In the short-run we are all dead: Non-Equilibrium Dynamics in a Computational General Equilibrium model.

January 20, 2019

Davoud Taghawi-Nejad

Abstract

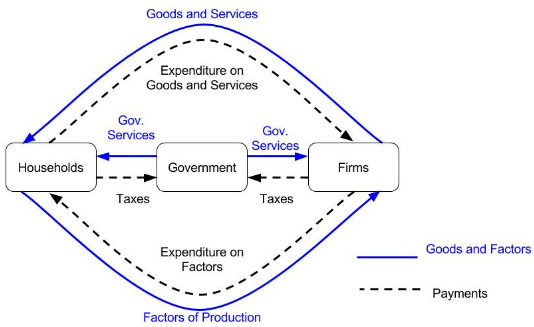
Studies of the economic impact and mitigation of climate change usually use computable general equilibrium models (CGE). Equilibrium models, as the name suggests, model the economy as in equilibrium, the transitions to the equilibrium are ignored. In the time spend outside equilibrium, the economy produces different quantities of goods and pollution as predicted by the equilibrium model. If the economy in this time outside of the equilibrium produces a different amount of climate gasses the predictions could be dangerously wrong. We present in this paper a computational generalization of the Arrow-Debreu general equilibrium model, which is not in equilibrium during the transitions, but converges to the same equilibrium as a CGE model with the same data and assumption. We call this new class of models Computational Complete Economy models. Computational Complete Economy models have other interesting applications for example in international trade, tax policy and macroeconomics. Keywords: GCE, Climate Change, International Trade

1 Introduction

Studies of the economic impact and mitigation of climate change usually use computable general equilibrium (CGE) models. Equilibrium models, as the name suggests, model the economy as in equilibrium; the out-of-equilibrium transitions to the equilibrium are ignored. In the time spend outside equilibrium, the economy produces different quantities of goods and pollution as predicted by the equilibrium model. If the economy in this time outside of the equilibrium produces more climate gasses the predictions are dangerously wrong. We present in this paper a computational generalization of the Arrow-Debreu general equilibrium which is not in equilibrium during the transitions, but converges to the same equilibrium as a CGE model with the same data and assumptions about production and consumption functions. The aim of this paper is to demonstrate how to transform a Computable General Equilibrium model into a Computational Complete Economy (CCE) model - an agent-based model with transitions. In order to achieve this we use a widely cited CGE model and recreate in as a CCE model. The emphasis here is not only on the model itself but also on the strategy to calibrate the model. Purpose of this paper is to generalize CGE models. Therefore we will lay out how to build a CGE model and then explain how to use the same system of equations in a non-equilibrium CCE model. In section two we will recapture the underlying CGE model. In section three we will explain how a CGE model is calibrated. In section four we will explain how the CCE model works and converges to equilibrium. In section five we will run policy experiments we will compare the asymptotic results with the CGE model and discuss the transitory effects of the experiment.

2 The underlying CGE model - firm, consumer and government behavior

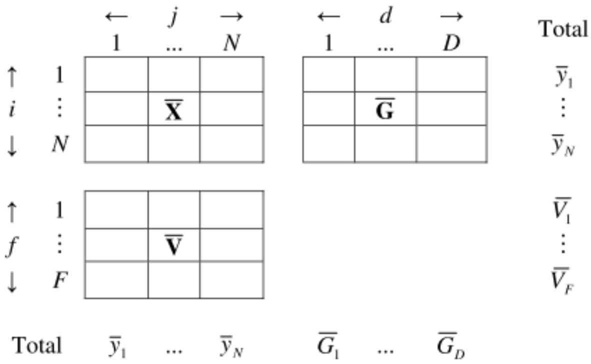
2.1 Walrasian Equilibrium and the Circular flow of the economy



