# CSC236 Homework Assignment #3

Language Regularity, Regular Expressions, and DFAs/NFAs

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## Question #1

Let $\Sigma = \{0, 1\}$.

**(a):**
**Claim:** $\Sigma^*$ is a regular language.

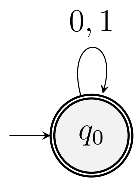
*Proof.*
Let $L_1 = \{0\}$ and $L_2 = \{1\}$ be regular languages of $\Sigma$.
Define $L_3 = L_1 \cup L_2 = \{0, 1\}$ as the regular language obtained by the union of $L_1$ and $L_2$.

By definition, ${L_3}^*$ is a regular language. Since $L_3 = \Sigma$, $\Sigma^*$ is also a regular language.

While this proof now is complete, the assignment encourages the use of determinstic finite automaton (or DFA) proofs to show that languages are regular.

So, as an alternative to *Part (a)* (this part's proof), and for all relevant subsequent parts, adopt a DFA proof.

Consider the **transition function** $\delta$ associated with the following DFA:



| **Old State** | **Symbol** | **New State** |
|---|---|---|
| $q_0$ | 0 | $q_0$ |
| $q_0$ | 1 | $q_0$ |

*Table 1: State Transition Table*

Using $\delta$, define the DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$, where

$$Q = \{q_0\} \text{ is the set of states in } \mathcal{D}$$

$$\Sigma = \{0,1\} \text{ is the alphabet of symbols used by } \mathcal{D}$$

$$\delta : Q \times \Sigma \to Q \text{ is the transition function defined by } \textit{Table 1}$$

$$s = q_0 \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_0\} \subseteq Q \text{ is the set of accepting states of } \mathcal{D}.$$

For the sole state $q_0$ in $\mathcal{D}$, define its **invariant** for strings $x \in \Sigma^* = \{0,1\}^*$:

$$P_{q_0}(x) : x \text{ is } \epsilon \text{ or consists of 0s and 1s.}$$

As the only state, $q_0$ is trivially **mutually exclusive**. As well, every string in $\Sigma^* = \{0,1\}^*$ is either $\epsilon$ or consists of some number of 0s and 1s, which $q_0$ clearly satisfies; **exhaustivity** is satisfied.

Let $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0,1\}^*$ both be arbitrary (here, $q = q_0$ always).
Denote the predicate:

$$P_{\delta(q_0,x)}(x) := P_q(x) \text{ is the state invariant.}$$

Perform structural induction on $P_{\delta(q_0,x)}(x)$ for all strings $x \in \Sigma^* = \{0,1\}^*$, as follows:

Base Case:
Let $q = q_0$ and $w = \epsilon$.
$P_q(w) = P_{q_0}(\epsilon)$ is true as $w = \epsilon$.

Induction Hypothesis:
Assume that $P_q(w)$ is true for all $q \in \{q_0\}$ and some $w \in \{0,1\}^*$.
This means $w$ is either the empty string or consists of 0s and 1s.

Induction Step:
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_0\}$ and some $w \in \{0,1\}^*$.

Demonstrate that the invariant of $q_0$ holds when processing all possible strings from $\Sigma^*$.

This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary.

Let $q = q_0$, $z \in \{0,1\}$, and assume $P_{q_0}(w)$ is true.
Consider that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,z)}(wz) = P_{q_0}(wz)$. Since $w$ is either the empty string or consists of 0s and 1s, while $z \in \{0,1\}$, $wz$ remains to consist of 0s and 1s. This matches the definition of $P_{q_0}(wz)$. Thus, $P_{\delta(q,z)}(wz)$ holds.

There are no other state invariants in $\mathcal{D}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0,1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language $L = \Sigma^*$ over $\Sigma = \{0,1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0,1\}^*$ is arbitrary, $x$ is a member of $L$ if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the sole state invariant, $P_{q_0}(x) : x$ is $\epsilon$ or consists of 0s and 1s.
If $x \in L$, then $x$ is either the empty string or consists of 0s and 1s. Clearly, $P_{q_0}(x)$ is true.
On the other hand, choose the accepting state $q_0 \in F$ and assume $P_{q_0}(x)$ is true. Clearly, $x \in L$, due to matching definitions.

Therefore, $P_{\delta(q_0,x)}(x)$ is true for all strings $x \in \Sigma^* = \{0,1\}^*$.

While this DFA proof is redundant in nature, it is clear that $\mathcal{D}$ accepts $L = \Sigma^*$ over $\Sigma = \{0,1\}$. Again, it has been demonstrated that $L$ is a regular language.

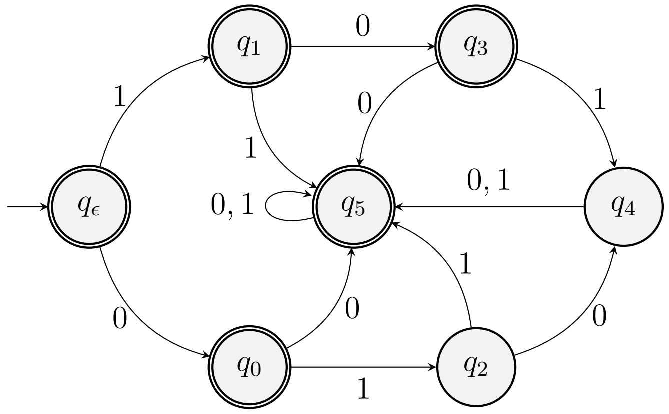
$\square$

**(b):**
**<u>Claim:</u>** $\Sigma^* \setminus K, K = \{01, 101, 010\}$ is a regular language.

*Proof.*
This proof aims to show that the language $\Sigma^* \setminus K$ is a regular language, by constructing

a DFA (with proof) that accepts all strings over $\Sigma^* = \{0,1\}^*$ except the literal strings in $K = \{01, 101, 010\}$.

Consider the **transition function** $\delta$ associated with the following DFA:



| **Old State** | **Symbol** | **New State** |
|---|---|---|
| $q_\epsilon$ | 0 | $q_0$ |
| $q_\epsilon$ | 1 | $q_1$ |
| $q_0$ | 0 | $q_5$ |
| $q_0$ | 1 | $q_2$ |
| $q_1$ | 0 | $q_3$ |
| $q_1$ | 1 | $q_5$ |
| $q_2$ | 0 | $q_4$ |
| $q_2$ | 1 | $q_5$ |
| $q_3$ | 0 | $q_5$ |
| $q_3$ | 1 | $q_4$ |
| $q_4$ | 0 | $q_5$ |
| $q_4$ | 1 | $q_5$ |
| $q_5$ | 0 | $q_5$ |
| $q_5$ | 1 | $q_5$ |

*Table 2: State Transition Table*

Using $\delta$, define the DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$, where

$$Q = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\} \text{ is the set of states in } \mathcal{D}$$

$$\Sigma = \{0, 1\} \text{ is the alphabet of symbols used by } \mathcal{D}$$

$$\delta : Q \times \Sigma \to Q \text{ is the transition function define by } \textit{Table 2}$$

$$s = q_\epsilon \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_\epsilon, q_0, q_1, q_3, q_5\} \subseteq Q \text{ is the set of accepting states of } \mathcal{D}.$$

For each state $q_i \;|_{i \in \{\epsilon,0,1,2,3,4,5\}}$ in $\mathcal{D}$, define a **state invariant** $P_q(x)$ for strings $x \in \Sigma^* = \{0,1\}^*$:

