Fast Approximation of Sine and Cosine Hyperbolic Functions for the Calculation of the Transmission Matrix of a Multiconductor Transmission Line

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***Abstract*—A fast and stable algorithm for approximation of sine and cosine hyperbolic functions is presented in this paper. The algorithm can be used for *S*-parameter calculation from RLGC parameters. The idea is to construct the recurrent relation for the approximate solution of sine and cosine hyperbolic complex value matrix functions. The stability of the proposed algorithm is shown and convergence theorem is proved. In the last section, different numerical simulations are made and compared with the already existing algorithm in terms of calculation time given.**

***Index Terms*—ABCD parameters, MATLAB, RLGC parameters, scattering parameters, sine and cosine hyperbolic functions, transmission lines.**

I. Introduction

VARIOUS modern computer-aided design tools can simulate 2-D cross section geometries to obtain RLGC parameters. Since real engineering applications deal with transmission lines (TL) based on these cross sections, it is important to have an algorithm that exports network parameters based on RLGC parameters and certain lengths of the TL. In cases when large amounts of different 2-D cross section geometries and TL lengths are considered—for example, for statistical design optimization—conversion operation is used for every set of parameters and it is crucial to have a very fast conversion algorithm. This paper proposes an algorithm based on approximation of sine and cosine hyperbolic functions that is 15–35 times faster than the one currently implemented in MATLAB.

As it is known, the solving of N-dimensional TL networks or communication circuits is related to computing the hyperbolic trigonometric functions of N-dimensional complex matrices. ABCD parameter matrices can be expressed by the following formulas (see [1], [2], [4]):

$$\begin{aligned}\mathbf{A} &= \cosh\left(l\mathbf{\Gamma}\right)\\ \mathbf{B} &= \sinh\left(l\mathbf{\Gamma}\right)\mathbf{\Gamma}^{-1}\mathbf{Z}\\ \mathbf{C} &= \mathbf{Z}^{-1}\mathbf{\Gamma}\sinh\left(l\mathbf{\Gamma}\right)\\ \mathbf{D} &= \mathbf{Z}^{-1}\cosh\left(l\mathbf{\Gamma}\right)\mathbf{Z}\end{aligned} \tag{1}$$

where $\mathbf{\Gamma} = (\mathbf{ZY})^{1/2}$, l is a length of TL and $\mathbf{Z}$ (series impedance) and $\mathbf{Y}$ (shunt admittance) matrices are defined using $\mathbf{R}$ (resistance), $\mathbf{L}$ (inductance), $\mathbf{G}$ (conductivity), and $\mathbf{C}$ (capacitance) matrices

$$\begin{aligned}\mathbf{Z} &= \mathbf{R} + j\omega\mathbf{L}\\ \mathbf{Y} &= \mathbf{G} + j\omega\mathbf{C}.\end{aligned} \tag{2}$$

Here ω is angular frequency and $j = \sqrt{-1}$.

Here and below bold letters are used for matrices and prime letters for scalar values.

As it is well-known, scattering parameters with Z_0 characteristic impedance (*S*-parameters) of linear electrical networks can be easily calculated from ABCD generalized parameter matrices (see [3], [4, Section IX-II-3]).

The main computation complexity for ABCD parameters is to calculate sine and cosine hyperbolic functions with a complex element matrix argument.

Using the Taylor expansion directly in (1) for the ABCD calculation is ineffective, especially if $\|l(\mathbf{ZY})^{1/2}\|$ is a large number. In this case, to achieve sufficient accuracy it is necessary to preserve $m$number of terms in the Taylor expansion, such that

$$\frac{\left[l\left(\mathbf{ZY}\right)^{1/2}\right]^m}{m!} < \varepsilon \tag{3}$$

where ε is a small parameter. Very large values of $m$are related to rounding errors and manipulation of large numbers in complex value matrices. From (2) we see that $\|l(\mathbf{ZY})^{1/2}\|$ becomes larger for high frequencies. From this it follows that this algorithm practically does not work for long electrical networks and high frequency values.

Also, in [2], there is a proposed computer-based algorithm for computing the N-dimensional generalized ABCD parameter matrices. The proposed method in this paper uses Cayley–Hamilton's theorem (see [5]) and requires calculation of eigenvalues of complex element matrices and also taking the square root from the complex value matrix, which makes the algorithm too complicated.

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Another algorithm proposed in [6, Section IV-C] is related with matrix diagonalization. One should find such similarity transform that makes $(ZY)^{-1/2}$ matrix diagonal. This algorithm also requires calculation of eigenvalues of complex element matrices and puts additional restrictions on matrix $(ZY)^{-1/2}$. Below are given some known cases of matrix M,whose diagonalization is assured (see [6], [7]):

1) All eigenvalues of M are distinct;
2) $M$is complex but normal matrix, i.e., $MM^{t*} = M^{t*}M$, where we denote by t transpose and its conjugate by $*$;
3) $M$is complex and Hermitian, i.e., $M = M^*$.

