CHAPTER TWO

The Multiconductor Transmission-Line Equations

The previous chapter discussed the general properties of all transmission-line-equation characterizations. *The TEM field structure and associated mode of propagation is the fundamental, underlying assumption in the representation of a transmission line structure with the transmission-line equations.* In this chapter we will extend those notions to *multiconductor transmission lines* or *MTLs* consisting of $(n + 1)$ conductors. In general, we will restrict the class of lines to those that are *uniform lines* consisting of $(n + 1)$ conductors of uniform cross section that are parallel to each other. However, *the conductors as well as the surrounding medium may be lossless or lossy.* Lossless conductors are perfect conductors, while lossless media are media with zero conductivity, $\sigma = 0$. The surrounding medium may be *homogeneous* or *inhomogeneous*. The development and derivation of the MTL equations parallel the developments for two-conductor lines considered in the previous chapter. In fact, the developed MTL equations have, using matrix notation, a *form* identical to those equations. There are some new concepts concerning the important per-unit-length parameters which contain the cross-sectional dimensions of the particular line.

2.1 DERIVATION FROM THE INTEGRAL FORM OF MAXWELL'S EQUATIONS

Figure 2.1 shows the general $(n + 1)$-conductor line to be considered. It consists of n conductors and a *reference conductor* (denoted as the zeroth conductor) to which the n line voltages will be referenced. This choice of the reference conductor is not unique. Recall Faraday's law in integral form:

$$\oint_c \vec{\mathscr{E}} \cdot d\vec{l} = -\mu \frac{d}{dt} \int_s \vec{\mathscr{H}} \cdot d\vec{s} \tag{2.1}$$

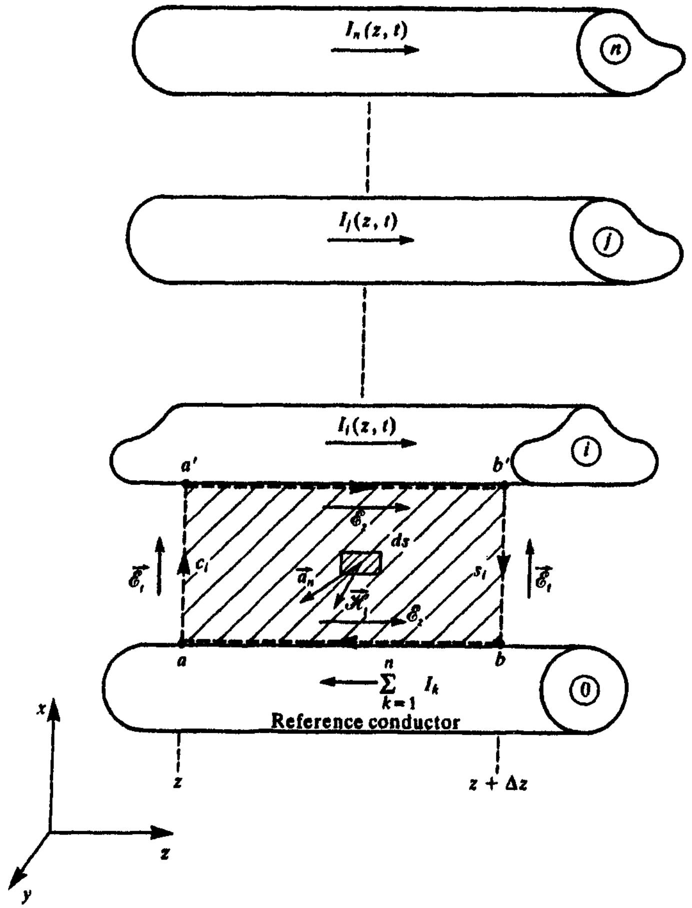

FIGURE 2.1 Definition of the contour for derivation of the first MTL equation.

Applying this to the contour c_i which encloses surface s_i shown between the reference conductor and the i-th conductor and encircles it in the clockwise direction gives

$$\int_a^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_{a'}^{b'} \vec{\mathscr{E}}_l \cdot d\vec{l} + \int_{b'}^{b} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_b^{a} \vec{\mathscr{E}}_l \cdot d\vec{l} = \mu \frac{d}{dt} \int_{s_i} \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \qquad (2.2)$$

where $\vec{\mathscr{E}}_t$ denotes the transverse electric field (in the x–y cross-sectional plane) and $\vec{\mathscr{E}}_l$ denotes the *longitudinal* or *z-directed* electric field (along the surfaces of the conductors). Observe, once again, that because of the choice of the direction of the contour, the direction of $\vec{a}_n$, and the right-hand rule, the minus sign on the right-hand side of Faraday's law is absent in (2.2). Once again, because of the assumption of a TEM field structure, we may *uniquely* define voltage between the i-th conductor and the reference conductor (positive on the i-th conductor) as

$$V_i(z, t) = -\int_a^{a'} \vec{\mathscr{E}}_t(x, y, z, t) \cdot d\vec{l} \tag{2.3a}$$

$$V_i(z + \Delta z, t) = -\int_b^{b'} \vec{\mathscr{E}}_t(x, y, z + \Delta z, t) \cdot d\vec{l} \tag{2.3b}$$

