

# MAT232 - Lecture 9

[Lesson Topic(s)]

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# Definitions and Theorems

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*Straight from the textbook — no fluff, just what we need.*

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**Quick recap before diving into the lecture.**



# Let's Get Started

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*Time to dive into the lecture notes.*

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Grab your pen or pencil, and let's break this down step by step.

**Example**

If  $\bar{r}''(t) = \langle 2t, 3 \rangle$ , find  $\bar{r}'(t)$  with  $\bar{r}'(0) = \langle 1, 0 \rangle$  and  $\bar{r}(0) = \langle -1, 1 \rangle$ .

**Solution**

We know that  $\bar{r}''(t) = \bar{r}'(t)$ . Therefore, we have

$$\begin{aligned}\bar{r}'(t) &= \int \bar{r}''(t) dt \\ &= \int \langle 2t, 3 \rangle dt \\ &= \langle t^2 + C_1, 3t + C_2 \rangle.\end{aligned}$$

We can now use the initial conditions to solve for  $C_1$  and  $C_2$ . We have

$$\begin{aligned}\bar{r}'(0) &= \langle 0 + C_1, 0 + C_2 \rangle = \langle 1, 0 \rangle \\ \bar{r}(0) &= \langle -1, 1 \rangle.\end{aligned}$$

Therefore, we have

$$\begin{aligned}C_1 &= 1 \\ C_2 &= 0.\end{aligned}$$

Therefore, the solution is

$$\bar{r}'(t) = \langle t^2 + 1, 3t \rangle.$$

**Answer**

The solution is  $\bar{r}'(t) = \langle t^2 + 1, 3t \rangle$ .

So,

$$\int \bar{r}'(t) dt = \int \langle t^2 + 1, 3t \rangle dt = \left\langle \frac{t^3}{3} + t + C_3, \frac{3t^2}{2} + C_4 \right\rangle.$$

We have that

$$\bar{r}(t) = \left\langle \frac{t^3}{3} + t + C_3, \frac{3t^2}{2} + C_4 \right\rangle.$$

We can now use the initial conditions to solve for  $C_3$  and  $C_4$ . We have

$$\bar{r}(0) = \left\langle \frac{0}{3} + 0 + C_3, \frac{3 \cdot 0^2}{2} + C_4 \right\rangle = \langle -1, 1 \rangle.$$

Therefore, we have

$$C_3 = -1$$

$$C_4 = 1.$$

Therefore, the solution is

$$\vec{r}(t) = \left\langle \frac{t^3}{3} + t - 1, \frac{3t^2}{2} + 1 \right\rangle.$$

### Find the Length of a Vector-Valued Function

#### Definition

Parametric Equation:

$$x = f(t), \quad y = g(t), \quad \alpha \leq t \leq$$

$$L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

$$\text{Length of the curve} = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

## Now, for Vector-Valued Functions

### Definition

Vector-Valued Functions:

$$\begin{aligned}\bar{r}(t) &= \langle x(t), y(t), z(t) \rangle, \quad \alpha \leq t \leq \beta \\ &= \langle f(t), g(t), h(t) \rangle \\ &= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}\end{aligned}$$

$$\begin{aligned}\bar{r}'(t) &= \langle f'(t), g'(t), h'(t) \rangle = \frac{d\bar{r}}{dt} = \frac{d}{dt} \langle f(t), g(t), h(t) \rangle \\ &= \frac{df}{dt} \hat{i} + \frac{dg}{dt} \hat{j} + \frac{dh}{dt} \hat{k}\end{aligned}$$

Thus,

$$\begin{aligned}\|\bar{r}'(t)\| &= \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \\ &= \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2} \\ &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.\end{aligned}$$

So, the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2} dt.$$



**Example**

Find the length of the curve defined by the vector-valued function

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle, \quad 0 \leq t \leq \pi.$$

$$x = \cos(t), \quad y = \sin(t), \quad z = t, \quad 0 \leq t \leq \pi$$

**Solution**

We have

$$\begin{aligned}\vec{r}'(t) &= \langle -\sin(t), \cos(t), 1 \rangle \\ \|\vec{r}'(t)\| &= \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} \\ &= \sqrt{\sin^2(t) + \cos^2(t) + 1} \\ &= \sqrt{1 + 1} \\ &= \sqrt{2}.\end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned}L &= \int_0^\pi \sqrt{2} \, dt \\ &= \sqrt{2} \int_0^\pi dt \\ &= \sqrt{2} [t]_0^\pi \\ &= \sqrt{2}\pi.\end{aligned}$$

**Answer**

The length of the curve is  $\sqrt{2}\pi$ .



