

# MAT232 - Lecture 2

Parametric Equations and Calculus: Concepts, Applications, and Derivations

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Prepared for January 9, 2025

## Lecture Title

### Note

This template is designed for MAT232 lecture notes. Replace this content with your specific lecture details.

## Key Concepts

### Definition

A **parametric equation** is a set of equations that express the coordinates of the points of a curve as functions of a variable, called a parameter.

## Examples

### Example

**Example:** Sketch the graph, using a table of values:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

$t$	$1/t$	$x$	$y$
0.01	$\frac{1}{0.01} = \frac{1}{\frac{1}{100}} = 100$	100.01	$0.01 - 100 = -99.99$
0.1	$\frac{1}{0.1} = \frac{1}{\frac{1}{10}} = 10$	10.1	$-9.9$
0.2	$\frac{1}{0.2} = \frac{1}{\frac{20}{100}} = \frac{1}{\frac{2}{10}} = 5$	5.2	4.8
1	$\frac{1}{1}$	2	0
5.0	0.2	5.2	4.8
10	0.1	10.1	9.9
10	0.01	100.01	99.99

This describes a hyperbolic curve.



Figure 1: Sample image illustrating the concept.

**Example**

**Example:** Sketch the graph (this is the same one), using the elimination method:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

$LHS = A^2 - B^2 = (A - B)(A + B) = RHS$   $X = A$  and  $y = B$ .  $LHS : x^2 - y^2$ .  $A - B = x - y = (t + \frac{1}{t}) - (t - \frac{1}{t}) = \frac{2}{t}$ .  $A + B = x + y = (t + \frac{1}{t}) + (t - \frac{1}{t}) = 2t$ .  $RHS : (A - B)(A + B) = (x - y)(x + y) = (\frac{2}{t})(2t) = 4$ . Therefore,  $x^2 - y^2 = 4$ ,  $y \in \mathbb{R}$  will work,  $x > 0$ .

This describes a hyperbolic curve.

## Theorems and Proofs

### Theorem

**Theorem:** If  $x(t)$  and  $y(t)$  are differentiable functions, the slope of the curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$



Figure 2: Graphical representation of the theorem.



## Practice Questions

### Note

Try this question at home!

Sketch and eliminate  $t$  if possible:

$$x = t^2, \quad y = t^3, \quad -2 \leq t \leq 2$$

Note that this is a closed interval. The starting point is the smallest value of  $t$ . This highlights where the graph should begin. The finishing point should be the largest value of  $t$ .

Using an arrow, make sure to indicate the direction of the graph as  $t \rightarrow \infty$ .

### Note

Try another question at home!

Sketch and eliminate  $t$  if possible:

$$c_1 : x = -\cos\left(\frac{t}{4}\right), y = \sin\left(\frac{t}{4}\right), \text{ for } 0 \leq t \leq 4\pi$$

$$c_2 : x = -\sin(t), y = -\cos(t), \text{ for } \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$

$$c_3 : x = \cos(t), y = \sin(t), \text{ for } t \in [0, \pi]$$

Hint:  $x = r\cos(\theta), y = r\sin(\theta), x^2 + y^2 = r^2$ . Also, for these curves, it follows that  $r = 1$ .

## The Elimination Method Does NOT Always Work

### Note

Consider the following case where  $t$  cannot be eliminated:

$$x = e^t - \sin^2(t), \quad y = \ln(t) + \frac{1}{t}, \quad t > 0$$

## Further Visualization



Figure 3: Additional visualization for parametric curves.

## Section 1.2: Calculus on Parametric Equations

### Key Concepts

Recall the concept from 1<sup>st</sup> year calculus:

#### Definition

If  $y = f(x)$  is given, then the slope of the tangent line to the curve of  $y = f(x)$  is:

$$y' = f'(x) = \frac{dy}{dx}$$

Now, for MAT232, we have:

**Definition**

Given  $x = f(t)$ ,  $y = g(t)$ ,  $t \in \mathbb{R}$ , these are differentiable w.r.t. (w.r.t. = “with respect to”)  $t$ . This is such that:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \frac{dx}{dt} \neq 0$$

This will also be provided in the formula sheet.

$$x = f(t), \quad y = g(t), \quad t \in \mathbb{R}$$

Because the chain rule must follow through, always!

