MAT232 - Lecture 5

Advanced Curve Analysis: Polar Derivatives and Conic Sections

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Contents

Definitions and Theorems

Straight from the textbook — lots of fluff this time, more than what we need!

Quick recap before diving into the lecture.

Conic Sections

Concept

Definition of Conic Sections: Conic sections are the curves formed by the intersection of a plane with a double-napped cone. The type of curve depends on the angle of the plane relative to the cone:

- *Circle:* The plane is perpendicular to the cone's axis.
- *Ellipse:* The plane intersects one nappe of the cone but is not perpendicular to the axis.
- *Parabola:* The plane is parallel to a generator of the cone.
- *Hyperbola:* The plane intersects both nappes of the cone.

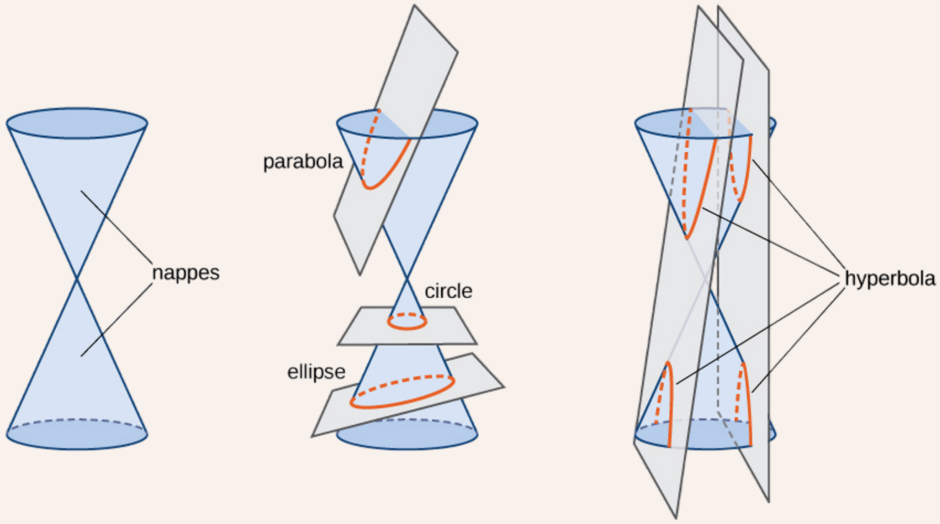

Figure 1: Conic sections formed by the intersection of a plane with a double-napped cone.

Ellipse

Definition

An **ellipse** is the set of all points in a plane such that the sum of their distances to two fixed points (called the *foci*) is constant.

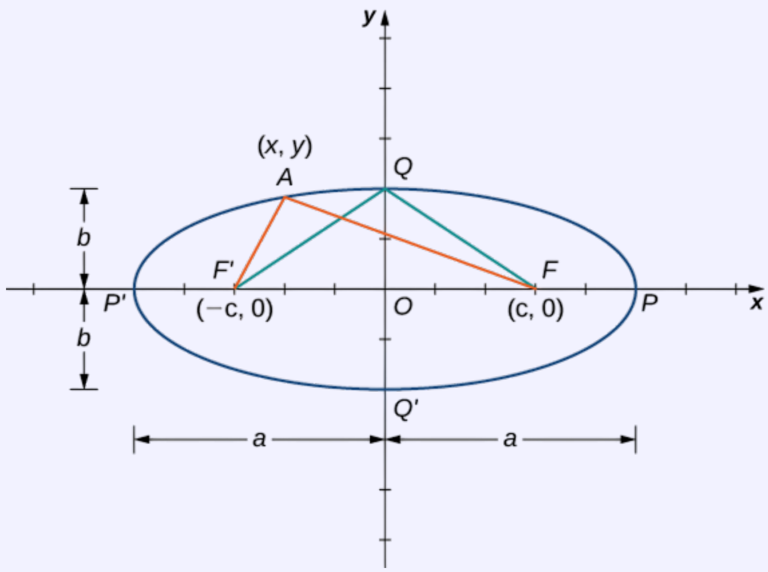

Figure 2: Diagram of an ellipse.

Intuition

Imagine looping a circular string around two fixed points F_1 and F_2 on a plane and pulling it taut (fully stretched without slack) with a pencil. As you move the pencil while keeping the string tight, the traced shape forms an ellipse. This method is commonly used for drawing ellipses with nails and string.

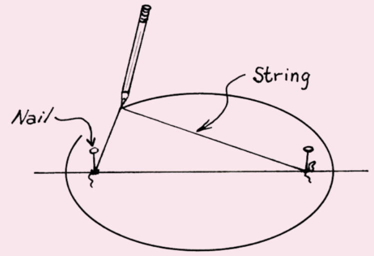

Figure 3: Drawing an ellipse with nails and string.

Standard Forms of an Ellipse

Definition

The equation of an ellipse depends on the orientation of its major axis:

- **Horizontal Major Axis:**
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
where:
  - (h, k) is the center,
  - $a > b$ (semi-major axis a, semi-minor axis b),
  - $c^2 = a^2 - b^2$, where c is the focal distance.
- **Vertical Major Axis:**
$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$
with the same parameters as above.

Remark

Properties of Ellipses:

- *Vertices:* Located a units from the center along the major axis.
- *Foci:* Located c units from the center along the major axis, where $c^2 = a^2 - b^2$.
- *Eccentricity:* Defined as $e = \frac{c}{a}$, with $0 < e < 1$.

Verifying an Ellipse

Example

Show that the equation

$$4x^2 + 9y^2 = 36$$

represents an ellipse and determine its key features.