The integrals along the surfaces of the conductors are zero if the conductors are considered to be perfect conductors. It was pointed out that the TEM mode cannot exist if the conductors are not perfect conductors. This is because a component of electric field will be directed in the z direction due to the voltage drop along the conductors. However, small losses can be accommodated as an approximation under the *quasi-TEM* mode assumption. To allow for imperfect conductors, we define the per-unit-length conductor resistance, $r\Omega/\text{m}$. Thus

$$-\int_{a'}^{b'} \vec{\mathscr{E}}_l \cdot d\vec{l} = -\int_{a'}^{b'} \vec{\mathscr{E}}_z \, dz = -r_i \Delta z I_i(z, t) \tag{2.4a}$$

$$-\int_b^a \vec{\mathscr{E}}_l \cdot d\vec{l} = -\int_b^a \mathscr{E}_z \, dz = -r_0 \Delta z \sum_{k=1}^{n} I_k(z, t) \tag{2.4b}$$

where, along the top of i-th conductor, $\vec{\mathscr{E}}_l = \mathscr{E}_z \vec{a}_z$ and $d\vec{l} = dz\vec{a}_z$, and along the bottom conductor, $\vec{\mathscr{E}}_l = -\mathscr{E}_z \vec{a}_z$ and $d\vec{l} = -dz\vec{a}_z$. The current is *uniquely defined*, because of the assumption of a TEM field structure, as

$$I_i(z, t) = \oint_{c_i} \vec{\mathscr{H}}_t \cdot d\vec{l} \tag{2.5}$$

and contour c_i is a contour just off the surface of and encircling the i-th conductor in the transverse plane as shown in Fig. 2.2. Because of this definition of current and the TEM field structure assumption it can be shown, as was the case for two conductor lines, that *the sum of the currents on all* (n + 1) *conductors in the z direction at any cross section is zero*. This is the basis for saying that the currents of the n conductors *return through the reference conductor*. Substituting (2.3) to (2.5) into (2.2) yields

$$-V_i(z, t) + r_i \Delta z I_i(z, t) + V_i(z + \Delta z, t) + r_0 \Delta z \sum_{k=1}^{n} I_k(z, t) = \mu \frac{d}{dt} \int_{s_i} \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \tag{2.6}$$

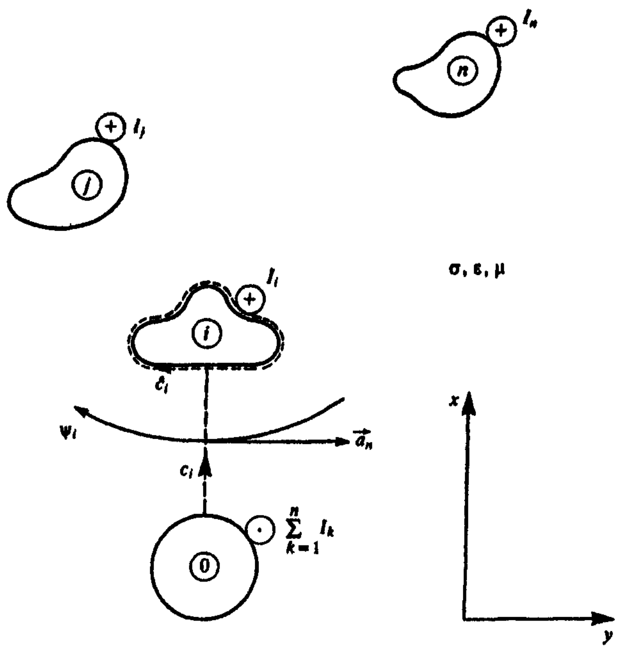

FIGURE 2.2 Illustration of the definitions of magnetic flux through a circuit for derivation of the per-unit-length inductances.

Dividing both sides by Δz, and rearranging gives

$$\frac{V_i(z + \Delta z, t) - V_i(z, t)}{\Delta z} = -r_0 I_1 - r_0 I_2 - \cdots - (r_0 + r_i)I_i - \cdots - r_0 I_n \tag{2.7}$$

$$+ \mu \frac{1}{\Delta z} \frac{d}{dt} \int_{s_i} \vec{\mathcal{H}}_t \cdot \hat{a}_n \, ds$$

Before taking the limit as $\Delta z \to 0$, let us make some observations similar to the case of two-conductor lines. Clearly, the total magnetic flux penetrating the surface s_i in Fig. 2.1 will be a linear combination of the fluxes due to the currents on the conductors. Consider a cross-sectional view of the line *looking in the direction of increasing z* shown in Fig. 2.2. The currents on the n conductors are implicitly defined in the positive z direction according to (2.5) since contour $\hat{c}_i$ is defined to be clockwise looking in the direction of increasing z. Therefore the magnetic fluxes due to the currents on the n conductors will also be in the clockwise direction looking in the direction of increasing z. The total magnetic flux, ψ_i, penetrating the surface s_i between the reference conductor and the ith conductor is therefore *defined* to be in this clockwise direction *when looking in the direction of increasing z* as shown in Fig. 2.2. Therefore, this total magnetic flux penetrating surface s_i can be written as

$$\begin{aligned}\psi_i &= -\mu \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{s_i} \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \\ &= l_{i1} I_1 + l_{i2} I_2 + \cdots + l_{ii} I_i + \cdots + l_{in} I_n\end{aligned} \tag{2.8}$$

