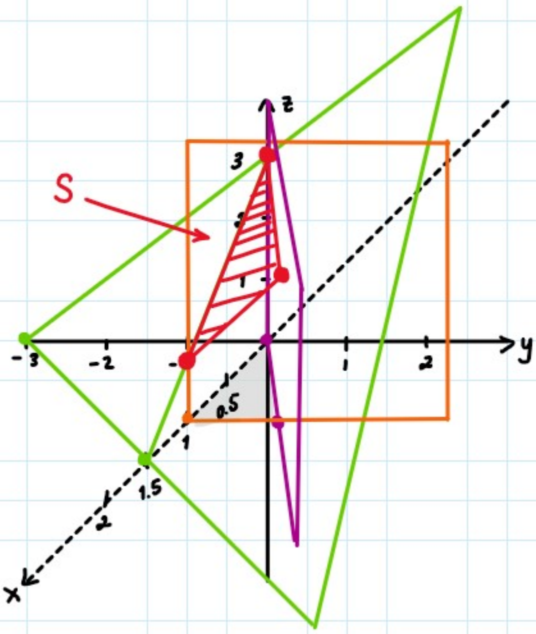
# Integrales de superficie.

EJEMPLO: Evalue $\iint_S (xy+z)\,dS$ ; dónde S es la parte del plano: $\underbrace{2x-y+z=3}$ •

que esta sobre el borde limitado en el I octante por: $y=x \wedge x=1$. Grafique S.




P1. Graficar y enmarcar S.

$2x-y+z=3$

I. con ejes: "x" $(y=z=0)$

$2x=3 \rightarrow x=3/2=1.5$ •

"y" $(x=z=0)$

$-y=3 \rightarrow y=-3$ •

"z" $(x=y=0)$

$z=3$ •

$y=x \rightarrow$ recta

$(0,0) \wedge (1,1)$

$x=1 \rightarrow$ recta.

$2\overset{1}{x} - \overset{1}{y} + z = 3$

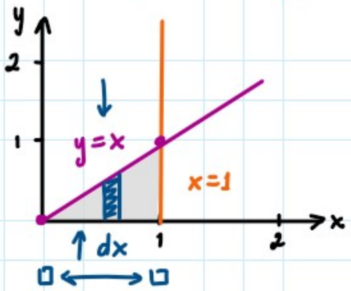
$1+z=3$

$z=3-1$

$z=2$

$(1,1,2)$

P2. Proyección "xy" $\rightsquigarrow$ despejar "z" a las superficie S.




$2x-y+z=3 \longrightarrow z=3-2x+y \longleftarrow$

$f_x=(-2)$ $\quad f_x^2=4$

$f_y=+1$ $\quad f_y^2=1$

$dS=\sqrt{1+f_x^2+f_y^2}\;dA$

$dS=\sqrt{1+4+1}\;dA=\sqrt{6}\,dA$

P3. Plantear la integral.

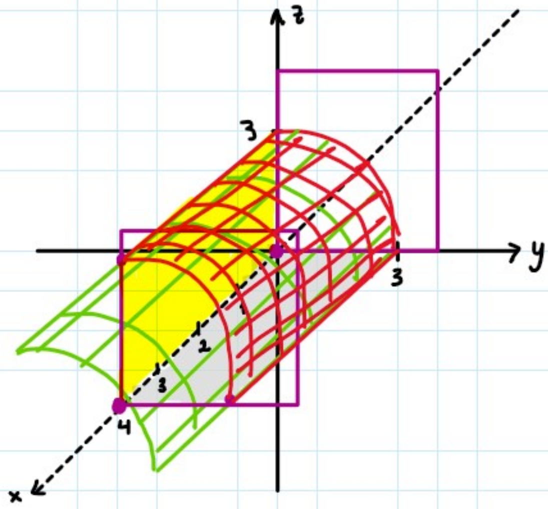
$$\iint_S (xy+z)\,dS \rightarrow \iint_R (xy+3-2x+y)\sqrt{6}\,dA = \sqrt{6}\int_0^1\int_0^x xy+3-2x+y\;.\;dy\cdot dx$$

P4. Resolver. $\sqrt{6}\int_0^1 \left(\frac{xy^2}{2}+3y-2xy+\frac{y^2}{2}\right)\Big|_0^x dx$

$$\sqrt{6}\int_0^1 \frac{x^3}{2}+3x-2x^2+\frac{x^2}{2}\;dx = \sqrt{6}\int_0^1 \frac{x^3}{2}-\frac{3}{2}x^2+3x\;dx = \sqrt{6}\left[\frac{x^4}{8}-\frac{x^3}{2}+\frac{3x^2}{2}\right]\Big|_0^1$$

$$=\sqrt{6}\left[\frac{1}{8}\underbrace{-\frac{1}{2}+\frac{3}{2}}\right]=\frac{9}{8}\sqrt{6}$$

EJEMPLO: Evaluar la integral $\iint_S (x+z)\,ds$ dónde S es la porción de: $y^2+z^2=9$ en el primer octante, limitado por $x=0 \wedge x=4$.