In the present study, a very simple and fast algorithm, which uses just matrix multiplication and does not put any additional requirements for matrix $(ZY)^{-1/2}$, is proposed for computing the N-dimensional generalized ABCD parameter matrices. The algorithm is based on the approximate calculation of the sine and cosine hyperbolic functions and uses only multiplication and summation of square matrices. For the proposed method, stability is shown, as well as convergence theorem proved and errors estimated for approximate solution.

II. Statement of the Problem and Construction of the Algorithm

As shown in (1), the main complexity of the ABCD matrix computation is to calculate sine and cosine hyperbolic functions with the complex element matrix argument. Sine and cosine hyperbolic functions are defined by the following formulas:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \tag{4}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots. \tag{5}$$

In the present section, a recurrent two-layer scheme will be constructed for approximate calculation of (4) and (5) sine and cosine hyperbolic matrix functions. The main idea of the algorithm is to calculate $\cosh(x_0\mathbf{\Gamma})$ and $\mathbf{\Gamma}^{-1}\sinh(x_0\mathbf{\Gamma})$for small values of $x_0 = \frac{l}{2^n}$ and then reconstruct $\cosh(l\mathbf{\Gamma})$ and $\mathbf{\Gamma}^{-1}\sinh(l\mathbf{\Gamma})$ using the following well-known double angle trigonometric formulas (see [8], Appendix B):

$$\cosh(l\mathbf{\Gamma}) = 2\cosh^2\left(\mathbf{\Gamma}\frac{l}{2}\right) - \mathbf{I} \tag{6}$$

$$(l\mathbf{\Gamma})^{-1}\sinh(l\mathbf{\Gamma}) = \left(\frac{l}{2}\mathbf{\Gamma}\right)^{-1}\sinh\left(\frac{l}{2}\mathbf{\Gamma}\right)\cosh\left(\frac{l}{2}\mathbf{\Gamma}\right). \tag{7}$$

The following notations are then introduced:

$$\mathbf{U}(x) = \cosh(x\mathbf{\Gamma}) = \mathbf{I} + \frac{x^2}{2!}\mathbf{\Gamma}^2 + \frac{x^4}{4!}\mathbf{\Gamma}^4 + \ldots \tag{8}$$

$$\mathbf{V}(x) = (x\mathbf{\Gamma})^{-1}\sinh(x\mathbf{\Gamma}) = \mathbf{I} + \frac{x^2}{3!}\mathbf{\Gamma}^2 + \frac{x^4}{5!}\mathbf{\Gamma}^4 + \ldots \tag{9}$$

where $x \in [0, l]$.

In (9), $\mathbf{V}(x)$ in its Taylor expansion, similar to $\mathbf{U}(x)$, contains only even powers of $x\mathbf{\Gamma} = x(\mathbf{ZY})^{-1/2}$, which allows calculation of the square root from complex value matrices to be avoided. Using (8) and (9), ABCD parameters from (1) can be calculated by the following formulas:

$$\begin{aligned} \mathbf{A} &= \mathbf{U}(l) \\ \mathbf{B} &= l\mathbf{V}(l)\mathbf{Z} \\ \mathbf{C} &= l\mathbf{Y}\mathbf{V}(l) \\ \mathbf{D} &= \mathbf{Z}^{-1}\mathbf{U}(l)\mathbf{Z}. \end{aligned} \tag{10}$$

According to (6) and (7), $\mathbf{U}(x)$ and $\mathbf{V}(x)$ satisfy the following recurrent relations:

$$\mathbf{U}(x_{k+1}) = 2\mathbf{U}^2(x_k) - \mathbf{I} \tag{11}$$

$$\mathbf{V}(x_{k+1}) = \mathbf{V}(x_k)\mathbf{U}(x_k) \tag{12}$$

where $x_k = \frac{2^k}{2^n}l, \quad k = 0, 1, \ldots, n.$

For the numerical calculation of (11) and (12), the initial values of $\mathbf{U}(x_0)$ and $\mathbf{V}(x_0)$ need to be obtained. The approximate values of these matrices can be calculated by the following formulas:

$$\mathbf{U}_0 = P_0(x_0\mathbf{\Gamma}) = \mathbf{I} + \frac{x_0^2}{2!}\mathbf{\Gamma}^2 + \frac{x_0^4}{4!}\mathbf{\Gamma}^4 + \ldots + \frac{x_0^{2p}}{(2p)!}\mathbf{\Gamma}^{2p} \tag{13}$$

$$\mathbf{V}_0 = \mathbf{I} + \frac{x_0^2}{3!}\mathbf{\Gamma}^2 + \frac{x_0^4}{5!}\mathbf{\Gamma}^4 + \ldots + \frac{x_0^{2p}}{(2p+1)!}\mathbf{\Gamma}^{2p}. \tag{14}$$