Taking the limit of (2.7) as $\Delta z \to 0$ and substituting (2.8) yields

$$\begin{aligned}\frac{\partial V_i(z, t)}{\partial z} = &-r_0 I_1(z, t) - r_0 I_2(z, t) - \cdots - (r_0 + r_i) I_i(z, t) - \cdots - r_0 I_n(z, t) \\ &- l_{i1} \frac{\partial I_1(z, t)}{\partial t} - l_{i2} \frac{\partial I_2(z, t)}{\partial t} - \cdots - l_{ii} \frac{\partial I_i(z, t)}{\partial t} - \cdots - l_{in} \frac{\partial I_n(z, t)}{\partial t}\end{aligned} \tag{2.9}$$

This first MTL equation can be written in a compact form using matrix notation as

$$\frac{\partial}{\partial z} \mathbf{V}(z, t) = -\mathbf{R}\mathbf{I}(z, t) - \mathbf{L} \frac{\partial}{\partial t} \mathbf{I}(z, t) \tag{2.10}$$

where the voltage and current vectors are defined as

$$\mathbf{V}(z, t) = \begin{bmatrix} V_1(z, t) \\ \vdots \\ V_i(z, t) \\ \vdots \\ V_n(z, t) \end{bmatrix} \tag{2.11a}$$

$$\mathbf{I}(z, t) = \begin{bmatrix} I_1(z, t) \\ \vdots \\ I_i(z, t) \\ \vdots \\ I_n(z, t) \end{bmatrix} \tag{2.11b}$$

The *per-unit-length inductance matrix* is defined from (2.8) as

$$\mathbf{\Psi} = \mathbf{L}\mathbf{I} \tag{2.12a}$$

where $\mathbf{\Psi}$ is an $n \times 1$ vector containing the total magnetic flux per unit length, ψ_i, penetrating the i-th circuit which is defined between the i-th conductor and the reference conductor:

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_i \\ \vdots \\ \psi_n \end{bmatrix} \tag{2.12b}$$

and the *per-unit-length inductance matrix*, **L**, contains the individual per-unit-length self-inductances, l_{ii}, of the circuits and the per-unit-length mutual inductances between the circuits, l_{ij}, as

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \tag{2.12c}$$

Similarly, from (2.9) we define the *per-unit-length resistance matrix* as

$$\mathbf{R} = \begin{bmatrix} (r_1 + r_0) & r_0 & \cdots & r_0 \\ r_0 & (r_2 + r_0) & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & (r_n + r_0) \end{bmatrix} \tag{2.13}$$

Observe that this first transmission-line equation given in (2.10) is identical in form to the *scalar* first transmission-line equation for a two-conductor line.

Consider placing a closed surface $\mathscr{S}$ around the i-th conductor as shown in Fig. 2.3. The portion of the surface over the end caps is denoted as $\mathscr{S}_e$ while the portion over the sides is denoted as $\mathscr{S}_o$. Recall the continuity equation or equation of conservation of charge:

$$\oint\!\!\!\oint_{\mathscr{S}} \vec{\mathscr{J}} \cdot d\vec{s} = -\frac{\partial}{\partial t} Q_{\text{enc}} \tag{2.14}$$

Over the end caps we have

$$\iint_{\mathscr{S}_e} \vec{\mathscr{J}} \cdot d\vec{s} = I_i(z + \Delta z, t) - I_i(z, t) \tag{2.15}$$

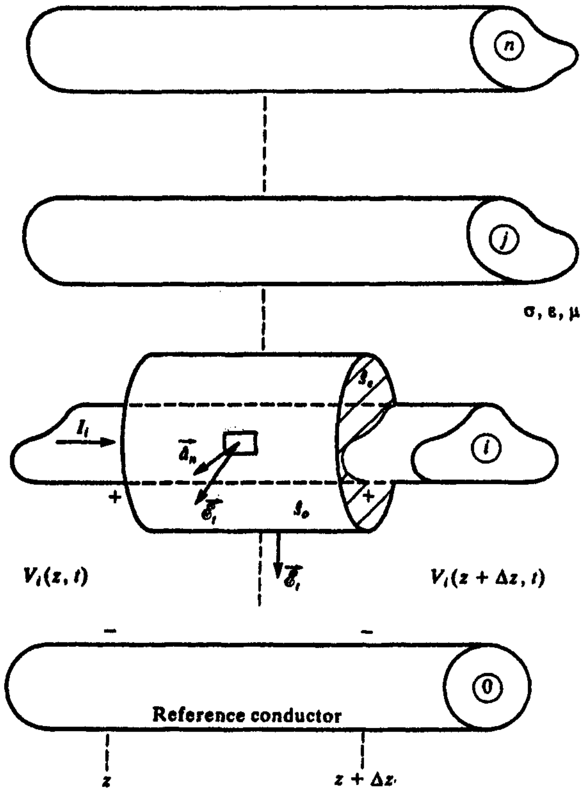

FIGURE 2.3 Definition of the surface for derivation of the second MTL equation.

