MAT232 - Lecture 4

Polar Coordinates and Curves

AlexanderTheMango

Prepared for January 16, 2025

Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Polar Coordinates - Key Theorems

Converting Points between Coordinate Systems

Theorem

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between rectangular and polar coordinates.

Uniqueness of Polar Coordinates

Proposition

Every point in the plane has an infinite number of representations in polar coordinates. Specifically, the polar coordinates (r, θ) of a point are not unique.

Remark

For example, the polar coordinates $(2, \pi/3)$ and $(2, 7\pi/3)$ both represent the same point in the rectangular coordinate system. Additionally, the value of r can be negative. Therefore, the point with polar coordinates $(-2, 4\pi/3)$ represents the same rectangular point as $(2, \pi/3)$.

Symmetry of Polar Curves

Theorem

Polar curves can exhibit symmetry similar to those in rectangular coordinates. The key symmetries to identify are:

- **Symmetry with respect to the polar axis:** A curve is symmetric with respect to the polar axis if replacing θ with $-\theta$ in its equation yields the same curve.
- **Symmetry with respect to the line $\theta = \frac{\pi}{2}$:** A curve is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if replacing θ with $\pi - \theta$ yields the same curve.
- **Symmetry with respect to the pole (origin):** A curve is symmetric with respect to the pole if replacing r with $-r$ yields the same curve.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Plotting Polar Coordinates

Recall the Content from Last Lecture

Note

Converting between Cartesian coordinates (x, y) and Polar coordinates (r, θ):

Algorithm

From Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

From Polar to Cartesian:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Converting Between Degrees and Radians:

Algorithm

- **Degrees to Radians:** Multiply by $\dfrac{\pi}{180^\circ}$

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180^\circ}$$

- **Radians to Degrees:** Multiply by $\dfrac{180^\circ}{\pi}$

$$\text{Degrees} = \text{Radians} \times \frac{180^\circ}{\pi}$$

Understanding the Convention for r in Polar Coordinates

Concept

In polar coordinates, a **point** is represented as (r, θ), where:

- r is the radial distance from the origin (how far the point is from the origin).
- θ is the angle, measured counterclockwise from the positive x-axis.

Note

Special Case: When r is Negative

- A negative r in $(-r, \theta)$ is interpreted as the point being reflected through the origin.
- The equivalent representation is:

$$(-r, \theta) = (r, \theta + 180^\circ)$$

or in radians:

$$(-r, \theta) = (r, \theta + \pi)$$

Intuition

- Reflecting (r, θ) through the origin is the same as rotating the point by 180° (or π radians).
- This property simplifies polar plots by offering alternate representations of the same point.

Tip

When plotting points, ensure to label points clearly on the polar grid, and verify angle conversions and reflections for accuracy.

Example: Plotting Points

Example

Let us plot the following points in polar coordinates:

$$(3, -45^\circ), \quad (3, 225^\circ), \quad (4, 330^\circ), \quad (1, -45^\circ)$$

Algorithm

Step-by-Step Process:

1. For each point, identify r and θ.
2. If θ is negative or exceeds 360°, convert it to a standard range:

$$\theta \in [0^\circ, 360^\circ)$$

   using $\theta = \theta + 360^\circ$ (for negative angles) or subtracting 360° (for angles over 360°).
3. Plot the point by measuring θ counterclockwise from the positive x-axis and placing it at a distance r from the origin.

Solution

- For $(3, -45^\circ)$: Add 360° to -45° to convert θ to 315°. Plot as $(3, 315^\circ)$.
- For $(3, 225^\circ)$: Already within the standard range, so plot directly.
- For $(4, 330^\circ)$: Angle is standard, so plot directly.
- For $(1, -45^\circ)$: Add 360° to -45°, yielding $(1, 315^\circ)$.