Then, (11) and (12) will be replaced by the following recurrent relation:

$$\mathbf{U}_k = 2\mathbf{U}_{k-1}^2 - \mathbf{I} \tag{15}$$

$$\mathbf{V}_k = \mathbf{V}_{k-1}\mathbf{U}_{k-1}, \quad k = 1, 2, ..., n. \tag{16}$$

Note that for $\cosh(l\mathbf{\Gamma}) \approx \mathbf{U}_n$ and $\mathbf{\Gamma}^{-1}\sinh(l\mathbf{\Gamma}) \approx \mathbf{V}_n$ calculations only matrix multiplication needs to be used.

In the next section, the stability of the scheme in (16) and (17) is shown, convergence theorem is proved, and approximation error is estimated.

III. Mathematical Foundation of the Algorithm

A. *Error Estimation for Cosine Hyperbolic Function* $\mathbf{U}(x)$

$\mathbf{U}_k$ and $\mathbf{V}_k$ matrices are declared as approximate values of $\mathbf{U}(x_k)$ and $\mathbf{V}(x_k)$ matrices. $\mathbf{U}(x_k) - \mathbf{U}_k$ and $\mathbf{V}(x_k) - \mathbf{V}_k$ errors need to be estimated. As recurrent relations (15) and (16) are nonlinear, it is important to show that $\mathbf{U}_k$ and $\mathbf{V}_k$are uniformly bounded. First, to show that the norms of$\mathbf{U}_k$ are uniformly bounded, for $\|\mathbf{U}_0\|$ we have

$$\|\mathbf{U}_0\| \le P_0\left(\frac{l}{2^n}a\right) \tag{17}$$

where $a = \|\mathbf{\Gamma}\|$ and

$$Q_0(x) = \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2P}}{(2p)!}. \tag{18}$$

Estimating the norm of matrix $\mathbf{U}_1$ according to formula (15) gives

$$\begin{aligned} \|\mathbf{U}_1\| &= \|2\mathbf{U}_0 - \mathbf{I}\| \\ &= \left\|2\left(\mathbf{I} + \mathbf{U}_0\left(\frac{l}{2^n}\mathbf{\Gamma}\right)\right)^2 - \mathbf{I}\right\| \le P_1\left(\frac{l}{2^n}a\right) \end{aligned} \tag{19}$$

where $P_1(x) = 1 + Q_1(x), \quad Q_1(x) = 4Q_0(x) + 2Q_0^2(x)$.

The following estimate can be obtained using the inductions rule:

$$\|\mathbf{U}_k\| \le \|P_k(\lambda)\|, \quad \lambda = \frac{l}{2^n}a \tag{20}$$

where scalar polynomial $P_k(\lambda)$ (k does not define the order of the polynomial) satisfies the following recurent relation:

$$\begin{aligned} &P_k(\lambda) = 2P_{k-1}^2(\lambda) - 1, \quad k = 1, \ldots, n \\ &P_0(\lambda) = 1 + c_0\lambda^2 \\ &c_0 = \tfrac{1}{2!} + \tfrac{\lambda^2}{4!} + \ldots + \tfrac{\lambda^{2p-2}}{(2p)!}. \end{aligned} \tag{21}$$

Let us estimate $P_k(\lambda)$, from (21) it follows that:

$$\begin{aligned} &P_0(\lambda) = 1 + \nu, \quad \nu = c_0\lambda^2 \\ &P_1(\lambda) = 2(1+\nu)^2 - 1 \le (1+2\nu)^2 \\ &P_2(\lambda) = 2P_1^2(\lambda) - 1 \le 2\left((1+2\nu)^2\right)^2 - 1 \le \left(1+2^2\nu\right)^{2^2}. \end{aligned} \tag{22}$$

Using induction, it is shown that

$$P_k(\lambda) \le \left(1 + 2^k v\right)^{2^k} \le e^{c_1 2^{-2(n-k)}} \tag{23}$$

where $c_1 = c_0(al)^2$.

