CHAPTER FOUR

Frequency-Domain Analysis

Having determined the entries in the per-unit-length parameter matrices of inductance, **L**, capacitance, **C**, conductance, **G**, and resistance, **R**, for the particular line cross-sectional dimensions as in Chapter 3, we now embark upon the solution of the resulting MTL equations. In this chapter, we consider the *frequency-domain* solution of the MTL equations where the excitation sources are sinusoids which have been applied a sufficient length of time so that the line voltages and currents are in *steady state*. In the next chapter, we consider the *time-domain* solution of the MTL equations wherein the sources (and, consequently, the time variations of the line voltages and currents) may have arbitrary time variation. The time-domain solution will be the sum of the transient and steady-state responses.

4.1 THE MTL EQUATIONS FOR SINUSOIDAL STEADY-STATE EXCITATION

We assume that the time variation of the sources is sinusoidal and the line is in steady state. Therefore the line voltages and currents are also sinusoidal having a magnitude and a phase angle. Thus we denote the line voltages and line currents in their *phasor* form [A.2]:

$$V_i(z, t) = \mathcal{Re}\{\hat{V}_i(z)e^{j\omega t}\} \tag{4.1a}$$

$$I_i(z, t) = \mathcal{Re}\{\hat{I}_i(z)e^{j\omega t}\} \tag{4.1b}$$

where $\mathcal{Re}\{\cdot\}$ denotes the *real part* of the enclosed complex quantity, and the *phasor* voltages and currents have a magnitude and phase angle as

$$\begin{aligned}\hat{V}_i(z) &= V_i(z)\underline{/\theta_i(z)} \\ &= V_i(z)e^{j\theta_i(z)}\end{aligned} \tag{4.2a}$$

$$\hat{I}_i(z) = I_i(z)\underline{/\phi_i(z)} \qquad (4.2b)$$
$$= I_i(z)e^{j\phi_i(z)}$$

We will denote all complex (phasor) quantities with ^ over the quantity. The radian frequency of excitation (as well as the radian frequency of the resulting line voltages and currents) is denoted by ω where $\omega = 2\pi f$ and f is the cyclic frequency of excitation. Applying (4.1) to (4.2) gives the resulting *time-domain* forms as [A.1, A.2]

$$V_i(z, t) = V_i(z)\cos(\omega t + \theta_i(z)) \qquad (4.3a)$$

$$I_i(z, t) = I_i(z)\cos(\omega t + \phi_i(z)) \qquad (4.3b)$$

The time-domain MTL equations are given, in matrix form, in equation (2.27):

$$\frac{\partial}{\partial z}\mathbf{V}(z, t) = -\mathbf{R}\mathbf{I}(z, t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z, t) \qquad (4.4a)$$

$$\frac{\partial}{\partial z}\mathbf{I}(z, t) = -\mathbf{G}\mathbf{V}(z, t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z, t) \qquad (4.4b)$$

For sinusoidal variation of the sources and line voltages and currents, the time variation is assumed to be $e^{j\omega t}$ as in (4.1) so that derivatives with respect to time, t, in (4.4) are replaced by $j\omega$. Substituting the phasor forms for the line voltages and currents given in (4.1) into (4.4) gives the MTL equations for sinusoidal, steady-state excitation as

$$\frac{d}{dz}\hat{\mathbf{V}}(z) = -\hat{\mathbf{Z}}\hat{\mathbf{I}}(z) \qquad (4.5a)$$

$$\frac{d}{dz}\hat{\mathbf{I}}(z) = -\hat{\mathbf{Y}}\hat{\mathbf{V}}(z) \qquad (4.5b)$$

where the per-unit-length *impedance matrix*, $\hat{\mathbf{Z}}$, and *admittance matrix*, $\hat{\mathbf{Y}}$, are given by

$$\hat{\mathbf{Z}} = \mathbf{R} + j\omega\mathbf{L} \qquad (4.6a)$$

$$\hat{\mathbf{Y}} = \mathbf{G} + j\omega\mathbf{C} \qquad (4.6b)$$

In taking the time derivatives to produce (4.5), we have assumed that the per-unit-length parameter matrices, **R**, **L**, **G**, and **C** are independent of time, t, i.e., the cross-sectional dimensions and surrounding media properties do not change with time. This is a local assumption but should be explicitly stated. The resulting equations in (4.5) to be solved are a set of *coupled-first-order, ordinary differential equations* with *complex coefficients*. They can be put in a more compact matrix form as

$$\frac{d}{dz}\hat{\mathbf{X}}(z) = \hat{\mathbf{A}}\hat{\mathbf{X}}(z) \tag{4.7a}$$

where

$$\hat{\mathbf{X}}(z) = \begin{bmatrix} \hat{\mathbf{V}}(z) \\ \hat{\mathbf{I}}(z) \end{bmatrix} \tag{4.7b}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & -\hat{\mathbf{Z}} \\ -\hat{\mathbf{Y}} & \mathbf{0} \end{bmatrix} \tag{4.7c}$$

Observe that for an $(n+1)$-conductor line, $\hat{\mathbf{V}}(z)$ and $\hat{\mathbf{I}}(z)$ are $n \times 1$ and contain the *phasor* line voltages and currents, respectively, and $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$ are $n \times n$. Therefore $\hat{\mathbf{X}}(z)$ is $2n \times 1$ and $\hat{\mathbf{A}}$ is $2n \times 2n$. Our task in this chapter will be to solve (4.7) and incorporate the terminal constraints. (The terminal constraints contain the lumped voltage and current source excitations and the load impedances.) Equations (4.7), being a set of coupled, first-order, differential equations, are similar in form to the *state-variable equations* found in the analysis of lumped systems wherein the independent variable is time, t, [A.2], whereas in (4.7) the independent variable is position along the line, z. Because of the direct similarity between the frequency-domain MTL equations and the state-variable equations, we will adapt the known solution properties for the state-variable equations *directly* to the solution of the frequency-domain MTL equations by making the simple analogy of time, t, in the state-variable solutions to position along the line, z, in the frequency-domain MTL equation solutions. This simple observation will obviate the necessity to obtain redundant solutions and will illuminate a number of interesting properties of the phasor MTL solution which are drawn by direct parallel from the state-variable solution.

Alternatively, the coupled, first-order phasor MTL equations in (4.5) can be placed in the form of *uncoupled, second-order, ordinary differential equations* by differentiating both with respect to line position, z, and substituting the first-order equations given in (4.5) as

$$\frac{d^2}{dz^2}\hat{\mathbf{V}}(z) = \hat{\mathbf{Z}}\hat{\mathbf{Y}}\hat{\mathbf{V}}(z) \tag{4.8a}$$

$$\frac{d^2}{dz^2}\hat{\mathbf{I}}(z) = \hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{I}}(z) \tag{4.8b}$$

Ordinarily, the per-unit-length parameter matrices $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$ do not commute so that the proper order of multiplication in (4.8) must be observed. In differentiating (4.5) with respect to line position, z, we have assumed that the per-unit-length parameter matrices, $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{C}$, are independent of z. Thus we have assumed that the cross-sectional line dimensions and surrounding

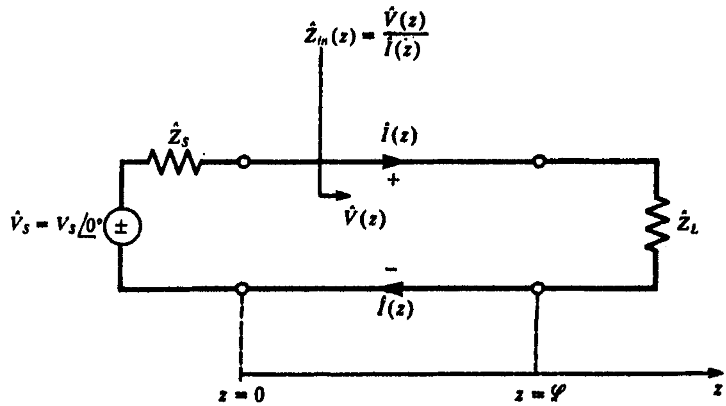


FIGURE 4.1 Definition of the parameters of a two-conductor line in the frequency domain.

media properties are constant along the line or, in other words, the line is a *uniform* line. Both the first-order, coupled forms of the MTL equations given in (4.5) or equivalently in (4.7) as well as the second-order, uncoupled forms given in (4.8) will be useful in obtaining the final solution.

4.2 SOLUTIONS FOR TWO-CONDUCTOR LINES

In this section we will summarize the well-known solutions for a two-conductor line shown in Fig. 4.1 [A.1, A.3]. This will be useful in the MTL solution since there are numerous analogies and parallels to this solution that appear in matrix form in the MTL solution.

For a two-conductor line, $n = 1$, the per-unit-length parameter matrices become scalars, r, l, g, and c, and the uncoupled second-order equations in (4.8) become

$$\frac{d^2}{dz^2}\hat{V}(z) = \hat{\gamma}^2\hat{V}(z) \tag{4.9a}$$

$$\frac{d}{dz^2}\hat{I}(z) = \hat{\gamma}^2\hat{I}(z) \tag{4.9b}$$

where the *propagation constant*, $\hat{\gamma}$, is

$$\begin{aligned}\hat{\gamma} &= \sqrt{\hat{z}\hat{y}} \qquad (4.10)\\ &= \sqrt{(r + j\omega l)(g + j\omega c)}\\ &= \alpha + j\beta\end{aligned}$$

where α is the *attenuation constant* whose units are nepers/m and β is the *phase constant* whose units are radians/m. The general form of the solution to these equations is [A.1, A.3]

$$\hat{V}(z) = \hat{V}^+ e^{-\hat{\gamma}z} + \hat{V}^- e^{\hat{\gamma}z} \tag{4.11a}$$

$$\begin{aligned}\hat{I}(z) &= \hat{I}^+ e^{-\hat{\gamma}z} + \hat{I}^- e^{\hat{\gamma}z} \\ &= \frac{\hat{V}^+}{\hat{Z}_C} e^{-\hat{\gamma}z} - \frac{\hat{V}^-}{\hat{Z}_C} e^{\hat{\gamma}z}\end{aligned} \tag{4.11b}$$

The terms $\hat{V}^+$, $\hat{V}^-$, $\hat{I}^+$, and $\hat{I}^-$ are complex-valued, *undetermined constants* which will be determined when we incorporate the terminal conditions at the two ends of the line. The quantity $\hat{Z}_C$ is the *characteristic impedance* of the line and is given, in terms of the per-unit-length parameters, as

$$\begin{aligned}\hat{Z}_C &= \sqrt{\frac{\hat{z}}{\hat{y}}} \\ &= \sqrt{\frac{(r + j\omega l)}{(g + j\omega c)}} \\ &= Z_C \angle \theta_z\end{aligned} \tag{4.12}$$

Substituting (4.10) and (4.12) into (4.11) gives

$$\begin{aligned}\hat{V}(z) &= V^+ e^{j\theta^+} e^{-\alpha z} e^{-j\beta z} + V^- e^{j\theta^-} e^{\alpha z} e^{j\beta z} \\ &= V^+ e^{-\alpha z} e^{-j(\beta z - \theta^+)} + V^- e^{\alpha z} e^{j(\beta z + \theta^-)}\end{aligned} \tag{4.13a}$$

$$\begin{aligned}\hat{I}(z) &= \frac{V^+}{Z_C} e^{j\theta^+} e^{-j\theta_Z} e^{-\alpha z} e^{-j\beta z} - \frac{V^-}{Z_C} e^{j\theta^-} e^{-j\theta_Z} e^{\alpha z} e^{j\beta z} \\ &= \frac{V^+}{Z_C} e^{-\alpha z} e^{-j(\beta z + \theta_Z - \theta^+)} - \frac{V^-}{Z_C} e^{\alpha z} e^{j(\beta z - \theta_Z + \theta^-)}\end{aligned} \tag{4.13b}$$

where the magnitudes and phases of the undetermined constants are noted by

$$\hat{V}^+ = V^+ \angle \theta^+ \tag{4.14a}$$

$$\hat{V}^- = V^- \angle \theta^- \tag{4.14b}$$

The time-domain expressions are obtained from (4.1) as

$$V(z, t) = V^+ e^{-\alpha z} \cos(\omega t - \beta z + \theta^+) + V^- e^{\alpha z} \cos(\omega t + \beta z + \theta^-) \tag{4.15a}$$

$$I(z, t) = \frac{V^+}{Z_C} e^{-\alpha z} \cos(\omega t - \beta z - \theta_Z + \theta^+) - \frac{V^-}{Z_C} e^{\alpha z} \cos(\omega t + \beta z - \theta_Z + \theta^-) \tag{4.15b}$$

These expressions are the sums of *forward-traveling waves*, traveling in the $+z$ direction:

$$V^+(z, t) = V^+ e^{-\alpha z} \cos(\omega t - \beta z + \theta^+) \tag{4.16a}$$

$$I^+(z, t) = \frac{V^+}{Z_C} e^{-\alpha z} \cos(\omega t - \beta z - \theta_Z + \theta^+) \tag{4.16b}$$

and *backward-traveling waves*, traveling in the $-z$ direction:

$$V^-(z, t) = V^- e^{\alpha z} \cos(\omega t + \beta z + \theta^-) \tag{4.17a}$$

$$I^-(z, t) = \frac{V^-}{Z_C} e^{\alpha z} \cos(\omega t + \beta z - \theta_Z + \theta^-) \tag{4.17b}$$

as

$$V(z, t) = V^+(z, t) + V^-(z, t) \tag{4.18a}$$

$$I(z, t) = I^+(z, t) - I^-(z, t) \tag{4.18b}$$

That these are traveling in the $+z$ and $-z$ directions can be seen from the observation that as time t progresses, z must either increase or decrease in order to keep the arguments of the cosine terms constant in order to track corresponding points on the waveforms [A.1]. The terms $e^{\pm \alpha z}$ represent *attenuation* of the amplitudes of the waves. The velocity of these waves is [A.1]:

$$v = \frac{\omega}{\beta} \tag{4.19a}$$

If the line is lossless (perfect conductors and lossless medium) then the velocity of these waves is the velocity of TEM waves in the surrounding (assumed homogeneous) medium:

$$v = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad r = g = 0 \tag{4.19b}$$

The ratios of backward-traveling and forward-traveling voltage waves at any point on the line is referred to as the *reflection coefficient* at that point. Taking the ratios of these in phasor form from (4.11) gives [A.1]

$$\begin{aligned} \hat{\Gamma}(z) &= \frac{\hat{V}^- e^{\hat{\gamma} z}}{\hat{V}^+ e^{-\hat{\gamma} z}} \\ &= \frac{\hat{V}^-}{\hat{V}^+} e^{2\hat{\gamma} z} \end{aligned} \tag{4.20a}$$

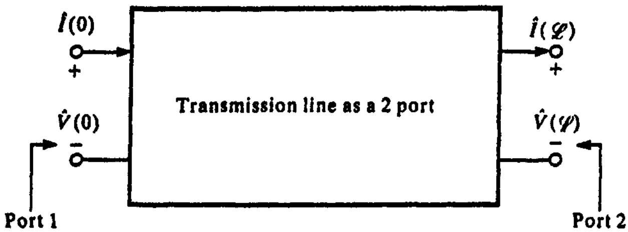


FIGURE 4.2 Illustration of viewing a two-conductor line as a two port in the frequency domain.

The reflection coefficients at two points on the line, z_2, z_1, are related as [A.1]

$$\hat{\Gamma}(z_2) = \hat{\Gamma}(z_1)e^{2\hat{\gamma}(z_2 - z_1)} \tag{4.20b}$$

Consider the two-conductor line shown in Fig. 4.1. The total line length is denoted by $\mathscr{L}$. The solutions given in (4.11) contain undetermined constants, $\hat{V}^+$ and $\hat{V}^-$. These can be eliminated by putting the solution in the form of the *chain parameter matrix* as

$$\begin{bmatrix} \hat{V}(\mathscr{L}) \\ \hat{I}(\mathscr{L}) \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{21} & \hat{\phi}_{22} \end{bmatrix} \begin{bmatrix} \hat{V}(0) \\ \hat{I}(0) \end{bmatrix} \tag{4.21}$$

This representation relates the line voltages at one end of the line, $z = \mathscr{L}$, to the line voltages and currents at the other end of the line, $z = 0$. In fact, the chain parameter matrix can be used to relate the voltage and current at any point on the line, z, to those at $z = 0$ by replacing $\mathscr{L}$ with z in (4.21) and the results that follow. Similarly, the chain parameter matrix can be used to relate the line voltages and currents at two interior points on the line, z_1 and z_2 with $z_2 \geq z_1$. by replacing $\mathscr{L}$ with z_2 and 0 with z_1 in (4.21) and the results that follow. The name, *chain parameter matrix*, is derived from the observation that the overall chain parameter matrix of several such lines in cascade is the product (in the appropriate order) of the chain parameter matrices of the individual lines in the chain. In fact, this observation provides an approximate method of modeling nonuniform lines such as twisted pairs of wires as a sequence or cascade of uniform lines [G.1–G.10]. The chain parameter matrix can be viewed as a way of characterizing the line at its end points as a *two port* as illustrated in Fig. 4.2 [A.2]. Evaluating the general solution in (4.11) at $z = \mathscr{L}$ and at $z = 0$ gives

$$\begin{bmatrix} \hat{V}(\mathscr{L}) \\ \hat{I}(\mathscr{L}) \end{bmatrix} = \begin{bmatrix} e^{-\hat{\gamma}\mathscr{L}} & e^{\hat{\gamma}\mathscr{L}} \\ \dfrac{1}{\hat{Z}_C} e^{-\hat{\gamma}\mathscr{L}} & -\dfrac{1}{\hat{Z}_C} e^{\hat{\gamma}\mathscr{L}} \end{bmatrix} \begin{bmatrix} \hat{V}^+ \\ \hat{V}^- \end{bmatrix} \tag{4.22a}$$

$$\begin{bmatrix} \hat{V}(0) \\ \hat{I}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \dfrac{1}{\hat{Z}_C} & -\dfrac{1}{\hat{Z}_C} \end{bmatrix} \begin{bmatrix} \hat{V}^+ \\ \hat{V}^- \end{bmatrix} \tag{4.22b}$$

Solving these gives the chain parameter matrix as

$$\begin{aligned} \hat{\Phi} &= \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} \\ \hat{\phi}_{21} & \hat{\phi}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \cosh(\hat{\gamma}\mathscr{L}) & -\hat{Z}_C \sinh(\hat{\gamma}\mathscr{L}) \\ -\dfrac{1}{\hat{Z}_C}\sinh(\hat{\gamma}\mathscr{L}) & \cosh(\hat{\gamma}\mathscr{L}) \end{bmatrix} \end{aligned} \tag{4.23}$$

where the hyperbolic cosine and sine are

$$\cosh(\hat{\gamma}\mathscr{L}) = \frac{e^{\hat{\gamma}\mathscr{L}} + e^{-\hat{\gamma}\mathscr{L}}}{2} \tag{4.24a}$$

$$\sinh(\hat{\gamma}\mathscr{L}) = \frac{e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}}}{2} \tag{4.24b}$$

Now that the general form of the solution has been obtained, we incorporate the terminal constraints in order to evaluate the undetermined constants in that general solution. Reconsider the two-conductor line shown in Fig. 4.1. The line is terminated at the load end, $z = \mathscr{L}$, with a load impedance $\hat{Z}_L$. At the source end, $z = 0$, an independent voltage source, $\hat{V}_S = V_S\angle 0°$, and source impedance, $\hat{Z}_S$, terminate the line. Thus the *terminal constraints* are:

$$\hat{V}(0) = \hat{V}_S - \hat{Z}_S\hat{I}(0) \tag{4.25a}$$

$$\hat{V}(\mathscr{L}) = \hat{Z}_L\hat{I}(\mathscr{L}) \tag{4.25b}$$

Substituting these constraints into the chain parameter form of the solution gives the explicit form of the solution for the line voltages and currents at any position along the line as [A.1, A.3]

$$\hat{V}(z) = \frac{1 + \hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}} e^{2\hat{\gamma}z}}{1 - \hat{\Gamma}_S\hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}}} \frac{\hat{Z}_C}{\hat{Z}_C + \hat{Z}_S} \hat{V}_S e^{-\hat{\gamma}z} \tag{4.26a}$$

$$\hat{I}(z) = \frac{1 - \hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}} e^{2\hat{\gamma}z}}{1 - \hat{\Gamma}_S\hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}}} \frac{1}{\hat{Z}_C + \hat{Z}_S} \hat{V}_S e^{-\hat{\gamma}z} \tag{4.26b}$$

where the *reflection coefficients* of the source ($\hat{\Gamma}_S$) and the load ($\hat{\Gamma}_L$) are given by

$$\hat{\Gamma}_S = \frac{\hat{Z}_S - \hat{Z}_C}{\hat{Z}_S + \hat{Z}_C} \tag{4.27a}$$

$$\hat{\Gamma}_L = \frac{\hat{Z}_L - \hat{Z}_C}{\hat{Z}_L + \hat{Z}_C} \tag{4.27b}$$

The *input impedance* at any point along the line can be obtained as the ratio of the line voltage and current at that point as

$$\begin{aligned}\hat{Z}_{\text{in}}(z) &= \hat{Z}_C \frac{1 + \hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}} e^{2\hat{\gamma} z}}{1 - \hat{\Gamma}_L e^{-2\hat{\gamma}\mathscr{L}} e^{2\hat{\gamma} z}} \\ &= \hat{Z}_C \frac{\hat{Z}_L + \hat{Z}_C \tanh(\hat{\gamma}(\mathscr{L} - z))}{\hat{Z}_C + \hat{Z}_L \tanh(\hat{\gamma}(\mathscr{L} - z))}\end{aligned} \tag{4.28}$$

If the line is *matched* at the load, i.e., $\hat{Z}_L = \hat{Z}_C$, then the reflection coefficient at the load is zero, $\hat{\Gamma}_L = 0$, and these relations simplify to

$$\hat{V}(z) = \frac{\hat{Z}_L}{\hat{Z}_L + \hat{Z}_S} \hat{V}_S e^{-\hat{\gamma} z} \qquad \hat{Z}_L = \hat{Z}_C \tag{4.29a}$$

$$\hat{I}(z) = \frac{1}{\hat{Z}_L + \hat{Z}_S} \hat{V}_S e^{-\hat{\gamma} z} \qquad \hat{Z}_L = \hat{Z}_C \tag{4.29b}$$

$$\hat{Z}_{\text{in}}(z) = \hat{Z}_C = \hat{Z}_L \qquad \hat{Z}_L = \hat{Z}_C \tag{4.29c}$$

The net flow of *average power* in the $+z$ direction is [A.1, A.2]

$$\begin{aligned}P_{\text{av}}(z) &= \tfrac{1}{2}\mathscr{Re}\{\hat{V}(z)\hat{I}^*(z)\} \text{ W} \\ &= \frac{1}{2}\frac{|\hat{V}^+|^2}{Z_C}(1 - |\hat{\Gamma}_L|^2)\end{aligned} \tag{4.30}$$

where $\hat{M}^*$ denotes the *conjugate* of the complex quantity $\hat{M}$ [A.2]. If the line is matched at the load, the reflection coefficient is zero and (4.30) shows that all the power is traveling in the $+z$ direction, i.e., there is no power reflected at the load and hence traveling in the $-z$ direction.