$P_{q_\epsilon}(x) : x$ is the empty string, $\epsilon$

$P_{q_0}(x) : x$ is 0

$P_{q_1}(x) : x$ is 1

$P_{q_2}(x) : x$ is 01

$P_{q_3}(x) : x$ is 10

$P_{q_4}(x) : x$ is either 101 or 010

$P_{q_5}(x) : x$ consists of 0s and 1s but $x \notin \{0, 1, 10, 01, 010, 101\}$

It is clear that $q_i$ for $i \in \{\epsilon, 0, 1, 2, 3, 4\}$ are mutually exclusive by their unique definitions. Moreover, the $q_5$ state is the direct complement of the **union of the previous** $q_i$ **states** (as a non-initial state, $q_5$ shall never process $\epsilon$), so $q_5$ must be mutually exclusive as well. Thus, all states in $\mathcal{D}$ are **mutually exclusive**.
For exhaustivity, notice that every string in $\Sigma^* = \{0,1\}^*$ is either $\epsilon$ or consists of some number of 0s and 1s. The $q_\epsilon$ state accounts for the empty string case, $q_i$ for $i \in \{0,1,2,3,4\}$ account for specific strings of 0s and 1s, and $q_5$ accounts for all the remaining cases of 0s and 1s not covered by the $q_i$. Thus, the states in $\mathcal{D}$ are **exhaustive**.

Let $q \in Q = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and $w \in \Sigma^* = \{0,1\}^*$ both be arbitrary.
Denote the predicate:

$$P_{\delta(q_0,w)}(w) \coloneqq P_q(w) \text{ is the state invariant.}$$

Perform structural induction as follows:

<u>Base Cases:</u>
Let $w = \epsilon$.

Let $q = q_\epsilon$. $P_q(w) = P_{q_\epsilon}(\epsilon)$ is true as $w = \epsilon$, matching the definition of $P_{q_\epsilon}(w)$.
Let $q \neq q_\epsilon$. $P_q(w) = P_{q_i}(\epsilon)\mid_{i \in \{0,1,2,3,4,5\}}$ are vacuously true as these corresponding states are non-initial and do not process $\epsilon$.

Induction Hypothesis:
Assume that $P_q(w)$ is true for all $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0,1\}^*$.

Induction Step:
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0,1\}^*$.

Demonstrate that all state invariants hold when processing strings from $\Sigma = \{0,1\}$.
This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary. Then, consider the following cases.

Case $(q = q_\epsilon, z = 0)$:
Assume $P_{q_\epsilon}(w)$ is true. Then, $w = \epsilon$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_\epsilon,0)}(\epsilon 0) = P_{q_0}(0)$, which is clearly true by definition.

Case $(q = q_\epsilon, z = 1)$:
Likewise, assume $P_{q_\epsilon}(w)$ is true. Then, $w = \epsilon$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_\epsilon,1)}(\epsilon 1) = P_{q_1}(1)$, which is also clearly true by definition.

Case $(q = q_0, z = 0)$:
Assume $P_{q_0}(w)$ is true. Then, $w = 0$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,0)}(00) = P_{q_5}(00)$, which is true as $w = 00$ indeed consists of 0s and $1s$ and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case $(q = q_0, z = 1)$:
Likewise, assume $P_{q_0}(w)$ is true. Then, $w = 0$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,1)}(01) = P_{q_2}(01)$, which is clearly true by definition.

Case $(q = q_1, z = 0)$:
Assume $P_{q_1}(w)$ is true. Then, $w = 1$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,0)}(10) = P_{q_3}(10)$, which is clearly true by definition.

Case $(q = q_1, z = 1)$:
Likewise, assume $P_{q_1}(w)$ is true. Then, $w = 1$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,1)}(11) = P_{q_5}(11)$, which is true as $w = 11$ indeed consists of 0s and $1s$ and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case $(q = q_2, z = 0)$:
Assume $P_{q_2}(w)$ is true. Then, $w = 01$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,0)}(010) = P_{q_4}(010)$, which is true as $w = 010$ is indeed either 101 or 010.

Case $(q = q_2, z = 1)$:
Likewise, assume $P_{q_2}(w)$ is true. Then, $w = 01$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,1)}(011) = P_{q_5}(011)$, which is true as $w = 011$ indeed consists of 0s and $1s$ and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case $(q = q_3, z = 0)$:
Assume $P_{q_3}(w)$ is true. Then, $w = 10$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_3,0)}(100) = P_{q_5}(100)$, which is true as $w = 100$ indeed consists of 0s and $1s$ and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case $(q = q_3, z = 1)$:
Assume $P_{q_3}(w)$ is true. Then, $w = 10$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_3,1)}(101) = P_{q_4}(101)$, which is true as $w = 101$ is indeed either 101 or 010.

Case $(q = q_4, z \in \{0, 1\})$:
Assume $P_{q_4}(w)$ is true. Then, $w \in \{101, 010\}$.

It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_4,z)}(wz) = P_{q_5}(wz)$. Notice that $wz$ is a string with 4 symbols, because $w$ and $z$ are strings with 3 symbols and 1 symbol, respectively. As well, $wz$ is constructed using only 0s and 1s. Thus, $wz$ consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$; $P_{\delta(q,z)}(wz)$ holds.

<u>Case $(q = q_5, z \in \{0, 1\})$:</u>
Assume $P_{q_5}(w)$ is true.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_5,z)}(wz) = P_{q_5}(wz)$. Recall that $z \in \{0, 1\}$.

Since $w$ consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$ by $P_{q_5}(w)$, concatenating $z$ (which is either 0 or 1) to $w$ does not result in $wz \in \{0, 1, 10, 01, 010, 101\}$. This is because $w \neq \epsilon$ (as $q_5$ does not process $\epsilon$), so $wz \notin \{0, 1\}$, which implies $wz \notin \{10, 01\}$ (as $q_5$ will not process 10 and 01 if does not process 1 and 0, respectively), which also implies $wz \notin \{010, 101\}$ (as $q_5$ will not process 010 and 101 if it does not process 01 and 10, respectively).

<u>End of Cases:</u>
Hence, it is demonstrated that if $P_q(w)$ is true for all $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0, 1\}^*$, then $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0, 1\}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and $x \in \Sigma^* = \{0, 1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language $L = \Sigma^* \setminus \{01, 101, 010\}$ over $\Sigma = \{0, 1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0, 1\}^*$ is arbitrary, $x$ is a member of $L$ if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the invariants of the accepting states, $q_i \in F \mid_{i\in\{\epsilon,0,1,3,5\}}$:

$$
\begin{aligned}
&P_{q_\epsilon}(x) : x \text{ is the empty string, } \epsilon\\
&P_{q_0}(x) : x \text{ is } 0\\
&P_{q_1}(x) : x \text{ is } 1\\
&P_{q_3}(x) : x \text{ is } 10\\
&P_{q_5}(x) : x \text{ consists of 0s and 1s but } x \notin \{0, 1, 10, 01, 010, 101\}
\end{aligned}
$$

*Implication*
If an arbitrary string $x \in L = \Sigma^* \setminus \{01, 101, 010\}$, then $x$ is either an empty string or consists of 0s and 1s, but must not be either of 01, 101, nor 010. Choose $q_\epsilon$ to cover the empty string case, and $q_5$ to cover all cases **except** $x \in \{0, 1, 10, 01, 010, 101\}$. Since $x$ must not be a member of $\{01, 010, 101\}$, it remains to account for $x \in \{0, 1, 10\}$. Conveniently, these cases are covered by $q_0$, $q_1$, and $q_3$. Thus, no matter which string $x \in L = \Sigma^* \setminus \{01, 101, 010\}$, there exists accepting states $q \in F$ such that $P_q(x)$ holds.