Finally from (20) and (23) the estimation for uniform boundedness of$\|\mathbf{U}_k\|$ is obtained

$$\|\mathbf{U}_k\| \le P_k(\lambda) \le e^{c_1 2^{-2(n-k)}}. \tag{24}$$

Now the approximation error $\mathbf{E}_k = \mathbf{U}(x_k) - \mathbf{U}_k$ is estimated. From (11) and (15) it follows that:

$$\mathbf{E}_k = 2\mathbf{E}_{k-1}\mathbf{W}_{k-1} \tag{25}$$

where

$$\mathbf{W}_k = \mathbf{U}(x_k) + \mathbf{U}_k. \tag{26}$$

From (25) the following is obtained:

$$\mathbf{E}_k = 2^k\mathbf{E}_0(\mathbf{W}_0\mathbf{W}_1 \ldots \mathbf{W}_k). \tag{27}$$

The norm of the initial approximation $\mathbf{E}_0$can be estimated. From expansions (4) and (13) we have

$$\begin{aligned} \mathbf{E}_0 = \ &\mathbf{U}(x_0) - \mathbf{U}_0 = \frac{x_0^{2p+2}}{(2p+2)!}\mathbf{\Gamma}^{2p+2} \\ &+ \frac{x_0^{2p+4}}{(2p+4)!}\mathbf{\Gamma}^{2p+4} + \cdots. \end{aligned} \tag{28}$$

From here, it follows that:

$$\begin{aligned} \|\mathbf{E_0}\| &\le \frac{x_0^{2p+2}}{(2p+2)!}a^{2p+2}\left(1 + \frac{ax_0^2}{(2p+3)(2p+4)} + \ldots\right) \\ &\le \frac{(al)^{2p+2}}{(2p+2)!2^{(2p+2)n}}e^{al2^{-n}} \end{aligned} \tag{29}$$

where $x_0 = \frac{l}{2^n}$.

According to the representation (4) and estimation (24), the following inequality is true for $\mathbf{W}_k$:

$$\|\mathbf{W}_k\| \le \|\mathbf{U}(x_k)\| + \|\mathbf{U}_k\| \le 2e^{2^{-(2n-k)}c} \tag{30}$$

where $c = \max\left\{c_0(al)^2, al\right\}$.

From (27), according to inequalities (29) and (30) the following estimation for approximation error $\mathbf{E}_k$is obtained:

$$\|\mathbf{E}_k\| \le \frac{2c^2(al)^{2p+2}}{(2p+2)!2^{(n-k)}}\frac{1}{2^{2pn}}. \tag{31}$$

From here, it follows that for $k = n$ the following is obtained:

$$\|\mathbf{E}_n\| = \|\cosh(l\mathbf{\Gamma}) - \mathbf{U}_n\| \le \frac{2c^2(al)^{2p+2}}{(2p+2)!}\frac{1}{2^{2pn}}. \tag{32}$$

B. Error Estimation for Sine Hyperbolic Function $\mathbf{V}(x)$

The approximation error for sine hyperbolic function needs to be estimated. According to formulas (12) and (16), we have

$$\mathbf{F}_k = \mathbf{F}_{k-1}\mathbf{U}(x_{k-1}) + \mathbf{V}_{k-1}\mathbf{E}_{k-1} \tag{33}$$

where $\mathbf{F}_k = \mathbf{V}(x_k) - \mathbf{V}_k$.

From (33), using induction, the following is obtained:

$$\mathbf{F}_{k+1} = \mathbf{F}_0\mathbf{S}_{0,k} + \mathbf{E}_0\mathbf{S}_{1,k} + \cdots + \mathbf{E}_{k-1}\mathbf{S}_{k,k} + \mathbf{E}_k\mathbf{S}_{k+1,k} \tag{34}$$

where

$$\begin{aligned} \mathbf{S}_{i,k} &= \mathbf{V}_{i-1}\mathbf{U}(x_i)\cdots\mathbf{U}(x_k) \\ &= \mathbf{V}_0\mathbf{U}_0\cdots\mathbf{U}_{i-2}\mathbf{U}(x_i)\cdots\mathbf{U}(x_k), \quad i = 0, 1, \ldots, k \\ \mathbf{V}_{-1} &= \mathbf{I}. \end{aligned} \tag{35}$$

The norm of the operator $\mathbf{S}_{i,k}$ can be estimated as

$$\|\mathbf{S}_{i,k}\| = \|\mathbf{V}_0\|\,\|\mathbf{U}_0\|\cdots\|\mathbf{U}_{i-1}\|\,\|\mathbf{U}(x_i)\|\cdots\|\mathbf{U}(x_k)\|. \tag{36}$$

