1.5 Campos escalares y vectoriales.

Un campo es una función $\vec{F}$ que asigna un valor o un parametro a cada punto del plano o del espacio.

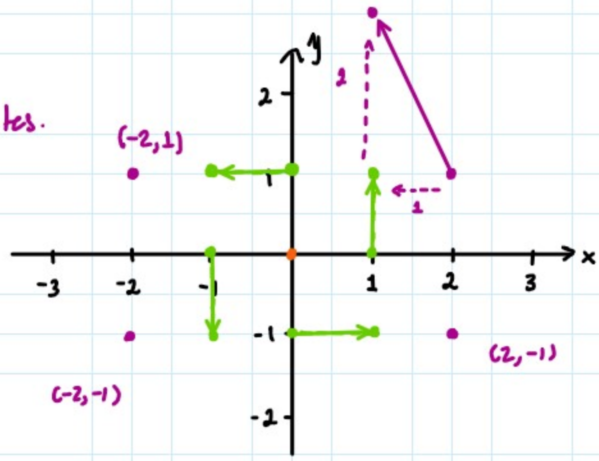
$$\vec{F}\underbrace{(x,y,z)}_{\text{en 2D ó 3D.}} = M\,\hat{i} + N\,\hat{j} + P\,\hat{k}$$

¿Cómo graficar los campos vectoriales?

- EJEMPLO: Bosqueje algunos vectores del campo: $\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}$.

1P. Origen.
2P. Evaluar en ejes.
3P. Evaluar en los cuadrantes.

(x,y)	$\vec{F} = -y\hat{i} + x\hat{j}$
$(0,0)$	$\vec{F} = -0\hat{i} + 0\hat{j} = \langle 0,0\rangle$
$(1,0)$	$\vec{F} = -0\hat{i} + 1\hat{j} = \langle 0,1\rangle$
$(0,1)$	$\vec{F} = -1\hat{i} + 0\hat{j} = \langle -1,0\rangle$
$(-1,0)$	$\vec{F} = \langle 0,-1\rangle$
$(0,-1)$	$\vec{F} = \langle 1,0\rangle$
$(\underset{x}{2},\underset{y}{1})$	$\vec{F} = -1\hat{i} + 2\hat{j} = \langle \overset{\leftarrow}{-1}, \overset{\uparrow}{2}\rangle$



1.5.1. Gradiente.

Fórmula: $\nabla f = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$.

$f \rightarrow$ es ESCALAR.

$\nabla f \rightarrow$ es VECTOR.

$\nabla \rightarrow$ operador NABLA.

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

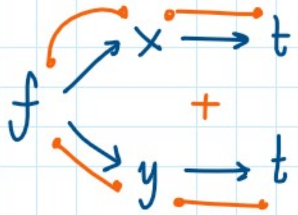
EJEMPLO: Determine: $\nabla f(x,y,z)$ si $f(x,y,z) = \underbrace{\arcsin(\overbrace{xy}^{u}) + \ln(z)}_{\text{ESCALAR.}}$ en el punto $(1,0,-3)$

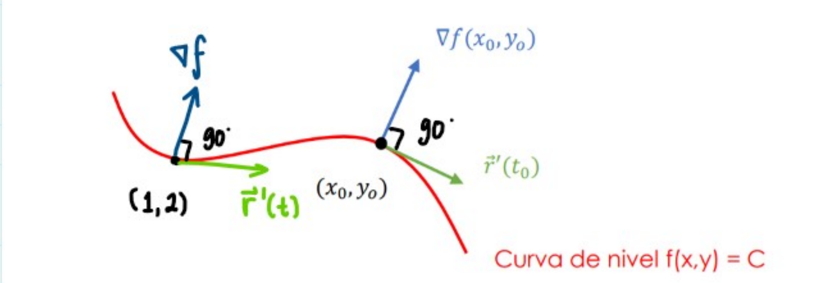
$$\nabla f = \frac{y}{\sqrt{1-(xy)^2}}\,\hat{i} + \frac{x}{\sqrt{1-(xy)^2}}\,\hat{j} + \frac{1}{z}\,\hat{k}\ \Bigg|_{(1,0,-3)}$$