Over the sides of the surface, there are two currents: *conduction current*, $\vec{\mathscr{J}}_c = \sigma\vec{\mathscr{E}}_t$, and *displacement current*, $\vec{\mathscr{J}}_d = \varepsilon(\partial\vec{\mathscr{E}}_t/\partial t)$, where the surrounding homogeneous medium is characterized by conductivity, σ, and permittivity, ε. These notions can be extended to an inhomogeneous medium surrounding the conductors in a similar but approximate manner. This is an approximation since an inhomogenous medium, uniform along the line or not, invalidates the TEM field structure assumption which requires that all waves propagate with the same velocity, that being the phase velocity of a plane wave in that medium. A portion of the left-hand side of (2.14) contains the *transverse conduction current* flowing between the conductors:

$$\iint_{s_o} \vec{\mathscr{J}}_c \cdot d\vec{s} = \sigma \iint_{s_o} \vec{\mathscr{E}}_t \cdot d\vec{s} \tag{2.16}$$

This can again be considered by defining *per-unit-length conductances*, g_{ij} S/m, between each pair of conductors as the ratio of conduction current flowing

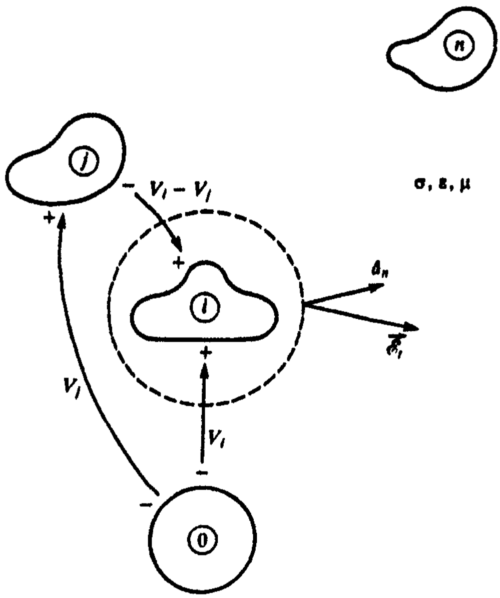

FIGURE 2.4 Illustration of the definitions used in computing the per-unit-length capacitances.

between the two conductors in the transverse plane to the voltage between the two conductors. (See Fig. 2.4.) Therefore,

$$\sigma \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s_o} \vec{\mathscr{E}}_t \cdot d\vec{s} = g_{i1}(V_i - V_1) + \cdots + g_{ii}V_i + \cdots + g_{in}(V_i - V_n) \quad (2.17)$$

$$= -g_{i1}V_1(z,t) - g_{i2}V_2(z,t) - \cdots + \sum_{k=1}^{n} g_{ik}V_k(z,t) - \cdots - g_{in}V_n(z,t)$$

Similarly, the charge enclosed by the surface (residing on the conductor surface) is, by Gauss' law,

$$Q_{enc} = \varepsilon \iint_{s_o} \vec{\mathscr{E}}_t \cdot d\vec{s} \quad (2.18)$$

The charge per unit of line length can be defined in terms of the *per-unit-length capacitances*, c_{ij}, between each pair of conductors as

$$\varepsilon \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s_o} \vec{\mathscr{E}}_t \cdot d\vec{s} = c_{i1}(V_i - V_1) + \cdots + c_{ii}V_i + \cdots + c_{in}(V_i - V_n) \quad (2.19)$$

$$= -c_{i1}V_1(z,t) - \cdots + \sum_{k=1}^{n} c_{ik}V_k(z,t) - \cdots - c_{in}V_n(z,t)$$

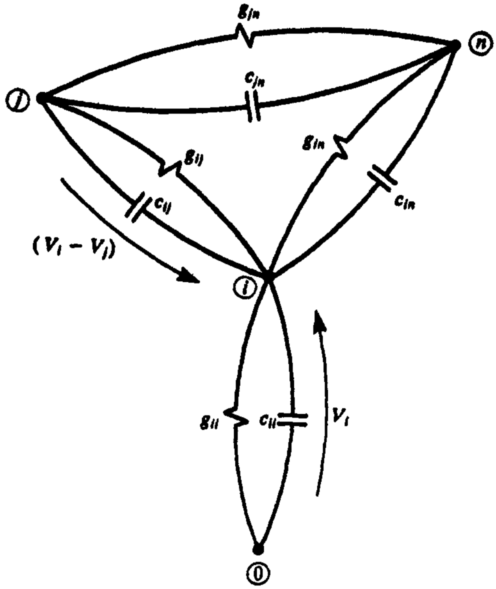

FIGURE 2.5 Two-dimensional illustration of the per-unit-length conductances and capacitances as an aid in the determination of the entries in **G** and **C**.

These concepts are illustrated in cross section in Fig. 2.5. Substituting (2.15), (2.16), and (2.18) into (2.14), and dividing both sides by Δz gives

$$\frac{I_i(z + \Delta z, t) + I_i(z, t)}{\Delta z} + \sigma \frac{1}{\Delta z} \iint_{s_0} \vec{\mathscr{E}}_t \cdot d\vec{s} = -\varepsilon \frac{1}{\Delta z} \iint_{s_0} \vec{\mathscr{E}}_t \cdot d\vec{s} \tag{2.20}$$

Taking the limit as $\Delta z \to 0$ and substituting (2.17) and (2.19) yields

$$\frac{\partial I_i(z, t)}{\partial z} = g_{i1} V_1(z, t) + g_{i2} V_2(z, t) + \cdots - \sum_{k=1}^{n} g_{ik} V_k(z, t) + \cdots + g_{in} V_n(z, t) \tag{2.21}$$
$$+ \frac{\partial}{\partial t} \left\{ c_{i1} V_1(z, t) + \cdots - \sum_{k=1}^{n} c_{ik} V_k(z, t) + \cdots + c_{in} V_n(z, t) \right\}$$

