MAT232 - Lecture 4

Polar Coordinates and Curves

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Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Polar Coordinates - Key Theorems

Converting Points between Coordinate Systems

Theorem

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between rectangular and polar coordinates.

Uniqueness of Polar Coordinates

Proposition

Every point in the plane has an infinite number of representations in polar coordinates. Specifically, the polar coordinates (r, θ) of a point are not unique.

Remark

For example, the polar coordinates $(2, \pi/3)$ and $(2, 7\pi/3)$ both represent the same point in the rectangular coordinate system. Additionally, the value of r can be negative. Therefore, the point with polar coordinates $(-2, 4\pi/3)$ represents the same rectangular point as $(2, \pi/3)$.

Symmetry of Polar Curves

Theorem

Polar curves can exhibit symmetry similar to those in rectangular coordinates. The key symmetries to identify are:

- **Symmetry with respect to the polar axis:** A curve is symmetric with respect to the polar axis if replacing θ with $-\theta$ in its equation yields the same curve.
- **Symmetry with respect to the line $\theta = \frac{\pi}{2}$:** A curve is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if replacing θ with $\pi - \theta$ yields the same curve.
- **Symmetry with respect to the pole (origin):** A curve is symmetric with respect to the pole if replacing r with $-r$ yields the same curve.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Plotting Polar Coordinates

Recall the Content from Last Lecture

Note

Converting between Cartesian coordinates (x, y) and Polar coordinates (r, θ):

Algorithm

From Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

From Polar to Cartesian:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Converting Between Degrees and Radians:

Algorithm

- **Degrees to Radians:** Multiply by $\dfrac{\pi}{180^\circ}$

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180^\circ}$$

- **Radians to Degrees:** Multiply by $\dfrac{180^\circ}{\pi}$

$$\text{Degrees} = \text{Radians} \times \frac{180^\circ}{\pi}$$

Understanding the Convention for r in Polar Coordinates

Concept

In polar coordinates, a **point** is represented as (r, θ), where:

- r is the radial distance from the origin (how far the point is from the origin).
- θ is the angle, measured counterclockwise from the positive x-axis.

Note

Special Case: When r is Negative

- A negative r in $(-r, \theta)$ is interpreted as the point being reflected through the origin.
- The equivalent representation is:

$$(-r, \theta) = (r, \theta + 180^\circ)$$

  or in radians:

$$(-r, \theta) = (r, \theta + \pi)$$

Intuition

- Reflecting (r, θ) through the origin is the same as rotating the point by 180° (or π radians).
- This property simplifies polar plots by offering alternate representations of the same point.

Tip

When plotting points, ensure to label points clearly on the polar grid, and verify angle conversions and reflections for accuracy.

Example: Plotting Points

Example

Let us plot the following points in polar coordinates:

$$(3, -45^\circ), \quad (3, 225^\circ), \quad (4, 330^\circ), \quad (1, -45^\circ)$$

Algorithm

Step-by-Step Process:

1. For each point, identify r and θ.
2. If θ is negative or exceeds 360°, convert it to a standard range:

$$\theta \in [0^\circ, 360^\circ)$$

   using $\theta = \theta + 360^\circ$ (for negative angles) or subtracting 360° (for angles over 360°).

3. Plot the point by measuring θ counterclockwise from the positive x-axis and placing it at a distance r from the origin.

Solution

- For $(3, -45^\circ)$: Add 360° to -45° to convert θ to 315°. Plot as $(3, 315^\circ)$.
- For $(3, 225^\circ)$: Already within the standard range, so plot directly.
- For $(4, 330^\circ)$: Angle is standard, so plot directly.
- For $(1, -45^\circ)$: Add 360° to -45°, yielding $(1, 315^\circ)$.

