Waves

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1 Wave Equation

The wave equation is one of the most fundamental partial differential equations, describing wave propagation in various physical settings. It arises in contexts such as mechanical vibrations (e.g., waves on strings or membranes), acoustics (sound waves in gases or liquids), elasticity (seismic waves in solids), and electromagnetism (light waves in vacuum or media).

In an n-dimensional domain, the wave equation is commonly written as:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi,$$

where c is the wave speed. The quantity ϕ could represent displacement (e.g., in a vibrating string), pressure fluctuations (in acoustics), or electric/magnetic field components (in electromagnetism).

Mathematical Interpretation The rate of change of the rate of change of ϕ over time at a point (the acceleration of the *value* of ϕ) is proportional to the rate of change of the rate of change of ϕ over space at that same point. For a greater positive curvature in space, the acceleration of ϕ's value increases. The initial value of ϕ and the rate of change of ϕ at all points are two the initial conditions that are needed to solve the wave equation.

Consider a sine wave IC for the initial values of ϕ, and zero for the initial velocity IC of ϕ. For a region with negative curvature, the value of ϕ is positive, so its value 'accelerates' to become negative, until it is negative, which then it has a positive curvature, causing the value of ϕ to 'accelerate' to become positive. Qualitatively, this manifests as a wave oscillating.

To get the wave to move rather that oscillate in one spot, the initial velocity IC of ϕ must be equal to the spatial gradient of the initial values IC of ϕ, which just so happens to qualitatively manifest as a moving wave.

Even with this mathematical intuition, the behavior of the wave equation with various ICs for both ϕ and the rate of change of ϕ over time can yield strange, yet correct, visuals when numerically solved and simulated in Python.

Derivation The wave equation can be derived easiest from the 1-dimension consideration of a horizontal string under uniform tension T from both ends, with linear mass density μ, and a vertical displacement given by y, which is a function of x and t, $y(x, t)$ more specifically. This 1-dimension derivation can be generalized to higher dimensions because the force in the x-direction depends only on x, and so on for higher dimensions.

Each string segment of length dx experiences a tension force that acts tangentially to the string: $\mathbf{T} = T\hat{\mathbf{t}}$, where $\hat{\mathbf{t}}$ is the unit tangent vector defined by:

$$\hat{\mathbf{t}} = \frac{1}{\sqrt{1 + (\frac{\partial y}{\partial x})^2}}\hat{\mathbf{i}} + \frac{\frac{\partial y}{\partial x}}{\sqrt{1 + (\frac{\partial y}{\partial x})^2}}\hat{\mathbf{j}}$$

Thus, the tension is:

$$\mathbf{T} = T\frac{1}{\sqrt{1 + (\frac{\partial y}{\partial x})^2}}\hat{\mathbf{i}} + T\frac{\frac{\partial y}{\partial x}}{\sqrt{1 + (\frac{\partial y}{\partial x})^2}}\hat{\mathbf{j}}$$

The vertical component of T at x and $x + dx$ is of interest:

$$T_y(x) = T\frac{\left.\frac{\partial y}{\partial x}\right|_x}{\sqrt{1 + (\left.\frac{\partial y}{\partial x}\right|_x)^2}},$$

$$T_y(x + dx) = T\frac{\left.\frac{\partial y}{\partial x}\right|_{x+dx}}{\sqrt{1 + (\left.\frac{\partial y}{\partial x}\right|_{x+dx})^2}}$$

The net vertical tension acting on the segment of length dx is given by $T_y = T_y(x+dx) - T_y(x)$, which resembles the definition of the derivative without dx in the denominator. In fact, $T_y = T_y(x+dx) - T_y(x) = \frac{dT_y}{dx}dx$, which is expanded to:

$$T_y = T\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial x}}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}\right)dx$$

Recall that tension is another way of saying force, so the upward force on the string at each segment of length dx is:

$$F_y = T\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial x}}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}\right)dx$$

By Newton's Second Law, the vertical motion of the string at each segment of length dx is governed by $F_y = dm\ a_y$, which can be written as $F_y = \mu dx\frac{\partial^2 y}{\partial t^2}$, where $\mu = \frac{dm}{dx}$. Thus, substituting for F_y simplifies to (the dx cancels out):