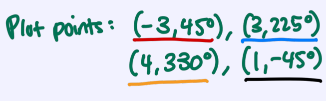

Figure 1: Colour Legend

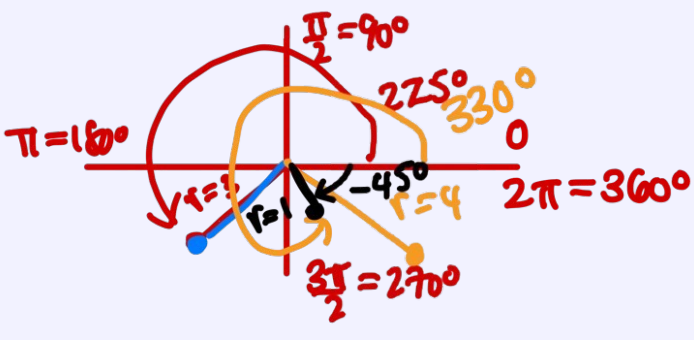

Figure 2: Polar Coordinates Plot and "Trajectories"

Converting Between Cartesian and Polar Coordinates

Example: Converting from Polar Coordinates to Cartesian Coordinates

Example

Find the **rectangular coordinates** of the point p with polar coordinates $(6, \dfrac{\pi}{3})$.

Solution

To convert from polar to Cartesian coordinates, use:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Substitute $r = 6$ and $\theta = \frac{\pi}{3}$:

$$x = 6\cos\left(\frac{\pi}{3}\right) = 6 \cdot \frac{1}{2} = 3, \quad y = 6\sin\left(\frac{\pi}{3}\right) = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}.$$

Thus, the Cartesian coordinates are:

$$(x, y) = (3, 3\sqrt{3}).$$

Answer

The rectangular coordinates are $(3, 3\sqrt{3})$.

Example: Converting from Cartesian Coordinates to Polar Coordinates

Example

Find the **polar coordinates** of the point p with rectangular coordinates $(-2, 2\sqrt{3})$.

Solution

To find the polar coordinates, use:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}.$$

Step 1: Solve for r:

$$r^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16 \implies r = 4.$$

Step 2: Solve for θ:

$$\tan(\theta) = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}.$$

The point $(-2, 2\sqrt{3})$ lies in Quadrant II. The reference angle for $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$. Thus:

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Tip

Alternatively, for a negative angle:

$$\theta = -\frac{\pi}{3}, \quad \text{adjust to Quadrant II: } -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Thus, $(r, \theta) = (4, \frac{2\pi}{3})$.

Answer

The polar coordinates are $(4, \dfrac{2\pi}{3})$ or $(4, 120^\circ)$.

Note

The professor recommends using graphing tools like Desmos or GeoGebra to enhance your understanding of the material. These tools are especially helpful for plotting lines and circles, which are key concepts in MAT232. Getting comfortable with them will make the course much easier.

Sketching Polar Curve Functions

Exercise

Consider $r = f(\theta)$. Sketch the following functions:

(a) $r = 1$

(b) $\theta = \frac{\pi}{4}$

(c) $r = \theta, \quad \theta \geqslant 0$

(d) $r = \sin(\theta)$

(e) $r = \cos(2\theta)$

(a) $r = 1$

Solution

Here, $r = 1$, and θ can take any value.
This means the point is always at a distance of 1 from the origin, regardless of the angle θ. Hence, the graph is a **circle** with radius 1, centred at the origin.

Concept

Cartesian Conversion
From the polar equation:

$$x^2 + y^2 = r^2 = 1$$

This confirms the equation of a unit circle in Cartesian coordinates.

Figure 3: The graph of $r = 1$.

(b) $\theta = \dfrac{\pi}{4}$

Solution

Here, $\theta = \dfrac{\pi}{4}$, and r can take any value.
This represents all points that lie along the line passing through the origin at an angle of $\frac{\pi}{4}$ (or 45°) with the positive x-axis. The graph is a **straight line** through the origin.

Concept

Cartesian Conversion

In polar coordinates:

$$\tan(\theta) = \frac{y}{x}$$

Substituting $\theta = \frac{\pi}{4}$, we get:

$$\tan\left(\frac{\pi}{4}\right) = 1 \quad \Rightarrow \quad y = x$$

Thus, the Cartesian equation is $y = x$.

