MAT232 - Lecture 6

vectors?

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Contents

Definitions and Theorems

Straight from the textbook — a thorough coverage.

Quick recap before diving into the lecture.

Introduction to Vectors

Definition

A **vector** is a quantity that has both magnitude and direction. Vectors can be optionally denoted in multiple ways:

- **Boldface Notation**: $\mathbf{v}$
- **Arrow Notation**: $\vec{v}$
- **Overline Notation**: $\overline{v}$

Note

In MAT232H5, the contents of a vector are typically written using angle bracket notation:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

For example, a 3D vector can be represented as:

$$\vec{v} = \langle 2, -1, 3 \rangle$$

Depending on the context, you might see $\mathbf{v} = \langle v_1, v_2 \rangle$ in 2D or $\mathbf{v} = \langle v_1, v_2, v_3, v_4 \rangle$ in higher dimensions.

Remark

Quantities such as velocity and force are examples of vectors because they require both magnitude and direction to be fully described.

Vector Representation

A **vector** in a plane is represented by a directed line segment (an arrow) with an **initial point** and a **terminal point**. The length of the segment represents its **magnitude**, denoted $\|\vec{v}\|$. A vector with the same initial and terminal point is called the **zero vector**, denoted $\vec{0}$.
Two vectors $\vec{v}$ and $\vec{w}$ are **equivalent** if they have the same magnitude and direction, written as $\vec{v} = \vec{w}$.

Exercise

Sketching Vectors
Sketch a vector in the plane from initial point $P(1, 1)$ to terminal point $Q(8, 5)$.

Basic Vector Operations

Scalar Multiplication

Multiplying a vector $\vec{v}$ by a scalar k results in a new vector $k\vec{v}$ with the following properties:

- Its magnitude is $|k|$ times the magnitude of $\vec{v}$.
- Its direction remains the same if $k > 0$.
- Its direction is reversed if $k < 0$.
- If $k = 0$ or $\vec{v} = \vec{0}$, then $k\vec{v} = \vec{0}$.

Note

The zero vector $\vec{0}$ is the vector with a magnitude of 0 and no direction (or any direction). It is the only vector that is orthogonal (perpendicular) to every vector, including itself.

Exercise

Scalar Multiplication

Given vector $\vec{v}$, sketch the vectors $3\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\vec{v}$.

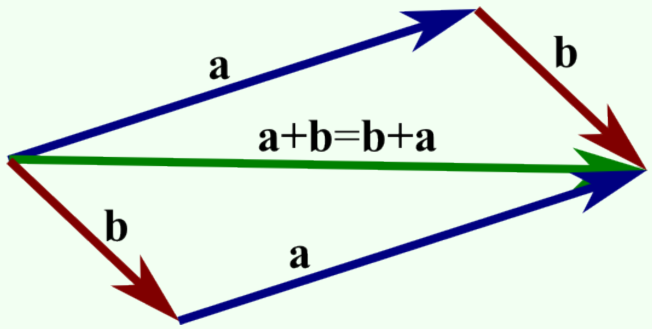
Vector Addition

The sum of two vectors $\vec{v}$ and $\vec{w}$ is constructed by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. The vector sum, $\vec{v} + \vec{w}$, is the vector from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.

Exercise

Vector Addition

Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v} + \vec{w}$ using both the triangle method and the parallelogram method.



Vector Subtraction

The difference $\vec{v} - \vec{w}$ is defined as $\vec{v} + (-\vec{w})$, where $-\vec{w}$ is the vector with the same magnitude as $\vec{w}$ but opposite direction.

Exercise

Vector Subtraction

Given vectors $\vec{v}$ and $\vec{w}$, sketch $\vec{v} - \vec{w}$.

Vector Components

A vector in standard position has its initial point at the origin $(0, 0)$. If the terminal point is (x, y), the vector is written in **component form** as $\vec{v} = \langle x, y \rangle$. The scalars x and y are called the **components** of $\vec{v}$.

Exercise

Expressing Vectors in Component Form

Express vector $\vec{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

Magnitude of a Vector

Definition

The magnitude of a vector $\vec{v} = \langle x, y \rangle$ is its length, and is given by:

$$\|\vec{v}\| = \sqrt{x^2 + y^2}.$$

Exercise

Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

Properties of Vector Operations

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let k and c be scalars. Then:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property)
3. $k(c\vec{v}) = (kc)\vec{v}$ (Associativity of Scalar Multiplication)
4. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (Distributive Property)

Proof

Proof of Commutative Property:

Proof.
Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$. Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle = \vec{v} + \vec{u}.$$

$\square$

Applications of Vectors

Example

Real-Life Applications

- A boat crossing a river experiences a force from its motor and a force from the river current. Both forces are vectors.
- A quarterback throwing a football applies a velocity vector to the ball, determining its speed and direction.