Analogous to the previous case, the following is obtained:

$$\|\mathbf{V}_0\| \le 1 + c_0\lambda^2 \le e^{2^{-n}c}. \tag{37}$$

From (36), according to the inequalities (24), (37) and $\|\mathbf{U}(x_k)\| \le e^{2^{-(2n-k)}c}$ the following estimation follows:

$$\|\mathbf{S}_{i,k}\| \le e^c. \tag{38}$$

According to (9) and (15)

$$\begin{aligned} \|\mathbf{F}_0\| &= \|\mathbf{V}(x_0) - \mathbf{V}_0\| \\ &= \left\|\frac{x_0^{2p+2}}{(2p+3)!}\mathbf{\Gamma}^{2p+2} + \frac{x_0^{2p+4}}{(2p+5)!}\mathbf{\Gamma}^{2p+4} + \cdots\right\| \\ &\le \frac{(al)^{2p+2}}{(2p+3)!2^{(2p+2)n}}e^{2^{-n}c}. \end{aligned} \tag{39}$$

From (34), according to (31), (37)–(39), the final estimation for approximate solution of the sine hyperbolic function can be obtained

$$\|\mathbf{F}_n\| \le \frac{4c^4(al)^{2p+2}}{(2p+2)!2^{2pn}}. \tag{40}$$

From (32) and (40) explicit estimations, it can be seen that approximation error converges fairly quickly. Increasing n- number of divisions of the length l, and p - number of terms preserved in the initial approximations (13) and (14) exponentially decreases the error.

IV. Algorithm Complexity Estimation

From (32) and (40) estimations it can be seen that the desired accuracy of calculations can be reached by increasing both n and p. Increasing n- number of divisions of the length l will imply calculating more steps in recurrent relations (15) and (16). In addition, increasing p makes it necessary to use more calculations for finding the initial approximations of (13) and (14). In this section, the algorithm is given for defining the optimal values for n and p, which will give the desired $\varepsilon > 0$ accuracy with a minimal number of operations.

If the approximate value $\mathbf{U}_n$ for the cosine hyperbolic function with error $\varepsilon > 0$ needs to be calculated, then according to (32), the following inequality should hold:

$$\frac{a_0 a_1^{2p}}{(2p+2)!} \cdot \frac{1}{2^{2pn}} \leq \varepsilon \tag{41}$$

where $a_0 = 2(cal)^2, \quad a_1 = al$.

Taking the base-two logarithm from inequality (41), the following is then obtained:

$$\lg_2 a_0 + 2p \lg_2 a_1 - \lg_2 (2p+2)! - 2pn \lg_2 2 \leq -\lg_2 \frac{1}{\varepsilon}. \tag{42}$$

Taking into account that

$$\begin{aligned}(2p+2)! = &\ 1 \cdot 2 \cdots p \cdot (p+1) \cdots (2p+2) \\ \geq &\ 2^p \cdot p^p = (2p)^p \end{aligned} \tag{43}$$

then from (42) the following is obtained:

$$n \geq \frac{1}{2p} \lg_2 \frac{a_0}{\varepsilon} + \lg_2 a_1 - \frac{1}{2} \lg_2 p - \frac{1}{2}. \tag{44}$$

From here, to achieve the desired accuracy ε, the following number of steps are required in the recurrent relation (15):

$$n \approx \frac{1}{2p} \lg_2 \frac{a_0}{\varepsilon} + \lg_2 a_1 - \frac{1}{2} \lg_2 p. \tag{45}$$

Assuming that the dimension of the matrix $\mathbf{\Gamma}$ is equal to m, then number of operations N_T can be calculated by the following formula:

$$N_T = (p+n) \cdot m^3 = N_0 \cdot m^3, \quad N_0 = p + n \tag{46}$$

where p is the number of terms preserved in the initial approximation (13). Finding the optimal value for p is the objective. For this purpose, the value of n from (44) is inserted into (45) to find the value p_0, where the function $N_0(p)$ reaches its minimum. Using simple calculations, the following is obtained:

$$p_0 \approx \frac{1}{2}\left(\lg_2 e + \sqrt{2 \lg_2 \frac{2(cal)^2}{\varepsilon}}\right). \tag{47}$$

To clarify c, a_0 and a_1 constants in formulas (44) and (46), it is assumed that $n \geq \log_2 (al)$. This will guarantee that λ is less than one [see formula (20)] and from here it follows that $c_0 < 1$ [see formula (20)]. From here and (30), $c = \max\{(al)^2, al\}$ and $a_0 = 2(al)^2$. Finally, according to these equalities, from (44) and (46) the following explicit formulas for optimal n and p can be obtained:

$$\begin{aligned} p = &\ 1 + \operatorname{ceil}\left(\sqrt{2\left|\lg_2 \frac{2(al)^3 \max(1, al)}{\varepsilon}\right|}\right) \\ n = &\ \operatorname{ceil}\left(\frac{1}{2p}\left|\lg_2 \frac{2(al)^4 \max(1, al)}{\varepsilon p^p}\right| + |\lg_2 (al)|\right). \end{aligned} \tag{48}$$

Here, $\operatorname{ceil}(x)$ is a function which gives the nearest integer number greater than x. Using formula (48) optimal n and p can be obtained for the computation of recurrent relations (13)–(16).

