MAT232 - Lecture 2

Parametric Equations and Calculus: Concepts, Applications, and Derivations

AlexanderTheMango

Prepared for January 9, 2025

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Parametric Equations and Parameters

Definition

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called **parametric equations**, and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the **graph of the parametric equations**. The graph of parametric equations is referred to as a **parametric curve** or **plane curve**, and is denoted by C.

Theorem 1.1: Derivative of Parametric Equations

Theorem

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

Proof

Proof.
This theorem can be proven using the Chain Rule. Assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule gives

$$y'(t) = F'(x(t))x'(t),$$

so

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 1.1 and Applications

Note

Equation 1.1 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 1.1 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Second-Order Derivatives

Theorem

The next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, it is possible to replace y on both sides of this equation with $\frac{dy}{dx}$. This yields

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Key Concepts

Definition

A **parametric equation** is a set of equations that express the coordinates of the points of a curve as functions of a variable, called a parameter.

Examples

Example

Example: Sketch the graph, using a table of values:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

t	$1/t$	x	y
0.01	${}^1/_{0.01} = {}^1/_{\frac{1}{100}} = 100$	100.01	$0.01 - 100 = -99.99$
0.1	${}^1/_{0.1} = {}^1/_{\frac{1}{10}} = 10$	10.1	-9.9
0.2	${}^1/_{0.2} = {}^1/_{\frac{20}{100}} = {}^1/_{\frac{2}{10}} = 5$	5.2	4.8
1	$\frac{1}{1}$	2	0
5.0	0.2	5.2	4.8
10	0.1	10.1	9.9
10	0.01	100.01	99.99

This describes a hyperbolic curve.

Figure 1: Sample image illustrating the concept.

Example

Example: Sketch the graph (this is the same one), using the elimination method:

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

$LHS = A^2 - B^2 = (A - B)(A + B) = RHS$ $X = A$ and $y = B$. $LHS : x^2 - y^2$. $A - B = x - y = (t + \frac{1}{t}) - (t - \frac{1}{t}) = \frac{2}{t}$. $A + B = x + y = (t + \frac{1}{t}) + (t - \frac{1}{t}) = 2t$. $RHS : (A - B)(A + B) = (x - y)(x + y) = (\frac{2}{t})(2t) = 4$. Therefore, $x^2 - y^2 = 4$, $y \in \mathbb{R}$ will work, $x > 0$.
This describes a hyperbolic curve.

Theorems and Proofs

Theorem

Theorem: If $x(t)$ and $y(t)$ are differentiable functions, the slope of the curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

Figure 2: Graphical representation of the theorem.

Practice Questions

Note

Try this question at home!

Sketch and eliminate t if possible:

$$x = t^2, \quad y = t^3, \quad -2 \leqslant t \leqslant 2$$

Note that this is a closed interval. The starting point is the smallest value of t. This highlights where the graph should begin. The finishing point should be the largest value of t.
Using an arrow, make sure to indicate the direction of the graph as $t \to \infty$.

Note

Try another question at home!

Sketch and eliminate t if possible:

$$c_1 : x = -cos(\frac{t}{4}), y = sin(\frac{t}{4}), for 0 \leqslant t \leqslant 4\pi$$

$$c_2 : x = -sin(t), y = -cos(t), for \frac{\pi}{2} \leqslant t \leqslant \frac{3\pi}{2}$$

$$c_3 : x = cos(t), y = sin(t), for t \in [0, \pi]$$

Hint: $x = rcos(\theta), y = rsin(\theta), x^2 + y^2 = r^2$. Also, for these curves, it follows that $r = 1$.

The Elimination Method Does NOT Always Work

Note

Consider the following case where t cannot be eliminated:

$$x = e^t - \sin^2(t), \quad y = ln(t) + \frac{1}{t}, \quad t > 0$$

Further Visualization

Figure 3: Additional visualization for parametric curves.

Section 1.2: Calculus on Parametric Equations

Key Concepts

Recall the concept from 1st year calculus:

Definition

If $y = f(x)$ is given, then the slope of the tangent line to the curve of $y = f(x)$ is:

$$y' = f'(x) = \frac{dy}{dx}$$

Now, for MAT232, we have:

Definition

Given $x = f(t), \quad y = g(t), \quad t \in \mathbb{R}$, these are defifferentiable w.r.t. (w.r.t. = "with respect to") t. This is such that:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \frac{dx}{dt} \neq 0$$

This will also be provided in the formula sheet.

$$x = f(t), \quad y = g(t), \quad t \in \mathbb{R}$$

Because the chain rule must follow through, always!

