Fluid Dynamics Mathematical Work

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1 Finite Volume Discretization

This section delineates the discretized solutions to the terms present in the Navier-Stokes equations employing the **finite volume method**. In this methodology, each term is integrated over the cell volumes constituting the mesh. The mesh under consideration is two-dimensional and structured, forming a regular grid. Due to the structured nature of the mesh, the finite volume method yields results equivalent to those of the finite difference method. Nevertheless, the finite volume method is elaborated in full detail to facilitate its application in more complex scenarios where it may diverge from the finite difference approach.

1.1 Transient Term

The transient term encapsulates the time-dependent behavior of the scalar field and is integrated over the area of a specific cell:

$$\iint_A \frac{\partial \phi}{\partial t}\, dA = \frac{\partial \phi_p}{\partial t} \iint_A dA = \frac{\partial \phi_p}{\partial t} \Delta x \Delta y = \frac{\phi_p^{n+1} - \phi_p^n}{\Delta t} \Delta x \Delta y$$

Here, ϕ represents any scalar field defined over the mesh. The subscript p denotes the particular cell, while the superscript n indicates the discrete time step. It is assumed that both the time derivative and the scalar field ϕ are constant within cell p, allowing them to be extracted from the integral. Consequently, the integral simplifies to the area of the cell, $\Delta x \Delta y$.

1.2 Diffusion Term

The diffusion term accounts for the spatial variation of the scalar field and is integrated over the cell's area as follows:

$$\alpha \iint_A \nabla^2 \phi\, dA = 2\alpha \left(\frac{\Delta x}{\Delta y}(\phi_T + \phi_B - 2\phi_p) + \frac{\Delta y}{\Delta x}(\phi_L + \phi_R - 2\phi_p) \right)$$

In this expression, α denotes the diffusion coefficient. The subscripts T, B, L, and R correspond to the top, bottom, left, and right faces of the cell, respectively. The derivation leverages the Divergence Theorem to transform the area integral into a line integral around the closed contour C of the cell, which is assumed to be a square with sides of lengths Δx and Δy.

Application of the Divergence Theorem Starting with the Laplacian of ϕ:

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi)$$

Applying the Divergence Theorem, where $\nabla \phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}}$ and $\hat{\mathbf{n}} = \frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}}$:

$$\iint_A \nabla^2 \phi\, dA = \iint_A \nabla \cdot (\nabla \phi)\, dA = \oint_C \nabla \phi \cdot \hat{\mathbf{n}}\, ds = \oint_C \left(\frac{\partial \phi}{\partial x}\frac{dy}{ds} - \frac{\partial \phi}{\partial y}\frac{dx}{ds} \right) ds = \oint_C \frac{\partial \phi}{\partial x}\, dy - \frac{\partial \phi}{\partial y}\, dx$$

Given the structured grid, the contour C comprises four straight sides with lengths Δx and Δy. For the top and bottom sides (C_T and C_B), $dy = 0$, and for the left and right sides (C_L and C_R), $dx = 0$. This simplification allows the line integrals to be evaluated individually:

$$\oint_C \frac{\partial \phi}{\partial x}\, dy - \frac{\partial \phi}{\partial y}\, dx = -\int_{C_T} \frac{\partial \phi}{\partial y}\Big|_{y_T} dx - \int_{C_B} \frac{\partial \phi}{\partial y}\Big|_{y_B} dx + \int_{C_L} \frac{\partial \phi}{\partial x}\Big|_{x_L} dy + \int_{C_R} \frac{\partial \phi}{\partial x}\Big|_{x_R} dy$$

