MAT232 - Lecture 5

Advanced Curve Analysis: Polar Derivatives and Conic Sections

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Contents

Definitions and Theorems

Straight from the textbook — lots of fluff this time, more than what we need!

Quick recap before diving into the lecture.

Conic Sections

Concept

Definition of Conic Sections: Conic sections are the curves formed by the intersection of a plane with a double-napped cone. The type of curve depends on the angle of the plane relative to the cone:

- *Circle:* The plane is perpendicular to the cone's axis.
- *Ellipse:* The plane intersects one nappe of the cone but is not perpendicular to the axis.
- *Parabola:* The plane is parallel to a generator of the cone.
- *Hyperbola:* The plane intersects both nappes of the cone.

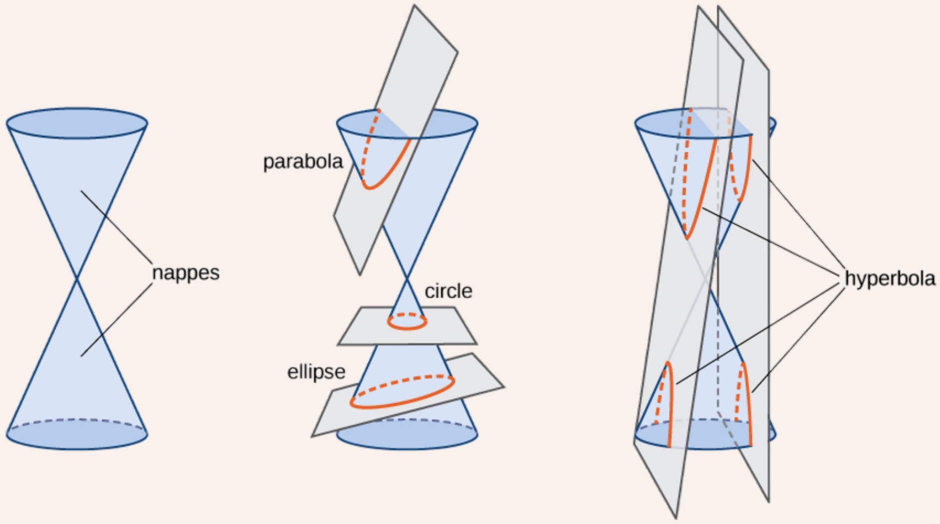

Figure 1: Conic sections formed by the intersection of a plane with a double-napped cone.

Ellipse

Definition

An **ellipse** is the set of all points in a plane such that the sum of their distances to two fixed points (called the *foci*) is constant.

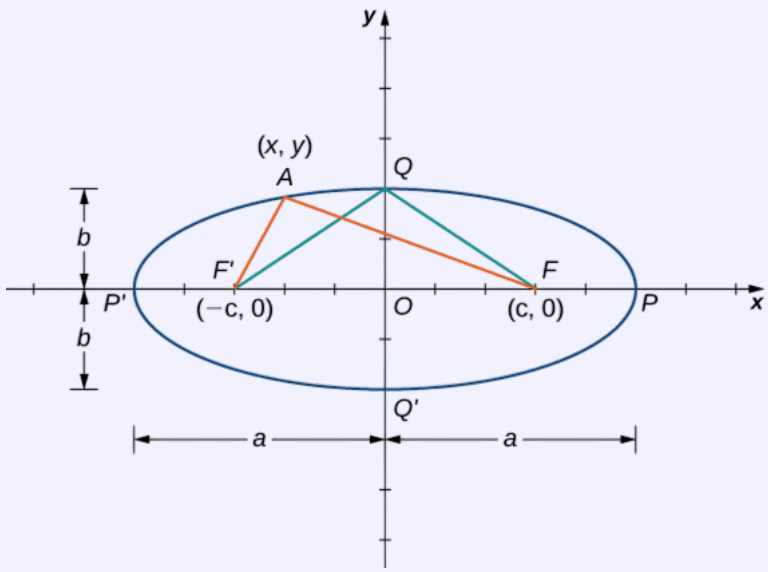

Figure 2: Diagram of an ellipse.

