HW5: if only grading 1 problem grade P2, else extra credit would be appreciated

1. Implement SQP to solve the following problem. Use $\mathbf{x}_0 = (1,1)^T$ as the initial guess. Use BFGS approximation for the Hessian of the Lagrangian. Use the merit function and Armijo Line Search to find the step size. Note: You can use an existing quadratic programming solver to solve the QP subproblem.

$$
\begin{aligned}
\min f &= x_1^2 + (x_2 - 3)^2 \\
\text{s.t. } g_1 &= x_2^2 - 2x_1 \le 0 \\
g_2 &= (x_2 - 1)^2 + 5x_1 - 15 \le 0
\end{aligned}
$$

from lecture notes:

① initialize $x_0, \lambda_0 = 0, \mu_0 = 0, W_0 = I, k = 0, \varepsilon = 10^{-3}$ $(h(x_0) = 0, g(x_0) \le 0), A = \phi$

② Calculate $\nabla_x L = \nabla_x f + \lambda^T \nabla_x h + \mu^T \nabla_x g$

③ while $|\nabla_x L| > \varepsilon$. do

3.1 $s_k, [\lambda_{k+1}, \mu_{k+1}] = QP(x_k, \lambda_k, \mu_k, f, h, g, A, W_k)$

3.2 α_k = line Search $(s_k, \lambda_{k+1}, \mu_{k+1}, f, h, g, A)$

3.3 $x_{k+1} = x_k + \alpha_k \cdot s_k$

3.4 W_{k+1} = BFGS$(W_k, \alpha_k s_k, x_k, \lambda_{k+1}, \mu_{k+1}, f, h, g, A)$

3.5 Calculate $\nabla_x L$.

$$\frac{\partial g_1}{\partial x} = \left[\frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2}\right] = [-2, 2x_2]$$

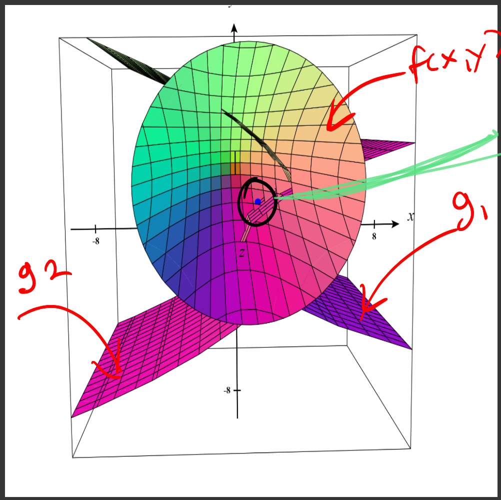
$$\frac{\partial g_2}{\partial x} = [5, 2(x_2 - 1)]$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \partial g_1/\partial x \\ \partial g_2/\partial x \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = [2x_1, 2(x_2 - 3)]$$

from code: Solution at $x = (1.06, 1.45)$

graphical solution



confirmed on graph & minimum value along f

code at:

https://github.com/Davwittma/MAE-598-Design-Optimization/blob/main/Homework5.py

2. Consider a moon lander with state $[h, v, m]^T$ to have the following dynamics:

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{a(t)}{m(t)} \\ \dot{m}(t) = -Ka(t). \end{cases}$$

Here h is the altitude, v is the velocity, and m is the mass of the moon lander. $a(t) \in [0,1]$ is the thrust, and K is a constant fuel burning rate. Let the initial state be $[h_0, v_0, m_0]^T$, and the target be $h(t^*) = 0$ and $v(t^*) = 0$ at terminal time t^*. Derive the optimal control policy for minimal fuel consumption.

Using pontryagin's Maximum principle to solve the optimal control prob.

$$\min_{\alpha(t)} \; C(\alpha) = \int_0^T \alpha(t)\,dt, \quad h(T)=0, \; v(T)=0$$

initial state:

$$\begin{cases} h(0) = h_0 \\ v(t) = V_0 \\ m(t) = m_0 \end{cases}$$

$$f = \begin{bmatrix} v \\ -g + \frac{\alpha(t)}{m(t)} \\ -k\,\alpha(t) \end{bmatrix} \qquad l = \alpha$$

$$H = -l + \lambda^T f = -\alpha + \lambda_1 v + \lambda_2\left(-g + \frac{\alpha(t)}{m(t)}\right) + \lambda_3(-k\,\alpha(t))$$

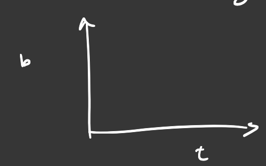
$$\alpha^* = \arg\max H = \arg\max\left(-\alpha + \lambda_1 v + \lambda_2\left(-g + \frac{\alpha(t)}{m(t)}\right) + \lambda_3(-k\,\alpha(t))\right)$$

for $\alpha \in [0,1]$

$$= \arg\max\left(v\lambda_1 - g\lambda_2 + \alpha\underbrace{\left(-1 + \frac{\lambda_2}{m(t)} - \lambda_3 k\right)}_{=b}\right)$$

$$\alpha^* = \begin{cases} 0 & b < 0 \\ 1 & b > 0 \end{cases}$$

Proving monotonic change



↳ if b is continuous and differentiable between

$t = [0, t^*]$

&

$t = [t^*, T]$

the monotonic change is expressed as:

$$\left.\frac{db}{dt}\right|_0^{t^*} \;\&\; \left.\frac{db}{dt}\right|_{t^*}^{T}$$

and $b = \frac{\lambda_2}{m(t)} - 1 - \lambda_2 k$

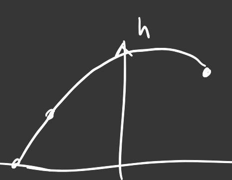
thus $\frac{db}{dt} = \frac{\lambda_2}{\dot{m}}$

where $\dot{m} = 0$ because there is no thrust between $\alpha \in [0, t^*]$ or for $\alpha \in [t^*, T]$ $\dot{m} = -k\alpha(t)$ where α is constant acceleration change from a negative # to a less negative number, thus this can be approximated between both $t = [0, t^*]$ & $t = [t^*, T]$ as monotonic decrease into monotonic inc

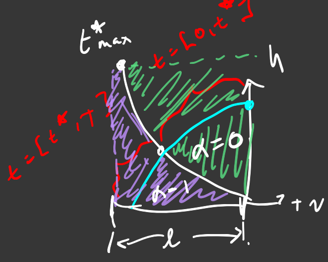
policy $\alpha^* = 0$, for $[0, t^*]$

$\alpha^* = 1$ for $[t^*, T]$

at $t\,[0, t^*]$ $\quad f = \begin{bmatrix} v \\ -g \\ 0 \end{bmatrix}$ $\quad \dot{h} = v$, $\dot{v} = -g$, $\dot{m} = 0$



$t\,[t^*, T]$ $\quad f = \begin{bmatrix} v \\ -g + \frac{\alpha}{m} \\ -k\alpha \end{bmatrix}$ $\quad \begin{cases} \dot{h} = v \\ \dot{v} = -g + \frac{1}{m} \\ \dot{m} = -k \end{cases}$ for $\alpha = 1$



aside: $\int_0^l f\begin{bmatrix} v \\ -g + \frac{\hat{\alpha}}{m} \\ -k\alpha \end{bmatrix} dv \leq m_{fuel}$

this makes sense when thinking about landing with $v(T) = 0$ at $h(T) = 0$