*Implied-by*
Conversely, $L = \Sigma^* \setminus \{01, 101, 010\}$ can be constructed by the same choices of accepting states. Choose from the accepting states $q_i \in F \mid_{i\in\{\epsilon,0,1,3,5\}}$ and assume $P_{q_i}(x)$ are true. Likewise, as demonstrated in the *implication* proof, reconstruct $L = \Sigma^* \setminus \{01, 101, 010\}$ using the accepting states $P_{q_i}(x)$, similarly. It is clear that the accepting state's invariants build a language consisting of the empty string ($P_{q_\epsilon}$), and any combination of 0s and 1s, excluding the specific strings 01, 101, and 010 ($P_{q_0}, P_{q_1}, P_{q_3}, P_{q_5}$). Thus, if there are accepting states $q \in F$ such that $P_q(x) \mid_{x\in\Sigma^*=\{0,1\}^*}$ holds, then $x \in L = \Sigma^* \setminus \{01, 101, 010\}$.

Therefore, $\mathcal{D}$ accepts $L = \Sigma^* \setminus \{01, 101, 010\}$ over $\Sigma = \{0, 1\}$. This means $L$ is a regular language.

$\square$

**(c):**
**Claim:** $\{w | w \text{ is a palindrome}\}$ is NOT a regular language.

*Proof.*
Seeking a contradiction, assume the language $L = \{w | w \text{ is a palindrome}\}$ is regular. Then, there exists a DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$ that recognizes $L$, where $|Q| = k, k \in \mathbb{N}$

Next, choose a string $w \in L$ such that $|w| \geq k + 1$.
Let $w = 0^k1^10^k \in L$. Notice that $|W| = 2k + 1 \geq k + 1$, and that the DFA must accept $w$.

As $\mathcal{D}$ processes the first $k$ symbols of $w = 0^k1^10^k$, some state $q \in Q$ must repeat, because the DFA only has $k$ states, but processes more than $k$ symbols. This repetition implies that $\mathcal{D}$ transitions into a loop at state $q$ while reading the initial segment $0^k$.

Let this loop correspond to some repeated substring $s$ of $0^k$. Because the DFA cannot remember how many 0s it has processed within this loop, it treats strings with any number of repetitions of $s$ the same way.

Now construct a string $w' = 0^{k-s}1^10^k$, where $0^{k-s}$ has $s$ less 0s than $0^k$. Since DFA processes $w'$ exactly as it processes $w$, then $\mathcal{D}$ must also accept $w'$.

However, $w'$ is not a palindrome: the number of 0s on the left $(k - s)$ does not match the number of 0s on the right $(k)$. This contradicts the assumption that $\mathcal{D}$ recognizes $L$, as $\mathcal{D}$ accepts $w' \notin L$.

By reaching a contradiction, the language $L = \{w | w \text{ is a palindrome}\}$ cannot be recognized by a DFA. Thus, $L$ is not a regular language.

□

**(d):**
**Claim:** $\{ww | w \in \Sigma^*\}$ is NOT a regular language.

*Proof.*
Seeking a contradiction, assume the language $L = \{ww | w \in \Sigma^*\}$ is regular. Then, there exists a DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$ that recognizes $L$, where $|Q| = k, k \in \mathbb{N}$

Next, choose a string $x \in L$ such that $|x| \geq k+1$.
Let $x = 0^k1^k0^k1^k \in L$, which is of length $2k+2 \geq k+1$. This string is in $L$ because it has the form $ww$, where $w = 0^k1^k$.

Since the DFA $\mathcal{D}$ has $k$ states, it must repeat (loop) at some state $q \in Q$ while processing the first $k+1$ symbols of $x = 0^k1^k0^k1^k$. That is

$$\delta(s, 0^i) = \delta(s, 0^{i+j}) \text{ for some } i \leq k \text{ and } j > 0.$$

Now consider modifying $x$ by increasing the looped segment. Let $x' = 0^{k+j}1^k0^k1^k$, where $j > 0$.
The DFA will process $x'$ in the same way as $x$, because of the looping on state $q$.

It follows that the string $x' = 0^{k+j}1^k0^k1^k \notin L$, because it does not have two identical halves. Specifically:

- $x$ splits evenly as $x = 0^k1^k0^k1^k = ww, w = 0^k1^k$, but
- $x'$ splits as $x' = 0^{k+j}1^k0^k1^k$, where $0^{k+j}1^k \neq 0^k1^k$.

Yet, the DFA $\mathcal{D}$ cannot distinguish between $x$ and $x'$ (due to the loop), and will still accept $w$. This contradicts the assumption that $\mathcal{D}$ recognizes $L$, as $\mathcal{D}$ accepts $x' \notin L$.

By reaching a contradiction, it follows that no DFA can recognize $L = \{ww | w \in \Sigma^*\}$, because DFAs cannot ensure both halves of a string are identical. Thus, $L$ is not regular.

□

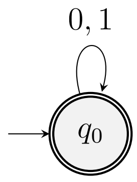
**(e):**
**<u>Claim:</u>** $\{w \mid ww \in \Sigma^*\}$ is a regular language.

*Proof.*
This proof aims to show that the language $\{w \mid ww \in \Sigma^*\}$ is a regular language, by constructing a DFA (with proof) that accepts all strings with its self-concatenation obtainable

over the alphabet $\Sigma = \{0, 1\}$.

Moreover, this proof recycles the DFA which was suggested in *Part (a)*, but with an adjustment to the state invariant. While it suffices to provide a double-subset inclusion proof to show an equivalence between $\{w \mid ww \in \Sigma^*\}$ and $\Sigma^*$ (from *Part (a)*), proceed with the modified DFA proof nonetheless.

Consider the **transition function** $\delta$ with the following DFA:



| **Old State** | **Symbol** | **New State** |
|---|---|---|
| $q_0$ | 0 | $q_0$ |
| $q_0$ | 1 | $q_0$ |

*Table 3: State Transition Table*

Using $\delta$, define the DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$, where

$$Q = \{q_0\} \text{ is the set of states in } \mathcal{D}$$

$$\Sigma = \{0, 1\} \text{ is the alphabet of symbols used by } \mathcal{D}$$

$$\delta : Q \times \Sigma \to Q \text{ is the transition function defined by } \textit{Table 1}$$

$$s = q_0 \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_0\} \subseteq Q \text{ is the set of accepting states of } \mathcal{D}.$$

For the sole state $q_0$ in $\mathcal{D}$, define its **invariant** for strings $x \in \Sigma^* = \{0, 1\}^*$:

$$P_{q_0}(x) : xx \in \Sigma^* = \{0, 1\}^*$$

As the only state, $q_0$ is trivially **mutually exclusive**. As well, every string in $\Sigma^* = \{0, 1\}^*$ is either $\epsilon$ or consists of some number of 0s and 1s. Concatenate the empty string, $\epsilon$, to itself and $\epsilon$ is the result. Concatenate a string consisting of 0s and 1s to itself, and the result is still a string consisting of 0s and 1s. There are no other strings in $\{0,1\}^*$ to test against this state invariant. Thus, $q_0$ satisfies **exhaustivity**.

