CHAPTER ONE

Introduction

This text concerns the analysis of transmission-line structures that serve to guide electromagnetic (EM) waves between two points. The analysis of transmission lines consisting of two parallel conductors of uniform cross section is a fundamental and well-understood subject in electrical engineering. However, the analysis of similar lines consisting of more than two conductors is not as well understood. The purpose of this text is to provide a concise, yet complete, description of the formulation and analysis of the transmission-line equations for lines consisting of more than two conductors (multiconductor transmission lines or MTLs).

The analysis of MTLs is somewhat more difficult than the analysis of two-conductor lines but the applications cover a broad frequency spectrum and extend from power transmission lines to microwave circuits [B.1, 1–16]. However, matrix methods and notation provide a straightforward extension of many, if not most, of the aspects of two-conductor lines to MTLs. Many of the concepts and performance measures of two-conductor lines require more elaborate concepts when extended to MTLs. For example, in order to eliminate reflections at terminations on a two-conductor line we simply terminate it in a matched load, i.e., a load impedance which equals the characteristic impedance of the line. In the case of MTLs, we must terminate the line in a *characteristic impedance matrix* or network of impedances in order to eliminate all reflections. It is not sufficient to simply insert a "characteristic impedance" between each conductor and the reference conductor; there must also be impedances between every pair of conductors. In order to describe the degree of mismatch of a particular load impedance on a two-conductor line, we compute a scalar reflection coefficient. In the case of a MTL, we can obtain the analogous quantity but it becomes a *reflection coefficient matrix*. On a two-conductor line there are forward- and backward-traveling waves each traveling in opposite directions with velocity v. In the case of a MTL consisting of $(n + 1)$ conductors, there exist n forward- and n backward-traveling waves each with its own velocity. Each pair of forward- and backward-traveling waves is referred to as a *mode*. If the MTL is immersed in a homogeneous medium, each mode velocity is identical to the velocity of light in that medium. Each mode velocity of a MTL that is immersed in an inhomogeneous medium (such as wires with dielectric insulations) will, in general, be different. The governing transmission-line equations for a two-conductor line will be a coupled set of two, first-order partial differential equations for the line voltage, $V(z, t)$, and line current, $I(z, t)$, where the line conductors are parallel to the z axis and time is denoted as t. Solution of these coupled, *scalar* equations is straightforward. In the case of a MTL consisting of $(n + 1)$ conductors parallel to the z axis, the corresponding governing equations are a coupled set of $2n$, first-order, *matrix* partial differential equations relating the n line voltages, $V_i(z, t)$, and n line currents, $I_i(z, t)$, for $i = 1, 2, \ldots, n$. The number of conductors may be quite large, e.g., $(n + 1) = 100$, in which case efficiency of solution of the $2n$ MTL equations becomes an important consideration. The efficiency of solution of the MTL equations depends upon the assumptions or approximations one is willing to make about the line, e.g., lossless vs. lossy line, homogeneous vs. inhomogeneous surrounding media, etc., as well as the solution technique chosen. Although it is tempting to dismiss the analysis of MTLs as simply being a special case of two-conductor lines thereby not requiring scrutiny, this is not the case. *The purpose of this text is to examine the common solution techniques for the MTL equations.* This makes it clear that a seemingly new solution technique may simply be a version of an existing technique.

The analysis of a MTL for the resulting n line voltages, $V_i(z, t)$, and n line currents, $I_i(z, t)$, is a three-step process.

STEP 1: *Determine the per-unit-length parameters of inductance, capacitance, conductance and resistance for the given line.* All cross-sectional information about the particular line that distinguishes it from some other line is contained in these per-unit-length parameters. The MTL equations are identical in form for all lines: only the per-unit-length parameters are different. Without a determination of the per-unit-length parameters for the specific line, one cannot solve the resulting MTL equations because the coefficients in those equations will be unknown.

STEP 2: *Solve the resulting MTL equations.* For a two-conductor line, the general solution consists of the sum of forward- and backward-traveling waves *with 2 unknown coefficients.* For a MTL consisting of $(n + 1)$ conductors the general solution consists of the sum of n forward- and n backward-traveling waves *with $2n$ unknown coefficients.*

STEP 3: *Incorporate the terminal conditions to determine the unknown coefficients in the general form of the solution.* A transmission line will have terminations at the left and right ends consisting of independent voltage and/or current sources and lumped elements such as resistors, capacitors, inductors, diodes, transistors, etc. These *terminal constraints* provide the additional $2n$ equations

(n for the left termination and n for the right termination) which can be used to explicitly determine the $2n$ undetermined coefficients in the general form of the MTL equation solution that was obtained in Step 2. The excitation for the MTL will have several forms. Independent lumped sources within the termination networks are one method of exciting the line. These sources are intended to be coupled to the endpoint of that line. However, the electromagnetic fields associated with the current and voltage on that line interact with neighboring lines inducing signals at those endpoints. This coupling is *unintentional* and is referred to as *crosstalk*. Another method of exciting a line is with an incident electromagnetic field as with a radio signal or a lightning pulse. This form of excitation produces sources that are distributed along the line and will also induce unintentional signals at the line endpoints that may cause *interference*. Lumped sources can occur at discrete points along the line as with the direct attachment of a lightning stroke. The effect of incident fields either distributed along the line or at discrete points will be included in the MTL equations. This type of excitation will be deferred to Chapter 7. Lumped sources in the termination networks will constitute the primary excitations up to that point, and their effect will be included in the terminal network characterizations.

In order to obtain the complete solution for the line voltages and currents, *each of the above three steps must be performed* and generally in the above order. Throughout our discussions this sequence of solution steps must be kept in mind and no steps can be omitted. It is as important to be able to determine the per-unit-length parameters for the particular line as it is to obtain the general form of the solution of the MTL equations!

Electromagnetic fields are, in reality, distributed continuously throughout space. If a structure's largest dimension is electrically small, i.e., much less than a wavelength, we can approximately lump the EM effects into circuit elements as in lumped-circuit theory and define alternative variables of interest such as voltages and currents. The transmission-line formulation views the line as a distributed-parameter structure along the structure axis and thereby extends the lumped-circuit analysis techniques to structures that are electrically large in this dimension. However, the cross-sectional dimensions, e.g., conductor separations, must be electrically small in order for the analysis to yield valid results. *The fundamental assumption for all transmission-line formulations and analyses, whether it be for a two-conductor line or a MTL, is that the field structure surrounding the conductors obeys a Transverse ElectroMagnetic or TEM structure.* A TEM field structure is one in which the electric and magnetic fields in the space surrounding the line conductors are transverse or perpendicular to the line axis which will be chosen to be the z axis of a rectangular coordinate system. The waves on such lines are said to propagate in the TEM mode. Transmission-line structures having electrically large cross-sectional dimensions have, in addition to the TEM mode of propagation, other higher-order modes of propagation [17–19]. An analysis of these structures using the transmission-line equation formulation would then only predict the TEM mode component and not represent a complete analysis. Other aspects, such as imperfect line conductors, also may invalidate the TEM mode transmission-line equation description. In addition, an assumption that is inherent in the MTL equation formulation is that *the sum of the line currents at any cross section of the line is zero.* In this sense we say that one of the conductors, the *reference conductor*, is the return for the other n currents. Even though the line cross section is electrically small, it may not be true that the currents sum to zero at any cross section; there may be other currents in existence on the line conductors [20–23]. Presence of nearby conductors or other metallic structures which are not included in the formulation may cause these additional currents [24]. Asymmetries in the physical terminal excitation such as offset source positions (which are implicitly ignored in the terminal representation) can also create these non-TEM currents [24]. It is important to understand these restrictions on the applicability of the representation and the validity of the results obtained from it, and those aspects will also be discussed in this text.

Although there is a voluminous base of references for this topic, important ones will be referenced, where appropriate, by [x]. These are grouped into two categories—those by the author (grouped by category) and other references. References consisting of publications on this topic by the author are listed at the end of the text and are grouped by category. Additional references will be listed at the end of each chapter. Limited numbers of problems are given at the end of each chapter to provide the reader with exercises for illumination of the important points and techniques.

It is important to remind the reader that the sole purpose of this text is to present a complete and concise description of *methods for solving the MTL equations that describe a MTL under the assumption of the TEM mode of propagation.* Therefore we will derive and solve *only* the MTL equations. A complete solution of the MTL structure which does not presuppose only the TEM mode can be obtained with so-called full-wave solutions of Maxwell's equations [17–19]. Generally these techniques require numerical methods for their solution. Our goal will be to examine methods for solving the MTL partial differential equations. So the effects of non-TEM field structures will not be considered. However, for parallel lines wherein the cross-sectional dimensions are much less than a wavelength, the solution of the MTL equations gives the significant contribution to the fields and resulting terminal voltages and currents. This is referred to as the quasi-TEM approximation and is an implicit assumption throughout this text.

1.1 EXAMPLES OF MULTICONDUCTOR TRANSMISSION-LINE STRUCTURES

There are a number of examples of wave-guiding structures that may be viewed as "transmission lines." Figure 1.1 shows examples of $(n + 1)$-conductor wire-type lines consisting of parallel wires. Throughout this text we will refer to conductors that have circular cylindrical cross sections as being *wires.* Figure

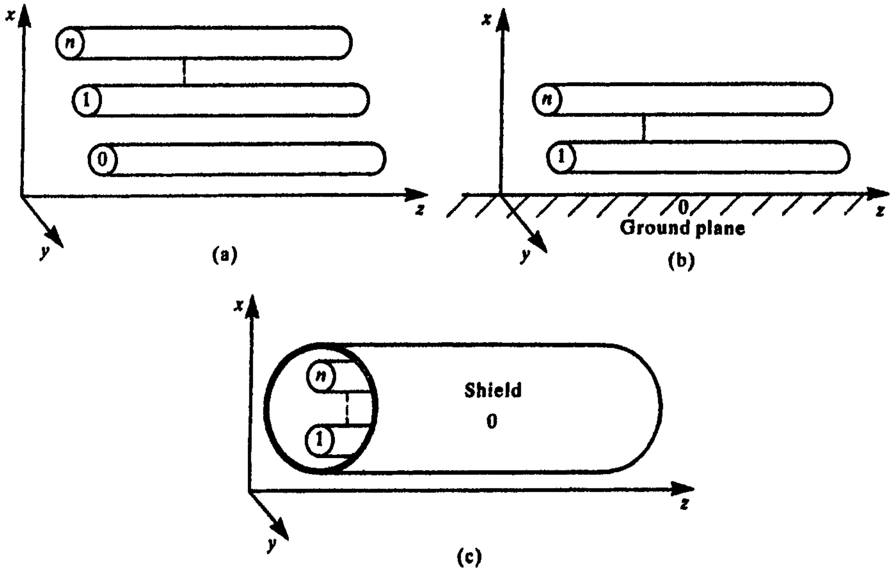

FIGURE 1.1 Multiconductor lines in homogeneous media: (a) $(n + 1)$-wire line, (b) n wires above a ground plane, (c) n wires within a cylindrical shield.

1.1(a) shows an example of $(n + 1)$ wires. Typical examples of such lines are flatpack or ribbon cables used to interconnect electronic systems. Normally these wires are surrounded by circular cylindrical dielectric insulations. However, these insulations are omitted from this figure, and, in some cases, may be ignored in the analysis of such lines. Figure 1.1(b) shows n wires above an infinite, perfectly conducting ground plane. Typical examples are cables which have a metallic structure as a return or high-voltage power distribution lines. In the case of high-voltage distribution lines, the return path is earth. Figure 1.1(c) shows n wires within an overall, cylindrical shield. Shields are often placed around cables in order to prevent or reduce the coupling of electromagnetic fields to the cable from adjacent cables (crosstalk) or from distant sources such as radar transmitters or radio and television stations. The wires in each of these structures are shown as being of *uniform cross section* along their length and parallel to each other (as well as the ground plane in Fig. 1.1(b) and the shield axis in Fig. 1.1(b)). Such lines are said to be *uniform lines.* Nonuniform lines in which either the conductors are not of uniform cross section along their length or are not parallel arise from either nonintentional or intentional reasons. For example, the conductors of a high-voltage power distribution line, because of their weight, sag and are not parallel to the ground. Tapered lines are intentionally designed to give certain desirable characteristics in microwave filters.