Equations (2.21) can be placed in compact form with matrix notation giving

$$\frac{\partial}{\partial z} \mathbf{I}(z, t) = -\mathbf{G}\mathbf{V}(z, t) - \mathbf{C} \frac{\partial}{\partial t} \mathbf{V}(z, t) \tag{2.22}$$

where **V** and **I** are defined by (2.11). The *per-unit-length conductance matrix*, **G**, represents the *conduction current flowing between the conductors in the transverse* *plane* and is defined from (2.21) as

$$\mathbf{G} = \begin{bmatrix} \sum_{k=1}^{n} g_{1k} & -g_{12} & \cdots & -g_{1n} \\ -g_{21} & \sum_{k=1}^{n} g_{2k} & \cdots & -g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{n1} & -g_{n2} & \cdots & \sum_{k=1}^{n} g_{nk} \end{bmatrix} \tag{2.23}$$

The *per-unit-length capacitance matrix*, **C**, represents the *displacement current flowing between the conductors in the transverse plane* and is defined from (2.21) as

$$\mathbf{C} = \begin{bmatrix} \sum_{k=1}^{n} c_{1k} & -c_{12} & \cdots & -c_{1n} \\ -c_{21} & \sum_{k=1}^{n} c_{2k} & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{n1} & -c_{n2} & \cdots & \sum_{k=1}^{n} c_{nk} \end{bmatrix} \tag{2.24}$$

Again observe that (2.22) is the matrix counterpart to the scalar second transmission-line equation for two-conductor lines. If we denote the total charge on the i-th conductor per unit of line length as q_i, then the fundamental definition of **C** which is the dual to (2.12) is

$$\mathbf{Q} = \mathbf{C}\mathbf{V} \tag{2.25a}$$

where

$$\mathbf{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{bmatrix} \tag{2.25b}$$

and **V** is given by (2.11a). Similarly, the fundamental definition of **G** is $\mathbf{I}_t = \mathbf{G}\mathbf{V}$, where $\mathbf{I}_t$ is the transverse conduction current between the conductors.

The above per-unit-length parameter matrices once again contain all the cross-sectional dimension information that distinguishes one MTL structure from another. Although these were shown as not being symmetric, it is logical

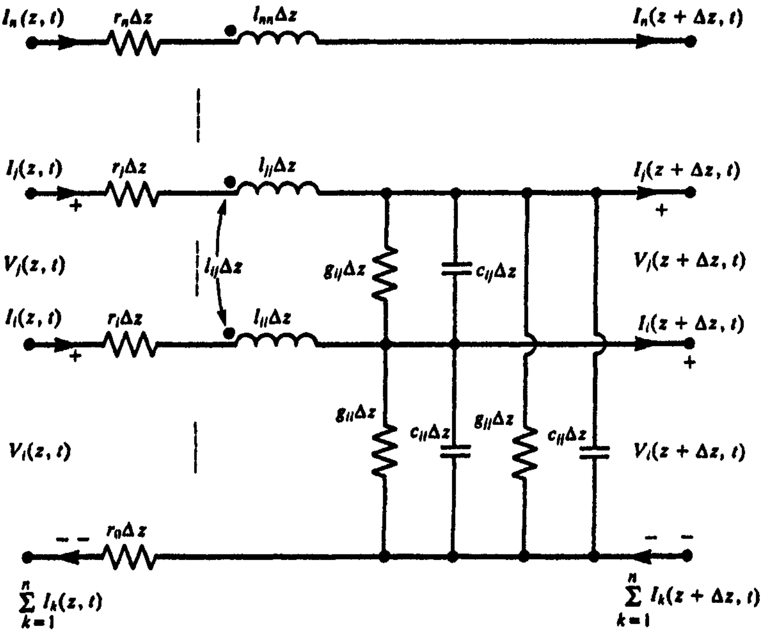

FIGURE 2.6 The per-unit-length MTL model for derivation of the MTL equations.

to expect that they are. This will be proven for isotropic surrounding media—the medium may be inhomogeneous.

2.2 DERIVATION FROM THE PER-UNIT-LENGTH EQUIVALENT CIRCUIT

As a final and alternative method we derive the MTL equations from the *per-unit-length equivalent circuit* shown in Fig. 2.6. Writing Kirchhoff's voltage law around the *i-th circuit* consisting of the *i*-th conductor and the reference conductor yields

$$-V_i(z, t) + r_i \Delta z I_i(z, t) + V_i(z + \Delta z, t) + r_0 \Delta z \sum_{k=1}^{n} I_k(z, t) \tag{2.26a}$$

$$= -l_{i1} \Delta z \frac{\partial I_1(z, t)}{\partial t} - l_{i2} \Delta z \frac{\partial I_2(z, t)}{\partial t} - \cdots - l_{ii} \Delta z \frac{\partial I_i(z, t)}{\partial t} - \cdots$$

$$- l_{in} \Delta z \frac{\partial I_n(z, t)}{\partial t}$$

Dividing both sides by Δz and taking the limit as $\Delta z \to 0$ once again yields the first transmission-line equation given in (2.9) with the collection for all i given in matrix form in (2.10).