Intuition

Imagine looping a circular string around two fixed points F_1 and F_2 on a plane and pulling it taut (fully stretched without slack) with a pencil. As you move the pencil while keeping the string tight, the traced shape forms an ellipse. This method is commonly used for drawing ellipses with nails and string.

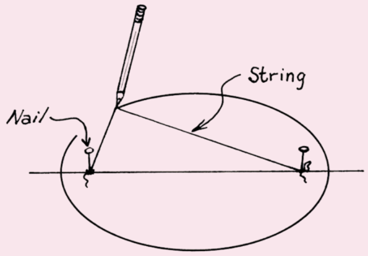

Figure 3: Drawing an ellipse with nails and string.

Standard Forms of an Ellipse

Definition

The equation of an ellipse depends on the orientation of its major axis:

- **Horizontal Major Axis:**
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
where:
  - (h, k) is the center,
  - $a > b$ (semi-major axis a, semi-minor axis b),
  - $c^2 = a^2 - b^2$, where c is the focal distance.
- **Vertical Major Axis:**
$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$
with the same parameters as above.

Remark

Properties of Ellipses:

- *Vertices:* Located a units from the center along the major axis.
- *Foci:* Located c units from the center along the major axis, where $c^2 = a^2 - b^2$.
- *Eccentricity:* Defined as $e = \frac{c}{a}$, with $0 < e < 1$.

Verifying an Ellipse

Example

Show that the equation

$$4x^2 + 9y^2 = 36$$

represents an ellipse and determine its key features.

Solution

- Rewrite the equation in standard form:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

- The ellipse is centered at $(0, 0)$ with $a = 3$, $b = 2$, and $c = \sqrt{a^2 - b^2} = \sqrt{5}$.
- The foci are $(\pm\sqrt{5}, 0)$, and the vertices are $(\pm 3, 0)$.

Parabolas

Definition

A **parabola** is the set of all points in a plane equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

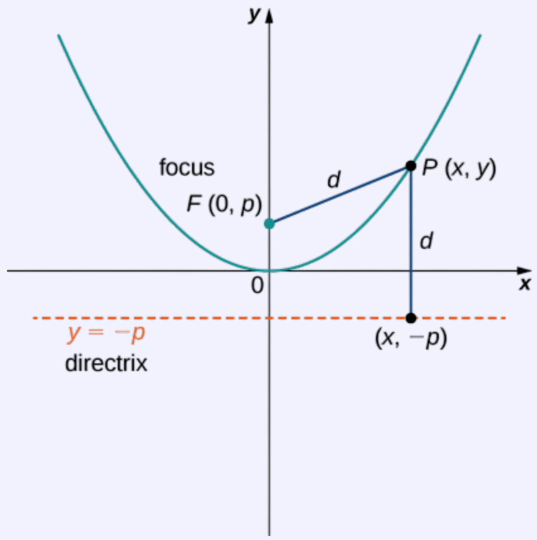

Figure 4: Diagram of a parabola.

Intuition

A parabola can be thought of as the trajectory of an object under uniform acceleration, such as the path of a ball thrown in the air.

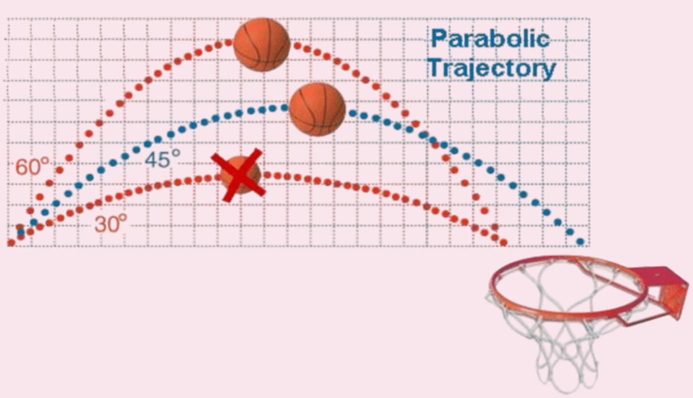

Figure 5: Parabolic trajectory of a ball.