Solution

- Rewrite the equation in standard form:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

- The ellipse is centered at $(0, 0)$ with $a = 3$, $b = 2$, and $c = \sqrt{a^2 - b^2} = \sqrt{5}$.
- The foci are $(\pm\sqrt{5}, 0)$, and the vertices are $(\pm 3, 0)$.

Parabola

Definition

A **parabola** is the set of all points in a plane equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

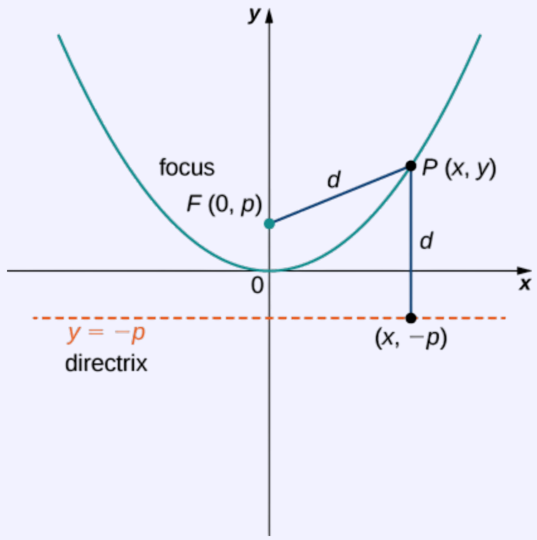

Figure 4: Diagram of a parabola.

Intuition

A parabola can be thought of as the trajectory of an object under uniform acceleration, such as the path of a ball thrown in the air.

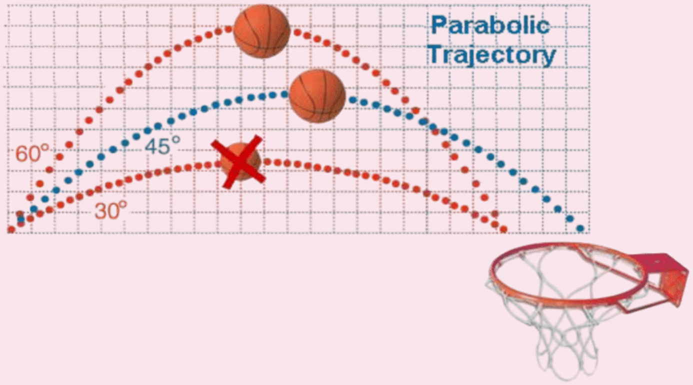

Figure 5: Parabolic trajectory of a ball.

Standard Forms of a Parabola

Definition

The equation of a parabola depends on whether it opens horizontally or vertically:

- **Opens Right or Left (Horizontal Axis):**

$$(y-k)^2 = 4p(x-h)$$

  - (h, k) is the vertex.
  - p is the directed distance from the vertex to the focus.
  - The focus is at $(h+p, k)$, and the directrix is the vertical line $x = h - p$.

- **Opens Up or Down (Vertical Axis):**

$$(x-h)^2 = 4p(y-k)$$

  - The vertex and p are the same as above.
  - The focus is at $(h, k+p)$, and the directrix is the horizontal line $y = k - p$.

Remark

Properties of Parabolas:

- *Focus:* Located p units from the vertex along the axis of symmetry.
- *Directrix:* A line perpendicular to the axis of symmetry at a distance p from the vertex.
- *Axis of Symmetry:* A line that passes through the focus and is perpendicular to the directrix.

Verifying a Parabola

Example

Show that the equation

$$y^2 = 12x$$

represents a parabola and determine its key features.

Solution

- The equation is in the standard form $y^2 = 4px$, with $4p = 12$, so $p = 3$.
- The parabola opens to the right, with vertex $(0, 0)$, focus $(3, 0)$, and directrix $x = -3$.

Hyperbola

Definition

A **hyperbola** is the set of all points in a plane such that the absolute difference of their distances to two fixed points (called the *foci*) is constant.

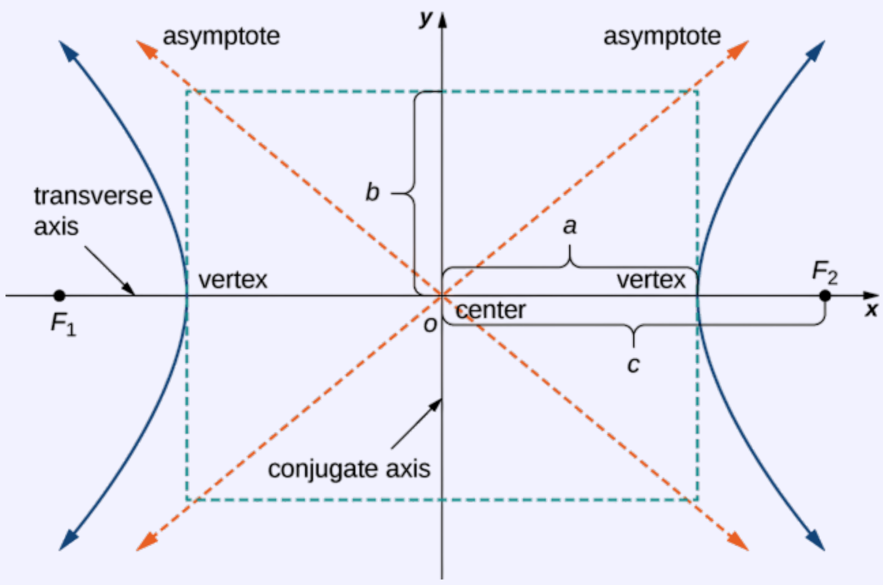

Figure 6: Diagram of a hyperbola.

Intuition

A hyperbola appears in real-world phenomena such as satellite orbits, radio wave propagation, and the paths of comets.