For many of the readers the circular flow model of the economy should be familiar. The circular flow of the economy shows the flows of products and factors and their counter transaction, the payment for these goods and products. Households supplies factors of production - capital and labor - to the firms, which in turn supply goods and services. In the counter direction money flows from the household to the firms as a payment for the goods and services. The firm in turn pays the household for its factor provision (profit and factor income). The government collects taxes and provides government services. The material flows must be balanced in the circular flow, that means that every factor provided by the household must be used by the firm and every good and service produced must reach the consumer. In equilibrium the value of the goods and services must balance the value of the factors. Otherwise value would just appear out of thin air. That implies also that the payments for factors balance the payments for goods and services. In other words the markets clear and there are zero economic profits. Owed to computational restrictions and data availability

CGE models typically assume representative agents. In the following we will assume that there is one representative household and on representative firm for each sector. We will also later on introduce a government agent, which only redistributes,an investment agent, that captures capital investment and a net-exports agent that captures international trade.

Figure 2. A Stylized Social Accounting Matrix



(a) No distortions

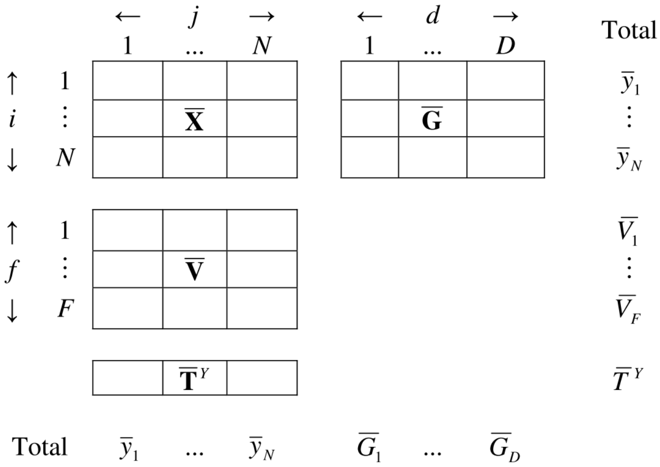


Figure 1, taken from Wing [2] represents this social accounting matrix. It depicts the naming of entries, rows and columns of the SAM. The SAM is a matrix where each cell is the value of goods transferred from the row to the column entry. As it is custom in CGE models we normalize the quantity of goods in such a way that the prices in the SAM are 1. The right hand side of figure 1 also contains taxes T. Taxes are a flow of money from industry j to the government agent. The matrix

- $\mathbf{X} \in \mathbb{R}^{N,N}$ is the input-output matrix of industries (x_{ij} is the quantity of goods transferred from the i-th industry to the j-th industry),
- $\mathbf{V} \in \mathbb{R}^{F,N}$ is the matrix of primary factor inputs to industries ($F = 2$: capital and labour; v_{fj} is the capital/labor provided by households/import-export/investments as a collective to the j-th industry),
- $\mathbf{G} \in \mathbb{R}^{N,D}$ is the matrix of commodity uses by final demand activities ($D = 3$: a column for households, a column for import-export, and a column for investments; g_{id} is the value of goods transferred from the i-th industry to households/import-export/investments).

The *social accounting matrix* (SAM) is defined as

$$\begin{bmatrix} \mathbf{X} & \mathbf{G} \\ \mathbf{V} & \mathbf{0} \end{bmatrix}.$$

The value of its i, j-component corresponds to the value of goods transferred from the i-th row to the j-th column entry.

Henceforth, we glue $\mathbf{X}$ and $\mathbf{V}$ together and denote it by $\mathbf{Z} = \binom{\mathbf{X}}{\mathbf{V}}$

Next, we introduce prices for goods and factors:

- by $p_i, i = 1, 2, \ldots, N$, is the price of each good
- by $p_i, i = N+1, N+2$ are the price of each factor.

