MAT232 - Lecture 11

[Lesson Topic(s)]

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Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Review of Last Lecture

(stuff goes here)

Section 4.4: Tangent Planes

Recall from 1st Year Calculus

Definition

Tangent lines are denoted by:

$$y = f(x) \quad \text{at} \quad x = x_0 \text{ (given)}$$

- Point $P = (x_0, f(x_0)) = (x_0, y_0)$
- Slope of tangent line: $m = f'(x_0)$
  - So at $x = x_0$, the slope of the tangent line is $f'(x_0) = m$

$$y - y_0 = m(x - x_0)$$

$$y = f'(x_0)(x - x_0) + f(x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Remark

Note: The tangent line is a linear approximation of the function $f(x)$ near $x = x_0$.
We will not be using this formula in this course, but it is good to know.

Now, in MAT232

Definition

Plane Equation:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

is the equation of a plane in $\mathbb{R}^3$ in **point-normal or scalar form**. Rearrange and notice:

$$z-z_0=-\frac{a}{c}(x-x_0)-\frac{b}{c}(y-y_0)$$

where $z=f(x,y)$ and $z_0=f(x_0,y_0)$.
So, $z_0=f(x_0,y_0)$ is the point on the surface of the function $z=f(x,y)$ at (x_0,y_0), and $-\frac{a}{c}=f_x(x_0,y_0)$ and $-\frac{b}{c}=f_y(x_0,y_0)$ are the partial derivatives of $f(x,y)$ at (x_0,y_0). This form is called the **tangent plane** to the surface of the function $z=f(x,y)$ at (x_0,y_0).

Note

This equation will be included on the formula sheet.

Concept

Set $y=y_0$:

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+\cancelto{0}{f_y(x_0,y_0)(y-y_0)}$$

Call this T_1, the tangent plane to the surface of the function $z=f(x,y)$ at (x_0,y_0) when $y=y_0$. Set $x=x_0$:

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$z=f(x_0,y_0)+\cancelto{0}{f_x(x_0,y_0)(x-x_0)}+f_y(x_0,y_0)(y-y_0)$$

Call this T_2, the tangent plane to the surface of the function $z=f(x,y)$ at (x_0,y_0) when $x=x_0$.

Let's Try an Example

Example

Find the equation of the tangent plane to the surface $z = f(x, y) = \ln(x - 2y)$ at the point $(x_0, y_0) = (3, 1)$.

Solution

Tangent plane equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- Point: $(x_0, y_0) = (3, 1)$

So...

$$\begin{aligned} z_0 = f(x_0, y_0) &= \ln(3 - 2(1)) = \ln(3 - 2) = \ln(1) = 0 \\ f_x(x, y) &= \frac{1}{x - 2y} \\ f_y(x, y) &= \frac{-2}{x - 2y} \end{aligned}$$

- Partial derivatives at $(x_0, y_0) = (3, 1)$:

$$\begin{aligned} f_x(3, 1) &= \frac{1}{3 - 2(1)} = \frac{1}{3 - 2} = 1 \\ f_y(3, 1) &= \frac{-2}{3 - 2(1)} = \frac{-2}{3 - 2} = -2 \end{aligned}$$

So, the equation of the tangent plane is:

$$\begin{aligned} z &= 0 + 1(x - 3) - 2(y - 1) \\ z &= x - 3 - 2y + 2 \\ z &= x - 2y - 1 \end{aligned}$$

Answer

The equation of the tangent plane to the surface $z = f(x, y) = \ln(x - 2y)$ at the point $(x_0, y_0) = (3, 1)$ is $z = x - 2y - 1$.

Another Example

Example

Find the equation of the tangent plane to the surface $z = f(x, y) = x^2 + y^2 + 1$ at the point $(x_0, y_0) = (2, 1)$.

Solution

Tangent plane equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- Point: $(x_0, y_0) = (2, 1)$

Partial derivatives:

$$f_x(x, y) = 2x$$
$$f_y(x, y) = 2y$$

Point:

$$z_0 = f(x_0, y_0) = 2^2 + 1^2 + 1 = 4 + 1 + 1 = 6$$

Partial derivatives at $(x_0, y_0) = (2, 1)$:

$$f_x(2, 1) = 2(2) = 4$$
$$f_y(2, 1) = 2(1) = 2$$

So, the equation of the tangent plane is:

$$z = 6 + 4(x - 2) + 2(y - 1)$$
$$z = 6 + 4x - 8 + 2y - 2$$
$$z = 4x + 2y - 4$$

Answer

The equation of the tangent plane to the surface $z = f(x, y) = x^2 + y^2 + 1$ at the point $(x_0, y_0) = (2, 1)$ is $z = 4x + 2y - 4$.

Section 4.5: Chain Rule

Recall from 1st Year Calculus

Let $y = f(u)$ and $u = g(x)$. Then, $y = f(g(x))$.

- Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example

Let $f(x) = \cos(x)$ and $g(x) = e^x$. Find $h(x) = f(g(x))$ and $h'(x)$.

Solution

Notice that $h(x) = f(g(x)) = \cos(e^x)$. So $h'(x) = -\sin(e^x) \cdot e^x$.

Answer

$h(x) = \cos(e^x)$ and $h'(x) = -\sin(e^x) \cdot e^x$.

Now, in MAT232

Definition

Given $w = f(x, y)$, $x = h(t)$, $y = g(t)$, and that they are all differentiable functions, then $w = f(x, y) = f(h(t), g(t))$.

Concept

Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Note

This is the chain rule in MAT232.

Let's illustrate how $w = f(x, y)$ breaks down:

$$w = f(x, y) = f(h(t), g(t))$$

Note that if $w = f(x, y)$ depends on x and y, and x and y depend on t, then w depends on t.

$$
\begin{aligned}
w &= f(x,y) = f(h(t), g(t)) \\
\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial x} = f_x(h(t), g(t)) \cdot h'(t) \\
\frac{\partial w}{\partial y} &= \frac{\partial f}{\partial y} = f_y(h(t), g(t)) \cdot g'(t)
\end{aligned}
$$

Example

Find the derivative of $w = f(x, y) = xy$ with respect to t if $x = \cos(t)$ and $y = \sin(t)$ at $t = \frac{\pi}{2}$.
Approach 1 (Direct Substitution):

Solution

$$\begin{aligned} w &= f(x, y) = xy \\ &= \cos(t) \cdot \sin(t) \\ &= \frac{1}{2}\sin(2t) \end{aligned}$$

Answer

The derivative of $w = f(x, y) = xy$ with respect to t if $x = \cos(t)$ and $y = \sin(t)$ at $t = \frac{\pi}{2}$ is $\cos(2t)$.

Approach 1 (Substitute $t = \frac{\pi}{2}$):

Solution

$$\begin{aligned} w &= f(x, y) = xy \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\mathrm{d}w}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}w}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= y \cdot (-\sin(t)) + x \cdot \cos(t) \\ &= \sin(t) \cdot (-\sin(t)) + \cos(t) \cdot \cos(t) \\ &= \sin\left(\frac{\pi}{2}\right) \cdot (-\sin\left(\frac{\pi}{2}\right)) + \cos\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\right) \\ &= 1 \cdot (-1) + 0 \cdot 0 = -1 \end{aligned}$$

Answer

The derivative of $w = f(x, y) = xy$ with respect to t if $x = \cos(t)$ and $y = \sin(t)$ at $t = \frac{\pi}{2}$ is -1.