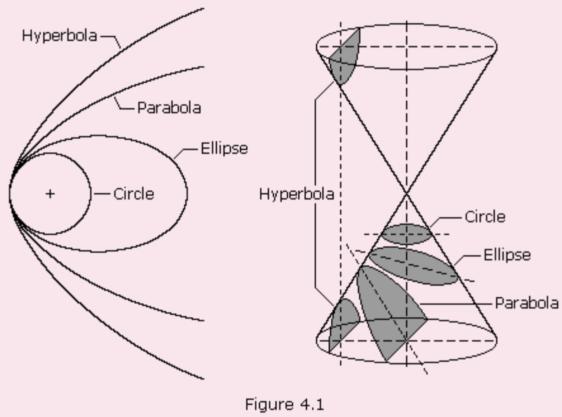

Figure 7: Hyperbolic orbits can have greater eccentricity than parabolic ones.

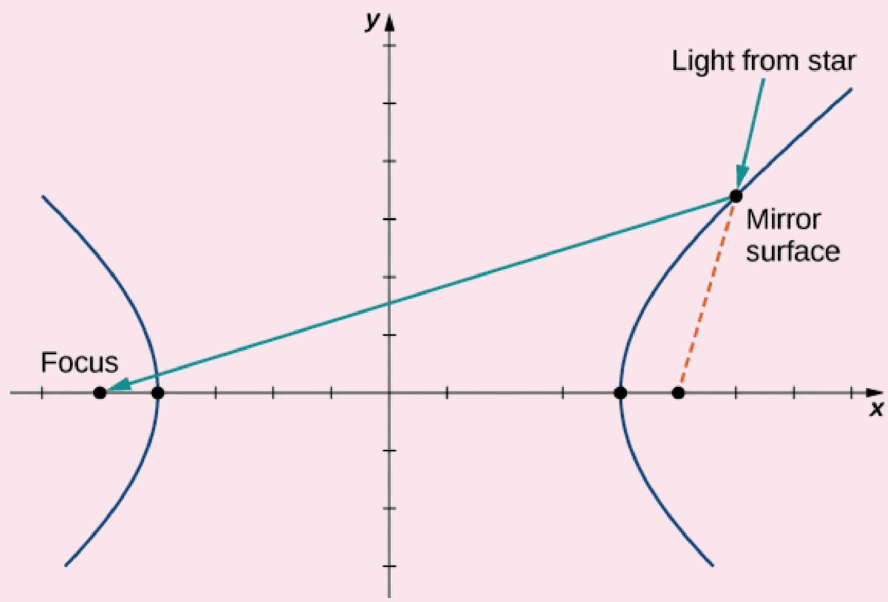

Figure 8: A hyperbolic mirror used to collect light from distant stars.

Standard Forms of a Hyperbola

Definition

A hyperbola is defined by the difference of distances to two fixed points (foci) being constant. Its standard equation depends on the orientation of its transverse axis:

- **Horizontal Transverse Axis:**

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

  where (h, k) is the center, a is the distance from the center to each vertex, and $c^2 = a^2 + b^2$ defines the distance from the center to each focus.

- **Vertical Transverse Axis:**

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

Remark

Properties of Hyperbolas:

- *Foci:* Located c units from the center along the transverse axis, where $c^2 = a^2 + b^2$.
- *Asymptotes:* Lines that the hyperbola approaches but never touches, given by:

$$y = k \pm \frac{b}{a}(x-h) \quad \text{(horizontal)}.$$

- *Vertices:* Located a units from the center along the transverse axis.

Verifying a Hyperbola

Example

Show that the equation

$$9x^2 - 16y^2 = 144$$

represents a hyperbola and determine its key features.

Solution

- Rewrite the equation in standard form:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

- The hyperbola is centered at $(0, 0)$ with $a = 4$, $b = 3$, and $c = \sqrt{a^2 + b^2} = 5$.
- The vertices are $(\pm 4, 0)$, the foci are $(\pm 5, 0)$, and the asymptotes are $y = \pm \frac{3}{4}x$.

Eccentricity and Directrix

Definition

The **eccentricity** e of a conic section is defined as the ratio of the distance from any point on the conic to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for a given conic and determines its type:

- If $e = 1$, the conic is a **parabola**.
- If $e < 1$, the conic is an **ellipse**.
- If $e > 1$, the conic is a **hyperbola**.

Remark

For a **circle**, the eccentricity is $e = 0$.

The **directrix** of a conic section is a fixed line that, together with the focus, helps define the conic.

- **Parabolas** have one focus and one directrix.
- **Ellipses** and **hyperbolas** (excluding circles) have two foci and two corresponding directrices.

Illustration

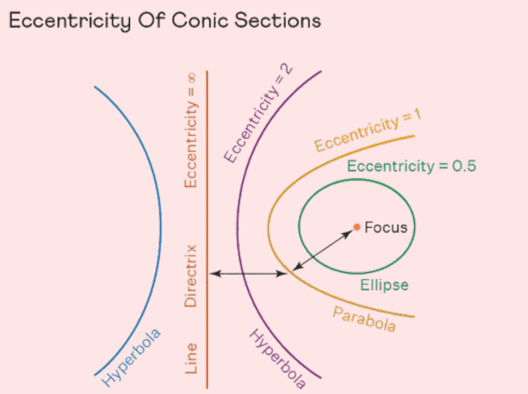

Figure 9: Eccentricity and directrix of conic sections.

