MAT232 - Lecture 13

after partial derivatives?

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Contents

Definitions and Theorems

Straight from the textbook — a thorough coverage.

Quick recap before diving into the lecture.

Definition

Given $z = f(x, y)$ at a point (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b\rangle$, the directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

Note

Note that the directional derivative can also be written as

$$\begin{aligned} D_{\vec{u}} f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle \\ &= \nabla f(x, y) \cdot \vec{u}. \end{aligned}$$

Definition

The gradient of f is the vector function $\nabla f = \langle f_x, f_y\rangle$.

Concept

The gradient of f is a vector that points in the direction of the greatest rate of increase of f at a point (x, y).

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = f_x(x, y)\hat{i} + f_y(x, y)\hat{j}.$$

The max rate of increase of f at (x, y) is $\|\nabla f(x, y)\|$.
The direction of the max rate of increase of f at (x, y) is $-\nabla f(x, y)$.

Definition

The max rate of decrease of f at (x, y) is $-\|\nabla f(x, y)\|$.

The min rate of increase of f at (x, y) is $-\|\nabla f(x, y)\|$.
The direction of the min rate of increase of f at (x, y) is $\nabla f(x, y)$.

Definition

The min rate of decrease of f at (x, y) is $\|\nabla f(x, y)\|$.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Term Test 2 Reminder!

Note

Term Test 2 will be held on March 6, 2025 (Week 8), from 6:00 PM to 8:00 PM. The test will cover the following topics:

- Double integrals
- Domain of integration
- Clairut's theorem
- Greene's theorem
- Level curves
- Vector fields
- Line integrals
- Surface integrals

Section 4.7: Max and Min Values

Recall from 1^{st} year calculus:

- Local max/min: $f'(c) = 0$ or $f'(c)$ does not exist.
- Absolute max/min: $f(c) \geqslant f(x)$ or $f(c) \leqslant f(x)$ for all x in the domain of f.
- Critical point: $f'(c) = 0$ or $f'(c)$ does not exist.

Illustration

Figure 1: Max and Min Values

Given $y = f(x), y' = f'(x) = 0$, find the critical numbers (cn):

- Find $f'(x)$.
- Solve $f'(x) = 0$ or $f'(x)$ does not exist.
- The solutions are the critical numbers.

Note

First derivative test:

- If $f'(c) = 0$ and $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then $f(c)$ is a local max.
- If $f'(c) = 0$ and $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then $f(c)$ is a local min.
- If $f'(c) = 0$ and $f'(x)$ does not change sign, then the test is inconclusive.

Given $y' = f'(x), y'' = f''(x) = 0$, find the critical numbers (cn):

- Find $f'(x)$ and $f''(x)$.
- Solve $f'(x) = 0$ or $f'(x)$ does not exist.

- The solutions are the critical numbers.

$f(x)$ is concave up if $f''(x) > 0$ and concave down if $f''(x) < 0$.

Note

Second derivative test:

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local min.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local max.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

Now in MAT232:

- Local max/min: $\nabla f = \langle 0, 0 \rangle$ or ∇f does not exist.
- Absolute max/min: $f(c) \geqslant f(x)$ or $f(c) \leqslant f(x)$ for all x in the domain of f.
- Critical point: $\nabla f = \langle 0, 0 \rangle$ or ∇f does not exist.
- Second derivative test: $D > 0$ and $f_{xx} > 0$ or $D > 0$ and $f_{xx} < 0$.

Suppose that the second partial derivatives of f are continuous in an open disk centered at (a, b). Moreover, suppose