Evaluation of Line Integrals Each line integral is evaluated under the assumption that ϕ is constant along each contour due to the discrete nature of the system (note that for the coordinate value on the given face relative to the given cell's center, we have $y_T = y_p + \frac{\Delta y}{2}$, $y_B = y_p - \frac{\Delta y}{2}$, $x_L = x_p - \frac{\Delta x}{2}$, and $x_R = x_p + \frac{\Delta x}{2}$):

$$C_T : -\int_{C_T} \frac{\partial \phi}{\partial y}\Big|_{y_T} dx = -\frac{\partial \phi}{\partial y}\Big|_{y_T} \int_{x_R}^{x_L} dx = -\frac{\partial \phi}{\partial y}\Big|_{y_T} (\Delta x) = \frac{\partial \phi}{\partial y}\Big|_{y_T} \Delta x = \frac{\phi_T - \phi_p}{y_T - y_p}\Delta x = \frac{2\Delta x}{\Delta y}(\phi_T - \phi_p),$$

$$C_B : -\int_{C_B} \frac{\partial \phi}{\partial y}\Big|_{y_B} dx = -\frac{\partial \phi}{\partial y}\Big|_{y_B} \int_{x_L}^{x_R} dx = -\frac{\partial \phi}{\partial y}\Big|_{y_B} (\Delta x) = -\frac{\phi_B - \phi_p}{y_B - y_p}\Delta x = \frac{2\Delta x}{\Delta y}(\phi_B - \phi_p),$$

$$C_L : \int_{C_L} \frac{\partial \phi}{\partial x}\Big|_{x_L} dy = \frac{\partial \phi}{\partial x}\Big|_{x_L} \int_{y_T}^{y_B} dy = \frac{\partial \phi}{\partial x}\Big|_{x_L} (-\Delta y) = -\frac{\phi_L - \phi_p}{x_L - x_p}\Delta y = \frac{2\Delta y}{\Delta x}(\phi_L - \phi_p),$$

$$C_R : \int_{C_R} \frac{\partial \phi}{\partial x}\Big|_{x_R} dy = \frac{\partial \phi}{\partial x}\Big|_{x_R} \int_{y_B}^{y_T} dy = \frac{\partial \phi}{\partial x}\Big|_{x_R} (\Delta y) = \frac{\phi_R - \phi_p}{x_R - x_p}\Delta y = \frac{2\Delta y}{\Delta x}(\phi_R - \phi_p).$$

Summation of Line Integrals Summing the contributions from each contour segment:

$$\begin{aligned}\oint_C \frac{\partial \phi}{\partial x}\, dy - \frac{\partial \phi}{\partial y}\, dx &= \frac{2\Delta x}{\Delta y}(\phi_T - \phi_p) + \frac{2\Delta x}{\Delta y}(\phi_B - \phi_p) + \frac{2\Delta y}{\Delta x}(\phi_L - \phi_p) + \frac{2\Delta y}{\Delta x}(\phi_R - \phi_p) \\ &= 2\left(\frac{\Delta x}{\Delta y}(\phi_T + \phi_B - 2\phi_p) + \frac{\Delta y}{\Delta x}(\phi_L + \phi_R - 2\phi_p)\right)\end{aligned}$$

Thus, the final expression for the diffusion term is:

$$\alpha \iint_A \nabla^2 \phi\, dA = 2\alpha \left(\frac{\Delta x}{\Delta y}(\phi_T + \phi_B - 2\phi_p) + \frac{\Delta y}{\Delta x}(\phi_L + \phi_R - 2\phi_p)\right)$$

1.3 Convection Term

The convection term represents the transport of the scalar field by the velocity field and is integrated over the cell's area as follows:

$$\iint_A \mathbf{u} \cdot \nabla \phi\, dA = (v_T \phi_T - v_B \phi_B)\Delta x + (u_R \phi_R - u_L \phi_L)\Delta y$$

Assuming an incompressible fluid, where the velocity field satisfies $\nabla \cdot \mathbf{u} = 0$, the following identity is employed:

$$\mathbf{u} \cdot \nabla \phi = \nabla \cdot (\mathbf{u}\phi) - \phi \nabla \cdot \mathbf{u} = \nabla \cdot (\mathbf{u}\phi)$$

