MAT232 - Lecture 3

The Arc Length and Area of Parametric Curves, and an Introduction to Polar Coordinates

AlexanderTheMango

Prepared for January 13, 2025

Contents

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Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

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Preliminary Definitions and Theorems

Definition

Polar Coordinates.

Each point in the Cartesian plane can be represented in polar coordinates as an ordered pair (r, θ), where r is the radial coordinate (distance from the origin), and θ is the angular coordinate (angle measured from the positive x-axis). The correspondence between Cartesian coordinates (x, y) and polar coordinates (r, θ) is given by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Theorem

Theorem 1.4. Converting Points Between Coordinate Systems.

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between Cartesian and polar coordinates.

Example

Example 1.10. Converting Between Rectangular and Polar Coordinates.

1. Convert $(1, 1)$ to polar coordinates: Use $x = 1$ and $y = 1$. Then:

$$r^2 = x^2 + y^2 = 1^2 + 1^2 = 2 \implies r = \sqrt{2}, \quad \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \implies \theta = \frac{\pi}{4}.$$

   Therefore, $(1, 1)$ can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

Concept

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates.

1. Create a table with two columns: one for θ values and one for r values.
2. Calculate the corresponding r values for each θ.
3. Plot each ordered pair (r, θ) on the polar coordinate axes.
4. Connect the points and observe the resulting graph.

Example

Example 1.12. Graphing a Function in Polar Coordinates.
Graph the curve defined by $r = 4\sin\theta$.

1. Create a table of values for θ and calculate r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	2π
$r = 4\sin\theta$	0	2	$2\sqrt{2}$	4	0	0

2. Plot the points and connect them to form the curve. The result is a circle with radius 2 centered at $(0, 2)$ in rectangular coordinates.

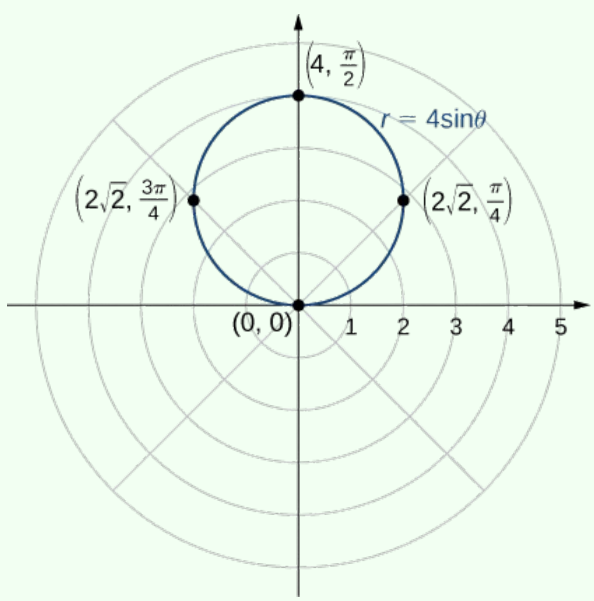

Figure 1: The graph of the function $r = 4\sin\theta$ is a circle.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

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Recall: 1st Year Calculus

Definition

The **definite integral** of a function $y = f(x)$, where $f(x) \geqslant 0$, represents the area under the curve from $x = a$ to $x = b$:

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx$$

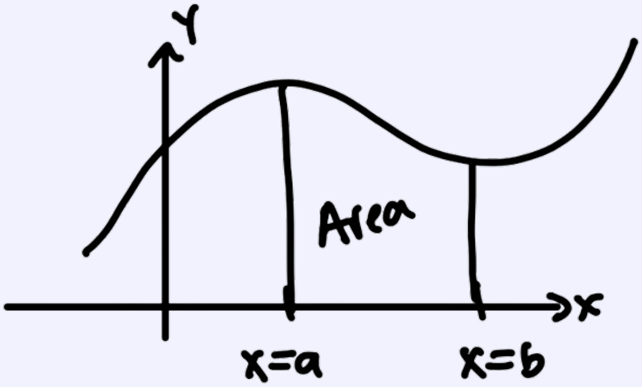

Figure 2: Illustration of the area under $y = f(x)$.

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Section 1.2: Area Enclosed by a Parametric Curve

Definition

A **parametric curve** is defined by:

$$x = f(t), \quad y = g(t), \quad \alpha \leqslant t \leqslant \beta$$

with the following properties:

- The curve lies above the x-axis.
- The curve does not self-intersect.