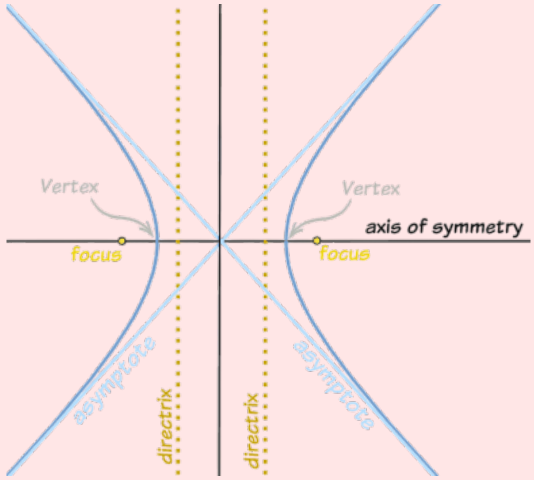

Figure 10: Directrix of a hyperbola.

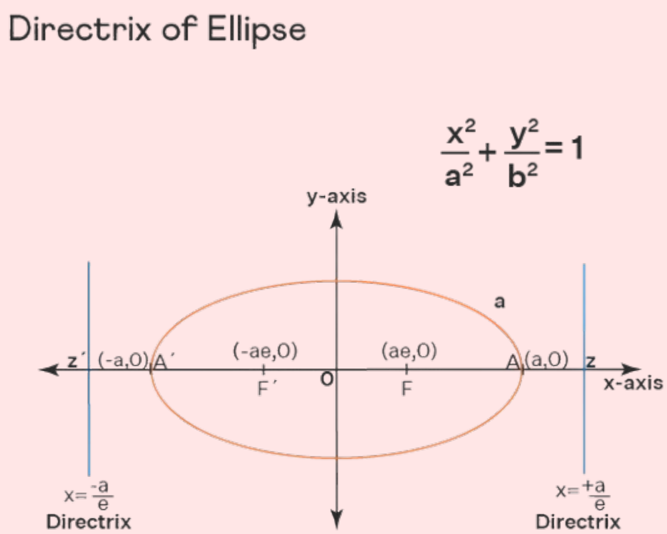

Figure 11: Directrix of an ellipse.

General Equations of Degree Two

Concept

A general second-degree equation is written as:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The nature of its graph (a conic section) is determined using the **discriminant**:

$$\Delta = 4AC - B^2.$$

- If $\Delta > 0$, the conic is an **ellipse**.
- If $\Delta = 0$, the conic is a **parabola**.
- If $\Delta < 0$, the conic is a **hyperbola**.

Remark

If $B \neq 0$, the coordinate axes are rotated.
To determine the rotation angle θ, use:

$$\cot 2\theta = \frac{A - C}{B}.$$

Distinguishing Between Conic Sections

Tip

To classify a conic section, follow these key steps:

1. **Check the discriminant** $\Delta = 4AC - B^2$:
   - $\Delta > 0$ indicates an **ellipse**.
   - $\Delta = 0$ indicates a **parabola**.
   - $\Delta < 0$ indicates a **hyperbola**.
2. **Identify the presence of an** xy**-term**:
   - If $B \neq 0$, the axes are rotated.
3. **Analyze the equation form**:
   - Ellipses and circles have **both** x^2 **and** y^2 **terms** with the same sign.
   - Hyperbolas have **both** x^2 **and** y^2 **terms** with opposite signs.
   - Parabolas have **only one squared term** (either x^2 or y^2, but not both).

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Review from the Previous Lecture

Remark

In the previous lecture, we covered important foundational concepts related to polar coordinates and their derivatives. Here's a brief summary:

- **Derivative of $r = f(\theta)$ in Cartesian Coordinates:**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

  This formula helps us compute the slope of the tangent line for polar curves when converted to Cartesian coordinates.

- **Equation of a Circle:**

$$(x-h)^2 + (y-k)^2 = r^2$$

  Here:

  - r: Radius of the circle
  - (h, k): Centre of the circle

Note

Reminder: Term Test 1 is scheduled for **Thursday, January 30th, 2025 (Week 4)**. Make sure to review polar derivatives, transformations, and conic sections!

Exploring Common Curve Shapes

Parabola

Definition

A **parabola** is a symmetric curve defined by the quadratic equation:

$$y = ax^2 + bx + c, \quad a \neq 0$$

To rewrite this equation in vertex form, we complete the square:

$$y = A(x - B)^2 + C$$

Here:

- A: Determines the direction and "width" of the parabola.

$$A > 0 \implies \text{The parabola opens upwards.}$$

$$A < 0 \implies \text{The parabola opens downwards.}$$

- (B, C): Represents the vertex of the parabola.
  - B: Horizontal position of the vertex.
  - C: Vertical position of the vertex.

Algorithm

Vertex Formula: To find the vertex when given the standard form $y = ax^2 + bx + c$, use the formulas:

$$B = -\frac{b}{2a}, \quad C = f(B)$$

where $f(B)$ is the value of the quadratic function evaluated at $x = B$.

Sketching the Region of a Set Defined by a Parabola

Example

Sketch the region of the set defined by

$$R = \{(x, y) \mid y \geqslant x^2 + 1\}.$$

Remark

To sketch the region defined by $y \geqslant x^2 + 1$, we first consider the graph of the parabola $y = x^2 + 1$:

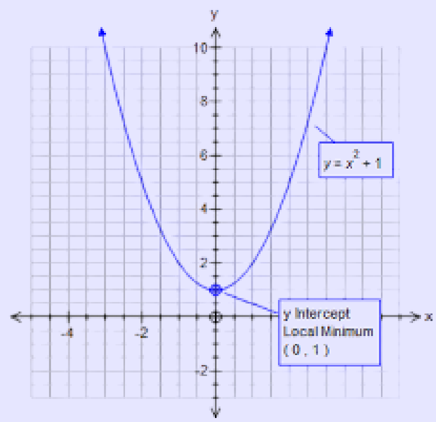

Figure 12: Graph of $y = x^2 + 1$.

