$\mathcal{U}$-Bootstrap percolation : critical probability, exponential decay and applications, by Ivailo HARTARSKY

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Update rules

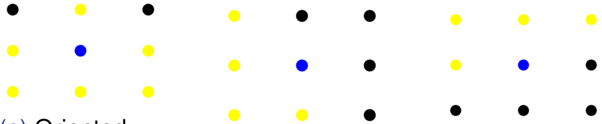
- An update rule is a finite set $X \subseteq \mathbb{Z}^2 - \{0\}$
- An update family is a finite collection of update rules $\mathcal{U} = \{X \subseteq \mathbb{Z}^2 - \{0\}\}$

$\mathcal{U}$-Bootstrap percolation initialized at A refers to the following process:

- $A_0 = A$
- $A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2 : x + X \subseteq A_t \textit{ for some } X \in \mathcal{U}\}$

- The set A is known as the set of initially infected sites
- The closure of A is defined as $[A] = \cup_{t \geq 0} A_t$
- The initialization is random i.e. each site (vertex) in $\mathbb{Z}^2$ is infected with probability p independently from the other vertices
- The process is monotone i.e. if a site gets infected, it stays infected forever
- After the initialization, the process is deterministic in the sense that a site will get infected if and only if there is some rule X in $\mathcal{U}$ such that $x + X$ is infected

Examples



(a) Oriented site percolation

(b) U_1 for Spiral

(c) U_2 for Spiral

- r-Neighbour models for r=1,2,3,4
- Oriented site $\mathcal{U} = \{(-1, 1), (1, 1)\}$
- Spiral $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$, where
  $U_1 = \{(1, -1), (1, 0), (1, 1), (0, 1)\}$
  $U_2 = \{(1, -1), (1, 0), (-1, -1), (0, -1)\}$
  $U_3 = -U_1, U_4 = -U_2$
- Directed triangular bootstrap percolation

Stable directions, basic properties

For a vector $u \in \mathbb{S}^1$, we define $\mathbb{H}_u = \{x \in \mathbb{Z}^2 | < x, u >< 0\}$.

Definition

Given an update family $\mathcal{U}$, a direction $u \in \mathbb{S}^1$ is

- stable if $[\mathbb{H}_u] = \mathbb{H}_u$. The set of stable directions is denoted by $\mathcal{S} = \mathcal{S}(\mathcal{U})$
- strongly stable if $u \in int\mathcal{S}$
- unstable if it is not stable

- Dichotomy $[\mathbb{H}_u] \in \{\mathbb{H}_u, \mathbb{Z}^2\}$
- $\mathcal{S} \subseteq \mathbb{S}^1$ is a set of stable directions for some update familit $\mathcal{U}$ if and only if it can be expressed as a union of closed intervals with rational endpoints[1] in $\mathbb{S}^1$

[1] A direction $u \in \mathbb{S}^1$ is said to be rational if there is a point in the grid $\mathbb{Z}^2 \cap \{\lambda u | \lambda \in \mathbb{R}\}$ 1

Classification of $\mathcal{U}$-Bootstrap percolation

$\mathcal{U}$-bootstrap percolation update families exhibit different properties based on their stable sets. Let $\mathcal{U}$ be an update family with a set of stable directions $\mathcal{S}$

- If there is a open semicircle C such that $\mathcal{S} \cap C = \emptyset$ then $\mathcal{U}$ is said to be *supercritical*
- If every open semicircle C intersects $\mathcal{S}$, but there is an open semicircle C_0 that doesn't intersect *int*$\mathcal{S}$ then $\mathcal{U}$ is said to be *critical*
- If every open semicircle C intersects *intS* then $\mathcal{U}$ is said to be *subcritical*

Supercritical and critical families

Infection time of the origin

The infection time of 0 is defined as $\tau_p = \inf\{t \in \mathbb{N} : 0 \in A_t\}$, given that $A_0 = A$ is sampled according to a Bernoulli p distribution

- For supercritical families, $\tau_p = p^{-\Theta(1)}$ as $p \to 0$ with high probability
- For critical families, $\tau_p = \exp(p^{-\Theta(1)})$ as $p \to 0$ with high probability

Corollary[2]: For supercritical and critical families, $p_c = \inf\{p > 0 | P_p([A] = \mathbb{Z}^2) = 1\} = 0$ i.e. for any $p > 0$ we have percolation.

