$\mathcal{U}$-Bootstrap percolation

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Outline of the presentation

1. Introduction to $\mathcal{U}$-bootstrap and examples
2. Universality classes and stable directions
3. critical densities
4. Applications of critical densities on Spiral and DTBP
5.
6. Conclusion and open questions

Definition

For $u \in \mathbb{S}^1$ and $\theta \in [-\pi, \pi]$

$$d_u^\theta = \inf\left\{ q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [((A \cup V_{u,u+\theta} \cap B_n)]) < \infty \right\}$$

Morally, it is the critical probability with infection of $V_{u,u+\theta} = \mathbb{H}_u \cap \mathbb{H}_{u+\theta}$.

Definition

We call $u \mapsto d_u = \max(d_u^+, d_u^-)$, where $d_u^\pm := \lim_{\theta \to 0^\pm}$, the *critical density function* of the model.

Theorem

$$\tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u$$

d_u and $\tilde{q}_c$

$$d_u^\theta := \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [((A \cup V_{u,u+\theta} \cap B_n)]) < \infty\}$$

$$\tilde{q}_c := \inf\{q \in [0,1], \sum_n n\theta_n(q) < \infty\}$$

Observing

$$\theta_n(q) := \mathbb{P}_q(0 \notin [A \cap B_n]) \geq \mathbb{P}_q(0 \notin [((A \cup V_{u,u+\theta} \cap B_n)])$$

Proposition

$$d_u^\theta \leq \tilde{q}_c$$

d_u and $\tilde{q}_c$

Using the previous observation we obtain

$$\tilde{q}_c \geq \sup_{u \in \mathbb{S}^1} \geq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$$

Thus, to prove the equality, it is left to prove

$$\tilde{q}_c \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$$

which means "at $q = \inf_C \sup_{u \in C} d_u$, it holds that $\sum_n n\theta_n(q)$ is finite". In even simple words, $\theta_n(q)$ decays sufficiently fast when q is the maximal critical density of any semi-circle.
A consequence of classification in universality classes is that for any $q > 0$, it is true that $\sum_n n\theta_n(q) < \infty$ for critical and super-critical models.

Proving $\inf \sup d_u = \tilde{q}_c$

Sketch of the proof

1. Pick a q' slightly larger than $\sup_u d_u$
2. At such a q', some infected sets with specific structure, will grow into larger infected sets of the same structure
3. Such sets will infect the origin after a sufficiently long time

Things to check

1. The proof must not depend on a specific choice of semi-circle
2. It must hold for every $q' > \sup d_u$

The "structured sets" will be droplets, with sides $(u_i)_{i=1}^n$ depending on $\inf \sup d_u$.
Then split q' as $q' = \varepsilon + \sup d_u$ and study the percolation as the union of two percolations

1. In the "ε-percolation", there exists a droplet of size L
2. In the "$\sup d_u$-percolation", a droplet of size L grows into a droplet of size $(1+\delta)L$, for some $\delta > 0$.

Theorem

For any update rules $\mathcal{U}$,

$$q_c \leq \tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

In particular, if $\mathcal{U}$ is not subcritical, then $\tilde{q}_c = 0$.

So, having knowledge on $u \mapsto d_u$ allows to upper bound q_c...

Proposition

For any sub-collection of rules $\mathcal{U}' \subset \mathcal{U}$

$$q_c(\mathcal{U}) \leq \tilde{q}_c(\mathcal{U}) \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u(\mathcal{U}')$$

... and it is not even necessary to know the critical density for the whole set of rules to get such bounds !

The basic bound, application to DTBP

Directed Triangular Bootstrap Percolation consists of 3 rules, each rule checking two vertices to infect a new vertex. The basic bound consists in applying the previous bound on one rule, e.g. $\mathcal{U}' = \{(-1,-1),(0,1)\}$.
Observing that $L(x,y) = (x, y-x)$ sends $\mathcal{U}'$ to $\{(-1,0),(0,1)\}$, the latter is Oriented Percolation, rotated by $\pi/4$.
Applying the basic bound gives

$$q_c(DTBP) \leq \sup_{u \in C} \min_i(\{U_i\}) \leq \sup_{u' \in C} d_{u'}^{OP}.$$

Once explicit values are computed for OP, we will obtain a bound on $q_c(DTBP)$.[1]

[1] Spoiler: This will improve the previous best known bound from 0.312 to $0.245\cdots$.

The second level bound, application to Spiral

However, the basic bound is not tight and it is even to find two rules U_1, U_2 such that $d(\{U_1, U_2\}$ is always strictly smaller that $\min(d(\{U_1\}), d(\{U_2\}))$.
A bright side of the general bound we obtained is that it can be applied for any sub-collection of rules. The Spiral model is an example where computing critical density for all pairs of rules can be done and gives a tight result.

Theorem

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$$q_c = 1 - p_c^{OP} = \tilde{q}_c$$