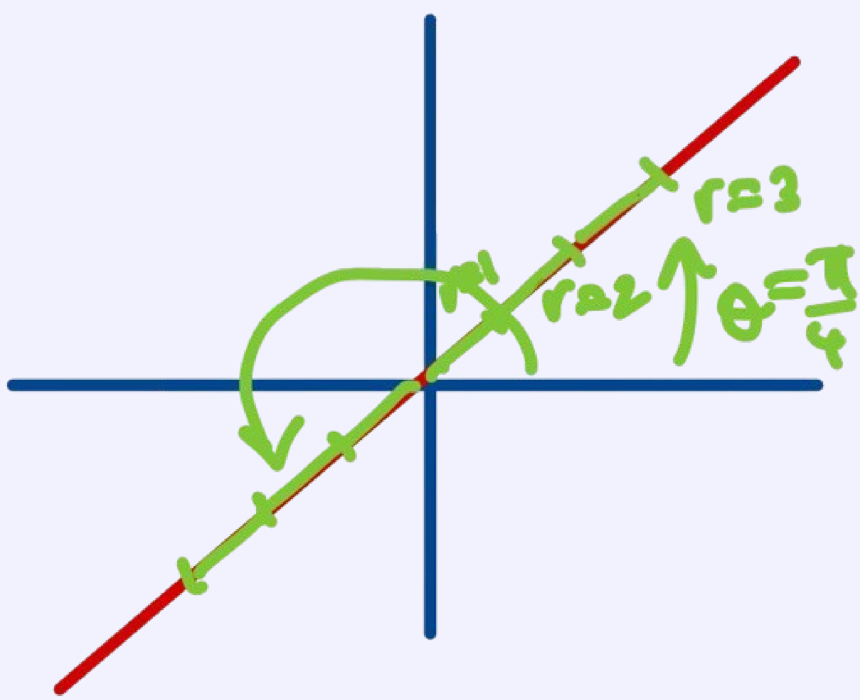

Figure 4: The graph of $\theta = \frac{\pi}{4}$.

(c) $r = \theta, \quad \theta \geqslant 0$

Solution

Here, r increases as θ increases. This creates a **spiral** that starts at the origin and winds outward as θ grows.

Concept

Table of Values

θ	r
0	0
$\frac{\pi}{6}$	$\frac{\pi}{6} \approx 0.52$
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx 0.79$
$\frac{\pi}{3}$	$\frac{\pi}{3} \approx 1.05$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$

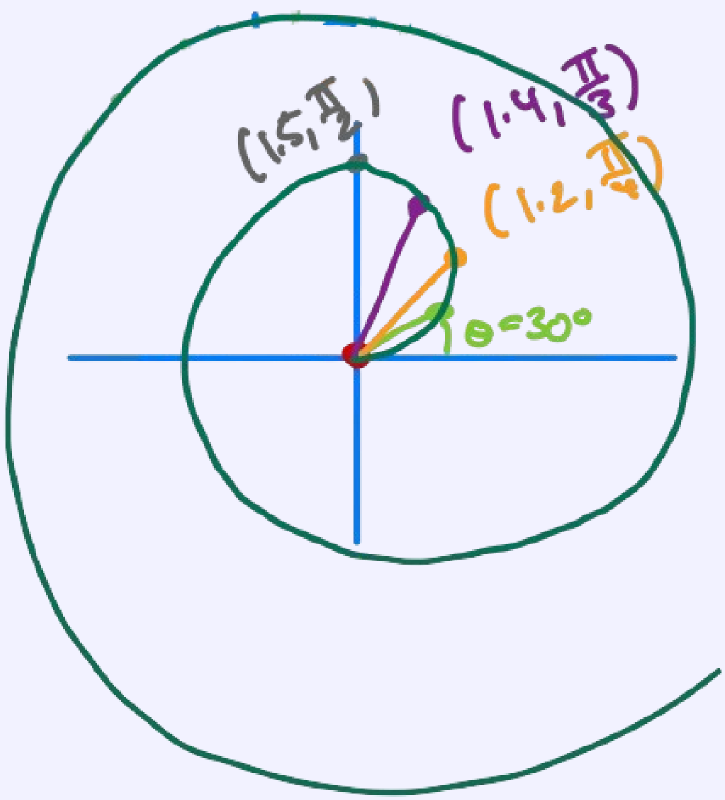

Figure 5: The graph of $r = \theta$.