## Section 4.1: Functions of Several Variables

### Concept

Function of one variable:

$$y = f(x), \quad \text{where } x \text{ is the independent variable and } y \text{ is the dependent variable.}$$

The image/range of the function is the set of all possible values of  $y$  as  $x$  varies over the domain  $\mathbb{R}$  of the function.

### Note

These functions are in 2D space.

- $f : x \in A \rightarrow y \in B$  is a function from set  $A$  to set  $B$ .
- The domain of the function is the set of all possible values of  $x$  for which the function is defined.
- The range of the function is the set of all possible values of  $y$  as  $x$  varies over the domain.
- The graph of the function is the set of all points  $(x, y)$  in the plane such that  $y = f(x)$ .
- The level curves of the function are the curves in the plane defined by  $f(x, y) = k$ , where  $k$  is a constant.
- The level surfaces of the function are the surfaces in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- $\mathbb{R} \rightarrow \mathbb{R}$ .

Now, we will extend this concept to functions of several variables.

**Definition**

Function of two variables:

$$z = f(x, y), \quad \text{where } x \text{ and } y \text{ are the independent variables and } z \text{ is the dependent variable.}$$

The domain of the function is the set of all possible values of  $(x, y)$  for which the function is defined.

**Note**

These functions are in 3D space.

- $f : (x, y) \in A \rightarrow z \in B$  is a function from set  $A$  to set  $B$ .
- The domain of the function is the set of all possible values of  $(x, y)$  for which the function is defined.
- The range of the function is the set of all possible values of  $z$  as  $(x, y)$  varies over the domain.
- The graph of the function is the set of all points  $(x, y, z)$  in space such that  $z = f(x, y)$ .
- The level curves of the function are the curves in the plane defined by  $f(x, y) = k$ , where  $k$  is a constant.
- The level surfaces of the function are the surfaces in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- $\mathbb{R}^2 \rightarrow \mathbb{R}$ .

There are even functions of three variables.

**Definition**

Function of three variables:

$w = f(x, y, z)$ , where  $x, y$ , and  $z$  are the independent variables and  $w$  is the dependent variable.

The domain of the function is the set of all possible values of  $(x, y, z)$  for which the function is defined.

**Note**

These functions are in 4D space.

- $f : (x, y, z) \in A \rightarrow w \in B$  is a function from set  $A$  to set  $B$ .
- The domain of the function is the set of all possible values of  $(x, y, z)$  for which the function is defined.
- The range of the function is the set of all possible values of  $w$  as  $(x, y, z)$  varies over the domain.
- The graph of the function is the set of all points  $(x, y, z, w)$  in space such that  $w = f(x, y, z)$ .
- The level curves of the function are the curves in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- $\mathbb{R}^3 \rightarrow \mathbb{R}$ .
- The level surfaces of the function are the surfaces in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- $\mathbb{R}^3 \rightarrow \mathbb{R}$ .
- The graph of the function is the set of all points  $(x, y, z, w)$  in space such that  $w = f(x, y, z)$ .
- The level curves of the function are the curves in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- The level surfaces of the function are the surfaces in space defined by  $f(x, y, z) = k$ , where  $k$  is a constant.
- $\mathbb{R}^3 \rightarrow \mathbb{R}$ .



## Sketching a Function in 2D

### Example

Let  $f(x, y) = 3x\sqrt{y} - 1$ . Sketch the graph of the function.

### Solution

#### Note

- $x \in \mathbb{R}$
- $\sqrt{y}$  so  $y \geq 0$

Consider some points in the domain of the function. We have

$$\begin{aligned}x, y &= f(x, y) \\f(1, 4) &= 3 \cdot 1 \cdot 2 - 1 = 5 = z \\f(0, 9) &= 3 \cdot 0 \cdot 3 - 1 = -1 = z \\f(t, t^2) &= 3 \cdot t \cdot \sqrt{t^2} - 1 = 3t \cdot t - 1 = 3t^2 - 1 = z.\end{aligned}$$

#### Note

$$\sqrt{t^2} = |t|.$$

Therefore, the function is defined for all  $(x, y)$  in the domain.

