CSC263 Tutorial #9 Exercises

Minimum Cycler and MST Uniqueness

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CSC263H5 – Tutorial 10: Minimum Cycler and MST Uniqueness

1. Minimum Cycler Algorithm

Problem: Given a weighted connected undirected graph $G = (V, E)$, find a set $S \subseteq E$ of minimum total weight such that every cycle in G contains at least one edge in S.

Hint: The algorithm can be described in one short sentence.

Algorithm:

Compute a **Maximum Weight Spanning Tree (MWST)** $T \subseteq E$, and set $S = E \setminus T$.

Correctness:

- Every cycle in G must contain at least one edge not in any spanning tree.
- Removing all non-tree edges eliminates all cycles, ensuring S covers all cycles.
- To minimize the total weight of S, maximize the weight of T, since:

$$\text{weight}(E) = \text{weight}(T) + \text{weight}(S)$$

- Thus, maximizing weight(T) minimizes weight(S), yielding a **Minimum Cycler**.

Runtime:

- Kruskal's Algorithm (modified for MWST): $O(m \log n)$
- Sorting edges: $O(m \log n)$
- Union-Find: $O(m\alpha(n))$ where $\alpha(n)$ is inverse Ackermann
- Total: $O(m \log n)$

2. Uniqueness of MSTs

(a) Existence of Exactly 2 Distinct MSTs

Answer: Yes, such a graph exists.

Example: Graph with 5 vertices forming a cycle plus one diagonal:

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1), (v_1, v_3)\}$$

Edge weights:

$$w(v_i, v_{i+1}) = 1 \pmod 5, \quad w(v_1, v_3) = 2$$

Two MSTs:

- MST 1: exclude (v_5, v_1)
- MST 2: exclude (v_2, v_3)

Both MSTs have total weight 4. No other spanning tree can omit the higher weight edge (v_1, v_3) and maintain minimum weight.

(b) Unique Weights $\Rightarrow$ Unique MST

Proof: Assume for contradiction $T_1 \neq T_2$ are two distinct MSTs.

Let $e \in T_1 \setminus T_2$ be the minimum weight edge among all differences. Then in $T_2 \cup \{e\}$, a cycle exists. Let $e' \in T_2 \setminus T_1$ be an edge in this cycle.

Since weights are unique and $w(e) < w(e')$, replacing e' with e in T_2 gives a lighter spanning tree – contradiction.

Conclusion: $T_1 = T_2$. MST is unique.

(c) Non-Unique Weights, Unique MST

Answer: Yes, possible.

Example: Graph: 5 vertices in a tree with 4 edges of weight 1, and one additional edge of weight 1 creating a cycle.

Reasoning: Although weights are not unique, only one spanning tree avoids creating a cycle and has minimal total weight. No alternative tree can be formed due to graph structure.

Conclusion: Duplicate weights $\not\Rightarrow$ multiple MSTs. Uniqueness can be preserved structurally.