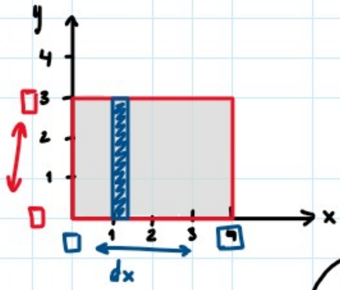



$y^2+z^2=9$  $x=0 \wedge x=4$

$C(0,0)$ $r=\sqrt{9}=3$

plano "yz"

Proyección: "xy" $\longrightarrow$ despejar "z"




$y^2+z^2=9$

$z^2=9-y^2$

$z=\pm\sqrt{9-y^2}$

$(9-y^2)^{1/2}$

$ds=\sqrt{1+\cancel{f_x^2}+f_y^2}\,dA$

$f_x=0$  $f_x^2=0$

$f_y=\frac{1}{\cancel{2}}\cdot(9-y^2)^{-1/2}(-\cancel{2}y)$

$f_y=\frac{-y}{\sqrt{9-y^2}}$

$f_y^2=\frac{y^2}{9-y^2}$

$ds=\sqrt{1+0+\frac{y^2}{9-y^2}}\,dA$

$\frac{1}{1}+\frac{y^2}{9-y^2}$

$\frac{9-\cancel{y^2}+\cancel{y^2}}{9-y^2}=\frac{9}{9-y^2}$

$ds=\sqrt{\frac{9}{9-y^2}}\,dA=\frac{3\,dA}{\sqrt{9-y^2}}$

$\iint_S (x+\underbrace{z}) \underbrace{ds} \longrightarrow \iint_R \left(x+\sqrt{9-y^2}\right)\underbrace{\frac{3\,dA}{\sqrt{9-y^2}}}$

$3\int_0^4\int_0^3 \left(x+\sqrt{9-y^2}\right)\frac{1}{\sqrt{9-y^2}}\cdot dy\,dx$

$3\int_0^4\int_0^3 \frac{x}{\sqrt{9-y^2}}+\cancel{\frac{\sqrt{9-y^2}}{\sqrt{9-y^2}}}^{\,1}\cdot dy\,dx$

$3\int_0^4 \lim_{h\to3^-}\int_0^h \frac{x}{\sqrt{9-y^2}}+1\cdot dy\,dx$

Integral impropia

R// = $36+12\pi$

1. Superficies abiertas.

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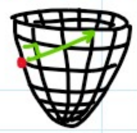
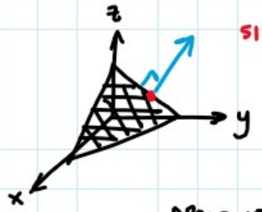
$\hat{n}$ es ascendente



$\hat{n}$ es descendente.




si el ejercicio no dicenada ='(

asumir: ascendente.

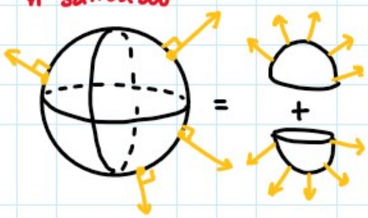
OPCIONES:

V. ascendentes y V. descendente.

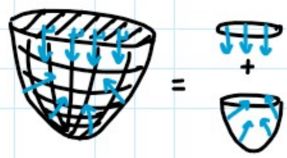
asumir.

2. Superficies cerradas.

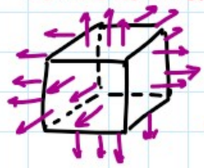
$\hat{n}$ saliendo




$\hat{n}$ entrando




no dicenada el ejercicio ='(

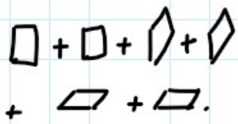


asumir:saliendo

OPCIONES:

V. saliendo y V. entrando.

asumir.