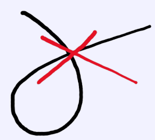

Figure 3: A curve that self-intersects.

Theorem

The area enclosed by the curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt$$

or equivalently:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} f(t) g'(t)\, dt$$

Alternative Forms for Area

Note

In specific cases, the area can also be calculated using:

$$\text{Area} = \int_{y=c}^{y=d} x(y)\, dy$$

or:

$$\text{Area} = \int_{x=a}^{x=b} y(x)\, dx$$

Example: Standard Parametric Curve

Example

Calculate the area enclosed by the parametric curve:

$$x = \cos(t), \quad y = \sin(t), \quad 0 \leqslant t \leqslant \pi$$

Solution

The area is calculated as:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$ and $y = g(t)$. Here, $f(t) = \cos(t)$, $g(t) = \sin(t)$, and $f'(t) = -\sin(t)$. Substituting:

$$\text{Area} = \int_{t=0}^{t=\pi} \sin(t)(-\sin(t))\, dt = \int_{t=0}^{t=\pi} -\sin^2(t)\, dt.$$

Using $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$, we get:

$$\text{Area} = -\int_{t=0}^{t=\pi} \frac{1}{2}(1 - \cos(2t))\, dt = -\frac{1}{2}\left[\int_{t=0}^{t=\pi} 1\, dt - \int_{t=0}^{t=\pi} \cos(2t)\, dt\right].$$

Evaluate the integrals:

$$\int_{t=0}^{t=\pi} 1\, dt = \pi, \quad \int_{t=0}^{t=\pi} \cos(2t)\, dt = \left[\frac{\sin(2t)}{2}\right]_0^{\pi} = 0.$$

Thus:

$$\text{Area} = -\frac{1}{2}(\pi - 0) = -\frac{\pi}{2}.$$

Taking the absolute value (since area is positive):

$$\text{Area} = \frac{\pi}{2}.$$

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Example: Cycloid

Example

Example: Find the area under the cycloid defined by:

$$x = t - \sin(t), \quad y = 1 - \cos(t), \quad 0 \leqslant t \leqslant 2\pi.$$

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt, \quad f'(t) = \frac{dx}{dt}.$$

Step 1: Substitution

From $x = t - \sin(t)$ and $y = 1 - \cos(t)$:

$$f'(t) = 1 - \cos(t), \quad g(t) = 1 - \cos(t).$$

Substitute into the formula:

$$\text{Area} = \int_0^{2\pi} (1 - \cos(t))^2\, dt.$$

Step 2: Expand and Separate Terms

Expand $(1 - \cos(t))^2$:

$$\text{Area} = \int_0^{2\pi} \left[1 - 2\cos(t) + \cos^2(t)\right] dt.$$

Split the integral:

$$\text{Area} = \int_0^{2\pi} 1\, dt - 2\int_0^{2\pi} \cos(t)\, dt + \int_0^{2\pi} \cos^2(t)\, dt.$$

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Example

Solution

...cont'd...

Step 3: Evaluate Each Term

1. First Term:

$$\int_0^{2\pi} 1\,dt = 2\pi.$$

2. Second Term:

$$\int_0^{2\pi} \cos(t)\,dt = [\sin(t)]_0^{2\pi} = 0.$$

3. Third Term:

Using $\cos^2(t) = \frac{1+\cos(2t)}{2}$:

$$\int_0^{2\pi} \cos^2(t)\,dt = \frac{1}{2}\int_0^{2\pi} 1\,dt + \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^{2\pi} 1\,dt = \pi, \quad \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt = 0.$$

Thus:

$$\int_0^{2\pi} \cos^2(t)\,dt = \pi.$$

Step 4: Combine Results

$$\text{Area} = 2\pi - 0 + \pi = 3\pi.$$

Answer

Area $= 3\pi$

Homework Practice Question: Area Under a Parametric Curve

Exercise

Find the area under the curve defined by

$$x = 3\cos(t) + \cos(3t), \quad y = 3\sin(t) - \sin(3t), \quad 0 \leqslant t \leqslant \pi.$$

Hint: Recall that $\sin^2(x) + \cos^2(x) = 1$.

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$, $y = g(t)$, and $f'(t) = \frac{dx}{dt}$.