The *budget constraint* asserts that

$$m := \sum_{f=N+1}^{N+2} \sum_{j=1}^{N} p_f z_{fj} = \sum_{i=1}^{N} \sum_{d=1}^{D} p_i g_{id}\,. \tag{1}$$

The Cobb-Douglas Economy

In order to model the economy, we need to assume production and utility functions that characterize the economy. In this first description of a generalized arrow-debreu equilibrium we choose simple cobb-douglas economy. This choice is by no means necessary the algorithm works also for other functional forms such as the CES production function. The matrix G has several consumers: the household, an investment agent and a net-export agent. For the purpose of this paper, we hold investment and net-export constant. That is regardless of the prices the quantity of investment and net-export is constant.

2.1.1 The households utility maximization

The Cobb-Douglas Economy asserts that households maximize their utility, that is, $\{g_{i1}\}_{i=1}^{N}$ satisfies

$$\{g_{i1}\}_{i=1}^{N} \in \operatorname{argmax} \prod\nolimits_{i=1}^{N} g_{i1}^{\alpha_i} \quad \text{s.t.} \sum_{i=1}^{N} \sum_{d=1}^{D} p_i g_{id} = m\,, \tag{2}$$

where m is given (according to the left-hand side(1)), and $\{g_{i2}\}_{i=1}^N$, $\{g_{i3}\}_{i=1}^N$ and the coefficients $\{\alpha_i\}_{i=1}^N$ are (chosen and) fixed a-priori. It can be shown that the solution of (2) is

$$g_{i1} = \frac{\alpha_i}{p_i}\left(m - \sum_{d=2}^{D}\sum_{i=1}^{N} p_i g_{id}\right) = \frac{\alpha_i}{p_i}\left(\sum_{i=1}^{N} p_i g_{i1}\right) . \tag{3}$$

By solving (3) for α, we are able to find the exponents of the cobb-douglas equation:

$$\alpha_{i1} = \frac{g_{i1}p_i}{\left(m - \sum_{d=2}^{D}\sum_{i=1}^{N} p_i g_{id}\right)} = \frac{g_{i1}p_i}{\sum_{i=1}^{N} p_i g_{i1}} \tag{4}$$

The industries' production functions

We model the (pre-taxes) "revenue" of a generic industry j with

$$f_j(\mathbf{Z}) := b_j \prod_{i=1}^{N+2} z_{ij}^{\beta_{ij}} , \tag{5}$$

where $b_j \in (0,\infty)$, $\{\beta_{ij}\}_{i=1}^{N+2}$, are physical parameters specific to the industry j ($\beta_{ij} \in [0,1]$). Industry maximize their profit if the N column vectors $\left(\{z_{ij}\}_{i=1}^{N+2}\right)$ satisfy

$$\left(\{z_{ij}\}_{i=1}^{N+2}\right) \in \operatorname{argmax}\left(\left((1-\tau_j)p_j - \tilde{\tau}_j\right) f_j(\mathbf{Z}) - \sum_{i=1}^{N+2} p_i z_{ij}\right) , \tag{6}$$

where τ_j refers to the taxes on output. The necessary optimality conditions of (6) reads

$$z_{ij} = \beta_{ij}\frac{\left((1-\tau_j)p_j - \tilde{\tau}_j\right) f_j(\mathbf{Z})}{p_i} . \tag{7}$$

That is, knowing the revenue y_j and the behavior $\left(\{x_{ij}\}_{i=1}^N, \{v_{fj}\}_{f=1}^F\right)$ of the j-th industry, we can infer the parameters $\{\beta_{ij}\}_{i=1}^N$ and $\{\beta_{fj}\}_{f=1}^F$. The parameters are parameters are

$$\beta_{ij} = \frac{p_i x_{ij}}{(1-\tau_j)p_j f_j(\mathbf{X},\mathbf{V})} \quad \text{and} \tag{8}$$

$$\beta_{fj} = \frac{p_i v_{fj}}{(1-\tau_j)p_j f_j(\mathbf{X},\mathbf{V})} . \tag{9}$$