4.3 GENERAL SOLUTION FOR AN ($n + 1$)-CONDUCTOR LINE

In the previous section we discussed the well-known solution for a two-conductor line. In this section we begin our study of the solutions for an

$(n + 1)$-conductor line or MTL. In many cases, the results and properties of the solution for a two-conductor line carry over, with matrix notation, to a MTL.

4.3.1 Analogy of the MTL Equations to the State-Variable Equations

Transmission lines are *distributed-parameter systems.* If the electrical dimensions of a structure are small, it can be approximately modeled as a *lumped-parameter system.* The independent variables for a distributed-parameter system are the spatial dimensions, x, y, z, and time, t. In the case of a lumped-parameter system, the quantities of interest are lumped rather than distributed throughout space so that they depend only on time, t. Lumped-parameter systems are characterized by ordinary differential equations, whereas distributed-parameter systems such as transmission lines are characterized by partial differential equations as we have seen. If our interest is only in sinusoidal, steady-state behavior of the MTL, the use of phasor quantities removes the time dependence. In the case of a transmission line the only spatial parameter is the line axis, z, and the partial differential equations become ordinary differential equations with complex-valued coefficients as is illustrated in equations (4.5), (4.7), and (4.8). So we may make a direct analogy between the sinusoidal, steady-state transmission-line equations and those of a lumped-parameter system by viewing the spatial parameter, z, in the distributed-parameter system phasor equations as the equivalent of time, t, in the lumped-parameter system-governing equations. This important observation will allow a considerable simplification of the necessary work to obtain the solution to the phasor, MTL equations. It will also allow considerable insight into the properties of that solution since we may draw, by analogy, from the abundance of known properties of the solution for the lumped-parameter system.

Consider a lumped-parameter system. One way of representing such a system is via the n coupled, ordinary differential equations in *state-variable form* as [A.2, 1, 2, 3]

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{W}(t) \tag{4.31a}$$

where

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_i(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{4.31b}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1l} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{l1} & \cdots & a_{ll} & \cdots & a_{ln} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nl} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1l} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{l1} & \cdots & b_{ll} & \cdots & b_{lp} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nl} & \cdots & b_{np} \end{bmatrix} \tag{4.31c}$$

$$\mathbf{W}(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_l(t) \\ \vdots \\ w_p(t) \end{bmatrix} \tag{4.31d}$$

The matrices **A** and **B** are assumed to be independent of the independent variable, t, in which case the system is said to be *stationary*, i.e., its parameters do not vary with time. This property is analogous to a *uniform* transmission line where $\hat{\mathbf{A}}$ in (4.7) is assumed independent of z, i.e., the line cross-sectional dimensions and media properties are constant along the line. The $w_i(t)$ are viewed as the p *inputs* to the *system*, and the $x_i(t)$ are viewed as the n *state variables* of the system. Any output of the system can be represented as a *linear combination* of the state variables and the system inputs [A.2, B.1]. In order to determine the response of this system to the (presumably known) inputs, we must prescribe the *initial conditions* on the state variables at some initial time t_0, $\mathbf{X}(t_0)$. As with any other set of ordinary differential equations, the total solution is the sum of the *zero input response*, with $\mathbf{W}(t) = \mathbf{0}$, and the *zero initial state response* with $\mathbf{X}(t_0) = \mathbf{0}$.

Let us begin this discussion of the solution to the state variable equations by considering a *first-order, lumped system, $n = 1$*, whose state-variable equations become the scalar equations

$$\frac{d}{dt}\, x(t) = ax(t) + bw(t) \tag{4.32}$$

with a prescribed initial state, $x(t_0)$. The solution of the homogeneous equations, $w(t) = 0$, is [A.2]

$$\begin{aligned} x_h(t) &= e^{a(t-t_0)}x(t_0) \\ &= \phi(t-t_0)x(t_0) \end{aligned} \tag{4.33}$$

The notation $\phi(t) = e^{at}$ is referred to as the *state-transition* function. Recall that the exponential is defined as the infinite series

$$e^{at} = 1 + \frac{t}{1!}a + \frac{t^2}{2!}a^2 + \frac{t^3}{3!}a^3 + \cdots \tag{4.34}$$

The solution to the original equation in (4.32) is referred to as the *particular solution*. It can be obtained from the above homogeneous solution via the *method of variation of parameters* [B.1] by replacing the initial state, $x(t_0)$, with an undetermined constant that is a function of t,

$$x_p(t) = e^{at}k(t) \tag{4.35}$$

Substituting this into the original equation, (4.32), gives

$$ae^{at}k(t) + e^{at}\frac{d}{dt}k(t) = ae^{at}k(t) + bw(t) \tag{4.36}$$

Equation (4.36) becomes

$$\frac{d}{dt}k(t) = e^{-at}bw(t) \tag{4.37}$$

which has the solution [A.2, B.1]

$$k(t) = \int_{t_0}^{t} e^{-a\tau}bw(\tau)\,d\tau \tag{4.38}$$

Substituting this into (4.35) gives the particular solution as

$$\begin{aligned} x_p(t) &= e^{at}\int_{t_0}^{t} e^{-a\tau}bw(\tau)\,d\tau \\ &= \int_{t_0}^{t} e^{a(t-\tau)}bw(\tau)\,d\tau \end{aligned} \tag{4.39}$$

Combining this with the homogeneous solution gives the total solution as

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)}bw(\tau)\,d\tau \end{aligned} \tag{4.40}$$

The homogeneous solution, $x_h(t)$, is referred to as the *zero input solution*, whereas the particular solution, $x_p(t)$, is referred to as the *zero state solution*. Given the input, $w(t)$, the key to obtaining the total response is obtaining the *state-transition function* $\phi(t) = e^{at}$, or exponential e^{at}. But this is simple for a first-order system.

Before we extend these results to the general n-th order system, it is worthwhile to examine some important properties of the *state-transition* *function*, $\phi(t) = e^{at}$. Perhaps the most important property is

$$\phi(0) = 1 \tag{4.41a}$$

Substituting $t = t_0$ into the total solution in (4.40) gives $x(t) = x(t_0)$. This also follows from the infinite-series definition of the exponential given in (4.34). The name *state-transition function* is used for $\phi(t) = e^{at}$ since it shows how the initial state, $x(t_0)$, *transitions* to the final state, $x(t)$. The second property is that in order to obtain the inverse of the state-transition function, we need only substitute $-t$ for t:

$$\begin{aligned}\phi^{-1}(t) &= e^{-at} \\ &= \phi(-t)\end{aligned} \tag{4.41b}$$

This property is rather obvious since we may obtain from the homogeneous solution in (4.33)

$$\begin{aligned}x(t_0) &= \phi^{-1}(t - t_0)x(t) \\ &= \phi(t_0 - t)x(t)\end{aligned} \tag{4.42}$$

which simply amounts to a reversal in time. We will find these important properties and the form of the general solution to the state-variable solution in (4.40) to carry over to the n-th order system considered next.

Now consider the general n-th order lumped system characterized by (4.31). If we carry through the above development for the first-order system in like fashion we obtain the general solution as [A.2, B.1, 1, 2, 3]

$$\mathbf{X}(t) = \Phi(t - t_0)\mathbf{X}(t_0) + \int_{t_0}^{t} \Phi(t - \tau)\mathbf{B}\mathbf{W}(\tau)\, d\tau \tag{4.43}$$

Given the vector of p inputs, $\mathbf{W}(t)$, and the vector of initial states, $\mathbf{X}(t_0)$, equation (4.43) allows a straightforward determination of the states at some future time, $\mathbf{X}(t)$. The $n \times n$ *state-transition matrix*, $\Phi(t)$, has the same important properties as the first-order system:

$$\Phi(0) = \mathbf{1}_n \tag{4.44a}$$

$$\Phi^{-1}(t) = \Phi(-t) \tag{4.44b}$$

and

$$\begin{aligned}\Phi(t) &= \mathbf{e}^{\mathbf{A}t} \\ &= \mathbf{1}_n + \frac{t}{1!}\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots\end{aligned} \tag{4.44c}$$

where the $n \times n$ *identity matrix* has ones on the main diagonal and zeros elsewhere:

$$\mathbf{1}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{4.44d}$$

Now consider the phasor, transmission-line equations for an $(n+1)$-conductor line given in (4.7):

$$\frac{d}{dz}\hat{\mathbf{X}}(z) = \hat{\mathbf{A}}\hat{\mathbf{X}}(z) \tag{4.7a}$$

where

$$\hat{\mathbf{X}}(z) = \begin{bmatrix} \hat{\mathbf{V}}(z) \\ \hat{\mathbf{I}}(z) \end{bmatrix} \tag{4.7b}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & -\hat{\mathbf{Z}} \\ -\hat{\mathbf{Y}} & \mathbf{0} \end{bmatrix} \tag{4.7c}$$

Comparing these to the lumped-parameter state-variable equations given in (4.31) with $\mathbf{W}(t) = \mathbf{0}$ shows that the general solution for the line voltages and currents are, by direct analogy,

$$\begin{bmatrix} \hat{\mathbf{V}}(z_2) \\ \hat{\mathbf{I}}(z_2) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{\Phi}}_{11}(z_2 - z_1) & \hat{\mathbf{\Phi}}_{12}(z_2 - z_1) \\ \hat{\mathbf{\Phi}}_{21}(z_2 - z_1) & \hat{\mathbf{\Phi}}_{22}(z_2 - z_1) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}(z_1) \\ \hat{\mathbf{I}}(z_1) \end{bmatrix} \tag{4.45a}$$

where the $\hat{\mathbf{\Phi}}_{ij}$ are $n \times n$. If we choose $z_2 = \mathscr{L}$ and $z_1 = 0$ we essentially obtain the *chain parameter matrix* $\hat{\mathbf{\Phi}}$ for the overall line as

$$\begin{aligned} \begin{bmatrix} \hat{\mathbf{V}}(\mathscr{L}) \\ \hat{\mathbf{I}}(\mathscr{L}) \end{bmatrix} &= \hat{\mathbf{\Phi}}(\mathscr{L}) \begin{bmatrix} \hat{\mathbf{V}}(0) \\ \hat{\mathbf{I}}(0) \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mathbf{\Phi}}_{11}(\mathscr{L}) & \hat{\mathbf{\Phi}}_{12}(\mathscr{L}) \\ \hat{\mathbf{\Phi}}_{21}(\mathscr{L}) & \hat{\mathbf{\Phi}}_{22}(\mathscr{L}) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}(0) \\ \hat{\mathbf{I}}(0) \end{bmatrix} \end{aligned} \tag{4.45b}$$

Because of the direct analogy between the state-variable equations for a lumped system and the phasor MTL equations we can immediately observe some important properties of the chain parameter matrix from comparison to (4.44):

$$\hat{\mathbf{\Phi}}(0) = \mathbf{1}_{2n} \tag{4.46a}$$

$$\hat{\mathbf{\Phi}}^{-1}(\mathscr{L}) = \hat{\mathbf{\Phi}}(-\mathscr{L}) \tag{4.46b}$$

and

$$\hat{\Phi}(\mathscr{L}) = e^{\hat{A}\mathscr{L}} \tag{4.46c}$$

$$= 1_{2n} + \frac{\mathscr{L}}{1!}\hat{A} + \frac{\mathscr{L}^2}{2!}\hat{A}^2 + \frac{\mathscr{L}^3}{3!}\hat{A}^3 + \cdots$$

Once again, the property of the inverse of the chain parameter matrix given in (4.46b) is logical to expect since the inverse of (4.45b) yields

$$\begin{aligned}\begin{bmatrix}\hat{V}(0)\\ \hat{I}(0)\end{bmatrix} &= \hat{\Phi}^{-1}(\mathscr{L})\begin{bmatrix}\hat{V}(\mathscr{L})\\ \hat{I}(\mathscr{L})\end{bmatrix}\\ &= \hat{\Phi}(-\mathscr{L})\begin{bmatrix}\hat{V}(\mathscr{L})\\ \hat{I}(\mathscr{L})\end{bmatrix}\end{aligned} \tag{4.47}$$

This follows as a simple reversal of the line axis scale (replacing z with $-z$) similar to the reversal in time for the state-transition matrix of lumped systems and the line is *reciprocal* (assuming the surrounding medium is linear and isotropic). We will find these properties to be important in obtaining insight into the interpretation of the MTL equation solution.

4.3.2 Decoupling the MTL Equations by Similarity Transformations

The essential task in solving the *phasor* MTL equations is to determine the *chain parameter matrix* $\hat{\Phi}(\mathscr{L})$. One obvious way of doing this is to use the matrix infinite series form given in (4.46c). Substituting the form of $\hat{A}$ given in (4.7c) gives

$$\hat{\Phi}_{11}(\mathscr{L}) = 1_n + \frac{\mathscr{L}^2}{2!}\hat{Z}\hat{Y} + \frac{\mathscr{L}^4}{4!}[\hat{Z}\hat{Y}]^2 + \cdots \tag{4.48a}$$

$$\hat{\Phi}_{12}(\mathscr{L}) = -\frac{\mathscr{L}}{1!}\hat{Z} - \frac{\mathscr{L}^3}{3!}[\hat{Z}\hat{Y}]\hat{Z} - \frac{\mathscr{L}^5}{5!}[\hat{Z}\hat{Y}]^2\hat{Z} + \cdots \tag{4.48b}$$

$$\hat{\Phi}_{21}(\mathscr{L}) = -\frac{\mathscr{L}}{1!}\hat{Y} - \frac{\mathscr{L}^3}{3!}[\hat{Y}\hat{Z}]\hat{Y} - \frac{\mathscr{L}^5}{5!}[\hat{Y}\hat{Z}]^2\hat{Y} + \cdots \tag{4.48c}$$

$$\hat{\Phi}_{22}(\mathscr{L}) = 1_n + \frac{\mathscr{L}^2}{2!}\hat{Y}\hat{Z} + \frac{\mathscr{L}^4}{4!}[\hat{Y}\hat{Z}]^2 + \cdots \tag{4.48d}$$

In theory, one could perform, using a digital computer, the various products of the per-unit-length parameter matrices and sum the terms in (4.48) for a sufficient number of terms to achieve convergence and truncate the series thereafter. However, a more practical, closed-form result can be obtained using the following idea.

The method of using a *similarity transformation* is perhaps the most frequently used technique for determining the chain parameter matrix [B.1, 5–10]. We will find this to be of equal use in the time-domain solution in the next chapter. Define a *change of variables* as

$$\hat{\mathbf{V}}(z) = \hat{\mathbf{T}}_V \hat{\mathbf{V}}_m(z) \tag{4.49a}$$

$$\hat{\mathbf{I}}(z) = \hat{\mathbf{T}}_I \hat{\mathbf{I}}_m(z) \tag{4.49b}$$

The $n \times n$ complex matrices $\hat{\mathbf{T}}_V$ and $\hat{\mathbf{T}}_I$ are said to be *similarity transformations* between the actual phasor line voltages and currents, $\hat{\mathbf{V}}$ and $\hat{\mathbf{I}}$, and the *mode* voltages and currents, $\hat{\mathbf{V}}_m$ and $\hat{\mathbf{I}}_m$. In order for this to be valid, these $n \times n$ transformation matrices must be nonsingular, i.e., $\hat{\mathbf{T}}_V^{-1}$ and $\hat{\mathbf{T}}_I^{-1}$ must exist where we denote the *inverse* of an $n \times n$ matrix $\mathbf{M}$ as $\mathbf{M}^{-1}$, in order to go between both sets of variables. Substituting these into the phasor MTL equations in (4.7) gives

$$\frac{d}{dz}\begin{bmatrix} \hat{\mathbf{V}}_m \\ \hat{\mathbf{I}}_m \end{bmatrix} = \begin{bmatrix} 0 & -\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I \\ -\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{V}}_m \\ \hat{\mathbf{I}}_m \end{bmatrix} \tag{4.50}$$

If we can obtain a $\hat{\mathbf{T}}_V$ and a $\hat{\mathbf{T}}_I$ such that $\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I$ and $\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V$ are *diagonal* as

$$\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I = \hat{\mathscr{z}} \tag{4.51a}$$
$$= \begin{bmatrix} \hat{\mathscr{z}}_1 & 0 & \cdots & 0 \\ 0 & \hat{\mathscr{z}}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\mathscr{z}}_n \end{bmatrix}$$

$$\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V = \hat{\mathscr{y}} \tag{4.51b}$$
$$= \begin{bmatrix} \hat{\mathscr{y}}_1 & 0 & \cdots & 0 \\ 0 & \hat{\mathscr{y}}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\mathscr{y}}_n \end{bmatrix}$$

then the phasor MTL equations are *uncoupled* as

$$\left.\begin{array}{l} \dfrac{d}{dz}\hat{V}_{m1}(z) = -\hat{\mathscr{z}}_1 \hat{I}_{m1}(z), \quad \dfrac{d}{dz}\hat{I}_{m1}(z) = -\hat{\mathscr{y}}_1 \hat{V}_{m1}(z) \\ \qquad\qquad\qquad\qquad\vdots \\ \dfrac{d}{dz}\hat{V}_{mn}(z) = -\hat{\mathscr{z}}_n \hat{I}_{mn}(z), \quad \dfrac{d}{dz}\hat{I}_{mn}(z) = -\hat{\mathscr{y}}_n \hat{V}_{mn}(z) \end{array}\right\} \tag{4.52}$$

If we can find two $n \times n$ matrices $\hat{\mathbf{T}}_V$ and $\hat{\mathbf{T}}_I$ which simultaneously *diagonalize both per-unit-length parameter matrices, $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$, then the solution essentially reduces to the solution of n coupled, first-order differential equations as in the case of two-conductor lines.* But the solution of the n first-order differential equations in (4.52) was obtained earlier in the analysis of two-conductor lines. Therefore obtaining a similarity transformation that simultaneously diagonalizes *both* per-unit-length parameter matrices essentially solves the problem of the solution to the n coupled MTL equations! We will use this technique of *decoupling the MTL equations* on numerous occasions.

The essential question becomes: When can we find a similarity transformation that diagonalizes a matrix? Before we address that question, let us examine the application of the similarity transformation to the *uncoupled, second-order MTL equations* given in (4.8):

$$\frac{d^2}{dz^2}\hat{\mathbf{V}}(z) = \hat{\mathbf{Z}}\hat{\mathbf{Y}}\hat{\mathbf{V}}(z) \tag{4.8a}$$

$$\frac{d^2}{dz^2}\hat{\mathbf{I}}(z) = \hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{I}}(z) \tag{4.8b}$$

Substituting the similarity transformations given in (4.49) gives

$$\begin{aligned}\frac{d^2}{dz^2}\hat{\mathbf{V}}_m(z) &= \hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V\hat{\mathbf{V}}_m(z)\\ &= \hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I\hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V\hat{\mathbf{V}}_m(z)\\ &= \hat{\mathbf{z}}\hat{\mathbf{y}}\hat{\mathbf{V}}_m(z)\end{aligned} \tag{4.53a}$$

$$\begin{aligned}\frac{d^2}{dz^2}\hat{\mathbf{I}}_m(z) &= \hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I\hat{\mathbf{I}}_m(z)\\ &= \hat{\mathbf{T}}_I^{-1}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{T}}_I\hat{\mathbf{I}}_m(z)\\ &= \hat{\mathbf{y}}\hat{\mathbf{z}}\hat{\mathbf{I}}_m(z)\end{aligned} \tag{4.53b}$$

Recall that $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$ are *symmetric*, i.e., $\hat{\mathbf{Z}}^t = \hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}^t = \hat{\mathbf{Y}}$, where the *transpose of a matrix* $\mathbf{M}$ *is denoted by* $\mathbf{M}^t$. Since the transpose of the product of two matrices is the product of the transposes of the matrices in reverse order, we see that

$$\begin{aligned}(\hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V)^t &= (\hat{\mathbf{T}}_V)^t(\hat{\mathbf{Y}})^t(\hat{\mathbf{Z}})^t(\hat{\mathbf{T}}_V^{-1})^t\\ &= (\hat{\mathbf{T}}_V)^t\hat{\mathbf{Y}}\hat{\mathbf{Z}}(\hat{\mathbf{T}}_V^{-1})^t\\ &= \mathbf{zy}\\ &= \mathbf{yz}\end{aligned} \tag{4.54}$$

where we have used the assumption that **z** and **y** are diagonal so that their product can be reversed. Comparing this to (4.53b) we observe that

$$\hat{\mathbf{T}}_I^t = \hat{\mathbf{T}}_V^{-1} \tag{4.55}$$

Therefore it suffices to diagonalize the product $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ *or* the product $\hat{\mathbf{Z}}\hat{\mathbf{Y}}$. Let us *arbitrarily* choose to decouple (4.53b) as

$$\frac{d^2}{dz^2}\hat{\mathbf{I}}_m(z) = \hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}}\hat{\mathbf{I}}_m(z) \tag{4.56a}$$

$$= \hat{\boldsymbol{\gamma}}^2\hat{\mathbf{I}}_m(z)$$

where

$$\hat{\mathbf{T}} = \hat{\mathbf{T}}_I \tag{4.56b}$$

and $\hat{\boldsymbol{\gamma}}^2$ is a diagonal matrix as

$$\hat{\boldsymbol{\gamma}}^2 = \begin{bmatrix} \hat{\gamma}_1^2 & 0 & \cdots & 0 \\ 0 & \hat{\gamma}_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\gamma}_n^2 \end{bmatrix} \tag{4.56c}$$

The general solution to these uncoupled equations is

$$\hat{\mathbf{I}}_m(z) = \mathbf{e}^{-\hat{\boldsymbol{\gamma}}z}\hat{\mathbf{I}}_m^+ - \mathbf{e}^{\hat{\boldsymbol{\gamma}}z}\hat{\mathbf{I}}_m^- \tag{4.57}$$

where the *matrix exponentials* are defined as

$$\mathbf{e}^{\pm\hat{\boldsymbol{\gamma}}z} = \begin{bmatrix} e^{\pm\hat{\gamma}_1 z} & 0 & \cdots & 0 \\ 0 & e^{\pm\hat{\gamma}_2 z} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\pm\hat{\gamma}_n z} \end{bmatrix} \tag{4.58a}$$

and the vectors of undetermined constants are

$$\hat{\mathbf{I}}_m^\pm = \begin{bmatrix} \hat{I}_{m1}^\pm \\ \hat{I}_{m2}^\pm \\ \vdots \\ \hat{I}_{mn}^\pm \end{bmatrix} \tag{4.58b}$$

The actual currents are obtained by multiplying these mode currents by the transformation matrix, $\hat{\mathbf{T}}_I = \hat{\mathbf{T}}$, to give

$$\begin{aligned} \hat{\mathbf{I}}(z) &= \hat{\mathbf{T}}\hat{\mathbf{I}}_m(z) \\ &= \hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ - \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \end{aligned} \tag{4.59}$$

Similarly, the uncoupled second-order differential equation in terms of the mode voltages is

$$\begin{aligned} \frac{d^2}{dz^2}\hat{\mathbf{V}}_m(z) &= \hat{\mathbf{T}}_V^{-1}\hat{\mathbf{Z}}\hat{\mathbf{Y}}\hat{\mathbf{T}}_V\hat{\mathbf{V}}_m(z) \\ &= \hat{\mathbf{T}}^t\hat{\mathbf{Z}}\hat{\mathbf{Y}}(\hat{\mathbf{T}}^t)^{-1}\hat{\mathbf{V}}_m(z) \\ &= \hat{\gamma}^2\hat{\mathbf{V}}_m(z) \end{aligned} \tag{4.60}$$

with the general solution

$$\hat{\mathbf{V}}_m(z) = \mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{V}}_m^+ + \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{V}}_m^- \tag{4.61}$$

The actual voltages can be obtained by multiplying this result by the transformation, $\hat{\mathbf{T}}_V = (\hat{\mathbf{T}}_I^{-1})^t = (\hat{\mathbf{T}}^{-1})^t$ to give

$$\hat{\mathbf{V}}(z) = (\hat{\mathbf{T}}^{-1})^t(\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{V}}_m^+ + \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{V}}_m^-) \tag{4.62}$$

The undetermined constants in these results are related. To determine this relation, substitute (4.59) into the second MTL equation, given in (4.5b) to give

$$\begin{aligned} \hat{\mathbf{V}}(z) &= -\hat{\mathbf{Y}}^{-1}\frac{d}{dz}\hat{\mathbf{I}}(z) \\ &= \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}(\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \\ &= (\hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1})\hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \\ &= \hat{\mathbf{Z}}_C\hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + \mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \end{aligned} \tag{4.63}$$

where we have defined the *characteristic impedance matrix* as

$$\hat{\mathbf{Z}}_C = \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1} \tag{4.64}$$

This can be placed in another form. From (4.56) we have

$$\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}} = \hat{\gamma}^2 \tag{4.65}$$

Thus

$$\hat{\mathbf{Z}}\hat{\mathbf{T}}\hat{\gamma}^{-1} = \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma} \tag{4.66}$$

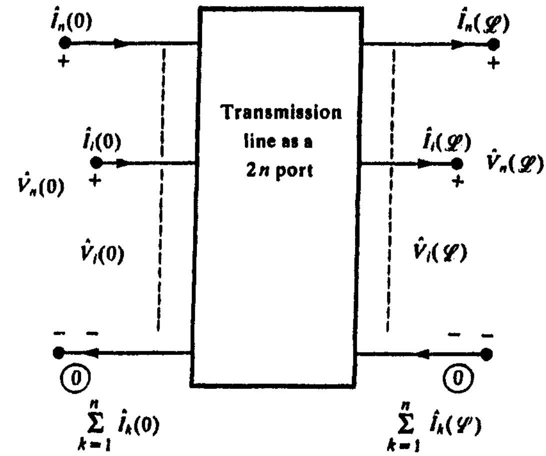


FIGURE 4.3 Illustration of viewing an $(n+1)$-conductor line as a $2n$ port in the frequency domain.