The lines in Fig. 1.1 are said to be immersed in a homogeneous medium

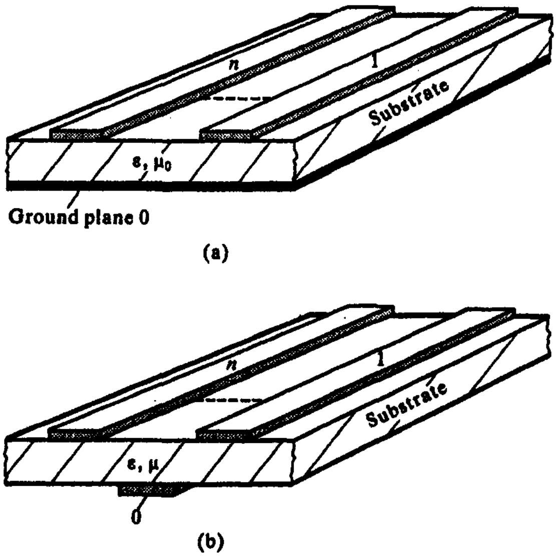

FIGURE 1.2 Multiconductor lines in inhomogeneous media, n lands on a printed circuit board (PCB): (a) n lands with a ground plane as reference, (b) $(n + 1)$ lands.

(logically free space since any dielectric insulations are not shown or ignored). There exist many useful transmission-line structures wherein the dielectric surrounding the conductors cannot be similarly ignored. Figure 1.2 shows examples of these. Figure 1.2(a) shows a structure having n conductors of rectangular cross section or *lands* supported on a dielectric substrate. A perfectly conducting, infinite ground plane covers the lower surface of the substrate. This is referred to in microwave literature as a *coupled microstrip* and is used to construct microwave filters. Figure 1.2(b) shows a similar structure where the ground plane is replaced by another land of rectangular cross section. This type of structure is common on printed circuit boards (PCB's) in modern electronic circuits. This type of structure is used to construct busses that carry digital data or control signals.

The structures in Fig. 1.1 are, by implication, immersed in a homogeneous medium. Therefore the velocity of propagation of the waves is equal to that of the medium in which it is immersed or $v = 1/\sqrt{\mu\varepsilon}$ where μ is the *permeability* of the surrounding medium and ε is the *permittivity* of the surrounding medium. For free space these become $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 \approx (1/36\pi) \times 10^{-9}$ F/m. For the structures shown in Fig. 1.2 which are immersed in an inhomogeneous medium (the fields exist partly in free space and partly in the substrate), there are n waves or *modes* whose velocities are, in general, different. This complicates the analysis of such structures as we will see.

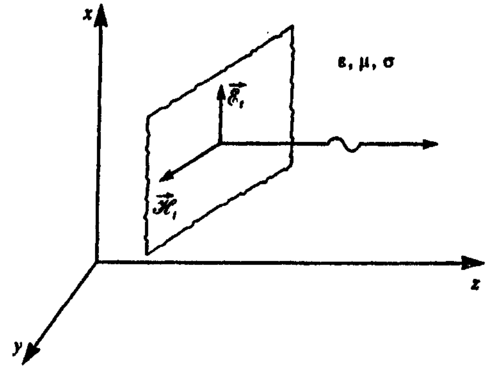

FIGURE 1.3 Illustration of the electromagnetic field structure of the transverse electromagnetic (TEM) mode of propagation.

1.2 PROPERTIES OF THE TRANSVERSE ELECTROMAGNETIC (TEM) MODE OF PROPAGATION

As mentioned previously, the fundamental assumption in any transmission-line formulation is that the electric field intensity vector, $\vec{\mathscr{E}}(x, y, z, t)$, and the magnetic field intensity vector, $\vec{\mathscr{H}}(x, y, z, t)$, satisfy the transverse electromagnetic (TEM) field structure, i.e., they lie in a plane (the x–y plane) transverse or perpendicular to the line axis (the z axis). Therefore it is appropriate to examine the general properties of this TEM mode of propagation or field structure.

Consider a rectangular coordinate system shown in Fig. 1.3 illustrating a propagating TEM wave in which the field vectors are assumed to lie in a plane (the x–y plane) that is transverse to the direction of propagation (the z axis). These field vectors are denoted with a t subscript to denote *transverse*. It is assumed that the medium is homogeneous, linear and isotropic and is characterized by the scalar parameters of *permittivity*, ε, *permeability*, μ, and *conductivity*, σ. Maxwell's equations become [A.1]

$$\nabla \times \vec{\mathscr{E}}_t = -\mu \frac{\partial \vec{\mathscr{H}}_t}{\partial t} \tag{1.1a}$$

$$\nabla \times \vec{\mathscr{H}}_t = \sigma \vec{\mathscr{E}}_t + \varepsilon \frac{\partial \vec{\mathscr{E}}_t}{\partial t} \tag{1.1b}$$

The del operator, ∇, can be broken into two components, one component, ∇_z, in the z direction and one component, ∇_t, in the transverse plane as

$$\nabla = \nabla_t + \nabla_z \tag{1.2a}$$

where

$$\nabla_t = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} \tag{1.2b}$$

$$\nabla_z = \hat{a}_z \frac{\partial}{\partial z} \tag{1.2c}$$

where $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ are unit vectors pointing in the appropriate directions. Separating (1.1) by equating those components in the z direction and in the transverse plane gives

$$\hat{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial z} = -\mu \frac{\partial \vec{\mathscr{H}}_t}{\partial t} \tag{1.3a}$$

$$\hat{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial z} = \sigma \vec{\mathscr{E}}_t + \varepsilon \frac{\partial \vec{\mathscr{E}}_t}{\partial t} \tag{1.3b}$$

$$\nabla_t \times \vec{\mathscr{E}}_t = 0 \tag{1.3c}$$

$$\nabla_t \times \vec{\mathscr{H}}_t = 0 \tag{1.3d}$$

Equations (1.3c) and (1.3d) are identical to those for *static fields.* This shows that *the electric and magnetic fields of a TEM field distribution satisfy a static distribution in the transverse plane.* Because of (1.3c) and (1.3d), we may define each of the transverse field vectors as the gradients of some auxiliary scalar fields or *potential functions,* ϕ and ψ, such as [A.1]

$$\vec{\mathscr{E}}_t = g(z, t)\nabla_t \phi(x, y) \tag{1.4a}$$

$$\vec{\mathscr{H}}_t = f(z, t)\nabla_t \psi(x, y) \tag{1.4b}$$

The scalar coefficients, $g(z, t)$ and $f(z, t)$, are to be determined. Gauss' laws become [A.1]

$$\nabla_t \cdot \vec{\mathscr{E}}_t = 0 \tag{1.5a}$$

$$\nabla_t \cdot \vec{\mathscr{H}}_t = 0 \tag{1.5b}$$

Applying (1.5) to (1.4) gives

$$\nabla_t \cdot \nabla_t \phi(x, y) = \nabla_t^2 \phi(x, y) = 0 \tag{1.6a}$$

$$\nabla_t \cdot \nabla_t \psi(x, y) = \nabla_t^2 \psi(x, y) = 0 \tag{1.6b}$$

Equations (1.6) show that the auxiliary scalar potential functions satisfy Laplace's equation in any transverse plane as they do for static fields. This permits the *unique* definition of voltage between two points in a transverse plane as the line integral of the transverse electric field between those two points:

$$V(z, t) = -\int_{P_1}^{P_2} \vec{\mathscr{E}}_t \cdot d\vec{l} \tag{1.7a}$$

This is a case where we may uniquely define voltage between two points for nonstatic time variation. Similarly, equation (1.6b) shows that we may uniquely define current in the z direction as the line integral of the transverse magnetic field around any closed contour lying solely in the transverse plane:

$$I(z, t) = \oint_{c_t} \vec{\mathscr{H}}_t \cdot d\vec{l} \tag{1.7b}$$

Ordinarily the line integral in (1.7b) contains conduction current, $\sigma\vec{\mathscr{E}}$, and any source current, J_S, as well as displacement current, $\varepsilon(\partial\vec{\mathscr{E}}/\partial t)$, due to the time rate-of-change of the electric field penetrating the surface bounded by the contour. Since the electric field is confined to the transverse plane and therefore has no z component, the conduction current, $\sigma\vec{\mathscr{E}}$, and displacement, $\varepsilon(\partial\vec{\mathscr{E}}/\partial t)$, penetrating the transverse contour, c_t, are zero thereby giving the current definition solely as source currents, such as may exist on the surfaces of conductors that penetrate the surface of this contour, as is the case for static fields. Once again, this permits a unique definition of current for nonstatic variation of the field vectors in a fashion similar to the static or dc case.

Now suppose we take the cross product of the z-directed unit vector with (1.3a) and (1.3b). This gives

$$\vec{a}_z \times \vec{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial z} = -\mu\left[\vec{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial t}\right] \tag{1.8a}$$

$$\vec{a}_z \times \vec{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial z} = \sigma(\vec{a}_z \times \vec{\mathscr{E}}_t) + \varepsilon\left[\vec{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial t}\right] \tag{1.8b}$$

However,

$$\vec{a}_z \times \vec{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial z} = -\frac{\partial \vec{\mathscr{E}}_t}{\partial z} \tag{1.9a}$$

$$\vec{a}_z \times \vec{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial z} = -\frac{\partial \vec{\mathscr{H}}_t}{\partial z} \tag{1.9b}$$

as illustrated in Fig. 1.4. Therefore, equations (1.8) become

$$-\frac{\partial \vec{\mathscr{E}}_t}{\partial z} = -\mu\left(\vec{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial t}\right) \tag{1.10a}$$

$$-\frac{\partial \vec{\mathscr{H}}_t}{\partial z} = \sigma(\vec{a}_z \times \vec{\mathscr{E}}_t) + \varepsilon\left(\vec{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial t}\right) \tag{1.10b}$$

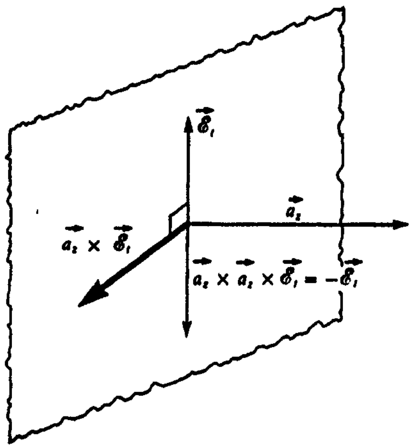

FIGURE 1.4 Illustration of the identity $\vec{a}_z \times \vec{a}_z \times \vec{\mathscr{E}}_t = -\vec{\mathscr{E}}_t$.

Taking the partial derivative of both sides of (1.10) with respect to z and substituting (1.3) gives

$$\frac{\partial^2 \vec{\mathscr{E}}_t}{\partial z^2} = \mu\sigma \frac{\partial \vec{\mathscr{E}}_t}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{\mathscr{E}}_t}{\partial t^2} \tag{1.11a}$$

$$\frac{\partial^2 \vec{\mathscr{H}}_t}{\partial z^2} = \mu\sigma \frac{\partial \vec{\mathscr{H}}_t}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{\mathscr{H}}_t}{\partial t^2} \tag{1.11b}$$

Now let us consider the case where the medium is *lossless*, i.e., $\sigma = 0$. In this case equations (1.11) reduce to

$$\frac{\partial^2 \vec{\mathscr{E}}_t}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \vec{\mathscr{E}}_t}{\partial t^2} \tag{1.12a}$$

$$\frac{\partial^2 \vec{\mathscr{H}}_t}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \vec{\mathscr{H}}_t}{\partial t^2} \tag{1.12b}$$

The solutions to these second-order differential equations are [A.1]

$$\vec{\mathscr{E}}_t(x, y, z, t) = \vec{\mathscr{E}}^+\left(x, y, t - \frac{z}{v}\right) + \vec{\mathscr{E}}^-\left(x, y, t + \frac{z}{v}\right) \tag{1.13a}$$

$$\vec{\mathscr{H}}_t(x, y, z, t) = \vec{\mathscr{H}}^+\left(x, y, t - \frac{z}{v}\right) + \vec{\mathscr{H}}^-\left(x, y, t + \frac{z}{v}\right) \tag{1.13b}$$

$$= \frac{1}{\eta} \vec{\mathscr{E}}^+\left(x, y, t - \frac{z}{v}\right) - \frac{1}{\eta} \vec{\mathscr{E}}^-\left(x, y, t + \frac{z}{v}\right)$$

where the *intrinsic impedance* of the medium is

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{1.13c}$$

and the velocity of propagation is

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{1.13d}$$

The function $\vec{\mathscr{E}}^+(x, y, t - z/v)$ represents a *forward-traveling wave* since as t progresses z must increase to keep the argument constant and track corresponding points on the waveform. Similarly, the function $\vec{\mathscr{E}}^-(x, y, t + z/v)$ represents a *backward-traveling wave*, a wave traveling in the $-z$ direction. Consequently we may indicate the vector relation between the electric and magnetic fields as

$$\vec{\mathscr{H}}_t = \pm \frac{1}{\eta} \vec{a}_z \times \vec{\mathscr{E}}_t \tag{1.14}$$

with the sign depending on whether we are considering the backward- or forward-traveling wave component. Equations (1.3a) and (1.3b) show that $\vec{\mathscr{E}}_t$ and $\vec{\mathscr{H}}_t$ are *orthogonal* so that (1.14) applies (with a different intrinsic impedance) even if the medium is lossy.

