MAT232 - Lecture 3

Polar Coordinates and the Arc Length of Parametric Curves

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Prepared for January 13, 2025

Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Preliminary Definitions and Theorems

Definition

Polar Coordinates.
Each point in the Cartesian plane can be represented in polar coordinates as an ordered pair (r, θ), where r is the radial coordinate (distance from the origin), and θ is the angular coordinate (angle measured from the positive x-axis). The correspondence between Cartesian coordinates (x, y) and polar coordinates (r, θ) is given by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Theorem

Theorem 1.4. Converting Points Between Coordinate Systems.
Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between Cartesian and polar coordinates.

Example

Example 1.10. Converting Between Rectangular and Polar Coordinates.

1. Convert $(1, 1)$ to polar coordinates: Use $x = 1$ and $y = 1$. Then:

$$r^2 = x^2 + y^2 = 1^2 + 1^2 = 2 \implies r = \sqrt{2}, \quad \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \implies \theta = \frac{\pi}{4}.$$

   Therefore, $(1, 1)$ can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

Concept

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates.

1. Create a table with two columns: one for θ values and one for r values.
2. Calculate the corresponding r values for each θ.
3. Plot each ordered pair (r, θ) on the polar coordinate axes.
4. Connect the points and observe the resulting graph.

Example

Example 1.12. Graphing a Function in Polar Coordinates.
Graph the curve defined by $r = 4\sin\theta$.

1. Create a table of values for θ and calculate r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	2π
$r = 4\sin\theta$	0	2	$2\sqrt{2}$	4	0	0

2. Plot the points and connect them to form the curve. The result is a circle with radius 2 centered at $(0, 2)$ in rectangular coordinates.

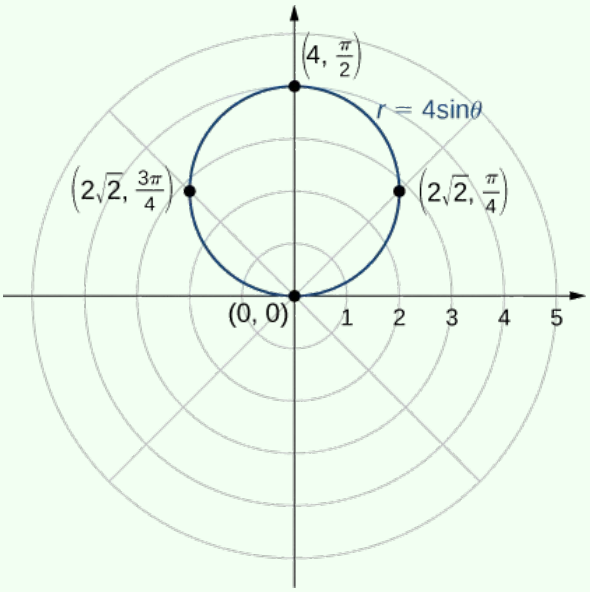

Figure 1: The graph of the function $r = 4\sin\theta$ is a circle.

Let’s Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let’s break this down step by step.

Recall: 1st Year Calculus

Definition

The **definite integral** of a function $y = f(x)$, where $f(x) \geqslant 0$, represents the area under the curve from $x = a$ to $x = b$:

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx$$

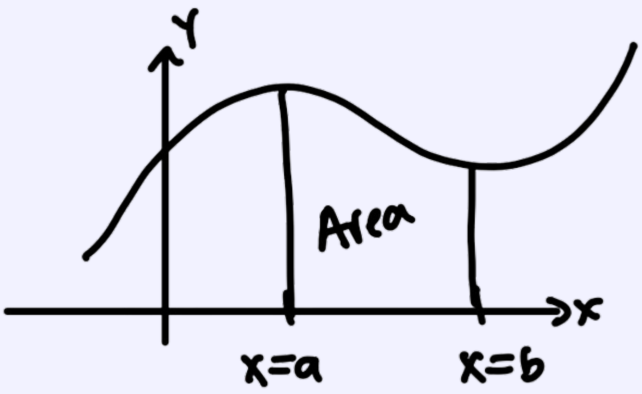

Figure 2: Illustration of the area under $y = f(x)$.

Section 1.2: Area Enclosed by a Parametric Curve

Definition

A **parametric curve** is defined by:

$$x = f(t), \quad y = g(t), \quad \alpha \leqslant t \leqslant \beta$$

with the following properties:

- The curve lies above the x-axis.
- The curve does not self-intersect.

