MAT232 - Lecture 6

vectors?

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Contents

Definitions and Theorems

Straight from the textbook — a thorough coverage.

Quick recap before diving into the lecture.

Introduction to Vectors

Definition

A **vector** is a quantity that has both magnitude and direction. Vectors can be optionally denoted in multiple ways:

- **Boldface Notation**: $\mathbf{v}$
- **Arrow Notation**: $\vec{v}$
- **Overline Notation**: $\overline{v}$

Note

In MAT232H5, the contents of a vector are typically written using angle bracket notation:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

For example, a 3D vector can be represented as:

$$\vec{v} = \langle 2, -1, 3 \rangle$$

Depending on the context, you might see $\mathbf{v} = \langle v_1, v_2 \rangle$ in 2D or $\mathbf{v} = \langle v_1, v_2, v_3, v_4 \rangle$ in higher dimensions.

Remark

Quantities such as velocity and force are examples of vectors because they require both magnitude and direction to be fully described.

Vector Representation

A **vector** in a plane is represented by a directed line segment (an arrow) with an **initial point** and a **terminal point**. The length of the segment represents its **magnitude**, denoted $\|\vec{v}\|$. A vector with the same initial and terminal point is called the **zero vector**, denoted $\vec{0}$.
Two vectors $\vec{v}$ and $\vec{w}$ are **equivalent** if they have the same magnitude and direction, written as $\vec{v} = \vec{w}$.

Exercise

Sketching Vectors
Sketch a vector in the plane from initial point $P(1, 1)$ to terminal point $Q(8, 5)$.

Basic Vector Operations

Scalar Multiplication

Multiplying a vector $\vec{v}$ by a scalar k results in a new vector $k\vec{v}$ with the following properties:

· Its magnitude is $|k|$ times the magnitude of $\vec{v}$.

· Its direction remains the same if $k > 0$.

· Its direction is reversed if $k < 0$.

· If $k = 0$ or $\vec{v} = \vec{0}$, then $k\vec{v} = \vec{0}$.

Note

The zero vector $\vec{0}$ is the vector with a magnitude of 0 and no direction (or any direction). It is the only vector that is orthogonal (perpendicular) to every vector, including itself.

Exercise

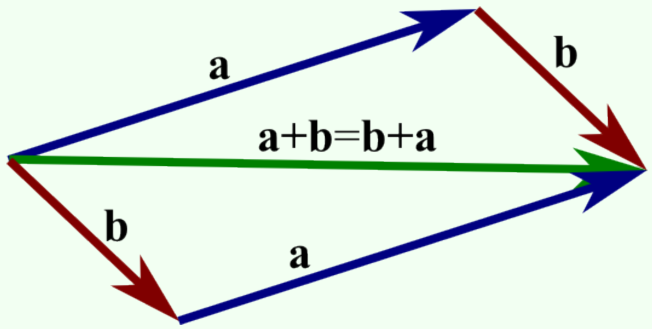
Scalar Multiplication
Given vector $\vec{v}$, sketch the vectors $3\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\vec{v}$.

Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ is constructed by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. The vector sum, $\vec{v} + \vec{w}$, is the vector from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.

Exercise

Vector Addition
Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v} + \vec{w}$ using both the triangle method and the parallelogram method.



Vector Subtraction

The difference $\vec{v} - \vec{w}$ is defined as $\vec{v} + (-\vec{w})$, where $-\vec{w}$ is the vector with the same magnitude as $\vec{w}$ but opposite direction.

> Exercise
>
> **Vector Subtraction**
> Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v} - \vec{w}$.

Vector Components

A vector in standard position has its initial point at the origin $(0, 0)$. If the terminal point is (x, y), the vector is written in **component form** as $\vec{v} = \langle x, y \rangle$. The scalars x and y are called the **components** of $\vec{v}$.