If the time variation of the field vectors is sinusoidal, we use phasor notation [A.2]:

$$\vec{\mathscr{E}}_t(x, y, z, t) = \mathscr{R}e\{\vec{E}_t(x, y, z)e^{j\omega t}\} \tag{1.15a}$$

$$\vec{\mathscr{H}}_t(x, y, z, t) = \mathscr{R}e\{\vec{H}_t(x, y, z)e^{j\omega t}\} \tag{1.15b}$$

Replacing time derivatives with $\partial/\partial t = j\omega$ in (1.12) gives the *phasor form* of the differential equations:

$$\frac{\partial^2 \vec{E}_t}{\partial z^2} = -\omega^2 \mu\varepsilon \vec{E}_t \tag{1.16a}$$

$$\frac{\partial^2 \vec{H}_t}{\partial z^2} = -\omega^2 \mu\varepsilon \vec{H}_t \tag{1.16b}$$

The solutions to these equations become [A.1]

$$\vec{E}_t(x, y, z) = \vec{E}^+(x, y)e^{-j\beta z} + \vec{E}^-(x, y)e^{j\beta z} \tag{1.17a}$$

$$\vec{H}_t(x, y, z) = \vec{H}^+(x, y)e^{-j\beta z} + \vec{H}^-(x, y)e^{j\beta z} \tag{1.17b}$$

$$= \frac{1}{\eta} \vec{E}^+(x, y)e^{-j\beta z} - \frac{1}{\eta} \vec{E}^-(x, y)e^{j\beta z}$$

where

$$\vec{E}^+ = -\frac{1}{\eta}\,\vec{a}_z \times \vec{H}^+ \tag{1.17c}$$

$$\vec{E}^- = \frac{1}{\eta}\,\vec{a}_z \times \vec{H}^- \tag{1.17d}$$

and the phase constant is denoted as

$$\beta = \omega\sqrt{\mu\varepsilon} \tag{1.17e}$$

The time-domain expressions are obtained by multiplying (1.17a) and (1.17b) by $e^{j\omega t}$ and taking the real part of the result [A.1]. For example, the x components of the transverse field vectors are:

$$\mathscr{E}_x(x, y, z, t) = E_{mx}^+ \cos(\omega t - \beta z + \theta_x^+) + E_{mx}^- \cos(\omega t + \beta z + \theta_x^-) \tag{1.18a}$$

$$\mathscr{H}_y(x, y, z, t) = \frac{1}{\eta} E_{mx}^+ \cos(\omega t - \beta z + \theta_x^+) - \frac{1}{\eta} E_{mx}^- \cos(\omega t + \beta z + \theta_x^-) \tag{1.18b}$$

where the x components of the complex components of $\vec{E}^\pm$ are denoted as $E_x^\pm = E_{mx}^\pm \angle \theta_x^\pm$.

If we now consider adding conductive losses to the medium, $\sigma \neq 0$, this adds a transverse conductive current term, $\vec{\mathscr{J}}_t = \sigma\vec{\mathscr{E}}_t$, to Ampere's law, equation (1.1b). The second-order differential equations become as shown in (1.11). In the case of sinusoidal excitation we obtain

$$\frac{d^2\vec{E}_t}{dz^2} = \gamma^2 \vec{E}_t \tag{1.19a}$$

$$\frac{d^2\vec{H}_t}{dz^2} = \gamma^2 \vec{H}_t \tag{1.19b}$$

where the *propagation constant* is

$$\begin{aligned}\gamma &= \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \\ &= \alpha + j\beta\end{aligned} \tag{1.19c}$$

The phasor solutions in (1.17) for the lossless medium case become

$$\vec{E}_t(x, y, z) = \vec{E}^+(x, y)e^{-\alpha z}e^{-j\beta z} + \vec{E}^-(x, y)e^{\alpha z}e^{j\beta z} \tag{1.20a}$$

$$\vec{H}_t(x, y, z) = \frac{1}{\eta}\,\vec{E}^+(x, y)e^{-\alpha z}e^{-j\beta z} - \frac{1}{\eta}\,\vec{E}^-(x, y)e^{\alpha z}e^{j\beta z} \tag{1.20b}$$

where the intrinsic impedance now becomes

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \tag{1.20c}$$

Thus, in addition to a phase shift represented by $e^{\pm j\beta z}$, the waves suffer an *attenuation* represented by $e^{\pm \alpha z}$. We find these properties of the TEM mode of propagation arising throughout our examination of MTLs in various guises.

1.3 DERIVATION OF THE TRANSMISSION-LINE EQUATIONS FOR TWO-CONDUCTOR LINES

The transmission-line equations are usually derived from a representation of the line as lumped circuit elements distributed along the line. While this gives the desired equations, a number of subtle aspects are obscured. In this section, we will derive the transmission-line equations for a general two-conductor line by three methods:

1. *From the integral forms of Maxwell's equations.*
2. *From the differential forms of Maxwell's equations.*
3. *From the usual distributed parameter, per-unit-length equivalent circuit.*

1.3.1 Derivation from the Integral Form of Maxwell's Equations

Consider a two-conductor transmission line shown in Fig. 1.5(a). We assume that:

1. The conductors are parallel to each other and the z axis.
2. The conductors are perfect conductors.
3. The conductors have uniform cross sections along the line axis.

Because of the first and third properties this is said to be a *uniform line.* The medium surrounding the conductors may be lossy which is represented by a nonzero conductivity, σ, and is homogeneous in σ, ε, and μ. Maxwell's equations in integral form are [A.1]

$$\oint_c \vec{\mathscr{E}} \cdot d\vec{l} = -\mu \frac{d}{dt} \iint_s \vec{\mathscr{H}} \cdot d\vec{s} \tag{1.21a}$$

$$\oint_c \vec{\mathscr{H}} \cdot d\vec{l} = \iint_s \vec{\mathscr{J}} \cdot d\vec{s} + \varepsilon \frac{d}{dt} \iint_s \vec{\mathscr{E}} \cdot d\vec{s} \tag{1.21b}$$

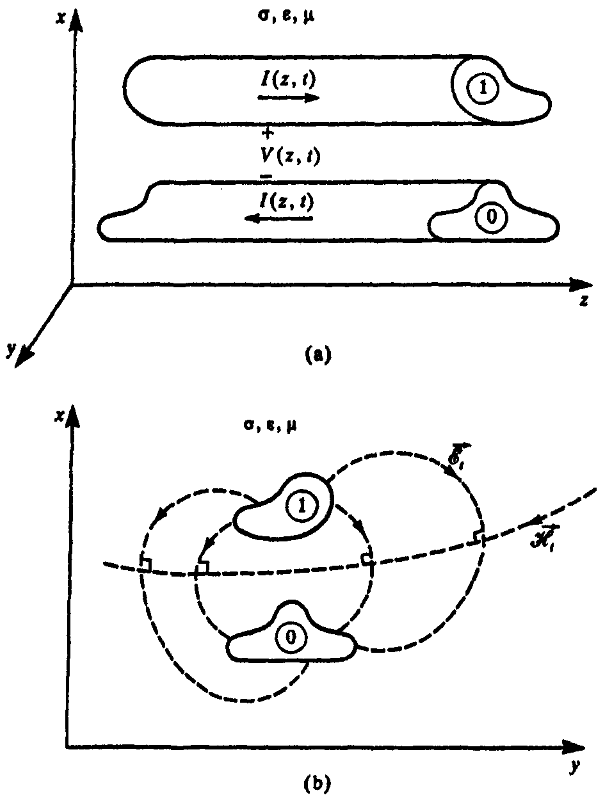

FIGURE 1.5 Illustration of (a) the current and voltage and (b) the TEM fields for a two-conductor line.

Equation (1.21a) is referred to as *Faraday's law*, and equation (1.21b) is referred to as *Ampere's law*. Open surface s is enclosed by the closed contour c and the directions are related by the right-hand rule [A.1]. The quantity $\vec{\mathscr{J}}$ is a current density in A/m and contains conduction current, $\vec{\mathscr{J}}_c = \sigma\vec{\mathscr{E}}$, as well as any source current, $\vec{\mathscr{J}}_s$, as $\vec{\mathscr{J}} = \vec{\mathscr{J}}_c + \vec{\mathscr{J}}_s$.

We will assume the TEM field structure about the conductors in any cross-sectional plane as indicated in Fig. 1.5(b). If we choose the contour in (1.21) to lie solely in the cross-sectional plane, c_{xy}, and the surface enclosed to be a flat surface in the transverse plane, s_{xy}, then (1.21) becomes

$$\oint_{c_{xy}} (\mathscr{E}_x\,dx + \mathscr{E}_y\,dy) = -\mu\frac{d}{dt}\iint_{s_{xy}} \mathscr{H}_z\,dx\,dz \qquad (1.22a)$$
$$= 0$$

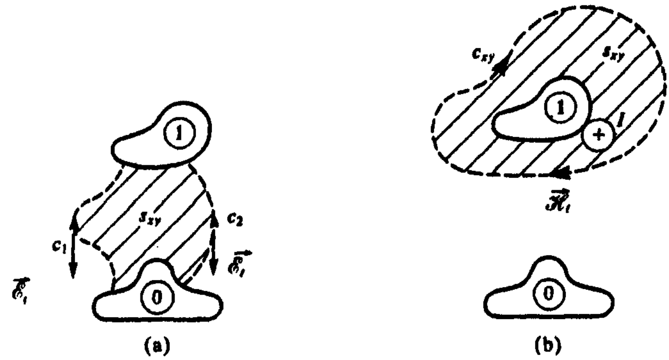

FIGURE 1.6 Definitions of (a) voltage and (b) current for a two-conductor line.

$$\oint_{c_{xy}} (\mathscr{H}_x\,dx + \mathscr{H}_y\,dy) = \iint_{s_{xy}} \mathscr{J}_z\,dx\,dy + \frac{d}{dt}\iint_{s_{xy}} \mathscr{E}_z\,dx\,dy \qquad (1.22b)$$
$$= \iint_{s_{xy}} \mathscr{J}_{sz}\,dx\,dy$$

Observe that the right-hand side of Faraday's law, (1.22a), is zero because there are, by the TEM assumption, no z-directed fields so that $\mathscr{H}_z = 0$. Similarly, by the TEM assumption, $\mathscr{E}_z = 0$, and there is no z-directed conduction or displacement current, only z-directed source currents, $\mathscr{J}_{sz}$. Thus Ampere's law, (1.22b), simplifies as shown. Observe that *equations* (1.22) *are identical to those for static* (*dc*) *time variation*. Therefore, we may uniquely define voltage between the two conductors, independent of path, so long as we take the path to lie in a transverse plane:

$$V(z, t) = -\int_0^1 \vec{\mathscr{E}}_t \cdot d\vec{l} \qquad (1.23)$$

Figure 1.6(a) illustrates this point. We can choose either contour c_1 or c_2 for the definition of voltage. Since the conductors are perfect conductors, their surfaces are equipotential surfaces so that the contours can terminate at different points on them. Furthermore, by the TEM assumption, there is no component of the magnetic field penetrating the surface bounded by the contour enclosed by these two paths and the conductor surfaces which makes the voltage definition in (1.23) unique. Similarly, (1.22b) allows the unique definition of current as illustrated in Fig. 1.6(b). Choosing a closed contour in the transverse plane encircling one of the conductors gives the current on that conductor as

$$I(z, t) = \oint_{c_{xy}} \vec{\mathscr{H}}_t \cdot d\vec{l} \qquad (1.24)$$

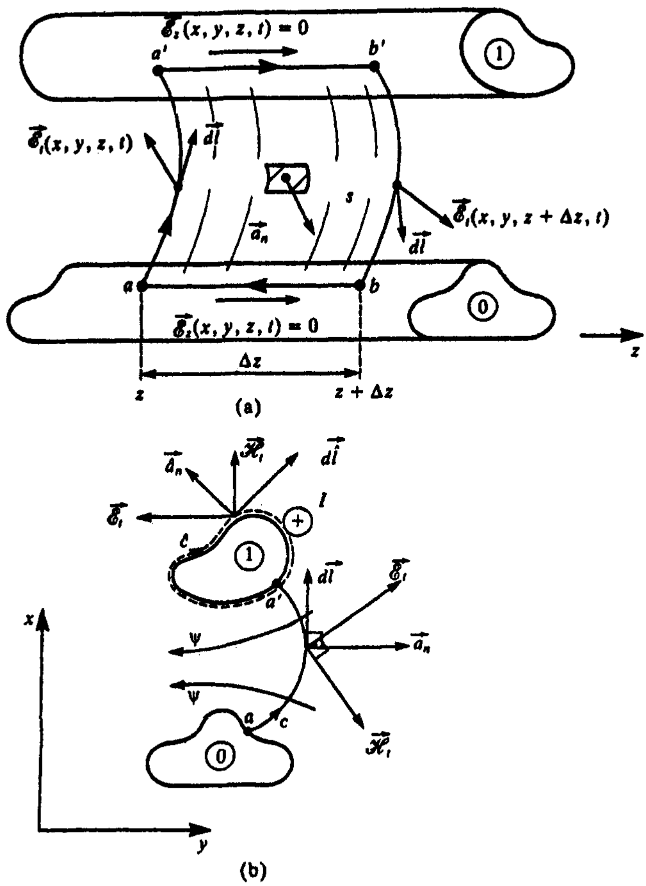

FIGURE 1.7 Contours for the derivation of the first transmission-line equation: (a) longitudinal plane, (b) transverse plane.