Therefore, the characteristic impedance matrix can be written as

$$\hat{\mathbf{Z}}_C = \hat{\mathbf{Z}}\hat{\mathbf{T}}\hat{\boldsymbol{\gamma}}^{-1}\hat{\mathbf{T}}^{-1} \tag{4.67}$$

We will find this seemingly arbitrary definition of the characteristic impedance matrix to have physical significance in terms of backward- and forward-traveling waves on the line in the following sections.

4.3.3 Characterizing the Line as a 2n Port with the Chain Parameter Matrix

The phasor voltages and currents at the two ends of the line can be related with the chain parameter matrix as in (4.45b):

$$\begin{aligned}\begin{bmatrix}\hat{\mathbf{V}}(\mathscr{L})\\ \hat{\mathbf{I}}(\mathscr{L})\end{bmatrix} &= \hat{\boldsymbol{\Phi}}(\mathscr{L})\begin{bmatrix}\hat{\mathbf{V}}(0)\\ \hat{\mathbf{I}}(0)\end{bmatrix} \\ &= \begin{bmatrix}\hat{\boldsymbol{\Phi}}_{11}(\mathscr{L}) & \hat{\boldsymbol{\Phi}}_{12}(\mathscr{L})\\ \hat{\boldsymbol{\Phi}}_{21}(\mathscr{L}) & \hat{\boldsymbol{\Phi}}_{22}(\mathscr{L})\end{bmatrix}\begin{bmatrix}\hat{\mathbf{V}}(0)\\ \hat{\mathbf{I}}(0)\end{bmatrix}\end{aligned} \tag{4.68}$$

This corresponds to viewing the $(n+1)$-conductor line as a $2n$ port as illustrated in Fig. 4.3. The essential task in solving the phasor MTL equations is to determine the entries in the $n \times n$ submatrices, $\hat{\boldsymbol{\Phi}}_{ij}$. This section is devoted to that task.

The general solutions of the phasor MTL equations are given, via similarity transformations, in (4.59) and (4.63):

$$\hat{\mathbf{V}}(z) = \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}(e^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + e^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \tag{4.69a}$$

$$\hat{\mathbf{I}}(z) = \hat{\mathbf{T}}(e^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ - e^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \tag{4.69b}$$

Evaluating these at $z = 0$ and $z = \mathscr{L}$ and eliminating $\hat{\mathbf{I}}_m^{\pm}$ gives the chain parameter matrix submatrices as [B.1]:

$$\hat{\Phi}_{11}(\mathscr{L}) = \tfrac{1}{2}\hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} + e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}} \tag{4.70a}$$

$$\begin{aligned}\hat{\Phi}_{12}(\mathscr{L}) &= -\tfrac{1}{2}\hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1} \\ &= -\tfrac{1}{2}\hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1}[\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}] \\ &= -\tfrac{1}{2}\hat{\mathbf{Z}}_C[\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}]\end{aligned} \tag{4.70b}$$

$$\begin{aligned}\hat{\Phi}_{21}(\mathscr{L}) &= -\tfrac{1}{2}\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\gamma}^{-1}\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}} \\ &= -\tfrac{1}{2}[\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}]\hat{\mathbf{T}}\hat{\gamma}^{-1}\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}} \\ &= -\tfrac{1}{2}[\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} - e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}]\hat{\mathbf{Y}}_C\end{aligned} \tag{4.70c}$$

$$\hat{\Phi}_{22}(\mathscr{L}) = \tfrac{1}{2}\hat{\mathbf{T}}(e^{\hat{\gamma}\mathscr{L}} + e^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1} \tag{4.70d}$$

and $\hat{\mathbf{Y}}_C = \hat{\mathbf{Z}}_C^{-1}$. As a check on this result, observe that the identity in (4.46a), $\hat{\Phi}(0) = \mathbf{1}_n$, is satisfied.

4.3.4 Properties of the Chain Parameter Matrix

In this section we will define certain matrix analogies to the two-conductor solution [B.1, B.4]. Although these will place the results in a form directly analogous to the two-conductor case, their use in numerical computation is limited.

First let us define the *square root of a matrix.* In scalar algebra, the square root is defined as *any quantity which when multiplied by itself gives the original quantity,* i.e., $\sqrt{a}\sqrt{a} = a$. The square root of a matrix can be similarly defined as a matrix which when multiplied by itself gives the original matrix, i.e., $\sqrt{\mathbf{M}}\sqrt{\mathbf{M}} = \mathbf{M}$. Recall the basic diagonalization in (4.65):

$$\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}} = \hat{\gamma}^2 \tag{4.71}$$

From this we may define the square root of the matrix product as

$$\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} = \hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1} \tag{4.72}$$

This can be verified by taking the product and using (4.71):

$$\begin{aligned}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} &= \hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1}\hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1} \\ &= \hat{\mathbf{T}}\hat{\gamma}^2\hat{\mathbf{T}}^{-1} \\ &= \hat{\mathbf{Y}}\hat{\mathbf{Z}}\end{aligned} \tag{4.73}$$

Alternatively,

$$\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} = \hat{\mathbf{Y}}\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}\,\hat{\mathbf{Y}}^{-1} \tag{4.74}$$

as multiplication by itself shows. Similarly, we can define $\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}$ as

$$\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}} = \hat{\mathbf{Y}}^{-1}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\,\hat{\mathbf{Y}} \tag{4.75}$$

as a multiplication by itself shows. Therefore, the characteristic impedance matrix in (4.64) or (4.67) can be written, symbolically, as

$$\begin{aligned}\hat{\mathbf{Z}}_C &= \hat{\mathbf{Y}}^{-1}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} \\ &= \hat{\mathbf{Z}}(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}})^{-1}\end{aligned} \tag{4.76}$$

Observe that this result reduces to the scalar characteristic impedance for two-conductor lines.

Additional symbolic definitions can be obtained for direct analogy to the two-conductor case by defining the matrix hyperbolic functions. First define the matrix exponentials as

$$\mathbf{e}^{\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}} = \mathbf{1}_n + \frac{\mathscr{L}}{1!}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} + \frac{\mathscr{L}^2}{2!}(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}})^2 + \cdots \tag{4.77a}$$

$$\mathbf{e}^{\hat{\gamma}\mathscr{L}} = \mathbf{1}_n + \frac{\mathscr{L}}{1!}\hat{\gamma} + \frac{\mathscr{L}^2}{2!}\hat{\gamma}^2 + \cdots \tag{4.77b}$$

In terms of these matrix exponentials, we may define the matrix hyperbolic functions as

$$\begin{aligned}\cosh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}) &= \tfrac{1}{2}(\mathbf{e}^{\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}} + \mathbf{e}^{-\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}}) \\ &= \mathbf{1}_n + \frac{\mathscr{L}^2}{2!}(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}})^2 + \frac{\mathscr{L}^4}{4!}(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}})^4 + \cdots \\ &= \tfrac{1}{2}\hat{\mathbf{T}}(\mathbf{e}^{\hat{\gamma}\mathscr{L}} + \mathbf{e}^{-\hat{\gamma}\mathscr{L}})\hat{\mathbf{T}}^{-1}\end{aligned} \tag{4.78a}$$

$$\begin{aligned}\sinh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}) &= \tfrac{1}{2}(e^{\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}} - e^{-\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}}) \\ &= \frac{\mathscr{L}}{1!}\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}} + \frac{\mathscr{L}^3}{3!}(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}})^3 + \cdots \\ &= \tfrac{1}{2}\mathbf{T}(e^{\hat{\boldsymbol{\gamma}}\mathscr{L}} - e^{-\hat{\boldsymbol{\gamma}}\mathscr{L}})\hat{\mathbf{T}}^{-1}\end{aligned} \tag{4.78b}$$

In terms of these symbolic definitions, the chain parameter matrix submatrices can be written, symbolically, as

$$\begin{aligned}\hat{\mathbf{\Phi}}_{11}(\mathscr{L}) &= \cosh(\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}\mathscr{L}) \\ &= \hat{\mathbf{Y}}^{-1}\cosh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L})\hat{\mathbf{Y}}\end{aligned} \tag{4.79a}$$

$$\begin{aligned}\hat{\mathbf{\Phi}}_{12}(\mathscr{L}) &= -\hat{\mathbf{Z}}_C\sinh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}) \\ &= -\sinh(\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}\mathscr{L})\hat{\mathbf{Z}}_C\end{aligned} \tag{4.79b}$$

$$\begin{aligned}\hat{\mathbf{\Phi}}_{21}(\mathscr{L}) &= -\hat{\mathbf{Z}}_C^{-1}\sinh(\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}\mathscr{L}) \\ &= -\sinh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L})\mathbf{Z}_C^{-1}\end{aligned} \tag{4.79c}$$

$$\begin{aligned}\hat{\mathbf{\Phi}}_{22}(\mathscr{L}) &= \cosh(\sqrt{\hat{\mathbf{Y}}\hat{\mathbf{Z}}}\mathscr{L}) \\ &= \hat{\mathbf{Y}}\cosh(\sqrt{\hat{\mathbf{Z}}\hat{\mathbf{Y}}}\mathscr{L})\hat{\mathbf{Y}}^{-1}\end{aligned} \tag{4.79d}$$

Observe that these reduce to the scalar results obtained for the two-conductor line in (4.23).

The final chain parameter identity has to do with the inverse of the chain parameter matrix given in (4.46b). Multiplying the chain parameter matrix by its inverse and using the identity for the inverse given in (4.47b) gives

$$\begin{aligned}\hat{\mathbf{\Phi}}(\mathscr{L})\hat{\mathbf{\Phi}}^{-1}(\mathscr{L}) &= \mathbf{1}_{2n} \\ &= \hat{\mathbf{\Phi}}(\mathscr{L})\hat{\mathbf{\Phi}}(-\mathscr{L})\end{aligned} \tag{4.80a}$$

Substituting the form of the chain parameter matrix gives

$$\begin{bmatrix}\hat{\mathbf{\Phi}}_{11}(\mathscr{L}) & \hat{\mathbf{\Phi}}_{12}(\mathscr{L}) \\ \hat{\mathbf{\Phi}}_{21}(\mathscr{L}) & \hat{\mathbf{\Phi}}_{22}(\mathscr{L})\end{bmatrix}\begin{bmatrix}\hat{\mathbf{\Phi}}_{11}(-\mathscr{L}) & \hat{\mathbf{\Phi}}_{12}(-\mathscr{L}) \\ \hat{\mathbf{\Phi}}_{21}(-\mathscr{L}) & \hat{\mathbf{\Phi}}_{22}(-\mathscr{L})\end{bmatrix} = \begin{bmatrix}\mathbf{1}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n\end{bmatrix} \tag{4.80b}$$

Multiplying this out gives the following identities for the chain parameter submatrices:

$$\hat{\mathbf{\Phi}}_{11}(\mathscr{L})\hat{\mathbf{\Phi}}_{11}(-\mathscr{L}) + \hat{\mathbf{\Phi}}_{12}(\mathscr{L})\hat{\mathbf{\Phi}}_{21}(-\mathscr{L}) = \mathbf{1}_n \tag{4.81a}$$

$$\hat{\mathbf{\Phi}}_{11}(\mathscr{L})\hat{\mathbf{\Phi}}_{12}(-\mathscr{L}) + \hat{\mathbf{\Phi}}_{12}(\mathscr{L})\hat{\mathbf{\Phi}}_{22}(-\mathscr{L}) = \mathbf{0} \tag{4.81b}$$

$$\hat{\Phi}_{21}(\mathscr{L})\hat{\Phi}_{11}(-\mathscr{L}) + \hat{\Phi}_{22}(\mathscr{L})\hat{\Phi}_{21}(-\mathscr{L}) = 0 \tag{4.81c}$$

$$\hat{\Phi}_{21}(\mathscr{L})\hat{\Phi}_{12}(-\mathscr{L}) + \hat{\Phi}_{22}(\mathscr{L})\hat{\Phi}_{22}(-\mathscr{L}) = \mathbf{1}_n \tag{4.81d}$$

From the series expansions of the chain parameter submatrices in (4.48) we see that

$$\hat{\Phi}_{11}(-\mathscr{L}) = \hat{\Phi}_{11}(\mathscr{L}) \tag{4.82a}$$

$$\hat{\Phi}_{12}(-\mathscr{L}) = -\hat{\Phi}_{12}(\mathscr{L}) \tag{4.82b}$$

$$\hat{\Phi}_{21}(-\mathscr{L}) = -\hat{\Phi}_{21}(\mathscr{L}) \tag{4.82c}$$

$$\hat{\Phi}_{22}(-\mathscr{L}) = \hat{\Phi}_{22}(\mathscr{L}) \tag{4.82d}$$

Substituting these into (4.81) yields the desired identities:

$$\hat{\Phi}_{12}(\mathscr{L})\hat{\Phi}_{22}(\mathscr{L})\hat{\Phi}_{12}^{-1}(\mathscr{L})\hat{\Phi}_{11}(\mathscr{L}) - \hat{\Phi}_{12}(\mathscr{L})\hat{\Phi}_{21}(\mathscr{L}) = \mathbf{1}_n \tag{4.83a}$$

$$\hat{\Phi}_{21}(\mathscr{L})\hat{\Phi}_{11}(\mathscr{L})\hat{\Phi}_{21}^{-1}(\mathscr{L})\Phi_{22}(\mathscr{L}) - \hat{\Phi}_{21}(\mathscr{L})\hat{\Phi}_{12}(\mathscr{L}) = \mathbf{1}_n \tag{4.83b}$$

$$\hat{\Phi}_{12}(\mathscr{L})\hat{\Phi}_{22}(\mathscr{L})\hat{\Phi}_{12}^{-1}(\mathscr{L}) = \hat{\Phi}_{11}(\mathscr{L}) \tag{4.83c}$$

$$\hat{\Phi}_{21}(\mathscr{L})\hat{\Phi}_{11}(\mathscr{L})\hat{\Phi}_{21}^{-1}(\mathscr{L}) = \hat{\Phi}_{22}(\mathscr{L}) \tag{4.83d}$$

$$\hat{\Phi}_{22}^{t}(\mathscr{L}) = \hat{\Phi}_{11}(\mathscr{L}) \tag{4.83e}$$

The last identity follows from the series expansions in (4.48) and the fact that $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$ are symmetric. These identities have proven of considerable value in reducing large matrix expressions that result from the solution of the MTL equations [B.1, B.4].

4.3.5 Incorporating the Terminal Conditions

The general solutions to the phasor MTL equations given in (4.69) involve $2n$ undetermined constants in the $n \times 1$ vectors $\hat{\mathbf{I}}_m^+$ and $\hat{\mathbf{I}}_m^-$. Therefore we need $2n$ additional constraint equations in order to evaluate these. These additional constraint equations are provided by the terminal conditions at $z = 0$ and $z = \mathscr{L}$ illustrated in Fig. 4.4. The driving sources and load impedances are contained in these terminal networks that are attached to the two ends of the line. The terminal constraint network at $z = 0$ shown in Fig. 4.4(a) provides n equations relating the n phasor voltages $\hat{\mathbf{V}}(0)$ and n phasor currents $\hat{\mathbf{I}}(0)$. The terminal constraint network at $z = \mathscr{L}$ shown in Fig. 4.4(b) provides n equations relating the n phasor voltages $\hat{\mathbf{V}}(\mathscr{L})$ and n phasor currents $\hat{\mathbf{I}}(\mathscr{L})$.

Alternatively, the chain parameter matrix given in (4.68) relates the phasor voltages at $z = 0$ and at $z = \mathscr{L}$. The chain parameter matrix does not explicitly determine these voltages and currents. Essentially then we still need $2n$ relations to explicitly determine the terminal voltages and currents from the chain parameter matrix relation. These again will be provided by the terminal

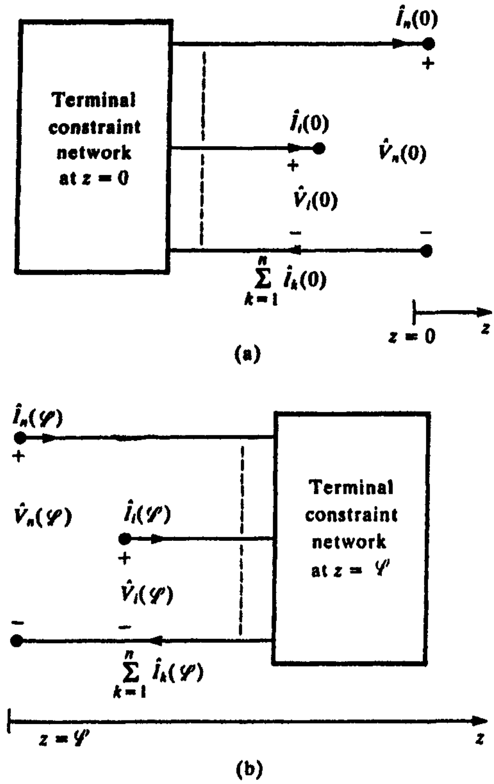


FIGURE 4.4 Modeling the terminal constraints of an $(n + 1)$-conductor line.

constraints. The purpose of this section is to incorporate these terminal constraints to explicitly determine the terminal voltages and currents and complete this final but important last step in the solution.

4.3.5.1 The Generalized Thévenin Equivalent There are many ways of relating the voltages and currents at the terminals of an n port. If the network is *linear*, this relationship will be a *linear combination* of the port voltages and currents. One obvious way is to generalize the Thévenin equivalent representation of a 1 port as [A.2]

$$\hat{\mathbf{V}}(0) = \hat{\mathbf{V}}_S - \hat{\mathbf{Z}}_S\hat{\mathbf{I}}(0) \tag{4.84a}$$

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{V}}_L + \hat{\mathbf{Z}}_L\hat{\mathbf{I}}(\mathscr{L}) \tag{4.84b}$$

The $n \times 1$ vectors $\hat{\mathbf{V}}_S$ and $\hat{\mathbf{V}}_L$ contain the effects of the independent voltage and current sources in the termination networks at $z = 0$ and $z = \mathscr{L}$, respectively. The $n \times n$ matrices, $\hat{\mathbf{Z}}_S$ and $\hat{\mathbf{Z}}_L$ contain the effects of the impedances and any controlled sources in the terminal networks at $z = 0$ and $z = \mathscr{L}$, respectively.

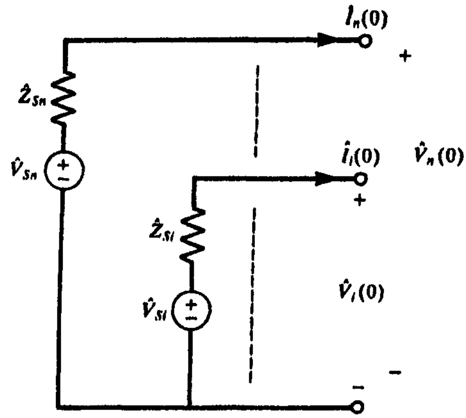


FIGURE 4.5 The generalized Thévenin representation of a termination with no cross coupling.