> Exercise
>
> **Expressing Vectors in Component Form**
> Express vector $\vec{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Magnitude of a Vector

> Definition
>
> The magnitude of a vector $\vec{v} = \langle x, y \rangle$ is its length, and is given by:
>
> $$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

> Exercise
>
> Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

Properties of Vector Operations

> Theorem
>
> Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let k and c be scalars. Then:
>
> 1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property)
> 2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property)
> 3. $k(c\vec{v}) = (kc)\vec{v}$ (Associativity of Scalar Multiplication)
> 4. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (Distributive Property)

Proof

Proof of Commutative Property:
Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$. Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \vec{v} + \vec{u}.$$

Applications of Vectors

Example

Real-Life Applications

- A boat crossing a river experiences a force from its motor and a force from the river current. Both forces are vectors.
- A quarterback throwing a football applies a velocity vector to the ball, determining its speed and direction.

Introduction to Three-Dimensional Space

The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis, with an origin at the point of intersection $(0, 0, 0)$. This system is often denoted by $\mathbb{R}^3$.

Tip

The three-dimensional coordinate system follows the **right-hand rule**. If you align your right hand's fingers with the positive x-axis and curl them toward the positive y-axis, your thumb points in the direction of the positive z-axis.

Remark

This can also be visualized by holding a screwdriver with your right hand. If you rotate the screwdriver from the positive x-axis to the positive y-axis, the direction of the screwdriver represents the positive z-axis.

Note

The right-hand rule can serve as a visual aid for determining the direction of the cross product of two vectors.

Locating Points in Space

A point in three-dimensional space is represented by coordinates (x, y, z), where:

- x is the distance along the x-axis,
- y is the distance along the y-axis,
- z is the distance along the z-axis.

Exercise

Sketch the points $(-2, 3, -1)$ and $(1, -2, 3)$ in three-dimensional space.

Coordinate Planes in $\mathbb{R}^3$

The three coordinate planes in $\mathbb{R}^3$ are:

- The xy-plane: $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$,
- The xz-plane: $\{(x, 0, z) \mid x, z \in \mathbb{R}\}$,
- The yz-plane: $\{(0, y, z) \mid y, z \in \mathbb{R}\}$.

Note

The coordinate planes divide space into eight regions called **octants**. The first octant is where $x > 0$, $y > 0$, and $z > 0$; the other octants are numbered counterclockwise. It's like quadrants in 2D, but with that extra dimension!

Distance Formula in Three Dimensions

Theorem

The distance d between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Exercise

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Equations of Planes

A plane parallel to one of the coordinate planes can be described by:

- $z = c$ for a plane parallel to the xy-plane,

- $y = b$ for a plane parallel to the xz-plane,
- $x = a$ for a plane parallel to the yz-plane.

Exercise

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Equations of Spheres

Definition

A **sphere** is the shape described by the set of all points in space equidistant from a fixed point, called the **centre**. The distance from the centre to any point on the sphere is called the **radius**.

Theorem

Equation of a Sphere:
The sphere with centre (a, b, c) and radius r is given by:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Exercise

Find the standard equation of the sphere with center $(-2, 4, -5)$ and passing through point $(4, 4, -1)$.

Exercise

Find the equation of the sphere with diameter PQ, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

Graphing Equations in Three Dimensions

Exercise

Describe the set of points that satisfies $(y + 2)(z - 3) = 0$, and graph the set.

Exercise

Describe the set of points in three-dimensional space that satisfies $x^2 + (z - 2)^2 = 16$, and graph the surface.

Working with Vectors in $\mathbb{R}^3$

Definition

A **three-dimensional vector** is a quantity with both magnitude and direction, represented by a directed line segment (arrow) in $\mathbb{R}^3$. A vector $\vec{v} = \langle x, y, z \rangle$ has its initial point at the origin $(0, 0, 0)$ and its terminal point at (x, y, z). The zero vector is $\vec{0} = \langle 0, 0, 0 \rangle$.

Exercise

Checkpoint 2.18:
Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\overrightarrow{ST}$ in component form and in standard unit form.