Let $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0,1\}^*$ both be arbitrary (here, $q = q_0$ always).
Denote the predicate:

$$P_{\delta(q_0,x)}(x) \coloneqq P_q(x) \text{ is the state invariant.}$$

Perform structural induction on $P_{\delta(q_0,x)}(x)$ for all strings $x \in \Sigma^* = \{0,1\}^*$, as follows:

Base Case:
Let $q = q_0$ and $w = \epsilon$.
$P_q(w) = P_{q_0}(\epsilon)$ is true as $\epsilon\epsilon = \epsilon \in \Sigma^* = \{0,1\}^*$.

Induction Hypothesis:
Assume that $P_q(w)$ is true for all $q \in \{q_0\}$ and some $w \in \{0,1\}^*$.
This means $ww \in \Sigma^* = \{0,1\}^*$.

Induction Step:
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_0\}$ and some $w \in \{0,1\}^*$.

Demonstrate that the invariant of $q_0$ holds when processing all possible strings from $\Sigma^*$. This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary.

Let $q = q_0$, $z \in \{0,1\}$, and assume $P_{q_0}(w)$ is true.
Consider that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,z)}(wz) = P_{q_0}(wz)$. Since $ww \in \Sigma^* = \{0,1\}^*$ and $z \in \{0,1\}$, it follows that $wzwz \in \Sigma^* = \{0,1\}^*$ as well. Namely, this is because strings consisting of 0s and 1s are members of $\Sigma^* = \{0,1\}^*$, and $wzwz$ matches this description as $w \in \{\epsilon,0,1\}$ while $z \in \{0,1\}$. By definition, $P_{q_0}(wz)$ holds. Thus, $P_{\delta(q,z)}(wz)$ holds.

There are no other state invariants in $\mathcal{D}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0,1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language $L = \{w \mid ww \in \Sigma^*\}$ over $\Sigma = \{0,1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0,1\}^*$ is arbitrary, $x$ is a member of $L$ if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the sole state invariant, $P_{q_0}(x) : xx \in \Sigma^* = \{0,1\}^*$.
If $x \in L$, then $xx \in \Sigma^*$. Clearly, $P_{q_0}(x)$ is true, due to matching definitions.
On the other hand, choose the accepting state $q_0 \in F$ and assume $P_{q_0}(x)$ is true. Then $xx \in \Sigma^*$, and $x \in L$ is, likewise, true due to matching definitions.

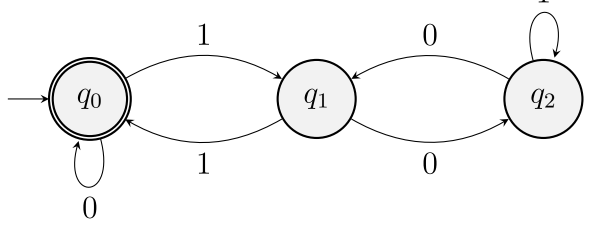
Therefore, $P_{\delta(q_0,x)}(x)$ is true for all strings $x \in \Sigma^* = \{0,1\}^*$. It has been demonstrated that $L = \{w \mid ww \in \Sigma^*\}$ is a regular language.

$\square$

**(f):**
<u>**Claim:**</u> $\{w \mid w$ is a binary representation of a multiple of $3\}$ is a regular language.

*Proof.*
This proof aims to show that the language $\{w \mid w$ is a binary representation of a multiple of $3\}$ is a regular language, by constructing a DFA (with proof) that accepts all strings over $\Sigma^* = \{0,1\}^*$ that are also a binary representation of multiples of 3.

Consider the **transition function** $\delta$ associated with the following DFA:



| **Old State** | **Symbol** | **New State** |
|---|---|---|
| $q_0$ | 0 | $q_0$ |
| $q_0$ | 1 | $q_1$ |
| $q_1$ | 0 | $q_2$ |
| $q_1$ | 1 | $q_0$ |
| $q_2$ | 0 | $q_1$ |
| $q_2$ | 1 | $q_2$ |

*Table 4: State Transition Table*

Using $\delta$, define the DFA $\mathcal{D} = (Q, \Sigma, \delta, s, F)$, where

$$Q = \{q_0, q_1, q_2\} \text{ is the set of states in } \mathcal{D}$$

$$\Sigma = \{0, 1\} \text{ is the alphabet of symbols used by } \mathcal{D}$$

$$\delta : Q \times \Sigma \to Q \text{ is the transition function define by } \textit{Table 4}$$

$$s = q_0 \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_0\} \subseteq Q \text{ is the set of accepting states of } \mathcal{D}.$$

For each state $q_i \mid_{i \in \{0,1,2\}}$ in $\mathcal{D}$, define a **state invariant** $P_q(x)$ for strings $x \in \Sigma^* = \{0,1\}^*$:

$$P_{q_0}(x) : (0 \equiv (\text{integer representation of } x) \pmod 3) \text{ or } (x = \epsilon)$$
$$P_{q_1}(x) : 1 \equiv (\text{integer representation of } x) \pmod 3$$
$$P_{q_2}(x) : 2 \equiv (\text{integer representation of } x) \pmod 3$$

When performing division by 3 on integers, the result either yields a remainder of 0, 1, or 2. Clearly, these remainders are unique, so $P_{q_0}(x)$, $P_{q_1}(x)$, and $P_{q_2}(x)$ are **mutually exclusive** states. Note that $\epsilon$ is considered solely by the initial state, $q_0$.
Moreover, all strings in $\Sigma^* = \{0,1\}^*$ are integer representations, if not $\epsilon$. After a division by 3, the possible remainders for these integer representations are cases directly covered by the three state invariants. Therefore, some state invariant must hold for an arbitrary string's integer representation. This means the collection of states in $\mathcal{D}$ must be **exhaustive**.

Let $q \in Q = \{q_0, q_1, q_2\}$ and $w \in \Sigma^* = \{0,1\}^*$ both be arbitrary.

Denote the predicate:

$$P_{\delta(q_0,w)}(w) \coloneqq P_q(w) \text{ is the state invariant.}$$

Perform structural induction as follows:

Base Cases:
Let $w = \epsilon$.
Let $q = q_0$. $P_q(w) = P_{q_0}(\epsilon)$ is true as $w = \epsilon$, satisfying the definition of the state invariant.
Let $q \neq q_0$. $P_q(w) = P_{q_i}(\epsilon)\ |_{i \in \{1,2\}}$ are vacuously true as these corresponding states are non-initial and do not process $\epsilon$.

Induction Hypothesis:
Assume that $P_q(w)$ is true for all $q \in \{q_0, q_1, q_2\}$ and some $w \in \{0,1\}^*$.

Induction Step:
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_0, q_1, q_2\}$ and some $w \in \{0,1\}^*$.

Demonstrate that all state invariants hold when processing strings from $\Sigma = \{0,1\}$.
This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary. Let $W$ be the integer representation of the string $w$. Then, consider the following cases.