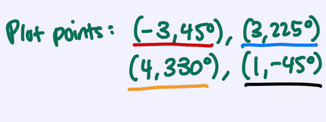

Figure 1: Colour Legend

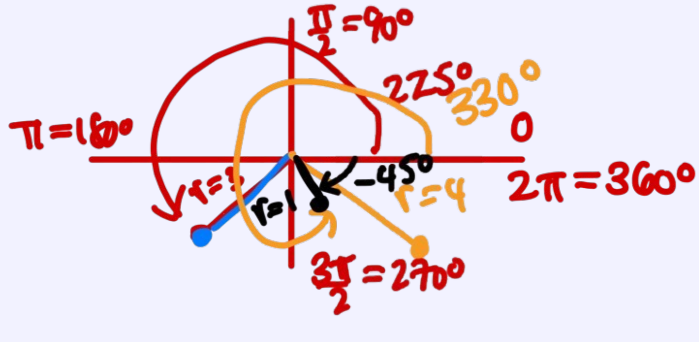

Figure 2: Polar Coordinates Plot and "Trajectories"

Converting Between Cartesian and Polar Coordinates

Example: Converting from Polar Coordinates to Cartesian Coordinates

Example

Find the **rectangular coordinates** of the point p with polar coordinates $(6, \dfrac{\pi}{3})$.

Solution

To convert from polar to Cartesian coordinates, use:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Substitute $r = 6$ and $\theta = \frac{\pi}{3}$:

$$x = 6\cos\left(\frac{\pi}{3}\right) = 6 \cdot \frac{1}{2} = 3, \quad y = 6\sin\left(\frac{\pi}{3}\right) = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}.$$

Thus, the Cartesian coordinates are:

$$(x, y) = (3, 3\sqrt{3}).$$

Answer

The rectangular coordinates are $(3, 3\sqrt{3})$.

Example: Converting from Cartesian Coordinates to Polar Coordinates

Example

Find the **polar coordinates** of the point p with rectangular coordinates $(-2, 2\sqrt{3})$.

Solution

To find the polar coordinates, use:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}.$$

Step 1: Solve for r:

$$r^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16 \implies r = 4.$$

Step 2: Solve for θ:

$$\tan(\theta) = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}.$$

The point $(-2, 2\sqrt{3})$ lies in Quadrant II. The reference angle for $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$. Thus:

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Tip

Alternatively, for a negative angle:

$$\theta = -\frac{\pi}{3}, \quad \text{adjust to Quadrant II: } -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Thus, $(r, \theta) = (4, \frac{2\pi}{3})$.

Answer

The polar coordinates are $(4, \dfrac{2\pi}{3})$ or $(4, 120^\circ)$.

Note

The professor recommends using graphing tools like Desmos or GeoGebra to enhance your understanding of the material. These tools are especially helpful for plotting lines and circles, which are key concepts in MAT232. Getting comfortable with them will make the course much easier.

Sketching Polar Curve Functions

Exercise

Consider $r = f(\theta)$. Sketch the following functions:

(a) $r = 1$

(b) $\theta = \frac{\pi}{4}$

(c) $r = \theta, \quad \theta \geqslant 0$

(d) $r = \sin(\theta)$

(e) $r = \cos(2\theta)$

(a) $r = 1$

Solution

Here, $r = 1$, and θ can take any value.

This means the point is always at a distance of 1 from the origin, regardless of the angle θ. Hence, the graph is a **circle** with radius 1, centred at the origin.

Concept

Cartesian Conversion

From the polar equation:

$$x^2 + y^2 = r^2 = 1$$

This confirms the equation of a unit circle in Cartesian coordinates.

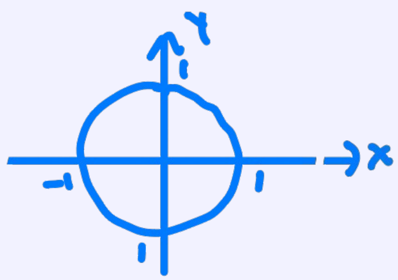

Figure 3: The graph of $r = 1$.

(b) $\theta = \dfrac{\pi}{4}$

Solution

Here, $\theta = \dfrac{\pi}{4}$, and r can take any value.
This represents all points that lie along the line passing through the origin at an angle of $\frac{\pi}{4}$ (or $45°$) with the positive x-axis. The graph is a **straight line** through the origin.

Concept

Cartesian Conversion

In polar coordinates:

$$\tan(\theta) = \frac{y}{x}$$

Substituting $\theta = \frac{\pi}{4}$, we get:

$$\tan\left(\frac{\pi}{4}\right) = 1 \quad \Rightarrow \quad y = x$$

Thus, the Cartesian equation is $y = x$.