Market clearing condition

The market clearing condition asserts that the value of input of each j-th industry equals its value of output, that is,

$$\sum_{i=1}^{N+2} p_i z_{ij} = p_j\left(\sum_{i=1}^{N} z_{ji} + \sum_{d=1}^{D} g_{ji}\right) . \tag{10}$$

Let

$$y_j := \sum_{i=1}^{N} p_i x_{ij} + \sum_{f=1}^{F} p_f v_{fj} \tag{11}$$

be the *value of input* of the j-th industry. By the zero profit condition, the value of input equals the *value of output* that is,

$$y_i = \sum_{j=1}^{N} p_i x_{ij} + \sum_{d=1}^{D} p_i g_{ij} \,. \tag{12}$$

General equilibrium

We say that the SAM is in *general equilibrium* if Equations (1), (3), (7), and (12) are simultaneously satisfied, that is,

$$\begin{cases} g_{i1} = \frac{\alpha_i}{p_i}\left(\sum_{i=1}^{N} p_i g_{i1}\right) \\ z_{ij} = \beta_{ij}\frac{((1-\tau_j)p_j - \tilde{\tau}_j)f_j(\mathbf{Z})}{p_i} \\ \sum_{f=N+1}^{N+2}\sum_{j=1}^{N} p_f z_{fj} = \sum_{i=1}^{N}\sum_{d=1}^{D} p_i g_{id} \\ \sum_{i=1}^{N+2} p_i z_{ij} = p_j\left(\sum_{i=1}^{N} z_{ji} + \sum_{d=1}^{D} g_{ji}\right) \,. \end{cases} \tag{13}$$

In the CGE approach, we take the system and find its equilibrium, where the equilibrium is defined as a situation where prices and quantities are such that there is no excess demand for goods, production factors and zero profit.
Following to Varian [5] there are two properties our general equilibrium system must obey to ensure the uniqueness of the general equilibrium. First the household must satisfy weak axiom of revealed preference - households need a stable preference ordering space of all possible prices and income levels. Secondly, the aggregate demand for any commodity or factor is non-decreasing in the prices of all other goods and factors (gross substitutability). Mas-Colell [6] proves that constant elasticity of substitution utility and production functions, whose elasticities of substitution are greater than or equal to one, have a unique equilibrium in the absence of taxes and other distortions. As a cobb-douglas economy is a CES economy with an elasticity of substitution of one this also holds here. Foster and Sonnenschein [7] and Hatta [8] on the other side find evidence that distortions can lead to multiple equilibria even in this setting. For a detailed treatment of taxes and distortions see also Kehoe [9]. In conclusion we can say that for our model existence and uniqueness is guaranteed until we introduce taxes or subsidies.

Numerical Calibration

In order to calibrate our model we first apply standard CGE trick. We define one unit of each good so that the price of each unit is one. With this definition each entry in the social accounting matrix represents both the value and the quantity of each good traded. With this definition equations (4), (8) and (9) reduce to the following calibration definitions:

$$\alpha_{i1} = \frac{g_{i1}}{\left(m - \sum_{d=2}^{D}\sum_{i=1}^{N} g_{id}\right)} = \frac{g_{i1}}{\sum_{i=1}^{N} g_{i1}} \tag{14}$$

$$\beta_{ij} = \frac{x_{ij}}{(1-\tau_j)f_j(\mathbf{X},\mathbf{V})} \quad \text{and} \tag{15}$$

$$\beta_{fj} = \frac{v_{fj}}{(1-\tau_j)f_j(\mathbf{X},\mathbf{V})}\ . \tag{16}$$

The cobb-douglas multiplier can be derived from the definition of y_j:

$$b_j = \frac{y_j}{\prod_i x_{ij}^{\beta_{ij}} \prod_i v_{fj}^{\beta_{fj}}} \tag{17}$$