Case $(q = q_0, z = 0)$:
Assume $P_{q_0}(w)$ is true. Then, $0 \equiv W \pmod 3$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,0)}(w0) = P_{q_0}(w0)$. Notice that, in treating the string $wz = w0$ as a binary integer representation, concatenating 0 to $w$ performs a logical left shift (a multiplication by 2).

Recall that $W$ is the integer representation of the string $w$; $2W$ shall be the integer representation of the string $wz = w0$ (due to the logical left shift). Since $0 \equiv W \pmod 3$, it follows that $3 \mid W$, meaning $\exists n \in \mathbb{N}^+$ such that $W = 3n$.

Perform the multiplication by 2 and notice, $2W = 2 \times 3n = 3(2n)$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W = 3(2n) = 3m$, meaning $3 \mid 2W$. It has been clearly demonstrated that $0 \equiv 2W \pmod 3$, where $2W$ is the integer representation of $wz = w0$. Thus, $P_{\delta(q,z)}(wz) = P_{q_0}(w0)$ is true.

<ins>Case $(q = q_0, z = 1)$:</ins>
Likewise, assume $P_{q_0}(w)$ is true. Then, $0 \equiv W \pmod 3$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,1)}(w1) = P_{q_1}(w1)$.

Declare $2W + 1$ as the integer representation of the string $wz = w1$, due to a logical left shift and addition by 1 (the result of concatenating 1 to $w$ instead of 0). Since $0 \equiv \pmod 3$, there exists $n \in \mathbb{N}^+$ such that $W = 3n$.

Perform the multiplication by 2, followed by an addition of 1, and notice that $2W + 1 = 2 \times 3n + 1 = 3(2n) + 1$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W = 3(2n) + 1 = 3m + 1$, meaning $2W$ yields a remainder of 1 after division by 3.

It follows that $1 \equiv 2W + 1 \pmod 3$. This means $P_{\delta(q,z)}(wz) = P_{q_1}(w1)$ holds, as $2W + 1$ is the integer representation of $wz = w1$.

<ins>Case $(q = q_1, z = 0)$:</ins>
Assume $P_{q_1}(w)$ is true. Then, $1 \equiv W \pmod 3$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,0)}(w0) = P_{q_2}(w0)$.

Declare $2W$ to be the integer representation of the string $wz = w0$, due to a logical left shift in concatenating 0 to $w$. Since $1 \equiv W \pmod 3$, it follows that $\exists n \in \mathbb{N}^+$ such that $W = 3n + 1$.

Perform the multiplication by 2 and notice, $2W = 2 \times (3n + 1) = 3(2n) + 2$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W = 3(2n) + 2 = 3m + 2$, meaning $2W$ yields a remainder of 2 after division by 3. It has been clearly demonstrated that $2 \equiv 2W \pmod 3$, where $2W$ is the integer representation of $wz = w0$. Thus, $P_{\delta(q,z)}(wz) = P_{q_2}(w0)$ is true.

Case $(q = q_1, z = 1)$:
Likewise, assume $P_{q_1}(w)$ is true. Then, $1 \equiv W \pmod 3$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,1)}(w1) = P_{q_0}(w1)$. Note that, in treating $wz = w1$ as a binary integer representation, concatenating 1 to $w$ performs a logical left shift (a multiplication by 2) followed by an addition by 1.

Declare $2W + 1$ to be the integer representation of the string $wz = w1$. Since $1 \equiv W \pmod 3$, it follows that $\exists n \in \mathbb{N}^+$ such that $W = 3n + 1$.

Perform the multiplication by 2, followed by an addition of 1, and notice that $2W + 1 = 2 \times (3n + 1) + 1 = 3(2n) + 2 + 1 = 3(2n) + 3 = 3(2n + 1)$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W + 1 = 3(2n + 1) = 3m$, meaning $3 \mid 2W + 1$.

It follows that $0 \equiv 2W + 1 \pmod 3$. This means, $P_{\delta(q,z)}(wz) = P_{q_0}(w1)$ holds, as $2W + 1$ is the integer representation of $wz = w1$.

Case $(q = q_2, z = 0)$:
Assume $P_{q_2}(w)$ is true. Then, $2 \equiv W \pmod 3$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,0)}(w0) = P_{q_1}(w0)$.

Declare $2W$ as the integer representation of the string $wz = w0$, due to the logical left shift. Since $2 \equiv W \pmod 3$, it follows that $\exists n \in \mathbb{N}^+$ such that $W = 3n + 2$.

Perform the multiplication by 2 and notice, $2W = 2 \times (3n+2) = 3(2n)+4 = 3(2n)+3+1 = 3(2n + 1) + 1$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W = 3(2n + 1) + 1 = 3m + 1$, meaning $2W$ yields a remainder of 1 after division by 3. It has been demonstrated that $1 \equiv 2W \pmod 3$, where $2W$ is the integer represesntation of $wz = w0$. Thus, $P_{\delta(q,z)}(wz) = P_{q_1}(w0)$ is true.

Case $(q = q_2, z = 1)$:
Likewise, assume $P_{q_2}(w)$ is true. Then, $2 \equiv W \pmod 3$.

It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,1)}(w1) = P_{q_2}(w1)$.

Declare $2W + 1$ to be the integer representation of the string $wz = w1$ (one shift logically left yields multiplication by 2, add 1 as $z = 1$). Since $2 \equiv W \pmod 3$, it follows that $\exists n \in \mathbb{N}^*$ such that $W = 3n + 2$.

Perform the multiplication by 2, followed by an addition of 1, and notice that $2W + 1 = 2\times(3n+2)+1 = 3(2n)+4+1 = 3(2n)+3+2 = 3(2n+1)+2$. Declare that $\exists m \in \mathbb{N}^+$ such that $2W+1 = 3(2n+1)+2 = 3m+2$, meaning $2W+1$ yields a remainder of 2 after division by 3.

It follows that $1 \equiv 2W + 1 \pmod 3$. This means $P_{\delta(q,z)}(wz) = P_{q_2}(w1)$ holds, as $2W + 1$ is the integer representation of $wz = w1$.

<u>End of Cases:</u>
Hence, it is demonstrated that if $P_q(w)$ is true for all $q \in \{q_0, q_1, q_2\}$ and some $w \in \{0, 1\}^*$, then $P_{\delta(q,z)}(wz)$ holds for all $z \in \{0, 1\}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_0, q_1, q_2\}$ and $x \in \Sigma^* = \{0, 1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language

$$L = \{w \mid w \text{ is a binary representation of a multiple of 3}\}$$

over $\Sigma = \{0, 1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0, 1\}^*$ is arbitrary, $x$ is a member of $L$ if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the sole accepting state invariant,

$$P_{q_0}(x) : (0 \equiv (\text{integer representation of } x) \pmod 3) \text{ or } (x = \epsilon).$$

<u>*Implication*</u>

If $x \in L$ is arbitrary, then $x$ can be interpreted as a binary representation of a multiple of 3. Notice that the definition of $P_{q_0}(x)$ directly fits the definition of $L$, so $P_{q_0}(x)$ is indeed true for $q_0 \in F$.

*Implied-by*
Consider $q_0 \in F$ and assume $P_{q_0}(x)$ is true for arbitrary $x \in \Sigma^* = \{0,1\}^*$. Clearly, $x$ is also a member of $L$ as $P_{q_0}(x)$ suggests that the integer representation of $x$ is indeed a multiple of 3.