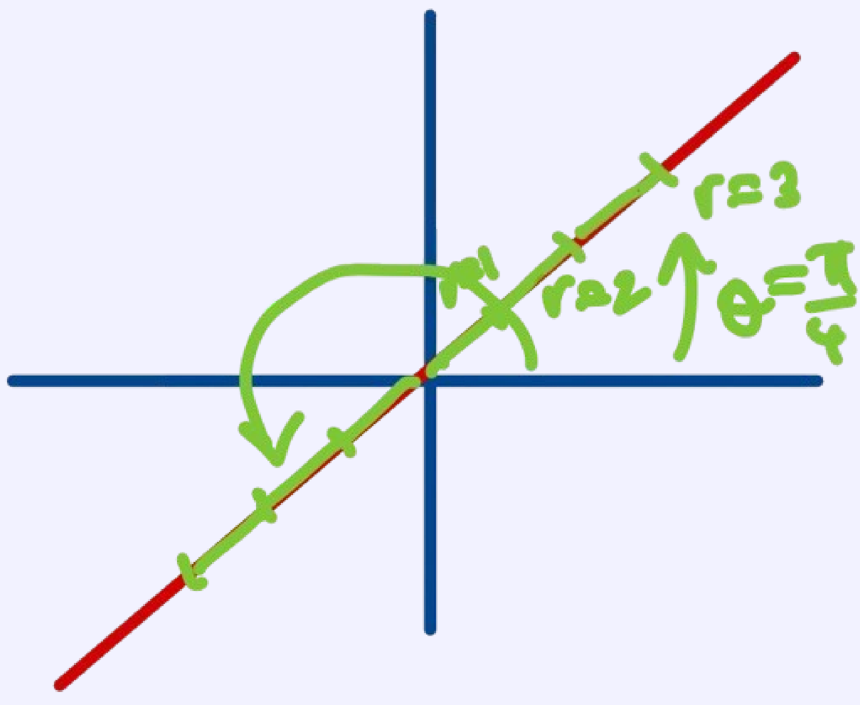

Figure 4: The graph of $\theta = \frac{\pi}{4}$.

(c) $r = \theta, \quad \theta \geqslant 0$

Solution

Here, r increases as θ increases. This creates a **spiral** that starts at the origin and winds outward as θ grows.

Concept

Table of Values

θ	r
0	0
$\frac{\pi}{6}$	$\frac{\pi}{6} \approx 0.52$
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx 0.79$
$\frac{\pi}{3}$	$\frac{\pi}{3} \approx 1.05$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$

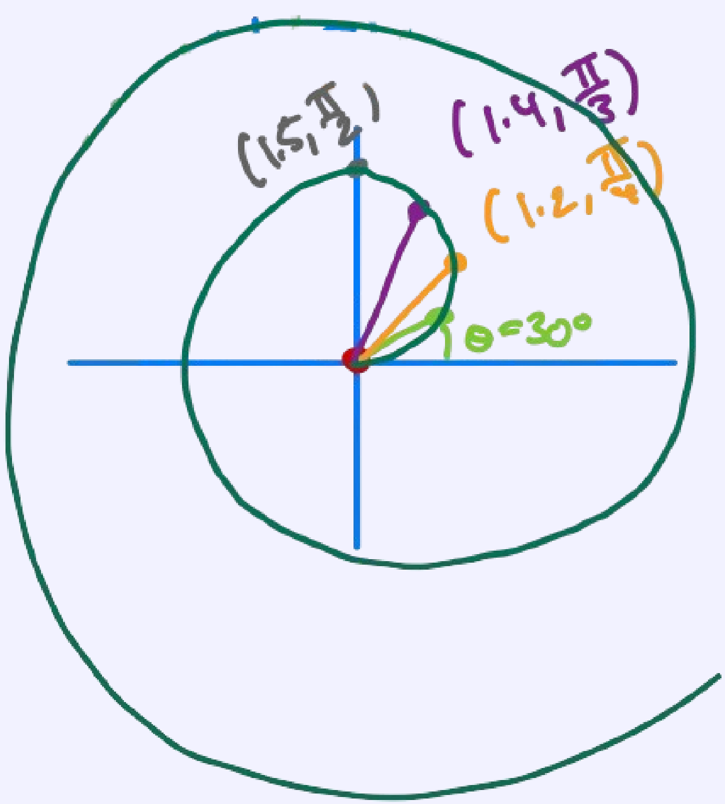

Figure 5: The graph of $r = \theta$.

