1. (20 points) Show that the stationary point (zero gradient) of the function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is a saddle (with indefinite Hessian).

Find the directions of downslopes away from the saddle. To do this, use Taylor's expansion at the saddle point to show that

$$f(x_1, x_2) = f(1, 1) + (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2),$$

with some constants a, b, c, d and $\partial x_i = x_i - 1$ for $i = 1, 2$. Then the directions of downslopes are such $(\partial x_1, \partial x_2)$ that

$$f(x_1, x_2) - f(1, 1) = (a\partial x_1 - b\partial x_2)(c\partial x_1 - d\partial x_2) < 0.$$

Q1)

Stationary pt = $\frac{\partial f}{\partial x_1} = 0$ & $\frac{\partial f}{\partial x_2} = 0$

$\frac{\partial f}{\partial x_1} = 4x_1 - 4x_2 = 0$ $\qquad$ $\frac{\partial f}{\partial x_2} = 0 - 4x_1 + 3x_2 + 1 = 0$

$x_1 = x_2$, $\quad -4x_1 + 3x_1 = -1$

$-1x_1 = -1$

$x_1 = 1$ ✓

Stationary pt at $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$f(1,1) = 2 - 4 + 1.5 + 1 = 0.5$

$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

$g(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 + 0 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$ $\qquad$ $g(1,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} \\ - & \frac{\partial^2 f}{(\partial x_2)^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$, $\qquad |H - \lambda I| = 0$

$\begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix} = 0$

$(4-\lambda)(3-\lambda) - (-4)(-4)$

$12 - 7\lambda + \lambda^2 - 16$ $\qquad$ $\sqrt{65} > 7 \therefore \lambda_2 > 0$

$\lambda^2 - 7\lambda - 4$

$\lambda = \frac{7 + \sqrt{65}}{2}, \frac{7 - \sqrt{65}}{2}$

Since λ_1 & λ_2 have + & − values the point $[1, 1]^T$ is a saddle pt.

Using the general Taylor Series,

$f(x_s) = f(x_s) + g(x_s)^T (x - x_s) + \frac{1}{2}(x - x_s)^T H|_{x_s} (x - x_s)$

$= 0.5 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \frac{1}{2}\left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \left(x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Continuing, let $X - X_s = \partial x_s$

$$f(x) = f(x_s) + 0 + \frac{1}{2} \partial x_s^T H \partial x_s$$

let $f(x) - f(x_s) = \partial f(x_s)$

$$\partial f(x_s) = \frac{1}{2} \partial x_s^T H \partial x_s$$

$$\frac{1}{2} \underbrace{\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}^T}_{1 \times 2} \underbrace{\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}}_{2 \times 1}$$

Q2a) $f(x) = x_1 + 2x_2 + 3x_3 = 1$, $f(x) \in \mathbb{R}^3$

nearest to $(-1, 0, 1)^T = (x_1, x_2, x_3)^T$

Q2b)

Q3) Prove hyperplane is a convex set

$$a^T x = c \quad \text{for } x \in \mathbb{R}^n$$

S is convex iff & only if $\forall\, x_1, x_2 \in S$

& $\forall\, \lambda \in [0,1]$

$$\lambda x_1 + (1-\lambda) x_2 \in S$$

$$S = \{ x \in \mathbb{R}^n \mid a^T x = c \}$$