Introduction to Three-Dimensional Space

The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis, with an origin at the point of intersection $(0, 0, 0)$. This system is often denoted by $\mathbb{R}^3$.

Tip

The three-dimensional coordinate system follows the **right-hand rule**. If you align your right hand's fingers with the positive x-axis and curl them toward the positive y-axis, your thumb points in the direction of the positive z-axis.

Remark

This can also be visualized by holding a screwdriver with your right hand. If you rotate the screwdriver from the positive x-axis to the positive y-axis, the direction of the screwdriver represents the positive z-axis.

Note

The right-hand rule can serve as a visual aid for determining the direction of the cross product of two vectors.

Locating Points in Space

A point in three-dimensional space is represented by coordinates (x, y, z), where:

- x is the distance along the x-axis,
- y is the distance along the y-axis,
- z is the distance along the z-axis.

Exercise

Sketch the points $(-2, 3, -1)$ and $(1, -2, 3)$ in three-dimensional space.

Coordinate Planes in $\mathbb{R}^3$

The three coordinate planes in $\mathbb{R}^3$ are:

- The xy-plane: $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$,
- The xz-plane: $\{(x, 0, z) \mid x, z \in \mathbb{R}\}$,
- The yz-plane: $\{(0, y, z) \mid y, z \in \mathbb{R}\}$.

Note

The coordinate planes divide space into eight regions called **octants**. The first octant is where $x > 0$, $y > 0$, and $z > 0$; the other octants are numbered counterclockwise. It's like quadrants in 2D, but with that extra dimension!

Distance Formula in Three Dimensions

Theorem

The distance d between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Exercise

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Equations of Planes

A plane parallel to one of the coordinate planes can be described by:

- $z = c$ for a plane parallel to the xy-plane,
- $y = b$ for a plane parallel to the xz-plane,
- $x = a$ for a plane parallel to the yz-plane.

Exercise

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Equations of Spheres

A **sphere** is the shape described by the set of all points in space equidistant from a fixed point, called the **centre**. The distance from the centre to any point on the sphere is called the **radius**.

Theorem

Equation of a Sphere:
The sphere with centre (a, b, c) and radius r is given by:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Example

Find the standard equation of the sphere with center $(-2, 4, -5)$ and passing through point $(4, 4, -1)$.

Solution

Since the sphere passes through $(4, 4, -1)$, the distance between the center and this point is the radius:

$$r = \sqrt{(-2-4)^2 + (4-4)^2 + (-5+1)^2} = \sqrt{36+0+16} = \sqrt{52}.$$

Thus, the equation of the sphere is:

$$(x+2)^2 + (y-4)^2 + (z+5)^2 = 52.$$

Exercise

Find the equation of the sphere with diameter PQ, where $P = (2, -1, -3)$ and $Q = (-2, 5, -1)$.

Example: Describing and Graphing a Set of Points

Example

Describe and graph the set of points satisfying $(y+2)(z-3)=0$.

Solution

The equation implies $y+2=0$ or $z-3=0$, giving the solution set:

$$y=-2 \quad \text{or} \quad z=3.$$

Geometrically, this represents two planes:

- $y=-2$: A plane parallel to the xz-plane.
- $z=3$: A plane parallel to the xy-plane.

These planes intersect along the line $y=-2, z=3$, parallel to the x-axis. The graph consists of these two planes meeting along this line.

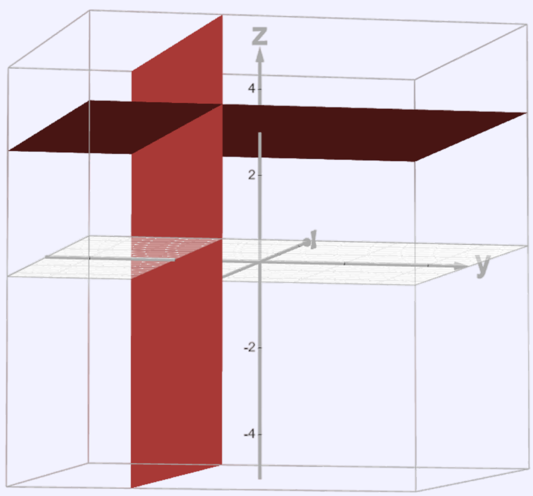


Figure 1: Graph of $(y+2)(z-3)=0$, represented by two planes intersecting along the line where $y=-2$ and $z=3$.