(d) $r = \sin(\theta)$

Solution

Here, $r = \sin(\theta)$. Since $\sin(\theta)$ oscillates between 0 and 1, the graph forms a **cardioid** (which happens to be a perfect circle in this case).

Concept

Cartesian Conversion

Using $r^2 = x^2 + y^2$ and $r = \sin(\theta)$, we get:

$$x^2 + y^2 = y \quad \Rightarrow \quad \left(x^2 + y^2\right) - y = 0$$

Completing the square for y:

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

This is a circle centred at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$.

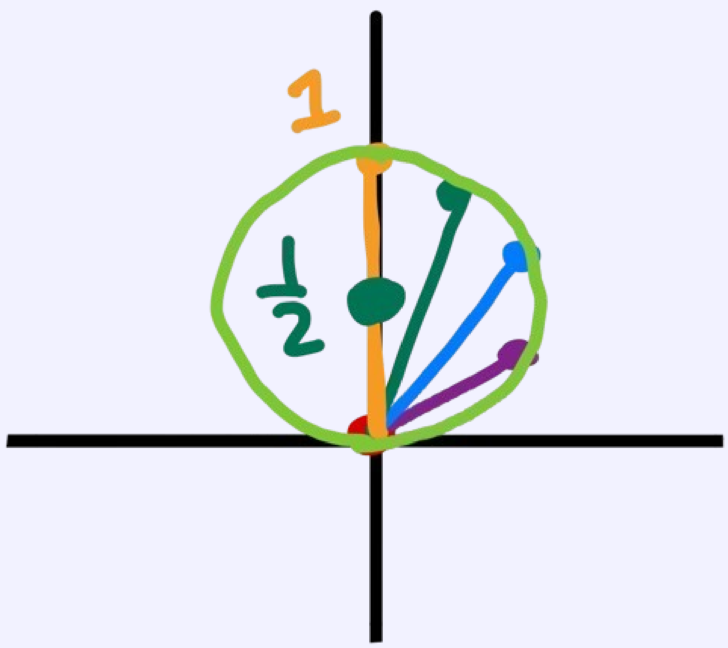

Figure 6: The graph of $r = \sin(\theta)$.

(e) $r = \cos(2\theta)$

Solution

The equation $r = \cos(2\theta)$ represents a **four-petaled rose**. As θ varies from 0 to 2π, r oscillates between -1 and 1, creating the petals.

Concept

Table of Values

Evaluate r at critical angles:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	1	0	-1	0	1	0	-1	0	1

The curve is symmetric about the polar axis, with petals centered at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

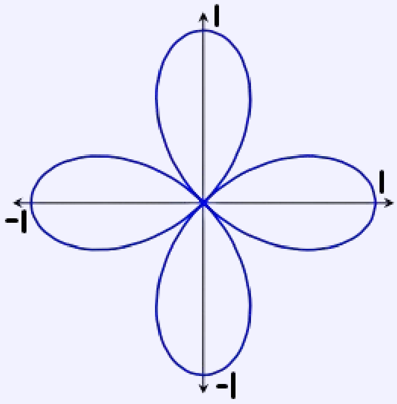

Figure 7: The graph of $r = \cos(2\theta)$.

Exercise

Try sketching the curve:

$$r = \cos\theta$$

under the same context as the previous questions.

Solution

Here, $r = \cos\theta$. Since $\cos\theta$ oscillates between -1 and 1, the graph will form a **limacon** with an inner loop.

Table of Values To visualize the curve, create a table of key points:

θ	r
0	1
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
π	-1

Note

Plot the points in the table and connect them smoothly to form the graph. Note that for negative r, the points are plotted in the opposite direction from the origin.

Cartesian Conversion Substitute $r = \cos\theta$ into the Cartesian equations:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Substituting $r = \sqrt{x^2 + y^2}$ and eliminating r, we derive:

$$x^2 + y^2 = x$$

Complete the square for x:

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

This represents a circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

Figure 8: Sample image illustrating the concept.