Therefore, $\mathcal{D}$ accepts $L = \{w \mid w \text{ is a binary representation of a multiple of } 3\}$ over $\Sigma = \{0,1\}$. This means $L$ is a regular language.

$\square$

## Question #2

**Claim:** Regular expressions that also have access to complement can still only express the same class of languages (i.e. the class of regular langauges) as regular expressions without the complement operation.

*Proof.*

**Remarks**

The proof proceeds by demonstrating that the addition of the complement operation to regular expressions does not expand their expressive power beyond the class of regular languages. The argument is structured as follows:

- First, establish the foundational setup and definitions;
- Second, use structural induction to show that regular expressions with complement can be transformed into equivalent regular expressions without complement;
- Finally, conclude that regular expressions with complement describe no more than the regular languages, the same class that regular expressions without complement describe.

**Definitions and Setup**

Suppose $r$ is an arbitrary regular expression over the alphabet $\Sigma$, so that $L = \mathcal{L}(r)$ is the language described by $r$. Assume $r$ has access to the complement operation.

Let $\overline{L} = \{x \in \Sigma^* \mid x \notin L\}$ represent the *complement* of $L$. By assumption, there exists a regular expression $\overline{r}$ such that $\mathcal{L}(\overline{r}) = \overline{\mathcal{L}(r)}$. The goal is to show that $\overline{r}$ and any regular expression containing complement can be rewritten as a regular expression without complement.

By definition, there exists a DFA $\mathcal{M}$ that accepts $L$. Let:

$$\mathcal{M} = (Q, \Sigma, \delta, s, F),$$

where

- $Q$ is the set of finite states;
- $\Sigma$ is the alphabet;
- $\delta : Q \times \Sigma \to Q$ is the transition function;
- $s$ is the initial state;
- $F \subseteq Q$ is the set of accepting states.

Define $\overline{L} = \Sigma^* \setminus L$, which contains all strings over $\Sigma$ not in $L$. Construct a new DFA $\overline{\mathcal{M}}$ to accept $\overline{L}$. This DFA $\overline{\mathcal{M}}$ is identical to $\mathcal{M}$ except for the set of accepting states:

- $Q$, $\Sigma$, $\delta$, and $s$ remain the same.
- The accepting states are now $Q \setminus F \subseteq Q$, i.e., all states not in $F$.

Notice that $\overline{\mathcal{M}}$ and $\mathcal{M}$ share the same structure but differ in their set of accepting states.

**Claim:** $\overline{\mathcal{M}}$ recognizes $\overline{L}$. This follows from the definition of complementation: any string $x \in \Sigma^*$ that is rejected by $\mathcal{M}$ (i.e., $x \notin L$) is accepted by $\overline{\mathcal{M}}$, and vice versa. Since $\overline{\mathcal{M}}$ is a DFA, $\overline{L}$ is regular, and there exists a regular expression $\overline{r}'$ (without complement) that describes $\overline{L}$.

**Structural Induction on Regular Expressions**

Now generalize this result to show that any regular expression $r$ with complement can be rewritten as a regular expression $r'$ without complement, using structural induction on the form of $r$.

Predicate

Let $P(r)$ denote the property: "Any regular expression $r$, possibly with complement, describes a regular language and can be rewritten as a regular expression $r'$ without complement."

Base Cases:

If $r$ is a basic regular expression ($\epsilon, \emptyset, a$ for $a \in \Sigma$), it does not use complement and is already a regular expression without complement. Thus, $P(r)$ holds for the base cases.

Induction Hypothesis:
Assume $P(r_1)$ and $P(r_2)$ hold for some regular expressions $r_1$ and $r_2$ (i.e., $r_1$ and $r_2$ can be rewritten without complement).

Inductive Step:
By the Induction Hypothesis, there exist regular expressions $r_1, r_2$ which can be rewritten without complement.

Proceed to demonstrate that $P(r)$ holds for any regular expression $r$ formed using the operations union ($r_1 + r_2$), concatenation ($r_1 r_2$), Kleene star ($r_1^*$), or complement ($\overline{r_1}$):

- If $r = r_1 + r_2$ (union) or $r = r_1 r_2$ (concatenation), $r$ can be rewritten using the Induction Hypothesis, as union and concatenation do not introduce complement.
- If $r = r_1^*$, $r$ can also be rewritten directly using the Induction Hypothesis, as Kleene star does not introduce complement.
- If $r = \overline{r_1}$, the complement of $r_1$ can be represented by the DFA $\overline{\mathcal{M}_1}$, as shown in *Definitions and Setup*. Since $\overline{\mathcal{M}_1}$ describes a regular language, there exists a regular expression $r_1'$ without complement such that $\mathcal{L}(r_1') = \overline{\mathcal{L}(r_1)}$.
  - Note that the regex $r'$ that describes $\overline{L}$ does not need to use the complement operation explicitly. Instead, it is derived from the structure of the DFA $\overline{\mathcal{M}}$, which was obtained by swapping accepting and non-accepting states in $\mathcal{M}$.
  - This means the complement operation at the language level (i.e. going from $L$ to $\overline{L}$) is reflected in the DFA construction, but the resulting regular expression $r'$ for $\overline{L}$ is still a standard regular expression without explicit use of complement.

By structural induction, $P(r)$ holds for all regular expressions $r$.

## Conclusion

Since any regular expression $r$ with complement can be rewritten as an equivalent regular

expression $r'$ without complement, the addition of the complement operation does not increase the expressive power of regular expressions.
Therefore, regular expressions with complement describe the same class of languages as regular expressions without complement: the class of regular languages.

$\square$

## Question #3

**Counter-free languages** are a subset of languages that satisfy the condition:

$$(\exists n \in \mathbb{N})(\forall x, y, z \in \Sigma^*)(\forall m \geq n)(xy^m z \in L \iff xy^n z \in L).$$

**Star-free regular expressions** are regular expressions without the Kleene star, but with complementation.

It is known in formal language theory that counter-free languages are equivalent to the languages that can be expressed as **star-free regular expressions**.

### (a):

**Claim:** $(ab)^*$ can be matched with a star-free regular expression, where $\Sigma = \{a, b\}$.

*Proof.*
The expression $(ab)^*$ represents strings in the set $\{\epsilon, ab, abab, ababab, \dots\}$. In other words, strings in $(ab)^*$ consist of zero or more repetitions of the substring $ab$. This means matching strings are strings where every occurrence of $a$ is immediately followed by a $b$ and the string must not have any extraneous characters or mismatches.

Strings in $(ab)^*$ **must not**:

- Start with $b$,
- End with $a$,
- Contain adjacent $a$s ($aa$),
- Contain adjacent $b$s ($bb$).

Note that if $L$ is a language, then its *complement* is the language $\overline{L} = \{x \in \Sigma^* \mid x \notin L\}$. For an equivalent notation in regular expressions, if $r$ is a regex matching $\mathcal{L}(r)$, then $\overline{r}$ is a regular expression such that $\mathcal{L}(\overline{r}) = \overline{\mathcal{L}(r)}$. This makes $\overline{\varnothing}$ is the star-free regex of $\Sigma^*$.

Let the complement regex of $(ab)^*$ be $\overline{(ab)^*} = b\overline{\varnothing} + \overline{\varnothing}a + \overline{\varnothing}aa\overline{\varnothing} + \overline{\varnothing}bb\overline{\varnothing}$.

