MAT232 - Lecture 3

Polar Coordinates and the Arc Length of Parametric Curves

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Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Preliminary Definitions and Theorems

Definition

Polar Coordinates.

Each point in the Cartesian plane can be represented in polar coordinates as an ordered pair (r, θ), where r is the radial coordinate (distance from the origin), and θ is the angular coordinate (angle measured from the positive x-axis). The correspondence between Cartesian coordinates (x, y) and polar coordinates (r, θ) is given by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Theorem

Theorem 1.4. Converting Points Between Coordinate Systems.

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between Cartesian and polar coordinates.

Example

Example 1.10. Converting Between Rectangular and Polar Coordinates.

1. Convert $(1, 1)$ to polar coordinates: Use $x = 1$ and $y = 1$. Then:

$$r^2 = x^2 + y^2 = 1^2 + 1^2 = 2 \implies r = \sqrt{2}, \quad \tan\theta = \frac{y}{x} = \frac{1}{1} = 1 \implies \theta = \frac{\pi}{4}.$$

   Therefore, $(1, 1)$ can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

Concept

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates.

1. Create a table with two columns: one for θ values and one for r values.
2. Calculate the corresponding r values for each θ.
3. Plot each ordered pair (r, θ) on the polar coordinate axes.
4. Connect the points and observe the resulting graph.

Example

Example 1.12. Graphing a Function in Polar Coordinates.
Graph the curve defined by $r = 4\sin\theta$.

1. Create a table of values for θ and calculate r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	2π
$r = 4\sin\theta$	0	2	$2\sqrt{2}$	4	0	0

2. Plot the points and connect them to form the curve. The result is a circle with radius 2 centered at $(0, 2)$ in rectangular coordinates.

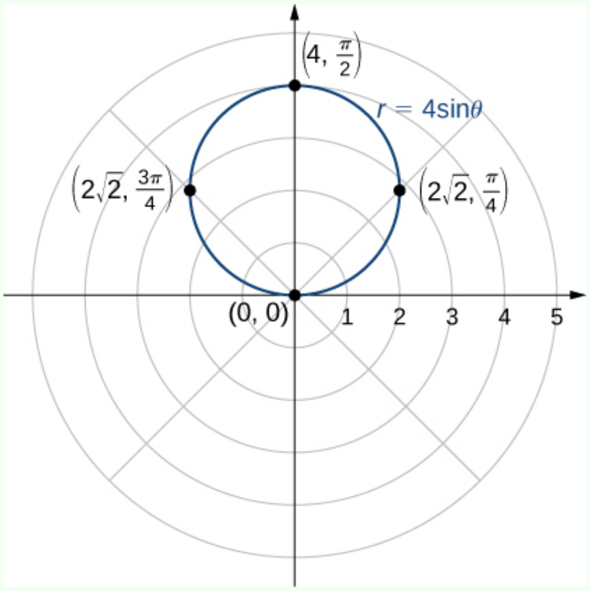


Figure 1: The graph of the function $r = 4\sin\theta$ is a circle.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

Recall: 1st Year Calculus

Definition

The **definite integral** of a function $y = f(x)$, where $f(x) \geqslant 0$, represents the area under the curve from $x = a$ to $x = b$:

$$\text{Area} = \int_{x=a}^{x=b} f(x)\, dx$$

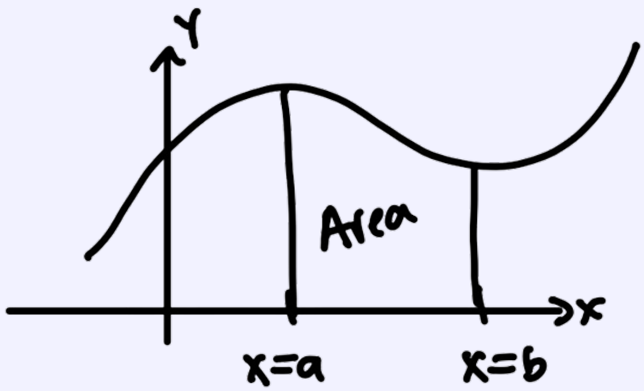


Figure 2: Illustration of the area under $y = f(x)$.

Section 1.2: MAT232 Perspective

Definition

A **parametric curve** is defined by:

$$x = f(t), \quad y = g(t), \quad \alpha \leqslant t \leqslant \beta$$

with the following properties:

- The curve lies above the x-axis.
- The curve does not self-intersect.

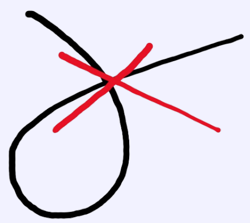


Figure 3: A curve that self-intersects.