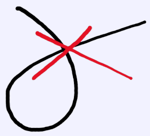

Figure 3: A curve that self-intersects.

Theorem

The area enclosed by the curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt$$

or equivalently:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} f(t) g'(t)\, dt$$

Alternative Forms for Area

Note

In specific cases, the area can also be calculated using:

$$\text{Area} = \int_{y=c}^{y=d} x(y)\, dy$$

or:

$$\text{Area} = \int_{x=a}^{x=b} y(x)\, dx$$

Example: Standard Parametric Curve

Example

Calculate the area enclosed by the parametric curve:

$$x = \cos(t), \quad y = \sin(t), \quad 0 \leqslant t \leqslant \pi$$

Solution

The area is calculated as:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$ and $y = g(t)$. Here, $f(t) = \cos(t)$, $g(t) = \sin(t)$, and $f'(t) = -\sin(t)$. Substituting:

$$\text{Area} = \int_{t=0}^{t=\pi} \sin(t)(-\sin(t))\, dt = \int_{t=0}^{t=\pi} -\sin^2(t)\, dt.$$

Using $\sin^2(t) = \frac{1}{2}(1-\cos(2t))$, we get:

$$\text{Area} = -\int_{t=0}^{t=\pi} \frac{1}{2}(1-\cos(2t))\, dt = -\frac{1}{2}\left[\int_{t=0}^{t=\pi} 1\, dt - \int_{t=0}^{t=\pi} \cos(2t)\, dt\right].$$

Evaluate the integrals:

$$\int_{t=0}^{t=\pi} 1\, dt = \pi, \quad \int_{t=0}^{t=\pi} \cos(2t)\, dt = \left[\frac{\sin(2t)}{2}\right]_0^{\pi} = 0.$$

Thus:

$$\text{Area} = -\frac{1}{2}(\pi - 0) = -\frac{\pi}{2}.$$

Taking the absolute value (since area is positive):

$$\text{Area} = \frac{\pi}{2}.$$

Example: Cycloid

Example

Example: Find the area under the cycloid defined by:

$$x = t - \sin(t), \quad y = 1 - \cos(t), \quad 0 \leqslant t \leqslant 2\pi.$$

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt, \quad f'(t) = \frac{dx}{dt}.$$

Step 1: Substitution

From $x = t - \sin(t)$ and $y = 1 - \cos(t)$:

$$f'(t) = 1 - \cos(t), \quad g(t) = 1 - \cos(t).$$

Substitute into the formula:

$$\text{Area} = \int_0^{2\pi} (1 - \cos(t))^2\, dt.$$

Step 2: Expand and Separate Terms

Expand $(1 - \cos(t))^2$:

$$\text{Area} = \int_0^{2\pi} \left[1 - 2\cos(t) + \cos^2(t)\right] dt.$$

Split the integral:

$$\text{Area} = \int_0^{2\pi} 1\, dt - 2\int_0^{2\pi} \cos(t)\, dt + \int_0^{2\pi} \cos^2(t)\, dt.$$

...cont'd...

Example

Solution

...cont'd...

Step 3: Evaluate Each Term

1. First Term:

$$\int_0^{2\pi} 1\,dt = 2\pi.$$

2. Second Term:

$$\int_0^{2\pi} \cos(t)\,dt = [\sin(t)]_0^{2\pi} = 0.$$

3. Third Term:

Using $\cos^2(t) = \dfrac{1+\cos(2t)}{2}$:

$$\int_0^{2\pi} \cos^2(t)\,dt = \frac{1}{2}\int_0^{2\pi} 1\,dt + \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^{2\pi} 1\,dt = \pi, \quad \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt = 0.$$

Thus:

$$\int_0^{2\pi} \cos^2(t)\,dt = \pi.$$

Step 4: Combine Results

$$\text{Area} = 2\pi - 0 + \pi = 3\pi.$$

Answer

Area $= 3\pi$

Homework Practice Question: Area Under a Parametric Curve

Exercise

Find the area under the curve defined by

$$x = 3\cos(t) + \cos(3t), \quad y = 3\sin(t) - \sin(3t), \quad 0 \leqslant t \leqslant \pi.$$

Hint: Recall that $\sin^2(x) + \cos^2(x) = 1$.

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$, $y = g(t)$, and $f'(t) = \frac{dx}{dt}$.