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial x}}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}\right)$$

The chain rule is used to expand the derivative on the right hand side:

$$\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial x}}{\sqrt{1+(\frac{\partial y}{\partial x})^2}}\right) = \frac{\frac{\partial^2 y}{\partial x^2}}{(1+(\frac{\partial y}{\partial x})^2)^{3/2}},$$

which is then substituted to give the final wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}\cdot\frac{1}{(1+(\frac{\partial y}{\partial x})^2)^{3/2}},$$

where $c^2 = \frac{T}{\mu}$ and c is the velocity of the string's vertical motion. This is the exact non-linear wave equation that governs the motion of the string. It is commonly assumed that the vertical change in displacement is very small, so it is then approximated by $\frac{\partial y}{\partial x} \approx 0$, which simplifies the wave equation to its more familiar linear form:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

Note that this equation is technically incorrect due to the approximation step, but this form is much easier to solve. For situations where $\frac{\partial y}{\partial x}$ is large, the non-linear form of the wave equation would be better.

The non-linear and linear wave equations generalized to n-dimensions are:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\nabla^2\phi\cdot\frac{1}{(1+|\nabla\phi|^2)^{3/2}},$$

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\nabla^2\phi$$

2 Schrodinger Equation

I am glad I took a class on complex variables, because otherwise I wouldn't understand just how fascinating this equation is. I am highly skeptical of the philosophical stance that mathematics is discovered and not invented. This equation substantiates that skepticism for me. The complex plane is utilized here to explain the wavelike behavior of electrons (particles) during the double-slit experiment.

I like writing the Schrodinger equation in this form:

$$i\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m}\nabla^2\Psi,$$

which is easiest to interpret. Ψ is called the wavefunction, which is a complex scalar field used to describe the probability/likely-hood of of finding a particle for all points within the same domain (the probability density). $\hbar$ is the reduced Planck constant. If the complex unit i were not present, this would a simple diffusion equation.

Mathematical Interpretation Since Ψ is a complex number, it can be expressed as its real and imaginary components on the complex plane: $\Psi = \Psi_R + i\Psi_I$, where $\forall \Psi_R, \Psi_I \in \mathbb{R}$ (Ψ_I by itself is a real number, but it is the complex component since it is multiplied by i). then:

$$i\frac{\partial}{\partial t}(\Psi_R + i\Psi_I) = -\frac{\hbar}{2m}\nabla^2(\Psi_R + i\Psi_I) \implies i\frac{\partial \Psi_R}{\partial t} - \frac{\partial \Psi_I}{\partial t} = -\frac{\hbar}{2m}\nabla^2\Psi_R - \frac{i\hbar}{2m}\nabla^2\Psi_I$$

Because the equation has real and imaginary terms on opposite sides of the equation, the real components must equal each other and the imaginary components must equal each other. This makes sense because a real number can't be equal to an imaginary number, $a \neq ib$, $\forall a, b \in \mathbb{R}$, so the equation can be separated into the real and imaginary components, where the following must be true:

$$\underbrace{-\frac{\partial \Psi_I}{\partial t} = -\frac{\hbar}{2m}\nabla^2\Psi_R,}_{\text{Real Components (no } i\text{)}}$$

$$\underbrace{i\frac{\partial \Psi_R}{\partial t} = -\frac{i\hbar}{2m}\nabla^2\Psi_I}_{\text{Imaginary Components (} i \text{ is present)}} \qquad \implies \frac{\partial \Psi_R}{\partial t} = -\frac{\hbar}{2m}\nabla^2\Psi_I$$

It is clear now that Ψ_I influences Ψ_R and vice versa. The observable wavefunction

Since Ψ is complex, two separate wavefunctions interacting may cause destructive interference rather than only constructive interference due to the complex numbers being out of phase and canceling out. Mathematically, this makes sense if studied with the appropriate visuals (see Welch Labs on YouTube). In terms of the physical reality, I think this is simply a logic trick that just so happens to describe well enough what we observe in our experiments, just like the Navier–Stokes equations with fluids.