Tip

To graph these equations, consider visualizing the planes and their intersections in three-dimensional space. This can help you understand the relationships between the variables and the geometric shapes they represent.

Example

Describe and graph the set of points satisfying $x^2 + (z-2)^2 = 16$.

Solution

The given equation represents a **circular cylinder** in three-dimensional space:

$$x^2 + (z-2)^2 = 16.$$

This describes a cylinder with:

- **Center:** $(0, 2)$ in the xz-plane.
- **Radius:** 4.
- **Axis:** Parallel to the y-axis, meaning it extends infinitely along y.

Graphing:

- Draw a circle of radius 4 centered at $(0, 2)$ in the xz-plane.
- Extend this shape infinitely in the y-direction to form the cylinder.

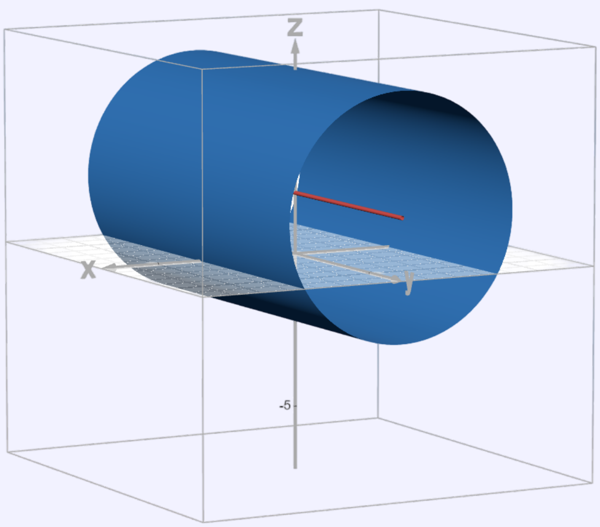


Figure 2: Graph of $x^2 + (z-2)^2 = 16$, representing a circular cylinder with center $(0, 2)$ and radius 4.

Working with Vectors in $\mathbb{R}^3$

A **three-dimensional vector** is a quantity with both magnitude and direction, represented by a directed line segment (arrow) in $\mathbb{R}^3$. A vector $\vec{v} = \langle x, y, z\rangle$ has its initial point at the origin $(0, 0, 0)$ and its terminal point at (x, y, z). The zero vector is $\vec{0} = \langle 0, 0, 0\rangle$.

Exercise

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\overrightarrow{ST}$ in component form and in standard unit form.

Vector Operations in $\mathbb{R}^3$

Definition

Let $\vec{v} = \langle x_1, y_1, z_1\rangle$ and $\vec{w} = \langle x_2, y_2, z_2\rangle$ be vectors in $\mathbb{R}^3$, and let k be a scalar. Then:

- **Vector Addition:** $\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2\rangle$
- **Scalar Multiplication:** $k\vec{v} = \langle kx_1, ky_1, kz_1\rangle$
- **Vector Subtraction:** $\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle$
- **Magnitude:** $\|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$
- **Unit Vector:** The unit vector in the direction of $\vec{v}$ is $\frac{1}{\|\vec{v}\|}\vec{v}$, provided $\vec{v} \neq \vec{0}$.

Example

Vector Operations in Three Dimensions

Let $\vec{v} = \langle -2, 9, 5 \rangle$ and $\vec{w} = \langle 1, -1, 0 \rangle$. Find the following vectors:

- $3\vec{v} - 2\vec{w}$
- $5\|\vec{w}\|$
- $\|5\vec{w}\|$
- A unit vector in the direction of $\vec{v}$

Solution

- $3\vec{v} - 2\vec{w} = 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle = \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle = \langle -8, 29, 15 \rangle$
- $5\|\vec{w}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}$
- $\|5\vec{w}\| = \|5\langle 1, -1, 0 \rangle\| = \|5, -5, 0\| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$
- A unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle -2, 9, 5 \rangle}{\sqrt{(-2)^2 + 9^2 + 5^2}} = \dfrac{\langle -2, 9, 5 \rangle}{\sqrt{110}} = \left\langle -\dfrac{2}{\sqrt{110}}, \dfrac{9}{\sqrt{110}}, \dfrac{5}{\sqrt{110}} \right\rangle$

Exercise

Let $\vec{v} = \langle -1, -1, 1 \rangle$ and $\vec{w} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\vec{v} + 3\vec{w}$.