V. Numerical Applications and Calculation Speed Analysis

In the proposed algorithm, the main objective is to find the approximate value of sine and cosine hyperbolic complex value matrix functions (8) and (9). To validate the correctness of the result, we can use the following well-known trigonometric formula:

$$\cosh^2(x) - \sinh^2(x) = 1. \tag{49}$$

According to this trigonometric formula, the accuracy of (13)–(16) recurrent relations was checked. The error $\mathbf{Err}$ is defined as

$$\mathbf{Err} = \mathbf{U}_n^2 - (l\mathbf{\Gamma})^2 \mathbf{V}_n^2 - \mathbf{I} \tag{50}$$

where $\mathbf{I}$ is an identity matrix. Corresponding RLGC parameter matrices from the quadraxial cable simulation were obtained. 2-D cross-sectional analysis tool from Fast Electromagnetic Analysis Suite (FEMAS) software was used for simulation (see [10]). Default parameters of the software were used for Quadraxial cable geometry. Geometry of the cable is given in Fig. 1.

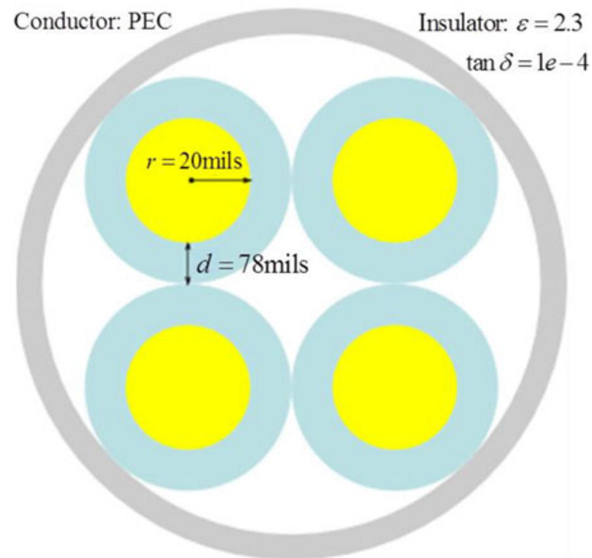

Fig. 1. Geometry description for quadraxial cable.

The length of TL was taken as 10 m and S-parameters for interval $[0, 20\text{MHz}]$ with step 5MHz in the frequency samples

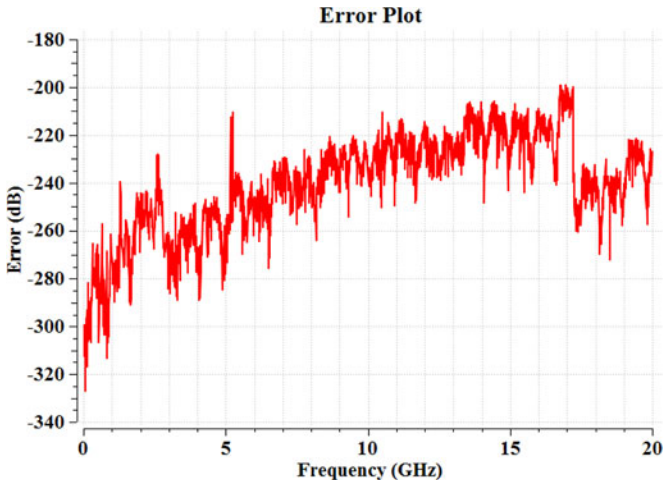

Fig. 2. Error plot for $\|\mathbf{Err}\| = \left\|\mathbf{U}_n^2 - (l\mathbf{\Gamma})^2\, \mathbf{V}_n^2 - \mathbf{I}\right\|$.