To sketch the graph of the function, we can sketch the level curves of the function. We have

$$f(x, y) = 3x\sqrt{y} - 1 = k.$$

We can now sketch the level curves of the function by setting  $k = 0$ . We have

$$3x\sqrt{y} - 1 = 0.$$

Therefore, the level curve is

$$3x\sqrt{y} = 1.$$



## Sketching the Domain of a Function in 2D

### Example

Sketch the domain of  $f(x, y) = \ln(x^2 - y)$ .

### Solution

#### Note

- $\ln(x^2 - y)$  implies that  $x^2 - y > 0$  meaning that  $x^2 > y$ .
- $x^2 - y > 0$  so  $x^2 > y$ .
- $x^2 - y$  is in the domain of  $\ln(x^2 - y)$  so  $x^2 - y > 0$ .
- $x^2 - y > 0$  so  $y < x^2$ .
- The domain of the function is  $y < x^2$ .

Therefore, the domain of the function is  $y < x^2$ . Pick some points in the domain of the function. We have

$$f(2, 0) = \ln(2^2 - 0) = \ln(4)$$

$$f(0, 2) = \ln(0^2 - 2) = \ln(-2).$$

Notice that  $f(0, 2)$  is not defined because  $\ln(-2)$  is not defined.

### Illustration



### Answer

The domain of the function is  $y < x^2$ .

## Describing the Domain of a Function in 3D

### Example

Describe the domain of the function  $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$ .

### Solution

#### Note

- $\sqrt{1 - x^2 - y^2 - z^2}$  is defined if  $1 - x^2 - y^2 - z^2 \geq 0$ .
- $1 - x^2 - y^2 - z^2 \geq 0$  so  $1 \geq x^2 + y^2 + z^2$ .
- The domain of the function is  $x^2 + y^2 + z^2 \leq 1$ .
- The domain of the function is the unit ball.
- The domain of the function is the set of all points  $(x, y, z)$  such that  $x^2 + y^2 + z^2 \leq 1$ .
- The domain of the function is the unit ball.

Notice that

$$\begin{aligned}\sqrt{1 - x^2 - y^2 - z^2} &> 0 \\ 1 - x^2 - y^2 - z^2 &> 0 \\ 1 &> x^2 + y^2 + z^2.\end{aligned}$$

This means, any point  $(x, y, z)$  on the edge of the unit ball/sphere centered at the origin  $(0, 0, 0)$  with radius 1 and inside the unit ball is in the domain of the function.

$$\begin{aligned}f\left(0, \frac{1}{2}, -\frac{1}{2}\right) &= \sqrt{1 - 0 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2} \\ &= \sqrt{1 - \frac{1}{4} - \frac{1}{4}} \\ &= \sqrt{1 - \frac{1}{2}} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}.\end{aligned}$$

Therefore, the domain of the function is  $x^2 + y^2 + z^2 \leq 1$ .

The domain of the function is  $x^2 + y^2 + z^2 \leq 1$ .



## Graphing and Describing Functions

### Example

Graph the following functions and describe them:

1.  $f(x, y) = 1 - x - \frac{y}{2}$
2.  $f(x, y) = \sqrt{1 - x^2 - y^2}$
3.  $f(x, y) = -\sqrt{x^2 + y^2}$

### Solution

1. The function is a plane. The graph of the function is a plane with a slope of -1 in the x-direction and -1/2 in the y-direction.
2. The function is a cone. The graph of the function is a cone with a vertex at the origin and a base of radius 1 in the xy-plane.
3. The function is a cone. The graph of the function is a cone with a vertex at the origin and a base of radius 1 in the xy-plane.

### Answer

1. The graph of the function is a plane with a slope of -1 in the x-direction and -1/2 in the y-direction.
2. The graph of the function is a cone with a vertex at the origin and a base of radius 1 in the xy-plane.
3. The graph of the function is a cone with a vertex at the origin and a base of radius 1 in the xy-plane.