This is unique because there is no z-directed electric field, $\mathscr{E}_z = 0$, so that no conduction or displacement current penetrates the flat surface enclosed by this contour. The current so defined by (1.24) lies solely on the surface of the perfect conductor. If we enclose both conductors with the contour, it can be shown, because of (1.22b), that the net current is zero, i.e., the current at any cross section on the lower conductor is equal and opposite to the current on the upper conductor. (See Problem 1.3 at the end of the chapter.)

We now turn to the derivation of the transmission-line equations in terms of the voltage and current defined above. Consider Fig. 1.7(a) where we have chosen an open surface, s, of uniform cross section in the z direction around which we integrate Faraday's law. The unit normal to this surface lies in the x–y (transverse) plane and is denoted by $\vec{a}_n$. Integrating Faraday's law around this contour gives

$$\int_a^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_{a'}^{b'} \vec{\mathscr{E}}_z \cdot d\vec{l} + \int_{b'}^{b} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_b^{a} \vec{\mathscr{E}}_z \cdot d\vec{l} = \mu \frac{d}{dt} \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \qquad (1.25)$$

Observe that the second and fourth integrals on the left-hand side are zero since these are along the surfaces of the perfectly conducting conductors. Also note that the negative sign usually present on the right-hand side of Faraday's law is absent here. This is because of the direction chosen for the line integral, the choice of direction for the unit normal vector, $\vec{a}_n$, and the right-hand rule. Defining the voltages between the two conductors as in (1.23) gives

$$V(z + \Delta z, t) = -\int_b^{b'} \vec{\mathscr{E}}_t(x, y, z + \Delta z, t) \cdot d\vec{l} \qquad (1.26a)$$

$$V(z, t) = -\int_a^{a'} \vec{\mathscr{E}}_t(x, y, z, t) \cdot d\vec{l} \qquad (1.26b)$$

Therefore, (1.25) becomes

$$-V(z, t) + V(z + \Delta z, t) = \mu \frac{d}{dt} \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \qquad (1.27)$$

Rewriting this gives

$$\frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = \frac{1}{\Delta z} \mu \frac{d}{dt} \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \qquad (1.28)$$

Taking the limit as $\Delta z \to 0$ gives

$$\frac{\partial V(z, t)}{\partial z} = \mu \frac{d}{dt} \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds \qquad (1.29)$$

The right-hand side of (1.29) can be interpreted as an *inductance* of the loop formed between the two conductors. In order to do this, consider Fig. 1.7(b). The current, $I(z, t)$, is again defined by

$$I(z, t) = \oint_c \vec{\mathscr{H}}_t \cdot d\vec{l} \qquad (1.30)$$

Therefore, the inductance for a Δz section is

$$L = \frac{\psi}{I} \tag{1.31}$$

$$= \frac{-\mu \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds}{I}$$

A *per-unit-length inductance*, l, can be defined at any cross section (since the line is uniform) as

$$l = \lim_{\Delta z \to 0} \frac{L}{\Delta z} \tag{1.32}$$

$$= -\frac{\mu}{I} \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds$$

$$= -\mu \frac{\int_a^{a'} \vec{\mathscr{H}}_t \cdot \vec{a}_n \, dl}{\oint_{\hat{c}} \vec{\mathscr{H}}_t \cdot d\vec{l}}$$

This, combined with (1.29) gives the first *transmission-line equation*:

$$\frac{\partial V(z, t)}{\partial z} = -l \frac{\partial I(z, t)}{\partial t} \tag{1.33}$$

We now turn our attention to the derivation of the second and remaining transmission-line equation. Recall the *continuity equation* which states that *the net outflow of current from some closed surface equals the time rate of decrease of the charge enclosed by that surface*:

$$\oint\!\!\!\oint_s \vec{\mathscr{J}} \cdot d\vec{s} = -\frac{d}{dt} \iiint_v \rho \, dv \tag{1.34}$$

$$= -\frac{d}{dt} Q_{enc}$$

Enclose each conductor with a closed surface, $\hat{s}$, of length Δz just off the surface of the conductor as shown in Fig. 1.8(a). Integrating the continuity equation over this closed surface gives

$$\iint_{\hat{s}_e} \vec{\mathscr{J}} \cdot d\hat{s} + \iint_{\hat{s}_0} \vec{\mathscr{J}} \cdot d\hat{s} = -\frac{d}{dt} Q_{enc} \tag{1.35}$$

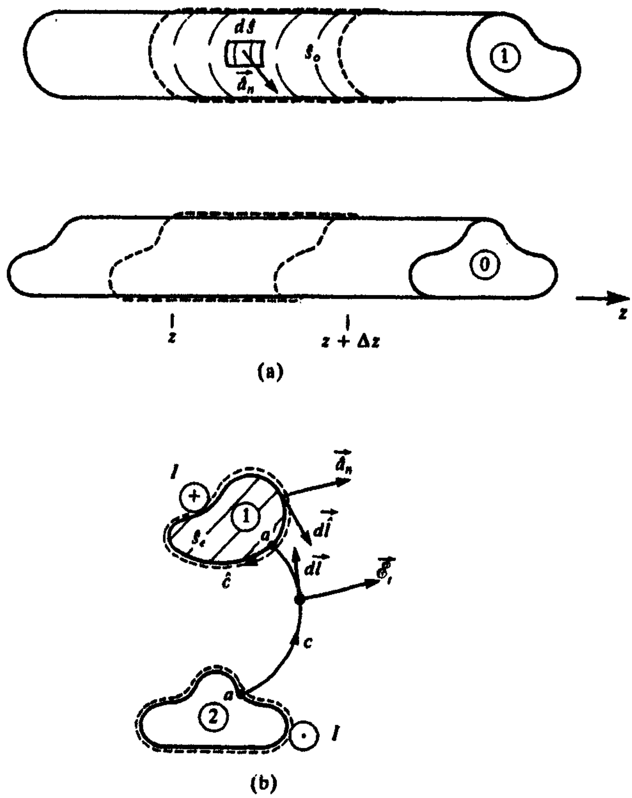

FIGURE 1.8 Contours for the derivation of the second transmission-line equation: (a) longitudinal plane, (b) transverse plane.

The portion of this closed surface over the ends is denoted by $\mathcal{S}_e$ whereas the portion of the surface over the sides is denoted by $\mathcal{S}_0$. The terms in (1.35) become

$$\iint_{\mathcal{S}_e} \vec{\mathcal{J}} \cdot d\vec{s} = I(z + \Delta z, t) - I(z, t) \tag{1.36a}$$

$$\iint_{\mathcal{S}_0} \vec{\mathcal{J}} \cdot d\vec{s} = \sigma \iint_{\mathcal{S}_0} \vec{\mathcal{E}}_t \cdot d\vec{s} \tag{1.36b}$$

The right-hand side of (1.35) can be defined in terms of a *per-unit-length capacitance*. The total charge enclosed by the surface is, according to Gauss' law, [A.1]

$$Q_{\text{enc}} = \varepsilon \oiint_{\mathcal{S}} \vec{\mathcal{E}}_t \cdot d\vec{s} \tag{1.37}$$

The capacitance between the conductors for a Δz section of the line is

$$C = \frac{Q_{enc}}{V} \tag{1.38}$$

and the *per-unit-length capacitance* is

$$c = \lim_{\Delta z \to 0} \frac{C}{\Delta z} \tag{1.39}$$

Substituting (1.37) and observing Fig. 1.8(b) gives

$$c = \varepsilon \frac{\oint_{c} \vec{\mathscr{E}}_t \cdot \vec{a}_n \, dl}{-\int_{a}^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l}} \tag{1.40}$$

Similarly, a conductance between the two conductors for a length of Δz may be defined as

$$G = \frac{\iint_{s_o} \vec{\mathscr{J}} \cdot d\vec{s}}{V(z, t)} \tag{1.41}$$

This leads, from (1.36b), to the definition of a *per-unit-length conductance* as

$$g = \lim_{\Delta z \to 0} \frac{G}{\Delta z} \tag{1.42}$$

$$= \sigma \frac{\oint_{c} \vec{\mathscr{E}}_t \cdot \vec{a}_n \, dl}{-\int_{a}^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l}}$$

Substituting (1.36a), (1.40) and (1.42) into (1.35), dividing both sides by Δz, and taking the limit as $\Delta z \to 0$ gives the second and last *transmission-line equation*:

$$\frac{\partial I(z, t)}{\partial z} = -gV(z, t) - c\,\frac{\partial V(z, t)}{\partial t} \tag{1.43}$$

Equations (1.33) and (1.43) are referred to as the *transmission-line equations* and represent a coupled set of first-order, partial differential equations in the line voltage, $V(z, t)$, and line current, $I(z, t)$. Solution of these equations will be one of our goals.

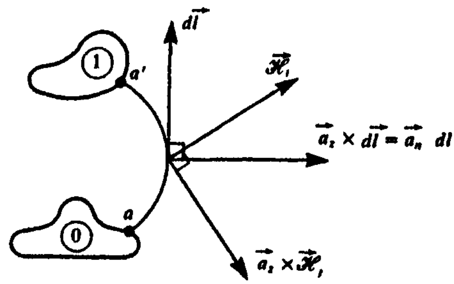

FIGURE 1.9 Illustration of the derivation of certain vector identities.

1.3.2 Derivation from the Differential Form of Maxwell's Equations

We now obtain the transmission-line equations from the differential forms of Maxwell's equations. We showed previously in (1.10a) and (1.10b) that, for the TEM mode of propagation, the transverse field vectors satisfy

$$\frac{\partial \vec{\mathscr{E}}_t}{\partial z} = \mu\left[\vec{a}_z \times \frac{\partial \vec{\mathscr{H}}_t}{\partial t}\right] \tag{1.44a}$$

$$\frac{\partial \vec{\mathscr{H}}_t}{\partial z} = -\sigma(\vec{a}_z \times \vec{\mathscr{E}}_t) - \varepsilon\left[\vec{a}_z \times \frac{\partial \vec{\mathscr{E}}_t}{\partial t}\right] \tag{1.44b}$$

Performing the line integral of both sides of (1.44a) between points a and a' on the conductors along a path in the transverse plane and recalling the definition of voltage given in (1.26b) yields

$$\frac{\partial V(z,t)}{\partial z} = -\mu \frac{\partial}{\partial t}\int_a^{a'} (\vec{a}_z \times \vec{\mathscr{H}}_t)\cdot \vec{dl} \tag{1.45}$$

From Fig. 1.9 we see the following identities:

$$(\vec{a}_z \times \vec{\mathscr{H}}_t)\cdot \vec{dl} = -\vec{\mathscr{H}}_t \cdot (\vec{a}_z \times \vec{dl}) \tag{1.46}$$

$$\vec{a}_n = \vec{a}_z \times \frac{\vec{dl}}{dl} \tag{1.47}$$

Substituting (1.46) and (1.47) into (1.45) and recalling the definition of per-unit-length inductance given in (1.32) yields the first transmission-line equation given in (1.33).

