MAT232 - Lecture 2

Parametric Equations and Calculus: Concepts, Applications, and Derivations

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Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Parametric Equations and Parameters

Definition

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called **parametric equations**, and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the **graph of the parametric equations**. The graph of parametric equations is referred to as a **parametric curve** or **plane curve**, and is denoted by C.

Theorem 1.1: Derivative of Parametric Equations

Theorem

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

Proof

Proof.
This theorem can be proven using the Chain Rule. Assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule gives

$$y'(t) = F'(x(t))x'(t),$$

so

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem.

$\square$

Equation 1.1 and Applications

Note

Equation 1.1 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 1.1 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Second-Order Derivatives

Theorem

The next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, it is possible to replace y on both sides of this equation with $\frac{dy}{dx}$. This yields

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Key Concept

Definition

A **parametric equation** is a set of equations that express the coordinates of the points of a curve as functions of a variable, called a parameter.

Sketching Parametric Equations

Example

Example: Sketch the graph, using a table of values:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

Solution

Table of Values:

t	$1/t$	x	y
0.01	${}^{1}/_{0.01} = {}^{1}/_{\frac{1}{100}} = 100$	100.01	$0.01 - 100 = -99.99$
0.1	${}^{1}/_{0.1} = {}^{1}/_{\frac{1}{10}} = 10$	10.1	-9.9
0.2	${}^{1}/_{0.2} = {}^{1}/_{\frac{20}{100}} = {}^{1}/_{\frac{2}{10}} = 5$	5.2	4.8
1	$\frac{1}{1}$	2	0
5.0	0.2	5.2	4.8
10	0.1	10.1	9.9
10	0.01	100.01	99.99

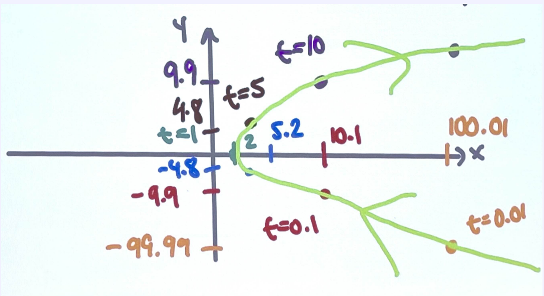

Figure 1: The corresponding hyperbolic graph.

Example

Example: Sketch the graph of the same parametric equation, using the elimination method:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

Solution

We start by simplifying the expressions for x and y:

$$\text{LHS} = A^2 - B^2 = (A - B)(A + B) = \text{RHS}.$$

Let $A = x$ and $B = y$, so we can rewrite this as:

$$\text{LHS:} \quad x^2 - y^2.$$

Now, compute $A - B$ and $A + B$:

$$A - B = x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t},$$

$$A + B = x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

Now, calculate the RHS:

$$\text{RHS:} \quad (A - B)(A + B) = (x - y)(x + y) = \left(\frac{2}{t}\right)(2t) = 4.$$

Thus, we obtain the equation:

$$x^2 - y^2 = 4, \quad x > 0.$$

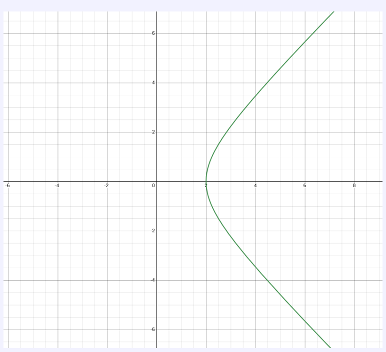

Figure 2: Precise graph of the parametric equation $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad x > 0$

The Slope of a Parametric Curve

Definition

Definition: If $x(t)$ and $y(t)$ are differentiable functions, the slope of the curve described by these parametric equations is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

Remark

This formula allows you to find the slope of the tangent line to the curve at any point where the derivative of $x(t)$ with respect to t is nonzero.

Here's a concrete example illustration!

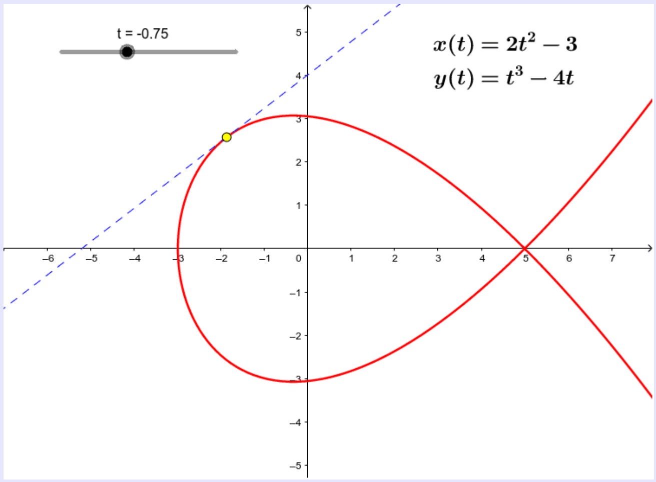

Figure 3: Graphical representation of the slope of a parametric curve.