Next, let's test some sample points to determine whether they lie in the region $y \geqslant x^2 + 1$:

- For the point $(-2, 0)$:

$$y \geqslant x^2 + 1 \implies 0 \geqslant (-2)^2 + 1 \implies 0 \geqslant 5,$$

  which is **false**. Therefore, $(-2, 0)$ is not in the region.

- For the point $(0, 2)$:

$$y \geqslant x^2 + 1 \implies 2 \geqslant 0^2 + 1 \implies 2 \geqslant 1,$$

  which is **true**. Therefore, $(0, 2)$ is in the region.

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Example

...cont'd...

Solution

The region defined by $y \geqslant x^2 + 1$ is shown below:

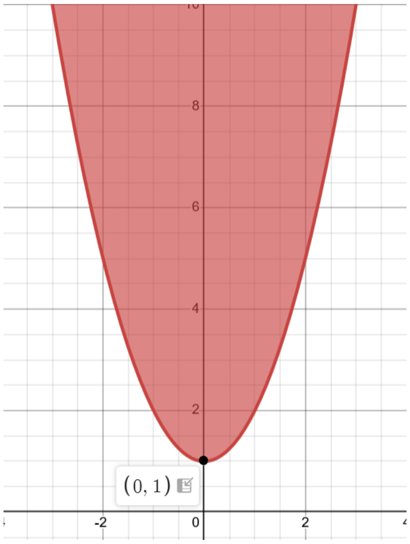

Figure 13: Shaded region satisfying $y \geqslant x^2 + 1$.

How to Determine the Region:

Concept

To determine the region for $y \geqslant x^2 + 1$:

- The parabola $y = x^2 + 1$ acts as a boundary. The inequality $y \geqslant x^2 + 1$ indicates that the region lies above or on this parabola.
- The graph of $y = x^2 + 1$ opens upwards, so the region R is the area above this curve, including the curve itself.
- The boundary curve $y = x^2 + 1$ is part of the region because the inequality includes equality ($\geqslant$).

Ellipse

Definition

The equation of an ellipse is defined by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Remark

Recall the equation of the circle, which is based on the equation of the ellipse when $a = b = 1$:

$$\text{Circle:} \quad (x-h)^2 + (y-k)^2 = r^2,$$

where (h, k) is the centre, a represents the x-axis radius, and b represents the y-axis radius.

Sketching the Region of a Set Defined by an Ellipse

Example

Sketch the region of the set defined by

$$A = \{(x, y) \mid x^2 + 4y^2 > 4\}.$$

Remark

To sketch the region defined by $x^2 + 4y^2 > 4$, we first consider the graph of the ellipse $x^2 + 4y^2 = 4$:

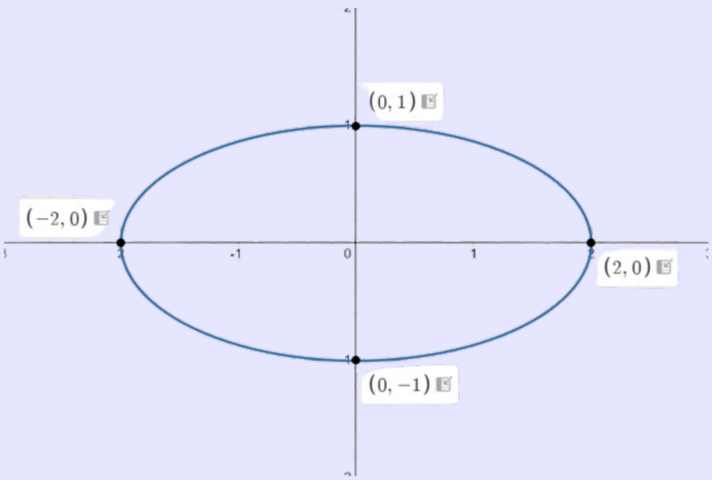

Figure 14: Graph of $x^2 + 4y^2 = 4$.

Next, let's test some sample points to determine whether they lie in the region $x^2 + 4y^2 > 4$:

- For the point $(0, 0)$:

$$x^2 + 4y^2 > 4 \implies 0^2 + 4(0)^2 > 4 \implies 0 > 4,$$

which is **false**. Therefore, $(0, 0)$ is not in the region.

- For the point $(3, 0)$:

$$x^2 + 4y^2 > 4 \implies 3^2 + 4(0)^2 > 4 \implies 9 > 4,$$

which is **true**. Therefore, $(3, 0)$ is in the region.

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Example

...cont'd...

Solution

The region defined by $x^2 + 4y^2 > 4$ is shown below:

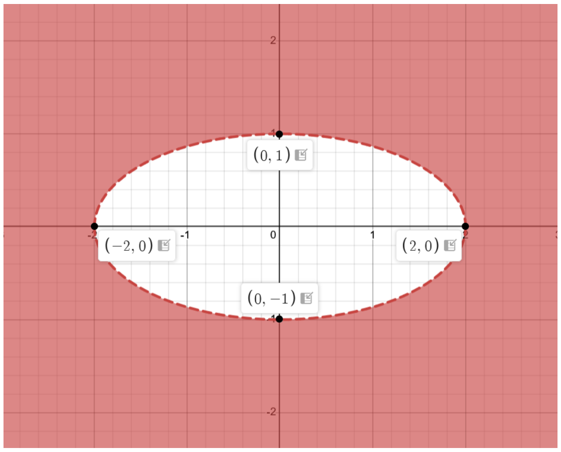

Figure 15: Shaded region satisfying $x^2 + 4y^2 > 4$.

