MAT232 - Lecture 17

[Lesson Topic(s)]

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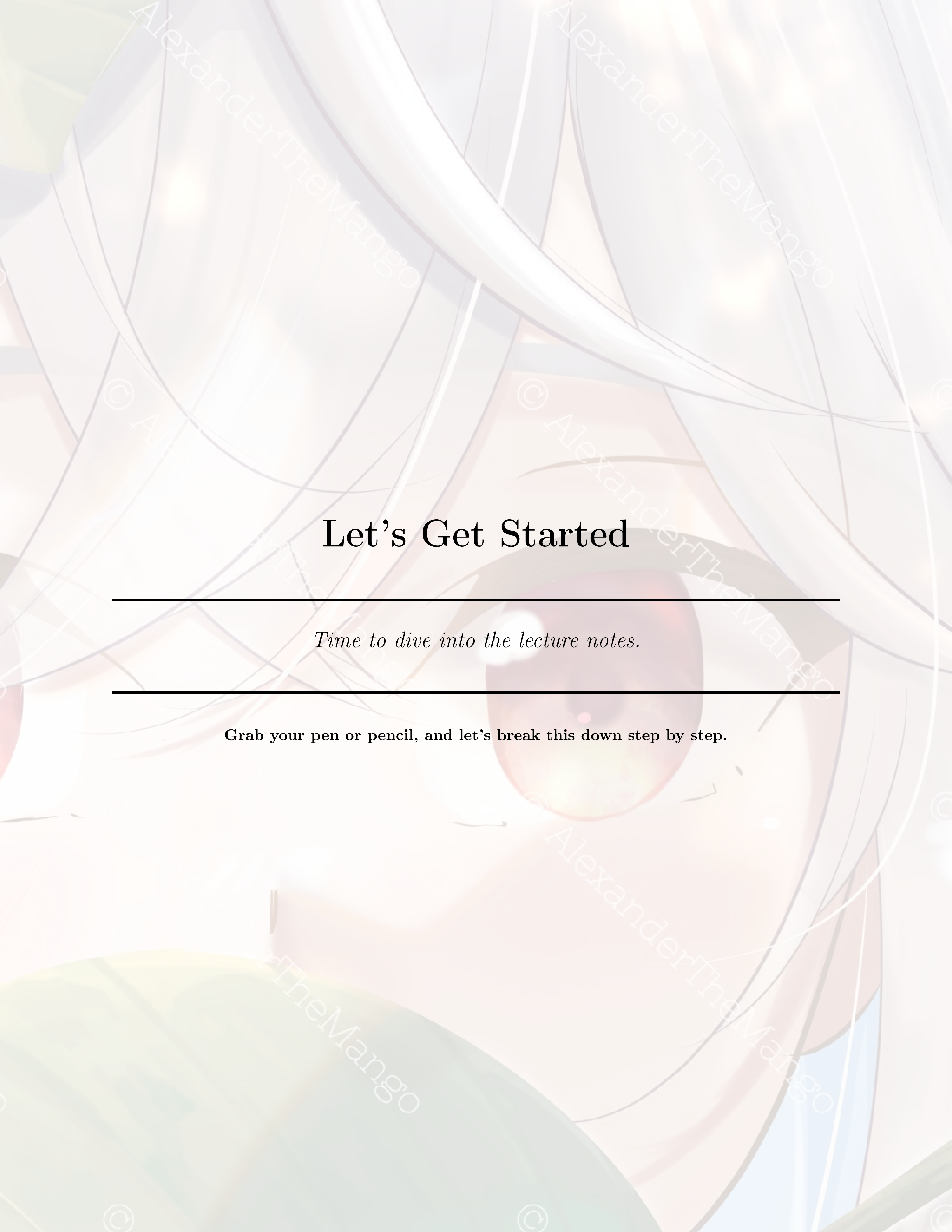
Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.



Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Reminders

Note

Term test 3 information:

Note

Final exam information:

Example

Set up $\iint(2x-y^2)dA$ where R is the triangular region bounded by $y=-x+1$ $y=x+1$, and $y=3$ in both ways (i.e. $dydx$ and $dxdy$).

Illustration

add stuff here lol

Solution

First, we need to find the intersection points of the lines. We have $-x+1=x+1$ which gives us $x=0$. So, the intersection points are $(0,1)$ and $(0,3)$.

