CSC236 Midterm Test Solutions

A correlation between the midterm test and exam has been confirmed. Please use this resource to study well!

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Question #1

Let $\mathcal{F}$ be the collection of all functions with domain $\mathbb{N}$ and co-domain $\mathbb{R}$.
Given $A, B \in \mathcal{P}(\mathcal{F})$, define addition on $\mathcal{P}(\mathcal{F})$ by

$$A + B \coloneqq \{f + g : f \in A, g \in B\}.$$

Recall that $f + g$ is the function with domain $\mathbb{N}$ and co-domain $\mathbb{R}$ such that

$$(f + g)(n) = f(n) + g(n).$$

Also recall that, if $h \in \mathcal{F}$, then

$$O(h) = \{q \in \mathcal{F} : (\exists n_0, c \in \mathbb{N})(\forall n \geq n_0)[|q(n)| \leq c|h(n)|]\}.$$

Claim: For arbitrary nonnegative $u, v \in \mathcal{F}$, it follows that $O(u) + O(v) = O(u + v)$.

Proof.
This proof demonstrates a double-subset inclusion to show equality.

Forward Inclusion — $O(u) + O(v) \subseteq O(u + v)$:
Let $h \in [O(u) + O(v)]$. Then, $h = f + g$, where $f \in O(u)$ and $g \in O(v)$.

By definition, there exists $c_1, c_2, n_1, n_2 > 0$ such that

- $|f(n)| \leq c_1|u(n)|$ for all $n \geq n_1$;
- $|g(n)| \leq c_2|v(n)|$ for all $n \geq n_2$.

Choose $n_0 = max(n_1, n_2)$ and $c = max(c_1, c_2)$.
Using the definition, triangle inequality, and assumption that u, v are nonnegative functions,

it follows that

$$\begin{aligned}|h(n)| = |f(n)+g(n)| \leq |f(n)|+|g(n)| \leq c_1|u(n)|+c_2|v(n)| \\ &\leq c|u(n)|+c|v(n)| \\ &\leq c(|u(n)|+|v(n)|) \\ &= c(u(n)+v(n)) = c(|u(n)+v(n)|) = c(|(u+v)(n)|).\end{aligned}$$

Thus, $|h(n)| \leq c|(u+v)(n)|$.

By definition, $h \in O(u+v)$. Therefore, $O(u)+O(v) \subseteq O(u+v)$.

Backward Inclusion — $O(u)+O(v) \supseteq O(u+v)$:
Let $h \in O(u+v)$.

By definition, there exists $c, n_0 > 0$ such that $|h(n)| \leq c|(u+v)(n)|$ for all $n \geq n_0$.

It follows that $|h(n)| \leq c|u(n)+v(n)| = c(u(n)+v(n))$, as u, v are nonnegative functions.

Let $w(n) = h(n) - cu(n)$. Consider the following cases for $w(n)$.

Case — $w(n) > 0$:
Notice that $w(n) > 0 \implies h(n) - cu(n) > 0 \implies h(n) > cu(n)$.
Since u is a nonnegative function, then $h(n)$ must be positive.

Recall $|h(n)| \leq c|(u+v)(n)| = c|u(n)+v(n)|$, and both u, v are nonnegative functions.

It follows that

$$\begin{aligned}|w(n)| &= |h(n) - cu(n)| = h(n) - cu(n) \\ &= |h(n)| - cu(n) \leq |cu(n)+cv(n)| - cu(n) \\ &= \cancel{cu(n)} + cv(n) - \cancel{cu(n)} = cv(n) = c|v(n)|.\end{aligned}$$

Thus, $|w(n)| \leq c|v(n)|$. This means $w(n) \in O(v)$.

Write $h(n) = cu(n) + w(n)$. It is obvious that $cu(n) \in O(u)$, and recall that $w(n) \in O(v)$.

Thus, $h(n) \in O(u + v)$.

Case — $w(n) \leq 0$:
Notice that $w(n) \leq 0 \implies h(n) - cu(n) \leq 0 \implies h(n) \leq cu(n)$. So, choose $h(n) = cu(n)$.
Then, $w(n) = h(n) - cu(n) = \cancel{cu(n)} - \cancel{cu(n)} = 0$.

Notice that $|w(n)| = |0| = 0 \leq c|v(n)|$, in fact, for any $c > 0$.
Clearly, $w(n) \in O(v)$.

Write $h(n) = cu(n) + w(n)$. It is obvious that $cu(n) \in O(u)$, and recall that $w(n) \in O(v)$.

Thus, $h(n) \in O(u + v)$.

Conclusion of Cases:
In all cases, $h(n) \in O(u + v)$ has been demonstrated.

Therefore, $O(u) + O(v) \subseteq O(u + v)$.

Conclusion:
Since both inclusions hold, $O(u) + O(v) = O(u + v)$.

$\square$

Question #X

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Proof.
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