The Derivative of a Polar Curve

Tangents to Polar Curves

Definition

Note

Recall that polar curves are defined by:

$$r = f(\theta).$$

The Cartesian coordinates x and y are expressed as:

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Intuition

The goal is to express the slope $\frac{dy}{dx}$ entirely in terms of θ, ensuring both x and y depend on this single parameter.

Finding $\frac{dy}{dx}$

To compute $\frac{dy}{dx}$, we use the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}, \quad \text{where} \quad \frac{dx}{d\theta} \neq 0.$$

Step 1: Compute $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\big(f(\theta)\cos\theta\big) = \frac{df(\theta)}{d\theta}\cos\theta - f(\theta)\sin\theta,$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\big(f(\theta)\sin\theta\big) = \frac{df(\theta)}{d\theta}\sin\theta + f(\theta)\cos\theta.$$

Step 2: Substitute into $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{df(\theta)}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df(\theta)}{d\theta}\cos\theta - f(\theta)\sin\theta}.$$

Step 3: Replace $f(\theta)$ with r For simplicity, we substitute $f(\theta)$ with r:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Examples

Example

Find the **vertical tangent** angles of the polar curve $r = 1 - \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$.

Solution

We aim to determine the angles θ where the polar curve has vertical tangents. This occurs when $\dfrac{dx}{d\theta} = 0$, provided $\dfrac{dy}{d\theta} \neq 0$.

Step 1: Compute $\dfrac{dy}{dx}$

The derivative of a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Given $r = 1 - \cos\theta$, compute $\dfrac{dr}{d\theta}$:

$$\frac{dr}{d\theta} = \sin\theta.$$

Substitute $r = 1 - \cos\theta$ and $\dfrac{dr}{d\theta} = \sin\theta$ into the formula for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\sin\theta\sin\theta + (1 - \cos\theta)\cos\theta}{\sin\theta\cos\theta - (1 - \cos\theta)\sin\theta}.$$

Simplify the numerator and denominator:

$$\frac{dy}{dx} = \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{\sin\theta(2\cos\theta - 1)}.$$

Step 2: Condition for Vertical Tangents

Vertical tangents occur when:

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1) = 0,$$

provided $\dfrac{dy}{d\theta} \neq 0$.

Solve $\dfrac{dx}{d\theta} = 0$:

$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta - 1 = 0.$$

1. When $\sin\theta = 0$:

$$\theta = 0, \pi \quad (\text{within } 0 \leqslant \theta \leqslant \pi).$$

2. When $2\cos\theta - 1 = 0$:

$$\cos\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}.$$

The candidates for vertical tangents are:

$$\theta \in \{0, \frac{\pi}{3}, \pi\}$$

Theorem

The equations below describe the geometry of a circle in the Cartesian plane:

- **Standard Circle:** A circle centered at the origin $(0, 0)$ with radius r is given by:

$$x^2 + y^2 = r^2.$$

- **General Circle:** A circle centered at (h, k) with radius r is described by:

$$(x - h)^2 + (y - k)^2 = r^2.$$

Figure 10: Graphical representation of a circle in the Cartesian plane.

Example

Sketch the curve described by the equation:

$$x^2 + y^2 - 2x = 10.$$

Solution

To sketch the curve, rewrite the given equation in the standard form of a circle by completing the square.

Step 1: Group x-terms and prepare to complete the square

$$x^2 - 2x + y^2 = 10.$$

Step 2: Complete the square for x

1. Divide the coefficient of x by 2 and square it:

$$-\frac{2}{2} = -1, \quad (-1)^2 = 1.$$

2. Add and subtract 1 to maintain equality:

$$x^2 - 2x + 1 + y^2 = 10 + 1.$$

Step 3: Rewrite as a perfect square

$$(x-1)^2 + y^2 = 11.$$

Final Form and Interpretation

The equation represents a circle centered at $(1,0)$ with radius $\sqrt{11}$.

Answer

Solution: The circle is centered at $(1,0)$ and has a radius of $\sqrt{11}$.

Note

Reminder: Next week is *Vector Week*! We will dive into vectors, a fundamental concept that bridges linear algebra and multivariable calculus (MAT232).