Parametric Curve Sketching Practice Questions

Exercise

Question 1: Sketch and eliminate t if possible:

$$x = t^2, \quad y = t^3, \quad -2 \leqslant t \leqslant 2$$

Note that this is a closed interval, which means the graph starts at $t = -2$ and ends at $t = 2$. The starting point is where $t = -2$, and the finishing point is where $t = 2$. The direction of the graph should be indicated using an arrow as $t \to 2$.
Express the relationship between x and y in Cartesian form.

Solution

Solution:
The parametric equations are $x = t^2$ and $y = t^3$. We aim to eliminate t.
From $x = t^2$, we can solve for t as:

$$t = \pm\sqrt{x}.$$

Substituting this into the equation for y:

$$y = (\pm\sqrt{x})^3 = \pm x^{3/2}.$$

Thus, the Cartesian form is:

$$y = \pm x^{3/2}.$$

This describes a curve that is symmetric about the x-axis. The graph starts at $t = -2$ with coordinates $(x, y) = (4, -8)$, and ends at $t = 2$ with coordinates $(x, y) = (4, 8)$. The graph is symmetric, opening upwards and downwards, as t moves from -2 to 2.
As t increases, the direction of the graph is indicated by the arrows as shown below.

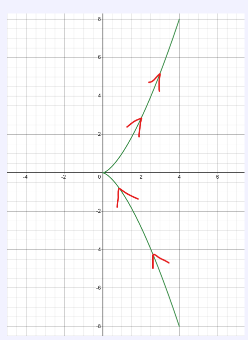

Figure 4: Graph of the curve for $x = t^2, y = t^3$.

Exercise

Question 2: Sketch and eliminate t if possible for each of the following parametric equations:

$$c_1 : x = -\cos\left(\frac{t}{4}\right), \quad y = \sin\left(\frac{t}{4}\right), \quad 0 \leqslant t \leqslant 4\pi$$

$$c_2 : x = -\sin(t), \quad y = -\cos(t), \quad \frac{\pi}{2} \leqslant t \leqslant \frac{3\pi}{2}$$

$$c_3 : x = \cos(t), \quad y = \sin(t), \quad t \in [0, \pi]$$

For each equation, express the relationship in Cartesian form and sketch the curve. The hint suggests that $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $x^2 + y^2 = r^2$. For these curves, $r = 1$.

Solution

Solution *(see illustration below)***:**
For c_1: The parametric equations are $x = -\cos\left(\frac{t}{4}\right)$ and $y = \sin\left(\frac{t}{4}\right)$. To eliminate t, use the identity $\cos^2\theta + \sin^2\theta = 1$, where $\theta = \frac{t}{4}$. This gives:

$$x^2 + y^2 = \cos^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{t}{4}\right) = 1.$$

This represents a circle with radius 1, centered at the origin.
For $t = 0$, we have $(x, y) = (-1, 0)$. For $t = 4\pi$, we have $(x, y) = (1, 0)$. The curve traces a semicircle in the clockwise direction, starting at $-1, 0$ and ending at $(1, 0)$.
For c_2: The parametric equations are $x = -\sin(t)$ and $y = -\cos(t)$. Using the identity $\sin^2\theta + \cos^2\theta = 1$, we have:

$$x^2 + y^2 = \sin^2(t) + \cos^2(t) = 1.$$

This describes a unit circle. The curve starts at $t = \frac{\pi}{2}$, corresponding to $(x, y) = (-1, 0)$, and ends at $t = \frac{3\pi}{2}$, corresponding to $(x, y) = (1, 0)$. The graph traces a semicircle in the clockwise direction.
For c_3: The parametric equations are $x = \cos(t)$ and $y = \sin(t)$. Again using the identity $\cos^2\theta + \sin^2\theta = 1$, we get:

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1.$$

This describes a unit circle. The curve starts at $t = 0$, corresponding to $(x, y) = (1, 0)$, and ends at $t = \pi$, corresponding to $(x, y) = (-1, 0)$. The graph traces the upper half of the unit circle, moving counterclockwise.

Illustration

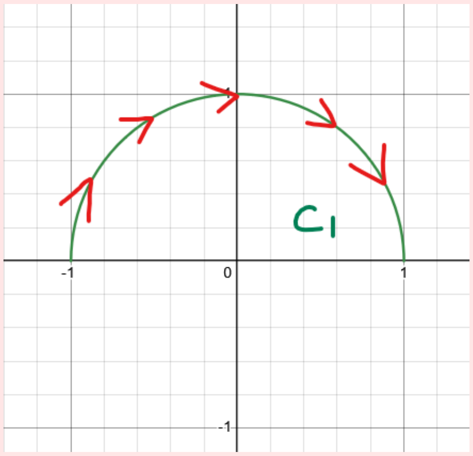

Figure 5: Graph of the curve for c_1.