Where g_i1 the spending of the household on good i; V_f is the factor income of the household; g_i is the net saving, investment - net exports; x_{ij} is the the spending of firm j on good i; and y_j is the total output of or equivalently the total spending on sector j. All this values can be readily read from the input output matrix.

$$\tau_j = \frac{t_j}{\sum_i x_{ij} + \sum_i v_{fj} + t_j} \tag{18}$$

Where t_j is the tax paid by industry j on its output.
As we are holding net investment and net-export constant:

$$s_i = g_{is}$$

And finally the factor endowments are the factor endowments from the SAM and the income of the household agent is its factor endowment:

$$V_f = V_f$$

$$m = \sum_f V_f \tag{19}$$

If we set the parameters of the model according to the equations (14) - (19) based on the SAM, and solve *(z)=0, z replicates the SAM. In other words our model economy parameterized according to the SAM tends in equilibrium to replicate the circular flow of the economy described in the SAM. However, as we will show in the next section and by numerical simulation the non-equilibrium computational complete economy model, this calibration does also lead to a CCE model that asymptotically replicates the SAM.

Dynamics of economy

We interpret economy as a dynamical system: SAMs are points in its phase space and evolve in time according to the laws of motion of this dynamical system. For consistency with previous works, dynamics must be modeled in such a way the general equilibrium (13) becomes a fixpoint of the system. Then, evolution of a SAM that lies in the basin of attraction of (13) of carries information on the transition from nonequilibrium to equilibrium,
In traditional CGE models we established our system of excess demand and profit functions *in order to reproduce the equilibrium result we just calibrated our functions and search, usually by non linear programming, for the z that satisfies *(z)=0 and z0. In this work we are interested in transition to equilibrium and the transitions between equilibria. We therefore take our economy -

equations (1) - (6) and calibrate them as described in section 2.4. With this description of the economy, we build a systemtem where time is modeled explicitly. Each time step firms trade goods at a price that assures market clearance and firms buying decision expressed in monetary terms is such that it corresponds to the excess profit function. The following algorithm is hold generically it can be combined with different functional forms to replicate different CGE models. The strategy follows [1]. Since we model time explicitly the simulation is a sequence of timesteps t. In each timestep the following happens:
We consider the following (discrete) model of economy. Fixed parameters are:

- the vector $b = \{b_1, \dots, b_N\} \in \mathbb{R}^N$, the matrix $\beta \in \mathbb{R}^{N,N}$, and the matrix $\gamma \in \mathbb{R}^{F,N}$ describe some fixed parameters related to the production function of the industries. In particular, the parameters related to the j-th industry are given by b_j and the column vectors $\{\beta_{i,j}\}_{i=1}^N$ and $\{\gamma_{i,j}\}_{i=1}^F$.
- the vector $\alpha = \{\alpha_i\}_{i=1}^N \in \mathbb{R}^N$ describes some fixed parameters of the household.
- investments and import-export have fixed requests $r \in \mathbb{R}^{N+2,2}$, with $r_{N+1,1} = r_{N+2,1} = r_{N+1,2} = r_{N+2,2} = 0$.
- a vector $\phi \in \mathbb{R}^{N+2}$ of fixed price stickiness (each entry is in the interval $[0, 1]$).

The initial condition reads:

- the vector $w^{(0)} = \{w_1^{(0)}, \dots, w_{N+1}^{(0)}\} \in \mathbb{R}^{N+1}$ is the *wealth* vector. The first N components describe the initial wealth of the industries (an entry for each industry). The last entry describes the initial wealth of the household.
- the vector $q^{(0)} = \{q_1^{(0)}, \dots, q_{N+2}^{(0)}\} \in \mathbb{R}^{N+2}$ is the initial *quantity-of-products* vector. The first N components describe the initial quantity for the industries (an entry for each industry). The entry $q_{N+1}^{(0)}$ is the initial quantity of labour, whilst $q_{N+2}^{(0)}$ is the initial quantity of capital.
- the vector $p^{(0)} = \{p_1^{(0)}, \dots, p_{N+2}^{(0)}\} \in \mathbb{R}^{N+2}$ is the initial price (of product) vector. The first N components describe the initial prices of the industries an entry for each industry. The entry $p_{N+1}^{(0)}$ is the initial price of labour, whilst $p_{N+2}^{(0)}$ is the initial price of capital.