How to Determine the Region:

Concept

To determine the region for $x^2 + 4y^2 > 4$:

- The ellipse $x^2 + 4y^2 = 4$ acts as a boundary. The inequality $x^2 + 4y^2 > 4$ indicates that the region lies outside this ellipse.
- The equation can be rewritten as $\frac{x^2}{4} + \frac{y^2}{1} = 1$, showing that it is an ellipse centered at $(0, 0)$ with a semi-major axis of 2 (along x-axis) and a semi-minor axis of 1 (along y-axis).
- The boundary curve $x^2 + 4y^2 = 4$ is **not** part of the region because the inequality is strict ($>$).
- A dashed boundary is used in the sketch to indicate that the ellipse itself is not included in the region.

Hyperbola

Definition

The equation of a hyperbola is defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

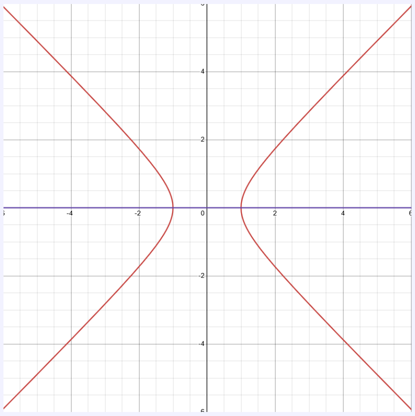

Figure 16: Graph of the hyperbola with a horizontal transverse axis.

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

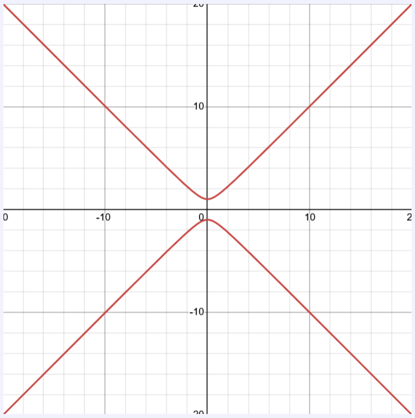

Figure 17: Graph of the hyperbola with a vertical transverse axis.

Sketching the Region of a Set Defined by a Hyperbola

Example

Sketch the region of the set defined by

$$H = \{(x, y) \mid \frac{x^2}{4} - \frac{y^2}{1} > 1\}.$$

Remark

To sketch the region defined by $\frac{x^2}{4} - \frac{y^2}{1} > 1$, we first consider the graph of the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$:

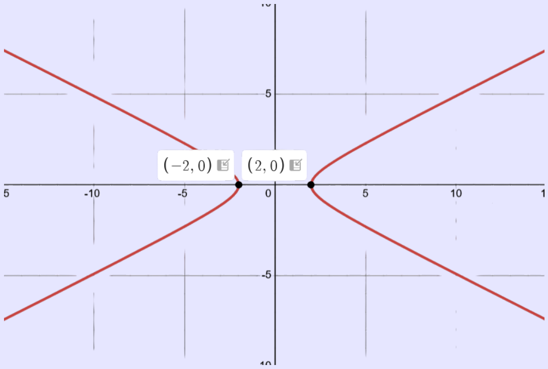

Figure 18: Graph of $\frac{x^2}{4} - \frac{y^2}{1} = 1$.

Next, let's test some sample points to determine whether they lie in the region $\frac{x^2}{4} - \frac{y^2}{1} > 1$:

- For the point $(0, 0)$:

$$\frac{x^2}{4} - \frac{y^2}{1} > 1 \implies \frac{0^2}{4} - \frac{0^2}{1} > 1 \implies 0 > 1,$$

  which is **false**. Therefore, $(0, 0)$ is not in the region.

- For the point $(3, 0)$:

$$\frac{x^2}{4} - \frac{y^2}{1} > 1 \implies \frac{3^2}{4} - \frac{0^2}{1} > 1 \implies \frac{9}{4} > 1,$$

  which is **true**. Therefore, $(3, 0)$ is in the region.

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Example

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Solution

The region defined by $\frac{x^2}{4} - \frac{y^2}{1} > 1$ is shown below:

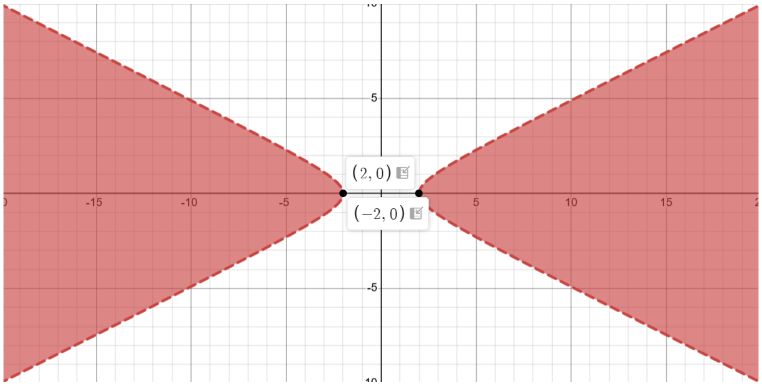

Figure 19: Shaded region satisfying $\frac{x^2}{4} - \frac{y^2}{1} > 1$.

How to Determine the Region:

Concept

To determine the region for $\frac{x^2}{4} - \frac{y^2}{1} > 1$:

- The hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ acts as a boundary. The inequality > 1 indicates that the region lies outside the branches of the hyperbola.
- The equation shows that the hyperbola has a center at $(0, 0)$, transverse axis along the x-axis, and asymptotes $y = \pm\frac{x}{2}$.
- The boundary curve $\frac{x^2}{4} - \frac{y^2}{1} = 1$ is **not** part of the region because the inequality is strict ($>$).
- A dashed boundary is used in the sketch to indicate that the hyperbola itself is not included in the region.

Section 2.1/2.2: Welcome to Linear Algebra...

Well... not really!

