$\mathcal{U}$-Bootstrap percolation

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Update rules

- An update rule is a finite set $X \subseteq \mathbb{Z}^2 - \{0\}$
- An update family is a finite collection of update rules $\mathscr{U} = \{X \subseteq \mathbb{Z}^2 - \{0\}\}$

$\mathscr{U}$-Bootstrap percolation initialized at A refers to the following process:

- $A_0 = A$
- $A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2 : x + X \subseteq A_t \textit{ for some } X \in \mathscr{U}\}$

- The set A is known as the set of initially infected sites
- The closure of A is defined as $[A] = \cup_{t \geq 0} A_t$
- The initialization is random i.e. each site (vertex) in $\mathbb{Z}^2$ is infected with probability p independently from the other vertices
- The process is monotone i.e. if a site gets infected, it stays infected forever
- After the initialization, the process is deterministic in the sense that a site will get infected if and only if there is some rule X in $\mathscr{U}$ such that $x + X$ is infected

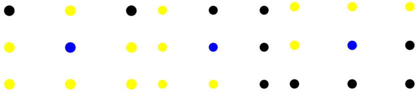

Figure: Oriented site, rules U_1 and U_2 for spiral model

- r-Neighbour models for r=1,2,3,4
- Oriented site $\mathscr{U} = \{(-1,1),(1,1)\}$
- Spiral $\mathscr{U} = \{U_1, U_2, U_3, U_4\}$, where
  $U_1 = \{(1,-1),(1,0),(1,1),(0,1)\}$
  $U_2 = \{(1,-1),(1,0),(-1,-1),(0,-1)\}$
  $U_3 = -U_1, U_4 = -U_2$
- Directed triangular bootstrap percolation

Stable directions, basic properties

For a vector $u \in \mathbb{S}^1$, we define $\mathbb{H}_u = \{x \in \mathbb{Z}^2 | < x, u >< 0\}$.

Definition

Given an update family $\mathscr{U}$, a direction $u \in \mathbb{S}^1$ is

- stable if $[\mathbb{H}_u] = \mathbb{H}_u$. The set of stable directions is denoted by $\mathscr{S} = \mathscr{S}U$
- strongly stable if $u \in int\mathscr{S}$
- unstable if it is not stable

- Dichotomy $[\mathbb{H}_u] \in \{\mathbb{H}_u, \mathbb{Z}^2\}$
- $\mathscr{S} \subseteq \mathbb{S}^1$ is a set of stable directions for some update familit $\mathscr{U}$ if and only if it can be expressed as a union of closed intervals with rational endpoints[1] in $\mathbb{S}^1$

[1] A direction $u \in \mathbb{S}^1$ is said to be rational if there is a point in the grid $\mathbb{Z}^2 \cap \{\lambda u | \lambda \in \mathbb{R}\}$ 1

Classification of $\mathscr{U}$-Bootstrap percolation

$\mathscr{U}$-bootstrap percolation update families exhibit different properties based on their stable sets. Let $\mathscr{U}$ be an update family with a set of stable directions $\mathscr{S}$

- If there is a open semicircle C such that $\mathscr{S} \cap C = \emptyset$ then $\mathscr{U}$ is said to be **supercritical**
- If every open semicircle C intersects $\mathscr{S}$, but there is an open semicircle C_0 that doesn't intersect $int \mathscr{S}$ then $\mathscr{U}$ is said to be **critical**
- If every open semicircle C intersects $intS$ then $\mathscr{U}$ is said to be **critical**

Supercritical and critical families

Infection time of the origin

The infection time of 0 is defined as $\tau_p = \inf\{t \in \mathbb{N} : 0 \in A_t\}$, given that $A_0 = A$ is sampled according to a Bernoulli p distribution

- For supercritical families, $\tau_p = p^{-\Theta(1)}$ as $p \to 0$ with high probability
- For critical families, $\tau_p = \exp(p^{-\Theta(1)})$ as $p \to 0$ with high probability

Corollary: For supercritical and critical families, $p_c = \inf\{p > 0 | P_p([A] = \mathbb{Z}^2) = 1\} = 0$ i.e. for any $p > 0$ we have percolation.
However, for subcritical families the situation is different.

d_u^θ measures directions that are difficult to infect

Critical densities with conic boundary conditions

For $u \in \mathbb{S}^1$ and $\theta \in [-\pi, \pi]$

$$d_u^\theta := \inf \left\{ q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]) < \infty \right\}$$

Morally, the critical probability with infection of $V_{u,u+\theta} = \mathbb{H}_u \cap \mathbb{H}_{u+\theta}$.

