MAT232 - Lecture 6

vectors?

AlexanderTheMango

Prepared for January 23, 2025

Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Introduction to Vectors

Definition

A **vector** is a quantity that has both magnitude and direction. Vectors can be optionally denoted in multiple ways:

- **Boldface Notation**: $\mathbf{v}$
- **Arrow Notation**: $\vec{v}$
- **Overline Notation**: $\overline{v}$

Note

In MAT232H5, the contents of a vector are typically written using angle bracket notation:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

For example, a 3D vector can be represented as:

$$\vec{v} = \langle 2, -1, 3 \rangle$$

Depending on the context, you might see $\mathbf{v} = \langle v_1, v_2 \rangle$ in 2D or $\mathbf{v} = \langle v_1, v_2, v_3, v_4 \rangle$ in higher dimensions.

Remark

Quantities such as velocity and force are examples of vectors because they require both magnitude and direction to be fully described.

Vector Representation

A **vector** in a plane is represented by a directed line segment (an arrow) with an **initial point** and a **terminal point**. The length of the segment represents its **magnitude**, denoted $\|\vec{v}\|$. A vector with the same initial and terminal point is called the **zero vector**, denoted $\vec{0}$.
Two vectors $\vec{v}$ and $\vec{w}$ are **equivalent** if they have the same magnitude and direction, written as $\vec{v} = \vec{w}$.

Exercise

Sketching Vectors
Sketch a vector in the plane from initial point $P(1, 1)$ to terminal point $Q(8, 5)$.

Basic Vector Operations

Scalar Multiplication

Multiplying a vector $\vec{v}$ by a scalar k results in a new vector $k\vec{v}$ with the following properties:

- Its magnitude is $|k|$ times the magnitude of $\vec{v}$.
- Its direction remains the same if $k > 0$.
- Its direction is reversed if $k < 0$.
- If $k = 0$ or $\vec{v} = \vec{0}$, then $k\vec{v} = \vec{0}$.

Note

The zero vector $\vec{0}$ is the vector with a magnitude of 0 and no direction (or any direction). It is the only vector that is orthogonal (perpendicular) to every vector, including itself.

Exercise

Scalar Multiplication
Given vector $\vec{v}$, sketch the vectors $3\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\vec{v}$.

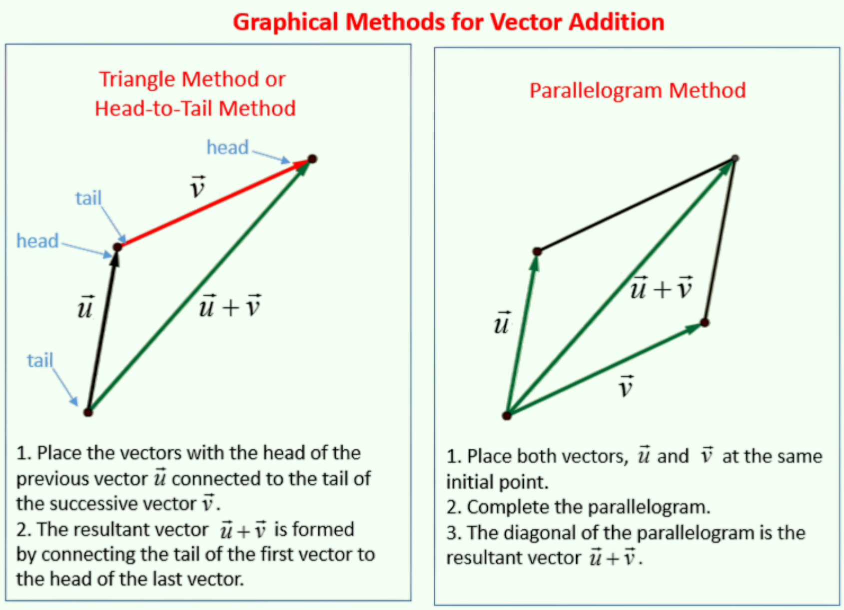
Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ is constructed by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. The vector sum, $\vec{v} + \vec{w}$, is the vector from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.

Exercise

Vector Addition

Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v}+\vec{w}$ using both the triangle method and the parallelogram method.



Vector Subtraction

The difference $\vec{v}-\vec{w}$ is defined as $\vec{v}+(-\vec{w})$, where $-\vec{w}$ is the vector with the same magnitude as $\vec{w}$ but opposite direction.

Exercise

Vector Subtraction

Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v}-\vec{w}$.