In general, the impedance matrices, $\hat{\mathbf{Z}}_S$ or $\hat{\mathbf{Z}}_L$, are *full*, i.e., there is *cross coupling* between all ports of a network. However, there may be terminal-network configurations wherein these impedance matrices are diagonal and the only coupling occurs along the MTL. Figure 4.5 shows such a case wherein each line at $z = 0$ is terminated directly to the chosen reference conductor with an impedance and a voltage source. In this case, the matrices in (4.84a) become

$$\hat{\mathbf{V}}_S = \begin{bmatrix} \hat{V}_{S1} \\ \vdots \\ \hat{V}_{Si} \\ \vdots \\ \hat{V}_{Sn} \end{bmatrix} \tag{4.85a}$$

$$\hat{\mathbf{Z}}_S = \begin{bmatrix} \hat{Z}_{S1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \hat{Z}_{Si} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \hat{Z}_{Sn} \end{bmatrix} \tag{4.85b}$$

The general forms of the solutions of the MTL equations for the line voltages and currents were obtained in (4.59) and (4.63) as

$$\hat{\mathbf{I}}(z) = \hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma} z}\hat{\mathbf{I}}_m^+ - \mathbf{e}^{\hat{\gamma} z}\hat{\mathbf{I}}_m^-) \tag{4.86a}$$

$$\begin{aligned}\hat{\mathbf{V}}(z) &= \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}(e^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + e^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-) \\ &= \hat{\mathbf{Z}}_C\hat{\mathbf{T}}(e^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ + e^{\hat{\gamma}z}\hat{\mathbf{I}}_m^-)\end{aligned} \tag{4.86b}$$

where the *characteristic impedance matrix* is defined as

$$\begin{aligned}\hat{\mathbf{Z}}_C &= \hat{\mathbf{Y}}^{-1}\hat{\mathbf{T}}\hat{\gamma}\hat{\mathbf{T}}^{-1} \\ &= \hat{\mathbf{Z}}\hat{\mathbf{T}}\hat{\gamma}^{-1}\hat{\mathbf{T}}^{-1}\end{aligned} \tag{4.86c}$$

In order to solve for the $2n$ undetermined constants in $\hat{\mathbf{I}}_m^+$ and $\hat{\mathbf{I}}_m^-$, we evaluate (4.86) at $z = 0$ and at $z = \mathscr{L}$ and substitute into the generalized Thévenin equivalent characterizations given in (4.84) to yield

$$\hat{\mathbf{Z}}_C\hat{\mathbf{T}}(\hat{\mathbf{I}}_m^+ + \hat{\mathbf{I}}_m^-) = \hat{\mathbf{V}}_S - \hat{\mathbf{Z}}_S\hat{\mathbf{T}}(\hat{\mathbf{I}}_m^+ - \hat{\mathbf{I}}_m^-) \tag{4.87a}$$

$$\hat{\mathbf{Z}}_C\hat{\mathbf{T}}(e^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^+ + e^{\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^-) = \hat{\mathbf{V}}_L + \hat{\mathbf{Z}}_L\hat{\mathbf{T}}(e^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^+ - e^{\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^-) \tag{4.87b}$$

Writing this in matrix form gives

$$\begin{bmatrix} (\hat{\mathbf{Z}}_C + \hat{\mathbf{Z}}_S)\hat{\mathbf{T}} & (\hat{\mathbf{Z}}_C - \hat{\mathbf{Z}}_S)\hat{\mathbf{T}} \\ (\mathbf{Z}_C - \hat{\mathbf{Z}}_L)\hat{\mathbf{T}}e^{-\hat{\gamma}\mathscr{L}} & (\hat{\mathbf{Z}}_C + \hat{\mathbf{Z}}_L)\hat{\mathbf{T}}e^{\hat{\gamma}\mathscr{L}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}_m^+ \\ \hat{\mathbf{I}}_m^- \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{V}}_S \\ \hat{\mathbf{V}}_L \end{bmatrix} \tag{4.88}$$

Once this set of $2n$ simultaneous equations is solved for $\hat{\mathbf{I}}_m^+$ and $\hat{\mathbf{I}}_m^-$, the line voltages and currents are obtained at any z along the line by substitution into (4.86).

An alternative method for incorporating the terminal conditions is to substitute the generalized Thévenin equivalent characterizations in (4.84) into the chain parameter matrix characterization given in (4.68):

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{11}(\mathscr{L})\hat{\mathbf{V}}(0) + \hat{\mathbf{\Phi}}_{12}(\mathscr{L})\hat{\mathbf{I}}(0) \tag{4.89a}$$

$$\hat{\mathbf{I}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{21}(\mathscr{L})\hat{\mathbf{V}}(0) + \hat{\mathbf{\Phi}}_{22}(\mathscr{L})\hat{\mathbf{I}}(0) \tag{4.89b}$$

to yield [B.1]

$$(\hat{\mathbf{\Phi}}_{12} - \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{Z}}_S - \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{22} + \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{21}\hat{\mathbf{Z}}_S)\hat{\mathbf{I}}(0) = \hat{\mathbf{V}}_L - (\hat{\mathbf{\Phi}}_{11} - \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{21})\hat{\mathbf{V}}_S \tag{4.90a}$$

$$\hat{\mathbf{I}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{21}\hat{\mathbf{V}}_S + (\hat{\mathbf{\Phi}}_{22} - \hat{\mathbf{\Phi}}_{21}\hat{\mathbf{Z}}_S)\hat{\mathbf{I}}(0) \tag{4.90b}$$

Equations (4.90a) are a set if n *simultaneous, algebraic equations* which can be solved for the n terminal currents at $z = 0$, $\hat{\mathbf{I}}(0)$. Numerous Gauss-elimination-type subroutines for digital computers are available to solve these equations [A.2, I.1]. Once these are solved, the n terminal currents at $z = \mathscr{L}$, $\hat{\mathbf{I}}(\mathscr{L})$, can

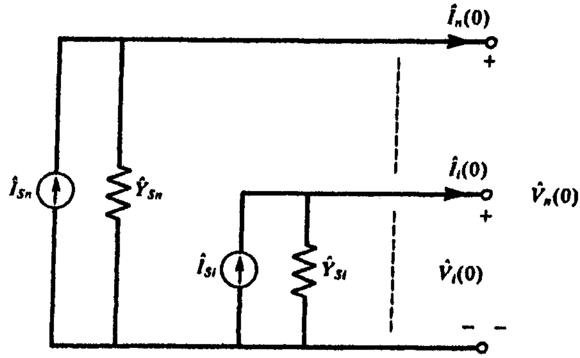


FIGURE 4.6 The generalized Norton representation of a termination with no cross coupling.

be obtained from (4.90b). The $2n$ terminal voltages, $\hat{\mathbf{V}}(0)$ and $\hat{\mathbf{V}}(\mathscr{L})$, can be obtained from the terminal relations in (4.84).

4.3.5.2 The Generalized Norton Equivalent The generalized Thévenin equivalent in the previous subsection is only one way of relating the terminal voltages and currents of a linear n port. An alternative representation is the generalized Norton equivalent wherein the voltages and currents are related by

$$\hat{\mathbf{I}}(0) = \hat{\mathbf{I}}_S - \hat{\mathbf{Y}}_S\hat{\mathbf{V}}(0) \tag{4.91a}$$

$$\hat{\mathbf{I}}(\mathscr{L}) = -\hat{\mathbf{I}}_L + \hat{\mathbf{Y}}_L\hat{\mathbf{V}}(\mathscr{L}) \tag{4.91b}$$

The $n \times 1$ vectors $\hat{\mathbf{I}}_S$ and $\hat{\mathbf{I}}_L$ again contain the effects of the independent voltage and current sources in the termination networks at $z = 0$ and $z = \mathscr{L}$, respectively. The $n \times n$ matrices, $\hat{\mathbf{Y}}_S$ and $\hat{\mathbf{Y}}_L$ again contain the effects of the impedances and any controlled sources in the terminal networks at $z = 0$ and $z = \mathscr{L}$, respectively. Again, the admittance matrices, $\hat{\mathbf{Y}}_S$ or $\hat{\mathbf{Y}}_L$, may be *full*, i.e., there is *cross coupling* between all ports of a terminal network. However, there may be terminal network configurations wherein these admittance matrices are diagonal and the only coupling occurs along the MTL. Figure 4.6 shows such a case wherein each line is terminated at $z = 0$ directly to the chosen reference conductor with an admittance in parallel with a current source. In this case, the matrices in (4.91a) become

$$\hat{\mathbf{I}}_S = \begin{bmatrix} \hat{I}_{S1} \\ \vdots \\ \hat{I}_{Si} \\ \vdots \\ \hat{I}_{Sn} \end{bmatrix} \tag{4.92a}$$

$$\hat{\mathbf{Y}}_S = \begin{bmatrix} \hat{Y}_{S1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \hat{Y}_{Si} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \hat{Y}_{Sn} \end{bmatrix} \tag{4.92b}$$

The $2n$ undetermined constants, $\hat{\mathbf{I}}_m^+$ and $\hat{\mathbf{I}}_m^-$, in the general solution given in (4.86) are once again found by evaluating these solutions at $z = 0$ and at $z = \mathscr{L}$ and substituting into the generalized Norton equivalent characterizations given in (4.91) to yield

$$\hat{\mathbf{T}}(\hat{\mathbf{I}}_m^+ - \hat{\mathbf{I}}_m^-) = \hat{\mathbf{I}}_S - \hat{\mathbf{Y}}_S \hat{\mathbf{Z}}_C \hat{\mathbf{T}}(\hat{\mathbf{I}}_m^+ + \hat{\mathbf{I}}_m^-) \tag{4.93a}$$

$$\hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^+ - \mathbf{e}^{\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^-) = -\hat{\mathbf{I}}_L + \hat{\mathbf{Y}}_L \hat{\mathbf{Z}}_C \hat{\mathbf{T}}(\mathbf{e}^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^+ + \mathbf{e}^{\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^-) \tag{4.93b}$$

Writing this in matrix form gives

$$\begin{bmatrix} (\hat{\mathbf{Y}}_S \hat{\mathbf{Z}}_C + \mathbf{1}_n)\hat{\mathbf{T}} & (\hat{\mathbf{Y}}_S \hat{\mathbf{Z}}_C - \mathbf{1}_n)\hat{\mathbf{T}} \\ (\hat{\mathbf{Y}}_L \hat{\mathbf{Z}}_C - \mathbf{1}_n)\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}\mathscr{L}} & (\hat{\mathbf{Y}}_L \hat{\mathbf{Z}}_C + \mathbf{1}_n)\hat{\mathbf{T}}\mathbf{e}^{\hat{\gamma}\mathscr{L}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}_m^+ \\ \hat{\mathbf{I}}_m^- \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{I}}_S \\ \hat{\mathbf{I}}_L \end{bmatrix} \tag{4.94}$$

Once this set of $2n$ simultaneous equations are solved for $\hat{\mathbf{I}}_m^+$ and $\hat{\mathbf{I}}_m^-$, the line voltages and currents are obtained at any z along the line by substitution into (4.86).

An alternative method of incorporating the terminal conditions is to again substitute the generalized Norton equivalent terminal relations given in (4.91) into the chain parameter representation given in (4.89) to yield

$$(\hat{\mathbf{\Phi}}_{21} - \hat{\mathbf{\Phi}}_{22}\hat{\mathbf{Y}}_S - \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{11} + \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{12}\hat{\mathbf{Y}}_S)\hat{\mathbf{V}}(0) = -\hat{\mathbf{I}}_L - (\hat{\mathbf{\Phi}}_{22} - \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{12})\hat{\mathbf{I}}_S \tag{4.95a}$$

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{12}\hat{\mathbf{I}}_S + (\hat{\mathbf{\Phi}}_{11} - \hat{\mathbf{\Phi}}_{12}\hat{\mathbf{Y}}_S)\hat{\mathbf{V}}(0) \tag{4.95b}$$

Equations (4.95a) are once again a set of n *simultaneous, algebraic equations* which can be solved for the n terminal voltages at $z = 0$, $\hat{\mathbf{V}}(0)$. Once these are solved, the n terminal voltages at $z = \mathscr{L}$, $\hat{\mathbf{V}}(\mathscr{L})$, can be obtained from (4.95b). The $2n$ terminal currents, $\hat{\mathbf{I}}(0)$ and $\hat{\mathbf{I}}(\mathscr{L})$, can be obtained from the terminal relations in (4.91).

4.3.5.3 Mixed Representations There are numerous cases where both terminations cannot be represented as generalized Thévenin equivalents or as generalized Norton equivalents [F.8, G.4]. For example, suppose some of the conductors are terminated at $z = 0$ to the reference conductor in short circuits. In this case, the generalized Norton equivalent in (4.91a) does not exist for this termination since the termination admittance is infinite. However, the generalized Thévenin equivalent in (4.84a) does exist since the short circuit is equivalent to a load impedance of 0 which is a legitimate entry in $\hat{\mathbf{Z}}_S$. Shielded wires in which the shield (one of the MTL conductors) is "grounded" to the reference conductor represent such a case. Conversely, one of the conductors may be unterminated, i.e., there is an open circuit between that conductor and the reference conductor. In this case we must use the generalized Norton equivalent (the termination has 0 admittance) since the generalized Thévenin equivalent does not exist (the termination has infinite impedance). An example of this is commonly found in *balanced* wire lines such as twisted pairs where neither wire is connected to the reference conductor [G.1–G.10]. This calls for a mixed representation of the terminal networks wherein one is represented with a generalized Thévenin equivalent whereas the other is represented with a generalized Norton equivalent.

We now obtain the equations to be solved for these mixed representations. Using (4.84a) and (4.91b) yields

$$\begin{bmatrix} (\hat{\mathbf{Z}}_C + \hat{\mathbf{Z}}_S)\hat{\mathbf{T}} & (\hat{\mathbf{Z}}_C - \hat{\mathbf{Z}}_S)\hat{\mathbf{T}} \\ (\hat{\mathbf{Y}}_L\hat{\mathbf{Z}}_C - \mathbf{1}_n)\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}\mathscr{L}} & (\hat{\mathbf{Y}}_L\hat{\mathbf{Z}}_C + \mathbf{1}_n)\hat{\mathbf{T}}\mathbf{e}^{\hat{\gamma}\mathscr{L}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}_m^+ \\ \hat{\mathbf{I}}_m^- \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{V}}_S \\ \hat{\mathbf{I}}_L \end{bmatrix} \tag{4.96}$$

or, via the chain parameter matrix,

$$(\hat{\mathbf{\Phi}}_{22} - \hat{\mathbf{\Phi}}_{21}\hat{\mathbf{Z}}_S - \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{12} + \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{11}\hat{\mathbf{Z}}_S)\hat{\mathbf{I}}(0) = -\hat{\mathbf{I}}_L - (\hat{\mathbf{\Phi}}_{21} - \hat{\mathbf{Y}}_L\hat{\mathbf{\Phi}}_{11})\hat{\mathbf{V}}_S \tag{4.97a}$$

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{V}}_S + (\hat{\mathbf{\Phi}}_{12} - \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{Z}}_S)\hat{\mathbf{I}}(0) \tag{4.97b}$$

Similarly, using (4.91a) and (4.84b) yields

$$\begin{bmatrix} (\hat{\mathbf{Y}}_S\hat{\mathbf{Z}}_C + \mathbf{1}_n)\hat{\mathbf{T}} & (\hat{\mathbf{Y}}_S\hat{\mathbf{Z}}_C - \mathbf{1}_n)\hat{\mathbf{T}} \\ (\hat{\mathbf{Z}}_C - \hat{\mathbf{Z}}_L)\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}\mathscr{L}} & (\hat{\mathbf{Z}}_C + \hat{\mathbf{Z}}_L)\hat{\mathbf{T}}\mathbf{e}^{\hat{\gamma}\mathscr{L}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}_m^+ \\ \hat{\mathbf{I}}_m^- \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{I}}_S \\ \hat{\mathbf{V}}_L \end{bmatrix} \tag{4.98}$$

or, using the chain parameter matrix,

$$(\hat{\mathbf{\Phi}}_{11} - \hat{\mathbf{\Phi}}_{12}\hat{\mathbf{Y}}_S - \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{21} + \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{22}\hat{\mathbf{Y}}_S)\hat{\mathbf{V}}(0) = \hat{\mathbf{V}}_L - (\hat{\mathbf{\Phi}}_{12} - \hat{\mathbf{Z}}_L\hat{\mathbf{\Phi}}_{22})\hat{\mathbf{I}}_S \tag{4.99a}$$

$$\hat{\mathbf{I}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{22}\hat{\mathbf{I}}_S + (\hat{\mathbf{\Phi}}_{21} - \hat{\mathbf{\Phi}}_{22}\hat{\mathbf{Y}}_S)\hat{\mathbf{V}}(0) \tag{4.99b}$$

The above mixed representation can characterize termination networks wherein short-circuit terminations exist within one termination network and open-circuit terminations exist within the other termination network. Terminal networks wherein both short-circuit and open-circuit terminations exist within the same network can be handled with a more general formulation such as

$$\hat{\mathbf{Y}}_S\hat{\mathbf{V}}(0) + \hat{\mathbf{Z}}_S\hat{\mathbf{I}}(0) = \hat{\mathbf{P}}_S \tag{4.100a}$$

$$\hat{\mathbf{Y}}_L\hat{\mathbf{V}}(\mathscr{L}) + \hat{\mathbf{Z}}_L\hat{\mathbf{I}}(\mathscr{L}) = \hat{\mathbf{P}}_L \tag{4.100b}$$

For example, (4.100a) can be written in partitioned form as

$$\underbrace{\begin{bmatrix} \hat{\mathbf{Y}}_{11} & \hat{\mathbf{Y}}_{12} \\ \hat{\mathbf{Y}}_{21} & \hat{\mathbf{Y}}_{22} \end{bmatrix}}_{\hat{\mathbf{Y}}_S} \underbrace{\begin{bmatrix} \hat{\mathbf{V}}_1(0) \\ \hat{\mathbf{V}}_2(0) \end{bmatrix}}_{\hat{\mathbf{V}}(0)} + \underbrace{\begin{bmatrix} \hat{\mathbf{Z}}_{12} & \hat{\mathbf{Z}}_{12} \\ \hat{\mathbf{Z}}_{21} & \hat{\mathbf{Z}}_{22} \end{bmatrix}}_{\hat{\mathbf{Z}}_S} \underbrace{\begin{bmatrix} \hat{\mathbf{I}}_1(0) \\ \hat{\mathbf{I}}_2(0) \end{bmatrix}}_{\hat{\mathbf{I}}(0)} = \underbrace{\begin{bmatrix} \hat{\mathbf{I}}_{S1} \\ \hat{\mathbf{I}}_{S2} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{V}}_{S1} \\ \hat{\mathbf{V}}_{S2} \end{bmatrix}}_{\hat{\mathbf{P}}_S} \tag{4.101}$$

Suppose there is no cross coupling within this termination network with the first set of terminals characterized as Norton equivalents as in Fig. 4.6 and the last set of terminals characterized as Thévenin equivalents as in Fig. 4.5. The partitioned general form becomes

$$\underbrace{\begin{bmatrix} \hat{\mathbf{Y}}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}}_{\hat{\mathbf{Y}}_S} \underbrace{\begin{bmatrix} \hat{\mathbf{V}}_1(0) \\ \hat{\mathbf{V}}_2(0) \end{bmatrix}}_{\hat{\mathbf{V}}(0)} + \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Z}}_{22} \end{bmatrix}}_{\hat{\mathbf{Z}}_S} \underbrace{\begin{bmatrix} \hat{\mathbf{I}}_1(0) \\ \hat{\mathbf{I}}_2(0) \end{bmatrix}}_{\hat{\mathbf{I}}(0)} = \underbrace{\begin{bmatrix} \hat{\mathbf{I}}_{S1} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{V}}_{S2} \end{bmatrix}}_{\hat{\mathbf{P}}_S} \tag{4.102}$$

where $\hat{\mathbf{Y}}_{11}$ and $\hat{\mathbf{Z}}_{22}$ are diagonal. If any of the admittances are zero we set the appropriate entry in $\hat{\mathbf{Y}}_{11}$ to zero, whereas if any of the impedances are zero we set the appropriate entry in $\hat{\mathbf{Z}}_{22}$ to zero. The more general terminal constraints accommodated by (4.100) can also be incorporated into the MTL descriptions given by either (4.69) or the chain parameter representation given by (4.68) by similarly partitioning those MTL descriptions to yield a set of $2n$ or n simultaneous equations to be solved for the phasor terminal line voltages or currents. The result is somewhat more complicated than a single Thévenin or Norton representation and will be considered in more detail in Chapter 8.

4.3.6 Approximating Nonuniform Lines

As discussed previously, nonuniform lines are lines whose cross-sectional dimensions (conductors and media) vary along the line axis [B.1, 13, 14]. For these types of lines, the per-unit-length parameter matrices will be functions of z, i.e., $\mathbf{R}(z)$, $\mathbf{L}(z)$, $\mathbf{G}(z)$, and $\mathbf{C}(z)$. In this case the MTL differential equations become *nonconstant-coefficient differential equations.* Although they remain linear (if the surrounding medium is linear), they are as difficult to solve as nonlinear differential equations. A simple but approximate way of solving the MTL equations for a nonuniform MTL is to approximate it as a *discretely uniform* MTL. To do this we break the line into a cascade of sections each of which can be modeled *approximately* as a uniform line characterized by a chain parameter matrix $\hat{\mathbf{\Phi}}_k$ as illustrated in Fig. 4.7. The *overall chain parameter matrix* of the entire line can be obtained as the product (in the appropriate order) of the chain parameter matrices of the individual uniform sections as

$$\begin{aligned} \hat{\mathbf{\Phi}}(\mathscr{L}) &= \hat{\mathbf{\Phi}}_N(\Delta z_N) \times \cdots \times \hat{\mathbf{\Phi}}_k(\Delta z_k) \times \cdots \times \hat{\mathbf{\Phi}}_1(\Delta z_1) \\ &= \prod_{k=1}^{N} \hat{\mathbf{\Phi}}_k(\Delta z_k) \end{aligned} \tag{4.103}$$

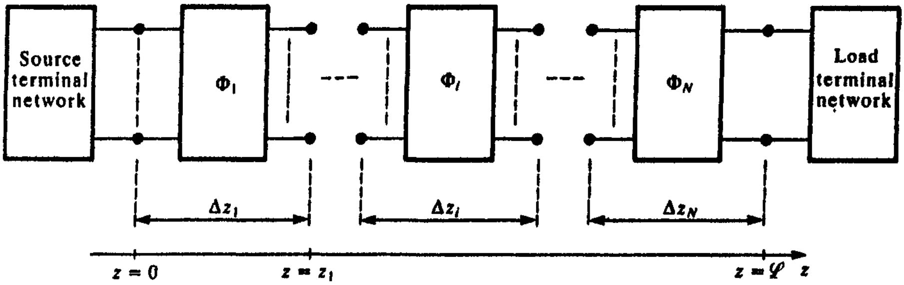


FIGURE 4.7 Representation of a line as a cascade of uniform sections each of which is represented by its chain parameter matrix.

Observe the important order of multiplication of the individual chain parameter matrices. This is a result of the definitions of the chain parameter matrices as

$$\left.\begin{aligned}\begin{bmatrix}\hat{\mathbf{V}}(z_{k+1})\\ \hat{\mathbf{I}}(z_{k+1})\end{bmatrix} &= \hat{\mathbf{\Phi}}_{k+1}(\Delta z_{k+1})\begin{bmatrix}\hat{\mathbf{V}}(z_k)\\ \hat{\mathbf{I}}(z_k)\end{bmatrix}\\ \begin{bmatrix}\hat{\mathbf{V}}(z_k)\\ \hat{\mathbf{I}}(z_k)\end{bmatrix} &= \hat{\mathbf{\Phi}}_k(\Delta z_k)\begin{bmatrix}\hat{\mathbf{V}}(z_{k-1})\\ \hat{\mathbf{I}}(z_{k-1})\end{bmatrix}\end{aligned}\right\} \tag{4.104}$$

Many nonuniform MTL's can be approximately modeled in this fashion. Once the overall chain parameter matrix of the entire line is obtained as in (4.103), the terminal constraints at the ends of the line may be incorporated as in the previous section then the model solved for terminal voltages and currents. Voltages and currents at interior points can also be determined from these terminal solutions by using the individual chain parameter matrices of the uniform sections. For example, the voltage at the right port of the second subsection can be obtained from the terminal voltages and currents as

$$\begin{bmatrix}\hat{\mathbf{V}}(z_2)\\ \hat{\mathbf{I}}(z_2)\end{bmatrix} = \hat{\mathbf{\Phi}}_2(\Delta z_2) \times \hat{\mathbf{\Phi}}_1(\Delta z_1)\begin{bmatrix}\hat{\mathbf{V}}(0)\\ \hat{\mathbf{I}}(0)\end{bmatrix} \tag{4.105}$$

One such application is the analysis of MTL's consisting of twisted pairs of wires [G.1–G.10]. Consider the case of two twisted wires shown in Fig. 4.8(a). This can be approximated as a sequence of abrupt loops in cascade as illustrated in Fig. 4.8(b). The chain parameter matrices of the uniform sections are then multiplied together along with an interchange of the voltages and currents at the junctions. If the lengths of the sections are assumed to be identical, then the overall chain parameter matrix is the N-th power of the chain parameter matrix of each section which can be computed quite efficiently using, for example, the Cayley–Hamilton theorem for powers of a matrix [A.2, 1, 2, 3].