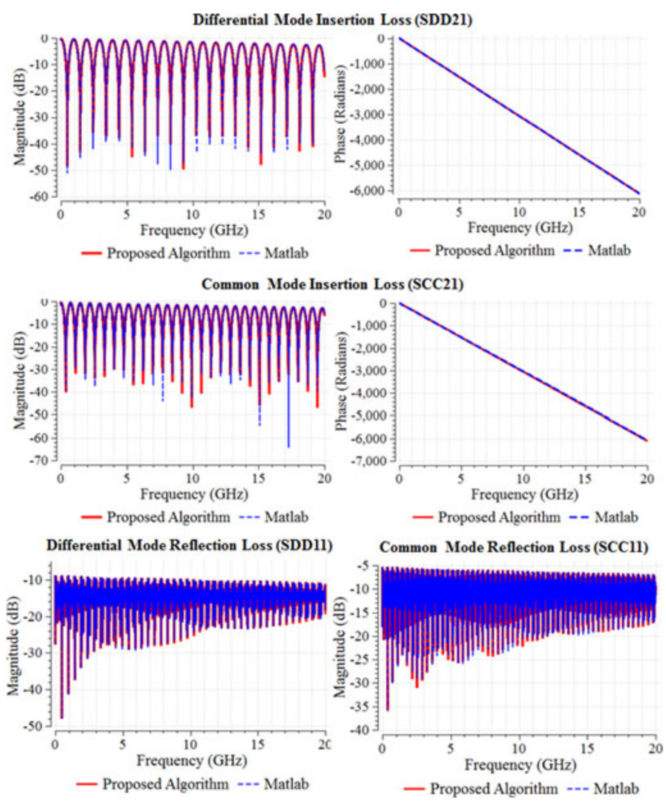

Fig. 3. Comparison of *S*-parameters calculated using proposed algorithm and MATLAB standard function rlgc2s().

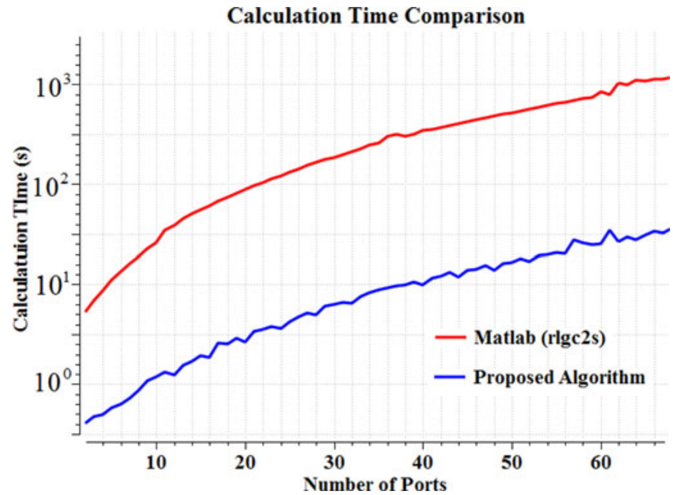

Fig. 4. Comparison of calculation times of MATLAB and proposed algorithm.

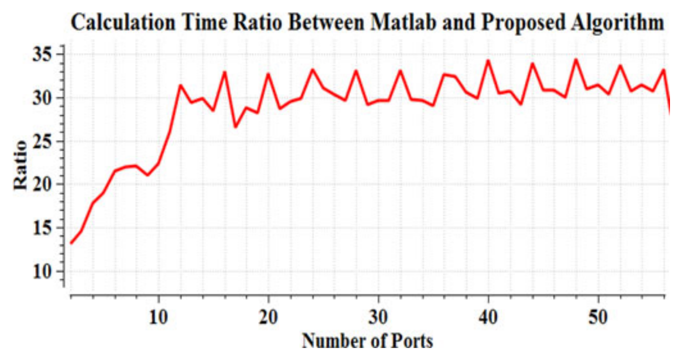

Fig. 5. Calculation time ratio between MATLAB and proposed algorithm.

were calculated. In Fig. 2 there is a given $\|\mathbf{Err}\|$ for all frequencies.

Fig. 2 shows that the calculation error is pretty small. *S*-parameters corresponding to geometry given in Fig. 1 were calculated using the algorithm proposed in the paper by means of MATLAB function *rlgc2s()*. Fig. 3 shows the comparison of the obtained results.

The main values of the proposed methodology are calculation speed and simplicity of the algorithm. Results obtained in Fig. 3 can be calculated in 3.5 s using MATLAB and in 0.27 s with the proposed algorithm. This figure shows that the proposed algorithm achieves the same accuracy as MATLAB much faster. A more detailed comparison of computation time is shown in Fig. 4.

In Fig. 4, on the x-axis is the given number of ports for *S*-parameters and on the y-axis is the given calculation speed. Real and imaginary parts of RLGC parameters matrices were taken randomly on the interval $[-100, 100]$ in such a way to obtain symmetric and diaganolly dominant matrices.

Symmetricity restrictions are necessary for standard MATLAB function *rlgc2s*. Fig. 5 shows the proposed algorithm for a different number of ports is approximately 15–35 times faster than the standard MATLAB algorithm *rlgc2s*.

