$\mathcal{U}$-Bootstrap percolation

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Outline of the presentation

1. Introduction to $\mathcal{U}$-bootstrap percolation
2. Universal classes and stable directions
3. Critical densities
4. Applications of critical densities on Spiral and DTBP
5.
6. Conclusion and open questions

d_u^θ measures directions that are difficult to infect

Critical densities with conic boundary conditions

For $u \in \mathbb{S}^1$ and $\theta \in [-\pi, \pi]$

$$d_u^\theta := \inf \left\{ q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]) < \infty \right\}$$

Morally, the critical probability with infection of $V_{u,u+\theta} = \mathbb{H}_u \cap \mathbb{H}_{u+\theta}$.

- The summand decays slowly in n when it is hard to infect the origin using only infections at distance less than n. So, when it is hard to infect 0, d_u^θ is large[1].
- When $\theta \sim \pm\pi$, few sites are infected, so it is easy for the origin not to be infected, the summand can be large. Hence, d_u^θ decreases when $\theta \to 0$
- $d_u^\pm := \lim_{\theta \to 0^\pm} d_u^\theta$ can be large when a small number of infections is not enough to infect the origin, even with a

Theorem

For any $\mathcal{U}$-bootstrap percolation model, its critical probability

$$\tilde{q}_c = \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [A \cap B_n]) < \infty\}$$

is equal to the maximal value of its critical density function

$$d_u = \max_{0^\pm} \inf\{q \in [0,1], \sum_n n\mathbb{P}_q(0 \notin [(A \cup V_{u,u+0^\pm}) \cap B_n] < \infty\}$$

for u in any semicircle C, i.e.,

$$\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

Let's denote $E_{u,\theta} = \{0 \notin [(A \cup V_{u,u+\theta}) \cap B_n]\}$. Then,

$$E_{u,\pm\pi} = \{0 \notin [A \cap B_n]\} \supset E_{u,\theta}$$

which gives that the following holds for any u

$$\tilde{q}_c \geq \sup_{\theta} \sup_{u} d_u^{\theta} \geq \limsup_{\theta \to 0} \sup_{u} d_u^{\theta} = \sup_{u} d_u.$$

The theorem states that all those quantities are equal.

Meaning of the theorem

The difficulty of the model is as hard as its most difficult direction. In this direction, infecting a half plane doesn't affect the infection of the origin.

Proving $\sup d_u \geq \tilde{q}_c$

The goal is to show, that for any $q' > \sup d_u$ it holds that

$$\sum_n n\mathbb{P}_{q'}(0 \notin [A \cap B_n]) < \infty.$$

The idea is to show that, at q', the origin is infected most of the time.

2-step percolation : $q' = \sup d_u + \varepsilon$

1. Infect sites with probability ε to find some structures
2. Infecting new sites with probability q allows structures to grow

Some details on the proof

- The structures that grow are droplets, with sides $(u_i)_{i=1}^n$ depending on $\sup d_u$.
- In the second percolation, droplets of size L grow into droplets of size $\geq (1+\delta)L$, for some $\delta > 0$.
- The proof can be done in any semi-circle, so we can get $\tilde{q}_c = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u$
- The proof contains that $\forall q > \sup d_u$, there exists a constant $c(q) > 0$ such that

$$\theta_n(q) \leq e^{-c(q)n}$$

Applying the theorem

Theorem

For any update rules $\mathcal{U}$*,*

$$q_c \leq \tilde{q}_c = \sup_{u \in \mathbb{S}^1} d_u = \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u.$$

In particular, if $\mathcal{U}$ *is not subcritical, then* $\tilde{q}_c = q_c = 0$

So, having knowledge on $u \mapsto d_u$ allows to upper bound q_c...

Proposition : (It's harder for submodels to infect)

For any sub-collection of rules $\mathcal{U}' \subset \mathcal{U}$

$$q_c(\mathcal{U}) \leq \tilde{q}_c(\mathcal{U}) \leq \inf_{C} \sup_{u \in C} d_u(\mathcal{U}')$$

... and it is not even necessary to know the critical density for the whole set of rules to get such bounds.

First level bound

$DTBP$: Directed Triangular Bootstrap Percolation

Let $\mathcal{U}' = \{(-1,-1),(0,1)\}$, one of the rules of DTBP, then

$$q_c(DTBP) \leq \tilde{q}_c(DTBP) \leq \inf_{C \in \mathcal{C}} \sup_{u \in C} d_u(\mathcal{U}').$$

Applying a general formula for one rule families (using OP) gives $q_c(DTBP) \leq 0.245...$[a]

[a] Previous known bound was 0.312

However, knowing one rule subfamilies is not enough.

Spiral

For spiral, it is possible to compute d_u for all pairs of rules, such that the difficulty on pairs is the same as the difficulty of some Bidirectional OP :

$$q_c(Spiral) \leq \tilde{q}_c(Spiral) \leq 1 - p_c^{OP}.$$

And the result is tight.