1. (20 points) Show that the stationary point (zero gradient) of the function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is a saddle (with indefinite Hessian).

Find the directions of downslopes away from the saddle. To do this, use Taylor's expansion at the saddle point to show that

$$f(x_1, x_2) = f(1, 1) + (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2),$$

with some constants a, b, c, d and $\partial x_i = x_i - 1$ for $i = 1, 2$. Then the directions of downslopes are such $(\partial x_1, \partial x_2)$ that

$$f(x_1, x_2) - f(1, 1) = (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2) < 0.$$

Q1)

Stationary pt = $\frac{\partial f}{\partial x_1} = 0$ & $\frac{\partial f}{\partial x_2} = 0$

$\frac{\partial f}{\partial x_1} = 4x_1 - 4x_2 = 0$ $\qquad$ $\frac{\partial f}{\partial x_2} = 0 - 4x_1 + 3x_2 + 1 = 0$

$x_1 = x_2$, $\quad -4x_1 + 3x_1 = -1$

$-1x_1 = -1$

$x_1 = 1$ ✓

Stationary pt at $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$f(1, 1) = 2 - 4 + 1.5 + 1 = 0.5$

$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$g(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 + 0 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$ $\qquad$ $g(1, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} \\ - & \frac{\partial^2 f}{(\partial x_2)^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$, $\qquad$ $|H - \lambda I| = 0$

$\begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix} = 0$

$(4-\lambda)(3-\lambda) - (-4)(-4)$

$12 - 7\lambda + \lambda^2 - 16$ $\qquad$ $\sqrt{65} > 7 \therefore \lambda_2 > 0$

$\lambda^2 - 7\lambda - 4$

$\lambda = \frac{7+\sqrt{65}}{2}, \frac{7-\sqrt{65}}{2}$

Since λ_1 & λ_2 have + & − values the point $[1, 1]^T$ is a saddle pt.

Using the general Taylor Series,

$f(x_s) = f(x_s) + g(x_s)^T(x - x_s) + \frac{1}{2}(x - x_s)^T H|_{x_s}(x - x_s)$

$= 0.5 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \frac{1}{2}\left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

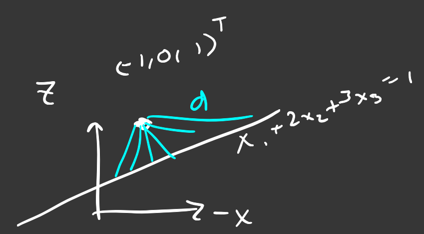
Continuing, let $X - X_s = \partial x_s$

$$f(x) = f(x_s) + 0 + \frac{1}{2}\partial x_s^T H \partial x_s$$

let $f(x) - f(x_s) = \partial f(x_s)$

$$\partial f(x_s) = \frac{1}{2}\partial x_s^T H \partial x_s$$

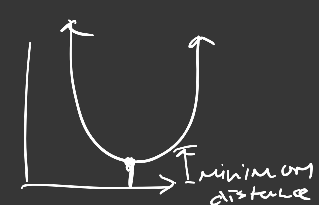
$$\frac{1}{2}\underbrace{\begin{bmatrix}\partial x_1\\ \partial x_2\end{bmatrix}^T}_{1\times 2}\underbrace{\begin{bmatrix}4 & -4\\ -4 & 3\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}\partial x_1\\ \partial x_2\end{bmatrix}}_{2\times 1}$$



2. (a) (10 points) Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in $\mathbb{R}^3$ that is nearest to the point $(-1, 0, 1)^T$. Is this a convex problem? Hint: Convert the problem into an unconstrained problem using $x_1 + 2x_2 + 3x_3 = 1$.

   (b) (40 points) Implement the gradient descent and Newton's algorithm for solving the problem. Attach your codes in the report, along with a short summary of your findings. The summary should include: (1) The initial points tested; (2) corresponding solutions; (3) A log-linear convergence plot.

$x_1 + 2x_2 + 3x_3 = 1$



Q2a) $f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3 = 1$, $f(x) \in \mathbb{R}^3$

nearest to $(-1, 0, 1)^T = (x_1, x_2, x_3)^T$

$Ax = b$ where $x = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ & $b = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

Q2b)

Q3) Prove hyperplane is a convex set

S is convex iff & only if $\forall x_1, x_2 \in S$ & $\forall \lambda \in [0,1]$

Let $a^T x = c$ for $x \in \mathbb{R}^n$

& $H = \{ x \in \mathbb{R}^n \mid a^T x = c \}$ be a hyperplane in $\mathbb{R}^n$

let $x_1, x_2 \in H$

$a^T x_1 = c$

$a^T x_2 = c$

$a^T(\lambda x_1 + (1-\lambda) x_2) =$

$(\lambda c + (1-\lambda) c) = c$

thus

$(\lambda c + (1-\lambda) c) \in H$

4)

4. (15 points) Consider the following illumination problem:

$$\min_{\mathbf{p}} \quad \max_{k}\{h(\mathbf{a}_k^T\mathbf{p}, I_t)\}$$

$$\text{subject to:} \quad 0 \le p_i \le p_{\max},$$

where $\mathbf{p} := [p_1, ..., p_n]^T$ are the power output of the n lamps, $\mathbf{a}_k$ for $k = 1, ..., , m$ are fixed parameters for the m mirrors, I_t the target intensity level. $h(I, I_t)$ is defined as follows:

$$h(I, I_t) = \begin{cases} I_t/I & \text{if } I \le I_t \\ I/I_t & \text{if } I_t \le I \end{cases}$$

(a) show convex: $I = a_k^T P$

5) Let $c(x)$ be the cost of Producing x amount of product A & assume that $c(x)$ is differentiable everywhere. Let y = Price set for product. Assuming product is sold out. Total profit

$$c^*(y) = \max_x \{xy - c(x)\}$$

Show $c^*(y)$ is a convex func. w.r.t y

Assuming $c^*(y) \in C^2$