Properties of Vectors in $\mathbb{R}^3$

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$, and let k and c be scalars. Then:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative Property)
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative Property)
3. $\vec{u} + \vec{0} = \vec{u}$ (Additive Identity Property)
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (Additive Inverse Property)
5. $k(c\vec{v}) = (kc)\vec{v}$ (Associativity of Scalar Multiplication)
6. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (Distributive Property)
7. $(k + c)\vec{u} = k\vec{u} + c\vec{u}$ (Distributive Property)
8. $1\vec{u} = \vec{u}$ and $0\vec{u} = \vec{0}$ (Identity and Zero Properties)

Proof

Proof of Commutative Property:

Proof.
Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$. Then:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle = \vec{v} + \vec{u}.$$

□

The Dot Product and Its Properties

The **dot product** of vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is given by:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

For two-dimensional vectors $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$, the dot product is:

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$

Exercise

Find $\vec{u} \cdot \vec{v}$, where $\vec{u} = \langle 2, 9, -1\rangle$ and $\vec{v} = \langle -3, 1, -4\rangle$.

Properties of the Dot Product

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let c be a scalar. Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutative Property)
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributive Property)
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ (Associative Property)
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ (Property of Magnitude)

Proof

Proof of Commutative Property:

Proof.
Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}.$$

□

Example

Let $\vec{a} = \langle 1, 2, -3 \rangle$, $\vec{b} = \langle 0, 2, 4 \rangle$, and $\vec{c} = \langle 5, -1, 3 \rangle$. Find each of the following products:

- $(\vec{a} \cdot \vec{b})\vec{c}$
- $\vec{a} \cdot (2\vec{c})$
- $\|\vec{b}\|^2$

Solution

- $(\vec{a} \cdot \vec{b})\vec{c} = (1 \cdot 0 + 2 \cdot 2 + (-3) \cdot 4)\langle 5, -1, 3 \rangle = -8\langle 5, -1, 3 \rangle = \langle -40, 8, -24 \rangle$
- $\vec{a} \cdot (2\vec{c}) = \langle 1, 2, -3 \rangle \cdot \langle 10, -2, 6 \rangle = 1 \cdot 10 + 2 \cdot (-2) + (-3) \cdot 6 = 10 - 4 - 18 = -12$
- $\|\vec{b}\|^2 = \vec{b} \cdot \vec{b} = 0 \cdot 0 + 2 \cdot 2 + 4 \cdot 4 = 0 + 4 + 16 = 20$

Exercise

Find the following products for $\vec{p} = \langle 7, 0, 2\rangle$, $\vec{q} = \langle -2, 2, -2\rangle$, and $\vec{r} = \langle 0, 2, -3\rangle$:

- $(\vec{r} \cdot \vec{p})\vec{q}$
- $\|\vec{p}\|^2$

Using the Dot Product to Find the Angle Between Two Vectors

Theorem

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta.$$

Example

Find the measure of the angle between each pair of vectors:

- $\vec{i} + \vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} - 3\vec{k}$
- $\langle 2, 5, 6\rangle$ and $\langle -2, -4, 4\rangle$

Solution

Concept

In each case, use the formula $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$ to find the angle θ between the vectors.

When $\vec{u} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{v} = 2\vec{i} - \vec{j} - 3\vec{k}$:

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= (1)(2) + (1)(-1) + (1)(-3) = 2 - 1 - 3 = -2, \\
\|\vec{u}\| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \\
\|\vec{v}\| &= \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14}.
\end{aligned}$$

Then:

$$-2 = \sqrt{3}\sqrt{14}\cos\theta \implies \cos\theta = -\frac{2}{\sqrt{42}} \implies \theta \approx 107.98^\circ \approx 1.88 \text{ radians.}$$

When $\vec{u} = \langle 2, 5, 6\rangle$ and $\vec{v} = \langle -2, -4, 4\rangle$:

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= (2)(-2) + (5)(-4) + (6)(4) = -4 - 20 + 24 = 0, \\
\|\vec{u}\| &= \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}, \\
\|\vec{v}\| &= \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{36} = 6.
\end{aligned}$$

Then:

$$0 = \sqrt{65} \cdot 6\cos\theta \implies \cos\theta = 0 \implies \theta = 90^\circ = \frac{\pi}{2} \text{ radians.}$$

Exercise

Find the measure of the angle, in radians, formed by vectors $\vec{a} = \langle 1, 2, 0\rangle$ and $\vec{b} = \langle 2, 4, 1\rangle$. Round to the nearest hundredth.

