MAT232 - Lecture 7

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Contents

Section 2.5: Lines and Planes **1**

Section 2.5: Lines and Planes . . . 1

Now, in MAT232, exploring the 3D world... . . . 1

2 Lines in 3D . . . 4

Planes **6**

Idea . . . 6

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Note

Remember that term test 1 is on Thursday, January 30$^{\text{th}}$, 2025 — from 6-8pm!

Good luck studying!

Section 2.5: Lines and Planes

Recall from high school...

The line equation is defined by

$$y = mx + b$$

where m is the slope and b is the y-intercept. The slope is defined as the change in y over the change in x. The y-intercept is the point where the line crosses the y-axis.

Alternatively, there was point-slope form, which is defined as

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line.

Now, in MAT232, exploring the 3D world...

Definition

In 3D, we have a line equation defined by

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

where (x_0, y_0, z_0) is a point on the line and (a, b, c) is the direction vector. The parameter t is a scalar.

Definition

Vector equation:

$$\overline{r} = \overline{r_0} + t\overline{v}$$

$$< x, y, z >=< x_0, y_0, z_0 > +t < v_1, v_2, v_3 >$$

Definition

This is also written as:

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Example

What does $< x, y, z >=< -1, 0, 2 > +t^2 < 2, 10, -8 >$ represent? Note that $t \in \mathbb{R}$ is a scalar.

Solution

It represents the line in 3D space that passes through the point $(-1, 0, 2)$ and has direction vector $< 2, 10, -8 >$ (or is parallel to this direction vector).

Try this at home: what about this one?

Exercise

What does $< x, y, z >=< -1, 0, 2 > +2t^3 < 1, 5, -4 >$ represent?

Example

(A): Find the parametric equations of the line L that pass through the points $A(2, 4-1)$ and $B(5, 0, 7)$.
(B): Does this line intersect the xy-plane? If so, where? (Hint: $z = 0$.)

Solution

(A): The direction vector is $< 5-2, 0-4, 7-(-1) >=< 3, -4, 8 >$. The parametric equations are

$$x = 2 + 3t$$
$$y = 4 - 4t$$
$$z = -1 + 8t$$

Vector equation:

$$\overline{r} = \overline{r_0} + t\overline{v}$$

$$< x, y, z >=< 2, 4, -1 > +t < 3, -4, 8 >$$

(B): To find the intersection with the xy-plane, we set $z = 0$ and solve for t:

$$-1 + 8t = 0 \implies t = \frac{1}{8}$$

Substitute $t = \frac{1}{8}$ into the parametric equations to find the intersection point:

$$x = 2 + 3\left(\frac{1}{8}\right) = \frac{19}{8}$$
$$y = 4 - 4\left(\frac{1}{8}\right) = \frac{7}{2}$$
$$z = -1 + 8\left(\frac{1}{8}\right) = 0$$

Thus, the line intersects the xy-plane at the point $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$.

Answer

The parametric equations of the line are

$$x = 2 + 3t$$
$$y = 4 - 4t$$
$$z = -1 + 8t$$

The line intersects the xy-plane at the point $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$.

2 Lines in 3D

Remark

Two lines in 3D are either parallel, intersecting, or skew. Skew lines are lines that are not parallel and do not intersect.

- Parallel lines have the same direction vector.
- Intersecting lines have the same direction vector and a point in common.
- Skew lines have different direction vectors.

Note

1. Can be parallel?
2. intersect at a point?
3. Skewed?

Tip

To determine if two lines are parallel, intersecting, or skew, we can compare the direction vectors and points on the lines.

Let's try an example:

Example

Let L_1 and L_2 be the lines defined as:

$$L_1 : x = 1 + 4t$$
$$y = 5 - 4t$$
$$z = -1 + 6t$$
$$L_2 :< x, y, z > =< 2, 4, 5 > +s < 8, -3, 1 >$$

(A): Are the lines parallel, intersecting, or skew?
(B): If they intersect, find the point of intersection.

Solution

(A): The direction vector of L_1 is $< 4, -4, 6 >$ and the direction vector of L_2 is $< 8, -3, 1 >$. Since the direction vectors are not the same, the lines are skew.
Specifically,

$$\overline{v_1} \stackrel{?}{=} k\overline{v_2}$$

$$\langle 4, -4, 6\rangle \stackrel{?}{\neq} k\langle 8, -3, 1\rangle$$

No, L_1 is not $\parallel$ to L_2.

Note

There are two ways to check if two lines are parallel:

1. $\overline{v_1} = k\overline{v_2}$, k is a scalar.
2. $\overline{v_1} \times \overline{v_2} = \overline{0}$.

(B): Notice that

$$L_1 : x = 1 + 4t, y = 5 - 4t, z = -1 + 6t$$

$$L_2 : x = 2 + 8s, y = 4 - 3s, z = 5 + s$$

Equate the x, y, and z components of the two lines to find the point of intersection:

$$1 + 4t = 2 + 8s \quad (1)$$
$$5 - 4t = 4 - 3s \quad (2)$$
$$-1 + 6t = 5 + s \quad (3)$$

So,

$$6 = 6 + 5s \implies s = 0$$

Using (1),

$$1 + 4t = 2 \implies t = \frac{1}{4}.$$

Now, check $s = 0$ and $t = \frac{1}{4}$ in (3) ($LHS = RHS$):

$$-1 + 5\left(\frac{1}{4}\right) = 5 + 0$$

Planes

Definition

In 3D, we can define a plane using the equation

$$Ax + By + Cz = D$$

or alternative, we can define the plane using the parametric form

$$x = x_0 + su + tv$$
$$y = y_0 + su + tv$$
$$z = z_0 + su + tv$$

where (x_0, y_0, z_0) is a point on the plane and (u, v) are the direction vectors. The parameter s and t are scalars.

Check out the x plane:

$$x = a \text{ fixed}$$

Check out the y plane:

$$y = b \text{ fixed}$$

Check out the z plane:

$$z = c \text{ fixed}$$

Idea

Illustration

self-note: add the illustration here from the camera roll

Note

1. Point on plane - $P(x_0, y_0, z_0)$
2. Vector living on the plane:

$$\overline{r} - \overline{r_0}$$

3. Normal vector (90 degrees to the plane): $\overline{n} = \langle A, B, C \rangle$

Definition

The equation of the plane is defined by

$$\overline{n} \cdot (\overline{r} - \overline{r_0}) = 0.$$

$$\langle A, B, C \rangle \cdot \langle x, y, z \rangle - \langle < x_0, y_0, z_0 \rangle = 0$$

$$\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

This is the scalar equation of the plane.

In vector form, we can define the line as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where (x_0, y_0, z_0) is a point on the line and (a, b, c) is the direction vector. The parameter t is a scalar.

The vector equation can also be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\vec{d}$$

where $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the direction vector.

We can also define the line using the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point on the line and (a, b, c) is the direction vector.

In 3D, we can also define a plane using the equation

$$Ax + By + Cz = D$$

where (A, B, C) is the normal vector to the plane. The normal vector is perpendicular to the plane. The point (x, y, z) is a point on the plane. The scalar D is the distance from the origin to the plane.

Alternatively, we can define the plane using the parametric form

$$x = x_0 + su + tv$$
$$y = y_0 + su + tv$$
$$z = z_0 + su + tv$$

where (x_0, y_0, z_0) is a point on the plane and (u, v) are the direction vectors. The parameter s and t are scalars.