- The summand decays slowly in n when it is hard to infect the origin using only infections at distance less than n. So, when it is hard to infect 0, d_u^θ is large[2].
- When $\theta \sim \pm\pi$, few sites are infected, so it is easy for the origin not to be infected, the summand can be large. Hence, d_u^θ decreases when $\theta \to 0$
- $d_u^\pm := \lim_{\theta \to 0^\pm} d_u^\theta$ can be large when a small number of infections is not enough to infect the origin, even with a

Theorem

For any $\mathcal{U}$-bootstrap percolation model, its critical probability

$$\tilde{q}_c = \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [A \cap B_n]) < \infty\}$$

is equal to the maximal value of its critical density function

$$d_u = \max_{0^\pm} \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [(A \cup V_{u,u+0^\pm}) \cap B_n] < \infty\}$$

for u in any semicircle C, i.e.,

$$\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

Let's denote $E_{u,\theta} = \{0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]\}$. Then,

$$E_{u,\pm\pi} = \{0 \notin [A \cap B_n]\} \supset E_{u,\theta}$$

which gives that the following holds for any u

$$\tilde{q}_c \geq \sup_{\theta} \sup_{u} d_u^{\theta} \geq \lim_{\theta \to 0} \sup_{u} \sup d_u^{\theta} = \sup_{u} d_u.$$

The theorem states that all those quantities are equal.

Meaning of the theorem

The difficulty of the model is as hard as its most difficult direction. In this direction, infecting a half plane doesn't affect the infection of the origin.

Proving $\sup d_u \geq \tilde{q}_c$

The goal is to show, that for any $q' > \sup d_u$ it holds that

$$\sum_n n\mathbb{P}_{q'}(0 \notin [A \cap B_n]) < \infty.$$

The idea is to show that, at q', the origin is infected most of the time.

2-step percolation : $q' = \sup d_u + \varepsilon$

1. Infect sites with probability ε to find some structures
2. Infecting new sites with probability q allows structures to grow

Some details on the proof

- The structures that grow are droplets, with sides $(u_i)_{i=1}^n$ depending on $\sup d_u$.
- In the second percolation, droplets of size L grow into droplets of size $\geq (1+\delta)L$, for some $\delta > 0$.
- The proof can be done in any semi-circle, so we can get $\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$
- The proof contains that $\forall q > \sup d_u$, there exists a constant $c(q) > 0$ such that

$$\theta_n(q) \leq e^{-c(q)n}$$

Applying the theorem

Theorem

For any update rules $\mathcal{U}$,

$$q_c \leq \tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

In particular, if $\mathcal{U}$ is not subcritical, then $\tilde{q}_c = q_c = 0$

So, having knowledge on $u \mapsto d_u$ allows to upper bound q_c...

Proposition : (It's harder for submodels to infect)

For any sub-collection of rules $\mathcal{U}' \subset \mathcal{U}$

$$q_c(\mathcal{U}) \leq \tilde{q}_c(\mathcal{U}) \leq \inf_{C} \sup_{u \in C} d_u(\mathcal{U}')$$

... and it is not even necessary to know the critical density for the whole set of rules to get such bounds.

First level bound

$DTBP$: Directed Triangular Bootstrap Percolation

Let $\mathcal{U}' = \{(-1,-1),(0,1)\}$, one of the rules of DTBP, then

$$q_c(DTBP) \leq \tilde{q}_c(DTBP) \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u(\mathcal{U}').$$

Applying a general formula for one rule families (using OP) gives $q_c(DTBP) \leq 0.245...$[a]

[a] Previous known bound was 0.312

Second level bound

However, knowing one rule subfamilies is not enough.

Spiral

For spiral, it is possible to compute d_u for all pairs of rules, such that the difficulty on pairs is the same as the difficulty of some Bidirectional OP :

$$q_c(Spiral) \leq \tilde{q}_c(Spiral) \leq 1 - p_c^{OP}.$$

And the result is tight.