Similarly, the second MTL equation can be obtained by applying Kirchhoff's current law to the i-th conductor in the per-unit-length equivalent circuit in Fig. 2.6 to yield

$$\begin{aligned} I_i(z+\Delta z, t) - I_i(z, t) = &-g_{i1}\Delta z(V_i - V_1) - \cdots - g_{ij}\Delta z(V_i - V_j) - \cdots \\ &-g_{in}\Delta z(V_i - V_n) - c_{i1}\Delta z \frac{\partial}{\partial t}(V_i - V_1) - \cdots \\ &- c_{ii}\Delta z \frac{\partial}{\partial t} V_i - \cdots - c_{in}\Delta z \frac{\partial}{\partial t}(V_i - V_n) \end{aligned} \tag{2.26b}$$

Dividing both sides by Δz, taking the limit $\Delta z \rightarrow 0$, and collecting terms once again yields the second transmission-line equation given in (2.21) with the collection for all i given in matrix form in (2.22). Strictly speaking, the voltages in (2.26b) are at $z + \Delta z$ so that (2.26a) should be substituted before taking the limit. However, as was shown for two-conductor lines in the previous chapter, this yields the same result as when we take the limit $\Delta z \rightarrow 0$ in (2.26b) directly.

2.3 SUMMARY OF THE MTL EQUATIONS

In summary, the *MTL equations* are given by the collection

$$\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t) \tag{2.27a}$$

$$\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\mathbf{V}(z,t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t) \tag{2.27b}$$

The structures of the per-unit-length resistance matrix, **R**, in (2.13), inductance matrix, **L**, in (2.12), conductance matrix, **G**, in (2.23), and capacitance matrix, **C**, in (2.24) are very important as are the definitions of the per-unit-length entries in those matrices. The precise definitions of these elements are rather intuitive and lead to many ways of computing them for a particular MTL type. These computational methods will be considered in detail in Chapter 3. The important properties of the per-unit-length parameter matrices will be obtained in the next section. Again, these bear striking parallels to their scalar counterparts for the two-conductor line considered in the previous chapter.

The MTL equations in (2.27) are a set of $2n$, *coupled, first-order, partial differential equations.* They may be put in a more compact form as

$$\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} = -\begin{bmatrix}\mathbf{0} & \mathbf{R}\\ \mathbf{G} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{L}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} \tag{2.28}$$

We will find this first-order form to be especially helpful when we set out to solve them in later chapters. If the conductors are perfect conductors, $\mathbf{R} = 0$, whereas if the surrounding medium is lossless ($\sigma = 0$), $\mathbf{G} = 0$. The line is said to be *lossless* if both the conductors and the medium are lossless in which case the MTL equations simplify to

$$\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} = -\begin{bmatrix}\mathbf{0} & \mathbf{L}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} \tag{2.29}$$

The first-order, coupled forms in (2.27) can be placed in the form of *second-order, uncoupled* equations by differentiating (2.27a) with respect to z and differentiating (2.27b) with respect to t to yield

$$\frac{\partial^2}{\partial z^2}\mathbf{V}(z,t) = -\mathbf{R}\frac{\partial}{\partial z}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial^2}{\partial z\,\partial t}\mathbf{I}(z,t) \tag{2.30a}$$

$$\frac{\partial^2}{\partial z\,\partial t}\mathbf{I}(z,t) = -\mathbf{G}\frac{\partial}{\partial t}\mathbf{V}(z,t) - \mathbf{C}\frac{\partial^2}{\partial t^2}\mathbf{V}(z,t) \tag{2.30b}$$

Substituting (2.30b) and (2.27b) into (2.30a) and reversing the process yields the *uncoupled, second-order equations*:

$$\frac{\partial^2}{\partial z^2}\mathbf{V}(z,t) = (\mathbf{RG})\mathbf{V}(z,t) + (\mathbf{RC} + \mathbf{LG})\frac{\partial}{\partial t}\mathbf{V}(z,t) + \mathbf{LC}\frac{\partial^2}{\partial t^2}\mathbf{V}(z,t) \tag{2.31a}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{I}(z,t) = (\mathbf{GR})\mathbf{I}(z,t) + (\mathbf{GL} + \mathbf{CR})\frac{\partial}{\partial t}\mathbf{I}(z,t) + \mathbf{CL}\frac{\partial^2}{\partial t^2}\mathbf{I}(z,t) \tag{2.31b}$$

Observe that the various matrix products in (2.31) do not generally commute so that the proper order of multiplication must be observed.

2.4 PROPERTIES OF THE PER-UNIT-LENGTH PARAMETER MATRICES L, C, G

In the previous chapter we showed that, for a two-conductor line *immersed in a homogeneous medium characterized by permeability, μ, conductivity, σ, and permittivity, ε*, the per-unit-length inductance, l, conductance, g, and capacitance, c, are related by $lc = \mu\varepsilon$ and $lg = \mu\sigma$. For the case of a MTL consisting of $(n + 1)$ conductors *immersed in a homogeneous medium characterized by permeability, μ, conductivity, σ, and permittivity, ε*, the per-unit-length parameter matrices are similarly related by

$$\mathbf{LC} = \mathbf{CL} = \mu\varepsilon\mathbf{1}_n \tag{2.32a}$$

$$\mathbf{LG} = \mathbf{GL} = \mu\sigma\mathbf{1}_n \tag{2.32b}$$

where the $n \times n$ *identity matrix* is defined as having unity entries on the main diagonal and zeros elsewhere:

$$\mathbf{1}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.33}$$

Other important properties such as our logical assumption that these per-unit-length matrices are symmetric will also be shown.