(d) $r = \sin(\theta)$

Solution

Here, $r = \sin(\theta)$. Since $\sin(\theta)$ oscillates between 0 and 1, the graph forms a **cardioid** (which, in this case, is a perfect circle).

Concept

Cartesian Conversion

Using $r^2 = x^2 + y^2$ and $r = \sin(\theta)$, we get:

$$x^2 + y^2 = y \quad \Rightarrow \quad \left(x^2 + y^2\right) - y = 0$$

Completing the square for y:

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

This is a circle centred at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$.

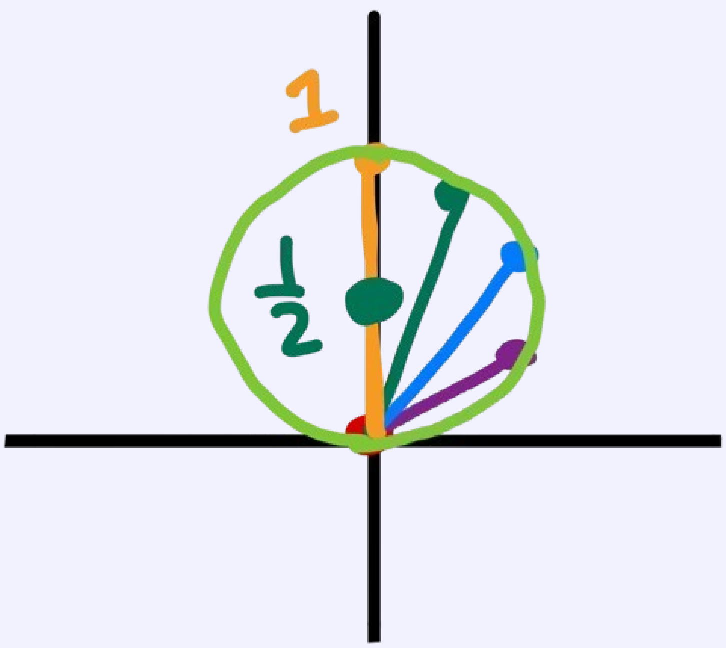

Figure 6: The graph of $r = \sin(\theta)$.

(e) $r = \cos(2\theta)$

Solution

The equation $r = \cos(2\theta)$ represents a **four-petaled rose**. As θ varies from 0 to 2π, r oscillates between -1 and 1, creating the petals.

Concept

Table of Values

Evaluate r at critical angles:

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	1	0	-1	0	1	0	-1	0	1

The curve is symmetric about the polar axis, with petals centered at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

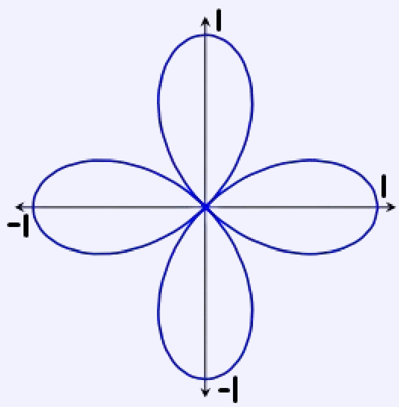

Figure 7: The graph of $r = \cos(2\theta)$.

Homework Practice Problem: Polar Graph Sketching

Exercise

Try sketching the curve:

$$r = \cos\theta$$

under the same context as the previous questions.

Solution

Here, $r = \cos\theta$. Since $\cos\theta$ oscillates between -1 and 1, the graph will form a **limacon** (which, in this case, simplifies to a standard circle) with an inner loop.

Concept

Table of Values

To visualize the curve, create a table of key points:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

Note

Plot the points in the table and connect them smoothly to form the graph. Note that for negative r, the points are plotted in the opposite direction from the origin.

...cont'd...

Exercise

Solution

...cont'd...