Vector Operations in $\mathbb{R}^3$

Definition

Let $\vec{v} = \langle x_1, y_1, z_1 \rangle$ and $\vec{w} = \langle x_2, y_2, z_2 \rangle$ be vectors in $\mathbb{R}^3$, and let k be a scalar. Then:

- **Vector Addition:** $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$
- **Scalar Multiplication:** $k\vec{v} = \langle kx_1, ky_1, kz_1 \rangle$
- **Vector Subtraction:** $\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$
- **Magnitude:** $\|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$
- **Unit Vector:** The unit vector in the direction of $\vec{v}$ is $\frac{1}{\|\vec{v}\|}\vec{v}$, provided $\vec{v} \neq \vec{0}$.

Exercise

Vector Operations in Three Dimensions
Let $\vec{v} = \langle -2, 9, 5 \rangle$ and $\vec{w} = \langle 1, -1, 0 \rangle$. Find the following vectors:

- $3\vec{v} - 2\vec{w}$
- $5\|\vec{w}\|$
- $\|5\vec{w}\|$
- A unit vector in the direction of $\vec{v}$

Exercise

Let $\vec{v} = \langle -1, -1, 1 \rangle$ and $\vec{w} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\vec{v} + 3\vec{w}$.

Properties of Vectors in $\mathbb{R}^3$

Theorem

Properties of Vectors in Space:
Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$, and let k and c be scalars. Then:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property)
3. $\vec{u} + \vec{0} = \vec{u}$ (Additive Identity Property)
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse Property)
5. $k(c\vec{v}) = (kc)\vec{v}$ (Associativity of Scalar Multiplication)
6. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (Distributive Property)
7. $(k + c)\vec{u} = k\vec{u} + c\vec{u}$ (Distributive Property)
8. $1\vec{u} = \vec{u}$ and $0\vec{u} = \vec{0}$ (Identity and Zero Properties)

Proof

Proof of Commutative Property:
Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle = \vec{v} + \vec{u}.$$

□

self-note: embellish the following

The Dot Product and Its Properties

Definition

Definition: The **dot product** of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

For two-dimensional vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, the dot product is:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$

Example

Example 2.21: Calculating Dot Products

- Find the dot product of $\vec{u} = \langle 3, 5, 2\rangle$ and $\vec{v} = \langle -1, 3, 0\rangle$.
- Find the scalar product of $\vec{p} = 10\vec{i} - 4\vec{j} + 7\vec{k}$ and $\vec{q} = -2\vec{i} + \vec{j} + 6\vec{k}$.

Exercise

Checkpoint 2.21:
Find $\vec{u} \cdot \vec{v}$, where $\vec{u} = \langle 2, 9, -1\rangle$ and $\vec{v} = \langle -3, 1, -4\rangle$.

Properties of the Dot Product

Theorem

Theorem 2.3: Properties of the Dot Product
Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let c be a scalar. Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutative Property)
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributive Property)
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ (Associative Property)
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ (Property of Magnitude)

Proof

Proof of Commutative Property:
Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$. Then:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v} \cdot \vec{u}.$$

$\square$

Example

Example 2.22: Using Properties of the Dot Product

Let $\vec{a} = \langle 1, 2, -3 \rangle$, $\vec{b} = \langle 0, 2, 4 \rangle$, and $\vec{c} = \langle 5, -1, 3 \rangle$. Find each of the following products:

- $(\vec{a} \cdot \vec{b})\vec{c}$
- $\vec{a} \cdot (2\vec{c})$
- $\|\vec{b}\|^2$

Exercise

Checkpoint 2.22:

Find the following products for $\vec{p} = \langle 7, 0, 2 \rangle$, $\vec{q} = \langle -2, 2, -2 \rangle$, and $\vec{r} = \langle 0, 2, -3 \rangle$:

- $(\vec{r} \cdot \vec{p})\vec{q}$
- $\|\vec{p}\|^2$

Using the Dot Product to Find the Angle Between Two Vectors

Theorem

Theorem 2.4: Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos \theta.$$

Example

Example 2.23: Finding the Angle Between Two Vectors

Find the measure of the angle between each pair of vectors:

- $\vec{i} + \vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} - 3\vec{k}$
- $\langle 2, 5, 6 \rangle$ and $\langle -2, -4, 4 \rangle$

Exercise

Checkpoint 2.23:

Find the measure of the angle, in radians, formed by vectors $\vec{a} = \langle 1, 2, 0 \rangle$ and $\vec{b} = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

Orthogonal Vectors

Theorem

Theorem 2.5: Orthogonal Vectors
The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

Example

Example 2.24: Identifying Orthogonal Vectors
Determine whether $\vec{p} = \langle 1, 0, 5 \rangle$ and $\vec{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Exercise

Checkpoint 2.24:
For which value of x is $\vec{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{q} = \langle x, -1, 2 \rangle$?

Direction Cosines

Definition

Definition: The **direction angles** of a nonzero vector $\vec{v}$ are the angles α, β, and γ that $\vec{v}$ makes with the positive x-, y-, and z-axes, respectively. The cosines of these angles are called the **direction cosines**:

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_2}{\|\vec{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\vec{v}\|}.$$

Example

Example 2.25: Measuring the Angle Formed by Two Vectors
Let $\vec{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by:

- $\vec{v}$ and $\vec{i}$
- $\vec{v}$ and $\vec{j}$
- $\vec{v}$ and $\vec{k}$

Exercise

Checkpoint 2.25:
Let $\vec{v} = \langle 3, -5, 1\rangle$. Find the measure of the angles formed by:

- $\vec{v}$ and $\vec{i}$
- $\vec{v}$ and $\vec{j}$
- $\vec{v}$ and $\vec{k}$

Vector Projections

Definition

Definition: The **vector projection** of $\vec{v}$ onto $\vec{u}$ is:

$$\mathrm{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}.$$

The **scalar projection** of $\vec{v}$ onto $\vec{u}$ is:

$$\mathrm{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

Example

Example 2.27: Finding Projections
Find the projection of $\vec{v}$ onto $\vec{u}$:

- $\vec{v} = \langle 3, 5, 1\rangle$ and $\vec{u} = \langle -1, 4, 3\rangle$
- $\vec{v} = 3\vec{i} - 2\vec{j}$ and $\vec{u} = \vec{i} + 6\vec{j}$

Example

Example 2.28: Resolving Vectors into Components
Express $\vec{v} = \langle 8, -3, -3\rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = \langle 2, 3, 2\rangle$.

Exercise

Checkpoint 2.27:
Express $\vec{v} = 5\vec{i} - \vec{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = 4\vec{i} + 2\vec{j}$.

Work

Definition

Definition: When a constant force $\vec{F}$ is applied to an object so the object moves in a straight line from point P to point Q, the **work** W done by the force is:

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\| \|\overrightarrow{PQ}\| \cos\theta,$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

Example

Example 2.30: Calculating Work
A conveyor belt generates a force $\vec{F} = 5\vec{i} - 3\vec{j} + \vec{k}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters, and the force is measured in newtons.

Exercise

Checkpoint 2.29:
A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?

self-note: continue from here with section 2.4 in the textbook

Let’s Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let’s break this down step by step.

Previous Lecture Recap

Recall the Circle, Ellipse, Parabola, and Hyperbola

Recall the four shapes (and their equations) derived from taking cross sections of double-naped cones:

- Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Circle: $x^2 + y^2 = r^2$
- Parabola: $y = ax^2 + bx + c = A(x + B)^2 + C, \quad a, A \neq 0$
- Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Graph Differences in 2D and 3D

- In 2D, a graph is a curve.
- In 3D, a graph is a surface.