Application of the Divergence Theorem Applying the Divergence Theorem to the convection term:

$$\iint_A \mathbf{u} \cdot \nabla \phi\, dA = \iint_A \nabla \cdot (\mathbf{u}\phi)\, dA = \oint_C (\mathbf{u}\phi) \cdot \hat{\mathbf{n}}\, ds = \oint_C \left(u\phi \frac{dy}{ds} - v\phi \frac{dx}{ds}\right) ds = \oint_C (u\phi\, dy - v\phi\, dx)$$

Given the structured grid, the contour C consists of four straight sides. Each integral along the contour is evaluated individually:

Evaluation of Line Integrals

$$C_T : \int_{C_T} (u\phi\, dy - v\phi\, dx)\Big|_{y_T} = -\int_{x_R}^{x_L} v\phi\, dx\Big|_{y_T} = -v\phi \int_{x_R}^{x_L} dx\Big|_{y_T} = -v\phi\Delta x\Big|_{y_T} = v_T \phi_T \Delta x,$$

$$C_B : \int_{C_B} (u\phi\, dy - v\phi\, dx)\Big|_{y_B} = -\int_{x_L}^{x_R} v\phi\, dx\Big|_{y_B} = -v\phi \int_{x_L}^{x_R} dx\Big|_{y_B} = -v\phi\Delta x\Big|_{y_B} = -v_B \phi_B \Delta x,$$

$$C_L : \int_{C_L} (u\phi\, dy - v\phi\, dx)\Big|_{x_L} = \int_{y_T}^{y_B} u\phi\, dy\Big|_{x_L} = u\phi \int_{y_T}^{y_B} dy\Big|_{x_L} = u\phi(-\Delta y)\Big|_{x_L} = -u\phi\Delta y\Big|_{x_L} = -u_L \phi_L \Delta y,$$

$$C_R : \int_{C_R} (u\phi\, dy - v\phi\, dx)\Big|_{x_R} = \int_{y_B}^{y_T} u\phi\, dy\Big|_{x_R} = u\phi \int_{y_B}^{y_T} dy\Big|_{x_R} = u\phi\Delta y\Big|_{x_R} = u\phi\Delta y\Big|_{x_R} = u_R \phi_R \Delta y.$$

Summation of Line Integrals Aggregating the contributions from each contour segment:

$$\oint_C (u\phi\, dy - v\phi\, dx) = v_T\phi_T\Delta x - v_B\phi_B\Delta x - u_L\phi_L\Delta y + u_R\phi_R\Delta y = (v_T\phi_T - v_B\phi_B)\Delta x + (u_R\phi_R - u_L\phi_L)\Delta y$$

Thus, the final expression for the convection term is:

$$\iint_A \mathbf{u}\cdot\nabla\phi\, dA = (v_T\phi_T - v_B\phi_B)\Delta x + (u_R\phi_R - u_L\phi_L)\Delta y$$

1.4 Body Force Terms

Body forces, such as gravity, are integrated over the cell's area similarly to the other terms. Considering gravity and buoyancy effects, the body force terms are expressed as:

$$\rho\mathbf{g} - \rho\mathbf{g}\beta T : \iint_A \rho g\, dA - \iint_A \rho g\beta T\, dA = \Delta x\Delta y(\rho g - \rho g\beta T_p)$$

Here, ρ is the fluid density, $\mathbf{g}$ is the gravitational acceleration vector, and β is the thermal expansion coefficient. Typically, the gravitational vector $\mathbf{g}$ has only one non-zero component, $\mathbf{g} = g\hat{\mathbf{j}}$, simplifying the integration process. The gravitational force is assumed to be uniform and constant across the grid, allowing the body force to be evaluated at the cell center T_p.

1.5 Pressure Gradient Term

The pressure gradient term is a critical component of the momentum Navier-Stokes equations, representing the force exerted by pressure variations within the fluid. Unlike other terms, it does not possess an explicit time-dependent formulation. Instead, it is incorporated indirectly through the solution process of the Navier-Stokes equations.

