MAT232 - Lecture 9

[Lesson Topic(s)]

AlexanderTheMango

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Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Example

If $\overline{r}''(t) = \langle 2t, 3\rangle$, find $\overline{r}'(t)$ with $\overline{r}'(0) = \langle 1, 0\rangle$ and $\overline{r}(0) = \langle -1, 1\rangle$.

Solution

We know that $\overline{r}''(t) = \overline{r}'(t)$. Therefore, we have

$$\begin{aligned}\overline{r}'(t) &= \int \overline{r}''(t)\,\mathrm{d}t \\ &= \int \langle 2t, 3\rangle\,\mathrm{d}t \\ &= \langle t^2 + C_1, 3t + C_2\rangle.\end{aligned}$$

We can now use the initial conditions to solve for C_1 and C_2. We have

$$\begin{aligned}\overline{r}'(0) &= \langle 0 + C_1, 0 + C_2\rangle = \langle 1, 0\rangle \\ \overline{r}(0) &= \langle -1, 1\rangle.\end{aligned}$$

Therefore, we have

$$\begin{aligned}C_1 &= 1 \\ C_2 &= 0.\end{aligned}$$

Therefore, the solution is

$$\overline{r}'(t) = \langle t^2 + 1, 3t\rangle.$$

Answer

The solution is $\overline{r}'(t) = \langle t^2 + 1, 3t\rangle$.

So,

$$\int \overline{r}'(t)dt = \int \langle t^2 + 1, 3t\rangle dt = \langle \frac{t^3}{3} + t + C_3, \frac{3t^2}{2} + C_4\rangle.$$

We have that

$$\overline{r}(t) = \langle \frac{t^3}{3} + t + C_3, \frac{3t^2}{2} + C_4\rangle.$$

We can now use the initial conditions to solve for C_3 and C_4. We have

$$\overline{r}(0) = \langle \frac{0}{3} + 0 + C_3, \frac{3 \cdot 0^2}{2} + C_4\rangle = \langle -1, 1\rangle.$$

Therefore, we have

$$C_3 = -1$$
$$C_4 = 1.$$

Therefore, the solution is

$$\bar{r}(t) = \langle \frac{t^3}{3} + t - 1, \frac{3t^2}{2} + 1 \rangle.$$

Find the Length of a Vector-Valued Function

Definition

Parametric Equation:

$$x = f(t), \quad y = g(t), \quad \alpha \leqslant t \leqslant$$

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t$$

$$L = \int_\alpha^\beta \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,\mathrm{d}t$$

$$\text{Length of the curve} = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t$$

Now, for Vector-Valued Functions

Definition

Vector-Valued Functions:

$$\begin{aligned}\bar{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad \alpha \leqslant t \leqslant \beta \\ &= \langle f(t), g(t), h(t)\rangle \\ &= f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\end{aligned}$$

$$\begin{aligned}\bar{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = \frac{d\bar{r}}{dt} &= \frac{d}{dt}\langle f(t), g(t), h(t)\rangle \\ &= \frac{df}{dt}\hat{\imath} + \frac{dg}{dt}\hat{\jmath} + \frac{dh}{dt}\hat{k}\end{aligned}$$

Thus,

$$\begin{aligned}|| \ \bar{r}'(t) \ || &= \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \\ &= \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2} \\ &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.\end{aligned}$$

So, the length of the curve is

$$L = \int_\alpha^\beta \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, \mathrm{d}t\,.$$

Example

Find the length of the curve defined by the vector-valued function

$$\overline{r}(t) = \langle cos(t), sin(t), t \rangle, \quad 0 \leqslant t \leqslant \pi.$$

$$x = cos(t), \quad y = sin(t), \quad z = t, \quad 0 \leqslant t \leqslant \pi$$

Solution

We have

$$\begin{aligned} \overline{r}'(t) &= \langle -sin(t), cos(t), 1 \rangle \\ ||\, \overline{r}'(t)\, || &= \sqrt{(-sin(t))^2 + (cos(t))^2 + 1^2} \\ &= \sqrt{sin^2(t) + cos^2(t) + 1} \\ &= \sqrt{1+1} \\ &= \sqrt{2}. \end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{2}\, \mathrm{d}t \\ &= \sqrt{2} \int_0^{\pi} \mathrm{d}t \\ &= \sqrt{2}\, [t]_0^{\pi} \\ &= \sqrt{2}\pi. \end{aligned}$$

Answer

The length of the curve is $\sqrt{2}\pi$.

Section 4.1: Functions of Several Variables

Concept

Function of one variable:

$$y = f(x), \quad \text{where } x \text{ is the independent variable and } y \text{ is the dependent variable.}$$

The image/range of the function is the set of all possible values of y as x varies over the domain $\mathbb{R}$ of the function.

Note

These functions are in 2D space.

- $f : x \in A \to y \in B$ is a function from set A to set B.
- The domain of the function is the set of all possible values of x for which the function is defined.
- The range of the function is the set of all possible values of y as x varies over the domain.
- The graph of the function is the set of all points (x, y) in the plane such that $y = f(x)$.
- The level curves of the function are the curves in the plane defined by $f(x, y) = k$, where k is a constant.
- The level surfaces of the function are the surfaces in space defined by $f(x, y, z) = k$, where k is a constant.
- $\mathbb{R} \to \mathbb{R}$.

Now, we will extend this concept to functions of several variables.

Definition

Function of two variables:

$$z = f(x, y), \quad \text{where } x \text{ and } y \text{ are the independent variables and } z \text{ is the dependent variable.}$$

The domain of the function is the set of all possible values of (x, y) for which the function is defined.

Note

These functions are in 3D space.