[2] BOLLOBÁS, SMITH, UZZEL, Monotone Cellular automata in a random environment, *Combinatorics, Probability and Computing*, 2015

Subcritical Families

- For subcritical families $p_c > 0$[3]
- Percolation at $p = p_c$ is open
- Behaviour of τ_p as $p \to p_c$ from above is open
- Exponential decay answered
- Infinite component without percolation answered

[3] Balister, Bollobás, Przykucki, Smith, Subcritical $\mathcal{U}$-Bootstrap percolation models have non-trivial phase transitions, *arXiv*,2019

d_u^θ measures directions that are difficult to infect

Critical densities with conic boundary conditions

For $u \in \mathbb{S}^1$ and $\theta \in [-\pi, \pi]$

$$d_u^\theta := \inf \left\{ q \in [0,1], \sum_n n \mathbb{P}_q(0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]) < \infty \right\}$$

Morally, the critical probability with infection of $V_{u,u+\theta} = \mathbb{H}_u \cap \mathbb{H}_{u+\theta}$.

- The summand decays slowly in n when it is hard to infect the origin using only infections at distance less than n. So, when it is hard to infect 0, d_u^θ is large[4].
- When $\theta \sim \pm\pi$, few sites are infected, so it is easy for the origin not to be infected, the summand can be large. Hence, d_u^θ decreases when $\theta \to 0$.

[4] non zero...

Theorem

For any $\mathcal{U}$-bootstrap percolation model, its critical probability

$$\tilde{q}_c = \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [A \cap B_n]) < \infty\}$$

is equal to the maximal value of its critical density function

$$d_u = \max_{0^\pm} \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [(A \cup V_{u,u+0^\pm}) \cap B_n] < \infty\}$$

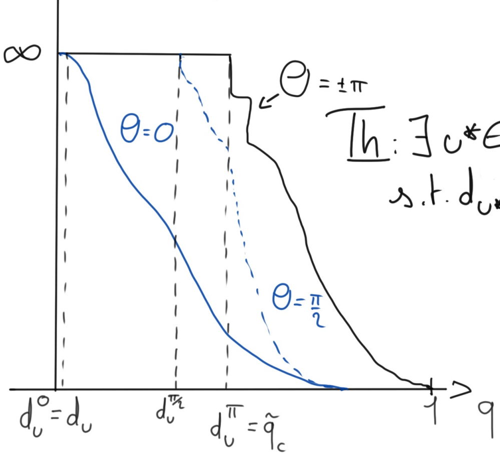
for u in any semicircle C, i.e.,

$$\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

Phase diagram in a fixed direction u

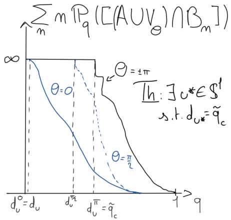
$$0 \notin$$

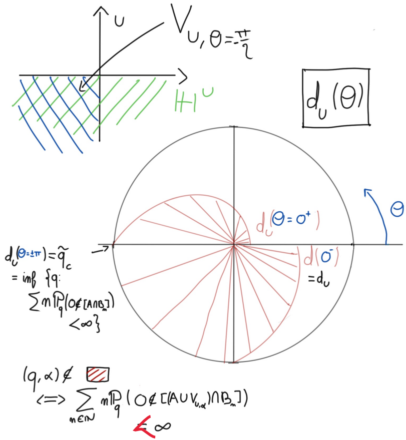
$$\sum_n n \, \mathbb{P}_q([A \cup V_\theta] \cap B_n])$$