Step 1: Differentiate $x(t)$
Given $x = 3\cos(t) + \cos(3t)$, compute:

$$f'(t) = \frac{d}{dt}[3\cos(t) + \cos(3t)] = -3\sin(t) - 3\sin(3t).$$

Step 2: Substitute into the Formula
The parametric area formula becomes:

$$\text{Area} = \int_0^{\pi} \big[3\sin(t) - \sin(3t)\big]\big[-3\sin(t) - 3\sin(3t)\big]\, dt.$$

Step 3: Simplify the Expression
Expand the product:

$$\big[3\sin(t) - \sin(3t)\big]\big[-3\sin(t) - 3\sin(3t)\big] = -9\sin^2(t) - 9\sin(t)\sin(3t) + 3\sin(3t)\sin(t) + 3\sin^2(3t).$$

Combine terms:

$$-9\sin^2(t) + 3\sin^2(3t) - 6\sin(t)\sin(3t).$$

...cont'd...

Exercise

Solution

...cont'd..

Using the product-to-sum identity for $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$:

$$\sin(t)\sin(3t) = \frac{1}{2}[\cos(2t) - \cos(4t)].$$

Substitute this back:

$$\text{Area} = \int_0^{\pi} \big[-9\sin^2(t) + 3\sin^2(3t) - 3[\cos(2t) - \cos(4t)]\big]\, dt.$$

Step 4: Break the Integral into Separate Terms

Split the integral:

$$\text{Area} = -9\int_0^{\pi} \sin^2(t)\, dt + 3\int_0^{\pi} \sin^2(3t)\, dt - 3\int_0^{\pi} \cos(2t)\, dt + 3\int_0^{\pi} \cos(4t)\, dt.$$

Step 5: Evaluate Each Integral

1. First Term ($-9\int_0^{\pi} \sin^2(t)\, dt$):

Use the identity $\sin^2(t) = \dfrac{1-\cos(2t)}{2}$:

$$\int_0^{\pi} \sin^2(t)\, dt = \int_0^{\pi} \frac{1-\cos(2t)}{2}\, dt = \frac{1}{2}\int_0^{\pi} 1\, dt - \frac{1}{2}\int_0^{\pi} \cos(2t)\, dt.$$

Evaluate:

$$\frac{1}{2}\int_0^{\pi} 1\, dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^{\pi} \cos(2t)\, dt = \frac{1}{2}[0] = 0.$$

So:

$$\int_0^{\pi} \sin^2(t)\, dt = \frac{\pi}{2}.$$

Multiply by -9:

$$-9\int_0^{\pi} \sin^2(t)\, dt = -9 \cdot \frac{\pi}{2} = -\frac{9\pi}{2}.$$

...cont'd...

Exercise

Solution

...cont'd...

2. Second Term ($3\int_0^\pi \sin^2(3t)\,dt$):

Similarly, $\sin^2(3t) = \dfrac{1-\cos(6t)}{2}$:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{1}{2}\int_0^\pi 1\,dt - \frac{1}{2}\int_0^\pi \cos(6t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^\pi 1\,dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^\pi \cos(6t)\,dt = 0.$$

So:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{\pi}{2}.$$

Multiply by 3:

$$3\int_0^\pi \sin^2(3t)\,dt = 3\cdot\frac{\pi}{2} = \frac{3\pi}{2}.$$

3. Third Term ($-3\int_0^\pi \cos(2t)\,dt$):

Since $\int_0^\pi \cos(2t)\,dt = 0$:

$$-3\int_0^\pi \cos(2t)\,dt = 0.$$

4. Fourth Term ($3\int_0^\pi \cos(4t)\,dt$):

Similarly, $\int_0^\pi \cos(4t)\,dt = 0$:

$$3\int_0^\pi \cos(4t)\,dt = 0.$$

Step 6: Combine Results

Add the evaluated terms:

$$\text{Area} = -\frac{9\pi}{2} + \frac{3\pi}{2} + 0 + 0 = -\frac{6\pi}{2} = -3\pi.$$

However, the area is always positive, so:

$$\text{Area} = 3\pi.$$

Exercise

Solution

...cont'd...

Answer

Area $= 3\pi$

The Arc Length of a Parametric Curve

Theorem

Let a curve be parameterized by t, such that:

$$x = x(t) \quad \text{and} \quad y = y(t), \quad \text{for } t \in [\alpha, \beta].$$

The **arc length** L of the curve between $t = \alpha$ and $t = \beta$ is given by:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Concept

Consider two points (x_1, y_1) and (x_2, y_2) on a curve. The differences in their coordinates are defined as:

$$\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1$$

The distance D between the two points is given by the Pythagorean theorem:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substituting Δx and Δy:

$$D = \sqrt{\Delta x^2 + \Delta y^2}$$

Now, consider the curve parameterized by t, where $x = x(t)$ and $y = y(t)$. Dividing Δx and Δy by the parameter Δt:

$$D \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$$

Taking the limit as $\Delta t \to 0$, this becomes a Riemann sum. Therefore, the arc length L of the curve is:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note

L represents the total length of the curve between $t = \alpha$ and $t = \beta$. This formula is provided on the formula sheet for term test 1.