Step 1: Differentiate $x(t)$
Given $x = 3\cos(t) + \cos(3t)$, compute:

$$f'(t) = \frac{d}{dt}[3\cos(t) + \cos(3t)] = -3\sin(t) - 3\sin(3t).$$

Step 2: Substitute into the Formula
The parametric area formula becomes:

$$\text{Area} = \int_0^\pi \big[3\sin(t) - \sin(3t)\big]\big[-3\sin(t) - 3\sin(3t)\big]\, dt.$$

Step 3: Simplify the Expression
Expand the product:

$$\big[3\sin(t) - \sin(3t)\big]\big[-3\sin(t) - 3\sin(3t)\big] = -9\sin^2(t) - 9\sin(t)\sin(3t) + 3\sin(3t)\sin(t) + 3\sin^2(3t).$$

Combine terms:

$$-9\sin^2(t) + 3\sin^2(3t) - 6\sin(t)\sin(3t).$$

...cont'd...

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Exercise

Solution

...cont'd..

Using the product-to-sum identity for $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$:

$$\sin(t)\sin(3t) = \frac{1}{2}[\cos(2t) - \cos(4t)].$$

Substitute this back:

$$\text{Area} = \int_0^\pi \big[-9\sin^2(t) + 3\sin^2(3t) - 3[\cos(2t) - \cos(4t)]\big]\,dt.$$

Step 4: Break the Integral into Separate Terms

Split the integral:

$$\text{Area} = -9\int_0^\pi \sin^2(t)\,dt + 3\int_0^\pi \sin^2(3t)\,dt - 3\int_0^\pi \cos(2t)\,dt + 3\int_0^\pi \cos(4t)\,dt.$$

Step 5: Evaluate Each Integral

1. First Term ($-9\int_0^\pi \sin^2(t)\,dt$):

Use the identity $\sin^2(t) = \dfrac{1-\cos(2t)}{2}$:

$$\int_0^\pi \sin^2(t)\,dt = \int_0^\pi \frac{1-\cos(2t)}{2}\,dt = \frac{1}{2}\int_0^\pi 1\,dt - \frac{1}{2}\int_0^\pi \cos(2t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^\pi 1\,dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^\pi \cos(2t)\,dt = \frac{1}{2}[0] = 0.$$

So:

$$\int_0^\pi \sin^2(t)\,dt = \frac{\pi}{2}.$$

Multiply by -9:

$$-9\int_0^\pi \sin^2(t)\,dt = -9\cdot\frac{\pi}{2} = -\frac{9\pi}{2}.$$

...cont'd...

Exercise

Solution

...cont'd...

2. Second Term ($3\int_0^\pi \sin^2(3t)\,dt$):

Similarly, $\sin^2(3t) = \dfrac{1-\cos(6t)}{2}$:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{1}{2}\int_0^\pi 1\,dt - \frac{1}{2}\int_0^\pi \cos(6t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^\pi 1\,dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^\pi \cos(6t)\,dt = 0.$$

So:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{\pi}{2}.$$

Multiply by 3:

$$3\int_0^\pi \sin^2(3t)\,dt = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}.$$

3. Third Term ($-3\int_0^\pi \cos(2t)\,dt$):

Since $\int_0^\pi \cos(2t)\,dt = 0$:

$$-3\int_0^\pi \cos(2t)\,dt = 0.$$

4. Fourth Term ($3\int_0^\pi \cos(4t)\,dt$):

Similarly, $\int_0^\pi \cos(4t)\,dt = 0$:

$$3\int_0^\pi \cos(4t)\,dt = 0.$$

Step 6: Combine Results

Add the evaluated terms:

$$\text{Area} = -\frac{9\pi}{2} + \frac{3\pi}{2} + 0 + 0 = -\frac{6\pi}{2} = -3\pi.$$

However, the area is always positive, so:

$$\text{Area} = 3\pi.$$

Exercise

Solution

...cont'd...

Answer

Area $= 3\pi$

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The Arc Length of a Parametric Curve

Theorem

Let a curve be parameterized by t, such that:

$$x = x(t) \quad \text{and} \quad y = y(t), \quad \text{for } t \in [\alpha, \beta].$$

The **arc length** L of the curve between $t = \alpha$ and $t = \beta$ is given by:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Concept

Consider two points (x_1, y_1) and (x_2, y_2) on a curve. The differences in their coordinates are defined as:

$$\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1$$

The distance D between the two points is given by the Pythagorean theorem:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substituting Δx and Δy:

$$D = \sqrt{\Delta x^2 + \Delta y^2}$$

Now, consider the curve parameterized by t, where $x = x(t)$ and $y = y(t)$. Dividing Δx and Δy by the parameter Δt:

$$D \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$$

Taking the limit as $\Delta t \to 0$, this becomes a Riemann sum. Therefore, the arc length L of the curve is:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note

L represents the total length of the curve between $t = \alpha$ and $t = \beta$. This formula is provided on the formula sheet for term test 1.