- $f : (x, y) \in A \to z \in B$ is a function from set A to set B.
- The domain of the function is the set of all possible values of (x, y) for which the function is defined.
- The range of the function is the set of all possible values of z as (x, y) varies over the domain.
- The graph of the function is the set of all points (x, y, z) in space such that $z = f(x, y)$.
- The level curves of the function are the curves in the plane defined by $f(x, y) = k$, where k is a constant.
- The level surfaces of the function are the surfaces in space defined by $f(x, y, z) = k$, where k is a constant.
- $\mathbb{R}^2 \to \mathbb{R}$.

There are even functions of three variables.

Definition

Function of three variables:

$$w = f(x, y, z), \quad \text{where } x, y, \text{ and } z \text{ are the independent variables and } w \text{ is the dependent variable.}$$

The domain of the function is the set of all possible values of (x, y, z) for which the function is defined.

Note

These functions are in 4D space.

- $f : (x, y, z) \in A \to w \in B$ is a function from set A to set B.
- The domain of the function is the set of all possible values of (x, y, z) for which the function is defined.
- The range of the function is the set of all possible values of w as (x, y, z) varies over the domain.
- The graph of the function is the set of all points (x, y, z, w) in space such that $w = f(x, y, z)$.
- The level curves of the function are the curves in space defined by $f(x, y, z) = k$, where k is a constant.
- $\mathbb{R}^3 \to \mathbb{R}$.
- The level surfaces of the function are the surfaces in space defined by $f(x, y, z) = k$, where k is a constant.
- $\mathbb{R}^3 \to \mathbb{R}$.
- The graph of the function is the set of all points (x, y, z, w) in space such that $w = f(x, y, z)$.
- The level curves of the function are the curves in space defined by $f(x, y, z) = k$, where k is a constant.
- The level surfaces of the function are the surfaces in space defined by $f(x, y, z) = k$, where k is a constant.
- $\mathbb{R}^3 \to \mathbb{R}$.

Sketching a Function in 2D

Example

Let $f(x, y) = 3x\sqrt{y} - 1$. Sketch the graph of the function.

Solution

Note

- $x \in \mathbb{R}$
- $\sqrt{y}$ so $y \geqslant 0$

Consider some points in the domain of the function. We have

$$\begin{aligned} x, y &= f(x, y) \\ f(1, 4) &= 3 \cdot 1 \cdot 2 - 1 = 5 = z \\ f(0, 9) &= 3 \cdot 0 \cdot 3 - 1 = -1 = z \\ f(t, t^2) &= 3 \cdot t \cdot \sqrt{t^2} - 1 = 3t \cdot t - 1 = 3t^2 - 1 = z. \end{aligned}$$

Note

$\sqrt{t^2} = \mid t \mid$.

Therefore, the function is defined for all (x, y) in the domain.
To sketch the graph of the function, we can sketch the level curves of the function. We have

$$f(x, y) = 3x\sqrt{y} - 1 = k.$$

We can now sketch the level curves of the function by setting $k = 0$. We have

$$3x\sqrt{y} - 1 = 0.$$

Therefore, the level curve is

$$3x\sqrt{y} = 1.$$

Sketching the Domain of a Function in 2D

Example

Sketch the domain of $f(x,y) = \ln\big(x^2 - y\big)$.

Solution

Note

- $\ln\big(x^2 - y\big)$ implies that $x^2 - y > 0$ meaning that $x^2 > y$.
- $x^2 - y > 0$ so $x^2 > y$.
- $x^2 - y$ is in the domain of $\ln\big(x^2 - y\big)$ so $x^2 - y > 0$.
- $x^2 - y > 0$ so $y < x^2$.
- The domain of the function is $y < x^2$.

Therefore, the domain of the function is $y < x^2$. Pick some points in the domain of the function. We have

$$
\begin{aligned}
f(2,0) &= \ln\big(2^2 - 0\big) = \ln(4) \\
f(0,2) &= \ln\big(0^2 - 2\big) = \ln(-2).
\end{aligned}
$$

Notice that $f(0,2)$ is not defined because $\ln(-2)$ is not defined.

Illustration

Answer

The domain of the function is $y < x^2$.

Describing the Domain of a Function in 3D

Example

Describe the domain of the function $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$.

Solution

Note

- $\sqrt{1 - x^2 - y^2 - z^2}$ is defined if $1 - x^2 - y^2 - z^2 \geqslant 0$.
- $1 - x^2 - y^2 - z^2 \geqslant 0$ so $1 \geqslant x^2 + y^2 + z^2$.
- The domain of the function is $x^2 + y^2 + z^2 \leqslant 1$.
- The domain of the function is the unit ball.
- The domain of the function is the set of all points (x, y, z) such that $x^2 + y^2 + z^2 \leqslant 1$.
- The domain of the function is the unit ball.

Notice that

$$\begin{aligned}\sqrt{1 - x^2 - y^2 - z^2} &> 0\\ 1 - x^2 - y^2 - z^2 &> 0\\ 1 &> x^2 + y^2 + z^2.\end{aligned}$$

This means, any point (x, y, z) on the edge of the unit ball/sphere centered at the origin (0, 0, 0) with radius 1 and inside the unit ball is in the domain of the function.

$$\begin{aligned}f(0, \frac{1}{2}, -\frac{1}{2}) &= \sqrt{1 - 0 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2}\\ &= \sqrt{1 - \frac{1}{4} - \frac{1}{4}}\\ &= \sqrt{1 - \frac{1}{2}}\\ &= \sqrt{\frac{1}{2}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}.\end{aligned}$$

Therefore, the domain of the function is $x^2 + y^2 + z^2 \leqslant 1$.

Answer

The domain of the function is $x^2 + y^2 + z^2 \leqslant 1$.