$x=1,\ y=0,\ z=-3$

$$\nabla f = \cancelto{0}{\frac{0}{\sqrt{1-0^2}}}\,\hat{i} + \cancelto{1}{\frac{1}{\sqrt{1-0^2}}}\,\hat{j} + \frac{1}{-3}\,\hat{k} = 0\hat{i} + 1\hat{j} - \frac{1}{3}\hat{k} = \underbrace{\hat{j} - \frac{1}{3}\hat{k} \quad ó \quad \langle 0, 1, -\tfrac{1}{3} \rangle}_{\text{VECTOR.}}$$

Regla de la cadena:





EJEMPLO: Determinar la curva de nivel de la función: $f(x,y) = -x^2 + y^2$ que pasa por $P(2,3)$. Además, trace el gradiente en dicho punto.

- Curva de nivel. $f(x,y) = -x^2 + y^2$ $\quad P(\underset{x}{2}, \underset{y}{3})$ $\quad \Rightarrow f(2,3) = -2^2 + 3^2 = -4 + 9 = \underbrace{+5}$

$-x^2 + y^2 = 5 \quad \leftarrow$ (Hip.)

$$-\frac{x^2}{5} + \frac{y^2}{5} = 1$$

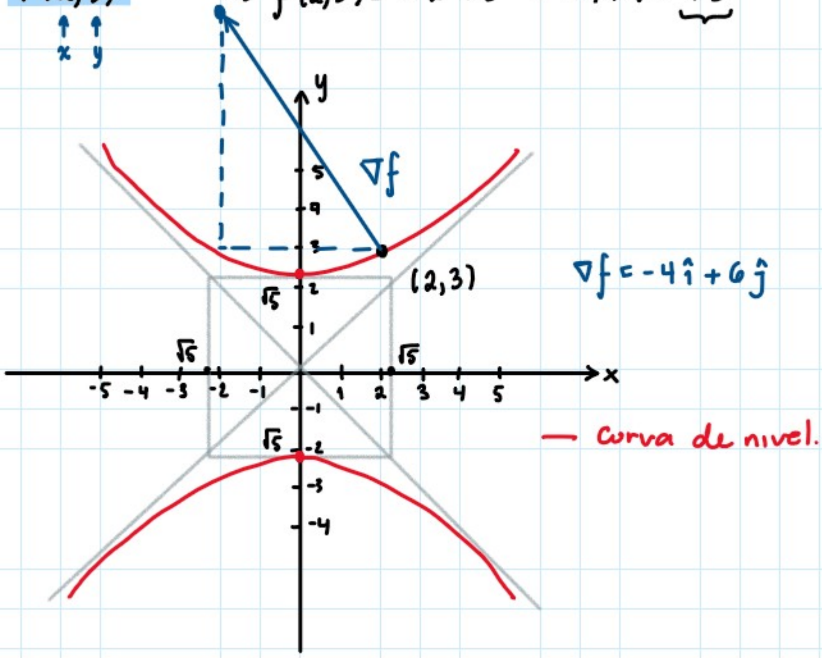
$\sqrt{5} \quad \sqrt{5}$

$\sqrt{4} \quad \sqrt{5} \quad \sqrt{9}$

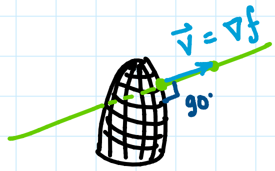
$2 \quad (\ldots) \quad 3$

$$\nabla f = -2x\,\hat{i} + 2y\,\hat{j}\ \Big|_{(2,3)}$$

$$= -2(2)\hat{i} + 2(3)\hat{j} = -4\hat{i} + 6\hat{j}$$



EJERCICIO: Encontrar las ecuaciones paramétricas de la recta normal a la superficie: $x^2+y^2-z^2=64$ en el punto $P(8,8,8)$



(...)

¿Ec. de una recta? ✓ Punto $P(8,8,8)$
✓ vector director.

$\vec{v}=\nabla f$

punto + vector . t

$$\begin{aligned} x &= 8+16t \\ y &= 8+16t \\ z &= 8-16t \end{aligned}$$

EJERCICIO SUGERIDO:

Suponga que un campo eléctrico en el plano xy causado por una línea de carga infinita a lo largo del eje x es un campo de gradiente con función de potencial eléctrico $V(x, y) = c \ln\left(\frac{y}{\sqrt{x^2+y^2}}\right)$ donde c > 0 es una constante.