Then, $(ab)^* = \overline{b\overline{\varnothing} + \overline{\varnothing}a + \overline{\varnothing}aa\overline{\varnothing} + \overline{\varnothing}bb\overline{\varnothing}}$.

Clearly, $r = \overline{b\overline{\varnothing} + \overline{\varnothing}a + \overline{\varnothing}aa\overline{\varnothing} + \overline{\varnothing}bb\overline{\varnothing}}$ is a star-free regex. Thus, $(ab)^*$ can be expressed as a star-free regular expression by using complements to describe its constraints.

□

**(b):**
**Claim:** $(ab)^*$ is a counter-free language, where $\Sigma = \{a, b\}$.

*Proof.*
By definition, if $(ab)^*$ is a counter-free language over $\Sigma = \{a, b\}$, there exists natural $n$ for all $x, y, z \in \{a, b\}^*$ and for all $m \geq n$ such that $xy^mz \in (ab)^* \iff xy^nz \in (ab)^*$.

Let $n = 2 \in \mathbb{N}$. Let $x, y, z \in \{a, b\}^*$ and $m \geq n$ both be arbitrary.

Show that $xy^mz \in (ab)^* \implies xy^nz \in (ab)^*$:
Suppose $xy^mz \in (ab)^*$.

The following cases may arise.

Case $(xy^mz = \epsilon)$
Then, $x, y, z = \epsilon$, so $y^m = y^n = \epsilon$. Clearly, $xy^nz \in (ab)^*$.

Case ($x$ ends with $a$)
Assume $x$ ends with $a$. If $x$ is not solely $a$, then it must be some sequence of $ab$ followed by $a$. Then, $xy^mz \in (ab)^*$ implies $z = b$, as strings in $(ab)^*$ must end with $b$, if not empty. This makes $y = ba$ so that $y^m = (ba)^m$ is a sequence of $ba$. It follows that $y^n$ is also a sequence of $ba$.

Therefore, the string $xy^nz$ starts with some sequence (if any) of $ab$ followed by $a$, continues with a sequence of $ba$, and ends with $b$. Collectively, this string is indeed some number of concatenations of $ab$, so $xy^nz \in (ab)^*$.

Case ($x$ ends with $b$)
Assume $x$ ends with $b$. If $x$ is not solely $b$, then it must be $a$ concatenated with some sequence $ba$ then concatenated with $b$. Then, $xy^mz \in (ab)^*$ implies $z = ab$ as well, where $y = ab$ so that $y^m = (ab)^m$ is a sequence of $ab$. It follows that $y^n$ is also a sequence of $ab$.

Therefore, the string $xy^nz$ starts with $a$ concatenated with some sequence (if any) of $ba$ then concatenated by $b$, continues with a sequence of $ab$, and ends with $ab$. Collectively, this string is indeed some number of concatenations of $ab$, so $xy^nz \in (ab)^*$.

Altogether:
Thus, $xy^nz \in (ab)^*$.

Show that $xy^mz \in (ab)^* \impliedby xy^nz \in (ab)^*$:
Suppose $xy^nz \in (ab)^*$.

Consider the same cases as previously noted.

Case ($xy^nz = \epsilon$)
Then, $x, y, z = \epsilon$, so $y^n = y^m = \epsilon$. Clearly, $xy^mz \in (ab)^*$.

Case ($x$ ends with $a$)
Assume $x$ ends with $a$. If $x$ is not solely $a$, then it must be some sequence of $ab$ followed by $a$. Then, $xy^nz \in (ab)^*$ implies $z = b$, as strings in $(ab)^*$ must end with $b$, if not empty. This makes $y = ba$ so that $y^m = (ba)^m$ is a sequence of $ba$. It follows that $y^m$ is also a sequence of $ba$ (having only a greater number of concatenations of $ba$).

Therefore, the string $xy^mz$ starts with some sequence (if any) of $ab$ followed by $a$, continues with a sequence of $ba$, and ends with $b$. Collectively, this string is indeed some number of concatenations of $ab$, so $xy^mz \in (ab)^*$.

Case ($x$ ends with $b$)
Assume $x$ ends with $b$. If $x$ is not solely $b$, then it must be $a$ concatenated with some sequence

$ba$ then concatenated with $b$. Then, $xy^nz \in (ab)^*$ implies $z = ab$ as well, where $y = ab$ so that $y^n = (ab)^m$ is a sequence of $ab$. It follows that $y^m$ is also a sequence of $ab$ (having only a greater number of concatenations of $ab$).

Therefore, the string $xy^mz$ starts with $a$ concatenated with some sequence (if any) of $ba$ then concatenated by $b$, continues with a sequence of $ab$, and ends with $ab$. Collectively, this string is indeed some number of concatenations of $ab$, so $xy^mz \in (ab)^*$.

Altogether:
Thus, $xy^mz \in (ab)^*$.

Conclusion:
It has been demonstrated that $xy^mz \in (ab)^* \implies xy^nz \in (ab)^*$ and $xy^mz \in (ab)^* \impliedby xy^nz \in (ab)^*$. Therefore, $xy^mz \in (ab)^* \iff xy^nz \in (ab)^*$. This makes $(ab)^*$ a counter-free language over $\Sigma = \{a, b\}$.

□

**(c):**
**Claim:** $(aa)^*$ is NOT a counter-free language, where $\Sigma = \{a\}$.

*Proof.*
Seeking a contradiction, assume $(aa)^*$ is a counter-free language, where $\Sigma = \{a\}$.

Then, by definition, there exists natural $n$ for all $x, y, z \in \{a\}^*$ and for all $m \geq n$ such that $xy^mz \in (aa)^* \iff xy^nz \in (aa)^*$. Moreover, note that strings in $(aa)^*$ must have an even number of substrings $a$.

Evaluate the parity of $n$ for $xy^mz \in (aa)^* \impliedby xy^nz \in (aa)^*$.

Case ($n$ is even)
Assume $n$ is even. Consider $xy^nz \mid_{x,y,z=a} = aa^na$.

By assumption, $xy^mz \in (aa)^*$ for arbitrary $m \geq n$. However, notice when $x, z = a$ and $y^m \mid_{m=n+1} = a^{n+1}$, it follows that $xy^mz = aa^{n+1}a = a^{n+3}$. This means there are $n+3$ substrings $a$, which is an odd quantity of substrings $a$ as $n$ is even. Thus, $xy^mz \notin (aa)^*$, contradicting the assumption that $xy^mz \in (aa)^*$.

Case ($n$ is odd)
Assume $n$ is odd. Consider $xy^nz \mid_{x=\epsilon, y^n=a^n, z=a} = \epsilon a^n a = a^n a$.

By assumption, $xy^mz \in (aa)^*$ for arbitrary $m \geq n$. However, notice when $x = \epsilon, y^m \mid_{m=n+1} = a^{n+1}, z = a$, it follows that $xy^mz = \epsilon a^{n+1}a = a^{n+2}$. This means there are $n+2$ substrings $a$, which is an odd quantity of substrings $a$ as $n$ is odd. Thus, $xy^mz \notin (aa)^*$, contradicting the assumption that $xy^mz \in (aa)^*$.

Altogether:
Regardless of the parity of $n$, $xy^mz \in (aa)^* \impliedby xy^nz \in (aa)^*$ is **false**, by contradiction. It follows that $xy^mz \in (aa)^* \iff xy^nz \in (aa)^*$ is **false** as well.