Standard Forms of a Parabola

Definition

The equation of a parabola depends on whether it opens horizontally or vertically:

- **Opens Right or Left (Horizontal Axis):**

$$(y-k)^2 = 4p(x-h)$$

  - (h,k) is the vertex.
  - p is the directed distance from the vertex to the focus.
  - The focus is at $(h+p,k)$, and the directrix is the vertical line $x = h - p$.

- **Opens Up or Down (Vertical Axis):**

$$(x-h)^2 = 4p(y-k)$$

  - The vertex and p are the same as above.
  - The focus is at $(h,k+p)$, and the directrix is the horizontal line $y = k - p$.

Remark

Properties of Parabolas:

- *Focus:* Located p units from the vertex along the axis of symmetry.
- *Directrix:* A line perpendicular to the axis of symmetry at a distance p from the vertex.
- *Axis of Symmetry:* A line that passes through the focus and is perpendicular to the directrix.

Verifying a Parabola

Example

Show that the equation

$$y^2 = 12x$$

represents a parabola and determine its key features.

Solution

- The equation is in the standard form $y^2 = 4px$, with $4p = 12$, so $p = 3$.
- The parabola opens to the right, with vertex $(0, 0)$, focus $(3, 0)$, and directrix $x = -3$.

Hyperbola

Definition

A **hyperbola** is the set of all points in a plane such that the absolute difference of their distances to two fixed points (called the *foci*) is constant.

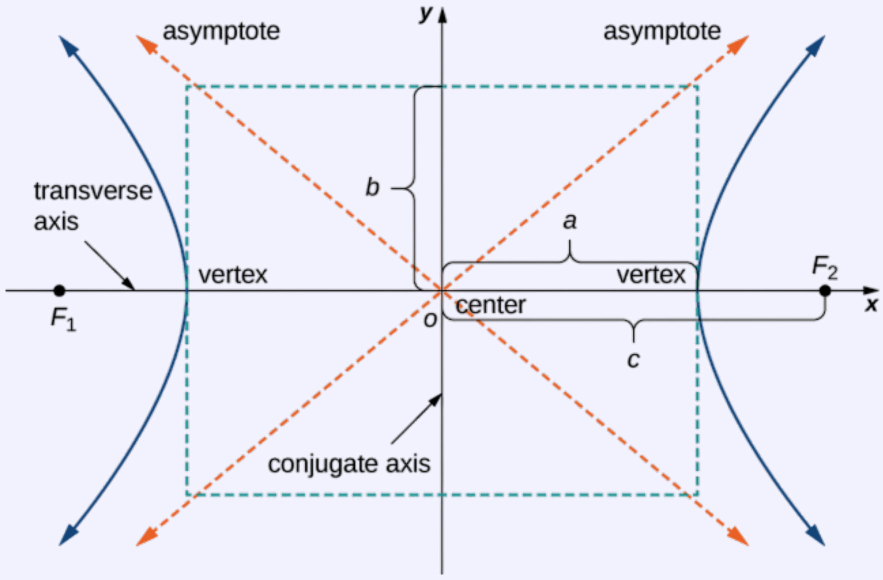

Figure 6: Diagram of a hyperbola.

Intuition

A hyperbola appears in real-world phenomena such as satellite orbits, radio wave propagation, and the paths of comets.