The second transmission-line equation is obtained from (1.3b). Performing the line integral over both sides between points a and a' on the two conductors yields

$$\frac{\partial}{\partial z}\int_a^{a'} (\vec{a}_z \times \vec{\mathscr{H}}_t)\cdot d\vec{l} = \sigma \int_a^{a'} \vec{\mathscr{E}}_t\cdot d\vec{l} + \varepsilon \frac{\partial}{\partial t}\int_a^{a'} \vec{\mathscr{E}}_t\cdot d\vec{l} \tag{1.48}$$

Using the identities in (1.46) and (1.47) in the definition of inductance in (1.32) and observing the definition of voltage given in (1.26b) yields

$$\frac{l}{\mu}\frac{\partial}{\partial z} I(z,t) = -\sigma V(z,t) - \varepsilon \frac{\partial V(z,t)}{\partial t} \tag{1.49}$$

Rewriting gives

$$\frac{\partial I(z,t)}{\partial t} = -\frac{\sigma\mu}{l} V(z,t) - \frac{\mu\varepsilon}{l}\frac{\partial V(z,t)}{\partial t} \tag{1.50}$$

We shall prove the following important identity relating the per-unit-length parameters, g, c, and l for a *homogeneous* surrounding medium as is assumed here:

$$lc = \mu\varepsilon \tag{1.51a}$$

$$gl = \sigma\mu \tag{1.51b}$$

Substituting these into (1.50) gives the second transmission-line equation given in (1.43).

1.3.3 Derivation from the Per-Unit-Length Equivalent Circuit

The previous two derivations of the transmission-line equations were rigorous and illustrated many important concepts and restrictions on the formulation. In this section we will show the usual derivation from a distributed-parameter, lumped circuit. The concept stems from the fact that lumped-circuit concepts are only valid for structures whose largest dimension is electrically small, i.e., much less than a wavelength, at the frequency of excitation. If a structural dimension is electrically large, we may break it into the union of electrically small substructures and can then represent each substructure with a lumped circuit model. In order to apply this to a transmission line, consider breaking it into small, Δz length subsections as illustrated in Fig. 1.10. The per-unit-length inductance, l, derived previously represents the magnetic flux passing between the conductors due to the current on those conductors. We may lump this in each Δz subsection by multiplying the per-unit-length parameter by Δz. Since the line is assumed to be a uniform one, this can be done for all such subsections as shown in Fig. 1.10. Similarly, the per-unit-length capacitance, c, represents the displacement current flowing between the two conductors and can be similarly lumped in each subsection. The per-unit-length conductance, g,

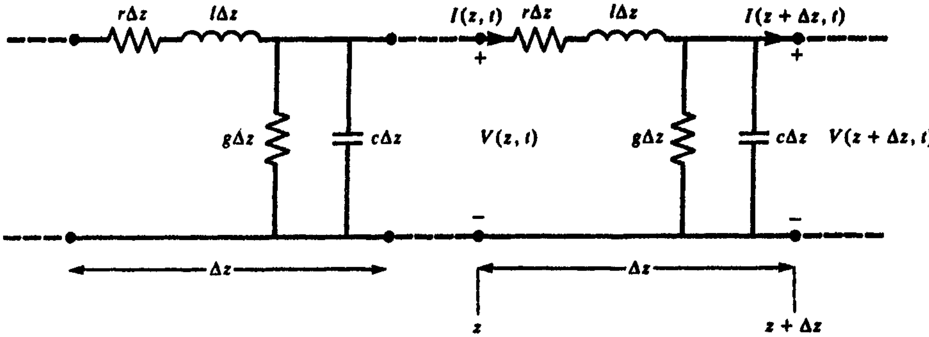

FIGURE 1.10 The per-unit-length model for use in deriving the transmission-line equations.

represents the transverse conduction current flowing between the two conductors and can be lumped in a similar fashion.

The previous derivations assumed that the two conductors are perfect conductors. Small conductor losses can be handled in this equivalent circuit in an approximate manner by including the per-unit-length resistance, r, (the total for both conductors) in series with the inductance element. The validity of this approximation will be discussed in a later section.

From the per-unit-length equivalent circuit shown in Fig. 1.10, we obtain

$$V(z + \Delta z, t) - V(z, t) = -r\Delta z\, I(z, t) - l\Delta z\, \frac{\partial I(z, t)}{\partial t} \tag{1.52a}$$

Similarly, we obtain

$$I(z + \Delta z, t) - I(z, t) = -g\Delta z\, V(z + \Delta z, t) - c\Delta z\, \frac{\partial V(z + \Delta z, t)}{\partial t} \tag{1.52b}$$

Dividing (1.52a) by Δz and taking the limit as $\Delta z \to 0$ gives the first transmission-line equation:

$$\lim_{\Delta z \to 0} \frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = \frac{\partial V(z, t)}{\partial z} = -r\, I(z, t) - l\, \frac{\partial I(z, t)}{\partial t} \tag{1.53}$$

The second transmission-line equation can be derived from (1.52b) in a similar manner. However, before taking the limit as $\Delta z \to 0$, we should substitute the result for $V(z + \Delta z)$ from (1.52a) into (1.52b) giving

$$\begin{aligned}\frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} &= -g V(z, t) - c\, \frac{\partial V(z, t)}{\partial t} \\ &\quad + \Delta z\left[grI(z, t) + (gl + rc)\, \frac{\partial I(z, t)}{\partial t} + lc\, \frac{\partial^2 I(z, t)}{\partial t^2}\right]\end{aligned} \tag{1.54}$$

Taking the limit of (1.54) as $\Delta z \to 0$ gives the second transmission-line equation:

$$\frac{\partial I(z,t)}{\partial z} = -gV(z,t) - c\,\frac{\partial V(z,t)}{\partial t} \tag{1.55}$$

1.3.4 Properties of the Per-Unit-Length Parameters

The per-unit-length parameters of inductance, l, conductance, g, and capacitance, c, share important properties with each other for a *homogeneous* surrounding medium. These are:

$$lc = \mu\varepsilon \tag{1.56a}$$

$$gl = \sigma\mu \tag{1.56b}$$

In this section we will prove these important identities.

In order to prove (1.56a) we multiply the definitions of l and c given in (1.32) and (1.40), respectively:

$$lc = -\mu\,\frac{\displaystyle\int_a^{a'} \vec{\mathscr{H}}_t \cdot \vec{a}_n\,dl}{\displaystyle\oint_{c} \vec{\mathscr{H}}_t \cdot d\vec{l}}\;\varepsilon\,\frac{\displaystyle\oint_{c'} \vec{\mathscr{E}}_t \cdot \bar{a}_n\,dl'}{\displaystyle-\int_a^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l}} \tag{1.57}$$

From Fig. 1.7(b) and Fig. 1.8(b) we have the following identities:

$$\vec{a}_n = \vec{a}_z \times \frac{d\vec{l}}{dl} \tag{1.58a}$$

$$\bar{a} = -\vec{a}_z \times \frac{d\vec{l}'}{dl'} \tag{1.58b}$$

so that

$$lc = -\mu\,\frac{\displaystyle\int_a^{a'} \vec{\mathscr{H}}_t \cdot (\vec{a}_z \times d\vec{l})}{\displaystyle\oint_{c} \vec{\mathscr{H}}_t \cdot d\vec{l}}\;\varepsilon\,\frac{\displaystyle\oint_{c'} -\vec{\mathscr{E}}_t \cdot (\vec{a}_z \times d\vec{l}')}{\displaystyle-\int_a^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l}} \tag{1.59}$$

We showed previously (see equation (1.14)) that

$$\vec{\mathscr{E}}_t = -\eta(\vec{a}_z \times \vec{\mathscr{H}}_t) \tag{1.60a}$$

$$\vec{\mathscr{H}}_t = \frac{1}{\eta}(\vec{a}_z \times \vec{\mathscr{E}}_t) \tag{1.60b}$$

Therefore [A.1]

$$\vec{\mathscr{E}}_t\cdot(\vec{a}_z\times d\vec{l}) = -\eta(\vec{a}_z\times\vec{\mathscr{H}}_t)\cdot(\vec{a}_z\times d\vec{l}) = -\eta\vec{\mathscr{H}}_t\cdot d\vec{l} \tag{1.61a}$$

and, similarly,

$$\vec{\mathscr{H}}_t\cdot(\vec{a}_z\times d\vec{l}) = \frac{1}{\eta}\vec{\mathscr{E}}_t\cdot d\vec{l} \tag{1.61b}$$

Substituting (1.61) into (1.59) yields

$$lc = -\mu\frac{\displaystyle\int_a^{a'}\frac{1}{\eta}\vec{\mathscr{E}}_t\cdot d\vec{l}}{\displaystyle\oint_c\vec{\mathscr{H}}_t\cdot d\vec{l}}\,\varepsilon\,\frac{\displaystyle\oint_c\eta\vec{\mathscr{H}}_t\cdot d\vec{l}}{\displaystyle-\int_a^{a'}\vec{\mathscr{E}}_t\cdot d\vec{l}} = \mu\varepsilon \tag{1.62}$$

The proof of the identity in (1.56b) follows an identical pattern using the expression for g given in (1.42).

A simpler method of proving these identities is from the general, second-order relations for the fields of a general TEM mode given in (1.11):

$$\frac{\partial^2\vec{\mathscr{E}}_t}{\partial z^2} = \mu\varepsilon\frac{\partial^2\vec{\mathscr{E}}_t}{\partial t^2} + \mu\sigma\frac{\partial\vec{\mathscr{E}}_t}{\partial t} \tag{1.63a}$$

$$\frac{\partial^2\vec{\mathscr{H}}_t}{\partial z^2} = \mu\varepsilon\frac{\partial^2\vec{\mathscr{H}}_t}{\partial t^2} + \mu\sigma\frac{\partial\vec{\mathscr{H}}_t}{\partial t} \tag{1.63b}$$

Performing the line integral between two points, a and a', on the conductors in a transverse plane on both sides of (1.63a) and recalling the definition of voltage given in (1.26b) yields

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \mu\sigma\frac{\partial V(z,t)}{\partial t} + \mu\varepsilon\frac{\partial^2 V(z,t)}{\partial t^2} \tag{1.64a}$$

Similarly, performing the contour integral around the top conductor in a transverse plane on both sides of (1.63b) and recalling the definition of current from (1.30) yields

$$\frac{\partial^2 I(z,t)}{\partial z^2} = \mu\sigma\frac{\partial I(z,t)}{\partial t} + \mu\varepsilon\frac{\partial^2 I(z,t)}{\partial t^2} \tag{1.64b}$$

If we differentiate the first transmission-line equation in (1.33) with respect to z, differentiate the second transmission-line equation in (1.43) with respect to

t and substitute we obtain

$$\frac{\partial^2 V(z,t)}{\partial z^2} = gl\,\frac{\partial V(z,t)}{\partial t} + lc\,\frac{\partial^2 V(z,t)}{\partial t^2} \tag{1.65a}$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = gl\,\frac{\partial I(z,t)}{\partial t} + lc\,\frac{\partial^2 I(z,t)}{\partial t^2} \tag{1.65b}$$

where the second equation was obtained by reversing the process. Comparing (1.65) to (1.64) we identify the two important identities given in (1.56).

As mentioned previously, one must be able to determine the per-unit-length parameters for a given cross-sectional line configuration as well as be able to solve the transmission-line equations. All structural differences between classes of lines are contained in the per-unit-length parameters and nowhere else. The above identities show that we only need to obtain one of the three per-unit-length parameters, g, l, or c. The transverse electric and magnetic fields satisfy Laplace's equation in any transverse plane (see equations (1.6)) so that *determination of each of the per-unit-length parameters is simply a static field problem in the transverse plane.* Numerous static-field-solution algorithms and computer codes can then be applied to this subproblem even though the eventual use of the parameters is in describing a problem whose voltages and currents vary with time!

1.4 CLASSIFICATION OF TRANSMISSION LINES

One of the primary tasks in obtaining the complete solutions for the voltage and current of a transmission line is the general solution of the transmission-line equations (Step 2). The type of line being considered significantly affects this solution. We are familiar with the difficulties in the solution of various ordinary differential equations encountered in the analysis of lumped circuits. Although the equations to be solved for lumped systems are ordinary differential equations (there is only one independent variable, time t) and are somewhat simpler to solve than the transmission-line equations which are partial differential equations (since the voltage and current are functions of two independent variables, time t and position along the line, z), the type of circuit strongly affects the solution difficulty. For example, if any of the circuit elements are functions of time (a time-varying circuit), then the coefficients of the ordinary derivatives will be functions of the independent variable, t. These equations, although linear, are said to be *nonconstant coefficient ordinary differential equations* which are considerably more difficult to solve than constant coefficient ones [A.4]. Suppose one or more of the circuit elements are *nonlinear*, i.e., the element voltage has a nonlinear relation to its current. In this case, the circuit differential equations become *nonlinear ordinary differential equations* which are equally difficult to solve [A.4]. So the class of lumped circuit being considered drastically affects the difficulty of solution of the governing differential equations.