Therefore, $(aa)^*$ is **not** a counter-free language over $\Sigma = \{a\}$.

$\square$

## Question #4

Let $k \in \mathbb{N}$ be arbitrary. Let $w \in \Sigma^*$, where $|\Sigma| \geq 2$ and has 1 as one of its symbols.

Consider the language $L = \{w \mid \text{the } k^{\text{th}} \text{ last character of } w \text{ is } 1\}$.

**(a):**

<u>**Claim:**</u> A DFA that accepts $L$ has to have at least $2^k$ number of states.

*Proof.*

Seeking a contradiction, assume there exists a DFA that accepts $L$ with less than $2^k$ states.

Now consider $w \in L = \{w \mid \text{the } k^{\text{th}} \text{ last character of } w \text{ is } 1\}$ such that $|w| = k$. Then, there would be at least $2^k$ unique strings with length $k$, as $|\Sigma| \geq 2$. For simplicity, consider only $|\Sigma| = 2$ to proceed with exactly $2^k$ unique strings of length $k$.

By the pigeonhole principle, since the DFA has fewer than $2^k$ states, at least two distinct strings $u, v \in \Sigma^k$ must end up in the same state after being processed by the DFA.

Let $u$ and $v$ differ at some position $i$, where the $i^{\text{th}}$ symbol of $u$ is 1, and the $i^{\text{th}}$ symbol of $v$ is $s \neq 1$. Since the DFA is in the same state after processing $u$ and $v$, it cannot distinguish between suffixes appended to $u$ and $v$.

Consider the strings $u'$ and $v'$, where $u' = ux$ and $v' = vx$, with $x \in \Sigma^*$. Let $x$ be a string of length $k - i$ consisting of only the symbol 1. Then, for $u'$, the $k^{\text{th}}$ last character is 1, so $u' \in L$. For $v'$, the $k^{\text{th}}$ last character is $s \neq 1$, so $v' \notin L$.

However, since the DFA reaches the same state after processing $u$ and $v$, it must accept or reject both $u'$ and $v'$, leading to a contradiction. Thus, the DFA cannot correctly recognize $L$ with fewer than $2^k$ states.

Therefore, a DFA that accepts $L$ must have at least $2^k$ states.

□

**(b):**

<u>**Claim:**</u> The smallest NFA that accepts $L$ has exactly $k+1$ states.

*Proof.*

Proceed by constructing an NFA with $k+1$ states that accepts $L$, followed by arguing why no NFA with fewer than $k+1$ states can correctly accept $L$.

<u>**Construction of the NFA:**</u>

Consider the following NFA:

- The NFA has $k+1$ states, labeled $q_0, q_1, q_2, \ldots, q_k$.
- The start state is $q_0$.
- For each character in the input string, the NFA transitions nondeterministically to either:
  - Remain in the current state (to ignore the current character), or
  - Move to the next state (to "count" the character).
- Once the NFA reaches $q_k$, it checks if the current character is 1 (the $k^{\text{th}}$ last character) and transitions to an accept state.

This NFA uses nondeterminism to track all possible positions in the string where the $k^{\text{th}}$ last character might be 1. It is minimal because it uses exactly $k+1$ states, one for each possible position from 0 to $k$. Thus, this NFA accepts precisely the language $L$.

<u>**Contradiction Argument:**</u>

Seeking a contradiction, assume that there exists an NFA with fewer than $k+1$ states that accepts $L$. Such an NFA would need to:

- Track whether the $k^{\text{th}}$ last character is 1, and
- Ensure exactly $k-1$ characters follow the $k^{\text{th}}$ last character.

However, with fewer than $k+1$ states, the NFA cannot distinguish between all possible scenarios where the $k^{\text{th}}$ last character occurs, as it would lose track of the required position information.

Specifically, any NFA with fewer than $k+1$ states would necessarily involve a loop in the portion of the state space responsible for verifying the $k^{\text{th}}$ last character. This loop would cause the NFA to lose track of whether it has correctly counted $k-1$ characters after the candidate $k^{\text{th}}$ last character. Consequently, the NFA would incorrectly accept or reject certain strings.

**Counterexample:**
Consider $k = 3$ and the alphabet $\Sigma = \{0, 1\}$. The language $L$ includes strings such as $w = 1101$ (where the $3^{\text{rd}}$ last character is 1) and excludes strings such as $w = 1001$ (where the $3^{\text{rd}}$ last character is 0). An NFA with fewer than $k+1 = 4$ states cannot distinguish between these strings:

- If the NFA has a loop to reduce its state count, it cannot reliably count the number of characters following the candidate $3^{\text{rd}}$ last character.
- As a result, the NFA might erroneously accept $w = 1001$ or reject $w = 1101$, violating the definition of $L$.

This shows that fewer than $k+1$ states are insufficient.

**Conclusion:**
The assumption that an NFA with fewer than $k+1$ states can accept $L$ leads to a contradiction. Thus, the smallest NFA that accepts $L$ must have exactly $k+1$ states.

$\square$

## Question #5

**Claim:** Every finite language can be represented by a regular expression (meaning all finite languages are regular).

*Proof.*
Let $\Sigma$ be an arbitrary alphabet. Let $L$ be an arbitrary finite language over $\Sigma$.

Let $n$ be an arbitrary natural number.
Denote the predicate:

$$P(n) \coloneqq |L_n| = n \implies L_n \text{ can be represented as a regular expression.}$$

This proof uses the principle of simple induction to show $P(n)$ for all $n \in \mathbb{N}$.

Base Cases:
Let $n = 0$.
This means $|L_n| = 0$, so $L_n = \emptyset$. By definition, the empty set is a regular expression.
Thus, $P(0)$.

Let $n = 1$.
Then $|L_n| = 1$, so $L_n = \{w\}$ for some string $w \in \Sigma^*$. By definition, any single string over an alphabet is a regular expression.
Thus, $P(1)$.

Induction Hypothesis
Assume that $P(k)$ holds for some natural $k$.

This means if $L_k$ has $k$ strings, then $L_k$ can be represented as a regular expression.

Induction Step:
Let $L_{k+1} = \{w_1, w_2, \ldots, w_k, w_{k+1}\}$, where $w_i \in \Sigma^*$ for $i \in [1, k+1] \cap \mathbb{N}$.

By the Induction Hypothesis, $L_k = L_{k+1} \setminus \{w_{k+1}\} = \{w_1, w_2, \ldots, w_k, w_{k+1}\} \setminus \{w_{k+1}\} = \{w_1, w_2, \ldots, w_k\}$ has language has regular expression $r_k$ such that $L_k = \mathcal{L}(r_k)$.

Notice that:

- The regex $r_k$ represents the language $L_k$;
- The regex $w_{k+1}$ represents the language $\{w_{k+1}\}$.

Then, $L_{k+1}$ can be constructed as a regex as follows:

$$L_{k+1} = L_k \cup \{w_{k+1}\}$$

By definition, the union of two regexes is a regex. Construct $r_{k+1}$:

$$r_{k+1} = r_k + w_{k+1}$$

As desired, the regex $r_{k+1}$ represents the language $L_{k+1}$.

Conclusion:
By the principle of simple induction, $P(n)$ holds for all $n \in \mathbb{N}$. It follows that all finite languages must be regular.

$\square$