

CSC236 Midterm Test Solutions

A correlation between the midterm test and exam has been confirmed. Please use this resource to study well!

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Question #1

Let \mathcal{F} be the collection of all functions with domain \mathbb{N} and co-domain \mathbb{R} .

Given $A, B \in \mathcal{P}(\mathcal{F})$, define addition on $\mathcal{P}(\mathcal{F})$ by

$$A + B := \{f + g : f \in A, g \in B\}.$$

Recall that $f + g$ is the function with domain \mathbb{N} and co-domain \mathbb{R} such that

$$(f + g)(n) = f(n) + g(n).$$

Also recall that, if $h \in \mathcal{F}$, then

$$O(h) = \{q \in \mathcal{F} : (\exists n_0, c \in \mathbb{N})(\forall n \geq n_0)[|q(n)| \leq c|h(n)|]\}.$$

Claim: For arbitrary nonnegative $u, v \in \mathcal{F}$, it follows that $O(u) + O(v) = O(u + v)$.

Proof.

This proof demonstrates a double-subset inclusion to show equality.

Forward Inclusion — $O(u) + O(v) \subseteq O(u + v)$:

Let $h \in [O(u) + O(v)]$. Then, $h = f + g$, where $f \in O(u)$ and $g \in O(v)$.

By definition, there exists $c_1, c_2, n_1, n_2 > 0$ such that

- $|f(n)| \leq c_1|u(n)|$ for all $n \geq n_1$;
- $|g(n)| \leq c_2|v(n)|$ for all $n \geq n_2$.

Choose $n_0 = \max(n_1, n_2)$ and $c = \max(c_1, c_2)$.

Using the definition, triangle inequality, and assumption that u, v are nonnegative functions,

it follows that

$$\begin{aligned} |h(n)| &= |f(n) + g(n)| \leq |f(n)| + |g(n)| \leq c_1|u(n)| + c_2|v(n)| \\ &\leq c|u(n)| + c|v(n)| \\ &\leq c(|u(n)| + |v(n)|) \\ &= c(u(n) + v(n)) = c(|u(n) + v(n)|) = c(|(u + v)(n)|). \end{aligned}$$

Thus, $|h(n)| \leq c|(u + v)(n)|$.

By definition, $h \in O(u + v)$. Therefore, $O(u) + O(v) \subseteq O(u + v)$.

Backward Inclusion — $O(u) + O(v) \supseteq O(u + v)$:

Let $h \in O(u + v)$.

By definition, there exists $c, n_0 > 0$ such that $|h(n)| \leq c|(u + v)(n)|$ for all $n \geq n_0$.

It follows that $|h(n)| \leq c|u(n) + v(n)| = c(u(n) + v(n))$, as u, v are nonnegative functions.

Case:

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Question #X

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Proof.

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