$\underline{\text{Th}}$: $\exists\, u^* \in S^1$ s.t. $d_{u^*} = \tilde{q}_c$

Phase diagram in a fixed direction u





Monotonicity

Let's denote $E_{u,\theta} = \{0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]\}$. Then,

$$E_{u,\pm\pi} = \{0 \notin [A \cap B_n]\} \supset E_{u,\theta}$$

which gives that the following holds for any u

$$\tilde{q}_c \geq \sup_{\theta} \sup_{u} d_u^{\theta} \geq \lim_{\theta \to 0} \sup_{u} d_u^{\theta} = \sup_{u} d_u.$$

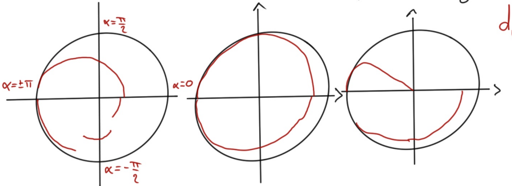
The theorem states that all those quantities are equal.

Meaning of the theorem

The difficulty of the model is as hard as its most difficult direction. In this direction, infecting a half plane doesn't affect the infection of the origin.

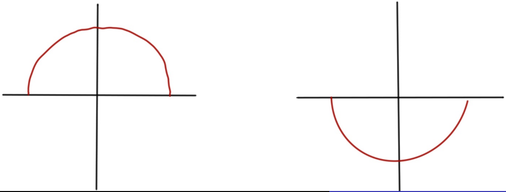
Consequence of monotonicity

For $u_1, u_2, \ldots \in S^1$, we have phase diagrams



$d_u(\alpha) = \inf\{q : \sum_n \mathbb{P}_n(0 \notin [A \cup V_{u,\alpha}, n]) < \infty\} \in [0, \tilde{q}_c]$

Th: $\exists\, u^* \in S^1$ with one of the following phase diagrams on $[0,\pi]$ or $[-\pi,0]$



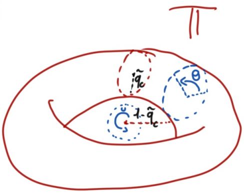
Th: (Bollobás-... & Hartarsky)

For any update family $\mathcal{U}$,

the family is:

- subcritical if and only if

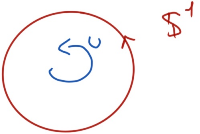
its phase diagram $(u,\theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \mapsto d_u^\theta$
is contained in a torus and contains a half disk
(of radius $1-\tilde{q}_c$ and $\tilde{q}_c$)



(of radius $\tilde{q}_c$)

- supercritical or critical if and only if

its phase diagram $(u,\theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \mapsto d_u^\theta$
is contained in a circle ($\cong \mathbb{S}^1$)



and it is supercritical if its phase diagram

$$u \in \mathbb{S}^1 \mapsto \mathbb{P}_{q=0}(0 \notin [V_{u,0}]) = \begin{cases} 1, & \text{if } u \text{ unstable} \\ 0, & \text{otherwise} \end{cases}$$

contains a half circle

Proving $\sup d_u \geq \tilde{q}_c$

The goal is to show, that for any $q' > \sup d_u$ it holds that

$$\sum_n n\mathbb{P}_{q'}(0 \notin [A \cap B_n]) < \infty.$$

The idea is to show that, at q', the origin is infected most of the time.

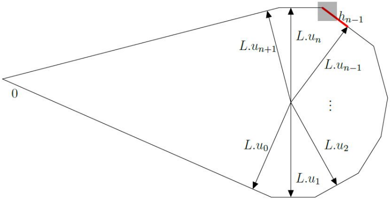
2-step percolation : $q' = \sup d_u + \varepsilon$

1. Infect sites with probability ε to find some structures
2. Infecting new sites with probability q allows structures to grow

Some details on the proof

- The structures that grow are droplets, with sides $(u_i)_{i=1}^{n}$ depending on $\sup d_u$.
- In the second percolation, droplets of size L grow into droplets of size $\geq (1+\delta)L$, for some $\delta > 0$.
- The proof can be done in any semi-circle, so we can get $\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$
- The proof contains that $\forall q > \sup d_u$, there exists a constant $c(q) > 0$ such that

$$\theta_n(q) \leq e^{-c(q)n}$$



Applying the theorem

Theorem

For any update rules $\mathcal{U}$,

$$q_c \leq \tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

In particular, if $\mathcal{U}$ is not subcritical, then $\tilde{q}_c = q_c = 0$

So, having knowledge on $u \mapsto d_u$ allows to upper bound q_c...