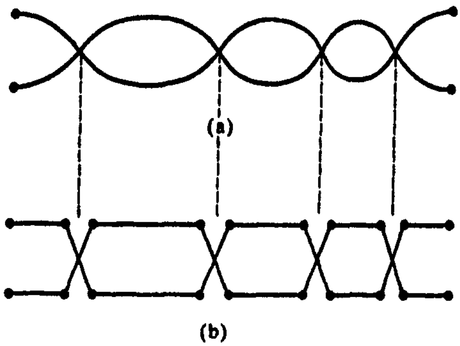


FIGURE 4.8 Approximate representation of a twisted-pair of wires as a cascade of uniform sections.

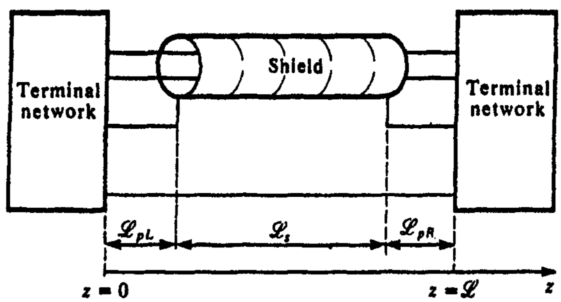


FIGURE 4.9 Representation of a shielded line having pigtails as a cascade of three uniform sections.

Another application is shown in Fig. 4.9. Shielded cables frequently have exposed sections at the ends to facilitate connection of the shield to the terminal networks [F.1–F.8]. The shield is connected to the terminations via a "pigtail" wire over the exposed sections. The overall chain parameter matrix can then be obtained as the product of the chain parameter matrices for the pigtail sections of lengths $\mathscr{L}_{pL}$ and $\mathscr{L}_{pR}$ and the chain parameter matrix of the shielded section of length $\mathscr{L}_S$ as

$$\hat{\Phi}(\mathscr{L}) = \hat{\Phi}_{pR}(\mathscr{L}_{pR}) \times \hat{\Phi}_s(\mathscr{L}_s) \times \hat{\Phi}_{pL}(\mathscr{L}_{pL}) \tag{4.106}$$

Once this overall chain parameter matrix is obtained, the terminal constraints are incorporated in the usual fashion in order to solve for the terminal voltages and currents.

4.4 SOLUTION FOR LINE CATEGORIES

One of the primary problems in this solution process is the determination of the chain parameter matrix, $\hat{\mathbf{\Phi}}$. The solution process for determining the submatrices described previously *assumes* that one can find an $n \times n$, nonsingular transformation matrix, $\hat{\mathbf{T}}$, which *diagonalizes* the product of per-unit-length parameter matrices, $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$, as

$$\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}} = \hat{\gamma}^2 \tag{4.107a}$$

where $\hat{\gamma}^2$ is diagonal as

$$\hat{\gamma}^2 = \begin{bmatrix} \hat{\gamma}_1^2 & 0 & \cdots & 0 \\ 0 & \hat{\gamma}_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\gamma}_n^2 \end{bmatrix} \tag{4.107b}$$

This is a classic problem in matrix analysis [A.4, 1–4]. The n values, $\hat{\gamma}_i^2$, are said to be the *eigenvalues* of the matrix $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$. Premultiplying (4.107a) by $\hat{\mathbf{T}}$ gives

$$\hat{\mathbf{Y}}\hat{\mathbf{Z}}\hat{\mathbf{T}} - \hat{\mathbf{T}}\hat{\gamma}^2 = \mathbf{0} \tag{4.108}$$

Let us denote the $n \times 1$ columns of $\hat{\mathbf{T}}$ as $\hat{\mathbf{T}}_i$ where

$$\hat{\mathbf{T}} = [\hat{\mathbf{T}}_1 \quad \cdots \quad \mathbf{T}_i \quad \cdots \quad \hat{\mathbf{T}}_n] \tag{4.109}$$

Substituting (4.109) and (4.107b) into (4.108) and expanding the result gives n sets of simultaneous equations as

$$\left.\begin{aligned} (\hat{\mathbf{Y}}\hat{\mathbf{Z}} - \hat{\gamma}_1^2\mathbf{1}_n)\hat{\mathbf{T}}_1 &= \mathbf{0} \\ &\vdots \\ (\hat{\mathbf{Y}}\hat{\mathbf{Z}} - \hat{\gamma}_i^2\mathbf{1}_n)\hat{\mathbf{T}}_i &= \mathbf{0} \\ &\vdots \\ (\hat{\mathbf{Y}}\hat{\mathbf{Z}} - \hat{\gamma}_n^2\mathbf{1}_n)\hat{\mathbf{T}}_n &= \mathbf{0} \end{aligned}\right\} \tag{4.110}$$

The columns of $\hat{\mathbf{T}}$, $\hat{\mathbf{T}}_i$, are said to be the *eigenvectors* of the matrix $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ [A.4]. Thus the question becomes whether we can find *n linearly independent eigenvectors of* $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ *which will diagonalize it as in* (4.107). Equations (4.110) are a homogeneous set of linear, algebraic equations. As such, they have:

1. The unique trivial solution $\hat{\mathbf{T}}_i = \mathbf{0}$.
2. An infinite number of solutions for the $\hat{\mathbf{T}}_i$ [A.4].

Clearly we want to determine the nontrivial solution which will exist only if the *determinant of the coefficient matrix is zero*; i.e.,

$$|\hat{\mathbf{Y}}\hat{\mathbf{Z}} - \hat{\gamma}_i^2 \mathbf{1}_n| = 0 \tag{4.111}$$

We now set out to investigate when this is possible and how to compute it.

There are a number of known cases of $n \times n$ matrices, $\hat{\mathbf{M}}$, whose diagonalization is assured. These are [A.4, 3]:

1. All eigenvalues of $\hat{\mathbf{M}}$ are *distinct*.
2. $\hat{\mathbf{M}}$ is *real*, and *symmetric*.
3. $\hat{\mathbf{M}}$ is *complex* but *normal*, i.e., $\hat{\mathbf{M}}\hat{\mathbf{M}}^{t*} = \hat{\mathbf{M}}^{t*}\hat{\mathbf{M}}$ where we denote the transpose of a matrix by t and its conjugate by *.
4. $\hat{\mathbf{M}}$ is *complex* and *Hermitian*, i.e., $\hat{\mathbf{M}} = \hat{\mathbf{M}}^{t*}$.

For *normal* or *Hermitian* $\hat{\mathbf{M}}$, the transformation matrix can be found such that $\hat{\mathbf{T}}^{-1} = (\hat{\mathbf{T}}^t)^*$. For a *real, symmetric* $\mathbf{M}$, the transformation matrix can be found such that $\mathbf{T}^{-1} = \mathbf{T}^t$. For other types of matrices, we are not assured that a nonsingular transformation can be found that diagonalizes it.

The matrix product to be diagonalized is expanded as

$$\begin{aligned}\hat{\mathbf{Y}}\hat{\mathbf{Z}} &= (\mathbf{G} + j\omega\mathbf{C})(\mathbf{R} + j\omega\mathbf{L}) \\ &= \mathbf{GR} + j\omega\mathbf{CR} + j\omega\mathbf{GL} - \omega^2\mathbf{CL}\end{aligned} \tag{4.112}$$

There exist digital computer subroutines that find the eigenvalues and eigenvectors of a general complex matrix. These can be used to attempt to diagonalize $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$. However, because the number of conductors, n, of the MTL can be quite large, it is important to investigate the conditions under which we can obtain an *efficient* and *numerically stable* diagonalization. The following sections address that point.

4.4.1 Perfect Conductors in Homogeneous Media

Consider the case of perfect conductors for which $\mathbf{R} = \mathbf{0}$. The matrix product becomes

$$\begin{aligned}\hat{\mathbf{Y}}\hat{\mathbf{Z}} &= (\mathbf{G} + j\omega\mathbf{C})(j\omega\mathbf{L}) \\ &= j\omega\mathbf{GL} - \omega^2\mathbf{CL}\end{aligned} \tag{4.113}$$

If the surrounding medium is *homogeneous* with parameters σ, ε, and μ, then we have the important identities:

$$\mathbf{CL} = \mathbf{LC} = \mu\varepsilon\mathbf{1}_n \tag{4.114a}$$

$$\mathbf{GL} = \mathbf{LG} = \mu\sigma\mathbf{1}_n \tag{4.114b}$$

and $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ is already diagonal. In this case we may choose

$$\hat{\mathbf{T}} = \mathbf{1}_n \tag{4.115a}$$

and all the eigenvalues are identical giving the propagation constants as

$$\begin{aligned} \hat{\gamma} &= \sqrt{j\omega\mu\sigma - \omega^2\mu\varepsilon} \\ &= \alpha + j\beta \end{aligned} \tag{4.115b}$$

In this case, the chain parameter submatrices in (4.70) become

$$\hat{\mathbf{\Phi}}_{11} = \cosh(\hat{\gamma}\mathscr{L})\mathbf{1}_n \tag{4.116a}$$

$$\hat{\mathbf{\Phi}}_{12} = -\sinh(\hat{\gamma}\mathscr{L})\hat{\mathbf{Z}}_C \tag{4.116b}$$

$$\hat{\mathbf{\Phi}}_{21} = -\sinh(\hat{\gamma}\mathscr{L})\hat{\mathbf{Z}}_C^{-1} \tag{4.116c}$$

$$\hat{\mathbf{\Phi}}_{22} = \cosh(\hat{\gamma}\mathscr{L})\mathbf{1}_n \tag{4.116d}$$

where

$$\hat{\mathbf{Z}}_C = \frac{j\omega}{\hat{\gamma}}\mathbf{L} \tag{4.117a}$$

$$\hat{\mathbf{Z}}_C^{-1} = \frac{\hat{\gamma}}{j\omega}\mathbf{L}^{-1} = \frac{\hat{\gamma}}{j\omega\mu\varepsilon}\mathbf{C} \tag{4.117b}$$

In the medium, in addition to being homogeneous, is also lossless, $\sigma = 0$, the propagation constant becomes

$$\hat{\gamma} = j\omega\sqrt{\mu\varepsilon} \tag{4.118}$$

so that the attenuation constant is zero, $a = 0$, and the phase constant is $\beta = \omega\sqrt{\mu\varepsilon}$. The velocity of propagation becomes

$$\begin{aligned} v &= \frac{\omega}{\beta} \\ &= \frac{1}{\sqrt{\mu\varepsilon}} \end{aligned} \tag{4.119}$$

The chain parameter submatrices simplify to

$$\hat{\mathbf{\Phi}}_{11} = \cos(\beta\mathscr{L})\mathbf{1}_n \tag{4.120a}$$

$$\begin{aligned} \hat{\mathbf{\Phi}}_{12} &= -j\sin(\beta\mathscr{L})\hat{\mathbf{Z}}_C \\ &= -jv\sin(\beta\mathscr{L})\mathbf{L} \end{aligned} \tag{4.120b}$$

$$\begin{aligned}\hat{\Phi}_{21} &= -j\sin(\beta\mathscr{L})\hat{\mathbf{Z}}_C^{-1} \qquad (4.120c)\\ &= -jv\sin(\beta\mathscr{L})\mathbf{C}\\ \hat{\Phi}_{22} &= \cos(\beta\mathscr{L})\mathbf{1}_n \qquad (4.120d)\end{aligned}$$

where the characteristic impedance becomes real given by

$$\mathbf{Z}_C = v\mathbf{L} \tag{4.121a}$$

$$\hat{\mathbf{Z}}_C^{-1} = v\mathbf{C} \tag{4.121b}$$

This case of perfect conductors in a homogeneous medium (lossless or lossy) forms the purest form of TEM waves on the line. In the following sections we investigate the *quasi-TEM* mode of propagation wherein the conductors can be lossy and/or the medium may be inhomogeneous.

4.4.2 Lossy Conductors in Homogeneous Media

Consider the case where we permit imperfect conductors, $\mathbf{R} \neq \mathbf{0}$, but assume a homogeneous medium. The matrix product $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ in (4.112) becomes, using the identities for a homogeneous medium given in (4.114),

$$\begin{aligned}\hat{\mathbf{Y}}\hat{\mathbf{Z}} &= \mathbf{GR} + j\omega\mathbf{CR} + (j\omega\mu\sigma - \omega^2\mu\varepsilon)\mathbf{1}_n \qquad (4.122)\\ &= \left(\frac{\sigma}{\varepsilon} + j\omega\right)\mathbf{CR} + (j\omega\mu\sigma - \omega^2\mu\varepsilon)\mathbf{1}_n\end{aligned}$$

where we have substituted the identity

$$\mathbf{G} = \frac{\sigma}{\varepsilon}\mathbf{C} \tag{4.123}$$

and have neglected the *internal inductance* of the wires, $\mathbf{L}_i = \mathbf{0}$, for reasons discussed in Chapter 3. From (4.122) we need only diagonalize $\mathbf{CR}$ as

$$\begin{aligned}\hat{\mathbf{T}}^{-1}\mathbf{CR}\hat{\mathbf{T}} &= \mathbf{\Lambda}^2 \qquad (4.124)\\ &= \begin{bmatrix} \Lambda_1^2 & 0 & \cdots & 0 \\ 0 & \Lambda_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_n^2 \end{bmatrix}\end{aligned}$$

The eigenvalues of $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ become

$$\hat{\gamma}_i^2 = \left(\frac{\sigma}{\varepsilon} + j\omega\right)\Lambda_i^2 + (j\omega\mu\sigma - \omega^2\mu\varepsilon) \tag{4.125}$$

The key problem here is finding the transformation matrix which diagonalizes **CR** as in (4.124). Thus we need to diagonalize the product of two matrices. A numerically stable transformation can be found to accomplish this in the following manner. Recall that **C** is *real, symmetric,* and *positive definite* and **R** is *real symmetric.* First consider diagonalizing **C**. Since **C** is real and symmetric, one can find a *real, orthogonal transformation* **U** which diagonalizes **C** where $\mathbf{U}^{-1} = \mathbf{U}^t$ [B.1, 3]:

$$\mathbf{U}^t\mathbf{C}\mathbf{U} = \boldsymbol{\theta}^2 \tag{4.126}$$

$$= \begin{bmatrix} \theta_1^2 & 0 & \cdots & 0 \\ 0 & \theta_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \theta_n^2 \end{bmatrix}$$

Since **C** is real, symmetric, and *positive definite,* its eigenvalues θ_i^2 are all real, nonzero, and positive [B.1]. Therefore the square roots of these, $\theta_i = \sqrt{\theta_i^2}$, will be *real, nonzero numbers.* Next form the *real, symmetric* matrix product

$$\boldsymbol{\theta}^{-1}\mathbf{U}^t\mathbf{C}\mathbf{U}\boldsymbol{\theta}^{-1} = \mathbf{1}_n \tag{4.127}$$

Now form

$$\begin{aligned}\boldsymbol{\theta}^{-1}\mathbf{U}^t\mathbf{C}\mathbf{R}\mathbf{U}\boldsymbol{\theta} &= \underbrace{\boldsymbol{\theta}^{-1}\mathbf{U}^t\mathbf{C}\mathbf{U}\boldsymbol{\theta}^{-1}}_{\mathbf{1}_n}\boldsymbol{\theta}\mathbf{U}^t\mathbf{R}\mathbf{U}\boldsymbol{\theta} \\ &= \boldsymbol{\theta}\mathbf{U}^t\mathbf{R}\mathbf{U}\boldsymbol{\theta}\end{aligned} \tag{4.128}$$

The matrix $\boldsymbol{\theta}\mathbf{U}^t\mathbf{R}\mathbf{U}\boldsymbol{\theta}$ is real and symmetric so it can also be diagonalized with a real, orthogonal transformation, **S**, as

$$\begin{aligned}\mathbf{S}^t[\boldsymbol{\theta}\mathbf{U}^t\mathbf{R}\mathbf{U}\boldsymbol{\theta}]\mathbf{S} &= \mathbf{S}^t[\boldsymbol{\theta}^{-1}\mathbf{U}^t\mathbf{C}\mathbf{R}\mathbf{U}\boldsymbol{\theta}]\mathbf{S} \\ &= \boldsymbol{\Lambda}^2\end{aligned} \tag{4.129}$$

This result shows that the desired transformation is

$$\mathbf{T} = \mathbf{U}\boldsymbol{\theta}\mathbf{S} \tag{4.130}$$

The inverse of **T** is $\mathbf{T}^{-1} = \mathbf{S}^t\boldsymbol{\theta}^{-1}\mathbf{U}^t = \mathbf{T}^t\mathbf{C}^{-1}$. There are numerous digital computer subroutines that implement the diagonalization of the product of two real, symmetric matrices, one of which is positive definite in the above fashion [I.1].

The entries in the per-unit-length resistance matrix, **R**, are functions of the square root of frequency, $\sqrt{f}$, at high frequencies due to the skin effect. This does not pose any problems in the phasor solution since we simply evaluate these at the frequency of interest and perform the above computation at that frequency. Reevaluate **R** at the next frequency and perform the above computation at that frequency. Reevaluate **R** at the next frequency and perform the above computations for that frequency and so forth. We will find in the next chapter that this dependence of **R** on $\sqrt{f}$ poses some significant problems in the time-domain analysis of MTL's.

4.4.3 Perfect Conductors in Inhomogeneous Media

We next turn our attention to the diagonalization of the matrix product $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ where we assume perfect conductors, $\mathbf{R} = \mathbf{0}$, and lossless media, $\mathbf{G} = \mathbf{0}$. The surrounding, lossless medium may be *inhomogeneous* in which case we no longer have the fundamental identity in (4.114a), i.e., $\mathbf{CL} \neq \mu\varepsilon\mathbf{1}_n$. In fact, the product of the per-unit-length inductance and capacitances matrices is generally not diagonal. The matrix product to be diagonalized becomes

$$\hat{\mathbf{Y}}\hat{\mathbf{Z}} = -\omega^2\mathbf{CL} \tag{4.131}$$

Once again, **C** and **L** are real, symmetric and **C** is positive definite. Therefore we can diagonalize **CL** as in the previous section with orthogonal transformations as

$$\hat{\mathbf{T}}^{-1}\mathbf{CL}\hat{\mathbf{T}} = \mathbf{\Lambda}^2 \tag{4.132}$$
$$= \begin{bmatrix} \Lambda_1^2 & 0 & \cdots & 0 \\ 0 & \Lambda_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_n^2 \end{bmatrix}$$

The transformation matrix, $\hat{\mathbf{T}}$, which accomplishes this is real and given by

$$\mathbf{T} = \mathbf{U\theta S} \tag{4.133a}$$

where **U** and **S** are obtained from

$$\mathbf{U}^t\mathbf{CU} = \mathbf{\theta}^2 \tag{4.133b}$$

$$= \begin{bmatrix} \theta_1^2 & 0 & \cdots & 0 \\ 0 & \theta_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \theta_n^2 \end{bmatrix}$$

$$\mathbf{S}^t(\boldsymbol{\theta}\mathbf{U}^t\mathbf{L}\mathbf{U}\boldsymbol{\theta})\mathbf{S} = \mathbf{S}^t(\boldsymbol{\theta}^{-1}\mathbf{U}^t\mathbf{C}\mathbf{L}\mathbf{U}\boldsymbol{\theta})\mathbf{S} \qquad (4.133c)$$
$$= \boldsymbol{\Lambda}^2$$

The inverse of T is $\mathbf{T}^{-1} = \mathbf{S}^t\boldsymbol{\theta}^{-1}\mathbf{U}^t = \mathbf{T}^t\mathbf{C}^{-1}$. The desired eigenvalues of $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ are

$$\hat{\gamma}_i^2 = -\omega^2\Lambda_i^2 \qquad (4.134)$$

4.4.4 The General Case: Lossy Conductors in Lossy Inhomogeneous Media

In the most general case wherein the line is lossy, $\mathbf{R} \neq \mathbf{0}$, $\mathbf{G} \neq \mathbf{0}$, and the medium is inhomogeneous so that $\mathbf{LC} \neq 1/v^2\mathbf{1}_n$, we have no other recourse but to seek a *frequency-dependent* transformation, $\mathbf{T}(\omega)$, such that

$$\mathbf{T}^{-1}(\omega)\underbrace{(\mathbf{G} + j\omega\mathbf{C})}_{\hat{\mathbf{Y}}(\omega)}\underbrace{(\mathbf{R} + j\omega\mathbf{L})}_{\hat{\mathbf{Z}}(\omega)}\mathbf{T}(\omega) = \hat{\boldsymbol{\gamma}}^2(\omega) \qquad (4.135)$$

Although we are not assured of a numerically stable diagonalization, a more important computational problem here is that the transformation matrix, $\mathbf{T}(\omega)$, is frequency dependent and *must be recomputed at each frequency*! Thus an eigenvector–eigenvalue subroutine for complex matrices must be called repeatedly at each frequency which can be quite time-consuming if the responses at a large number of frequencies are desired. In order to provide for this general case, a general frequency-domain FORTRAN program, **MTL.FOR**, which determines $\mathbf{T}(\omega)$ via (4.135) and incorporates the generalized Thévenin equivalent terminal representations in (4.84) is described in Appendix A. Other FORTRAN codes that efficiently implement the considerations in Sections 4.4.1, 4.4.2, or 4.4.3 which make assumptions about the losses of the line and/or the homogeneity of the surrounding medium are described in [I.1].

4.4.5 Cyclic Symmetric Structures

The MTL structures considered in Sections 4.4.1, 4.4.2, and 4.4.3 are such that the matrix product, $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$, can always be diagonalized with a numerically efficient and stable similarity transformation, $\mathbf{T}$, which is *frequency independent*. Not all structures can be diagonalized in this fashion. One can try to diagonalize $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ with a digital computer subroutine that determines the eigenvalues and eigenvectors of a general, complex matrix such as $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ but we are not assured that the eigenvectors will be linearly independent. Furthermore, the transformation matrix, $\mathbf{T}(\omega)$, will be *frequency dependent* as was demonstrated in the previous section. This section discusses MTL's which have certain *structural symmetry* so that a numerically stable (and trivial) transformation $\hat{\mathbf{T}}$ can always be found which diagonalizes $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$. Furthermore this transformation is *frequency independent* regardless of whether the line is lossy or the medium is inhomogeneous, i.e., the general case.