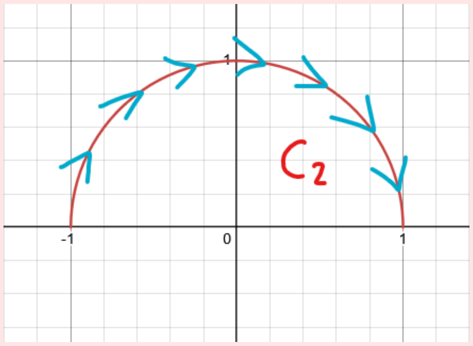

Figure 6: Graph of the curve for c_2.

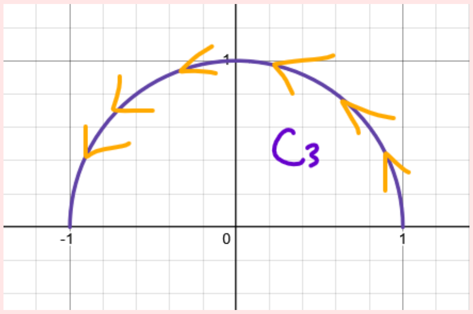

Figure 7: Graph of the curve for c_3.

The Elimination Method Does NOT Always Work

Warning

In some cases, it is not possible to eliminate the parameter t to express the relationship between x and y in a Cartesian form. While elimination works in many situations—especially when the parametric equations describe simpler curves, like circles or straight lines—there are cases where it's not possible to eliminate t algebraically, or it becomes extremely complicated to do so.

Remark

MAT232 is structured around problems where the **elimination method** is guaranteed to work effectively. However, the following section delves into extreme cases of equations where the elimination method may fail. These types of equations are not part of the curriculum for MAT232.

Example

Consider the following parametric equations:

$$x = e^t - \sin^2(t), \quad y = \ln(t) + \frac{1}{t}, \quad t > 0$$

Here, x and y are both expressed in terms of t, but neither x nor y is simply related to t in a way that allows for easy elimination of t.

Intuition

Let's break down why we can't eliminate t easily from these equations.

- **Equation for x:**

$$x = e^t - \sin^2(t)$$

  This equation combines an exponential and a trigonometric function, making it difficult to solve for t in terms of x. The exponential grows, while the trigonometric part oscillates, leading to a complex relationship.

- **Equation for y:**

$$y = \ln(t) + \frac{1}{t}$$

  This equation mixes a logarithmic and a rational function. The different forms of t (logarithmic and rational) make it challenging to eliminate t algebraically.

Warning

Note

To try and eliminate t, we would have to manipulate these equations in such a way that we express one variable in terms of the other, without involving t. However, due to the combination of different types of functions (exponential, trigonometric, logarithmic, and rational), it becomes extremely difficult to isolate t in either equation and then substitute it into the other equation.

In this case, even if we tried solving one equation for t and substituting it into the other, the resulting expressions would likely be too complicated or may not even have a simple closed form.

Concept

The **elimination method** relies on being able to algebraically manipulate the parametric equations into a form where we can solve for t and eliminate it. However, in cases like this one, where the parametric equations involve complex combinations of different types of functions (like exponentials, trigonometric functions, and logarithms), the elimination method becomes impractical or impossible.

This is why the elimination method does not always work, and in these situations, it's important to either:

- **Graph the parametric equations** to understand the curve visually, or
- Use other methods such as **numerical techniques** or **approximation methods** if a Cartesian form is needed.

Section 1.2: Calculus on Parametric Equations

The First Derivative

In first-year calculus, we learned that for a function $y = f(x)$, the slope of the tangent line to the curve is given by:

Corollary

The derivative of $y = f(x)$ with respect to x is:

$$y' = f'(x) = \frac{dy}{dx}$$

In MAT232, we deal with parametric equations. Given the parametric equations $x = f(t)$ and $y = g(t)$, where t is a real parameter, the derivative of y with respect to x is defined as:

Definition

For parametric equations $x = f(t)$ and $y = g(t)$, we differentiate with respect to t as follows:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{with} \quad \frac{dx}{dt} \neq 0$$

This formula is also provided on the formula sheet.
Derivation: Using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Rearranging gives:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided that} \quad \frac{dx}{dt} \neq 0$$

Second Derivative

Definition

Given that $x = f(t)$, $y = g(t)$, and $t \in \mathbb{R}$ are differentiable at t, and the first derivative $\frac{dy}{dx}$ exists and is differentiable, the second derivative of y with respect to x is given by:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Using the chain rule, we can express this as:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx}$$

Simplifying further:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

This result follows from the **inverse function theorem**.