Here is the derivation: So $\ldots y = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Chain rule.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}, \quad \text{provided that} \quad \frac{dx}{dt} \neq 0$$

Second Derivative

Theorem

Given $x = f(t), y = g(t), t \in \mathbb{R}$ are differentiable at t and $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ exists and is differentiable at t:

$$\frac{d^{2y}}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = dx(\frac{\frac{dy}{dt}}{\frac{dx}{dt}})$$

Notice that the expression of the innermost bracket is a derivative all in terms of t. Thus:

$$= \frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}) \cdot \frac{dt}{dx} = \frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}) = \frac{\frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}})}{\frac{dx}{dt}}.$$

This follows from the **inverse function theorem**.
Collectively, it follows that:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}})}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

This is not included on the formula sheet.

Examples

Example

Consider the following parametric curve:

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(A) Find the tangent line to the given curve at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$.
(B) Find the vertical tangent(s), if any.
(C) Find $\frac{d^2y}{dx^2}$.

Let's do this, one at a time!

(A) Find the tangent line to the given curve at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$.

Example

<u>Tangent Line:</u> Recall...

1. $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the curve;
2. $y = mx + b$, where m is the slope and b is the y-intercept.

Given point $(\sqrt{2}, 1) = (x_0, y_0)$, $\frac{dy}{dt} = \sec^2(t)$, and $\frac{dx}{dt} = \sec(t)\tan(t)$, it follows that:

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^{\cancelto{1}{2}}(t)\tan(t)}{\cancel{\sec(t)}\tan(t)} = \frac{\sec(t)}{\tan(t)}$$

Next, $\frac{dy}{dx}\,|_{t=\frac{\pi}{4}} = \frac{\sec\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}}{1} = \sqrt{2} = m.$

self-note: finish these notes (check the camera roll)

(B) Find the vertical tangent(s), if any.

Example

$$\frac{dy}{dt} = \sec^2(t)$$

$$\frac{dx}{dt} = \sec(t)\tan(t)$$

So...

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2(t)}{\sec(t)\tan(t)}$$

Recall from first year calculus:

Theorem

Given $y = f(x)$, it follows that $y' = f'(x) = 0$. That is, the roots of $y' = 0$ indicate the positions of the horizontal tangents.

So...
Horizonal Tangent: $\frac{dy}{dx} = 0$; find t values.

$$\frac{dy}{dt} = 0, \quad \text{but } \frac{dx}{dt} \neq 0$$

Vertical Tangent: $\frac{dy}{dx}$ is *undefined*; find t values.

$$\frac{dx}{dt} = 0, \quad \text{but} \frac{dy}{dt} \neq 0$$

In this case, there is a singular point:

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0$$

Vertical Tangents: $\frac{dx}{dt} = 0$, but $\frac{dy}{dt} \neq 0$.
So...

$$\frac{dx}{dt} = \sec(t)\tan(t) = 0, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Notice that

- $\sec(t) = \frac{1}{\cos(t)} = 0$ is impossible as $1 \neq 0$;
- $\tan(t) = 0$ occurs at $t = 0$.

Now, check $\frac{dy}{dt} = 0$ at $t = 0$.

$$\frac{dy}{dt} = \sec^2(t) = 0, \quad \text{for } t = 0$$

Is this true?

Therefore, the vertical tangent is at $t = 0$.

(C) Find $\frac{d^2y}{dx^2}$.

Example

Recall:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}})}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{\sec(t)}{\tan(t)} \quad \text{and} \quad \frac{dx}{dt} = \sec(t)\tan(t)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{\sec(t)}{\tan(t)})}{\sec(t)\tan(t)}$$

$$\frac{\sec(t)}{\tan(t)} = \frac{\frac{1}{cos(t)}}{\cdot}\frac{\sin(t)}{\cos(t)} = \frac{1}{\cos(t)}(\frac{\cos(t)}{\sin(t)})$$

$$= \frac{1}{\sin(t)}$$

$$\sec(t)\tan(t) = \frac{1}{\cos(t)} \cdot \frac{\sin(t)}{\cos(t)} = \frac{\sin(t)}{\cos^2(t)}$$

Now, find the derivative of $y = \frac{1}{\sin(t)}$:

$$y' = \frac{0 \cdot \sin(t) - \cos(t) \cdot 1}{\sin^2(t)} = -\frac{\cos(t)}{\sin^2(t)}$$

note to self: finish this off