Consider structures composed of *n identical conductors* and a reference conductor wherein the *n* conductors have structural symmetry with respect to the reference conductor so that the per-unit-length impedance and admittance matrices have the following structural symmetry [B.1, 11]:

$$\hat{\mathbf{Z}} = \begin{bmatrix} \hat{Z}_1 & \hat{Z}_2 & \hat{Z}_3 & \cdots & \hat{Z}_3 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_1 & \hat{Z}_2 & \hat{Z}_3 & \ddots & \hat{Z}_3 \\ \hat{Z}_3 & \hat{Z}_2 & \hat{Z}_1 & \hat{Z}_2 & \ddots & \vdots \\ \vdots & \hat{Z}_3 & \hat{Z}_2 & \ddots & \ddots & \hat{Z}_3 \\ \hat{Z}_3 & \ddots & \ddots & \ddots & \hat{Z}_1 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_3 & \cdots & \hat{Z}_3 & \hat{Z}_2 & \hat{Z}_1 \end{bmatrix} \tag{4.136a}$$

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 & \hat{Y}_3 & \cdots & \hat{Y}_3 & \hat{Y}_2 \\ \hat{Y}_2 & \hat{Y}_1 & \hat{Y}_2 & \hat{Y}_3 & \ddots & \hat{Y}_3 \\ \hat{Y}_3 & \hat{Y}_2 & \hat{Y}_1 & \hat{Y}_2 & \ddots & \vdots \\ \vdots & \hat{Y}_3 & \hat{Y}_2 & \ddots & \ddots & \hat{Y}_3 \\ \hat{Y}_3 & \ddots & \ddots & \ddots & \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_2 & \hat{Y}_3 & \cdots & \hat{Y}_3 & \hat{Y}_2 & \hat{Y}_1 \end{bmatrix} \tag{4.136b}$$

Examples of structures which result in these types of per-unit-length parameter matrices are shown in Fig. 4.10. Observe that in order for the main diagonal terms to be equal, the conductors and surrounding media (which may be inhomogeneous) must also exhibit symmetry. For example, if the *n* conductors are dielectric-insulated wires, the dielectric insulations of the *n* wires must have identical ε and thicknesses. The reference conductor need not share this property. A general cyclic symmetric matrix $\hat{\mathbf{M}}$ has the entries given by

$$[\hat{\mathbf{M}}]_{ij} = \hat{M}_{|i-j|+1} \tag{4.137a}$$

where

$$\hat{M}_{j\pm n} = \hat{M}_j \tag{4.137b}$$

$$\hat{M}_{n+2-j} = \hat{M}_j \tag{4.137c}$$

and indices greater than n or less than 1 are defined by the convention: $n + j = j$ and $n + i = i$ [B.1]. Because of this special structure of the per-unit-length

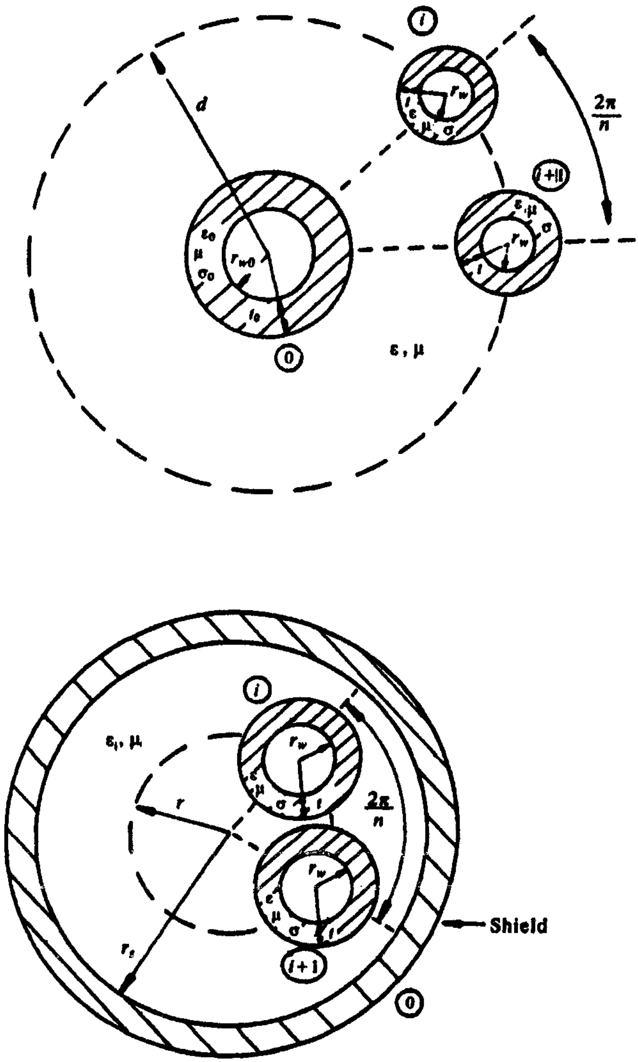


FIGURE 4.10 Cyclic symmetric structures which are diagonalizable by a frequency-independent transformation.

matrices, they are *normal* matrices, $\hat{\mathbf{Z}}(\hat{\mathbf{Z}}^t)^* = (\hat{\mathbf{Z}}^t)^*\hat{\mathbf{Z}}$, and we are guaranteed that each can be diagonalized as [B.1, 1–4]

$$\hat{\mathbf{T}}^{-1}\hat{\mathbf{Z}}\mathbf{T} = \hat{\gamma}_Z^2 \tag{4.138a}$$

$$\hat{\mathbf{T}}^{-1}\hat{\mathbf{Y}}\mathbf{T} = \hat{\gamma}_Y^2 \tag{4.138b}$$

where the $n \times n$ matrices $\hat{\gamma}_Z^2$ and $\hat{\gamma}_Y^2$ are *diagonal* [B.1]. In fact, the transformation is trivial to obtain [B.1]:

$$[\hat{\mathbf{T}}]_{ij} = \frac{1}{\sqrt{n}} \angle \left[\tfrac{2\pi}{n}(i-1)(j-1)\right] \tag{4.139}$$

and

$$\hat{\mathbf{T}}^{-1} = (\hat{\mathbf{T}}^t)^* \tag{4.140}$$

Similarly, the eigenvalues of $\hat{\mathbf{Y}}\hat{\mathbf{Z}}$ (the propagation constants) are easily determined as [B.1]

$$\hat{\gamma}_i^2 = \left\{\sum_{p=1}^{n} [\hat{\mathbf{Z}}]_{1p} \angle \left[\tfrac{2\pi}{n}(p-1)(i-1)\right]\right\}\left\{\sum_{q=1}^{n} [\hat{\mathbf{Y}}]_{1q} \angle \left[\tfrac{2\pi}{n}(q-1)(i-1)\right]\right\} \tag{4.141}$$

As an illustration of these results, consider a four-conductor ($n = 3$) line with a cyclic symmetric structure so that

$$\hat{\mathbf{Z}} = \begin{bmatrix} \hat{Z}_1 & \hat{Z}_2 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_1 & \hat{Z}_2 \\ \hat{Z}_2 & \hat{Z}_2 & \hat{Z}_1 \end{bmatrix} \tag{4.142a}$$

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 & \hat{Y}_2 \\ \hat{Y}_2 & \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_2 & \hat{Y}_2 & \hat{Y}_1 \end{bmatrix} \tag{4.142b}$$

The transformation matrix is

$$\hat{\mathbf{T}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & e^{j4\pi/3} & e^{j2\pi/3} \end{bmatrix} \tag{4.143a}$$

$$\hat{\mathbf{T}}^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} \\ 1 & e^{-j4\pi/3} & e^{-j2\pi/3} \end{bmatrix} \tag{4.143b}$$

and the propagation constants are

$$\hat{\gamma}_1^2 = (\hat{Z}_1 + \hat{Z}_2 + \hat{Z}_2)(\hat{Y}_1 + \hat{Y}_2 + \hat{Y}_2) \tag{4.144a}$$

$$\hat{\gamma}_2^2 = (\hat{Z}_1\underline{/0^\circ} + \hat{Z}_2\underline{/2\pi/3} + \hat{Z}_2\underline{/4\pi/3})(\hat{Y}_1\underline{/0^\circ} + \hat{Y}_2\underline{/2\pi/3} + \hat{Y}_2\underline{/4\pi/3}) \tag{4.144b}$$

$$\hat{\gamma}_3^2 = (\hat{Z}_1\underline{/0^\circ} + \hat{Z}_2\underline{/4\pi/3} + \hat{Z}_2\underline{/2\pi/3})(\hat{Y}_1\underline{/0^\circ} + \hat{Y}_2\underline{/4\pi/3} + \hat{Y}_2\underline{/2\pi/3}) \tag{4.144c}$$

There are a number of cases where a MTL can be *approximated* as a cyclic symmetric structure. A common case is a three-phase, high-voltage power transmission line consisting of three wires above earth. In order to reduce interference to neighboring telephone lines, the three conductors are *transposed* at regular intervals. As an approximation, we may assume that each of the three (identical) wires occupy, at regular intervals, each of the three possible positions along the line (all of which are at the same height above earth and produce identical separation distances between adjacent wires). With this assumption, the per-unit-length matrices, $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$, take on a cyclic symmetric structure:

$$\hat{\mathbf{Z}} = \begin{bmatrix} \hat{Z} & \hat{Z}' & \hat{Z}' \\ \hat{Z}' & \hat{Z} & \hat{Z}' \\ \hat{Z}' & \hat{Z}' & \hat{Z} \end{bmatrix} \tag{4.145a}$$

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y} & \hat{Y}' & \hat{Y}' \\ \hat{Y}' & \hat{Y} & \hat{Y}' \\ \hat{Y}' & \hat{Y}' & \hat{Y} \end{bmatrix} \tag{4.145b}$$

The transformation matrix is given by (4.143) and the propagation constants simplify to

$$\hat{\gamma}_1^2 = (\hat{Z} + 2\hat{Z}')(\hat{Y} + 2\hat{Y}') \tag{4.146a}$$

$$\hat{\gamma}_2^2 = (\hat{Z} - \hat{Z}')(\hat{Y} - \hat{Y}') \tag{4.146b}$$

$$\hat{\gamma}_3^2 = (\hat{Z} - \hat{Z}')(\hat{Y} - \hat{Y}') \tag{4.146c}$$

Two of the propagation constants, $\hat{\gamma}_2$ and $\hat{\gamma}_3$, are equal and these are associated with the *aerial mode* of propagation. The third propagation constant, $\hat{\gamma}_1$, is associated with the *ground mode* of propagation. This transformation is referred to in the power transmission literature as the *method of symmetrical components.* Such lines are said to be *balanced.* In the case of unbalanced lines where, for example, one phase may be shorted to ground, this transformation does not apply.

Other approximations of MTL's as cyclic symmetric structures are useful. Cable harnesses carrying tightly packed, insulated wires have been assumed to be cyclic symmetric structures on the notion that all wires occupy at some point along the line all possible positions. This leads to a cyclic symmetric structure of the $n \times n$ per-unit-length impedance and admittance matrices that is similar to the special case of transposed power distribution lines shown in (4.145). Other common cases are the cyclic symmetric, three-conductor lines shown in Fig. 4.11. Two, identical, dielectric-insulated wires are suspended at equal heights above a ground plane as shown in Fig. 4.11(a). The per-unit-length

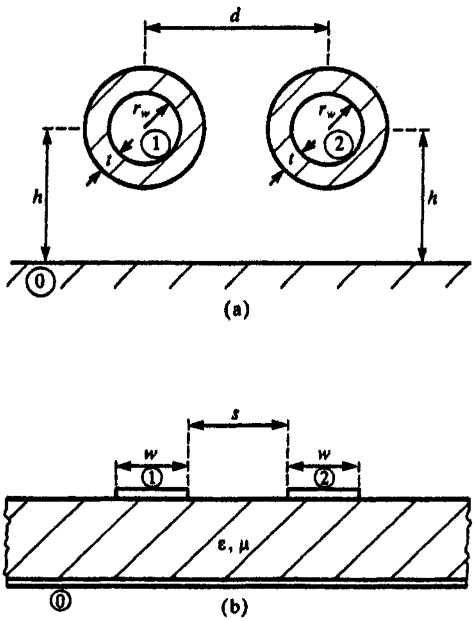


FIGURE 4.11 Certain three-conductors structures that are cyclic summetric: (a) two identical wires at identical heights above a ground plane, (b) two identical lands of a coupled microstrip configuration.

impedance and admittance matrices become

$$\hat{\mathbf{Z}} = \begin{bmatrix} \hat{Z} & \hat{Z}' \\ \hat{Z}' & Z \end{bmatrix} \tag{4.147a}$$

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y} & \hat{Y}' \\ \hat{Y}' & \hat{Y} \end{bmatrix} \tag{4.147b}$$

The transformation matrix and propagation constants simplify to

$$\hat{\mathbf{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{4.148a}$$

$$\hat{\gamma}_1^2 = (\hat{Z} + \hat{Z}')(\hat{Y} + \hat{Y}') \tag{4.148b}$$

$$\hat{\gamma}_2^2 = (\hat{Z} - \hat{Z}')(\hat{Y} - \hat{Y}') \tag{4.148c}$$

This transformation is referred to in the microwave literature as the *even–odd mode* transformation and has been applied to the symmetrical, coupled microstrip line shown in Fig. 4.11(b).

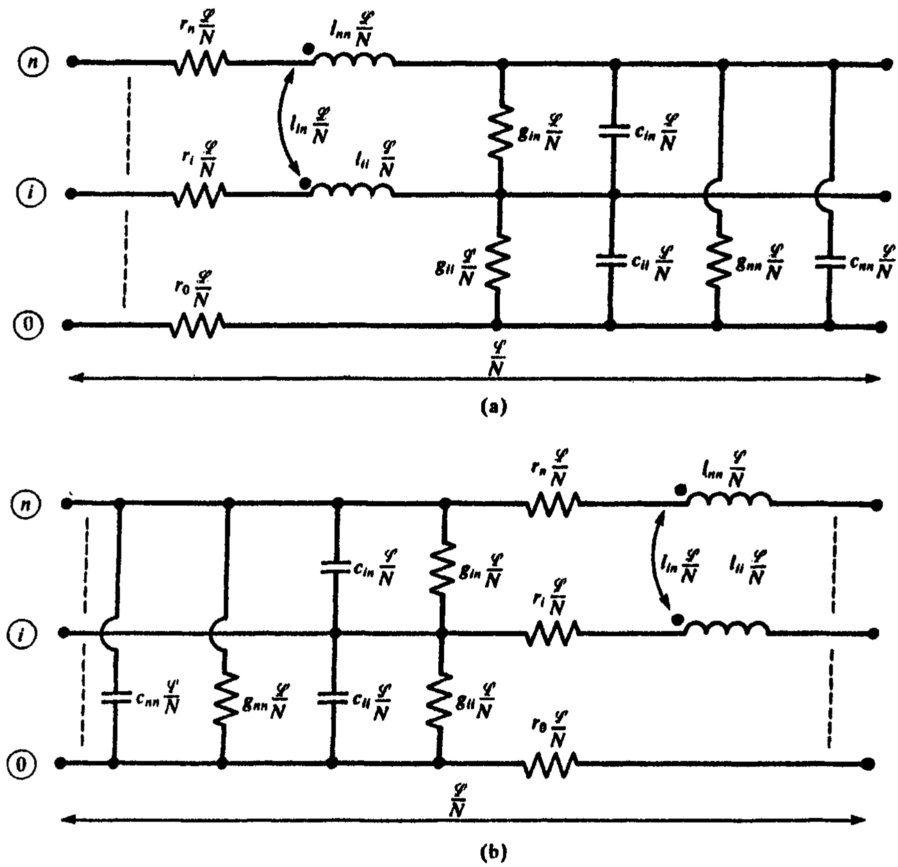
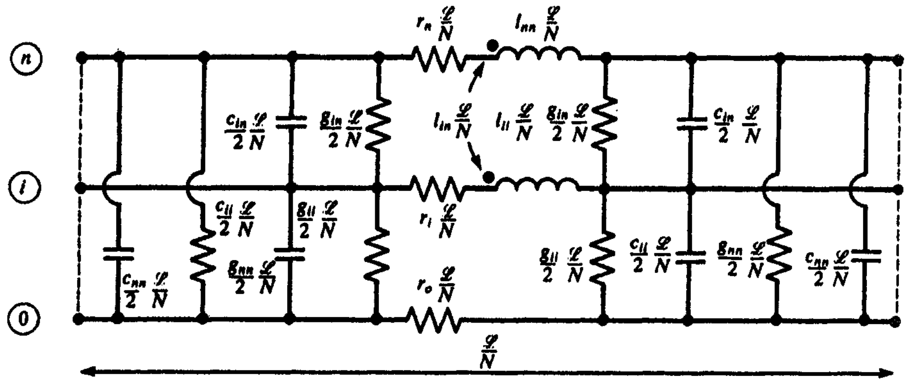


FIGURE 4.12 Lumped-circuit iterative approximate structures: (a) lumped ⅂, (b) lumped Γ.

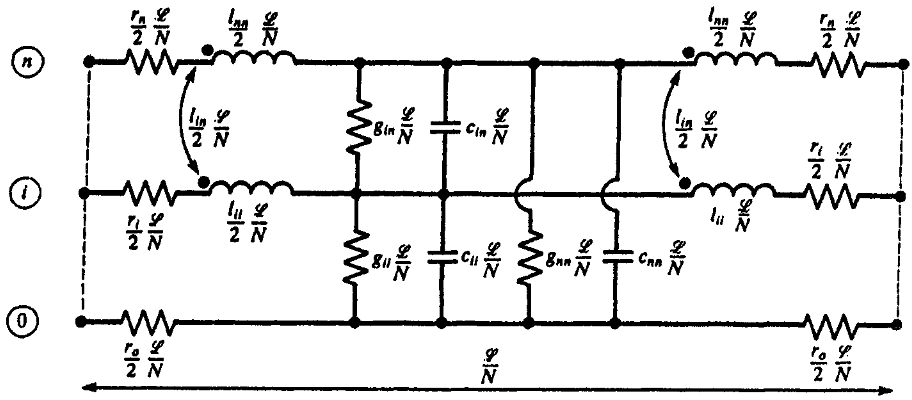
4.5 LUMPED-CIRCUIT ITERATIVE APPROXIMATE CHARACTERIZATIONS

Lumped-circuit notions apply to circuits whose largest dimension is *electrically small*, i.e., $\ll \lambda$, where $\lambda = v/f$ is a wavelength at the frequency of interest. In the derivation of the MTL equations we divided the line into sections of length Δz. In order to insure that the section length was electrically small for all frequencies, we allowed $\Delta z \to 0$ resulting in the MTL equations. This suggests another, frequently used approximation of a MTL. Divide the line into N sections of length $\mathcal{L}/N$ as illustrated in Fig. 4.7. If each of these section lengths is electrically short at the frequency of interest, $\Delta z = \mathcal{L}/N \ll \lambda$, then each section may be represented with a lumped model. These are referred to as *lumped, iterative* structures since the line must be more finely divided as frequency is increased.

Some typical lumped structures are shown in Fig. 4.12 [B.1, 12]. Observe



(c)



(d)

FIGURE 4.12 (*Continued*) Lumped-circuit iterative approximate structures: (c) lumped Pi, and (d) lumped T.

that the total parameter is the per-unit-length multiplied by the section length, $\mathscr{L}/N$. These structures are named for the symbols their structures represent: ⅂, Γ, π, T. The chain parameter matrices of these structures can be derived in a straightforward fashion as [B.1]

$$\hat{\Phi}_{⅂} = \begin{bmatrix} \{1_n\} & \left\{-\hat{\mathbf{Z}}\dfrac{\mathscr{L}}{N}\right\} \\ \left\{-\hat{\mathbf{Y}}\dfrac{\mathscr{L}}{N}\right\} & \left\{1_n + \hat{\mathbf{Y}}\hat{\mathbf{Z}}\left(\dfrac{\mathscr{L}}{N}\right)^2\right\} \end{bmatrix} \tag{4.149a}$$

$$\hat{\Phi}_{\Gamma} = \begin{bmatrix} \left\{1_n + \hat{\mathbf{Z}}\hat{\mathbf{Y}}\left(\dfrac{\mathscr{L}}{N}\right)^2\right\} & \left\{-\hat{\mathbf{Z}}\dfrac{\mathscr{L}}{N}\right\} \\ \left\{-\hat{\mathbf{Y}}\dfrac{\mathscr{L}}{N}\right\} & \{1_n\} \end{bmatrix} \tag{4.149b}$$

$$\hat{\Phi}_{\pi} = \begin{bmatrix} \left\{ 1_n + \frac{1}{2} \hat{Z}\hat{Y}\left(\frac{\mathscr{L}}{N}\right)^2 \right\} & \left\{ -\hat{Z}\frac{\mathscr{L}}{N} \right\} \\ \left\{ -\hat{Y}\frac{\mathscr{L}}{N} - \frac{1}{4}\hat{Y}\hat{Z}\hat{Y}\left(\frac{\mathscr{L}}{N}\right)^3 \right\} & \left\{ 1_n + \frac{1}{2}\hat{Y}\hat{Z}\left(\frac{\mathscr{L}}{N}\right)^2 \right\} \end{bmatrix} \tag{4.149c}$$

$$\hat{\Phi}_{T} = \begin{bmatrix} \left\{ 1_n + \frac{1}{2} \hat{Z}\hat{Y}\left(\frac{\mathscr{L}}{N}\right)^2 \right\} & \left\{ -\hat{Z}\frac{\mathscr{L}}{N} - \frac{1}{4}\hat{Z}\hat{Y}\hat{Z}\left(\frac{\mathscr{L}}{N}\right)^3 \right\} \\ \left\{ -\hat{Y}\frac{\mathscr{L}}{N} \right\} & \left\{ 1_n + \frac{1}{2}\hat{Y}\hat{Z}\left(\frac{\mathscr{L}}{N}\right)^2 \right\} \end{bmatrix} \tag{3.149d}$$

This overall chain parameter matrix of a line that is represented as a cascade of N such lumped sections is

$$\hat{\Phi} = \hat{\Phi}^{N}_{(\urcorner, \ulcorner, \pi, \mathrm{T})} \tag{4.150}$$

Once this overall chain parameter matrix is obtained, the terminal conditions are incorporated as described in Section 4.3.5 to give the terminal voltages and currents of the MTL. Lumped-circuit analysis programs such as SPICE can be used to analyze the resulting lumped circuit as an alternative to obtaining the overall chain parameter matrix via (4.150) and then incorporating the terminal conditions. Nonlinear terminations such as transistors and diodes can be readily incorporated into the terminations since these lumped-circuit programs include sophisticated models for them. We will find this notion of the lumped iterative model to be a useful approximation in the time-domain analysis of the MTL in the next chapter since these lumped-circuit programs can be used to perform the time-domain analysis with a simple change in a control statement. However, there are additional considerations in the use of this approximate model in the time-domain analysis of MTL's.