Concept

Cartesian Conversion

To convert the polar equation $r = \cos\theta$ into Cartesian form:

1. Recall the polar-to-Cartesian relationships:
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$
2. Substitute $r = \cos\theta$ into $x = r\cos\theta$:
$$x = (\cos\theta)(\cos\theta) = \cos^2\theta.$$
3. Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, write $\cos^2\theta$ as:
$$\cos^2\theta = 1 - \sin^2\theta.$$
4. Replace $\sin^2\theta$ with $\left(\frac{y}{r}\right)^2$ since $\sin\theta = \frac{y}{r}$:
$$x = 1 - \left(\frac{y}{r}\right)^2.$$
Multiply through by $r^2 = x^2 + y^2$ to eliminate r:
$$r^2 x = r^2 - y^2 \quad \Rightarrow \quad (x^2+y^2)x = x^2 + y^2 - y^2.$$

After simplifying, we find:
$$x^2 + y^2 = x.$$

...cont'd...

Exercise

Solution

...cont'd...

Concept

Completing the Square:
To identify the shape of this equation:

1. Isolate $x^2 + x$:
$$x^2 - x + y^2 = 0.$$

2. Complete the square for x:
$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + y^2 = 0 \quad \Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2.$$

This represents a circle with:

- **Centre:** $\left(\frac{1}{2}, 0\right)$,
- **Radius:** $\frac{1}{2}$.

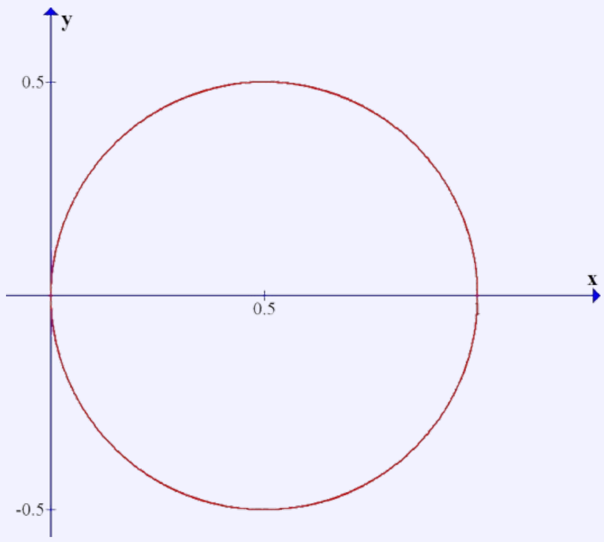

Figure 8: Graph of $r = \cos\theta$, showing the limacon with an inner loop.

Derivatives of Polar Curves and Their Applications

Tangents to Polar Curves: Finding Slopes and Key Conditions

Theorem

Tangent Slopes to Polar Curves

Let an arbitrary polar curve be given by $r = f(\theta)$, where r and θ are related. The slope of the tangent line to the curve at a given point can be expressed as:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Tangent Types:

- **Horizontal Tangents:** Occurs when

$$\frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} \neq 0.$$

- **Vertical Tangents:** Occurs when

$$\frac{dx}{d\theta} = 0 \quad \text{and} \quad \frac{dy}{d\theta} \neq 0.$$

- **Singular Points** (typically discarded in MAT232)**:** Occurs when both derivatives are zero...

$$\frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} = 0.$$

...cont'd...

Theorem

...cont'd...

Concept

Derivation of the Tangent Slope

To derive the formula for $\frac{dy}{dx}$, we use the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}, \quad \text{where} \quad \frac{dx}{d\theta} \neq 0.$$

Recall that $r = f(\theta)$. First, compute the derivatives of x and y with respect to θ:

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\cos\theta\right) = \frac{df(\theta)}{d\theta}\cos\theta - f(\theta)\sin\theta,$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(f(\theta)\sin\theta\right) = \frac{df(\theta)}{d\theta}\sin\theta + f(\theta)\cos\theta.$$

Substitute these into the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{df(\theta)}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df(\theta)}{d\theta}\cos\theta - f(\theta)\sin\theta}.$$

Finally, replace $f(\theta)$ with r, so we have the final expression:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Example: Finding the Vertical Tangent Angles on a Polar Curve

Example

Find the **vertical tangent** angles of the polar curve $r = 1 - \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$.