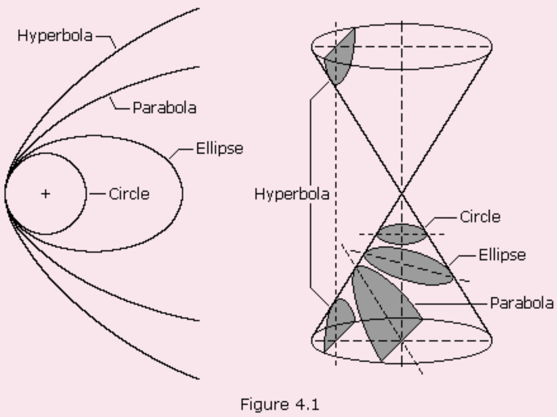

Figure 7: Hyperbolic orbits can have greater eccentricity than parabolic ones.

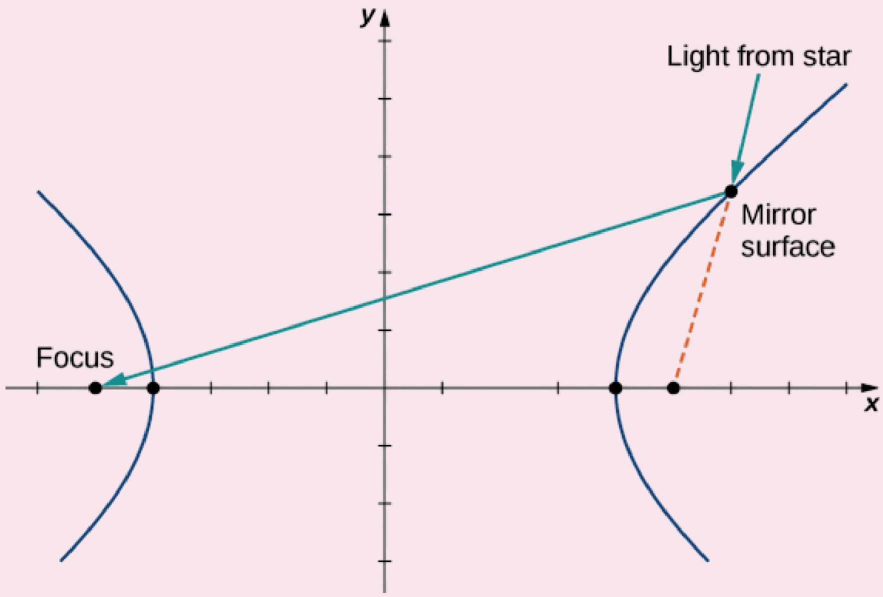

Figure 8: A hyperbolic mirror used to collect light from distant stars.

Standard Forms of a Hyperbola

Definition

A hyperbola is defined by the difference of distances to two fixed points (foci) being constant. Its standard equation depends on the orientation of its transverse axis:

- **Horizontal Transverse Axis:**

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

  where (h, k) is the center, a is the distance from the center to each vertex, and $c^2 = a^2 + b^2$ defines the distance from the center to each focus.

- **Vertical Transverse Axis:**

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

Remark

Properties of Hyperbolas:

- *Foci:* Located c units from the center along the transverse axis, where $c^2 = a^2 + b^2$.
- *Asymptotes:* Lines that the hyperbola approaches but never touches, given by:

$$y = k \pm \frac{b}{a}(x-h) \quad \text{(horizontal)}.$$

- *Vertices:* Located a units from the center along the transverse axis.

Verifying a Hyperbola

Example

Show that the equation

$$9x^2 - 16y^2 = 144$$

represents a hyperbola and determine its key features.

Solution

- Rewrite the equation in standard form:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

- The hyperbola is centered at $(0, 0)$ with $a = 4$, $b = 3$, and $c = \sqrt{a^2 + b^2} = 5$.
- The vertices are $(\pm 4, 0)$, the foci are $(\pm 5, 0)$, and the asymptotes are $y = \pm \frac{3}{4}x$.

Eccentricity and Directrix

Definition

The **eccentricity** e of a conic section is defined as the ratio of the distance from any point on the conic to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for a given conic and determines its type:

- If $e = 1$, the conic is a **parabola**.
- If $e < 1$, the conic is an **ellipse**.
- If $e > 1$, the conic is a **hyperbola**.