Illustration

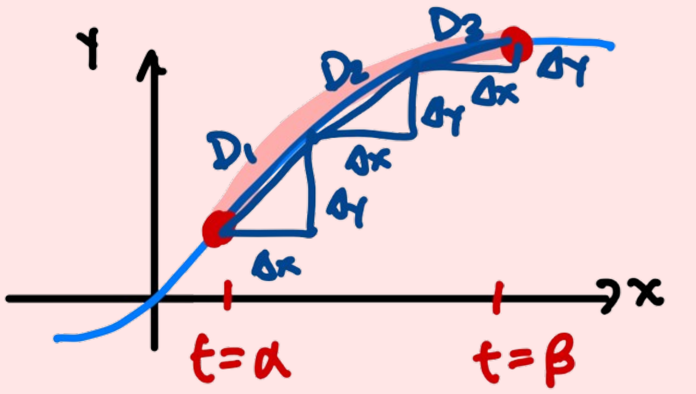

Figure 4: Visualizing the arc length theorem — the connection between Riemann sums and the integral for a curve's length.

Example: Parametric Curve

Example

Find the arc length of the curve defined by:

$$x = 3\cos(t), \quad y = 3\sin(t), \quad t \in [0, 2\pi].$$

Solution

The arc length is denoted by L. We evaluate it as follows:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2} dt \\ &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)} dt. \end{aligned}$$

Using the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, simplify the integrand:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9 \cdot (\sin^2(t) + \cos^2(t))} dt \\ &= \int_0^{2\pi} \sqrt{9 \cdot 1}\, dt \\ &= \int_0^{2\pi} 3\, dt. \end{aligned}$$

The integral simplifies to:

$$\begin{aligned} L &= 3 \int_0^{2\pi} 1\, dt \\ &= 3\,[t]_0^{2\pi} \\ &= 3 \cdot (2\pi - 0) \\ &= 6\pi. \end{aligned}$$

Answer

The arc length of the curve is $L = 6\pi$.

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Homework Practice Problem: Parametric Curve

Note

Find the arc length of the curve defined by:

$$x = 3t^2, \quad y = 2t^3, \quad 1 \leqslant t \leqslant 3.$$

Solution

The arc length is denoted by L, which is evaluated as follows:

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Compute the derivatives:

$$\frac{dx}{dt} = \frac{d}{dt}(3t^2) = 6t, \quad \frac{dy}{dt} = \frac{d}{dt}(2t^3) = 6t^2.$$

Substitute these into the arc length formula:

$$\begin{aligned} L &= \int_1^3 \sqrt{(6t)^2 + (6t^2)^2} dt \\ &= \int_1^3 \sqrt{36t^2 + 36t^4} dt \\ &= \int_1^3 \sqrt{36t^2(1+t^2)} dt \\ &= \int_1^3 6t\sqrt{1+t^2} dt. \end{aligned}$$

Use substitution to simplify the integral. Let:

$$u = 1 + t^2, \quad \frac{du}{dt} = 2t, \quad dt = \frac{du}{2t}.$$

Change the limits of integration:

$$\text{When } t = 1,\ u = 1 + 1^2 = 2; \quad \text{When } t = 3,\ u = 1 + 3^2 = 10.$$

Substitute into the integral:

$$\begin{aligned} L &= \int_2^{10} 6t \cdot \sqrt{u} \cdot \frac{du}{2t} \\ &= \int_2^{10} 3\sqrt{u}\, du. \end{aligned}$$

Note

Solution

Evaluate the integral:

$$\begin{aligned} L &= 3\int_2^{10} u^{1/2}\,du \\ &= 3\left[\frac{2}{3}u^{3/2}\right]_2^{10} \\ &= 2\left[u^{3/2}\right]_2^{10} \\ &= 2\left[(10)^{3/2} - (2)^{3/2}\right]. \end{aligned}$$

Simplify the result:

$$L = 2\left[10\sqrt{10} - 2\sqrt{2}\right].$$

Answer

The arc length of the curve is $L = 2\left(10\sqrt{10} - 2\sqrt{2}\right)$.

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Section 1.3: Polar Coordinates

Definition

Polar coordinates represent a point in the plane by specifying its distance from the origin (r) and the angle (θ) it makes with the positive x-axis, measured counterclockwise.