Welcome to MAT232! While the name might suggest a course in linear algebra, this course remains focused on multivariable calculus. However, linear algebra concepts will be integrated into our discussions, particularly when we explore vectors and their applications.

Review of Cartesian Coordinates in Two Dimensions

Remark

Before expanding into three dimensions, let's recall the familiar Cartesian coordinate system in $\mathbb{R}^2$, where every point is represented as an ordered pair (x, y) on the xy-plane.

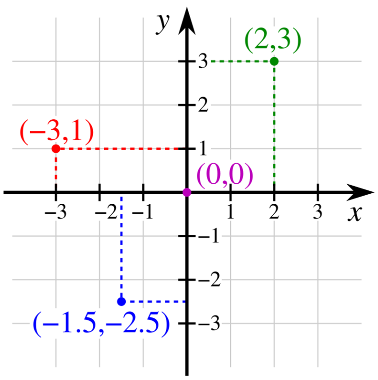

Figure 20: The Cartesian coordinate plane in $\mathbb{R}^2$.

Introducing Three-Dimensional Cartesian Coordinates

Concept

Now, we step into the three-dimensional space, $\mathbb{R}^3$, by introducing a third coordinate, z. Each point in $\mathbb{R}^3$ is now represented as an ordered triple (x, y, z). The additional z-axis extends perpendicular to the xy-plane, allowing for depth perception in our coordinate system.

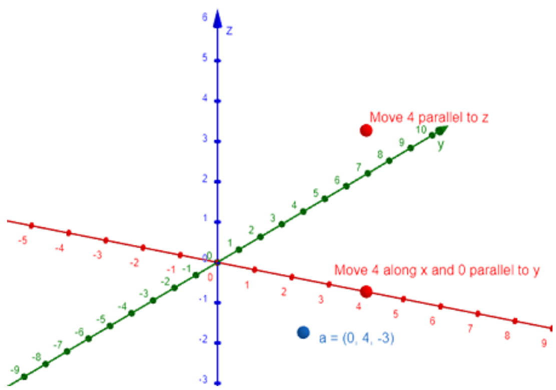

Figure 21: The Cartesian coordinate system in $\mathbb{R}^3$, including the z-axis.

Understanding this extension is crucial for working with vectors, planes, and other geometric structures in higher dimensions.

Note

<u>In 2D:</u>

Notice that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where the first $\mathbb{R}$ represents the x-values and the second $\mathbb{R}$ represents the y-values.

<u>Now, in 3D:</u>

Notice that $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

- The first $\mathbb{R}$ represents the x-values;
- The second $\mathbb{R}$ represents the y-values;
- The third $\mathbb{R}$ represents the z-values.

Example of Plotting in a 3D Cartesian Plane

Example

Plot the points $(-1, 2, -3)$ and $(2, -4, 2)$.

Illustration

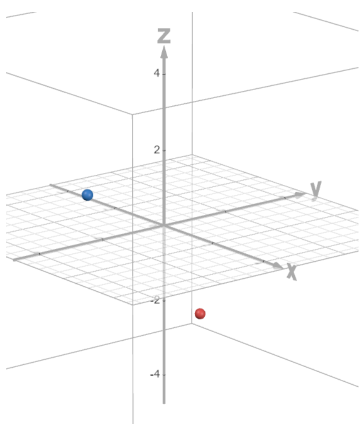

Figure 22: Illustration from Desmos.

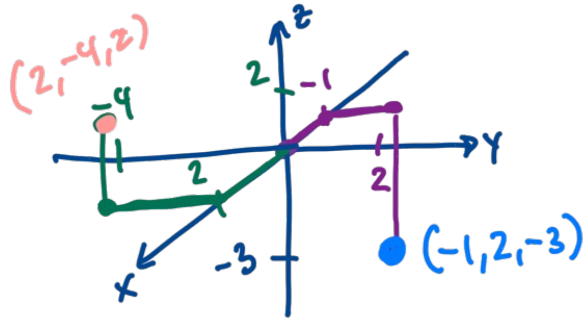

Figure 23: Illustration from lecture.

Concept

To plot a point in 3D space, locate the corresponding x, y, and z values on the axes. The point is then represented by the intersection of the three coordinate planes.

Tip

Trace the path from the origin to the point to visualize its position in 3D space. This approach helps in understanding the spatial relationships between points.

Understanding Planes in 3D

Concept

In a 2D world, there is no notion of height when considering the xy-plane. However, in a 3D world, we introduce the z-coordinate.

Here are the fundamental planes in a 3D Cartesian coordinate system:

The xy-Plane ($z = 0$)

In the 3D space, the xy-plane is defined by the equation $z = 0$, where the z-coordinate is always zero. This plane extends infinitely along the x and y-axes.

$$(x, y, 0)$$

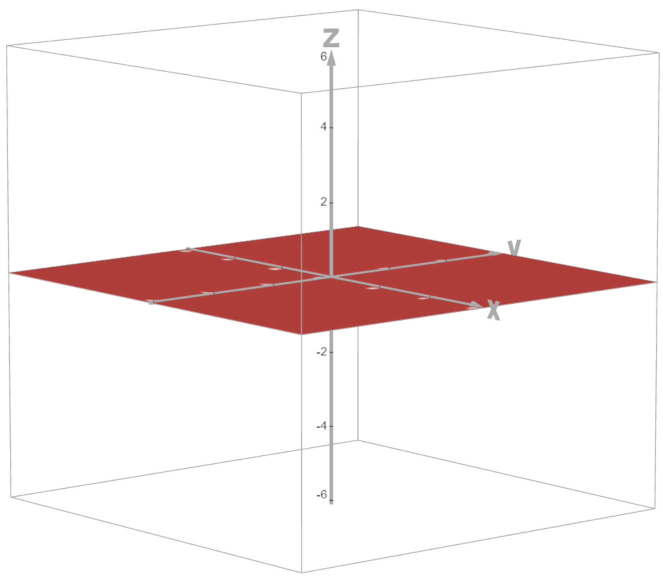

Figure 24: The xy-plane where $z = 0$.