Remark

For a **circle**, the eccentricity is $e = 0$.

The **directrix** of a conic section is a fixed line that, together with the focus, helps define the conic.

- **Parabolas** have one focus and one directrix.
- **Ellipses** and **hyperbolas** (excluding circles) have two foci and two corresponding directrices.

Illustration

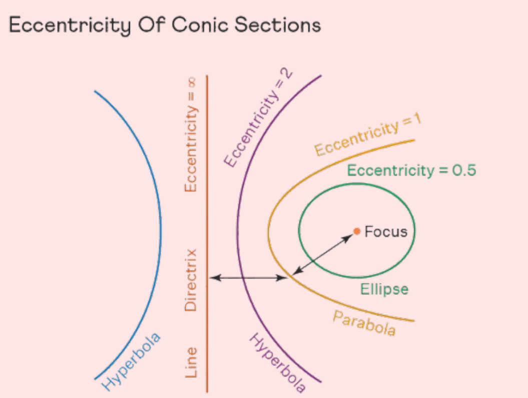

Figure 9: Eccentricity and directrix of conic sections.

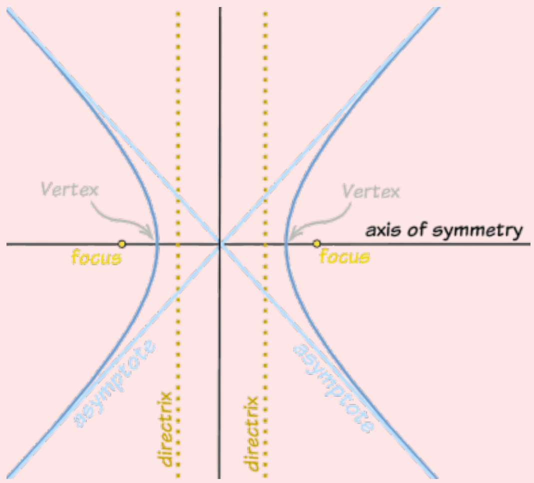

Figure 10: Directrix of a hyperbola.

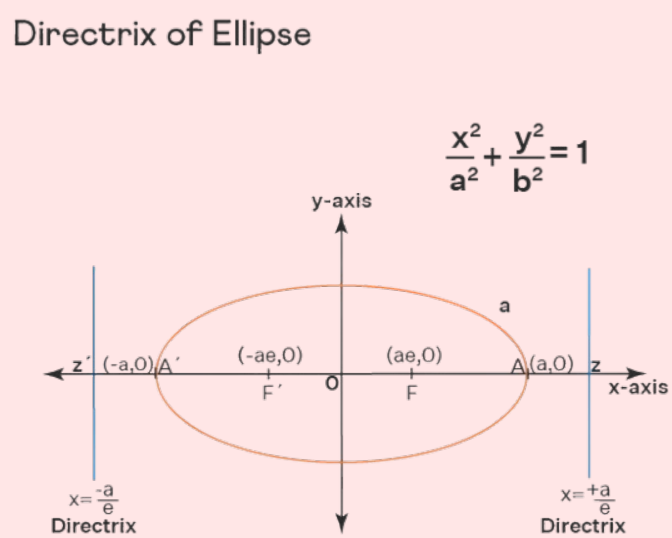

Figure 11: Directrix of an ellipse.

General Equations of Degree Two

Concept

A general second-degree equation is written as:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The nature of its graph (a conic section) is determined using the **discriminant**:

$$\Delta = 4AC - B^2.$$

- If $\Delta > 0$, the conic is an **ellipse**.
- If $\Delta = 0$, the conic is a **parabola**.
- If $\Delta < 0$, the conic is a **hyperbola**.