Formulation Without Pressure Gradient To solve the Navier-Stokes equations, an intermediate velocity vector field, denoted as $\mathbf{u}^* = u^*\hat{\mathbf{i}} + v^*\hat{\mathbf{j}}$, is computed by temporarily excluding the pressure gradient term. This intermediate velocity does not inherently satisfy the incompressibility condition ($\nabla\cdot\mathbf{u} = 0$), necessitating subsequent correction.

The relationship between the intermediate velocity and the updated velocity field $\mathbf{u}^{n+1}$ using the finite volume method is given by:

$$\rho\iint_A \frac{\partial\mathbf{u}^*}{\partial t}\, dA = -\iint_A \nabla P\, dA,$$

which (due to area terms canceling out) becomes:

$$\rho\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\nabla P \implies \mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho}\nabla P,$$

Enforcing Incompressibility To ensure that the updated velocity field $\mathbf{u}^{n+1}$ satisfies the incompressibility condition ($\nabla\cdot\mathbf{u}^{n+1} = 0$), we substitute the expression for $\mathbf{u}^{n+1}$:

$$\nabla\cdot\mathbf{u}^{n+1} = \nabla\cdot\left(\mathbf{u}^* - \frac{\Delta t}{\rho}\nabla P\right) = 0 \implies \nabla\cdot\mathbf{u}^* - \frac{\Delta t}{\rho}\nabla\cdot\nabla P = 0 \implies \nabla^2 P = \frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^*$$

This results in the Poisson equation for pressure:

$$\nabla^2 P = \frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^*$$

Discretizing the Poisson Equation The central difference method is used to discretize the Poisson equation. The neighboring values are evaluated at their respective cell centers, not the faces of the given cell. They could be solved at the faces, but ultimately the solution is equivalent on a structured grid:

$$\nabla^2 P = \frac{(P_T + P_B - 2P_p)}{\Delta y^2} + \frac{(P_L + P_R - 2P_p)}{\Delta x^2}$$

Thus, we have:

$$\frac{(P_T + P_B - 2P_p)}{\Delta y^2} + \frac{(P_L + P_R - 2P_p)}{\Delta x^2} = \frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^*,$$

which is used to solve for P_p:

$$P_p = -\frac{\Delta x^2\Delta y^2\frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^* - \Delta x^2(P_T + P_B) - \Delta y^2(P_L + P_R)}{2(\Delta x^2 + \Delta y^2)}$$

Iterative Solution Approach The Poisson equation for pressure, being a global equation, requires solving a system where the pressure at each cell depends on the pressures of its neighboring cells. Due to the interconnected nature of the pressure field, an exact solution is computationally expensive. Instead, iterative methods such as the Gauss-Seidel or Successive Over-Relaxation (SOR) are employed to approximate the solution efficiently by updating the pressure values multiple time based on neighboring pressures until convergence is achieved.

The method that I used assumed the initial pressure field to be zero. The pressure field will still iteratively converge to the correct pressure field because the divergence of the intermediate velocity, $\nabla \cdot \mathbf{u}^*$, is not zero.

After the desired number of iterations, the corrected velocity field is solved for with:

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho} \nabla P,$$

which becomes the following two equations when separating the velocity components:

$$u^{n+1} = u^* - \frac{\Delta t}{\rho} \, ,$$

1.6 Summary of Discretized Navier-Stokes Equations

By amalgamating all the discretized terms—the transient, diffusion, convection, pressure gradient, and body force terms—the finite volume formulation of the Navier-Stokes equations for each cell in the structured grid is obtained. This formulation underpins the numerical solution of fluid flow problems by systematically accounting for the various physical phenomena influencing the scalar and velocity fields. The pressure gradient term, in particular, is addressed through the introduction of an intermediate velocity field and the solution of a Poisson equation to enforce incompressibility, ensuring a divergence-free velocity field in the final solution.