1. (20 points) Show that the stationary point (zero gradient) of the function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is a saddle (with indefinite Hessian).

Find the directions of downslopes away from the saddle. To do this, use Taylor's expansion at the saddle point to show that

$$f(x_1, x_2) = f(1, 1) + (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2),$$

with some constants a, b, c, d and $\partial x_i = x_i - 1$ for $i = 1, 2$. Then the directions of downslopes are such $(\partial x_1, \partial x_2)$ that

$$f(x_1, x_2) - f(1, 1) = (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2) < 0.$$

Q1)

$$f(x_1,x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$g(x_1,x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 + 0 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

for $g(x_1, x_2) = 0$

$= 0$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} \\ - & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix},$$

$$|H - \lambda I| = 0$$

$$\begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - (-4)(-4)$$

$$12 - 7\lambda + \lambda^2 - 16$$

$$\lambda^2 - 7\lambda - 4$$

$$\lambda = \frac{7+\sqrt{65}}{2}, \quad \frac{7-\sqrt{65}}{2}$$

Q2a) $f(x) = x_1 + 2x_2 + 3x_3 = 1$, $f(x) \in \mathbb{R}^3$

nearest to $(-1, 0, 1)^T = (x_1, x_2, x_3)^T$

Q2b)

Q3) Prove hyperplane is a convex set

$a^T x = c$ for $x \in \mathbb{R}^n$

S is convex iff & only if $\forall\, x_1, x_2 \in S$

& $\forall \lambda \in [0,1]$

$\lambda x_1 + (1-\lambda) x_2 \in S$

$S = \{ x \in \mathbb{R}^n \mid a^T x = c \}$