*Here is the derivation:* So  $\dots y = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

*Chain rule.*

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}, \quad \text{provided that } \frac{dx}{dt} \neq 0$$

## Second Derivative

### Theorem

Given  $x = f(t), y = g(t), t \in \mathbb{R}$  are differentiable at  $t$  and  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$  exists and is differentiable at  $t$ :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = dx\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

Notice that the expression of the innermost bracket is a derivative all in terms of  $t$ . Thus:

$$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}.$$

This follows from the **inverse function theorem**.

Collectively, it follows that:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

*This is not included on the formula sheet.*

## Examples

### Example

Consider the following parametric curve:

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

- (A) Find the tangent line to the given curve at the point  $(\sqrt{2}, 1)$  where  $t = \frac{\pi}{4}$ .
- (B) Find the vertical tangent(s), if any.
- (C) Find  $\frac{d^2y}{dx^2}$ .

Let's do this, one at a time!

- (A) Find the tangent line to the given curve at the point  $(\sqrt{2}, 1)$  where  $t = \frac{\pi}{4}$ .



**Example**

Tangent Line: Recall...

1.  $y - y_0 = m(x - x_0)$ , where  $m$  is the slope and  $(x_0, y_0)$  is a point on the curve;
2.  $y = mx + b$ , where  $m$  is the slope and  $b$  is the y-intercept.

Given point  $(\sqrt{2}, 1) = (x_0, y_0)$ ,  $\frac{dy}{dt} = \sec^2(t)$ , and  $\frac{dx}{dt} = \sec(t) \tan(t)$ , it follows that:

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2(t) \tan(t)}{\sec(t) \tan(t)} = \frac{\sec(t)}{\tan(t)}$$

Next,  $\frac{dy}{dx} \big|_{t=\frac{\pi}{4}} = \frac{\sec(\frac{\pi}{4})}{\tan(\frac{\pi}{4})} = \frac{\sqrt{2}}{1} = \sqrt{2} = m$ .

**self-note: finish these notes (check the camera roll)**

(B) Find the vertical tangent(s), if any.

## Example

$$\frac{dy}{dt} = \sec^2(t)$$

$$\frac{dx}{dt} = \sec(t) \tan(t)$$

So...

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2(t)}{\sec(t) \tan(t)}$$

Recall from first year calculus:

## Theorem

Given  $y = f(x)$ , it follows that  $y' = f'(x) = 0$ . That is, the roots of  $y' = 0$  indicate the positions of the horizontal tangents.

So...

Horizontal Tangent:  $\frac{dy}{dx} = 0$ ; find  $t$  values.

$$\frac{dy}{dt} = 0, \quad \text{but} \quad \frac{dx}{dt} \neq 0$$

Vertical Tangent:  $\frac{dx}{dx}$  is *undefined*; find  $t$  values.

$$\frac{dx}{dt} = 0, \quad \text{but} \quad \frac{dy}{dt} \neq 0$$

In this case, there is a singular point:

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$

Vertical Tangents:  $\frac{dx}{dt} = 0$ , but  $\frac{dy}{dt} \neq 0$ .

So...

$$\frac{dx}{dt} = \sec(t) \tan(t) = 0, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Notice that

- $\sec(t) = \frac{1}{\cos(t)} = 0$  is impossible as  $1 \neq 0$ ;
- $\tan(t) = 0$  occurs at  $t = 0$ .

Now, check  $\frac{dy}{dt} = 0$  at  $t = 0$ .

$$\frac{dy}{dt} = \sec^2(t) = 0, \quad \text{for } t = 0$$

Is this true?

Therefore, the vertical tangent is at  $t = 0$ .

(C) Find  $\frac{d^2y}{dx^2}$ .

### Example

Recall:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{\sec(t)}{\tan(t)} \quad \text{and} \quad \frac{dx}{dt} = \sec(t) \tan(t)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\sec(t)}{\tan(t)}\right)}{\sec(t) \tan(t)}$$

$$\begin{aligned} \frac{\sec(t)}{\tan(t)} &= \frac{\frac{1}{\cos(t)}}{\cdot} \frac{\sin(t)}{\cos(t)} = \frac{1}{\cos(t)} \left( \frac{\cos(t)}{\sin(t)} \right) \\ &= \frac{1}{\sin(t)} \end{aligned}$$

$$\sec(t) \tan(t) = \frac{1}{\cos(t)} \cdot \frac{\sin(t)}{\cos(t)} = \frac{\sin(t)}{\cos^2(t)}$$

Now, find the derivative of  $y = \frac{1}{\sin(t)}$ :

$$y' = \frac{0 \cdot \sin(t) - \cos(t) \cdot 1}{\sin^2(t)} = -\frac{\cos(t)}{\sin^2(t)}$$

**note to self: finish this off**