$$-f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k}$$

$\hat{n}$ arriba

$$f_x\hat{\imath} + f_y\hat{\jmath} - \hat{k}$$

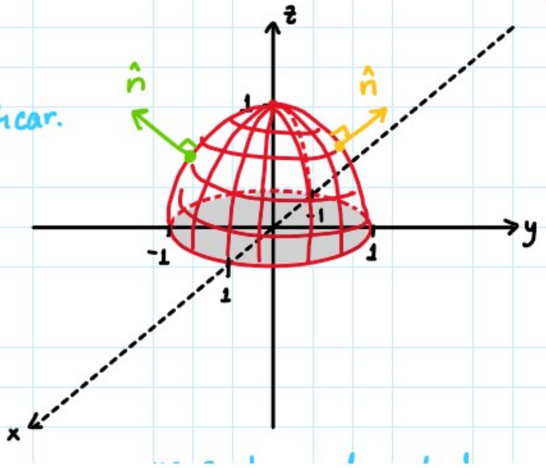
$\hat{n}$ abajo.

$$\text{Flujo} = \iint_R \langle M, N, P\rangle \cdot \langle -f_x, -f_y, 1\rangle \, dA$$

$$\text{Flujo} = \iint_R \langle M, N, P\rangle \cdot \langle f_x, f_y, -1\rangle \, dA$$

EJEMPLO 1. Calcule el flujo del campo vectorial $\vec{F}(x,y,z) = \underbrace{x}_{M}\hat{\imath} + \underbrace{y}_{N}\hat{\jmath} + \underbrace{z}_{P}\hat{k}$ a traves de la superficie S dada por: $\underbrace{z = 1 - x^2 - y^2}_{\text{paraboloide eliptico.}}$ que esta sobre el plano "xy." haciendo $\hat{n}$ normal ascendente.

P1. Graficar.




eje "z".

V(0,0,1)

Intersep con z=0 ← "xy"

$0 = 1 - x^2 - y^2$

$x^2 + y^2 = 1$

C(0,0) r=1

¿Cómo obtener $\hat{n}$?

Obligación: [despejar z]

$z = 1 - x^2 - y^2$

$f_x = -2x \rightarrow -f_x = 2x$

$f_y = -2y \rightarrow -f_y = 2y$

▶ Proyección "xy"

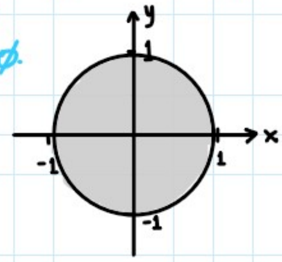
P2. Encontrar

y

Proyección "xy"

P3. Realizar el producto punto y sustituir "z".

P2. Encontrar $\hat{n}$ y proyeø.




$$\text{Flujo} = \iint_R \langle M, N, P\rangle \cdot \langle -f_x, -f_y, 1\rangle \, dA$$

$$\langle x, y, z\rangle \cdot \langle 2x, 2y, 1\rangle$$

$0 \leq r \leq 1$ ✓

$0 \leq \theta \leq 2\pi$ ✓

$$2x^2 + 2y^2 + z$$

$$= \iint_R 2x^2 + 2y^2 + \cancel{z}^{\,1-x^2-y^2} \, . \, dA$$

$$2x^2 + 2y^2 + 1 - x^2 - y^2$$

$$x^2 + y^2 + 1$$

$$\underbrace{x^2+y^2}_{r^2} + 1$$

$$r^2 + 1 \; ✓$$

$$dA = r \, dr \, d\theta \; ✓$$

$$= \iint_R x^2 + y^2 + 1 \, . \, dA$$

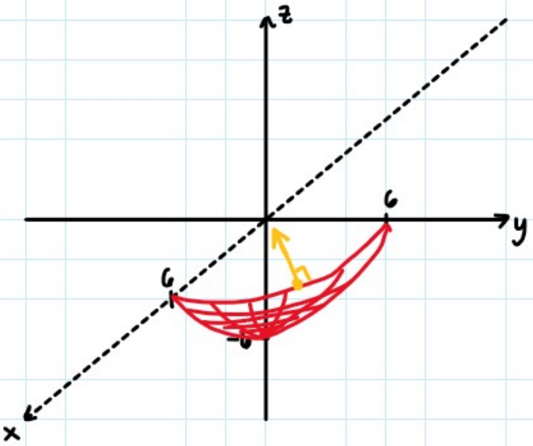
P4. Plantear y resolver la I.D.

$$= \int_0^{2\pi} \int_0^1 \underbrace{(r^2+1)\, r}_{r^3 + r} \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4} + \frac{r^2}{2} \Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2}\right) \cdot d\theta = \frac{3}{4} \int_0^{2\pi} d\theta$$

$$= \frac{3}{4} \theta \Big|_0^{2\pi} = \frac{3}{\cancel{4}_{2}} (\cancel{2}^{1}\pi) = \frac{3\pi}{2} \text{ u.V.}$$

EJEMPLO 2. Plantee la integral de flujo si $\vec{F}$ y S estan dados por:

$\vec{F}(x,y,z) = \underbrace{x}_{M}\hat{\imath} + \underbrace{y}_{N}\hat{\jmath} + \underbrace{z}_{P}\hat{k}$, S: $\underbrace{x^2+y^2+z^2 = 36}_{\text{Esfera } (0,0,0) \;\; r = \sqrt{36} = 6.}$ en el V octante.




asumir: $\hat{n}$ ascendente.

$$x^2 + y^2 + z^2 = 36$$

$$z = -\sqrt{36 - x^2 - y^2} = -(36 - x^2 - y^2)^{1/2}$$

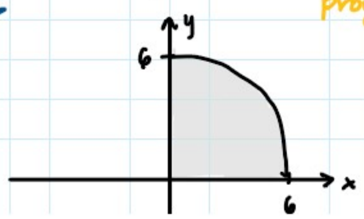
$$f_x = \cancel{-}\frac{1}{\cancel{2}}(36 - x^2 - y^2)^{-1/2}(\cancel{-}\cancel{2}x) = \frac{x}{\sqrt{36 - x^2 - y^2}}$$

$$f_y = \frac{y}{\sqrt{36 - x^2 - y^2}} \text{ (por analogía)}$$

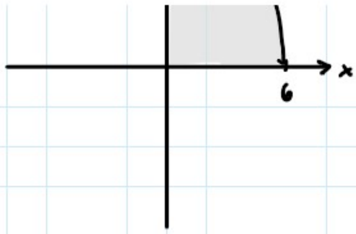
Proyección "xy"

$$\text{Flujo} = \iint_R \langle M, N, P\rangle \cdot \langle -f_x, -f_y, 1\rangle \, dA$$

$$\langle x, y, z\rangle \cdot \left\langle \frac{-x}{\sqrt{36-x^2-y^2}}, \frac{-y}{\sqrt{36-x^2-y^2}}, 1 \right\rangle$$



$$\langle x, y, z\rangle \cdot \left\langle \frac{-x}{\sqrt{36-x^2-y^2}}, \frac{-y}{\sqrt{36-x^2-y^2}}, 1 \right\rangle$$




$$\frac{-x^2-y^2}{\sqrt{36-x^2-y^2}} + z$$

$$\iint_R \frac{-x^2-y^2}{\sqrt{36-x^2-y^2}} + \cancel{z} \cdot dA \qquad z \to -\sqrt{36-x^2-y^2}$$

$$\frac{-x^2-y^2}{\sqrt{36-x^2-y^2}} - \frac{\sqrt{36-x^2-y^2}}{1}$$

$$= \frac{-x^2-y^2-(36-x^2-y^2)}{\sqrt{36-x^2-y^2}}$$

$$= \frac{\cancel{-x^2}-\cancel{y^2}-36+\cancel{x^2}+\cancel{y^2}}{\sqrt{36-x^2-y^2}} = \frac{-36}{\sqrt{36-x^2-y^2}}$$

$$-36 \iint_R \frac{dA}{\sqrt{36-x^2-y^2}}$$

$0 \le r \le 6$
$0 \le \theta \le \pi/2$

$$\sqrt{36-x^2-y^2} = \sqrt{36-(\underbrace{x^2+y^2}_{r^2})} = \sqrt{36-r^2}$$

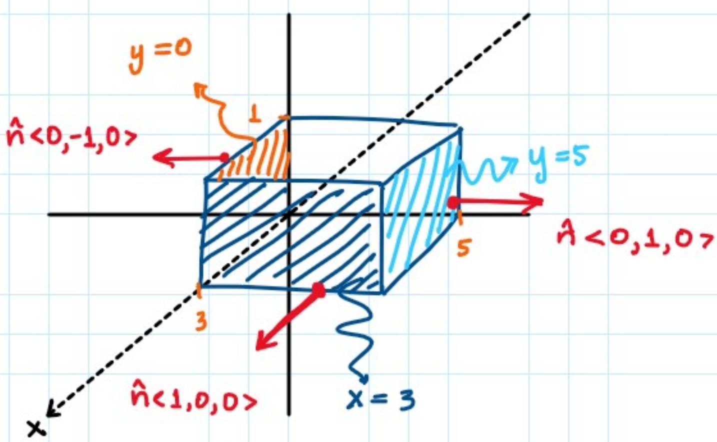
$$dA = r.dr.d\theta$$

$$-36 \int_0^{\pi/2} \lim_{m\to 6^-} \int_0^m \frac{r\,dr\,d\theta}{\sqrt{36-r^2}}$$

CASO ESPECIAL.

$x = \#$, $y = \#$ ó $z = \#$

asumir vectores...




vector se asume: