CSC236 Homework Assignment #3

Language Regularity, Regular Expressions, and DFA/NFA Complexity

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Prepared for November 25, 2024

Question #1

Let $\Sigma = \{0, 1\}$.

(a):

Claim: Σ^* is a regular language.

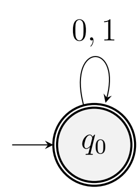
Proof.
Let $L_1 = \{0\}$ and $L_2 = \{1\}$ be regular languages of Σ.
Define $L_3 = L_1 \cup L_2 = \{0, 1\}$ as the regular language obtained by the union of L_1 and L_2.

By definition, ${L_3}^*$ is a regular language. Since $L_3 = \Sigma$, Σ^* is also a regular language.

While this proof now is complete, the assignment encourages the use of determinstic finite automaton (or DFA) proofs to show that languages are regular.

So, as an alternative to *Part (a)* (this part's proof), and for all relevant subsequent parts, adopt a DFA proof.

Consider the **transition function** δ associated with the following DFA:



Old State	**Symbol**	**New State**
q_0	0	q_0
q_0	1	q_0

Table 1: State Transition Table

Using δ, define the DFA $\mathcal{D} = (\mathcal{Q}, \Sigma, \delta, s, F)$, where

$$Q = \{q_0\} \text{ is the set of states in } \mathcal{D}$$

$$\Sigma = \{0, 1\} \text{ is the alphabet of symbols used by } \mathcal{D}$$

$$\delta : Q \times \Sigma \to Q \text{ is the transition function defined by } \textit{Table 1}$$

$$s = q_0 \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_0\} \subseteq Q \text{ is the set of accepting states of } \mathcal{D}.$$

For the sole state q_0 in $\mathcal{D}$, define its **invariant** for strings $x \in \Sigma^* = \{0, 1\}^*$:

$$P_{q_0}(x) : x \text{ is } \epsilon \text{ or consists of 0s and 1s.}$$

As the only state, q_0 is trivially **mutually exclusive**. As well, every string in $\Sigma^* = \{0, 1\}^*$ is either ϵ or consists of some number of 0s and 1s, which q_0 clearly satisfies; **exhaustivity** is satisfied.

Let $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0, 1\}^*$ both be arbitrary (here, $q = q_0$ always).
Denote the predicate:

$$P_{\delta(q_0, x)}(x) \coloneqq P_q(x) \text{ is the state invariant.}$$

Perform structural induction on $P_{\delta(q_0,x)}(x)$ for all strings $x \in \Sigma^* = \{0, 1\}^*$, as follows:

<u>Base Case:</u>
Let $q = q_0$ and $w = \epsilon$.
$P_q(w) = P_{q_0}(\epsilon)$ is true as $w = \epsilon$.

<u>Induction Hypothesis:</u>
Assume that $P_q(w)$ is true for all $q \in \{q_0\}$ and some $w \in \{0, 1\}^*$.
This means w is either the empty string or consists of 0s and 1s.

<u>Induction Step:</u>
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_0\}$ and some $w \in \{0, 1\}^*$.

Demonstrate that the invariant of q_0 holds when processing all possible strings from Σ^*.

This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for some $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary.

Let $q = q_0$, $z \in \{0,1\}$, and assume $P_{q_0}(w)$ is true.
Consider that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,z)}(wz) = P_{q_0}(wz)$. Since w is either the empty string or consists of 0s and 1s, while $z \in \{0,1\}$, wz remains to consist of 0s and 1s. This matches the definition of $P_{q_0}(wz)$. Thus, $P_{\delta(q,z)}(wz)$ holds.

There are no other state invariants in $\mathcal{D}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_0\}$ and $x \in \Sigma^* = \{0,1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language $L = \Sigma^*$ over $\Sigma = \{0,1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0,1\}^*$ is arbitrary, x is a member of L if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the sole state invariant, $P_{q_0}(x) : x$ is ϵ or consists of 0s and 1s.
If $x \in L$, then x is either the empty string or consists of 0s and 1s. Clearly, $P_{q_0}(x)$ is true.
On the other hand, choose the accepting state $q_0 \in F$ and assume $P_{q_0}(x)$ is true. Clearly, $x \in L$, due to matching definitions.

Therefore, $P_{\delta(q_0,x)}(x)$ is true for all strings $x \in \Sigma^* = \{0,1\}^*$.

While this DFA proof is redundant in nature, it is clear that $\mathcal{D}$ accepts $L = \Sigma^*$ over $\Sigma = \{0,1\}$. Again, it has been demonstrated that L is a regular language.

□

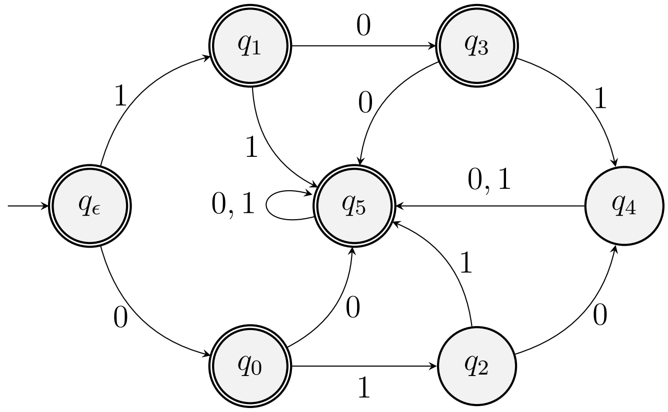
(b):

Claim: $\Sigma^* \setminus K, K = \{01, 101, 010\}$ is a regular language.

Proof.
This proof aims to show that the language $\Sigma^* \setminus K$ is a regular language by constructing a

DFA that accepts all strings over $\Sigma^* = \{0,1\}^*$ except the literal strings in $K = \{01, 101, 010\}$.

Consider the **transition function** δ associated with the following DFA:



Old State	**Symbol**	**New State**
q_ϵ	0	q_0
q_ϵ	1	q_1
q_0	0	q_5
q_0	1	q_2
q_1	0	q_3
q_1	1	q_5
q_2	0	q_4
q_2	1	q_5
q_3	0	q_5
q_3	1	q_4
q_4	0	q_5
q_4	1	q_5
q_5	0	q_5
q_5	1	q_5

Table 2: State Transition Table

Using δ, define the DFA $\mathcal{D} = (\mathcal{Q}, \Sigma, \delta, s, F)$, where

$\mathcal{Q} = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ is the set of states in $\mathcal{D}$

$\Sigma = \{0, 1\}$ is the alphabet of symbols used by $\mathcal{D}$

$$\delta : \mathcal{Q} \times \Sigma \to \mathcal{Q} \text{ is the transition function define by } \textit{Table 2}$$

$$s = q_\epsilon \text{ is the initial state of } \mathcal{D}$$

$$F = \{q_\epsilon, q_0, q_1, q_3, q_5\} \subseteq \mathcal{Q} \text{ is the set of accepting states of } \mathcal{D}.$$

For each state $q_i \mid_{i \in \{\epsilon,0,1,2,3,4,5\}}$ in $\mathcal{D}$, define a **state invariant** $P_q(x)$ for strings $x \in \Sigma^* = \{0,1\}^*$:

$P_{q_\epsilon}(x) : x$ is the empty string, ϵ

$P_{q_0}(x) : x$ is 0

$P_{q_1}(x) : x$ is 1

$P_{q_2}(x) : x$ is 01

$P_{q_3}(x) : x$ is 10

$P_{q_4}(x) : x$ is either 101 or 010

$P_{q_5}(x) : x$ consists of 0s and 1s but $x \notin \{0, 1, 10, 01, 010, 101\}$

It is clear that q_i for $i \in \{\epsilon, 0, 1, 2, 3, 4\}$ are mutually exclusive by their unique definitions. Moreover, the q_5 state is the direct complement of the **union of the previous q_i states** (as a non-initial state, q_5 shall never process ϵ), so q_5 must be mutually exclusive as well. Thus, all states in $\mathcal{D}$ are mutually exclusive.
For exhaustivity, notice that every string in $\Sigma^* = \{0,1\}^*$ is either ϵ or consists of some number of 0s and 1s. The q_ϵ state accounts for the empty string case, q_i for $i \in \{0, 1, 2, 3, 4\}$ account for specific strings of 0s and 1s, and q_5 accounts for all the remaining cases of 0s and 1s not covered by the q_i. Thus, the states in $\mathcal{D}$ are exhaustive.

Let $q \in Q = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and $w \in \Sigma^* = \{0,1\}^*$ both be arbitrary.
Denote the predicate:

$$P_{\delta(q_0,w)}(w) \coloneqq P_q(w) \text{ is the state invariant.}$$

Perform structural induction as follows:

Base Cases:

Let $w = \epsilon$.

Let $q = q_\epsilon$. $P_q(w) = P_{q_\epsilon}(\epsilon)$ is true as $w = \epsilon$, matching the definition of $P_{q_\epsilon}(w)$.
Let $q \neq q_\epsilon$. $P_q(w) = P_{q_i}(\epsilon) \mid_{i \in \{0,1,2,3,4,5\}}$ are vacuously true as these corresponding states are non-initial and do not process ϵ.

Induction Hypothesis:
Assume that $P_q(w)$ is true for all $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0,1\}^*$.

Induction Step:
By the Induction Hypothesis, $P_q(w)$ is true for arbitrary $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0,1\}^*$.

Demonstrate that all state invariants hold when processing strings from $\Sigma = \{0,1\}$.
This can be achieved by showing that $P_{\delta(q,z)}(wz)$ holds for some $z \in \{0,1\}$.

Let $w \in \{0,1\}^*$ be arbitrary. Then, consider the following cases.

Case $(q = q_\epsilon, z = 0)$:
Assume $P_{q_\epsilon}(w)$ is true. Then, $w = \epsilon$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_\epsilon,0)}(\epsilon 0) = P_{q_0}(0)$, which is clearly true by definition.

Case $(q = q_\epsilon, z = 1)$:
Likewise, assume $P_{q_\epsilon}(w)$ is true. Then, $w = \epsilon$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_\epsilon,1)}(\epsilon 1) = P_{q_1}(1)$, which is also clearly true by definition.

Case $(q = q_0, z = 0)$:
Assume $P_{q_0}(w)$ is true. Then, $w = 0$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,0)}(00) = P_{q_5}(00)$, which is true as $w = 00$ indeed consists of 0s and $1s$ and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case $(q = q_0, z = 1)$:
Likewise, assume $P_{q_0}(w)$ is true. Then, $w = 0$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_0,1)}(01) = P_{q_2}(01)$, which is clearly true by definition.

Case ($q = q_1, z = 0$):
Assume $P_{q_1}(w)$ is true. Then, $w = 1$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,0)}(10) = P_{q_3}(10)$, which is clearly true by definition.

Case ($q = q_1, z = 1$):
Likewise, assume $P_{q_1}(w)$ is true. Then, $w = 1$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_1,1)}(11) = P_{q_5}(11)$, which is true as $w = 11$ indeed consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case ($q = q_2, z = 0$):
Assume $P_{q_2}(w)$ is true. Then, $w = 01$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,0)}(010) = P_{q_4}(010)$, which is true as $w = 010$ is indeed either 101 or 010.

Case ($q = q_2, z = 1$):
Likewise, assume $P_{q_2}(w)$ is true. Then, $w = 01$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_2,1)}(011) = P_{q_5}(011)$, which is true as $w = 011$ indeed consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case ($q = q_3, z = 0$):
Assume $P_{q_3}(w)$ is true. Then, $w = 10$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_3,0)}(100) = P_{q_5}(100)$, which is true as $w = 100$ indeed consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$.

Case ($q = q_3, z = 1$):
Assume $P_{q_3}(w)$ is true. Then, $w = 10$.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_3,1)}(101) = P_{q_4}(101)$, which is true as $w = 101$ is indeed either 101 or 010.

Case ($q = q_4, z \in \{0, 1\}$):
Assume $P_{q_4}(w)$ is true. Then, $w \in \{101, 010\}$.

It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_4,z)}(wz) = P_{q_5}(wz)$. Notice that wz is a string with 4 symbols, because w and z are strings with 3 symbols and 1 symbol, respectively. As well, wz is constructed using only 0s and 1s. Thus, wz consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$; $P_{\delta(q,z)}(wz)$ holds.

Case ($q = q_5, z \in \{0, 1\}$):
Assume $P_{q_5}(w)$ is true.
It follows that $P_{\delta(q,z)}(wz) = P_{\delta(q_5,z)}(wz) = P_{q_5}(wz)$. Recall that $z \in \{0, 1\}$.

Since w consists of 0s and 1s and $w \notin \{0, 1, 10, 01, 010, 101\}$ by $P_{q_5}(w)$, concatenating z (which is either 0 or 1) to w does not result in $wz \in \{0, 1, 10, 01, 010, 101\}$. This is because $w \neq \epsilon$ (as q_5 does not process ϵ), so $wz \notin \{0, 1\}$, which implies $wz \notin \{10, 01\}$ (as q_5 will not process 10 and 01 if does not process 1 and 0, respectively), which also implies $wz \notin \{010, 101\}$ (as q_5 will not process 010 and 101 if it does not process 01 and 10, respectively).

Hence, it is demonstrated that if $P_q(w)$ is true for all $q \in \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and some $w \in \{0, 1\}^*$, then $P_{\delta(q,z)}(wz)$ holds for some $z \in \{0, 1\}$. By the principle of structural induction, $P_{\delta(q,x)}$ is true for all $q \in Q = \{q_\epsilon, q_0, q_1, q_2, q_3, q_4, q_5\}$ and $x \in \Sigma^* = \{0, 1\}^*$.

Finally, demonstrate that $\mathcal{D}$ accepts exactly the language $L = \Sigma^* \setminus \{01, 101, 010\}$ over $\Sigma = \{0, 1\}$.
This is achievable by showing that if $x \in \Sigma^* = \{0, 1\}^*$ is arbitrary, x is a member of L if and only if there exists an accepting state $q \in F$ such that $P_q(x)$ holds.

Recall the invariants of the accepting states, $q_i \in F \,|_{i \in \{\epsilon, 0, 1, 3, 5\}}$:

$$P_{q_\epsilon}(x) : x \text{ is the empty string, } \epsilon$$
$$P_{q_0}(x) : x \text{ is } 0$$
$$P_{q_1}(x) : x \text{ is } 1$$
$$P_{q_3}(x) : x \text{ is } 10$$
$$P_{q_5}(x) : x \text{ consists of 0s and 1s but } x \notin \{0, 1, 10, 01, 010, 101\}$$

Implication

If an arbitrary string $x \in L = \Sigma^* \setminus \{01, 101, 010\}$, then x is either an empty string or consists of 0s and 1s, but must not be either of 01, 101, nor 010. Choose q_ϵ to cover the empty string case, and q_5 to cover all cases **except** $x \in \{0, 1, 10, 01, 010, 101\}$. Since x must not be a member of $\{01, 010, 101\}$, it remains to account for $x \in \{0, 1, 10\}$. Conveniently, these cases are covered by q_0, q_1, and q_3. Thus, no matter which string $x \in L = \Sigma^* \setminus \{01, 101, 010\}$, there exists accepting states $q \in F$ such that $P_q(x)$ holds.

Implied-by

Conversely, $L = \Sigma^* \setminus \{01, 101, 010\}$ can be constructed by the same choices of accepting states. Choose from the accepting states $q_i \in F \mid_{i \in \{\epsilon,0,1,3,5\}}$ and assume $P_{q_i}(x)$ are true. Likewise, as demonstrated in the *implication* proof, reconstruct $L = \Sigma^* \setminus \{01, 101, 010\}$ using the accepting states $P_{q_i}(x)$, similarly. It is clear that the accepting state's invariants build a language consisting of the empty string (P_{q_ϵ}), and any combination of 0s and 1s, excluding the specific strings 01, 101, and 010 ($P_{q_0}, P_{q_1}, P_{q_3}, P_{q_5}$). Thus, if there are accepting states $q \in F$ such that $P_q(x) \mid_{x \in \Sigma^* = \{0,1\}^*}$ holds, then $x \in L = \Sigma^* \setminus \{01, 101, 010\}$.

Therefore, $\mathcal{D}$ accepts $L = \Sigma^* \setminus \{01, 101, 010\}$ over $\Sigma = \{0, 1\}$. This means L is a regular language.

$\square$

(c):

Claim: $\{w | w \text{ is a palindrome}\}$ is NOT a regular language.

Proof.
proofgoeshere

$\square$

(d):

Claim: $\{ww | w \in \Sigma*\}$ is NOT a regular language.

Proof.

proofgoeshere

□

(e):

Claim: $\{w|ww \in \Sigma*\}$ is a regular language.

Proof.
proofgoeshere

□

(f):

Claim: $\{w|w$ is a binary representation of a multiple of $3\}$ is a regular language.

Proof.
proofgoeshere

□

Question #2

<u>Claim:</u> Regular expressions that also have access to complement can still only express the same class of languages (i.e. the class of regular langauges) as regular expressions without the complement operation.

Proof.
Suppose r is an arbitrary regular expression with alphabet Σ so that $L = \mathcal{L}(r)$ is a regular language.
Assume r has access to the complement operation.

Let $\overline{L} = \{x \in \Sigma^* \mid x \notin L\}$ be the regular language representing the *complement* of L.
Assume $\overline{r}$ is a regular expression, with $\mathcal{L}(\overline{r}) = \overline{\mathcal{L}(r)}$.

By definition, there exists a DFA $\mathcal{M}$ that accepts L.

Construct $\mathcal{M}$:
$\mathcal{M} = (Q, \Sigma, \delta, s, F)$, where

- Q is a set of finite states;
- Σ is the alphabet;
- $\delta : Q \times \Sigma \to Q$ is the transition function;
- s is the start state;
- $F \subseteq Q$ is the set of accepting states.

Next, define $\overline{L} = \Sigma^* \setminus L$ as the regular language containing everything obtainable from Σ except members of L.

Construct $\overline{M}$ to be identical to M, except for its accepting states:

- Q is a set of finite states;

- Σ is the alphabet;
- $\delta : Q \times \Sigma \to Q$ is the transition function;
- s is the start state;
- $Q \setminus F \subseteq Q$ is the set of accepting states.

Notice that $\overline{M}$ share the same states, alphabet, transition function, and start state as those of M. The difference is the set of accepting states in $\overline{M}$, which is designed to be mutually exclusive from that of M.

Self-Note: now add the proof that $\overline{M}$ accepts $\overline{L}$, and connect it back to regexes. Next, let $\overline{r}$ be some regular expression without the complement operation.

$\square$

Question #3

Counter-free languages are a subset of languages that satisfy the condition:

$$(\exists n \in \mathbb{N})(\forall x, y, z \in \Sigma^*)(\forall m \geq n)(xy^m z \in L \iff xy^n z \in L).$$

Star-free regular expressions are regular expressions without the Kleene star, but with complementation.

It is known in formal language theory that counter-free languages are equivalent to the languages that can be expressed as **star-free regular expressions**.

(a):

<u>**Claim:**</u> $(ab)^*$ can be matched with a star-free regular expression, where $\Sigma = \{a, b\}$.

Proof.
The expression $(ab)^*$ represents strings in the set $\{\epsilon, ab, abab, ababab, \ldots\}$.

This means matching strings are strings where every occurrence of a is immediately followed by a b.

Due to the definition of $\Sigma = \{a, b\}$, an equivalent star-free regular expression can be written in terms of the complement. The complement definition is highlighted below:

- The string starts with b if it is not empty;
- The string contains an a not immediately followed by a b.

As a regular expression, this is ?.

□

(b):

<u>**Claim:**</u> $(ab)^*$ is a counter-free language, where $\Sigma = \{a, b\}$.

Proof.
By definition, if $(ab)^*$ is a counter-free language, there exists natural n for all $x, y, z \in \{a, b\}^*$ and for all $m \geq n$ such that $xy^m z \in (ab)^* \iff xy^n z \in (ab)^*$.

Let $n \in \mathbb{N}$. Let $x, y, z \in \{a, b\}^*$ and $m \geq n$ both be arbitrary.

Show that $xy^m z \in (ab)^* \implies xy^n z \in (ab)^*$:
Suppose $xy^m z \in (ab)^*$.

Since $m \geq n$ is arbitrary,

a

Show that $xy^m z \in (ab)^* \impliedby xy^n z \in (ab)^*$:

□

(c):
Claim: $(aa)^*$ is NOT a counter-free language, where $\Sigma = \{a\}$.

Proof.
proofgoeshere

□

Question #4

Let $k \in \mathbb{N}$ be arbitrary. Let $w \in \Sigma^*$, where $|\Sigma| \geq 2$ and has 1 as one of its symbols.

Consider the language $L = \{w | \text{the } k^{\text{th}} \text{ to last character of } w \text{ is } 1\}$.

(a):

Claim: A DFA that accepts L has to have at least 2^k number of states.

Proof.

ACTUALLY, SHOW THIS BY CONTRADICTION: Suppose whatever... less than 2^k states.

A DFA is determinstic and requires states to remember the "history" of the input. For the language $L = \{w | \text{the } k^{\text{th}} \text{ to last character of } w \text{ is } 1\}$, the DFA must track the last k characters of the input string.

Notice that there are 2^k possible combinations of k-length binary substrings, and each of these combinations must map to a unique state in the DFA for accurate processing.

Any DFA with fewer than 2^k states cannot differentiate between all possible k-length suffixes, causing the automaton to classify strings incorrectly.

Any DFA with more than 2^k states either accepts L with a larger alphabet with more than 2 symbols, or is introducing repetitive and redundant states, but still works.

□

(b):

Claim: The smallest NFA that accepts L has to have exactly k number of states.

Proof.

In an NFA, non-determinism allows the automaton to "guess" when it is k-steps away from the end of the string.

The NFA for L needs only k states because: - The start state (initial state) represents the starting position; - The NFA transitions through $k-1$ intermediate states to track progress.

□

(c):
Claim: The smallest DFA that accepts L has to have exactly $2^{k+1}-1$ number of states.

Proof.
proofgoeshere

□

Question #5

<u>Claim:</u> Every finite language can be represented by a regular expression (meaning all finite languages are regular).

Proof.
Let Σ be an arbitrary alphabet. Let L be an arbitrary finite language over Σ.

Let n be an arbitrary natural number.
Denote the predicate:

$$P(n) \coloneqq |L_n| = n \implies L_n \text{ can be represented as a regular expression.}$$

This proof uses the principle of simple induction to show $P(n)$ for all $n \in \mathbb{N}$.

<u>Base Cases:</u>
Let $n = 0$.
This means $|L_n| = 0$, so $L_n = \emptyset$. By definition, the empty set is a regular expression.
Thus, $P(0)$.

Let $n = 1$.
Then $|L_n| = 1$, so $L_n = \{w\}$ for some string $w \in \Sigma^*$. By definition, any single string over an alphabet is a regular expression.
Thus, $P(1)$.

<u>Induction Hypothesis</u>
Assume that $P(k)$ holds for some natural k.

This means if L_k has k strings, then L_k can be represented as a regular expression.

<u>Induction Step:</u>
Let $L_{k+1} = \{w_1, w_2, \ldots, w_k, w_{k+1}\}$, where $w_i \in \Sigma^*$ for $i \in [1, k+1] \cap \mathbb{N}$.

By the Induction Hypothesis, $L_k = L_{k+1} \setminus \{w_{k+1}\} = \{w_1, w_2, \dots, w_k, w_{k+1}\} \setminus \{w_{k+1}\} = \{w_1, w_2, \dots, w_k\}$ has language has regular expression r_k such that $L_k = \mathcal{L}(r_k)$.

Notice that:

- The regex r_k represents the language L_k;
- The regex w_{k+1} represents the language $\{w_{k+1}\}$.

Then, L_{k+1} can be constructed as a regex as follows:

$$L_{k+1} = L_k \cup \{w_{k+1}\}$$

By definition, the union of two regexes is a regex. Construct r_{k+1}:

$$r_{k+1} = r_k + w_{k+1}$$

As desired, the regex r_{k+1} represents the language L_{k+1}.

<u>Conclusion:</u>
By the principle of simple induction, $P(n)$ holds for all $n \in \mathbb{N}$. It follows that all finite languages must be regular.

$\square$