Recall from Chapter 1 that the transverse electric and magnetic fields of the TEM field structure satisfy the following differential equations (see equations (1.11)):

$$\frac{\partial^2 \vec{\mathscr{E}}_t}{\partial z^2} = \mu\sigma \frac{\partial \vec{\mathscr{E}}_t}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{\mathscr{E}}_t}{\partial t^2} \tag{2.34a}$$

$$\frac{\partial^2 \vec{\mathscr{H}}_t}{\partial z^2} = \mu\sigma \frac{\partial \vec{\mathscr{H}}_t}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{\mathscr{H}}_t}{\partial t^2} \tag{2.34b}$$

Define voltage and current in the usual fashion as integrals in the transverse plane (see Fig. 2.2) as

$$V_i(z, t) = -\int_{c_i} \vec{\mathscr{E}}_t \cdot d\vec{l} \tag{2.35a}$$

$$I_i(z, t) = \oint_{c_i'} \vec{\mathscr{H}}_t \cdot d\vec{l} \tag{2.35b}$$

Applying (2.35) to (2.34) yields

$$\frac{\partial^2}{\partial z^2} V_i(z, t) = \mu\sigma \frac{\partial}{\partial t} V_i(z, t) + \mu\varepsilon \frac{\partial^2}{\partial t^2} V_i(z, t) \tag{2.36a}$$

$$\frac{\partial^2}{\partial z^2} I_i(z, t) = \mu\sigma \frac{\partial}{\partial t} I_i(z, t) + \mu\varepsilon \frac{\partial^2}{\partial t^2} I_i(z, t) \tag{2.36b}$$

Collecting equations (2.36) for all conductors in matrix form yields

$$\frac{\partial^2}{\partial z^2} \mathbf{V}(z, t) = \mu\sigma \frac{\partial}{\partial t} \mathbf{V}(z, t) + \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{V}(z, t) \tag{2.37a}$$

$$\frac{\partial^2}{\partial z^2} \mathbf{I}(z, t) = \mu\sigma \frac{\partial}{\partial t} \mathbf{I}(z, t) + \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{I}(z, t) \tag{2.37b}$$

Comparing (2.37) to (2.31) with $\mathbf{R} = \mathbf{0}$ gives the identities in (2.32). Because of the identities in (2.32), *valid only for a homogeneous medium*, we need to determine only one of the per-unit-length parameter matrices since (2.32) can be written, for example, as

$$\mathbf{L} = \mu\varepsilon\mathbf{C}^{-1} \tag{2.38a}$$

$$\mathbf{G} = \frac{\sigma}{\varepsilon}\mathbf{C} \tag{2.38b}$$

$$\mathbf{C} = \mu\varepsilon\mathbf{L}^{-1} \tag{2.38c}$$

The identities in (2.32) are *valid only for a homogeneous surrounding medium* as is the assumption of a TEM field structure and the resulting MTL equations. We will often extend the MTL equation representation, in an approximate manner, to include *inhomogeneous media* as well as imperfect conductors under the *quasi-TEM assumption*. Even in the case of an *inhomogeneous medium*, the per-unit-length parameter matrices, **L**, **C**, and **G**, have several important properties. The primary ones are that they are *symmetric* and *positive-definite* matrices. The proof that **C**, **L**, and **G** are *symmetric matrices* (*regardless of whether the surrounding medium is homogeneous or inhomogeneous*) can be accomplished from energy considerations [A.1, 1]. As an illustration, we will prove that **C** is symmetric; the proof that **L** and **G** are symmetric follows in a similar fashion. The basic relation for **C** is given in (2.24) and (2.25). Suppose we invert this relation to give

$$\begin{aligned} V_1 &= p_{11}q_1 + \cdots + p_{1n}q_n \\ &\vdots \\ V_n &= p_{n1}q_1 + \cdots + p_{nn}q_n \end{aligned} \tag{2.39}$$

If we can prove that $p_{ij} = p_{ji}$ then it follows that $c_{ij} = c_{ji}$. Suppose all conductors *except* the i-th and j-th are connected to the reference conductor (grounded) and all conductors are initially uncharged. Suppose we start charging the i-th conductor to a final per-unit-length charge of q_i. Charging the i-th conductor to an incremental charge q results in a voltage of the conductor, from (2.39), of $V_i = p_{ii}q$. The incremental energy required to do this is $dW = V_i\,dq$. The total energy required to place the charge q_i on the i-th conductor is $W = \int_0^{q_i} p_{ii}q\,dq = p_{ii}q_i^2/2$. Now if we charge the j-th conductor to an incremental charge of q in the presence of the charged i-th conductor, the voltage of the j-th conductor is $V_j = p_{ji}q_i + p_{jj}q$ and the incremental energy required is $dW = (p_{ji}q_i + p_{jj}q)\,dq$. The total energy required to charge the j-th conductor to a charge of q_j becomes $W = \int_0^{q_j} dW\,dq = p_{ji}q_iq_j + p_{jj}q_j^2/2$. Thus the total energy required to charge conductor i to q_i and conductor j to q_j is

$$W_{\text{total}} = p_{ji}q_iq_j + \frac{p_{ii}q_i^2}{2} + \frac{p_{jj}q_j^2}{2}$$

If we reverse this process charging conductor j to q_j and then charging conductor i to q_i we obtain

$$W_{\text{total}} = p_{ij} q_j q_i + \frac{p_{jj} q_j^2}{2} + \frac{p_{ii} q_i^2}{2}$$

Since the total energies must be the same regardless of the sequence in which the conductors are charged, we see, by comparing these two energy expressions, that

$$p_{ij} = p_{ji}$$

Therefore, it follows that

$$c_{ij} = c_{ji}$$

and therefore the capacitance matrix, **C**, is *symmetric*. Proof that **L** and **G** are also *symmetric* follows in a like fashion. Recall that this proof of symmetry relied on energy considerations and therefore is valid for *inhomogeneous media*.