Vector Components

A vector in standard position has its initial point at the origin $(0,0)$. If the terminal point is (x,y), the vector is written in **component form** as $\vec{v}=\langle x,y\rangle$. The scalars x and y are called the **components** of $\vec{v}$.

Exercise

Expressing Vectors in Component Form

Express vector $\vec{v}$ with initial point $(-3,4)$ and terminal point $(1,2)$ in component form.

Magnitude of a Vector

Definition

The magnitude of a vector $\vec{v} = \langle x, y \rangle$ is its length, and is given by:

$$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

Exercise

Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

Properties of Vector Operations

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let k and c be scalars. Then:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property)
3. $k(c\vec{v}) = (kc)\vec{v}$ (Associativity of Scalar Multiplication)
4. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (Distributive Property)

Proof

Proof of Commutative Property:
Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$. Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \vec{v} + \vec{u}.$$

Applications of Vectors

Example

Real-Life Applications

- A boat crossing a river experiences a force from its motor and a force from the river current. Both forces are vectors.
- A quarterback throwing a football applies a velocity vector to the ball, determining its speed and direction.

Learning Objectives

By the end of this section, you should be able to:

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Introduction to Three-Dimensional Space

Definition

Definition: The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis, with an origin at the point of intersection $(0, 0, 0)$. This system is often denoted by $\mathbb{R}^3$.

Note

Note: The three-dimensional coordinate system follows the **right-hand rule**. If you align your right hand's fingers with the positive x-axis and curl them toward the positive y-axis, your thumb points in the direction of the positive z-axis.

Locating Points in Space

Definition

Definition: A point in three-dimensional space is represented by coordinates (x, y, z), where:

- x is the distance along the x-axis,
- y is the distance along the y-axis,
- z is the distance along the z-axis.

Example

Example 2.11: Locating Points in Space
Sketch the point $(1, -2, 3)$ in three-dimensional space.

Exercise

Checkpoint 2.11:
Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Coordinate Planes in $\mathbb{R}^3$

Definition

Definition: The three coordinate planes in $\mathbb{R}^3$ are:

- The xy-plane: $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$,
- The xz-plane: $\{(x, 0, z) \mid x, z \in \mathbb{R}\}$,
- The yz-plane: $\{(0, y, z) \mid y, z \in \mathbb{R}\}$.

Note

Note: The coordinate planes divide space into eight regions called **octants**. The first octant is where $x > 0$, $y > 0$, and $z > 0$.

Distance Formula in Three Dimensions

Theorem

Theorem 2.2: Distance Between Two Points in Space
The distance d between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

Example 2.12: Distance in Space
Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

Exercise

Checkpoint 2.12:
Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Equations of Planes

Definition

Definition: A plane parallel to one of the coordinate planes can be described by:

- $z = c$ for a plane parallel to the xy-plane,
- $y = b$ for a plane parallel to the xz-plane,
- $x = a$ for a plane parallel to the yz-plane.

Example

Example 2.13: Writing Equations of Planes

- Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
- Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$.

Exercise

Checkpoint 2.13:
Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Equations of Spheres

Definition

Definition: A **sphere** is the set of all points in space equidistant from a fixed point, called the **center**. The distance from the center to any point on the sphere is called the **radius**.

Theorem

Equation of a Sphere:
The sphere with center (a, b, c) and radius r is given by:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example

Example 2.14: Finding an Equation of a Sphere
Find the standard equation of the sphere with center $(10, 7, 4)$ and passing through point $(-1, 3, -2)$.

Exercise

Checkpoint 2.14:
Find the standard equation of the sphere with center $(-2, 4, -5)$ and passing through point $(4, 4, -1)$.

Example

Example 2.15: Finding the Equation of a Sphere
Let $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$, and suppose line segment PQ forms the diameter of a sphere. Find the equation of the sphere.

Exercise

Checkpoint 2.15:
Find the equation of the sphere with diameter PQ, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

Graphing Equations in Three Dimensions

Example

Example 2.16: Graphing Other Equations
Describe the set of points that satisfies $(x - 4)(z - 2) = 0$, and graph the set.

Exercise

Checkpoint 2.16:
Describe the set of points that satisfies $(y + 2)(z - 3) = 0$, and graph the set.

Example

Example 2.17: Graphing Other Equations
Describe the set of points in three-dimensional space that satisfies $(x - 2)^2 + (y - 1)^2 = 4$, and graph the set.

Exercise

Checkpoint 2.17:
Describe the set of points in three-dimensional space that satisfies $x^2 + (z - 2)^2 = 16$, and graph the surface.

self-note: add the rest from deepseek alexandermenginquiries@gmail.com account

Let’s Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let’s break this down step by step.