- $dydx$: We have x going from 0 to 1 and y going from $-x+1$ to $x+1$. So, we have

$$\int_0^1\int_{-x+1}^{x+1}(2x-y^2)dydx$$

- $dxdy$: We have y going from 1 to 3 and x going from $-y+1$ to $y-1$. So, we have

$$\int_1^3\int_{-y+1}^{y-1}(2x-y^2)dxdy$$

Next, we may optionally evaluate both integrals to verify that they are equal.

Evaluating $\int_0^1\int_{-x+1}^{x+1}(2x-y^2)dydx$

We have

$$\begin{aligned}\int_0^1\int_{-x+1}^{x+1}(2x-y^2)dydx &= \int_0^1\left[2xy-\frac{y^3}{3}\right]_{-x+1}^{x+1}dx\\ &= \int_0^1\left[2x(x+1)-\frac{(x+1)^3}{3}-2x(-x+1)+\frac{(-x+1)^3}{3}\right]dx\\ &= \int_0^1\left[2x^2+2x-\frac{x^3+3x^2+3x+1}{3}+2x^2-2x+\frac{-x^3+3x^2-3x+1}{3}\right]dx\\ &= \int_0^1\left[4x^2-\frac{2}{3}\right]dx\\ &= \left[\frac{4}{3}x^3-\frac{2}{3}x\right]_0^1\\ &= \frac{4}{3}-\frac{2}{3}\\ &= \frac{2}{3}\end{aligned}$$

Evaluating $\int_1^3\int_{-y+1}^{y-1}(2x-y^2)dxdy$

We have

$$\begin{aligned}\int_1^3\int_{-y+1}^{y-1}(2x-y^2)dxdy &= \int_1^3\left[x^2-y^2x\right]_{-y+1}^{y-1}dy\\ &= \int_1^3\left[(y-1)^2-y^2(y-1)-((-y+1)^2-y^2(-y+1))\right]dy\end{aligned}$$

Example

Reverse the order of the integral defined by $\int_{-1}^{1}\int_{1+y^2}^{2y^2}(2x+y)dxdy$.

Solution

Consider that there are three integrals in the expression. We also have the following bounds:

- $-1 \leqslant x \leqslant 1$
- $1+y^2 \leqslant x \leqslant 2y^2$

We can reverse the order of the integral by considering the bounds of the integral. We have

$$\int_{y=-1}^{y=1}\int_{x=1+y^2}^{x=2y^2}(2x+y)dxdy$$
$$= \int_{x=0}^{x=1}\int_{y=-\sqrt{\frac{x}{2}}}^{y=\sqrt{\frac{x}{2}}}(2x+y)dxdy + \int_{x=1}^{x=2}\int_{y=-\sqrt{x-1}}^{y=\sqrt{\frac{x}{2}}}(2x+y)dxdy + \int_{x=1}^{x=2}\int_{y=-\sqrt{\frac{x}{2}}}^{y=-\sqrt{x-1}}(2x+y)dxdy$$

Example

Rewrite the following integral as a single double integral:

$$\int_0^{\frac{1}{2}} \int_0^{2y} (2 - x - 2y) dx dy + \int_{\frac{1}{2}}^{1} \int_0^{2-2y} (2 - x - 2y) dx dy$$

Solution

Illustration

- $x = 2 - 2y \implies y = 1 - \frac{x}{2}$

The triangular region is denoted by:

- $y = \frac{x}{2}$
- $x = 0$
- $y = 1 - \frac{x}{2}$

Thus, we consider y_{upper} and y_{lower} to find the bounds of the integral. We have

$$\begin{aligned} y_1 &= y_2 \\ \frac{x}{2} &= 1 - \frac{x}{2} \\ x &= 1 \end{aligned}$$

Tip

Soemtimes it is easier to evaluate the integral by reversing the order of integration.

Note

Sometimes it is the region that is difficult to describe, not the integral itself. In other times, it is the function that has a difficult integral.

Some functions are not integrable, such as $f(x) = \frac{1}{x}$ on $[0, 1]$.

However, feel free to always use u-substitution as done in first year calculus.

However, you can reverse the order of integration to actually solve the question from an integral that is not integrable.