Orthogonal Vectors

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

Example

Determine whether $\vec{p} = \langle 1, 0, 5\rangle$ and $\vec{q} = \langle 10, 3, -2\rangle$ are orthogonal vectors.

Solution

Concept

To determine if two vectors are orthogonal, find their dot product and check if it equals zero.

The dot product of $\vec{p}$ and $\vec{q}$ is:

$$\vec{p} \cdot \vec{q} = (1)(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Since $\vec{p} \cdot \vec{q} = 0$, the vectors $\vec{p}$ and $\vec{q}$ are orthogonal.

Exercise

For which value of x is $\vec{p} = \langle 2, 8, -1\rangle$ orthogonal to $\vec{q} = \langle x, -1, 2\rangle$?

Direction Cosines

The **direction angles** of a nonzero vector $\vec{v}$ are the angles α, β, and γ that $\vec{v}$ makes with the positive x-, y-, and z-axes, respectively. The cosines of these angles are called the **direction cosines**:

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_2}{\|\vec{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\vec{v}\|},$$

where $\vec{v} = \langle v_1, v_2, v_3\rangle$.

Example

Let $\vec{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by:

- $\vec{v}$ and $\vec{i}$
- $\vec{v}$ and $\vec{j}$
- $\vec{v}$ and $\vec{k}$.

Solution

Concept

To find the direction cosines of a vector, divide each component by the magnitude of the vector.

The direction cosines of $\vec{v}$ are:

$$\cos\alpha = \frac{2}{\sqrt{2^2+3^2+3^2}} = \frac{2}{\sqrt{22}}, \quad \cos\beta = \frac{3}{\sqrt{22}}, \quad \cos\gamma = \frac{3}{\sqrt{22}}.$$

Thus, the measures of the angles are:

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{22}}\right), \quad \beta = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right), \quad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right).$$

Exercise

Let $\vec{v} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by:

- $\vec{v}$ and $\vec{i}$
- $\vec{v}$ and $\vec{j}$
- $\vec{v}$ and $\vec{k}$

Vector Projections

The **vector projection** of $\vec{v}$ onto $\vec{u}$ is:

$$\mathrm{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}.$$

The **scalar projection** of $\vec{v}$ onto $\vec{u}$ is:

$$\mathrm{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

Example

Find the projection of $\vec{v}$ onto $\vec{u}$:

- $\vec{v} = \langle 3, 5, 1\rangle$ and $\vec{u} = \langle -1, 4, 3\rangle$
- $\vec{v} = 3\vec{i} - 2\vec{j}$ and $\vec{u} = \vec{i} + 6\vec{j}$

Solution

Concept

To find the vector projection of $\vec{v}$ onto $\vec{u}$, use the formula $\mathrm{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}$.

Thus, compute the projections as follows:

- $$\begin{aligned}\mathrm{proj}_{\vec{u}}\vec{v} &= \left(\frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\right)\langle -1,4,3\rangle \\ &= \left(\frac{(-3)+20+3}{(-1)^2+4^2+3^2}\right)\langle -1,4,3\rangle = \left(\frac{20}{26}\right)\langle -1,4,3\rangle = \left\langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13}\right\rangle.\end{aligned}$$

- $$\begin{aligned}\mathrm{proj}_{\vec{u}}\vec{v} &= \left(\frac{(\vec{i}+6\vec{j})\cdot(3\vec{i}-2\vec{j})}{\|\vec{i}+6\vec{j}\|^2}\right)(\vec{i}+6\vec{j}) \\ &= \left(\frac{3-12}{1^2+6^2}\right)(\vec{i}+6\vec{j}) = \left(\frac{-9}{37}\right)(\vec{i}+6\vec{j}) = \left\langle -\frac{9}{37}, -\frac{54}{37}\right\rangle.\end{aligned}$$

Example

Express $\vec{v} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = \langle 2, 3, 2 \rangle$.

Solution

Concept

To express $\vec{v}$ as a sum of orthogonal vectors, find the projection of $\vec{v}$ onto $\vec{u}$ and the orthogonal component of $\vec{v}$ with respect to $\vec{u}$.