Note: This formula is not provided on the formula sheet.

Applying the First and Second Derivatives

Example

Consider the following parametric curve:

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(A) Find the tangent line to the given curve at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$.
(B) Find the vertical tangent(s), if any.
(C) Find $\frac{d^2y}{dx^2}$.

Let's tackle each part one at a time.

(A) Find the tangent line to the given curve at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$.

Solution

Tangent Line:

The tangent line is given by:

$$y - y_0 = m(x - x_0)$$

where m is the slope and (x_0, y_0) is a point on the curve. Alternatively, it can be written as:

$$y = mx + b$$

We are given $(\sqrt{2}, 1) = (x_0, y_0)$. To find the slope m, we use the chain rule:

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

where $\frac{dy}{dt} = \sec^2(t)$ and $\frac{dx}{dt} = \sec(t)\tan(t)$. Substituting these gives:

$$m = \frac{\sec^2(t)}{\sec(t)\tan(t)} = \frac{\sec(t)}{\tan(t)}$$

At $t = \frac{\pi}{4}$, we have:

$$\sec\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad \tan\left(\frac{\pi}{4}\right) = 1$$

Thus, the slope is:

$$m = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Using the point-slope form:

$$y - 1 = \sqrt{2}(x - \sqrt{2})$$

Simplifying gives:

$$y = \sqrt{2}x - 1$$

Answer

The equation of the tangent line is $y = \sqrt{2}x - 1$.

(B) Find the vertical tangent(s), if any.

Solution

To find the vertical tangent, we need to analyze $\frac{dy}{dx}$, which is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2(t)}{\sec(t)\tan(t)} = \frac{\sec(t)}{\tan(t)}$$

Vertical tangents occur when $\frac{dy}{dx}$ is undefined. This happens when $\frac{dx}{dt} = 0$ while $\frac{dy}{dt} \neq 0$.
We have:

$$\frac{dx}{dt} = \sec(t)\tan(t)$$

To find the vertical tangent, we set $\frac{dx}{dt} = 0$:

$$\sec(t)\tan(t) = 0$$

Since $\sec(t) = \frac{1}{\cos(t)}$, it cannot be zero. Thus, the condition for a vertical tangent occurs when:

$$\tan(t) = 0$$

This happens at $t = 0$, so we need to check if $\frac{dy}{dt} \neq 0$ at $t = 0$.
We have:

$$\frac{dy}{dt} = \sec^2(t)$$

At $t = 0$, $\frac{dy}{dt} = \sec^2(0) = 1$, which is nonzero. Therefore, there is a vertical tangent at $t = 0$.
Thus, the vertical tangent occurs at $t = 0$.

Answer

The vertical tangent occurs at $t = 0$.

(C) Find $\frac{d^2y}{dx^2}$.

Solution

To find the second derivative, we use the formula:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

We already know that:

$$\frac{dy}{dx} = \frac{\sec(t)}{\tan(t)}, \quad \frac{dx}{dt} = \sec(t)\tan(t)$$

Thus, we need to compute the derivative of $\frac{dy}{dx}$ with respect to t. Start by differentiating:

$$\frac{dy}{dx} = \frac{\sec(t)}{\tan(t)}$$

Using the quotient rule, we get:

$$\frac{d}{dt}\left(\frac{\sec(t)}{\tan(t)}\right) = \frac{\tan(t) \cdot \frac{d}{dt}[\sec(t)] - \sec(t) \cdot \frac{d}{dt}[\tan(t)]}{\tan^2(t)}$$

Now, recall:

$$\frac{d}{dt}[\sec(t)] = \sec(t)\tan(t), \quad \frac{d}{dt}[\tan(t)] = \sec^2(t)$$

Substituting these into the derivative:

$$\frac{d}{dt}\left(\frac{\sec(t)}{\tan(t)}\right) = \frac{\tan(t) \cdot \sec(t)\tan(t) - \sec(t) \cdot \sec^2(t)}{\tan^2(t)} = \frac{\sec(t)(\tan^2(t) - \sec^2(t))}{\tan^2(t)}$$

Thus, the second derivative is:

$$\frac{d^2y}{dx^2} = \frac{\frac{\sec(t)(\tan^2(t)-\sec^2(t))}{\tan^2(t)}}{\sec(t)\tan(t)}$$

Simplifying the expression:

$$\frac{d^2y}{dx^2} = \frac{\tan^2(t) - \sec^2(t)}{\tan^3(t)}$$

This is the second derivative of y with respect to x.

Answer

$$\frac{d^2y}{dx^2} = \frac{\tan^2(t)-\sec^2(t)}{\tan^3(t)}$$