{(**)\, P^{-1}}$$

Asimismo, vemos que

$$P D_2 P^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = C$$

Así pues, $C = P D_2 P^{-1}$

7) Quedo ver $C\mu$

$$C\mu = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

que no cumple con la hipótesis del teorema.
Pero, si consideramos $X \sim N_3(\mu, \sigma^2 I)$ con $\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ tenemos que se cumplen todas las hipótesis, podemos decir que

$$Y = \frac{X^T A X}{\sigma^2} \sim \chi^2_{(1)} \quad y \quad q = \frac{X^T C X}{\sigma^2} \sim \chi^2_{(2)}$$

de donde resulta que son independientes.

4) $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + U_i$

con $SCReg = 160$, $SCT = 200$, si tenemos $n = 44$ observaciones con $\alpha = 0.05$, demuestre si el modelo es conjuntamente significativo. Especifica la hipótesis nula y alterna.

Hacemos la prueba de hipótesis

$H_0: \beta_1 = \beta_2 = \beta_3 = 0$ v.s. $H_1: \beta_1^2 + \beta_2^2 + \beta_3^2 \neq 0$

Calculamos el estadístico F^* y $F_{(3,40),0.05}$.

$$F^* = \frac{SCReg/k}{SCRes/(n-(k+1))} = \frac{160/3}{40/(44-4)} = \frac{160}{3} = 53.\bar{3}$$

$F_{(3,40),0.05} = 2.838745$

$SCT = SCReg + SCRes \iff SCRes = SCT - SCReg$

Entonces, $F^* = 53.\bar{3} \geq 2.838745 = F_{(3,40),0.05}$ por lo que se rechaza $H_0: \beta_1 = \beta_2 = \beta_3 = 0$

Asimismo si calculamos el valor-p (p-value

p-value = $\mathbb{P}(F_{(3,40)} \geq 53.333) = 4.84988 \times 10^{-14} \leq 0.05 = \alpha$

Como p-value $\approx 0 < 0.05 = \alpha$ rechazo Ho.

5) Tenemos los siguientes dos modelos:

I) $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + U_i$, SCRes = 20

II) $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_3 X_{3i} + U_i$, SCRes = 25

Con $n = 55$ y $\alpha = 0.01$ encuentra el mejor modelo. Especifica la hipótesis nula y alterna.

Ho: Modelo Restringido

H1: Modelo no Restringido

Tenemos la prueba de hipótesis

$H_0: \beta_2 = \beta_4 = 0$ v.s. $H_1: \beta_2^2 + \beta_4^2 \neq 0.$

$\underbrace{SCR_{NR}}_{\text{Modelo no Restringido}} = 20, \quad \underbrace{SCR_R}_{\text{Modelo Restringido}} = 25$

Si Ho es cierta entonces el modelo restringido es mejor

Si rechazo Ho, resulta que el modelo no restringido es mejor.

Calculemos F^* y $F_{(k-r,\, n-(k+1)),\alpha}$

$$F^* = \frac{(SCR_R - SCR_{NR})/(k-r)}{SCR_{NR}/(n-(k+1))} = \frac{(25-20)/(4-2)}{20/(55-5)} = \frac{5/2}{20/50} = \frac{25}{4} = 6.25$$

$F_{(k-r,\, n-(k+1)),\alpha} = F_{(2,50),0.01} = 0.01005236$

Así que $F^* = 6.25 \geq 0.01005236 = F_{(2,50),0.01}$ por lo que rechazamos Ho.

Para el p-value = $\mathbb{P}(F_{(2,50)} \geq 6.25) = 0.003777$

Como p-value $= 0.003777 \leq 0.01 = \alpha$ resulta que rechazo Ho por lo que el mejor modelo es el modelo no restringido porque X_{2i} y X_{4i} afectan a cada Y_i, $i = 1, \ldots, n$.