The projection of $\vec{v}$ onto $\vec{u}$ is:

$$\begin{aligned}
\text{proj}_{\vec{u}}\vec{v} &= \left(\frac{\langle 2,3,2\rangle \cdot \langle 8,-3,-3\rangle}{\|\langle 2,3,2\rangle\|^2} \right) \langle 2,3,2\rangle \\
&= \left(\frac{16-9-6}{2^2+3^2+2^2} \right) \langle 2,3,2\rangle \\
&= \left(\frac{1}{17} \right) \langle 2,3,2\rangle \\
&= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle.
\end{aligned}$$

Thus, the orthogonal component of $\vec{v}$ with respect to $\vec{u}$ is:

$$\begin{aligned}
\vec{v} - \text{proj}_{\vec{u}}\vec{v} &= \langle 8,-3,-3\rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\
&= \left\langle \frac{136}{17}, \frac{-51}{17}, \frac{-51}{17} \right\rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\
&= \left\langle \frac{134}{17}, \frac{-54}{17}, \frac{-53}{17} \right\rangle.
\end{aligned}$$

Exercise

Express $\vec{v} = 5\vec{i} - \vec{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = 4\vec{i} + 2\vec{j}$.

self-note: continue here!

Work

Definition

Definition: When a constant force $\vec{F}$ is applied to an object so the object moves in a straight line from point P to point Q, the **work** W done by the force is:

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\| \cos\theta,$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

Example

Example 2.30: Calculating Work
A conveyor belt generates a force $\vec{F} = 5\vec{i} - 3\vec{j} + \vec{k}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters, and the force is measured in newtons.

Exercise

Checkpoint 2.29:
A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?

The Cross Product

Definition

Definition: Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$. The **cross product** $\vec{u} \times \vec{v}$ is defined as:

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle.$$

This can also be written using the standard unit vectors:

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

Example

Example 2.31: Finding a Cross Product
Let $\vec{p} = \langle -1, 2, 5\rangle$ and $\vec{q} = \langle 4, 0, -3\rangle$. Find $\vec{p} \times \vec{q}$.

Exercise

Checkpoint 2.30:
Find $\vec{p} \times \vec{q}$ for $\vec{p} = \langle 5, 1, 2 \rangle$ and $\vec{q} = \langle -2, 0, 1 \rangle$. Express the answer using standard unit vectors.

Properties of the Cross Product

Theorem

Theorem 2.6: Properties of the Cross Product
Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$, and let c be a scalar. Then:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (Anticommutative Property)
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ (Distributive Property)
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$ (Multiplication by a Constant)
4. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$ (Cross Product with the Zero Vector)
5. $\vec{v} \times \vec{v} = \vec{0}$ (Cross Product of a Vector with Itself)
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ (Scalar Triple Product)

Example

Example 2.32: Anticommutativity of the Cross Product
Let $\vec{u} = \langle 0, 2, 1 \rangle$ and $\vec{v} = \langle 3, -1, 0 \rangle$. Calculate $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$, and graph them.

Exercise

Checkpoint 2.31:
Suppose vectors $\vec{u}$ and $\vec{v}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{u}$ and the y-component of $\vec{v}$ are all positive, whereas the x-component of $\vec{v}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{u} \times \vec{v}$ point?

Cross Product of Standard Unit Vectors

Example

Example 2.33: Cross Product of Standard Unit Vectors
Find $\vec{i} \times (\vec{j} \times \vec{k})$.

Exercise

Checkpoint 2.32:
Find $(\vec{i} \times \vec{j}) \times (\vec{k} \times \vec{i})$.

Magnitude of the Cross Product

Theorem

Theorem 2.7: Magnitude of the Cross Product
Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta.$$

Example

Example 2.35: Calculating the Cross Product
Use the properties of the cross product to find the magnitude of $\vec{u} \times \vec{v}$, where $\vec{u} = \langle 0, 4, 0\rangle$ and $\vec{v} = \langle 0, 0, -3\rangle$.

Exercise

Checkpoint 2.34:
Use the properties of the cross product to find the magnitude of $\vec{u} \times \vec{v}$, where $\vec{u} = \langle -8, 0, 0\rangle$ and $\vec{v} = \langle 0, 2, 0\rangle$.

Determinants and the Cross Product

Definition

Definition: The cross product of $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ can be calculated using the determinant of a 3×3 matrix:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example

Example 2.37: Using Determinant Notation to Find $\vec{p} \times \vec{q}$
Let $\vec{p} = \langle -1, 2, 5\rangle$ and $\vec{q} = \langle 4, 0, -3\rangle$. Find $\vec{p} \times \vec{q}$ using determinant notation.