Proposition : (It's harder for submodels to infect)

For any sub-collection of rules $\mathcal{U}' \subset \mathcal{U}$

$$q_c(\mathcal{U}) \leq \tilde{q}_c(\mathcal{U}) \leq \inf_{C} \sup_{u \in C} d_u(\mathcal{U}')$$

... and it is not even necessary to know the critical density for the whole set of rules to get such bounds.

First level bound

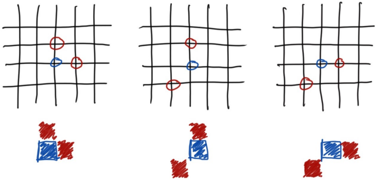
DTBP : Directed Triangular Bootstrap Percolation

Let $\mathcal{U}' = \{(-1,-1),(0,1)\}$, one of the rules of DTBP, then

$$q_c(DTBP) \leq \tilde{q}_c(DTBP) \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u(\mathcal{U}').$$

Applying a general formula for one rule families (using OP) gives $q_c(DTBP) \leq 0.245...$[a]

[a] Previous known bound was 0.312



Second level bound

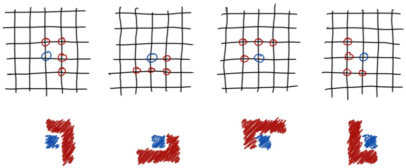
However, knowing subfamilies with a single rule is not enough.

Spiral

For spiral, it is possible to compute $d_{\mathcal{U}}$ for all pairs of rules, such that the difficulty on pairs is the same as the difficulty of some Bidirectional OP :

$$q_c(Spiral) \leq \tilde{q}_c(Spiral) \leq 1 - p_c^{OP}.$$

And the result is tight.



Oriented percolation as an example of bootstrap percolation

- Oriented percolation (OP) is one of the simplest subcritical BP model.
- It is a BP model with one-family rule $\mathcal{U} = \{U\} = \{\{-1, 1\}, \{1, 1\}\}$
- Some of the well-known results in the field are reviewed in Durrett's article [5]
- We will follow this article to introduce some non-trivial results about OP

Remark

In the article of Durrett bond percolation is considered rather than site. However, it can be shown that the results obtained also apply to site OP.

[5] R. DURRET, Oriented Percolation in Two Dimensions, *The Annals of Probability*, 1984

Edge speed

- We remind that the parameter of OP - $p = 1 - q$ stands for the intensity of **healthy** sites
- We let p_c^{OP} be the critical probability for OP
- We say that $x \to y$ if there exist $x_0 = x, x_1, \ldots, x_n = y$ such that $x_i - x_{i-1} \in \mathcal{U}$ and x_i open for $0 < i \leq n$, i.e. there exists an OP path between x and y.
- We are naturally interested in the event $\{0 \to \infty\} = \{\cap_{i=1}^{\infty} L_i \neq \emptyset\}$, where $L_n = \{x : (0,0) \to (x,n)\}$ - the set of sites on the height n connected to 0.
- Notice that saying that $0 \to \infty$ is equivalent to saying that 0 is not infected in the BP model (duality BP - OP).

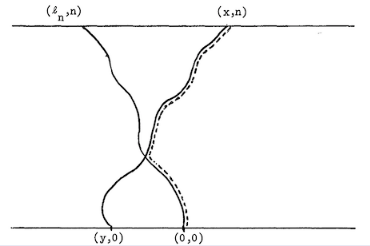
Edge Speed

- We denotel $r_n = \sup\{x \in \mathbb{Z}, \exists y \leq 0, (y,0) \to (x,n)\}$ - the rightmost edge with the convention $\sup\{\emptyset\} = -\infty$
- This definition may seem a bit artificial at first glance (why would we look at $(y,0) \to (x,n)$ for some $y \leq 0$ rather than $(0,0) \to (x,n)$)?