Remark

If $B \neq 0$, the coordinate axes are rotated.
To determine the rotation angle θ, use:

$$\cot 2\theta = \frac{A - C}{B}.$$

Distinguishing Between Conic Sections

Tip

To classify a conic section, follow these key steps:

1. **Check the discriminant** $\Delta = 4AC - B^2$:
   - $\Delta > 0$ indicates an **ellipse**.
   - $\Delta = 0$ indicates a **parabola**.
   - $\Delta < 0$ indicates a **hyperbola**.
2. **Identify the presence of an xy-term**:
   - If $B \neq 0$, the axes are rotated.
3. **Analyze the equation form**:
   - Ellipses and circles have **both x^2 and y^2 terms** with the same sign.
   - Hyperbolas have **both x^2 and y^2 terms** with opposite signs.
   - Parabolas have **only one squared term** (either x^2 or y^2, but not both).

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Review from the Previous Lecture

Remark

In the previous lecture, we covered important foundational concepts related to polar coordinates and their derivatives. Here's a brief summary:

- **Derivative of $r = f(\theta)$ in Cartesian Coordinates:**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

  This formula helps us compute the slope of the tangent line for polar curves when converted to Cartesian coordinates.

- **Equation of a Circle:**

$$(x-h)^2 + (y-k)^2 = r^2$$

  Here:

  - r: Radius of the circle
  - (h, k): Centre of the circle

Note

Reminder: Term Test 1 is scheduled for **Thursday, January 30th, 2025 (Week 4)**. Make sure to review polar derivatives, transformations, and conic sections!

Exploring Common Curve Shapes

Parabola

Definition

A **parabola** is a symmetric curve defined by the quadratic equation:

$$y = ax^2 + bx + c, \quad a \neq 0$$

To rewrite this equation in vertex form, we complete the square:

$$y = A(x - B)^2 + C$$

Here:

- A: Determines the direction and "width" of the parabola.

$$A > 0 \implies \text{The parabola opens upwards.}$$

$$A < 0 \implies \text{The parabola opens downwards.}$$

- (B, C): Represents the vertex of the parabola.
    - B: Horizontal position of the vertex.
    - C: Vertical position of the vertex.

Algorithm

Vertex Formula: To find the vertex when given the standard form $y = ax^2 + bx + c$, use the formulas:

$$B = -\frac{b}{2a}, \quad C = f(B)$$

where $f(B)$ is the value of the quadratic function evaluated at $x = B$.

...cont'd...

Definition

...cont'd...

Illustration

Below are examples of parabolas showcasing key features:

Figure 12: A parabola opening down, labeled with its vertex and axis of symmetry.

Figure 13: Generic parabolas showing upwards and downwards directions of opening.

Example: Sketching the Region of a Set

Example

Sketch the region of the set defined by

$$R = \{(x, y) \mid y \geqslant x^2 + 1\}$$

Solution

Consider the graph for the function $y = x^2 = 1$:

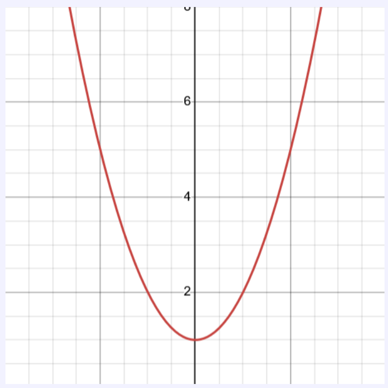

Figure 14: Graph of $y = x^2 + 1$.

Notice that

$$\begin{aligned} & y = x^2 + 1 \\ \implies & 0 \geqslant (-2)^2 + 1 \\ \implies & 0 \geqslant 5, \text{ which is not true.} \end{aligned}$$

Then, notice that

$$\begin{aligned} & 2 \geqslant 0^2 + 1 \\ \implies & 2 \geqslant 1, \text{ which is true!} \end{aligned}$$

Here is the region being considered:

Figure 15: Sample image illustrating the concept.

Ellipse

Definition

The equation of an ellipse is defined by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Note

Recall the equation of the circle, which is based on the equation of the ellipse when $a = b = 1$:

$$\text{Circle:} \quad (x-h)^2 + (y-k)^2 = r^2,$$

where (h, k) is the centre, a represents the x-axis radius, and b represents the y-axis radius.

Example of Skecthing an Ellipse

Example

Sketch the region of the set defined by

$$A = \{(x, y) \mid x^2 + 4y^2 > 4\}.$$

Solution

Notice that

$$x^2 + 4y^2 = 4.$$

This means the centre is at $(0, 0)$. Also,

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

provides that the x-axis radius is $a = 2$ and the y-axis radius is $b = 1$.

Here is the corresponding illustration:

self-note: add the illustratoin from the lecture note from your camera roll

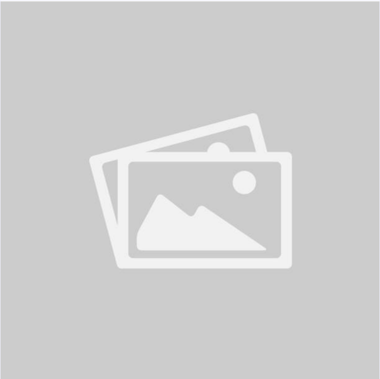

Figure 16: Illustration of ellipse.

Note

Note that dashed lines are used to denote that the edge of the ellipse is **not included** in the region A.

Check the point $(0, 0)$:

$$0^2 + 4 \cdot 0^2 > 4$$

$$\implies 0 > 4,$$

which is not true.

Therefore, the inside of the ellipse is **not** to be shaded in.

Check the point $(3, 0)$:

$$3^2 + 4 \cdot 0^2 > 4$$

Introducing the Hyperbola

Definition

The equation of a hyperbola is defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Illustration

self-note: add the image of the corresponding illustration here (see the lecture note)

Figure 17: Sample image illustrating the concept.

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Illustration

self-note: add the image of the corresponding illustration here (see the lecture note)

Figure 18: Sample image illustrating the concept.

Welcome to Linear Algebra...

well... not really!

Section 2.1/2.2: Welcome to 3D Space!

Remark

Recall that the cartesian coordinate system considers the 2-dimensional realm: a system in $\mathbb{R}^2$.

Illustration

self-note: add the cartesian plane — the typical one in 2D

Figure 19: Sample image illustrating the concept.

Now, check out the cartesian coordinate system being introduced in MAT232, considering the 3-dimensional realm; $\mathbb{R}^3$:

Illustration

self-note: add the illustration for the 3D cartesian plane, the z-axis in addition to the x- and y-axis.

Figure 20: Sample image illustrating the concept.

Note

<u>In 2D:</u>

Notice that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where the first $\mathbb{R}$ represents the x-values and the second $\mathbb{R}$ represents the y-values.

<u>Now, in 3D:</u>

Notice that $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

- The first $\mathbb{R}$ represents the x-values;
- The second $\mathbb{R}$ represents the y-values;
- The third $\mathbb{R}$ represents the z-values.

Example of Plotting in a 3D Cartesian Plane

Example

Plot the points $(-1, 2, -3)$ and $(2, -4, 2)$.

Illustration

self-note: add the illustration here!!

Figure 21: Sample image illustrating the concept.

*Follow the line segments denoted in **purple** for an interpretation guide of how the three components contribute to the final point destination, for* $(-1, 2, -3)$.

*Follow the line segments denoted in **green** for an interpretation guide of how the three components contribute to the final point destination, for* $(2, -4, 2)$.

Interpreting Planes

Concept

Notice that in a 2D world, there is no notion of height when considering the x, y-plane. In a 3D world, $z = 0$.