Illustration

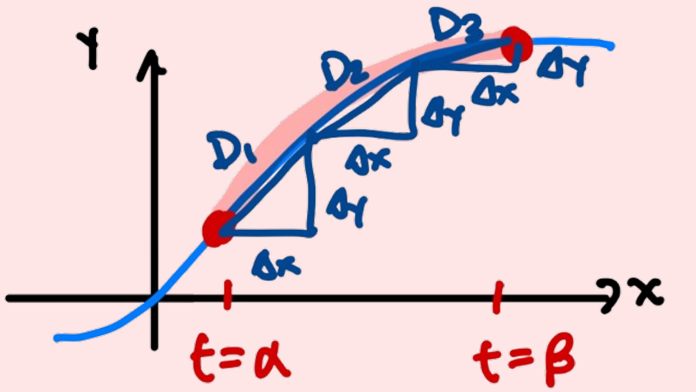

Figure 4: Visualizing the arc length theorem — the connection between Riemann sums and the integral for a curve's length.

Example: Parametric Curve

Example

Find the arc length of the curve defined by:

$$x = 3\cos(t), \quad y = 3\sin(t), \quad t \in [0, 2\pi].$$

Solution

The arc length is denoted by L. We evaluate it as follows:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\,dt \\ &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\,dt. \end{aligned}$$

Using the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, simplify the integrand:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9 \cdot (\sin^2(t) + \cos^2(t))}\,dt \\ &= \int_0^{2\pi} \sqrt{9 \cdot 1}\,dt \\ &= \int_0^{2\pi} 3\,dt. \end{aligned}$$

The integral simplifies to:

$$\begin{aligned} L &= 3\int_0^{2\pi} 1\,dt \\ &= 3\,[t]_0^{2\pi} \\ &= 3 \cdot (2\pi - 0) \\ &= 6\pi. \end{aligned}$$

Answer

The arc length of the curve is $L = 6\pi$.

Homework Practice Problem: Parametric Curve

Note

Find the arc length of the curve defined by:

$$x = 3t^2, \quad y = 2t^3, \quad 1 \leqslant t \leqslant 3.$$

Solution

The arc length is denoted by L, which is evaluated as follows:

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Compute the derivatives:

$$\frac{dx}{dt} = \frac{d}{dt}(3t^2) = 6t, \quad \frac{dy}{dt} = \frac{d}{dt}(2t^3) = 6t^2.$$

Substitute these into the arc length formula:

$$\begin{aligned} L &= \int_1^3 \sqrt{(6t)^2 + (6t^2)^2} dt \\ &= \int_1^3 \sqrt{36t^2 + 36t^4} dt \\ &= \int_1^3 \sqrt{36t^2(1 + t^2)} dt \\ &= \int_1^3 6t\sqrt{1 + t^2} dt. \end{aligned}$$

Use substitution to simplify the integral. Let:

$$u = 1 + t^2, \quad \frac{du}{dt} = 2t, \quad dt = \frac{du}{2t}.$$

Change the limits of integration:

$$\text{When } t = 1,\ u = 1 + 1^2 = 2; \quad \text{When } t = 3,\ u = 1 + 3^2 = 10.$$

Substitute into the integral:

$$\begin{aligned} L &= \int_2^{10} 6t \cdot \sqrt{u} \cdot \frac{du}{2t} \\ &= \int_2^{10} 3\sqrt{u}\, du. \end{aligned}$$

Note

Solution

Evaluate the integral:

$$\begin{aligned} L &= 3\int_2^{10} u^{1/2}\,du \\ &= 3\left[\frac{2}{3}u^{3/2}\right]_2^{10} \\ &= 2\left[u^{3/2}\right]_2^{10} \\ &= 2\left[(10)^{3/2} - (2)^{3/2}\right]. \end{aligned}$$

Simplify the result:

$$L = 2\left[10\sqrt{10} - 2\sqrt{2}\right].$$

Answer

The arc length of the curve is $L = 2\left(10\sqrt{10} - 2\sqrt{2}\right)$.