Exercise

Checkpoint 2.36:
Use determinant notation to find $\vec{a} \times \vec{b}$, where $\vec{a} = \langle 8, 2, 3 \rangle$ and $\vec{b} = \langle -1, 0, 4 \rangle$.

Applications of the Cross Product

Example

Example 2.38: Finding a Unit Vector Orthogonal to Two Given Vectors
Let $\vec{a} = \langle 5, 2, -1 \rangle$ and $\vec{b} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

Exercise

Checkpoint 2.37:
Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$, where $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle 1, 1, 4 \rangle$.

Area of a Parallelogram

Theorem

Theorem 2.8: Area of a Parallelogram
If $\vec{u}$ and $\vec{v}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by:

$$\text{Area} = \|\vec{u} \times \vec{v}\|.$$

Example

Example 2.39: Finding the Area of a Triangle
Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle. Find its area.

Exercise

Checkpoint 2.38:
Find the area of the parallelogram $PQRS$ with vertices $P = (1, 1, 0)$, $Q = (7, 1, 0)$, $R = (9, 4, 2)$, and $S = (3, 4, 2)$.

The Triple Scalar Product

Definition

Definition: The **triple scalar product** of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is:

$$\vec{u} \cdot (\vec{v} \times \vec{w}).$$

Theorem

Theorem 2.9: Calculating a Triple Scalar Product
The triple scalar product of $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Example 2.40: Calculating the Triple Scalar Product
Let $\vec{u} = \langle 1, 3, 5 \rangle$, $\vec{v} = \langle 2, -1, 0 \rangle$, and $\vec{w} = \langle -3, 0, -1 \rangle$. Calculate the triple scalar product $\vec{u} \cdot (\vec{v} \times \vec{w})$.

Exercise

Checkpoint 2.39:
Calculate the triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$, where $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle 0, 3, -1 \rangle$, and $\vec{c} = \langle 5, -3, 3 \rangle$.

Volume of a Parallelepiped

Theorem

Theorem 2.10: Volume of a Parallelepiped
The volume of a parallelepiped with adjacent edges given by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is:

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$

Example

Example 2.41: Calculating the Volume of a Parallelepiped
Let $\vec{u} = \langle -1, -2, 1 \rangle$, $\vec{v} = \langle 4, 3, 2 \rangle$, and $\vec{w} = \langle 0, -5, -2 \rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Exercise

Checkpoint 2.40:
Find the volume of the parallelepiped formed by the vectors $\vec{a} = 3\vec{i} + 4\vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$, and $\vec{c} = 3\vec{j} + \vec{k}$.

Applications of the Cross Product

Example

Example 2.42: Using the Triple Scalar Product
Use the triple scalar product to show that vectors $\vec{u} = \langle 2, 0, 5\rangle$, $\vec{v} = \langle 2, 2, 4\rangle$, and $\vec{w} = \langle 1, -1, 3\rangle$ are coplanar.

Exercise

Checkpoint 2.41:
Are the vectors $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$, and $\vec{c} = \vec{i} + \vec{j} + \vec{k}$ coplanar?

Torque

Definition

Definition: Torque (τ) measures the tendency of a force to produce rotation about an axis. If $\vec{r}$ is the position vector from the axis of rotation to the point where the force $\vec{F}$ is applied, then:

$$\tau = \vec{r} \times \vec{F}.$$

Example

Example 2.44: Evaluating Torque
A bolt is tightened by applying a force of 6 N to a 0.15-m wrench. The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

Exercise

Checkpoint 2.42:
Calculate the force required to produce $15\,\mathrm{N}\cdot\mathrm{m}$ torque at an angle of 30° from a 150-cm rod.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Previous Lecture Recap

Recall the Circle, Ellipse, Parabola, and Hyperbola

Recall the four shapes (and their equations) derived from taking cross sections of double-naped cones:

- Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Circle: $x^2 + y^2 = r^2$
- Parabola: $y = ax^2 + bx + c = A(x+B)^2 + C, \quad a, A \neq 0$
- Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Graph Differences in 2D and 3D

- In 2D, a graph is a curve.
- In 3D, a graph is a surface.