Solution

We aim to determine the angles θ where the polar curve has vertical tangents. This occurs when $\frac{dx}{d\theta} = 0$, provided $\frac{dy}{d\theta} \neq 0$.

Step 1: Compute $\frac{dy}{dx}$

The derivative of a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Given $r = 1 - \cos\theta$, compute $\frac{dr}{d\theta}$:

$$\frac{dr}{d\theta} = \sin\theta.$$

Substitute $r = 1 - \cos\theta$ and $\frac{dr}{d\theta} = \sin\theta$ into the formula for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\sin\theta\sin\theta + (1 - \cos\theta)\cos\theta}{\sin\theta\cos\theta - (1 - \cos\theta)\sin\theta}.$$

Simplify the numerator and denominator:

$$\frac{dy}{dx} = \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{\sin\theta(2\cos\theta - 1)}.$$

Step 2: Condition for Vertical Tangents

Vertical tangents occur when:

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1) = 0,$$

provided $\frac{dy}{d\theta} \neq 0$.

...cont'd...

Example

...cont'd...

Solution

Solve $\frac{dx}{d\theta} = 0$:

$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta - 1 = 0.$$

1. When $\sin\theta = 0$:

$$\theta \in \{0, \pi\} \quad (\text{within } 0 \leqslant \theta \leqslant \pi).$$

2. When $2\cos\theta - 1 = 0$:

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

The candidates for vertical tangents are:

$$\theta \in \{0, \frac{\pi}{3}, \pi\}.$$

Step 3: Verify Each Candidate

To confirm vertical tangents, check $\frac{dy}{dx} \neq 0$ at these angles:

- For $\theta = 0$:

$$\frac{dx}{d\theta} = \sin(0)(2\cos(0) - 1) = 0, \quad \frac{dy}{d\theta} = 0 \quad \Rightarrow \text{discard.}$$

- For $\theta = \frac{\pi}{3}$:

$$\frac{dx}{d\theta} \neq 0 \quad \Rightarrow \text{valid vertical tangent.}$$

- For $\theta = \pi$:

$$\frac{dx}{d\theta} = \sin(\pi)(2\cos(\pi) - 1) = 0, \quad \frac{dy}{d\theta} \neq 0 \quad \Rightarrow \text{valid vertical tangent.}$$

Answer

The vertical tangents are located at:

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \pi.$$

...cont'd...

Example

...cont'd...

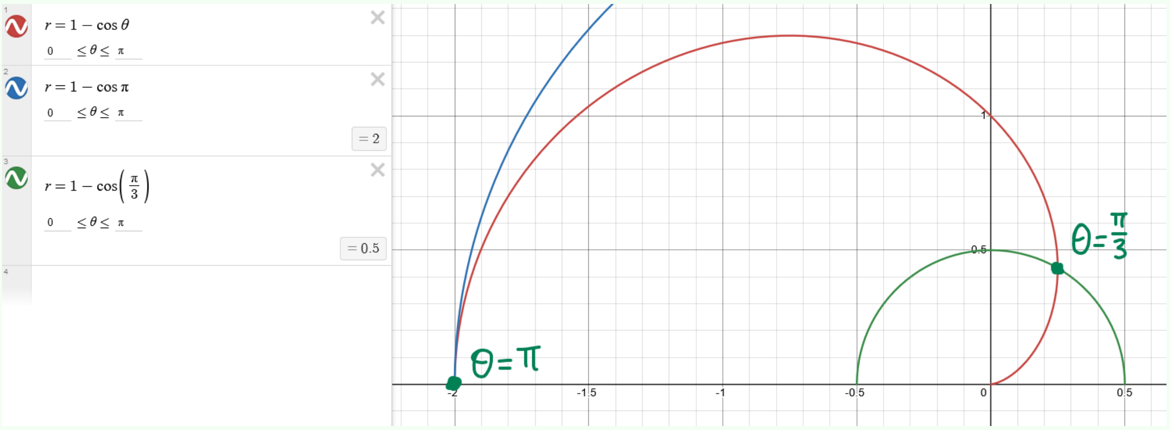

Figure 9: Illustration of vertical tangent positions for $r = 1 - \cos\theta$.

