MATH 690: Topics in Data Analysis and Computation	November 30, 2017

Deformation Stability in Convolutional neural network

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1 Deformation stability

In our discussion of deformation stability, without loss of generality, we discuss signals with one channel.

Definition 1. *The deformation of signal $x(u)$ can be written as*

$$D_\tau x(u) = x(u - \tau(u)),$$

where $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation of the signal.

There are many types of deformation including but not limited to translation, rotation and dilation.

1. Translation has a constant displacement. $\tau(u) = \tau$.
2. Rotation applies an orthogonal matrix to the signal. $u - \tau(u) = \theta_t(u)$, where

$$\theta_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

3. Dilation does not change the direction but the magnitude. $u - \tau(u) = \alpha(u),\ \alpha > 0$.

Since a slight perturbation in deformation may cause a big change in L^2 norm, it is necessary to discuss deformation stability.

Definition 2. *Let f, D_τ be the convolutional neural network and a deformation on the input respectively. We say the network has deformation stability if, for $\forall x$,*

$$||f[D_\tau x] - f[x]||_2 \le c||x||_2,$$

where $c = c(\tau)$ is a constant independent of x.

2 Deformation stability of one convolutional layer

Proposition 1. *Suppose ψ is absolute continuous, $||\psi||_1 < \infty$, $||\psi'||_1 < \infty$. Displacement $\tau : \mathbb{R} \to \mathbb{R}$, $|\tau|_\infty < \infty$, $|\tau'|_\infty < \frac{1}{2}$. Then, $\forall x \in L^2(\mathbb{R})$,*

$$||\psi * x - \psi * D_\tau x||_2 \le \Big(|\tau|_\infty \cdot ||\psi'||_1 + \frac{|\tau'|_\infty}{1 - |\tau'|_\infty}||\psi||_1\Big)||x||_2.$$

Remark 1. *By proposition 1, we can derive the deformation stability of one convolutional layer,*

$$||\sigma(x * \psi) - \sigma(D_\tau x * \psi)|| \le C(\tau, \psi)||x||$$

3 Deformation stability of multiple convolutional layers

For multiple convolutional layers, without loss of generality, we show that a CNN with 2 layers has deformation stability.

Suppose we have a CNN with two layers,

$$\begin{cases} x^{(2)} = \sigma(x^{(1)} * \psi^{(1)}), \\ x^{(1)} = \sigma(x^{(0)} * \psi^{(0)}), \\ x^{(0)}(u) = x(u) \quad \text{is the input.} \end{cases}$$

Then we apply a deformation on the input to the CNN,

$$\begin{cases} \widetilde{x}^{(2)} = \sigma(\widetilde{x}^{(1)} * \widetilde{\psi}^{(1)}), \\ \widetilde{x}^{(1)} = \sigma(\widetilde{x}^{(0)} * \widetilde{\psi}^{(0)}), \\ \widetilde{x}^{(0)}(u) = D_\tau x^{(0)}. \end{cases}$$

The idea for deriving the stability is equivariance relation.

Definition 3. *f and D_τ are defined as before. We say f has equivariant relation with respect to the transformation D_τ of x if*

$$f[D_\tau x] = D_\tau f[x]$$

Indeed, this is saying that the mapping f is D_τ transform invariant.

Our goal is to show that $||\widetilde{x}^{(2)} - x^{(2)}||_2$ is bounded. By triangle inequality, we know that

$$||\widetilde{x}^{(2)} - x^{(2)}||_2 \leq ||\widetilde{x}^{(2)} - D_\tau x^{(2)}||_2 + ||D_\tau x^{(2)} - x^{(2)}||_2$$

It can be shown that the second term is small when the network is deep. Thus if the first term is bounded, the multi-layer neural network has deformation stability.

Lemma 1. *Suppose that*

$$\int |s\psi'(s)| \, \mathrm{d}s = c_1 < \infty,$$

then, $\forall x \in C^2$

$$||(D_\tau x) * \psi - D_\tau(x * \psi)||_2 \leq c_1 \frac{|\tau'|_\infty}{1 - |\tau'|_\infty} ||x||_2$$

This lemma can be proved by Schur's test with a change of variable trick.

Summary

In brief, in this topic, we first saw a comparison between fully connected neural network and convolutional neural network. Then, for convolutional neural network, we focused on the stability. In particular, we discussed about the stability in L^2 norm and deformation. To derive the deformation stability, we went from single layer to multiple layer.