CSC236 Exam Review

Notes from CSC236 Lecture 12

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Typed on November 27, 2024

Question #1

Consider a program that takes an array of intervals `intervals` where `intervals[i]` $=$ [$\texttt{start}_i, \texttt{end}_i$] and returns an optimal schedule:

```
def optimalschedule(intervals):
    sort intervals by the end times
    S = []
    f = -infty
    for i in [1, ..., n]:
        if start_i >= f:
            S.append([start_i, end_i])
            f = end_i
    return S
```

Definitions, Notes, and Examples:

- An **optimal schedule** is a subarray of `intervals` in which all the intervals are non-overlapping, and the subarray has the maximum possible size.
- $[1, 2]$ and $[2, 3]$ are non-overlapping.
- There may be multiple optimal schedules for an arbitrary array of intervals.
- All optimal schedules have the same size.
- In general, `intervals` $= [[\texttt{start}_1, \texttt{end}_1], \ldots, [\texttt{start}_n, \texttt{end}_n]]$ for some $n \in \mathbb{N}^+$ and $\texttt{start}_i, \texttt{end}_i \in \mathbb{R}^+$.
- The length of `intervals` is at least 1 (`intervals` is non-empty).
- If S is the subarray (in the program) at the j^{th} iteration and there exists some optimal schedule Opt such that $[\texttt{start}_i, \texttt{end}_i] \in Opt \iff [\texttt{start}_i, \texttt{end}_i] \in S$, then S is looking good.
- Let S be the subarray on the j^{th} iteration of the program. Define the predicate, $P(S)$: S is looking good.

Claim: something

Proof.
proofgoeshere

□

Claim: something

Proof.
proofgoeshere

□

Claim: something

Proof.
proofgoeshere

□

Claim: something

Proof.
proofgoeshere

□

Claim: something

Proof.
proofgoeshere

□

Question #2

Prove that $f(n) = \lceil \sqrt{(n)} \rceil - \lfloor \sqrt{(n)} - 4 \rfloor$ is asymptotically constant (i.e. $\Theta(1)$).

Proof.
By definition, if x and y are arbitrary real numbers, then

$$(x \leq \lceil x \rceil < x + 1)$$

and

$$(y - 1 < \lfloor y \rfloor \leq y).$$

Rewrite the second inequality as $-y \leq -\lfloor y \rfloor < -(y-1)$.

By adding the two inequalities, it follows that $x - y \leq \lceil x \rceil - \lfloor y \rfloor < x + 1 - (y - 1) = x - y + 2$.

Let $x = \sqrt{n}$ and $y = \sqrt{n} - 4$, for arbitrary natural n.
Then, $\lceil x \rceil - \lfloor y \rfloor = \lceil \sqrt{n} \rceil - \lfloor \sqrt{n} - 4 \rfloor = f(n)$. As well, $x - y = \cancel{\sqrt{n}} - (\cancel{\sqrt{n}} - 4) = 4$.

This means $x - y \leq \lceil x \rceil - \lfloor y \rfloor < x - y + 2 \implies 4 \leq f(n) < 4 + 2 \implies 4 \leq f(n) < 6$.

Let $n_0 = 0, c = 4, d = 6$. Let $g(n) = 1$.
Notice that $4 \leq f(n) < 6 \implies cg(n) \leq f(n) \leq dg(n)$, for all $n \geq n_0 = 0$ with $c = 4, d = 6$.

Therefore, $f(n) \in \Theta(g(n)) \implies f(n) \in \Theta(1)$. Indeed, $f(n)$ is asymptotically constant.

□

Showing that a DFA does not accept a language

1. Show that there exists $x, y \in \Sigma^*$ such that $\hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$.

2. Show that there exists $z \in \Sigma^*$ such that $xz \in L \iff yz \notin L$.

3. Clarify the contradiction that $\hat{\delta}(q_0, xz) = \hat{\delta}(q_0, yz) \implies \hat{\delta}(q_0, xz) \in F$.

Question #3

Prove that $L = \{a^{n^2} \mid n \in \mathbb{N}\}$ is **not** a regular language.

Proof.
Seeking a contradiction, assume there exists a DFA $\mathcal{D} = \{Q, \Sigma, \delta, s, F\}$ that accepts $L = \{a^{n^2} \mid n \in \mathbb{N}\}$.
Let $|Q| = k$.

Then, choose $w \in L$ such that $|w| \geq k + 1$. For instance, let $w = a^{j^2}$.

Notice that through processing the first $k + 1$ symbols in w, some state q must repeat by the Pigeonhole principle. Namely, $\hat{\delta}(q_0, a^\beta = q)$ and $\hat{\delta}(q, a^\beta) = q$, where $\beta \geq 1$.

Since $\mathcal{D}$ accepts $w = a^{j^2}$, it follows that $\hat{\delta}(q_0, a^{j^2}) \in F$. Notice that $\hat{\delta}(q_0, a^{j^2+\beta}) = \hat{\delta}(q_0, a^{\alpha+\beta+\gamma+\beta}) = \hat{\delta}(\hat{\delta}(q_0, a^\alpha), a^{\beta+\beta+\gamma}) = \hat{\delta}(q, a^{\beta+\beta+\gamma}) = \hat{\delta}(\hat{\delta}(q, a^\beta), a^{\beta+\gamma}) = \hat{\delta}(q, a^{\beta+\gamma}) = \hat{\delta}(\hat{\delta}(q_0, a^{alpha}), a^{\beta+\gamma}) = \hat{\delta}(q_0, a^{\alpha+\beta+\gamma}) \in F$.

This equivalence shows that $\mathcal{D}$ accepts $w = a^{j^2+\beta}$. By the same argument, $\mathcal{D}$ also accepts $a^{j^2+2\beta}, a^{j^2+3\beta}$.

However, notice that one of $a^{j^2+2\beta}, a^{j^2+3\beta}$ is not a square, which is a contradiction. Therefore, $L = \{a^{n^2} \mid n \in \mathbb{N}\}$ must not be regular.

□

*Here's an alternative proof by using the **Pumping Lemma**.*

Proof.
To show that $L = \{a^{n^2} \mid n \in \mathbb{N}\}$ is not a regular language, show:

$$(\forall n \in \mathbb{N}^+)(\exists w = xyz \in L)(\exists i \in \mathbb{N}^+)(|y| \geq 1 \land |xy| \leq n + 1 \land xy^iz \notin L).$$

Let $n \in \mathbb{N}^+$ be arbitrary.
Notice that $|w| \geq n$. Choose arbitrary $x, y \in \Sigma^*$ where xy represents the start ($\leq n + 1$) of

w. This makes y the part in which state-looping occurs.

Let $n \in \mathbb{N}^+$. Choose $w = a^{b^2}$, where $b^2 > n$.

Now, write $w = xyz$.

Notice that $y = a^i$, where $1 \leq i \leq n + 1$.

Add some proof here to show that...

It follows that either $xyy^2 = a^{n^2+i} \notin L$ or $xyyy^2 = a^{n^2+2i} \notin L$.

Therefore, $L = \{a^{n^2} \mid n \in \mathbb{N}\}$ is not a regular language.

$\square$

Question #4

Prove that $L = \{0^n 1^n \mid n \in \mathbb{N}\}$ is **not** a regular language.

Proof.
Let $n \in \mathbb{N}$. Choose $w = 0^{n+300}1^{n+300}$

Only case: $y = 0$.

Then, $w = 0^{n+300}1^{n+300} \notin L$.

$\square$