Encuentre los componentes del campo eléctrico E en las direcciones x e y, donde: E (x, y) = - ∇V (x, y).

Respuesta: $\vec{E} = \frac{cx}{x^2+y^2}\hat{\imath} - \frac{cx^2}{x^2y+y^3}\hat{\jmath}$

▫ Divergencia y rotacional.

$$\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$$

$$\text{Div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Entrada: VECTOR ⟶ Salida: ESCALAR.

$$\text{Rot}\,\vec{F} = \nabla\times\vec{F} = \begin{array}{c} \\ \nabla \rightarrow \\ \vec{F} \rightarrow \end{array}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = (\quad)\hat{\imath} + (\quad)\hat{\jmath} + (\quad)\hat{k}$$

Entrada: VECTOR ⟶ Salida: VECTOR

EJEMPLO: Encontrar: a) Rot $\vec{F}$ ∧ b) Div $\vec{F}$ de los siguientes campos vectoriales.

i) $\vec{F}(x,y,z) = \underbrace{\frac{\sqrt{x}}{1+z}}_{M}\hat{\imath} + \underbrace{\frac{\sqrt{y}}{1+x}}_{N}\hat{\jmath} + \underbrace{\frac{\sqrt{z}}{1+y}}_{P}\hat{k}.$ } VECTOR

$$\frac{\partial P}{\partial y} = \left(\frac{\sqrt{z}}{1+y}\right)' = \sqrt{z}.\left((1+y)^{-1}\right)'$$
$$= \sqrt{z}\left((-1)(1+y)^{-2}.(1)\right)$$
$$= \frac{-\sqrt{z}}{(1+y)^2}$$

a) Rot $\vec{F} = \nabla\times\vec{F}$

⊕

$$\left| \hat{\imath} \quad \hat{\imath} \quad \hat{k} \right| = \left(\frac{-\sqrt{z}}{\ldots} - 0\right)\hat{\imath} - \left(0 - \frac{-\sqrt{x}}{\ldots}\right)\hat{\jmath}$$

por analogía

a) $Rot\,\vec{F} = \nabla \times \vec{F}$

$$\begin{array}{c} \\ \nabla \rightarrow \\ \vec{F} \rightarrow \end{array} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = \left(\frac{-\sqrt{z}}{(1+y)^2} - 0 \right)\hat{i} - \left(0 - \frac{-\sqrt{x}}{(1+z)^2} \right)\hat{j} + \left(\frac{-\sqrt{y}}{(1+x)^2} - 0 \right)\hat{k}$$

$$\frac{\quad}{(1+y)^2}$$

por analogía...

$$\frac{\partial M}{\partial z} = \left(\frac{\sqrt{x}}{(1+z)} \right)' = \frac{-\sqrt{x}}{(1+z)^2}$$

$$\frac{\partial N}{\partial x} = \left(\frac{\sqrt{y}}{(1+x)} \right)' = \frac{-\sqrt{y}}{(1+x)^2}$$

$$Rot\,\vec{F} = -\frac{\sqrt{z}}{(1+y)^2}\hat{i} - \frac{\sqrt{x}}{(1+z)^2}\hat{j} - \frac{\sqrt{y}}{(1+x)^2}\hat{k}. \quad \Big\}\ \text{VECTOR}$$

b) $$Div\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{1}{2(1+z)\sqrt{x}} + \frac{1}{2(1+x)\sqrt{y}} + \frac{1}{2(1+y)\sqrt{z}} \quad \Big\}\ \text{ESCALAR}$$

ii) $$\vec{H}(x,y,z) = \underbrace{x^2y^3 - z^4}_{M}\,\hat{i} + \underbrace{4x^5y^2z}_{N}\,\hat{j} \underbrace{- y^4z^6}_{P}\,\hat{k}.$$

a) $Rot\,\vec{H} =$

b) $Div\,\vec{H} =$

EJEMPLO: Si $\vec{F}(x,y,z) = \underbrace{x^2z}_{M}\,\hat{i} \underbrace{- 2xz}_{N}\,\hat{j} + \underbrace{yz}_{P}\,\hat{k}$. Determine: $\nabla \cdot \underbrace{(\nabla \times \vec{F})}$