We next set out to prove that **L**, **C**, and **G** are *positive definite*. The energy stored in the electric field per unit of line length is

$$\begin{aligned} w_e &= \frac{1}{2} \sum_{k=1}^{n} q_k V_k \\ &= \frac{1}{2} \mathbf{Q}^t \mathbf{V} \end{aligned} \tag{2.40}$$

where the *transpose* of a matrix **M** is denoted by $\mathbf{M}^t$. The vector **Q** contains the per-unit-length charges on the conductors and is given in (2.25b). Substituting the relation for **C** in (2.25a), $\mathbf{Q} = \mathbf{CV}$, into (2.40) gives

$$w_e = \frac{1}{2} \mathbf{V}^t \mathbf{C} \mathbf{V} > 0 \tag{2.41}$$

where we have used the matrix property that $(\mathbf{MN})^t = \mathbf{N}^t \mathbf{M}^t$. This total energy stored in the electric field must be positive and nonzero for all choices of the voltages (positive or negative). Thus we say that **C** is *positive definite* if

$$\mathbf{V}^t \mathbf{C} \mathbf{V} > 0 \tag{2.42}$$

for all possible values of the entries in **V**. It turns out that this implies that all of the *eigenvalues* of **C** must be *positive*; a property we will find very useful in our later developments. Proof that **L** and **G** are also positive definite follows in a similar fashion.

Finally we point out that *all of the per-unit-length parameter matrices can be obtained from capacitance calculations with and without the dielectric removed.*

Designate the capacitance matrix *with the surrounding medium (homogeneous or inhomogeneous) removed and replaced by free space having permeability* ε_o *and permeability* μ_o *as* $\mathbf{C}_o$. Since inductance depends on permeability of the surrounding medium and the permeability of dielectrics is typically that of free space, μ_o, the inductance matrix, $\mathbf{L}$, can be obtained from $\mathbf{C}_o$ (using the relations for a homogeneous medium (in this case, free space) given in (2.32)) as:

$$\mathbf{L} = \mu_o \varepsilon_o \mathbf{C}_o^{-1} \tag{2.43}$$

Therefore, $\mathbf{L}$ and $\mathbf{C}$ can be computed using only a capacitance calculation. This observation will be useful when we consider computing these parameters for inhomogeneous media in the next chapter.

REFERENCE

[1] R. Plonsey and R.E. Collin, *Principles and Applications of Electromagnetic Fields*, 2d ed., McGraw-Hill, NY, 1982.

PROBLEMS

2.1 Derive the MTL equations for the per-unit-length equivalent circuit of the four-conductor line shown in Fig. P2.1.

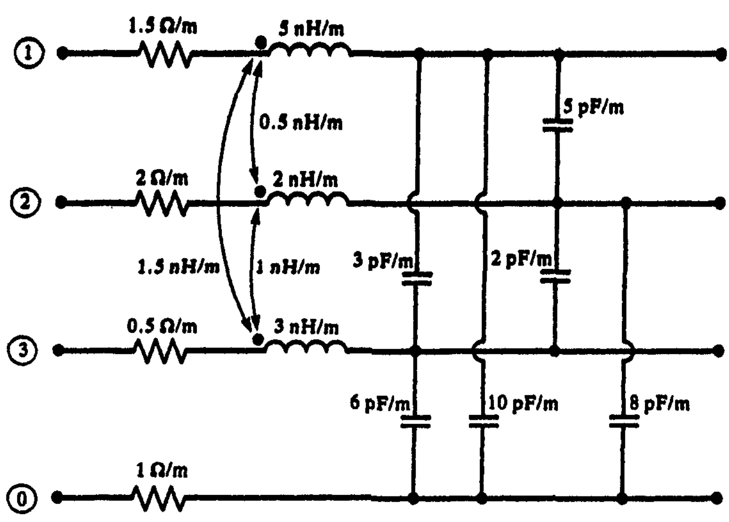

FIGURE P2.1

2.2 A four-conductor line immersed in free space has the following per-unit-length inductance matrix:

$$\mathbf{L} = \begin{bmatrix} 8 & 3 & 2 \\ 3 & 6 & 1 \\ 2 & 1 & 4 \end{bmatrix} \text{nH/m}$$

Determine the per-unit-length capacitance matrix. If the surrounding medium is homogeneous with conductivity $\sigma = 10^{-3}$ S/m, determine the per-unit-length conductance matrix.

2.3 Derive the uncoupled, second-order MTL equations in (2.31).

2.4 Show that the criterion for positive definiteness of a real, symmetric matrix is that its eigenvalues are all positive and nonzero. (**Hint**: Transform the matrix to another equivalent one with a transformation matrix that diagonalizes it as $\mathbf{T}^{-1}\mathbf{M}\mathbf{T} = \Lambda$ where Λ is diagonal with its eigenvalues on the main diagonal. It is always possible to diagonalize any real, symmetric matrix such that $\mathbf{T}^{-1} = \mathbf{T}^t$ where $\mathbf{T}^t$ is the transpose of $\mathbf{T}$.) Show that the per-unit-length inductance matrix in Problem 2.2 is positive definite.

2.5 A matrix with the structure of **G** in (2.23) or **C** in (2.24) whose off-diagonal terms are negative and the sum of the elements in a row or column are positive is said to be *hyperdominant*. Show that a hyperdominant matrix is always positive definite.