Concept

Equation	**2D Interpretation**	**3D Interpretation**
$y = 2$	Horizontal line	Plane
$x^2 + y^2 = 1$	Circle	Cylinder

Table 1: Comparison of 2D and 3D interpretations of equations

Circle vs Sphere

In 2D, a **circle** is given by the equation $(x-h)^2 + (y-k)^2 = r^2$, where (h,k) is the center and r is the radius. A **sphere** extends this concept to 3D, with the equation $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. Here, (h,k,l) is the center and r is the radius.

Illustration

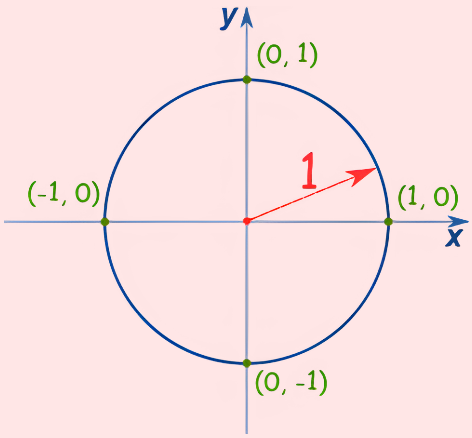


Figure 3: Unit Circle

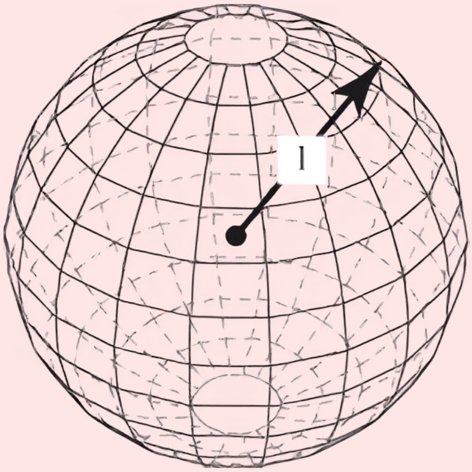


Figure 4: Unit Sphere

Example

The equation of a sphere centered at $(2,2,1)$ with radius $r=3$ is:

$$(x-2)^2 + (y-2)^2 + (z-1)^2 = 9$$

Example

The equation of a sphere centered at $(-1,0,-4)$ with radius $r=\sqrt{5}$ is:

$$(x+1)^2 + y^2 + (z+4)^2 = 5$$

Exercise

Given the equation:

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

Find the center and radius of the sphere.

Describing the Region Between Two Spheres

Example

Describe the following region in 3D space:

$$2 \leqslant x^2 + y^2 + z^2 < 5$$

Solution

Consider the two spheres:

- Sphere 1: $x^2 + y^2 + z^2 = 5$
- Sphere 2: $x^2 + y^2 + z^2 = 2$

The region is the set of points that lie inside the sphere of radius $\sqrt{5}$ (exclusive) and outside the sphere of radius $\sqrt{2}$ (inclusive).

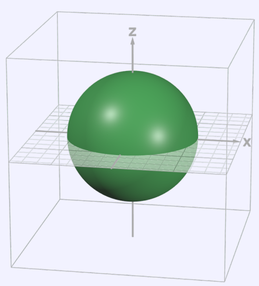


Figure 5: Region Between Spheres

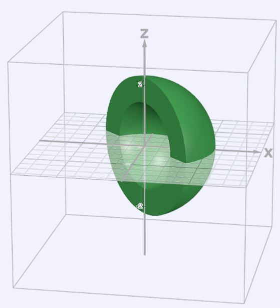


Figure 6: Region Between Spheres (Cut)

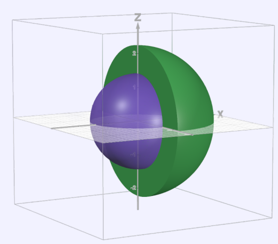


Figure 7: Region Between Spheres (Inner Sphere)

Introduction to Vectors

A **vector** is a quantity that has both magnitude and direction. It is represented by an arrow, with the length of the arrow representing the magnitude and the direction of the arrow representing the direction.

Note

A vector can additionally be denoted by its **initial** and **terminal** points. The **position vector** of a point P is the vector that starts at the origin and ends at P.

Example

Find the vector starting from point $A(-1,3)$ and ending at point $B(2,-1)$.

Solution

The vector is given by:

$$\vec{u} = \overrightarrow{AB} = B - A = \langle 2-(-1), -1-3 \rangle = \langle 3, -4 \rangle$$

Tip

There are two takeaways:

- Vectors have a length;
- Vectors have a direction.

Equivalent Vectors

Definition

Two vectors are **equivalent** if they have the same magnitude and direction, regardless of their initial points.