...cont'd...

Concept

...cont'd...

The yz-Plane ($x = 0$)

The yz-plane is defined by the equation $x = 0$, where the x-coordinate is always zero. This plane extends infinitely along the y and z-axes.

$$(0, y, z)$$

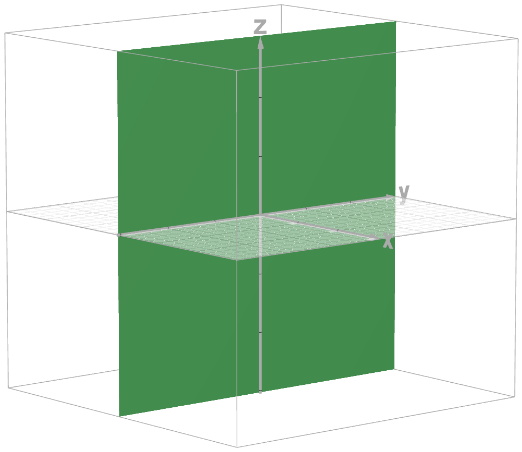

Figure 25: The yz-plane where $x = 0$.

The xz-Plane ($y = 0$)

The xz-plane is defined by the equation $y = 0$, where the y-coordinate is always zero. This plane extends infinitely along the x and z-axes.

$$(x, 0, z)$$

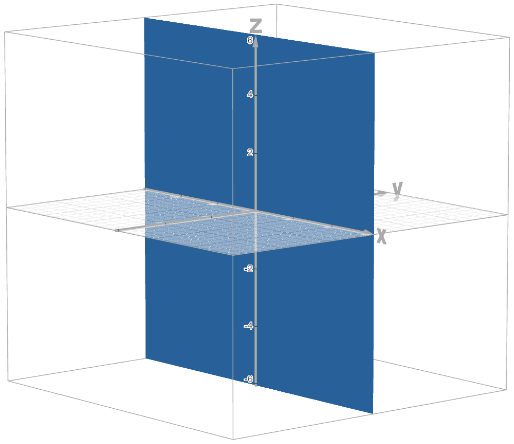

Figure 26: The xz-plane where $y = 0$.

Transitioning from 2D to 3D

Example

Visualizing a Line in 3D Consider the equation $y = 2$ on a 2D Cartesian plane:

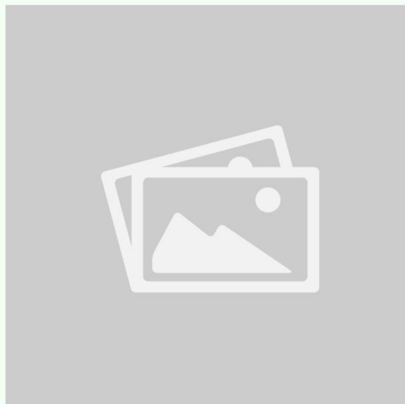

Figure 27: The line $y = 2$ in 2D.

In a 3D space, this extends infinitely along the z-axis, forming a vertical plane parallel to the xz-plane:

Figure 28: The same line extended into 3D space.

Example

Visualizing a Circle in 3D Consider the equation of a circle:

$$x^2 + y^2 = 4$$

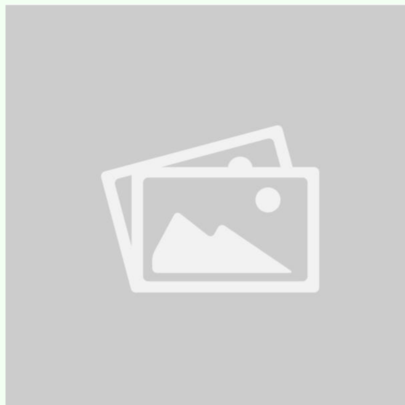

Figure 29: A circle in 2D defined by $x^2 + y^2 = 4$.

If we extend this into the third dimension by allowing any z-value, it forms a **cylinder**, where the circle acts as the cross-section:

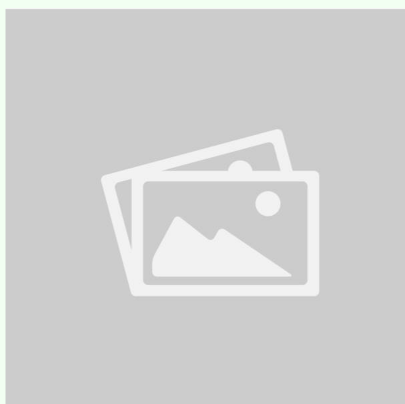

Figure 30: The circle extended into 3D space, forming a cylinder.

Next Lecture: We Discuss Vectors!

Lecture Title

Note

This template is designed for MAT232 lecture notes. Replace this content with your specific lecture details.

Key Concepts

Definition

A **parametric equation** is a set of equations that express the coordinates of the points of a curve as functions of a variable, called a parameter.

Examples

Example

Example 1: Consider the parametric equations:

$$x = t, \quad y = t^2, \quad t \in \mathbb{R}.$$

- At $t = 0$, $(x, y) = (0, 0)$.
- At $t = 1$, $(x, y) = (1, 1)$.

This describes a parabola.

Figure 31: Sample image illustrating the concept.

Theorems and Proofs

Theorem

Theorem: If $x(t)$ and $y(t)$ are differentiable functions, the slope of the curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

Figure 32: Graphical representation of the theorem.

Additional Notes

Note

Always check the domain of the parameter t when solving problems involving parametric equations.