It is interesting to compare the chain parameter matrices for the above lumped-circuit structures to the *exact* chain-parameter matrix given in series form in (4.48) for the entire line of length $\mathscr{L}$:

$$\hat{\Phi} = \begin{bmatrix} \left\{ 1_n + \frac{\mathscr{L}^2}{2!}\hat{Z}\hat{Y} + \frac{\mathscr{L}^4}{4!}[\hat{Z}\hat{Y}]^2 + \cdots \right\} \\ \left\{ -\frac{\mathscr{L}}{1!}\hat{Y} - \frac{\mathscr{L}^3}{3!}[\hat{Y}\hat{Z}]\hat{Y} - \frac{\mathscr{L}^5}{5!}[\hat{Y}\hat{Z}]^2\hat{Y} + \cdots \right\} \end{bmatrix} \tag{4.151}$$

$$\times \begin{bmatrix} \left\{ -\frac{\mathscr{L}}{1!}\hat{Z} - \frac{\mathscr{L}^3}{3!}[\hat{Z}\hat{Y}]\hat{Z} - \frac{\mathscr{L}^5}{5!}[\hat{Z}\hat{Y}]^2\hat{Z} + \cdots \right\} \\ \left\{ 1_n + \frac{\mathscr{L}^2}{2!}\hat{Y}\hat{Z} + \frac{\mathscr{L}^4}{4!}[\hat{Y}\hat{Z}]^2 + \cdots \right\} \end{bmatrix}$$

Comparing these submatrices to the submatrices in the chain parameter matrices for the lumped-iterative structures given in (4.149) we observe that the lumped π and lumped T structures give a better representation for $N = 1$, i.e., representing the total line with only one lumped section, than the lumped ˥ and lumped Γ structures. Representing the line with smaller sections and representing each section by one of the lumped iterative models gives the overall chain parameter matrix as the power of the lumped iterative chain parameter matrix as shown in (4.150). We are assuming that this would give a better approximation to (4.151). As a rudimentary investigation of this premise, let us represent the line with two lumped ˥ sections shown in Fig. 4.12(a) each of length $\mathscr{L}/2$. The overall chain parameter matrix is the chain parameter matrix in (4.149a) with N replaced by 2 and raised to the second power:

$$\hat{\Phi}_{\urcorner}^{2} = \begin{bmatrix} \left\{ \mathbf{1}_n + \frac{\mathscr{L}^2}{4} \hat{\mathbf{Z}}\hat{\mathbf{Y}} \right\} & \left\{ -\mathscr{L}\hat{\mathbf{Z}} - \frac{\mathscr{L}^3}{8} [\hat{\mathbf{Z}}\hat{\mathbf{Y}}]\hat{\mathbf{Z}} \right\} \\ \left\{ -\mathscr{L}\hat{\mathbf{Y}} - \frac{\mathscr{L}^3}{8} [\hat{\mathbf{Y}}\hat{\mathbf{Z}}]\hat{\mathbf{Y}} \right\} & \left\{ \mathbf{1}_n + 3\frac{\mathscr{L}^2}{4} \hat{\mathbf{Y}}\hat{\mathbf{Z}} + \frac{\mathscr{L}^4}{16} [\hat{\mathbf{Y}}\hat{\mathbf{Z}}]^2 \right\} \end{bmatrix} \tag{4.152}$$

Comparing this to the exact chain parameter matrix in (4.151) we do observe some convergence although the quantitative aspects of convergence are not clear.

4.6 ALTERNATIVE 2n-PORT CHARACTERIZATIONS

The chain parameter matrix is not the only way of relating the voltages and currents of the MTL viewed as a $2n$ port. Other obvious ways are the *impedance parameters* [A.2]:

$$\begin{bmatrix} \hat{\mathbf{V}}(0) \\ \hat{\mathbf{V}}(\mathscr{L}) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Z}}_{11} & \hat{\mathbf{Z}}_{12} \\ \hat{\mathbf{Z}}_{21} & \hat{\mathbf{Z}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{I}}(0) \\ -\hat{\mathbf{I}}(\mathscr{L}) \end{bmatrix} \tag{4.153}$$

and the *admittance parameters* [A.2]:

$$\begin{bmatrix} \hat{\mathbf{I}}(0) \\ -\hat{\mathbf{I}}(\mathscr{L}) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Y}}_{11} & \hat{\mathbf{Y}}_{12} \\ \hat{\mathbf{Y}}_{21} & \hat{\mathbf{Y}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}(0) \\ \hat{\mathbf{V}}(\mathscr{L}) \end{bmatrix} \tag{4.154}$$

The currents at both ends, $\hat{\mathbf{I}}(0)$ and $-\hat{\mathbf{I}}(\mathscr{L})$, are defined as being directed into the $2n$ port in accordance with the usual convention. These alternative representations can be obtained from the chain parameter submatrices. For example, the impedance parameter submatrices are defined, from (4.153), by setting currents equal to zero as

$$\hat{\mathbf{V}}(\mathscr{L}) = -\hat{\mathbf{Z}}_{22}\hat{\mathbf{I}}(\mathscr{L})\big|_{\hat{\mathbf{I}}(0)=0} \tag{4.155a}$$

$$\hat{\mathbf{V}}(0) = -\hat{\mathbf{Z}}_{12}\hat{\mathbf{I}}(\mathscr{L})|_{\hat{\mathbf{I}}(0)=\mathbf{0}} \tag{4.155b}$$

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{Z}}_{21}\hat{\mathbf{I}}(0)|_{\hat{\mathbf{I}}(\mathscr{L})=\mathbf{0}} \tag{4.155c}$$

$$\hat{\mathbf{V}}(0) = \hat{\mathbf{Z}}_{11}\hat{\mathbf{I}}(0)|_{\hat{\mathbf{I}}(\mathscr{L})=\mathbf{0}} \tag{4.155d}$$

From the chain parameter matrix, setting $\hat{\mathbf{I}}(0) = \mathbf{0}$, we obtain

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{V}}(0) \tag{4.156a}$$

$$\hat{\mathbf{I}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{21}\hat{\mathbf{V}}(0) \tag{4.156b}$$

from which we obtain

$$\hat{\mathbf{Z}}_{22} = -\hat{\mathbf{\Phi}}_{11}\hat{\mathbf{\Phi}}_{21}^{-1} = -\hat{\mathbf{\Phi}}_{21}^{-1}\hat{\mathbf{\Phi}}_{22} \tag{4.157a}$$

$$\hat{\mathbf{Z}}_{12} = -\hat{\mathbf{\Phi}}_{21}^{-1} \tag{4.157b}$$

Similarly setting $\hat{\mathbf{I}}(\mathscr{L}) = \mathbf{0}$ in the chain parameter matrix gives

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{V}}(0) + \hat{\mathbf{\Phi}}_{12}\hat{\mathbf{I}}(0) \tag{4.158a}$$

$$\mathbf{0} = \hat{\mathbf{\Phi}}_{21}\hat{\mathbf{V}}(0) + \hat{\mathbf{\Phi}}_{22}\hat{\mathbf{I}}(0) \tag{4.158b}$$

from which we obtain

$$\hat{\mathbf{Z}}_{21} = \hat{\mathbf{\Phi}}_{12} - \hat{\mathbf{\Phi}}_{11}\hat{\mathbf{\Phi}}_{21}^{-1}\hat{\mathbf{\Phi}}_{22} = -\hat{\mathbf{\Phi}}_{21}^{-1} \tag{4.159a}$$

$$\hat{\mathbf{Z}}_{11} = -\hat{\mathbf{\Phi}}_{21}^{-1}\hat{\mathbf{\Phi}}_{22} = -\hat{\mathbf{\Phi}}_{11}\hat{\mathbf{\Phi}}_{21}^{-1} \tag{4.159b}$$

We have used the matrix chain parameter identities given in (4.83) to give equivalent forms of these impedance parameters which demonstrate that the line is *reciprocal*, i.e., $\hat{\mathbf{Z}}_{11} = \hat{\mathbf{Z}}_{22}$ and $\hat{\mathbf{Z}}_{12} = \hat{\mathbf{Z}}_{21}$. The admittance parameters can also be derived from the chain parameters in like fashion or by realizing that the admittance parameter matrix is the inverse of the impedance parameter matrix. This yields

$$\hat{\mathbf{Y}}_{11} = \hat{\mathbf{Y}}_{22} = -\hat{\mathbf{\Phi}}_{12}^{-1}\hat{\mathbf{\Phi}}_{11} = -\hat{\mathbf{\Phi}}_{22}\hat{\mathbf{\Phi}}_{12}^{-1} \tag{4.160a}$$

$$\hat{\mathbf{Y}}_{12} = \hat{\mathbf{Y}}_{21} = \hat{\mathbf{\Phi}}_{12}^{-1} \tag{4.160b}$$

Although these appear to be viable alternatives to the chain parameter matrix representation, they have some unique problems. The primary problem is that *the impedance and admittance parameter matrices do not exist for certain frequencies where the line length is some multiple of a half-wavelength*, i.e., $\mathscr{L} = k\lambda/2$. For example, for the case of a lossless line, $\mathbf{R} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$, in a homogeneous medium, $\mathbf{LC} = \mu\varepsilon\mathbf{1}_n$, the impedance parameters are obtained from the chain parameter matrices given in (4.120) as

$$\hat{\mathbf{Z}}_{11} = \hat{\mathbf{Z}}_{22} = -\hat{\mathbf{\Phi}}_{11}\hat{\mathbf{\Phi}}_{21}^{-1} = -j\,\frac{\cos(\beta\mathscr{L})}{\sin(\beta\mathscr{L})}\,\hat{\mathbf{Z}}_C \tag{4.161a}$$

$$\hat{\mathbf{Z}}_{12} = \hat{\mathbf{Z}}_{21} = -\hat{\mathbf{\Phi}}_{21}^{-1} = -j\,\frac{1}{\sin(\beta\mathscr{L})}\,\hat{\mathbf{Z}}_C \tag{4.161b}$$

Recall that $\beta = 2\pi/\lambda$ so that the denominators of these expressions, $\sin(\beta\mathscr{L}) = \sin(2\pi\mathscr{L}/\lambda)$, are zero for frequencies where the line length is some multiple of a half-wavelength. Similar remarks apply to the admittance parameter matrix in (4.154). The chain parameter matrix exists and is nonsingular for all frequencies and all line configurations as was shown by analogy to the state-variable formulation in Section 4.3.

4.7 POWER AND THE REFLECTION COEFFICIENT MATRIX

As a final analogy to the two-conductor line let us define *the reflection coefficient matrix*, $\hat{\mathbf{\Gamma}}(z)$, and investigate the flow of power on the line. The phasor voltages and currents are written in the form of forward-traveling waves, $\hat{\mathbf{V}}^+(z)$ and $\hat{\mathbf{I}}^+(z)$, and backward-traveling waves, $\hat{\mathbf{V}}^-(z)$ and $\hat{\mathbf{I}}^-(z)$, from (4.59) and (4.63) as

$$\hat{\mathbf{V}}(z) = \hat{\mathbf{V}}^+(z) + \hat{\mathbf{V}}^-(z) \tag{4.162a}$$

$$\hat{\mathbf{I}}(z) = \hat{\mathbf{I}}^+(z) - \hat{\mathbf{I}}^-(z) \tag{4.162b}$$

where

$$\hat{\mathbf{V}}^+(z) = \hat{\mathbf{Z}}_C\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ \tag{4.163a}$$

$$\hat{\mathbf{V}}^-(z) = \hat{\mathbf{Z}}_C\hat{\mathbf{T}}\mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^- \tag{4.163b}$$

and

$$\hat{\mathbf{I}}^+(z) = \hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ \tag{4.164a}$$

$$\hat{\mathbf{I}}^-(z) = \hat{\mathbf{T}}\mathbf{e}^{\hat{\gamma}z}\hat{\mathbf{I}}_m^- \tag{4.164b}$$

Let us define the reflection coefficient matrix in a logical manner relating the reflected or backward-traveling voltage waves to the incident or forward-traveling voltage waves at any point on the line as

$$\hat{\mathbf{V}}^-(z) = \hat{\mathbf{\Gamma}}(z)\hat{\mathbf{V}}^+(z) \tag{4.165}$$

Substituting (4.163) gives

$$\begin{aligned}\hat{\mathbf{I}}_m^- &= \mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{T}}^{-1}\hat{\mathbf{Z}}_C^{-1}\hat{\mathbf{\Gamma}}(z)\hat{\mathbf{Z}}_C\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ \\ &= \mathbf{e}^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{T}}^{-1}\hat{\mathbf{Z}}_C^{-1}\hat{\mathbf{\Gamma}}_L\hat{\mathbf{Z}}_C\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}\mathscr{L}}\hat{\mathbf{I}}_m^+\end{aligned} \tag{4.166}$$

where the load reflection coefficient matrix is defined as $\hat{\mathbf{\Gamma}}_L = \hat{\mathbf{\Gamma}}(\mathscr{L})$. From this relation, the reflection coefficient matrix at any point on the line can be related to the load reflection coefficient matrix which we will show can be explicitly calculated knowing the termination impedance matrix and the characteristic impedance matrix. Thus, the voltage and current vectors can be written as

$$\hat{\mathbf{V}}(z) = (\mathbf{1}_n + \hat{\mathbf{\Gamma}}(z))\hat{\mathbf{Z}}_C\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ \tag{4.167a}$$

$$\hat{\mathbf{I}}(z) = (\mathbf{1}_n - \hat{\mathbf{Z}}_C^{-1}\hat{\mathbf{\Gamma}}(z)\hat{\mathbf{Z}}_C)\hat{\mathbf{T}}\mathbf{e}^{-\hat{\gamma}z}\hat{\mathbf{I}}_m^+ \tag{4.167b}$$

The *input impedance matrix* at any point on the line relates the voltages and currents at that point as

$$\hat{\mathbf{V}}(z) = \mathbf{Z}_{\text{in}}(z)\hat{\mathbf{I}}(z) \tag{4.168}$$

Substituting (4.167) yields

$$\hat{\mathbf{Z}}_{\text{in}}(z) = (\mathbf{1}_n + \mathbf{\Gamma}(z))(\mathbf{1}_n - \hat{\mathbf{\Gamma}}(z))^{-1}\hat{\mathbf{Z}}_C \tag{4.169}$$

Similarly, the reflection coefficient matrix can be written in terms of the input impedance matrix at a point on the line from (4.169) as

$$\begin{aligned}\hat{\mathbf{\Gamma}}(z) &= \hat{\mathbf{Z}}_C(\hat{\mathbf{Z}}_{\text{in}}(z) + \hat{\mathbf{Z}}_C)^{-1}(\hat{\mathbf{Z}}_{\text{in}}(z) - \hat{\mathbf{Z}}_C)\hat{\mathbf{Z}}_C^{-1} \\ &= (\hat{\mathbf{Z}}_{\text{in}}(z) - \hat{\mathbf{Z}}_C)(\hat{\mathbf{Z}}_{\text{in}}(z) + \hat{\mathbf{Z}}_C)^{-1}\end{aligned} \tag{4.170}$$

If the line is termined at $z = \mathscr{L}$ as

$$\hat{\mathbf{V}}(\mathscr{L}) = \hat{\mathbf{Z}}_L\hat{\mathbf{I}}(\mathscr{L}) \tag{4.171}$$

then the reflection coefficient matrix at the load is

$$\begin{aligned}\hat{\mathbf{\Gamma}}_L &= \hat{\mathbf{Z}}_C(\hat{\mathbf{Z}}_L + \hat{\mathbf{Z}}_C)^{-1}(\hat{\mathbf{Z}}_L - \hat{\mathbf{Z}}_C)\hat{\mathbf{Z}}_C^{-1} \\ &= (\hat{\mathbf{Z}}_L - \hat{\mathbf{Z}}_C)(\hat{\mathbf{Z}}_L + \hat{\mathbf{Z}}_C)^{-1}\end{aligned} \tag{4.172}$$

These formulae reduce to the corresponding scalar results for a two-conductor line. From this last result we observe that *in order to eliminate all reflections at the load, the line must be terminated in its characteristic impedance matrix, i.e.,* $\hat{\mathbf{Z}}_L = \hat{\mathbf{Z}}_C$. This is the meaning of a *matched line* in the MTL case. It is not sufficient to simply place impedances only between each line and the reference conductor. Impedances will need to be placed between all pairs of the n lines since the characteristic impedance matrix is, in general, full.

The total average power transmitted on the MTL in the $+z$ direction is [A.2]

$$
\begin{aligned}
P_{av}(z) &= \tfrac{1}{2}\mathcal{R}e\{\hat{\mathbf{V}}^t(z)\hat{\mathbf{I}}^*(z)\} \qquad (4.173)\\
&= \tfrac{1}{2}\left\{[\hat{V}_1(z) \quad \hat{V}_2(z) \quad \cdots \quad \hat{V}_n(z)]\begin{bmatrix}\hat{I}_1^*(z)\\ \hat{I}_2^*(z)\\ \vdots \\ \hat{I}_n^*(z)\end{bmatrix}\right\}\\
&= \tfrac{1}{2}\mathcal{R}e\{\hat{V}_1(z)\hat{I}_1^*(z) + \hat{V}_2(z)\hat{I}_2^*(z) + \cdots + \hat{V}_n(z)\hat{I}_n^*(z)\}
\end{aligned}
$$

where * again denotes the conjugate of the complex-valued quantity. Substituting the voltages and currents in terms of forward- and backward-traveling waves as in (4.162) gives

$$P_{av} = \tfrac{1}{2}\mathcal{R}e\{\hat{\mathbf{V}}^{+t}\hat{\mathbf{I}}^{+*} + \hat{\mathbf{V}}^{-t}\hat{\mathbf{I}}^{+*} - \hat{\mathbf{V}}^{+t}\hat{\mathbf{I}}^{-*} + \hat{\mathbf{V}}^{-t}\hat{\mathbf{I}}^{-*}\} \qquad (4.174)$$

The first term, $\hat{\mathbf{V}}^{+t}\hat{\mathbf{I}}^{+*}$, gives the average power carried by the forward-traveling waves, and the last term, $\hat{\mathbf{V}}^{-t}\hat{\mathbf{I}}^{-*}$, gives the average power carried by the backward-traveling waves. The middle two terms, $\hat{\mathbf{V}}^{-t}\hat{\mathbf{I}}^{+*}$ and $\hat{\mathbf{V}}^{+t}\hat{\mathbf{I}}^{-*}$, are cross-coupling terms between the waves. Suppose the line is matched at its load, i.e., $\hat{\mathbf{Z}}_L = \hat{\mathbf{Z}}_C$, so that the load reflection coefficient matrix is zero, i.e., $\hat{\mathbf{\Gamma}}_L = \mathbf{0}$. Equation (4.166) shows, as expected, that the reflection coefficient matrix is zero at all points on the line, i.e., $\hat{\mathbf{\Gamma}}(z) = \mathbf{0}$. Thus there are only forward-traveling waves on the line, and there is no power flow in the $-z$ direction. These properties are, of course, also directly analogous to the scalar, two-conductor line. The use of matrix notation allows a straightforward adaptation of the scalar results to the MTL case although there are some peculiarities unique to the MTL case. For example, equation (4.172) reduces to the familiar two-conductor case where the termination and characteristic impedance matrices become scalars.

4.8 COMPUTED RESULTS

In this section we will show some computed and experimental results that demonstrate the prediction methods of this chapter. The frequency-domain prediction model is implemented in the computer program **MTL.FOR** described in Appendix A. This program determines the $2n$ undetermined constants in the general form of solution in (4.86): $\mathbf{I}_m^{\pm}$. The terminal configurations for both structures are shown in Fig. 4.13. These are characterized as a generalized Thévenin equivalent as in (4.84) where

$$\hat{\mathbf{V}}_S = \begin{bmatrix} 0 \\ 1\underline{/0^\circ} \end{bmatrix}$$

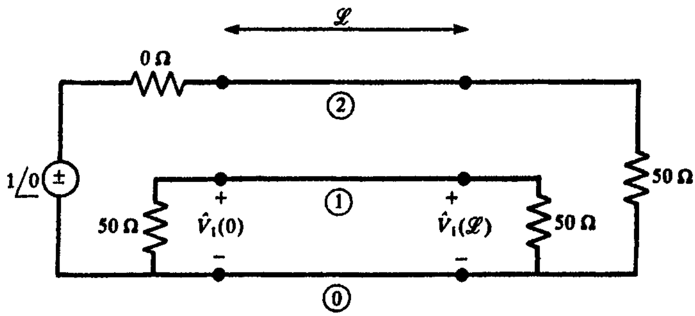


FIGURE 4.13 A three-conductor line for illustrating numerical results.

$$\hat{\mathbf{Z}}_S = \begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{V}}_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{Z}}_L = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}$$

Two configurations of a three-conductor line ($n = 2$) are considered: a three-wire ribbon cable and a three-conductor printed circuit board. Experimentally determined frequency responses will be compared to the predictions of the MTL model as well as those of the lumped-pi iterative approximation.

4.8.1 Ribbon Cables

The cross section of the three-wire ribbon cable is shown in Fig. 4.14. The total line length is $\mathscr{L} = 2$ m. The per-unit-length parameters for this configuration were computed using the computer program **RIBBON.FOR** described in Appendix A and are given in Chapter 3:

$$\mathbf{L} = \begin{bmatrix} 0.748\,50 & 0.507\,7 \\ 0.507\,7 & 1.015\,4 \end{bmatrix} \mu\text{H/m}$$

$$\mathbf{C} = \begin{bmatrix} 37.432 & -18.716 \\ -18.716 & 24.982 \end{bmatrix} \text{pF/m}$$

The experimental results are compared to the predictions of the MTL model using **MTL.FOR**, with and without losses, over the frequency range of 1 kHz to 100 MHz in Fig. 4.15. Observe that below 100 kHz, losses in the line conductors are important and cannot be ignored. The dc resistance was

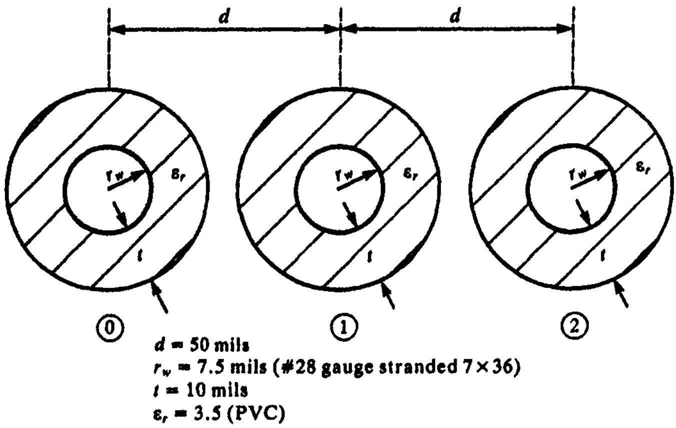


FIGURE 4.14 Dimensions of a three-conductor ribbon cable for illustrating numerical results.