Equations of Circles in the Cartesian Plane

Theorem

The equations below describe the geometry of a circle in the Cartesian plane:

- **Standard Circle:** A circle centered at the origin $(0,0)$ with radius r is given by:

$$x^2 + y^2 = r^2.$$

  This equation represents a circle with radius r, where every point on the circle is exactly r units away from the origin.

- **General Circle:** A circle centered at (h,k) with radius r is described by:

$$(x-h)^2 + (y-k)^2 = r^2.$$

  This equation represents a circle with center at the point (h,k) and radius r. Each point on this circle is at a distance of r from the center (h,k).

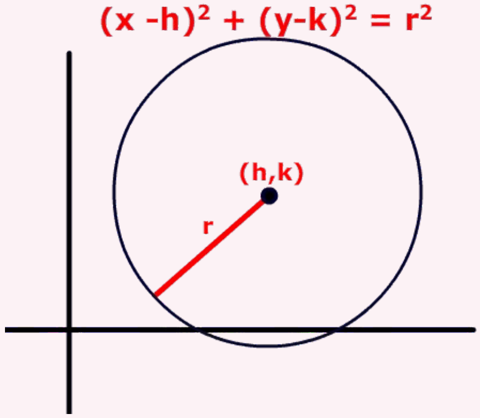

Figure 10: Graphical representation of a circle in the Cartesian plane.

Note

- In the case of the **standard circle**, the center of the circle is at the origin $(0,0)$, making the equation simple as $x^2 + y^2 = r^2$.
- For the **general circle**, the center is at (h,k), and the equation adjusts accordingly.
  - The terms $(x-h)$ and $(y-k)$ shift the circle to the new center.

Exercise: Sketching a Circle Curve

Exercise

Example: Sketch the curve described by the equation:

$$x^2 + y^2 - 2x = 10.$$

Solution

To sketch the curve, we need to rewrite the given equation in the standard form of a circle. We will do this by completing the square for the x-terms.

Step 1: Group x-terms and prepare to complete the square

First, we want to isolate the terms involving x on one side of the equation:

$$x^2 - 2x + y^2 = 10.$$

Notice that the y-terms are already in a simple form, and we will handle the x-terms separately.

Step 2: Complete the square for x

To complete the square, we focus on the expression $x^2 - 2x$:

- Divide the coefficient of x by 2:
$$-\frac{2}{2} = -1.$$
- Then square the result:
$$(-1)^2 = 1.$$
- Now, add and subtract this square, 1, to the equation to maintain equality:
$$x^2 - 2x + 1 + y^2 = 10 + 1.$$
This does not change the equation, but it allows us to rewrite the x-terms as a perfect square.

...cont'd...

Exercise

...cont'd...

Solution

Step 3: Rewrite as a perfect square

Now, we can express the x-terms as a perfect square:

$$(x-1)^2 + y^2 = 11.$$

This equation is now in the standard form of a circle, where:

$$(x-h)^2 + (y-k)^2 = r^2,$$

with $h = 1$, $k = 0$, and $r^2 = 11$.

Final Form and Interpretation

The equation $(x-1)^2 + y^2 = 11$ represents a circle. From the standard form, we can directly read the center and radius:

- The center of the circle is $(h, k) = (1, 0)$.
- The radius of the circle is $r = \sqrt{11}$, as the radius is the square root of r^2.

Tip

The circle is centered at $(1, 0)$ and has a radius of $\sqrt{11}$.

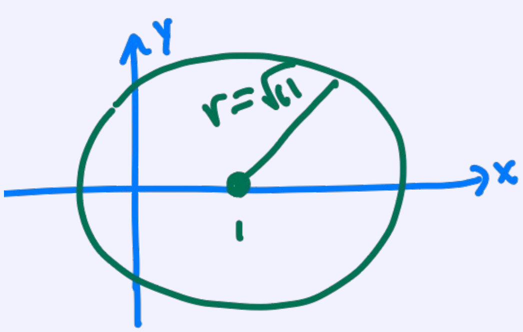

Figure 11: Illustration of the circle with center $(1, 0)$ and radius $\sqrt{11}$.