Evolution in time occurs in the following order ($t = 0, 1, \dots, T$):

1. First, a matrix $d \in \mathbb{R}^{N+2,N+3}$ of demands is generated. The first N columns of d are computed by maximing

$$f_j\left(\left\{\frac{d_{ij}}{p_i^{(t-1)}}\right\}_{i=1}^{N+2}\right) \quad \text{s.t.} \quad \sum_{i=1}^{N+2} d_{ij} = w_j^{(t)} \,,$$

where f_j is as in (5); the $(N+1)$-th column is computed by maximizing

$$\prod\nolimits_{i=1}^{N} (d_{i,N+1})^{\alpha_i} \quad \text{s.t.} \quad \sum_{i=1}^{N} d_{i,N+1} = w_{N+1}^{(t)}$$

and setting $d_{N+1,N+1} = d_{N+2,N+1} = 0$. The entries of $(N+2)$-th and the $(N+3)$-th columns are copied from the fixed request matrix r.

2. Then, the price vector is updated in the following way

$$p_j^{(t)} = \phi_j \left[\frac{\sum_{i=1}^{N+3} d_{j,i}}{q_j^{(t)}} \right] + (1-\phi_j) p_j^{(t-1)}$$

3. as a consequence, the first N entries of the quantity vector are updated according to the rule

$$q_i^{(t+1)} = q_i^{(t)} - \min(1, c_i)\, q_i^{(t)} + f_i \left(\left\{ \min\left(1, \frac{1}{c_j}\right) \frac{d_{j,i}}{p_j^{(t)}} \right\}_{j=1}^{N+2} \right),$$

where

$$c_i = \frac{\sum_{j=1}^{N+3} d_{i,j}}{p_i^{(t)} q_i^{(t)}}.$$

On the other hand, the remaining quantities are kept constant, that is,

$$q_{N+1}^{(t+1)} = q_{N+1}^{(t)} \quad \text{and} \quad q_{N+2}^{(t+1)} = q_{N+2}^{(t)}$$

4. and, again as a consequence, the first N entries of the wealth vector evolve according to

$$w_i^{(t+1)} = w_i^{(t)} + \min(1, c_i) \left((1-\tau_i) p_i^{(t)} - \tilde{\tau}_i \right) q_i^{(t)} - \sum_{j=1}^{N+2} \min\left(1, \frac{1}{c_j}\right) d_{j,i}$$

whilst the last entry evolves according to

$$w_{N+1}^{(t+1)} = w_{N+1}^{(t+1)} + \sum_{\ell=N+1}^{N+2} \min(1, c_\ell)\, p_\ell^{(t)} q_\ell^{(t)} - \sum_{j=1}^{N} \min\left(1, \frac{1}{c_j}\right) d_{j,N+1}$$