VI. Conclusion

A computer-based fast algorithm for calculation of *S*-parameters was presented. The algorithm is based on efficient numerical calculations of trigonometric hyperbolic functions with complex element matrix arguments. The main values of the proposed algorithm are calculation speed and simplicity. For numerical computation of the algorithm only matrix multiplication is needed, unlike the algorithm given in [2], [6] which

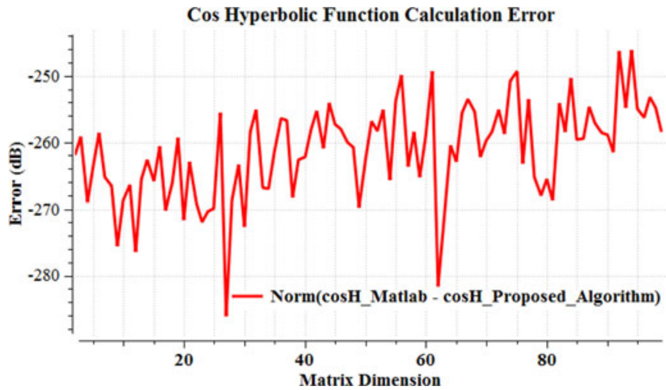

Fig. 6. Difference of values of hyperbolic cosine function calculated by MATLAB and the proposed algorithm for different matrix dimensions.

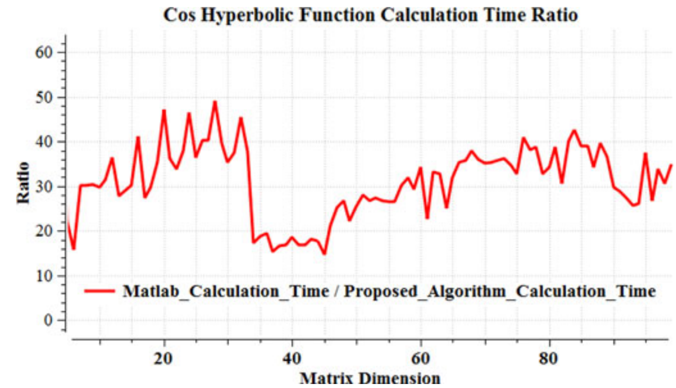

Fig. 7. Ratios of calculation times of hyperbolic cosine function by MATLAB and the proposed algorithm for different matrix dimensions.

is implemented in the MATLAB standard function *rlgc2s()* that requires root extraction from matrix and solution of eigenvalues. For this reason, the proposed algorithm does not have the restriction that RLGC parameters should be symmetric matrices. Various test cases were performed for validation of calculation accuracy and speed. Calculation speed comparison was made between the proposed algorithm and the algorithm implemented in MATLAB (see [2]). Test cases were done for different number of ports. Comparison showed that the proposed algorithm is approximately 15–35 times faster (see Fig. 5). Increasing number of conductors increases $(ZY)^{-1/2}$ matrix dimension. With larger dimension matrix eigenvalue solving problems becomes more complicated and time consuming. Because of this the proposed algorithm is more efficient for large number of conductors in the MTL. Fig. 5 shows that for small number of conductors (< 6) the proposed algorithm is about 15 times faster, but for conductors more than 15 the proposed algorithm is about 30–35 times faster. Both calculations have been performed on the same desktop computer with processor Intel i5-2400, and the same MATLAB R20013a version has been used.

The algorithm is very simple and can be easily implemented. From here it follows that the proposed algorithm is useful and can be used in practical applications. The proposed algorithm also has the independent value of being used for fast calculation of sine and cosine hyperbolic functions with matrix arguments. The same methodology can also be applied for fast calculation of other trigonometric functions with matrix arguments. Figs. 6 and 7 show comparisons of the proposed algorithm and MATLAB for hyperbolic cosine function with matrix argument in view of accuracy and calculation speed. Matrices with random values and dimensions from 1 to 100 were generated and hyperbolic cosine functions were calculated for these matrices with the proposed algorithm and MATLAB. Fig. 6 shows the difference between results obtained by the new algorithm and MATLAB for different matrix dimensions and Fig. 7 shows ratio of calculation times of both methods. It can be seen from Figs. 6 and 7 that the proposed algorithm gives almost the same results as MATLAB (difference is below –200 dB) and its calculation time is 15–50 times less than that of MATLAB for different matrix dimensions.

The proposed algorithm can be extended for nonuniform multiconductor transmission-line systems (NMTL) (see [9], Chapter 2). In this case NMTL is approximated by a set of cascaded segments, each representing a uniform MTL. For such an approximation it is plausible to assume a constant per-unit-length impedance matrix and per-unit-length admittance matrix within in each uniform segment. Using the proposed algorithm scattering parameters can be calculated for each uniform segment and after cascading we can obtain network parameters for the total nonuniform multiconductor transmission-line.

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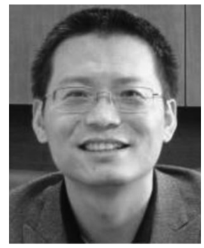

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