Property

Provided that $L_n \neq \emptyset$, we have $\mathrm{r}_n = \sup L_n = \sup\{x : (0,0) \to (x,n)\}$

Proof by a picture



Edge speed

- We now want to quantify the asymptotic behaviour of $\frac{r_n}{n}$

Theorem - definition

There exists a function $\alpha : [0,1] \to [-\infty, 1]$ called edge speed with the following property:

$$\frac{r_n}{n} \to \alpha(p) = \inf_n \mathbb{E}_p[r_n/n]$$

Moreover, we have that α is continuous and strictly increasing on $[p_c^{OP}, 1]$ with $\alpha(p_c^{OP}) = 0, \alpha(1) = 1$ and $\alpha(p) = -\infty$ for $p < p_c^{OP}$.

- Intuitively edge speed tells us how far the rightmost path (starting from height 0 and to the left of the origin) is expected to go.
- Edge speed has the following "criticality" properties which we state without proof

Properties of edge speed

Property 1

For $p > p_c^{OP}$ (above criticality for OP) and $\alpha_0 < \alpha(p)$ with positive probability there exists an infinite OP path $((a_i, i))_{i \in \mathcal{N}}$ with $a_0 = 0$ and $\inf_n \frac{a_n}{n} \geq \alpha_0$. Intuitively this means that we have chances to get an infinite path that goes sufficiently far (but below edge speed) to the right.

Property 2

For $\alpha_0 > \alpha(p)$, for some $\gamma > 0$ we have

$$\mathbb{P}_p(r_n \geq \alpha_0 n) \leq e^{-\gamma n}$$

This exponential decay shows that it is unlikely to find a path that goes too far (further than edge speed) to the right.

Critical densities of OP

- We let $\psi(u) = 1 - \alpha^{-1}(|tan(u)|)$, where α^{-1} is the inverse of the edge speed function α.
- The following theorem expresses the critical density of OP depending on the direction u which is parametrized by the angle it makes with the origin: $u \in [-\pi, \pi]$.

Theorem

The critical densities of the BP percolation with $\mathcal{U} = \{(1,1,)(-1,1)\}$ (dual to OP) are given by

$$d_u(\mathcal{U}) = \begin{cases} 0, & u \in [-3\pi/4, -\pi/4] \\ 1 - p_{\mathrm{c}}^{\mathrm{OP}} = q_{\mathrm{c}}, & u \in [0, \pi] \\ \psi(u), & \text{otherwise} \end{cases} \tag{1}$$

Outline of the proof

- If $u \in (-3\pi/4, -\pi/4)$, we have $[\mathbb{H}_u] = \mathbb{Z}^2$ (the directions are unstable), so in this case $d_u = 0$
- By symmetry it suffices to treat $u \in [-\pi/4, \pi/2]$
- The critical density d_u can be thought as the value $\tilde{q}$ above which a.s there is no oriented infinite path from the origin which does not pass by $\mathbb{H}_u$
- It suffices then to show that below $\tilde{q}$ there is such an infinite path with positive probability and above it does not exist a.s.
- For $q < \tilde{q}$ one can prove that it in order not to pass by $\mathbb{H}_u$ it suffices to go to the right with the speed below edge speed which is possible (with positive probability) by the property 2
- Conversely, for $q > \tilde{q}$, we would have to walk to the right above edge speed to get around $\mathbb{H}_u$ which is not possible (exponential decay)

Open questions and conjectures about OP

- We remind that $q_c = \inf\left\{q \in [0,1], \mathbb{P}_q\left([A] = \mathbb{Z}^2\right) = 1\right\}$, whereas $\tilde{q}_c = \inf\left\{q \in [0,1], \sum_n n\theta_n(q) < \infty\right\}$

Conjecture

For all BP models (all update families) we have: $q_c = \tilde{q}_c$.

- It would be practical to know if the complication of taking right and left limits to define the critical density $d_u = max(d_u^+, d_u^-)$ is necessary.

Question

What are the continuity properties of the function $(u, \theta) \to d_u^\theta$?