Section 1.3: Polar Coordinates

Definition

Polar coordinates represent a point in the plane by specifying its distance from the origin (r) and the angle (θ) it makes with the positive x-axis, measured counterclockwise.

Concept

Cartesian Coordinates:

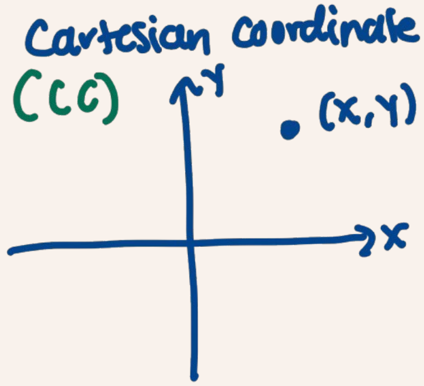

Figure 5: Graphical representation of Cartesian coordinates.

Polar Coordinates:

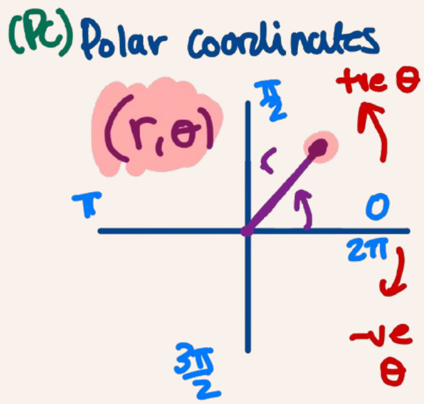

Figure 6: Graphical representation of Polar coordinates.

How to Work with Polar Coordinates

Tip

1. Start from the origin and move radially outward by a distance equal to r.
2. From this position, rotate the point counterclockwise by an angle θ (in radians or degrees).
3. The resultant position is the point represented by the polar coordinate (r, θ).

Converting from Cartesian to Polar Coordinates

Algorithm

1. Given a point (x, y) in Cartesian coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

2. r represents the radial distance from the origin.
3. θ is the angle measured from the positive x-axis to the line segment joining the origin to the point.
4. Adjust θ based on the quadrant of the point to ensure the correct angle.

Converting from Polar to Cartesian Coordinates

Algorithm

1. Given a point (r, θ) in polar coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

2. Compute x and y using trigonometric functions with r and θ.
3. The result (x, y) represents the Cartesian coordinates of the point.

Examples: Converting Between Rectangular and Polar Coordinates

Example

To convert a point from rectangular coordinates (x, y) to polar coordinates (r, θ), we use the following formulas:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, let's convert the following points:

Exercise

1. For the point $(1, 1)$:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

So, the polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Exercise

2. For the point $(-3, 4)$:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5, \quad \theta = \tan^{-1}\left(\frac{4}{-3}\right) = \tan^{-1}\left(-\frac{4}{3}\right)$$

Since the point is in the second quadrant, we add π to the angle:

$$\theta = \pi - \tan^{-1}\left(\frac{4}{3}\right)$$

The polar coordinates are approximately $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$.

Exercise

3. For the point $(0, 3)$:

$$r = \sqrt{0^2 + 3^2} = 3, \quad \theta = \frac{\pi}{2}$$

So, the polar coordinates are $(3, \frac{\pi}{2})$.

Examples: Converting Polar to Rectangular Coordinates

Example

To convert a point from polar coordinates (r, θ) to rectangular coordinates (x, y), we use the following formulas:

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Now, let's convert the following points:

Exercise

1. For the point $(3, \frac{\pi}{3})$:

$$x = 3\cos\left(\frac{\pi}{3}\right) = 3 \times \frac{1}{2} = \frac{3}{2}, \quad y = 3\sin\left(\frac{\pi}{3}\right) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

So, the rectangular coordinates are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

Exercise

2. For the point $(2, \frac{3\pi}{2})$:

$$x = 2\cos\left(\frac{3\pi}{2}\right) = 2 \times 0 = 0, \quad y = 2\sin\left(\frac{3\pi}{2}\right) = 2 \times (-1) = -2$$

So, the rectangular coordinates are $(0, -2)$.

Exercise

3. For the point $(6, -\frac{5\pi}{6})$:

$$x = 6\cos\left(-\frac{5\pi}{6}\right) = 6 \times (-\frac{\sqrt{3}}{2}) = -3\sqrt{3}, \quad y = 6\sin\left(-\frac{5\pi}{6}\right) = 6 \times (-\frac{1}{2}) = -3$$

So, the rectangular coordinates are $(-3\sqrt{3}, -3)$.