Concept

Cartesian Coordinates:

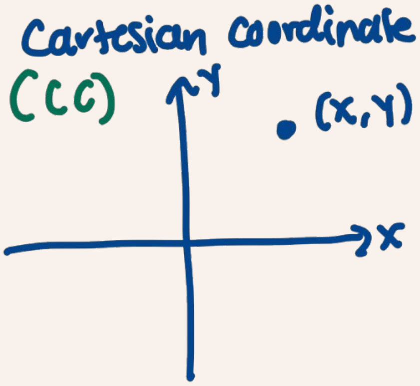

Figure 5: Graphical representation of Cartesian coordinates.

Polar Coordinates:

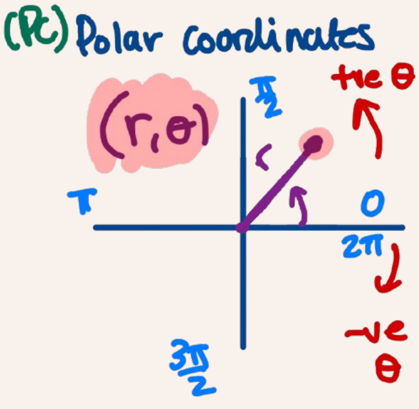

Figure 6: Graphical representation of Polar coordinates.

How to Work with Polar Coordinates

Tip

1. Start from the origin and move radially outward by a distance equal to r.
2. From this position, rotate the point counterclockwise by an angle θ (in radians or degrees).
3. The resultant position is the point represented by the polar coordinate (r, θ).

Converting from Cartesian to Polar Coordinates

Algorithm

1. Given a point (x, y) in Cartesian coordinates:
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$
2. r represents the radial distance from the origin.
3. θ is the angle measured from the positive x-axis to the line segment joining the origin to the point.
4. Adjust θ based on the quadrant of the point to ensure the correct angle.

Converting from Polar to Cartesian Coordinates

Algorithm

1. Given a point (r, θ) in polar coordinates:
$$x = r\cos(\theta), \quad y = r\sin(\theta).$$
2. Compute x and y using trigonometric functions with r and θ.
3. The result (x, y) represents the Cartesian coordinates of the point.

Examples: Converting Between Rectangular and Polar Coordinates

Example

To convert a point from rectangular coordinates (x, y) to polar coordinates (r, θ), we use the following formulas:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, let's convert the following points:

Exercise

1. For the point $(1, 1)$:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

So, the polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Exercise

2. For the point $(-3, 4)$:

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5, \quad \theta = \tan^{-1}\left(\frac{4}{-3}\right) = \tan^{-1}\left(-\frac{4}{3}\right)$$

Since the point is in the second quadrant, we add π to the angle:

$$\theta = \pi - \tan^{-1}\left(\frac{4}{3}\right)$$

The polar coordinates are approximately $\left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$.

Exercise

3. For the point $(0, 3)$:

$$r = \sqrt{0^2 + 3^2} = 3, \quad \theta = \frac{\pi}{2}$$

So, the polar coordinates are $(3, \frac{\pi}{2})$.

Examples: Converting Polar to Rectangular Coordinates

Example

To convert a point from polar coordinates (r, θ) to rectangular coordinates (x, y), we use the following formulas:

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Now, let's convert the following points:

Exercise

1. For the point $(3, \frac{\pi}{3})$:

$$x = 3\cos\left(\frac{\pi}{3}\right) = 3 \times \frac{1}{2} = \frac{3}{2}, \quad y = 3\sin\left(\frac{\pi}{3}\right) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

So, the rectangular coordinates are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

Exercise

2. For the point $(2, \frac{3\pi}{2})$:

$$x = 2\cos\left(\frac{3\pi}{2}\right) = 2 \times 0 = 0, \quad y = 2\sin\left(\frac{3\pi}{2}\right) = 2 \times (-1) = -2$$

So, the rectangular coordinates are $(0, -2)$.

Exercise

3. For the point $(6, -\frac{5\pi}{6})$:

$$x = 6\cos\left(-\frac{5\pi}{6}\right) = 6 \times (-\frac{\sqrt{3}}{2}) = -3\sqrt{3}, \quad y = 6\sin\left(-\frac{5\pi}{6}\right) = 6 \times (-\frac{1}{2}) = -3$$

So, the rectangular coordinates are $(-3\sqrt{3}, -3)$.