$\underbrace{\nabla \cdot Rot\,\vec{F}}$

$Div\,(Rot\,\vec{F})$

$$\nabla \times \vec{F} = \begin{array}{c} \\ \nabla \rightarrow \\ \vec{F} \rightarrow \end{array} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & -2xz & yz \end{vmatrix} = \left(z - (-2x) \right)\hat{i} - \left(0 - x^2 \right)\hat{j} + \left(-2z - 0 \right)\hat{k}$$

$$= \underbrace{(z + 2x)}_{M_2}\,\hat{i} + \underbrace{x^2}_{N_2}\,\hat{j} \underbrace{- 2z}_{P_2}\,\hat{k}$$

$$Div\,(Rot\,\vec{F}) = \frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z} = 2 + 0 - 2 = 0$$

$Div\,(Rot\,\vec{F}) = 0.$

- Formulas que involucran campos.

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

1. $$\underbrace{\nabla \cdot \nabla f}_{\nabla \cdot Grad\,f} = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \leftarrow \text{Laplaciano.}$$

Div (Grad f) Entrada: escalar $\longrightarrow$ Salida: escalar.

EJEMPLO: Encontrar el laplaciano de f si $f(x,y,z) = \underbrace{2x^2y - xz^3}_{\text{ESCALAR}}$

Grad.	$E \to V$
Div.	$V \to E$
Rot.	$V \to V$
Lap.	$E \to E$

$$\frac{\partial f}{\partial x} = 4xy - z^3 \qquad \frac{\partial f}{\partial y} = 2x^2 \qquad \frac{\partial f}{\partial z} = -3xz^2$$

$$\frac{\partial^2 f}{\partial x^2} = 4y \quad + \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad + \quad \frac{\partial^2 f}{\partial z^2} = -6xz$$

$$\nabla^2 f = 4y - 6xz \ \Big\} \text{ESCALAR}$$

$\longrightarrow$ NO ES ARMÓNICA.

al igualar el laplaciano a cero se obtiene una ecuación llamada: "Ecuación de Laplace"

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

sí eso se cumple: "f" toma el nombre de función armónica.

EJEMPLO: Demuestre que: $f(x,y) = \ln\left(\sqrt{x^2+y^2}\right)$ es una función armónica.

$$\ln(x^2+y^2)^{\boxed{1/2}}$$

$$f = \frac{1}{2}\ln\overbrace{(x^2+y^2)}^{u}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+y^2} = \frac{x}{x^2+y^2} \qquad \frac{\partial^2 f}{\partial x^2} = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}y}{x^2+y^2} = \frac{y}{x^2+y^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \ +$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\cancel{-x^2}+\cancel{y^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = 0$$

$\therefore$ $f(x,y)$ es armónica.

2) $\nabla \times \nabla f = 0$ (Rotacional del gradiente de $f = 0$)

3) $\nabla \cdot (\nabla \times \vec{F}) = 0$ (Divergencia del rotacional de $\vec{F} = 0$)

CAMPOS VECTORIALES CONSERVATIVOS.

- Plano: $\vec{F} = \underbrace{M}_{x}\,\hat{\imath} + \underbrace{N}_{y}\,\hat{\jmath}$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

1. $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \checkmark$$ $\vec{F}$ es conservativo.

- Espacio: $\vec{F} = \underbrace{M}_{x}\,\hat{\imath} + \underbrace{N}_{y}\,\hat{\jmath} + \underbrace{P}_{z}\,\hat{k}$

1. $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \checkmark$$
2. $$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \checkmark$$
3. $$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \checkmark$$

$\vec{F}$ es conservativo.

$$\vec{F} = \underbrace{2x}_{M}\,\hat{\imath} - \underbrace{yz^2}_{N}\,\hat{\jmath} + \underbrace{\sqrt{x+y}}_{P}\,\hat{k}$$

↳"x" ↳"y" ↳"z"

$$\frac{\partial M}{\partial \boxed{y}_?} = \frac{\partial N}{\partial \boxed{x}_?}$$

EJEMPLO: Determine si los siguientes campos son conservativos o no, en caso de serlo encuentre la función potencial f.

a) $\vec{F}(x,y) = \underbrace{2xy}_{M}\,\hat{\imath} + \underbrace{(x^2-y)}_{N}\,\hat{\jmath}$

1. $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial y} = 2x \checkmark \qquad \frac{\partial N}{\partial x} = 2x \checkmark$$

$\therefore \vec{F}$ es conservativo.