3 Simulation

The model has been implement the ABCE Agent-Based Computable Economy framework in python [5] and can be accessed online from http://52.90.210.1/. In order to test the validity of the model we run the simulation with the taxes - j - as implied by the calibration. The computational complete economy model must and does asymptotically reproduce the social accounting matrix. (Table 1 and 2 in the appendix) It is therefore asymptotically equivalent to a CGE model. We repeat this exercise with various output tax rates and compare the results with the outcomes of CGE model. It also here produces asymptotically the results CGE models produce as equilibrium results. The most interesting application of this CCE model and the underlying CGE model is the introduction of a tax on carbon. For this we modify (6) to include a further carbon tax: max j=(1-j)pjyj-carbejyj-ipixij-fpfvfj (6) subjected to yjx1j, ..., xij=bjixijijfvfjfj Where carbis the carbon tax, ejis the carbon emission in tons of CO2 per unit of output yj. It is important to note that -carbejyj, does not enter the maximization in step 1 of the CCE algorithm, but it implicitly enters the algorithm by modifying the capital available for buying inputs. The introduction of a tax on carbon produces equivalent results only in a limited range. From 0 to 82.4 dollars per ton of carbon, the result of the CCE model and the CGE model are asymptotically the same, but above this the CCE model starts to oscillate instead of converging to the CGE result. Interestingly the mean of the oscillation is still the same as the CGE result. We regard this as a weakness of the underlying CGE model rather than the CCE model. It has been demonstrated in [6], [7] and [8], the introduction to tax distortion leads to non uniqueness of the equilibrium. Exemplary we look at a 50$ per ton carbon-dioxide tax. In this stage, we have not empirically calibrated the adaptation speed of wage, prices and the technology. To illustrate out of equilibrium transitions we assume intermediary values of 0.5 for each. Asymptotically the model produces the same emissions as a CGE model produces in equilibrium - the total CO2 emissions decrease from 5834 million tons to 3814 million tons. But the transitions tell a richer story. While the CO2 emissions produced by the oil and the gas sector decrease, with fluctuations and temporarily produce less emissions than in equilibrium, the CO2 production of the coal sector do initially increase and then approximate the equilibrium output.

4 Conclusion

We have build a model that gives the same asymptotic results as a CGE model. This model has the same constraints and limitations as a CGE model: one representative agent per sector and multiple equilibria in face of tax distortions. The model demonstrates that there are transitional paths before equilibrium. Depending on the speed of adaptation of the production technology, prices and wages the transitory output of CO2 can be substantially different from the equilibrium result. The conclusion is that clearly CGE models are too simplistic and must be replaced to adequately assess environmental policy. We have demonstrated that CGE calibration techniques can be used to calibrate an agent-based model and that in an ABM that structurally similar to a CGE model the results are asymptotically identical. We do not propose to stop there. Agent-Based models have the potential to be much richer and allow to model more realistic assumptions than general equilibrium models. With this paper we hope to encourage researchers to build richer agent-based models that use CGE calibration techniques, but are not asymptotically equivalent.

References

1. Gualdi, S., Mandel, A.: On the emergence of scale-free production networks. Documents de travail du centre deconomie de la Sorbonne. 60 (2015)
2. Wing, Ian Sue: Computable General Equilibrium Models and Their Use in Economy-Wide Policy Analysis. Joint Program on the Science and Policy of Global Change Technical Note 6, September 2004
3. Hosoe, Nobuhiro, Kenji Gasawa, and Hideo Hashimoto. "The Standard CGE Model." Textbook of Computable General Equilibrium Modelling. Palgrave Macmillan UK, 2010. 87-121.
4. Taghawi-Nejad, D., Veetil, V. The Effects of Monetary Shocks on the

Distribution of Prices. Working paper. August 2016
5. Taghawi-Nejad, D. Modelling the economy as an agent-based process: Abce, a modelling platform and formal language for ace. Journal of Artificial Societies and Social Simulation 16.3 (2013): 1.
6. Foster, E., & Sonnenschein, H. (1970). Price Distortion and Economic Welfare. Econometrica, 38(2), 281297. http://doi.org/10.2307/1913010
7. Whalley, J., & Zhang, S. (2014). Tax induced multiple equilibria. Katalog BPS, XXXIII(2), 8187. http://doi.org/10.1007/s13398-014-0173-7.2
8. Kehoe, T. J. (1991). Chapter 38 Computation and multiplicity of equilibria. Handbook of Mathematical Economics, 4(C), 20492144. http://doi.org/10.1016/S1573-4382(05)80013-X