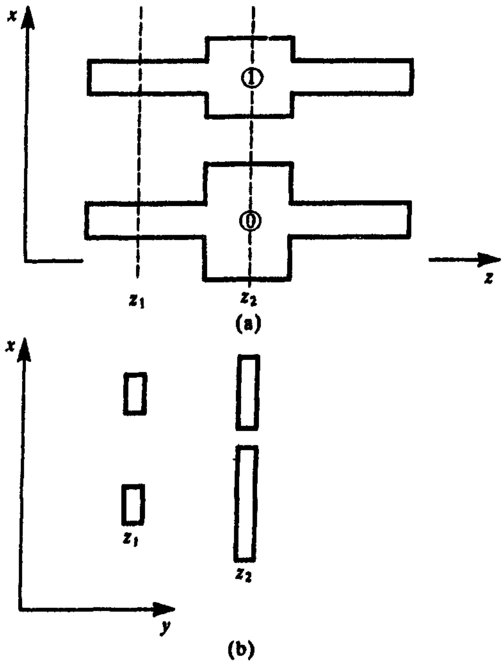

FIGURE 1.11 Illustration of a nonuniform line caused by variations in the conductor cross section.

Solution of the transmission-line partial differential equations has similar parallels. We have been implicitly assuming that the per-unit-length parameters are independent of time, t, and position along the line, z. The per-unit-length parameters contain all the cross-sectional structural dimensions of the line. If the cross-sectional dimensions of the line vary along the line axis, then the per-unit-length parameters will be functions of the position variable, z. This makes the resulting transmission-line equations very difficult to solve. Such transmission-line structures are said to be *nonuniform lines.* This includes both the cross-sectional dimensions of the line conductors as well as the cross-sectional dimensions of any inhomogeneous surrounding medium. If the cross-sectional dimensions of both the line conductors and the surrounding, perhaps inhomogeneous, medium are *constant along the line axis,* the line is said to be a *uniform line* whose resulting differential equations are simple to solve. An example of a nonuniform (in conductor cross section) line is shown in Fig. 1.11. Figure 1.11(a) shows the view along the line axis, while Fig. 1.11(b) shows the view in cross section. Because the conductor cross sections are different at z_1 and z_2, the per-unit-length parameters will be functions of one of the independent variables, in this case, position z. This type of structure occurs frequently on printed circuit boards (PCB's). A common way of handling

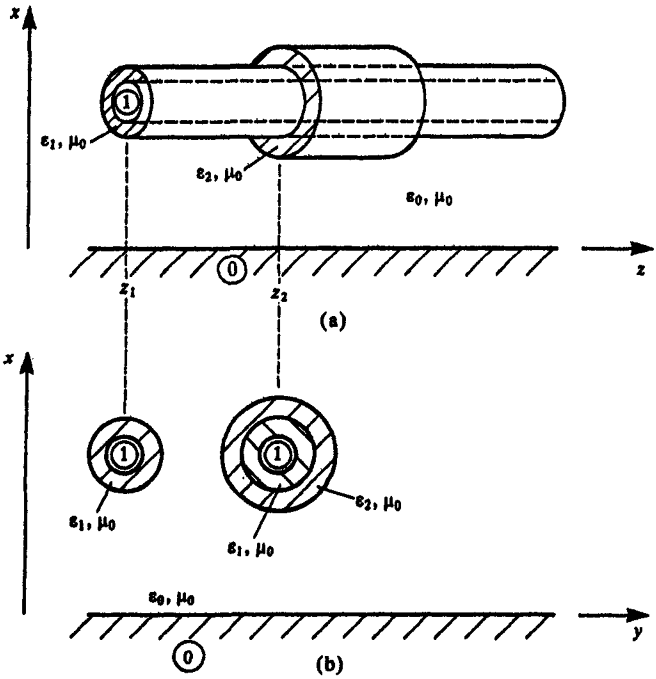

FIGURE 1.12 Illustration of a nonuniform line caused by variations in the surrounding medium cross section.

this is to divide the line into three uniform sections, analyze each separately and cascade the results. Figure 1.12 shows a nonuniform line where the nonuniformity is introduced by the inhomogeneous medium. A wire is surrounded by dielectric insulation. Along the two end segments the medium is inhomogeneous since in one part of the region the fields exist in the dielectric insulation, ε_1, μ_o, and in the other they exist in free space, ε_o, μ_o. In the middle region, the dielectric insulation is also inhomogeneous consisting of regions containing ε_1, μ_o, ε_2, μ_o and ε_o, μ_o. However, because of this change in the properties of the surrounding medium from one section, z_1, to the next, z_2, the total line is a *nonuniform* one and the resulting per-unit-length parameters will be functions of z. The resulting transmission line equations for Figs. 1.11 and 1.12 are difficult to solve because of the nonuniformity of the line. Again, a common way of solving this type of problem is to partition the line into a cascade of uniform subsections.

All of the previous derivations include losses in the medium through a per-unit-length conductance parameter, g. This loss does not invalidate the TEM field structure assumption. Most of the previous derivations assumed *perfect conductors.* In the derivation of the transmission-line equations from the distributed-parameter, lumped equivalent circuit shown in Fig. 1.10, we allowed

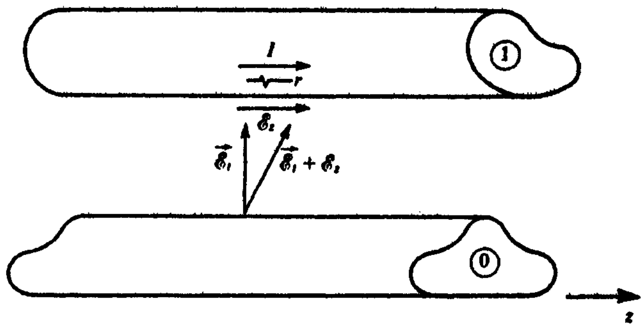

FIGURE 1.13 Illustration of the effect of conductor losses in creating non-TEM field structures.

the possibility of the line conductors being *imperfect conductors* with small losses through the per-unit-length resistance parameter, r. Unlike losses in the surrounding medium, *lossy conductors implicitly invalidate the TEM field structure assumption.* Figure 1.13 shows why this is the case. The line current flowing through the imperfect line conductor generates a nonzero electric field along the conductor surface, $\mathscr{E}_z(z, t) = rI(z, t)$, which is directed in the z direction violating the basic assumption of the TEM field structure in the surrounding medium. The total electric field is the sum of the transverse component and this z-directed component. However, if the conductor losses are small, this resulting field structure is *almost TEM*. This is referred to as the *quasi-TEM assumption* and, although the transmission-line equations are no longer valid, they are nevertheless assumed to represent the situation for small losses through the inclusion of the per-unit-length resistance parameter, r. An inhomogeneous surrounding medium, although nonuniform, also invalidates the basic assumption of a TEM field structure. The reason this is true is that a TEM field structure must have one and only one velocity of propagation of the waves in the medium. However, this cannot be the case for an inhomogeneous medium. If one portion of the inhomogeneous medium is characterized by ε_1, μ_o and the other is characterized by ε_2, μ_o, the velocities of TEM waves in these regions will be $v_1 = 1/\sqrt{\varepsilon_1, \mu_o}$ and $v_2 = 1/\sqrt{\varepsilon_2, \mu_o}$ which will be different. Nevertheless, the transmission-line equations are usually solved in spite of this observation and assumed to represent the situation so long as these velocities (and corresponding ε_1 and ε_2) are not substantially different. This is again referred to as the *quasi-TEM assumption.* A common way of characterizing this situation of an inhomogeneous medium is to obtain an *effective dielectric constant*, ε' [A.3]. This effective dielectric constant is defined such that if the line conductors are immersed in a *homogeneous dielectric* having this ε', the velocities of propagation and all other attributes of the solutions for the original inhomogeneous medium problem and this one will be the same.

In summary, *the TEM field structure and mode of propagation characterization* *of a transmission line is only valid for lines consisting of perfect conductors and surrounded by a homogeneous medium. Note that this medium may be lossy and not violate the TEM assumption so long as it is homogeneous (in σ, ε, and μ).* Violations of these assumptions (perfect conductors and a homogeneous medium surrounding the conductors) are considered under the *quasi-TEM* assumption so long as they are not extreme [17, 18].

1.5 RESTRICTIONS ON THE APPLICABILITY OF THE TRANSMISSION-LINE EQUATION FORMULATION

There are a number of additional, implicit assumptions in the TEM, transmission-line-equation characterization. It was pointed out in the derivation from the distributed-parameter, lumped circuit of Fig. 1.10 that distributing the lumped elements along the line and allowing the section length to go to zero, $\lim_{\Delta z \to 0} \Delta z$, means that line lengths that are *electrically long*, i.e., much greater than a wavelength λ, are properly handled with this lumped-circuit characterization. However, nothing was said about the ability of this lumped-circuit model to adequately characterize structures whose cross-sectional dimensions, e.g., conductor separations, are electrically large. Structures whose cross-sectional dimensions are electrically large at the frequency of excitation will have, in addition to the TEM field structure and mode of propagation, other higher-order TE and TM field structures and modes of propagation simultaneously with the TEM mode [A.1, 17–19]. Therefore, the solution of the transmission-line equations does not give the complete solution in the range of frequencies where these non-TEM modes coexist on the line. A comparison of the predictions of the TEM transmission-line-equation results with the results of a numerical code (which does not presuppose existence of only the TEM mode) for a two-wire line showed differences beginning with frequencies where the wire separations were as small as $\lambda/40$ [H.2]. Analytical solution of Maxwell's equations in order to consider the total effect of all modes is usually a formidable task. There are certain structures where an analytical solution is feasible and the next two sections consider these.

1.5.1 Higher-Order Modes

In the following two subsections we analytically solve Maxwell's equations for two *closed structures* to obtain the complete solution and demonstrate that the TEM formulation is complete up to some frequency where the conductor separations are some significant fraction of a wavelength above which higher-order modes begin to propagate.

1.5.1.1 The Infinite Parallel-Plate Transmission Line Consider the *infinite, parallel-plate transmission line* shown in Fig. 1.14. The two perfectly conducting plates lie in the y–z plane and are located at $x = 0$ and $x = a$. We will obtain

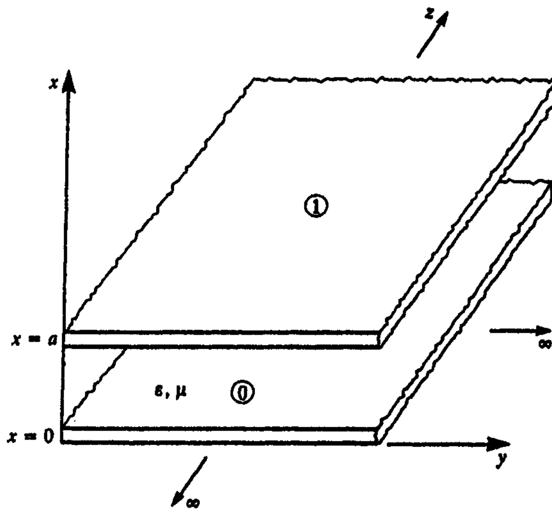

FIGURE 1.14 The infinite, parallel-plate waveguide for demonstrating the effect of cross-sectional dimensions on higher-order modes.

the complete solutions for the fields in the space between the two plates which is assumed to be homogeneous and characterized by ε and μ. Maxwell's equations for sinusoidal excitation become

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{1.66a}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \tag{1.66b}$$

Expanding these and noting that the plates are infinite in extent in the y direction so that $\partial/\partial y = 0$ gives [A.1]

$$\left.\begin{aligned}
\frac{\partial E_y}{\partial z} &= j\omega\mu H_x \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega\mu H_y \\
\frac{\partial E_y}{\partial x} &= -j\omega\mu H_z \\
\frac{\partial H_y}{\partial z} &= -j\omega\varepsilon E_x \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= j\omega\varepsilon E_y \\
\frac{\partial H_y}{\partial x} &= j\omega\varepsilon E_z
\end{aligned}\right\} \tag{1.67}$$