¿Cómo encontrar la función potencial? $\vec{F} = \nabla f$

$$\int M\,dx + \int N\,dy + \int P\,dz$$

[Suma concatenada]

$$\int 2xy.\,dx = \cancel{2}y.\frac{x^2}{\cancel{2}} = x^2y + \boxed{C(y)}$$ y es cte.

$$\int x^2 - y.\,dy = x^2y - \frac{y^2}{2} + \boxed{C(x)}$$ x es cte.

$$\boxed{f = x^2y - \frac{y^2}{2} + C}$$

b) $\vec{G}(x,y,z) = \langle \underbrace{2xy+z}_{M}, \underbrace{x^2yz+2}_{N}, \underbrace{x+2y}_{P} \rangle$

1. $$\frac{\partial M}{\partial y} = 2x \times \qquad \frac{\partial N}{\partial x} = 2xyz \times$$

$\therefore \vec{G}$ NO es conservativo (no existe función potencial.)

c) $\vec{H}(x,y,z) = yz\,(\cos(xyz) - z^2)\,\hat{\imath} + xz\,(\cos(xyz) - z^2)\,\hat{\jmath} + xy\,(\cos(xyz) - 3z^2)\,\hat{k}$

$$\vec{H} = [yz\cos(xyz) - yz^3]\,\hat{\imath} + [xz\cos(xyz) - xz^3]\,\hat{\jmath} + [xy\cos(xyz) - 3xyz^2]\,\hat{k}$$

$$\vec{H} = [\underbrace{yz\cos(xyz) - yz^3}_{M}]\,\hat{i} + [\underbrace{xz\cos(xyz) - xz^3}_{N}]\,\hat{j} + [\underbrace{xy\cos(xyz) - 3xyz^2}_{P}]\,\hat{k}$$

CONDICIÓN 1.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \checkmark$$

$$\frac{\partial M}{\partial y} = z\cos(xyz) - xyz^2\sin(xyz) - z^3 \checkmark$$

$$\frac{\partial N}{\partial x} = z\cos(xyz) - xyz^2\sin(xyz) - z^3 \checkmark$$

CONDICIÓN 2

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \checkmark$$

$$\frac{\partial M}{\partial z} = y\cos(xyz) - xy^2z\sin(xyz) - 3yz^2 \checkmark$$

$$\frac{\partial P}{\partial x} = y\cos(xyz) - xy^2z\sin(xyz) - 3yz^2 \checkmark$$

CONDICIÓN 3.

$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} \checkmark$$

$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y} = x\cos(xyz) - x^2yz\sin(xyz) - 3xz^2. \checkmark$$

$\therefore \vec{H}$ es conservativo.

$$\vec{H} = [\underbrace{yz\cos(xyz) - yz^3}_{M}]\,\hat{i} + [\underbrace{xz\cos(xyz) - xz^3}_{N}]\,\hat{j} + [\underbrace{xy\cos(xyz) - 3xyz^2}_{P}]\,\hat{k}$$

$$\int M\,dx + \int N\,dy + \int P\,dz$$

$$\int yz\cos(xyz) - yz^3 \, .dx = \cancel{yz}\,\frac{\sin(xyz)}{\cancel{yz}} - yz^3x = \sin(xyz) - xyz^3 + \boxed{C(y,z)}$$ $y \wedge z$ son ctes.

$$\int xz\cos(xyz) - xz^3 \, .dy = \cancel{xz}\,\frac{\sin(xyz)}{\cancel{xz}} - xz^3y = \sin(xyz) - xyz^3 + \boxed{C(x,z)}$$ $x \wedge z$ son ctes.

$$\int xy\cos(xyz) - 3xyz^2 \, .dz = \cancel{xy}\,\frac{\sin(xyz)}{\cancel{xy}} - \cancel{3}xy\cdot\frac{z^3}{\cancel{3}} = \sin(xyz) - xyz^3 + \boxed{C(x,y)}$$ $x \wedge y$ son ctes.

$$h = \sin(xyz) - xyz^3 + C$$