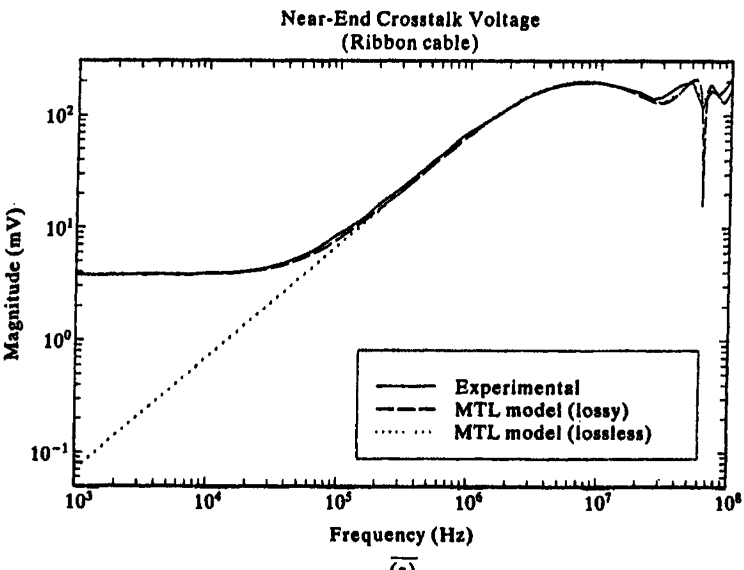
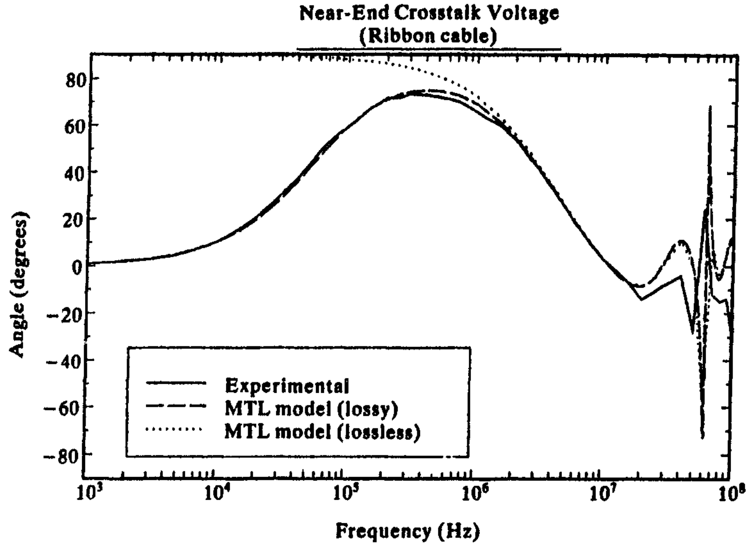


FIGURE 4.15 Comparison of the frequency response of the near-end crosstalk of the ribbon cable of Fig. 4.14 determined experimentally and via the MTL model with and without losses: (a) magnitude.

computed using (3.200) for one of the #36 gauge strands ($r_w = 2.5$ mils) and dividing this result by the number of strands (seven) to give 0.194 44 Ω/m. The skin effect was included by determining the frequency where the radius of one of the #36 gauge strands equal two skin depths, $r_w = 2\delta = 2/\sqrt{\pi f_0 \mu \sigma}$, according



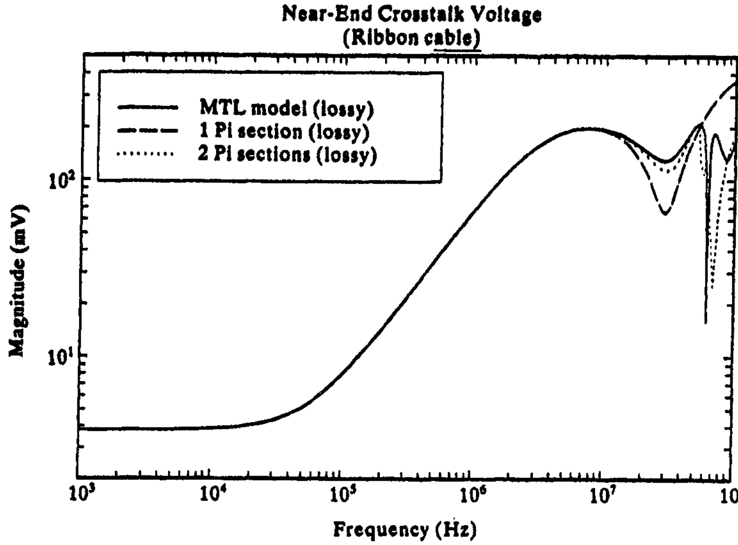
(b)

FIGURE 4.15 (*Continued*) (b) phase.

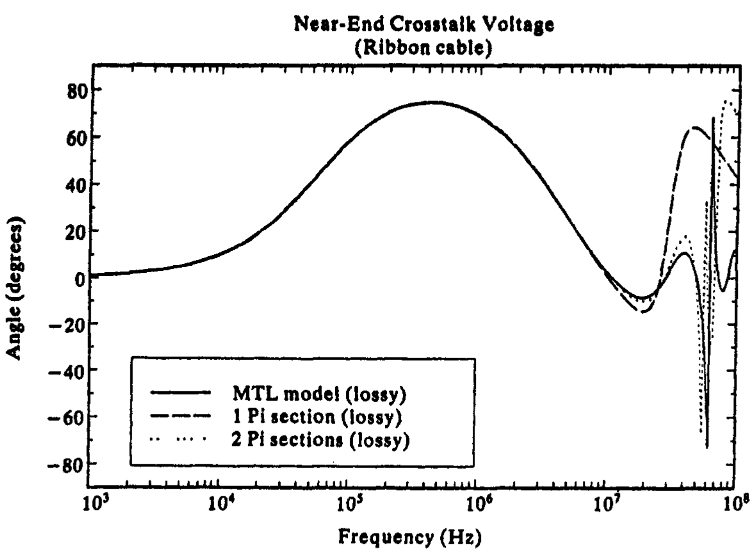
to Fig. 3.52 ($f_o = 4.332$ MHz) and taking the resistance to vary as $r(f) = r_{dc}\sqrt{f/f_o}$ above this. Both skin effect resistance and internal inductance are included according to (3.225). Both the magnitude and the phase are well predicted. Observe that the line is one wavelength (ignoring the dielectric insulation) at 150 MHz. So the line is electrically short below, say, 15 MHz. Observe that the magnitude of the crosstalk for the lossless case (and a significant portion of the lossy case) increases directly with frequency, i.e., 20 dB/decade. We will find this to be a general result in Chapter 6. Figure 4.16 shows the predictions of the lumped-pi (π) approximate model of Fig. 4.12(c) using one and two pi sections to represent the entire line. The wire resistances and internal inductances are assumed to be the dc values over the entire frequency range for these lumped-pi models since the skin effect ($\sqrt{f}$) dependency is difficult to model in the lumped-circuit program SPICE which was used to solve the resulting circuit. Both one and two pi sections give virtually identical predictions to the exact MTL model for frequencies below which the line is one-tenth of one wavelength long. Observe that two pi sections do not substantially improve the accuracy of the predicted frequency range even though the circuit complexity is double that for one pi section.

4.8.2 Printed Circuit Boards

The next configuration is a three-conductor printed circuit board whose cross section is shown in Fig. 4.17. The total line length is $\mathscr{L} = 10$ inches $= 0.254$ m.



(a)



(b)

FIGURE 4.16 Comparison of the frequency response of the near-end crosstalk of the ribbon cable of Fig. 4.14 determined via the MTL model with losses and via the lumped-pi model using one and two sections to represent the line: (a) magnitude, (b) phase.

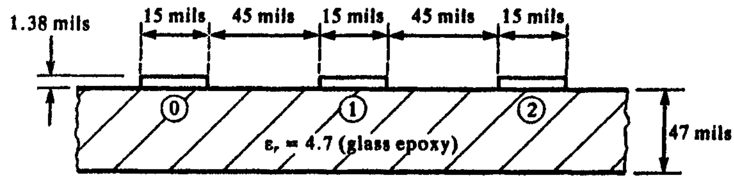
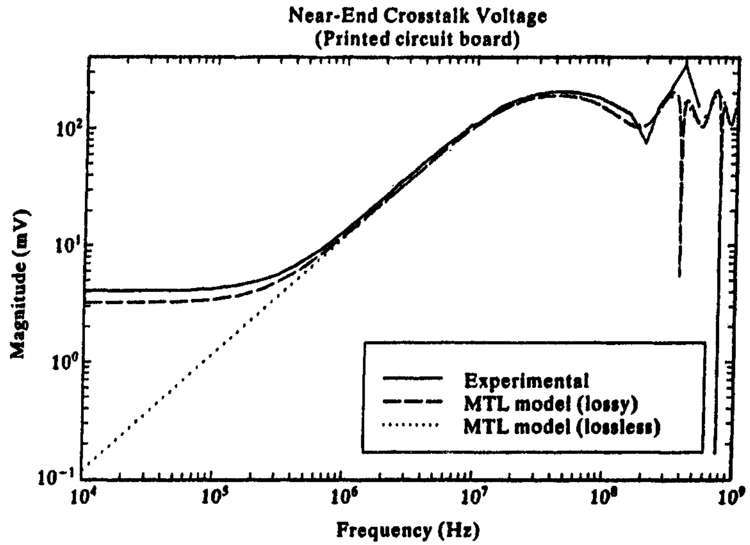


FIGURE 4.17 Dimensions of a three-conductor PCB for illustrating numerical results.



(a)

FIGURE 4.18 Comparison of the frequency response of the near-end crosstalk of the PCB of Fig. 4.17 determined experimentally and via the MTL model with and without losses: (a) magnitude.

The per-unit-length parameters were computed using the computer program **PCBGAL.FOR** described in Appendix A and are given in Chapter 3:

$$\mathbf{L} = \begin{bmatrix} 1.104\,18 & 0.690\,094 \\ 0.690\,094 & 1.380\,19 \end{bmatrix} \mu\text{H/m}$$

$$\mathbf{L} = \begin{bmatrix} 40.628\,0 & -20.314\,0 \\ -20.314\,0 & 29.763\,2 \end{bmatrix} \text{pF/m}$$

The experimentally obtained results are compared to the predictions of the MTL model using **MTL.FOR** with and without losses over the frequency range of 10 kHz to 1 GHz in Fig. 4.18. The conductor resistances and internal

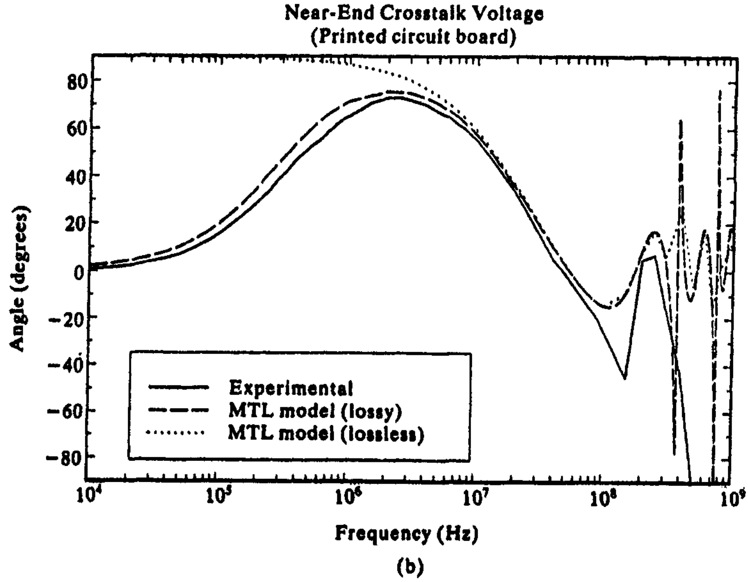


FIGURE 4.18 (*Continued*) (b) phase.

inductances were computed in a similar fashion to the ribbon cable. The dc resistance is $r_{dc} = 1/wt\sigma = 1.291\ \Omega/\text{m}$ where w is the land width ($w = 15$ mils) and t is the land thickness for 1 ounce copper ($t = 1.38$ mils). The frequency where this transitions to a $\sqrt{f}$ behavior was approximated as being where the land thickness equals two skin depths: $t = 2\delta$ or $f_o = 14.218$ MHz. Observe that the frequency where the conductor losses become important is of the order of 100 kHz. Both the magnitude and the phase are well predicted. The line is one wavelength (ignoring the board dielectric) at 1.18 GHz. Thus the line can be considered to be electrically short for frequencies below some 100 MHz. Again note that the magnitude of the frequency response increases directly with frequency, 20 dB/decade, where the line is electrically short for the lossless case (and a significant portion of the lossy case). Figure 4.19 shows the predictions of the lumped-pi approximate model of Fig. 4.12(c) using one and two pi sections to represent the entire line. The conductor resistances and internal inductances are again assumed to be the dc values over the entire frequency range for these lumped-pi models since the skin effect ($\sqrt{f}$) dependency is difficult to model in the lumped-circuit program SPICE which was used to solve the resulting circuit. Both one and two pi sections give virtually identical predictions to the exact MTL model for frequencies below which the line is one-tenth of one wavelength long. Observe that, as in the case of the ribbon cable, two pi sections do not substantially improve the accuracy of the predicted frequency range even though the circuit complexity is double that for one pi section.

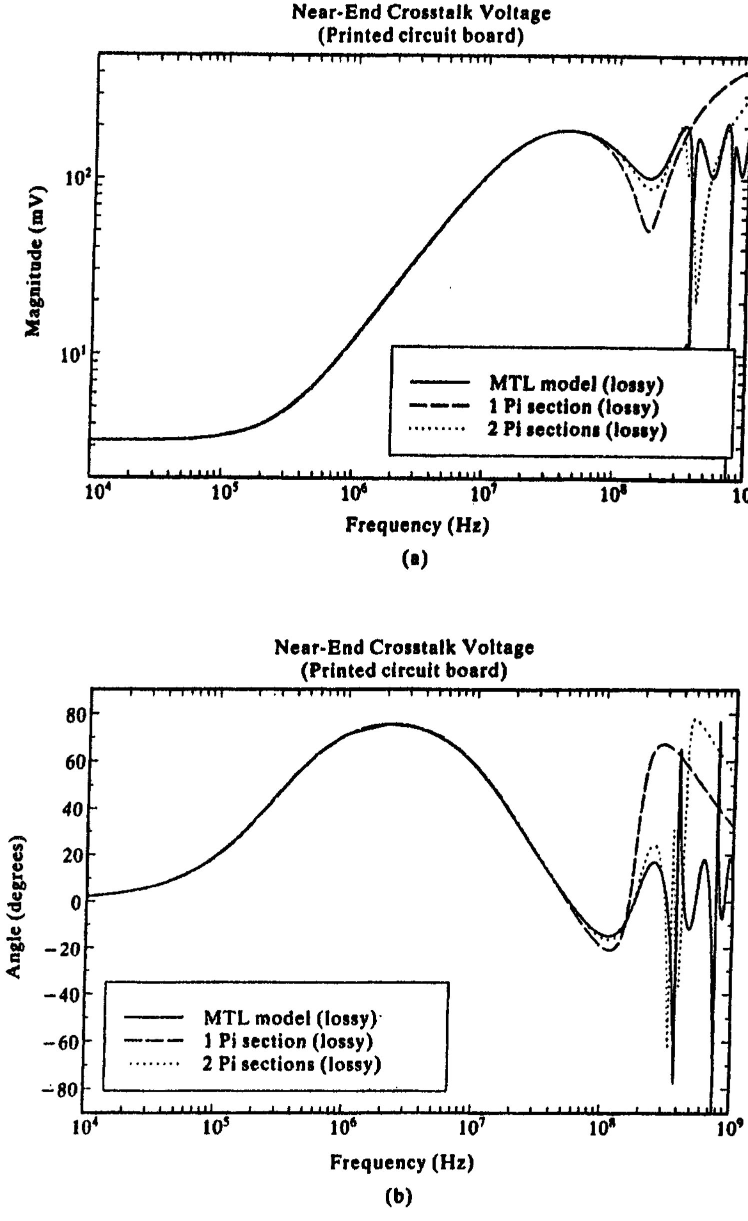


FIGURE 4.19 Comparison of the frequency response of the near-end crosstalk of the PCB of Fig. 4.17 determined via the MTL model with losses and via the lumped-pi model using one and two sections to represent the line: (a) magnitude, (b) phase.

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PROBLEMS

4.1 Determine the velocity of propagation and characteristic impedance for the following transmission lines:

1. $l = 0.25\ \mu\text{H/m}$, $c = 100$ pF/m.
2. Coaxial cable; $c = 50$ pF/m, $\varepsilon_r = 2.1$.
3. Two bare #28 gauge solid wires ($r_w = 6.3$ mils) separated by 100 mils.
4. One bare #16 gauge solid wire ($r_w = 25.4$ mils) 1/4 inch above a ground plane.

4.2 Consider a lossless transmission line operated in the sinusoidal steady state. For the following problem specifications determine:

1. The line length as a fraction of a wavelength.
2. The input impedance to the line.
3. The time-domain voltage at the line input and at the load.
4. The average power delivered to the load.

The specifications are:

(a) $\mathscr{L} = 1$ m, $f = 262.5$ MHz, $\hat{Z}_C = 50\,\Omega$, $\hat{Z}_L = (30 - j200)\,\Omega$, $\hat{Z}_S = (100 + j50)\,\Omega$, $v = 300$ m/μs, and $\hat{V}_S = 10\underline{/30^\circ}$ V.
(b) $\mathscr{L} = 36$ m, $f = 28$ MHz, $\hat{Z}_C = 150\,\Omega$, $\hat{Z}_L = -j30\,\Omega$, $\hat{Z}_S = 500\,\Omega$, $v = 300$ m/μs, and $\hat{V}_S = 100\underline{/0^\circ}$ V.
(c) $\mathscr{L} = 2$ m, $f = 175$ MHz, $\hat{Z}_C = 100\,\Omega$, $\hat{Z}_L = (200 - j30)\,\Omega$, $\hat{Z}_S = 50\,\Omega$, $v = 200$ m/μs, and $\hat{V}_S = 10\underline{/0^\circ}$ V.

4.3 Verify that (4.11) are solutions to (4.9).

4.4 Verify (4.14).

4.5 Verify (4.26).

4.6 Verify (4.28). Suppose the line is lossless and its length is one-quarter of one wavelength, $\mathscr{L} = \lambda/4$. If it is terminated in a short circuit, $\hat{Z}_L = 0$, determine the input impedance to the line. Repeat this for an open-circuit termination, $\hat{Z}_L = \infty$.

4.7 Verify (4.30).

4.8 Solve the first-order, ordinary differential equation given by (4.32) with $a = -2$, $b = 3$, and $w(t) = u(t)$ where $u(t)$ is the unit step function, $u(t) = 0$, $t < 0$, $u(t) = 1$, $t > 0$, and an initial state of $x(0) = 0$.

4.9 Solve the second-order, ordinary differential equation given by (4.31) with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\mathbf{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $w(t) = u(t)$.

4.10 Determine the state-transition matrices, $\Phi(t)$, for the following:

$$\mathbf{A} = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

4.11 Verify that (4.44c) satisfies the homogeneous n-th order state variable equations in (4.31).

4.12 Verify the chain parameter matrix entries given in (4.48).

4.13 Verify (4.57) and (4.63).

4.14 Verify the chain parameter submatrices in (4.70).

4.15 Verify (4.74), (4.75), and (4.76).

4.16 Determine a 2×2 matrix, $\mathbf{T}$, which will diagonalize the matrices in Problem 4.10.

4.17 Consider a ribbon cable consisting of three #28 gauge wires ($r_w = 7.5$ mils) lying in a plane with adjacent separations of 50 mils. Assume the wires are lossless and immersed in free space. Determine the characteristic impedance matrix if one of the outer wires is chosen as the reference conductor. Calculate the chain parameter matrix for this line.

4.18 Determine the generalized Thévenin equivalent representation, $\mathbf{V}(0) = \mathbf{V}_S - \mathbf{Z}_S\mathbf{I}(0)$, of the source termination network shown in Fig. P4.18.

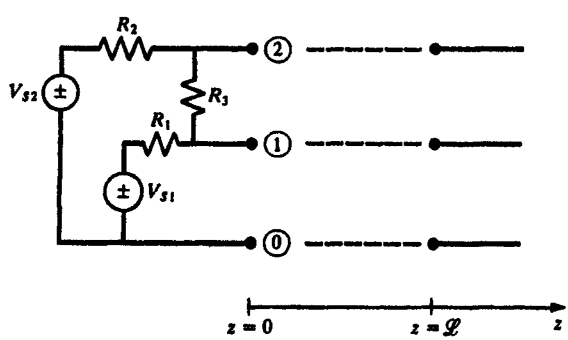

FIGURE P4.18

4.19 Repeat Problem 4.18 for the source termination network of Fig. P4.19.

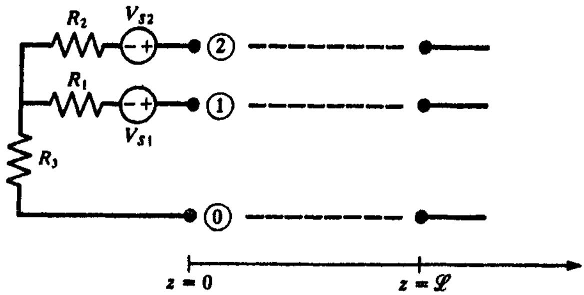

FIGURE P4.19

4.20 Characterize the source and load termination networks shown in Fig. P4.20.

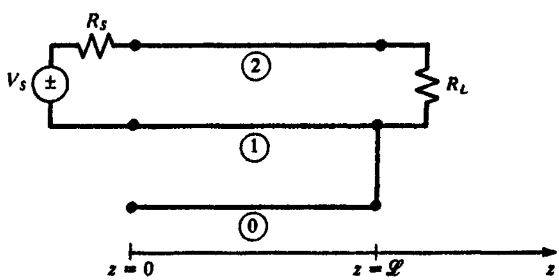

FIGURE P4.20

4.21 Derive the relations given in (4.90), (4.95), (4.97), and (4.99).

4.22 Verify the relations in (4.116) and (4.120).

4.23 Show that a 2×2 real, symmetric matrix, $\mathbf{M}$, can be diagonalized with the orthogonal transformation:

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where

$$\tan 2\theta = \frac{2M_{12}}{M_{11} - M_{22}}$$

Show that the eigenvalues are

$$\Lambda_1^2 = M_{11}\cos^2\theta + 2M_{12}\cos\theta\sin\theta + M_{22}\sin^2\theta$$
$$\Lambda_2^2 = M_{11}\sin^2\theta - 2M_{12}\cos\theta\sin\theta + M_{22}\cos^2\theta$$

4.24 Diagonalize the following product, CL, where

$$\mathbf{C} = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$
$$\mathbf{L} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

4.25 Diagonalize the following matrix which has a cyclic symmetric structure:

$$\mathbf{M} = \begin{bmatrix} 4 & 3 & 2 & 3 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 3 & 2 & 3 & 4 \end{bmatrix}$$

Show by direct calculation that the $\hat{\mathbf{T}}$ you obtain does in fact diagonalize this matrix and that $\hat{\mathbf{T}}^{-1} = \hat{\mathbf{T}}^{t*}$. Verify from this that the eigenvalues are as expected according to (4.141).

4.26 Verify the relations in (4.146) for the 3×3 cyclic symmetric matrices given in (4.145).

4.27 Verify the chain parameter matrices for the lumped-circuit iterative structures given in (4.149).

4.28 For the ribbon cable of Problem 4.17, construct lumped-π and lumped-T

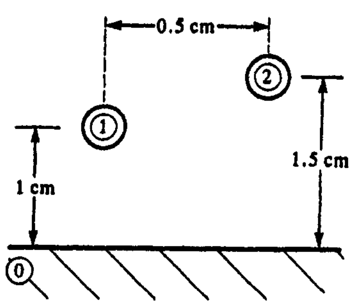

FIGURE P4.30.

iterative circuits for a line length of 5 m where the line is represented as one section, i.e., $N = 1$.

4.29 Derive the relations for the admittance parameter submatrices in (4.160). Evaluate these for a lossless line in a homogeneous medium to obtain the duals to (4.161).

4.30 Consider the MTL shown in Fig. P4.30 consisting of two #20 gauge, solid wires ($r_w = 16$ mils) above an infinite, perfectly conducting ground plane. Determine the per-unit-length inductance and capacitance matrices and the characteristic impedance matrix assuming the line to be lossless. Determine the structure and values for a termination network that will match the line. For a total line length of 5 m and the termination structure of Fig. 4.13 plot the crosstalk magnitude and phase from 100 kHz to 100 MHz. Compare these to the predictions of the lumped-π approximation. Show your results for the lossless and lossy cases.