In addition, we have the *wave equations* [A.1]:

$$\left.\begin{aligned}\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0\\ \nabla^2\vec{H} + \omega^2\mu\varepsilon\vec{H} = 0\end{aligned}\right\} \tag{1.68}$$

Expanding these and recalling that the plates are infinite in the y dimension so that $\partial/\partial y = 0$ gives [A.1]

$$\left.\begin{aligned}\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial z^2} + \omega^2\mu\varepsilon\vec{E} = 0\\ \frac{\partial^2\vec{H}}{\partial x^2} + \frac{\partial^2\vec{H}}{\partial z^2} + \omega^2\mu\varepsilon\vec{H} = 0\end{aligned}\right\} \tag{1.69}$$

Let us now look for waves propagating in the $+z$ direction. To do so we assume that *separation of variables* is valid where we separate the dependence on x, y and on z as

$$\vec{E}(x, y, z) = \vec{E}'(x, y)e^{-\gamma z} \tag{1.70}$$

where γ is the propagation constant (to be determined). Substituting this into the above equation yields

$$\left.\begin{aligned}\gamma E'_y &= -j\omega\mu H'_x\\ -\gamma E'_x - \frac{\partial E'_z}{\partial x} &= -j\omega\mu H'_y\\ \frac{\partial E'_y}{\partial x} &= -j\omega\mu H'_z\\ \gamma H'_y &= j\omega\varepsilon E'_x\\ -\gamma H'_x - \frac{\partial H'_z}{\partial x} &= j\omega\varepsilon E'_y\\ \frac{\partial H'_y}{\partial x} &= j\omega\varepsilon E'_z\end{aligned}\right\} \tag{1.71}$$

and

$$\left.\begin{aligned}\frac{\partial^2\vec{E}'}{\partial x^2} + (\gamma^2 + \omega^2\mu\varepsilon)\vec{E}' = 0\\ \frac{\partial^2\vec{H}'}{\partial x^2} + (\gamma^2 + \omega^2\mu\varepsilon)\vec{H}' = 0\end{aligned}\right\} \tag{1.72}$$

The equations in (1.71) can be manipulated to yield [A.1]

$$\left.\begin{aligned} H'_x &= -\frac{\gamma}{h^2}\frac{\partial H'_z}{\partial x} \\ E'_x &= -\frac{\gamma}{h^2}\frac{\partial E'_z}{\partial x} \\ H'_y &= -\frac{j\omega\varepsilon}{h^2}\frac{\partial E'_z}{\partial x} \\ E'_y &= \frac{j\omega\mu}{h^2}\frac{\partial H'_z}{\partial x} \\ h^2 &= \gamma^2 + \omega^2\mu\varepsilon \end{aligned}\right\} \tag{1.73}$$

Observe that $E'(x, y)$ and $H'(x, y)$ are functions of only x since there can be no variation in the y direction due to the infinite extent of the plates in this direction and also the z variation has been assumed. Thus the partial derivatives in (1.71), (1.72), and (1.73) can be replaced by ordinary derivatives. We now investigate the various modes of propagation.

The Transverse Electric (TE) Mode ($E_z = 0$) The transverse electric (TE) mode of propagation assumes that the electric field is confined to the transverse, x–y plane so that $E_z = 0$. Therefore, from (1.73) we see that $E'_x = H'_y = 0$. The wave equations in (1.72) reduce to

$$\frac{d^2E'_y}{dx^2} + h^2E'_y = 0 \tag{1.74}$$

whose general solution is

$$E'_y = C_1 \sin(hx) + C_2 \cos(hx) \tag{1.75}$$

The boundary conditions are that the electric field tangent to the surfaces of the plates are zero:

$$E_y = 0|_{x=0,x=a} \tag{1.76}$$

which, when applied to (1.75), yields $C_2 = 0$ and $ha = m\pi$ for $m = 0, 1, 2, 3, \ldots$ Thus, the solution becomes

$$E_y = C_1 \sin\left(\frac{m\pi x}{a}\right)e^{-\gamma z} \tag{1.77}$$

From (1.71) we obtain

$$\left.\begin{aligned} H_z &= j\frac{1}{\omega\mu}\frac{\partial E_y}{\partial x} \\ &= j\frac{m\pi}{\omega\mu a}C_1\cos\left(\frac{m\pi x}{a}\right)e^{-\gamma z} \end{aligned}\right\} \tag{1.78}$$

and

$$H_x = j\frac{\gamma}{\omega\mu}C_1\sin\left(\frac{m\pi x}{a}\right)e^{-\gamma z} \tag{1.79}$$

Since

$$h = \frac{m\pi}{a} \tag{1.80}$$

the propagation constant becomes

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon} \tag{1.81}$$

For the lowest-order mode, $m = 0$, all field components vanish. The next higher-order mode is the TE_1 mode for $m = 1$ whose nonzero components are E_y and H_x.

The Transverse Magnetic (TM) Mode ($H_z = 0$) The transverse magnetic (TM) mode has the magnetic field confined to the transverse, x–y plane so that $H_z = 0$. Carrying through a development similar to the above for this mode gives the nonzero field vectors as

$$\left.\begin{aligned} H_y &= D_2\cos\left(\frac{n\pi x}{a}\right)e^{-\gamma z} \\ E_x &= -j\frac{\gamma}{\omega\varepsilon}D_2\cos\left(\frac{n\pi x}{a}\right)e^{-\gamma z} \\ E_z &= j\frac{h}{\omega\varepsilon}D_2\sin\left(\frac{n\pi x}{a}\right)e^{-\gamma z} \end{aligned}\right\} \tag{1.82}$$

for $n = 0, 1, 2, \ldots$ The propagation constant is again given by (1.81) with m replaced by n. In contrast to the TE modes, the lowest-order TM mode is the TM_0 mode for $n = 0$. For this case the propagation constant reduces to the familiar

$$\begin{aligned} \gamma &= j\omega\sqrt{\mu\varepsilon} \\ &= j\beta \end{aligned} \tag{1.83}$$

and the field vectors in (1.82) reduce to

$$\left.\begin{aligned} H_y &= D_2 e^{-j\beta z} \\ E_x &= -j\frac{\gamma}{\omega\varepsilon} D_2 e^{-j\beta z} \\ &= \sqrt{\frac{\mu}{\varepsilon}} D_2 e^{-j\beta z} \\ E_z &= 0 \end{aligned}\right\} \tag{1.84}$$

However, this is the TEM mode!

Therefore, the lowest-order TM mode, TM_0, is equivalent to the TEM mode and the TE_0 mode is nonexistent. We must then ascertain when the next higher-order modes begin to propagate thus adding to the total picture. The propagation constant in (1.81) must be imaginary, or at least have a nonzero imaginary part. Clearly for $m = n = 0$, we have the propagation constant of a plane wave: $\gamma = j\omega\sqrt{\mu\varepsilon} = j\beta$. For higher-order modes to propagate, we require from (1.81) that $\omega^2\mu\varepsilon \geq h^2$ giving

$$\omega \geq \frac{1}{\sqrt{\mu\varepsilon}}\frac{n\pi}{a} \tag{1.85}$$

The cutoff frequency for the lowest-order TEM mode, TM_0, is clearly dc. The cutoff frequency of the next higher-order modes, TE_1 and TM_1, are from (1.85)

$$\begin{aligned} f_{TE_1,TM_1} &= \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a} \\ &= \frac{v}{2a} \end{aligned} \tag{1.86}$$

In terms of wavelength, $\lambda = \dfrac{v}{f}$, we find that *the TEM mode will be the only possible mode so long as the plate separation, a, is less than one-half of one wavelength*, i.e.,

$$a \leq \frac{\lambda}{2} \tag{1.87}$$

This illustrates that so long as the *cross-sectional dimensions of the line are electrically small, only the TEM mode can propagate!* This is illustrated in Fig. 1.15.

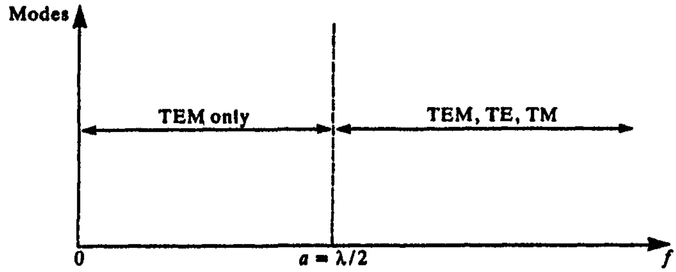

FIGURE 1.15 Illustration of the dependence of higher-order modes on cross-sectional electrical dimensions.

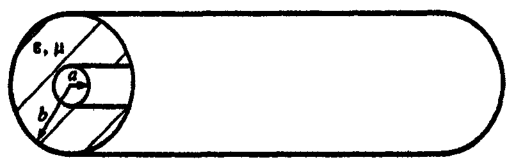

FIGURE 1.16 The coaxial cable for illustrating the dependence of higher-order modes on cross-sectional electrical dimensions.

1.5.1.2 The Coaxial Transmission Line Another *closed system* transmission line which is capable of supporting the TEM mode is the *coaxial transmission line* shown in Fig. 1.16. The general solution to Maxwell's equations for the fields and modes in the space between the inner wire and the outer shield was solved in [1]. Clearly this structure can support the TEM mode with a cutoff frequency of dc. The higher-order TE and TM modes have the following cutoff properties. The lowest order TE mode is cutoff for frequencies such that *the average circumference between the conductors is approximately less than one-half of one wavelength*, i.e.,

$$2\pi(a + b) \leq \frac{\lambda}{2} \tag{1.88}$$

Similarly, the lowest order TM mode is cutoff for frequencies such that the *difference between the two conductor radii is approximately less than one-half of one wavelength*, i.e.,

$$(b - a) \leq \frac{\lambda}{2} \tag{1.89}$$

These results again support the notion that *the TEM mode will be the only mode of propagation in closed systems so long as the conductor separation is electrically small.*

1.5.1.3 Two-Wire Lines The two previous transmission-line structures are closed systems. For *open systems* such as the two-wire line, the issue of higher-order modes is not so clear-cut. Numerical analysis of a two-wire line given in [H.2] showed that the predictions of the transmission-line formulation for the two-wire line begin to deviate from the complete solution when the cross-sectional dimensions such as wire separation are no longer electrically small. This supports our intuition. The problem was investigated in more detail in [19] where these notions are confirmed. Also certain "leaky modes" are capable of propagating with no clearly defined cutoff frequency. Thus, the TEM mode formulation and the resulting transmission-line equation representation for two-wire lines will be reasonably adequate so long as the wire separations are electrically small. Ordinarily, this is satisfied for practical transmission-line structures.

1.5.2 Transmission-Line Currents vs. Antenna Currents

There is one remaining restriction on the completeness of the TEM mode, transmission-line representation that needs to be discussed. It can be shown that under the TEM, transmission-line formulation for a two-conductor uniform line, the currents so determined on the two conductors at any cross section must be equal in magnitude and oppositely directed. Thus, *the total current at any cross section is zero.* This is the origin of the reference to the term that one of the conductors serves as a "return" for the current on the other conductor. Unless there is adherence to the following concepts, this will not be the case.

Consider the pair of parallel wires shown in Fig. 1.17(a) supporting on their surfaces, at the same cross section, currents I_1 and I_2. In general, we may decompose, or represent, these as a linear combination of two other currents. The so-called *differential-mode currents*, I_D, are equal in magnitude at a cross section and are oppositely directed as shown in Fig. 1.17(b). These correspond to the TEM mode, transmission-line currents that will be predicted by the transmission-line model. The other currents are the so-called *common-mode currents*, I_C, which are equal in magnitude at a cross section but are directed in the same direction as shown in Fig. 1.17(c). These are sometimes referred to as "antenna-mode" currents [20, 21]. This decomposition can be obtained by writing, from Fig. 1.17,

$$\begin{aligned} I_1 &= I_C + I_D \\ I_2 &= I_C - I_D \end{aligned} \tag{1.90}$$

In matrix form, these can be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} I_C \\ I_D \end{bmatrix} \tag{1.91}$$

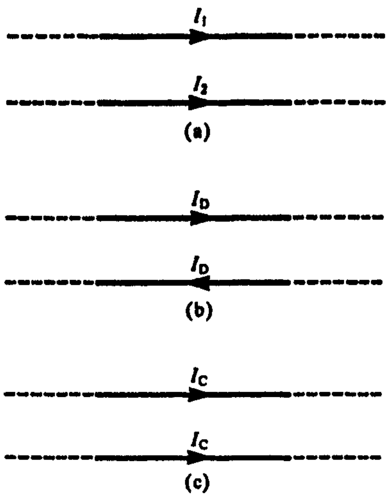

FIGURE 1.17 Illustration of the decomposition of total currents into differential-mode (transmission-line-mode) and common-mode (antenna-mode) components.