Concept

Equation	2D Interpretation	3D Interpretation
$y = 2$	Horizontal line	Plane
$x^2 + y^2 = 1$	Circle	Cylinder

Table 1: Comparison of 2D and 3D interpretations of equations

Circle vs Sphere

In 2D, a **circle** is given by the equation $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius. A **sphere** extends this concept to 3D, with the equation $(x-h)^2+(y-k)^2+(z-l)^2=r^2$. Here, (h,k,l) is the center and r is the radius.

Illustration

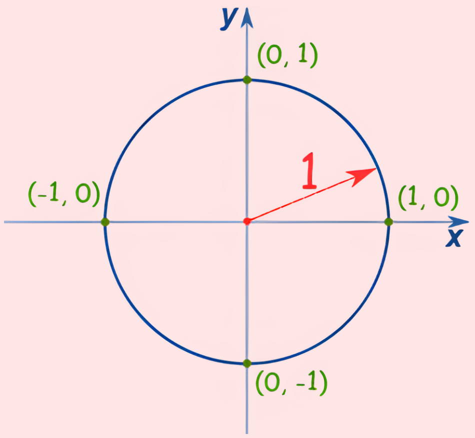


Figure 1: Unit Circle

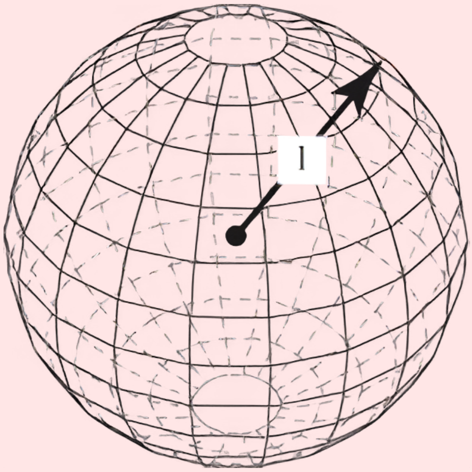


Figure 2: Unit Sphere

Example

The equation of a sphere centered at $(2,2,1)$ with radius $r=3$ is:

$$(x-2)^2+(y-2)^2+(z-1)^2=9$$

Example

The equation of a sphere centered at $(-1,0,-4)$ with radius $r=\sqrt{5}$ is:

$$(x+1)^2+y^2+(z+4)^2=5$$

Exercise

Given the equation:

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

Find the center and radius of the sphere.

Describing the Region Between Two Spheres

Example

Describe the following region in 3D space:

$$2 \leqslant x^2 + y^2 + z^2 < 5$$

Solution

Consider the two spheres:

- Sphere 1: $x^2 + y^2 + z^2 = 5$
- Sphere 2: $x^2 + y^2 + z^2 = 2$

The region is the set of points that lie inside the sphere of radius $\sqrt{5}$ (exclusive) and outside the sphere of radius $\sqrt{2}$ (inclusive).

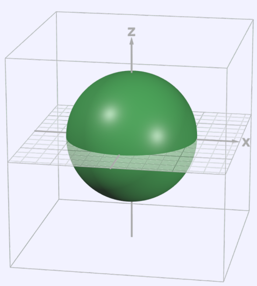


Figure 3: Region Between Spheres

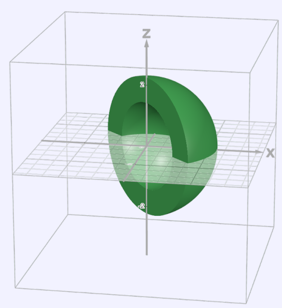


Figure 4: Region Between Spheres (Cut)

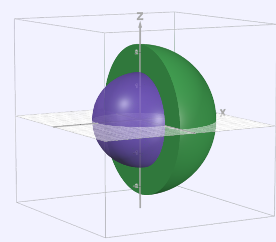


Figure 5: Region Between Spheres (Inner Sphere)

Introduction to Vectors

self-note: continue here from the lecture note