Now, have a look at the basic planes for a 3D cartesian graph:

The xy plane:

$$x = 0 \qquad (x, y, 0)$$

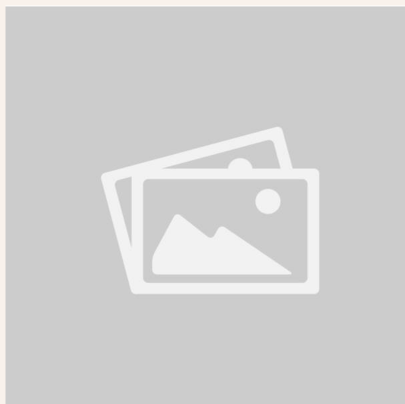

Figure 22: Sample image illustrating the concept.

The yz plane:

$$x = 0 \qquad (0, y, z)$$

Figure 23: Sample image illustrating the concept.

The xz plane:

$$x = 0 \qquad (x, 0, z)$$

Let's Try Going from 2D to 3D

Example

Consider the graph defined by $y = 2$ on a 2D cartesian graph:

Figure 25: Sample image illustrating the concept.

Here's how that would look like in a 3D cartesian space:

Figure 26: Sample image illustrating the concept.

Example

Consider the graph of a circle defined by

$$x^2 + y^2 = 4.$$

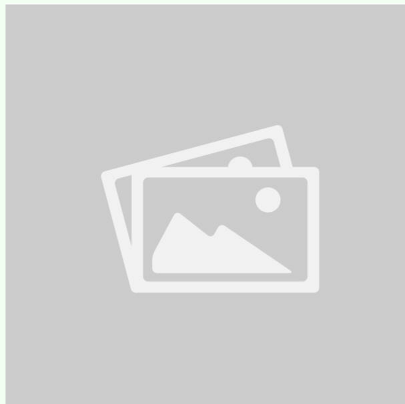

Figure 27: Sample image illustrating the concept.

If this circle is brought to the 3D world, stretched along the z-axis, for any values of z, then a cylinder is created (the cirle is the cross-section shape).

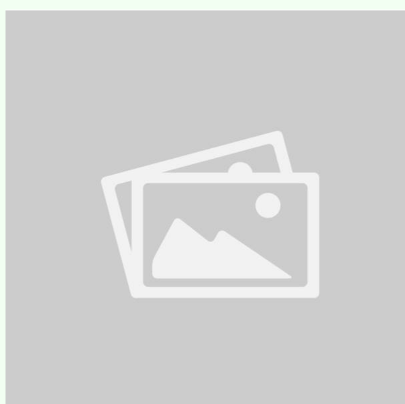

Figure 28: Sample image illustrating the concept.

Next Lecture: We Discuss Vectors!

Lecture Title

Note

This template is designed for MAT232 lecture notes. Replace this content with your specific lecture details.

Key Concepts

Definition

A **parametric equation** is a set of equations that express the coordinates of the points of a curve as functions of a variable, called a parameter.

Examples

Example

Example 1: Consider the parametric equations:

$$x = t, \quad y = t^2, \quad t \in \mathbb{R}.$$

- At $t = 0$, $(x, y) = (0, 0)$.
- At $t = 1$, $(x, y) = (1, 1)$.

This describes a parabola.

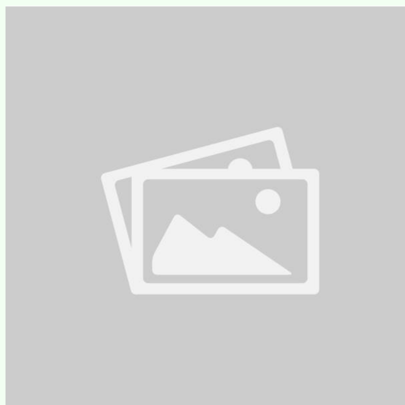

Figure 29: Sample image illustrating the concept.

Theorems and Proofs

Theorem

Theorem: If $x(t)$ and $y(t)$ are differentiable functions, the slope of the curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

Figure 30: Graphical representation of the theorem.

Additional Notes

Note

Always check the domain of the parameter t when solving problems involving parametric equations.