Equation (1.91) represents a *nonsingular transformation between the two sets of currents* since the transformation matrix is nonsingular. Therefore, its inverse can be taken and the transformation reversed to yield

$$\begin{bmatrix} I_C \\ I_D \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1.92}$$

This gives

$$\begin{aligned} I_C &= \frac{I_1 + I_2}{2} \\ I_D &= \frac{I_1 - I_2}{2} \end{aligned} \tag{1.93}$$

Ordinarily, the common-mode currents are much smaller in magnitude than the differential-mode currents so they do not substantially affect the results of an analysis of currents and voltages of a transmission line. However, the common-mode currents are significant, even though they are smaller in magnitude than the differential-mode currents, in the case of radiated emissions from this two-wire line. This is because the radiated electric fields from the differential-mode currents tend to subtract but those from the common-mode currents tend to add. Thus, a "small" common-mode current can give the same order of magnitude of radiated emission as a much larger differential-mode current. This was confirmed for cables and PCB's in [A.3, 22, 23]. The significant point here is that if one bases a prediction of the *radiated emissions* from a two-conductor line on the currents *obtained from a transmission-line-*

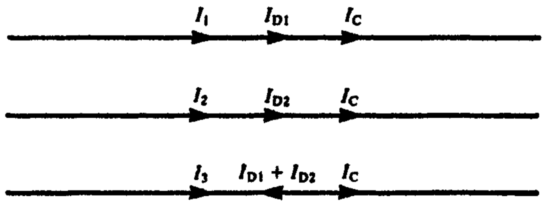

FIGURE 1.18 Decomposition of total currents of a three-conductor line into differential-mode and common-mode components.

equation analysis, the predicted emissions will generally lie far below those of the (unpredicted) emissions due to the common-mode currents. The common-mode currents can be ignored in a near-field, transmission-line analysis such as in determining *crosstalk.*

This decomposition can be extended to lines consisting of more than two conductors. Consider the three-conductor line shown in Fig. 1.18. There are three currents to be decomposed, I_1, I_2, and I_3. So we are free to redefine them in terms of three other currents such as is shown in Fig. 1.18 as I_{D1}, I_{D2}, and I_C. We have chosen two of the currents, I_{D1} and I_{D2} to be defined in the same fashion as the TEM mode currents in that they return through the lower conductor. The remainder current, I_C, is the same in magnitude and direction of all three conductors. The transformation becomes

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_{D1} \\ I_{D2} \\ I_C \end{bmatrix} \tag{1.94}$$

Inverting this transformation gives

$$\begin{bmatrix} I_{D1} \\ I_{D2} \\ I_C \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \tag{1.95}$$

from which the decomposition currents can be obtained.

There are a number of ways that these non-TEM mode currents can be created on a transmission line. Figure 1.19 illustrates one of these. It is important to remember that the TEM mode, transmission-line-equation formulation only characterizes the line and assumes that the two (or more) conductors of the line continue indefinitely along the z axis. The field analysis does not inherently consider the field effects of the eventual terminations for a finite-length line. This problem was investigated in [24]. It was found that *asymmetries*

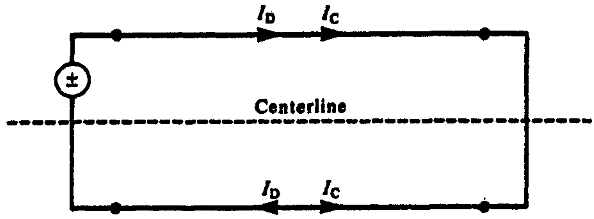

FIGURE 1.19 Illustration of an asymmetry that creates common-mode currents.

as well as the presence of nearby metallic obstacles create these "nonideal" currents. For example, consider the two-wire line shown in Fig. 1.19 which is driven by a voltage source at the left end and terminated in a short circuit at the right end. This was analyzed using a numerical solution of Maxwell's equations commonly referred to as a method of moments (MOM). This analysis gives the complete solution for the currents without presupposing the existence of only the TEM mode. It was found that if the voltage source was situated and modeled as being centered in the left segment on the centerline, then $I_2 = -I_1$; in other words, the currents on the wires are only differential-mode currents. However, if the voltage source was placed asymmetrically to the centerline such as shown and the resulting currents decomposed as in (1.93), common-mode currents appeared. This asymmetrical placement of the source, which is not explicitly considered in the transmission-line-equation formulation, was apparently the source.

The important point here is that the TEM mode, transmission-line-equation formulation that we will consider in this text only predicts the differential-mode currents. If the line cross section is electrically small and one is interested only in predicting the currents and voltages on the line for the purposes of predicting signal distortion and *crosstalk* (the primary goal of this text), this prediction will be reasonably accurate. On the other hand, if one is interested in predicting the radiated electric field from this line, then the predictions of that field using only the currents predicted by the transmission-line-equation formulation will most likely be inadequate since the contributions due to the common-mode currents are typically the dominant contributors to radiated emissions [22, 23].

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PROBLEMS

1.1 Two, perfectly-conducting, circular plates are separated a distance d as shown in Fig. P1.1. The plates have very large radii with respect to d,

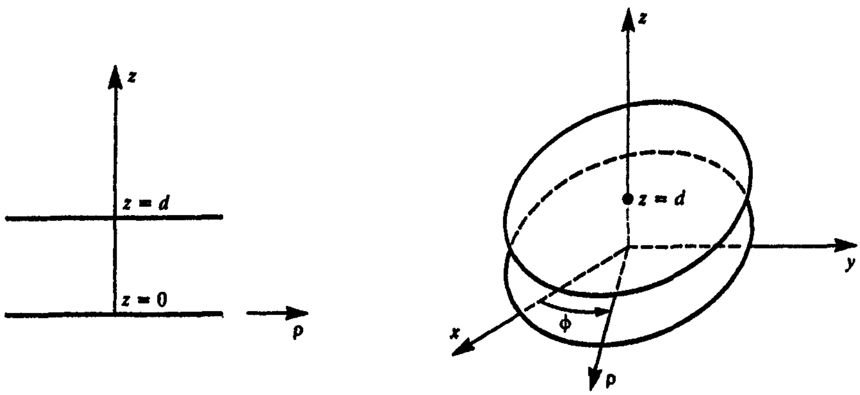

FIGURE P1.1

(ideally infinite), so that, in cylindrical coordinates, we may assume a "TEM-mode" field structure, $\vec{\mathscr{E}}(\rho, t) = \mathscr{E}_z(\rho, t)\vec{a}_z$, and $\vec{\mathscr{H}}(\rho, t) = \mathscr{H}_\phi(\rho, t)\vec{a}_\phi$. Define voltage and current as

$$V(\rho, t) = -\mathscr{E}_z(\rho, t)\, d$$
$$I(\rho, t) = 2\pi\rho\mathscr{H}_\phi(\rho, t)$$

Show, from Maxwell's equations in cylindrical coordinates, that V and I satisfy the transmission-line equations:

$$\frac{\partial V(\rho, t)}{\partial \rho} = -l\,\frac{\partial I(\rho, t)}{\partial t}$$
$$\frac{\partial I(\rho, t)}{\partial \rho} = -c\,\frac{\partial V(\rho, t)}{\partial t}$$

where l and c are static parameters defined by:

$$l = \frac{\mu d}{2\pi\rho}$$
$$c = \frac{2\pi\varepsilon\rho}{d}$$

Would it be appropriate to classify this as a *nonuniform* line? Could the mode of propagation to which these equations apply be classified as a "TEM mode"?

1.2 The infinite, biconical transmission line consists of two, perfectly conducting cones of half angle θ_h as shown in Fig. P1.2. Solve Maxwell's equations

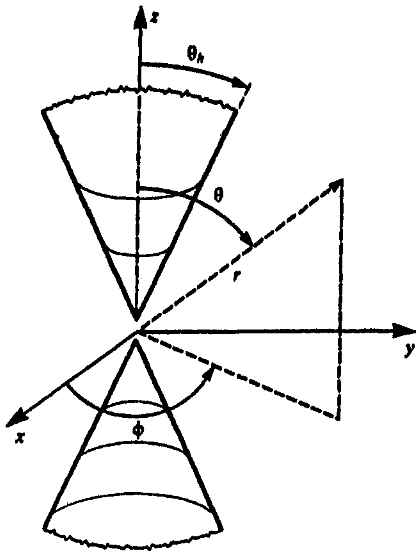

FIGURE P1.2

in spherical coordinates for this structure assuming that $\vec{\mathscr{E}}(r, \theta) = \mathscr{E}_\theta(r, \theta)\vec{a}_\theta$ and $\vec{\mathscr{H}}(r, \theta) = \mathscr{H}_\phi(r, \theta)\vec{a}_\phi$. Show that the following definitions of voltage and current are unique:

$$V(r, t) = \int_{\theta_h}^{\pi - \theta_h} \mathscr{E}_\theta r \, d\theta$$

$$I(r, t) = \int_0^{2\pi} \mathscr{H}_\phi r \sin(\theta) \, d\phi$$

where V and I satisfy the following "transmission-line equations":

$$\frac{\partial V(r, t)}{\partial r} = -l \frac{\partial I(r, t)}{\partial t}$$

$$\frac{\partial I(r, t)}{\partial r} = -c \frac{\partial V(r, t)}{\partial t}$$

Show that

$$l = \frac{\mu}{\pi} \ln\left(\cot \frac{\theta_h}{2}\right)$$

$$c = \frac{\pi\varepsilon}{\ln\left(\cot \frac{\theta_h}{2}\right)}$$

Would this be classified as a *uniform* or *nonuniform* line? Would it be appropriate to classify the propagation mode as TEM?

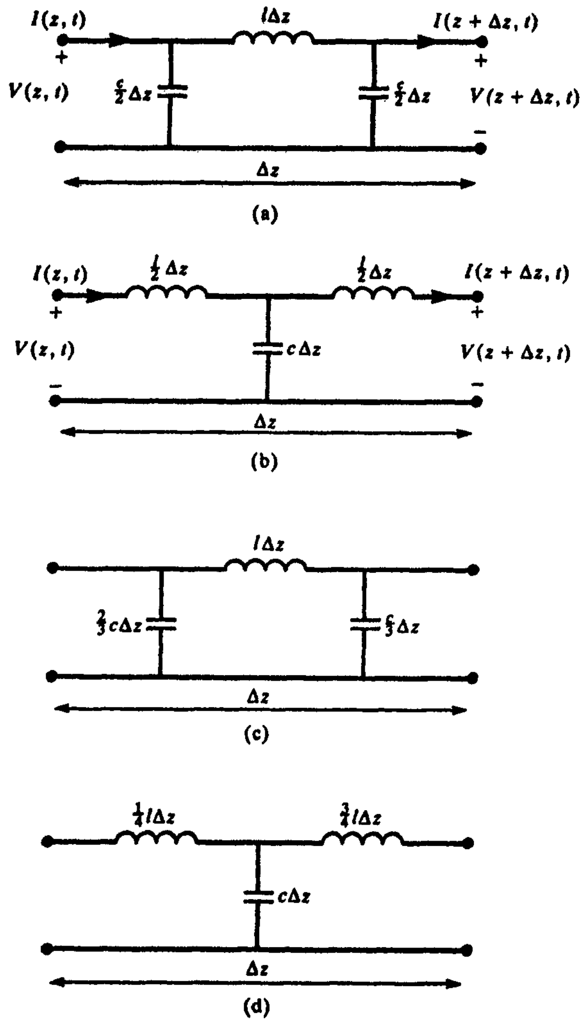

FIGURE P1.5

1.3 Show that, assuming a TEM field structure, the currents on the two conductors in Fig. 1.5 are equal in magnitude and oppositely directed at any cross section.

1.4 Show that, assuming a TEM field structure, the charge per unit length on one conductor in Fig. 1.5 is equal in magnitude and opposite in sign to the charge per unit length on the other conductor at any cross section.

1.5 Derive the transmission-line equations from each of the circuits in Fig. P1.5 in the limit as $\Delta z \rightarrow 0$. Observe that the total inductance (capacitance) in each structure is $l\Delta z$ ($c\Delta z$). This shows that the structure of the per-unit-length equivalent circuit is not important in obtaining the transmission line equations from it so long as the total per-unit-length inductance and capacitance is contained in the structure and we let $\Delta z \rightarrow 0$.