The area enclosed by the curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt$$

or equivalently:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} f(t) g'(t)\, dt$$

Alternative Forms for Area

Note

In specific cases, the area can also be calculated using:

$$\text{Area} = \int_{y=c}^{y=d} x(y)\, dy$$

or:

$$\text{Area} = \int_{x=a}^{x=b} y(x)\, dx$$

Area Enclosed by a Parametric Curve

Example

Calculate the area enclosed by the parametric curve:

$$x = \cos(t), \quad y = \sin(t), \quad 0 \leqslant t \leqslant \pi$$

Solution

The area is calculated as:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$ and $y = g(t)$. Here, $f(t) = \cos(t)$, $g(t) = \sin(t)$, and $f'(t) = -\sin(t)$. Substituting:

$$\text{Area} = \int_{t=0}^{t=\pi} \sin(t)(-\sin(t))\, dt = \int_{t=0}^{t=\pi} -\sin^2(t)\, dt.$$

Using $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$, we get:

$$\text{Area} = -\int_{t=0}^{t=\pi} \frac{1}{2}(1 - \cos(2t))\, dt = -\frac{1}{2}\left[\int_{t=0}^{t=\pi} 1\, dt - \int_{t=0}^{t=\pi} \cos(2t)\, dt\right].$$

Evaluate the integrals:

$$\int_{t=0}^{t=\pi} 1\, dt = \pi, \quad \int_{t=0}^{t=\pi} \cos(2t)\, dt = \left[\frac{\sin(2t)}{2}\right]_0^{\pi} = 0.$$

Thus:

$$\text{Area} = -\frac{1}{2}(\pi - 0) = -\frac{\pi}{2}.$$

Taking the absolute value (since area is positive):

$$\text{Area} = \frac{\pi}{2}.$$

Area Under the Curve of a Cycloid

Example

Example: Find the area under the cycloid defined by:

$$x = t - \sin(t), \quad y = 1 - \cos(t), \quad 0 \leqslant t \leqslant 2\pi.$$

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt, \quad f'(t) = \frac{dx}{dt}.$$

Step 1: Substitution
From $x = t - \sin(t)$ and $y = 1 - \cos(t)$:

$$f'(t) = 1 - \cos(t), \quad g(t) = 1 - \cos(t).$$

Substitute into the formula:

$$\text{Area} = \int_0^{2\pi} (1 - \cos(t))^2\, dt.$$

Step 2: Expand and Separate Terms
Expand $(1 - \cos(t))^2$:

$$\text{Area} = \int_0^{2\pi} [1 - 2\cos(t) + \cos^2(t)]\, dt.$$

Split the integral:

$$\text{Area} = \int_0^{2\pi} 1\, dt - 2\int_0^{2\pi} \cos(t)\, dt + \int_0^{2\pi} \cos^2(t)\, dt.$$

...cont'd...

Example

Solution

...cont'd...

Step 3: Evaluate Each Term

1. First Term:

$$\int_0^{2\pi} 1\,dt = 2\pi.$$

2. Second Term:

$$\int_0^{2\pi} \cos(t)\,dt = [\sin(t)]_0^{2\pi} = 0.$$

3. Third Term:

Using $\cos^2(t) = \frac{1+\cos(2t)}{2}$:

$$\int_0^{2\pi} \cos^2(t)\,dt = \frac{1}{2}\int_0^{2\pi} 1\,dt + \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^{2\pi} 1\,dt = \pi, \quad \frac{1}{2}\int_0^{2\pi} \cos(2t)\,dt = 0.$$

Thus:

$$\int_0^{2\pi} \cos^2(t)\,dt = \pi.$$

Step 4: Combine Results

$$\text{Area} = 2\pi - 0 + \pi = 3\pi.$$

Answer

Area $= 3\pi$

Homework Practice Question: Area Under a Parametric Curve

Exercise

Find the area under the curve defined by

$$x = 3\cos(t) + \cos(3t), \quad y = 3\sin(t) - \sin(3t), \quad 0 \leqslant t \leqslant \pi.$$

Hint: Recall that $\sin^2(x) + \cos^2(x) = 1$.

Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t)\, dt,$$

where $x = f(t)$, $y = g(t)$, and $f'(t) = \frac{dx}{dt}$.

Step 1: Differentiate $x(t)$
Given $x = 3\cos(t) + \cos(3t)$, compute:

$$f'(t) = \frac{d}{dt}[3\cos(t) + \cos(3t)] = -3\sin(t) - 3\sin(3t).$$

Step 2: Substitute into the Formula
The parametric area formula becomes:

$$\text{Area} = \int_0^{\pi} \big[3\sin(t) - \sin(3t)\big]\big[-3\sin(t) - 3\sin(3t)\big]\, dt.$$

Step 3: Simplify the Expression
Expand the product:

$$\big[3\sin(t)-\sin(3t)\big]\big[-3\sin(t)-3\sin(3t)\big] = -9\sin^2(t)-9\sin(t)\sin(3t)+3\sin(3t)\sin(t)+3\sin^2(3t).$$

Combine terms:

$$-9\sin^2(t) + 3\sin^2(3t) - 6\sin(t)\sin(3t).$$

...cont'd...

Exercise

Solution

...cont'd..

Using the product-to-sum identity for $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$:

$$\sin(t)\sin(3t) = \frac{1}{2}[\cos(2t) - \cos(4t)].$$

Substitute this back:

$$\text{Area} = \int_0^\pi \big[-9\sin^2(t) + 3\sin^2(3t) - 3[\cos(2t) - \cos(4t)] \big]\, dt.$$

Step 4: Break the Integral into Separate Terms

Split the integral:

$$\text{Area} = -9\int_0^\pi \sin^2(t)\, dt + 3\int_0^\pi \sin^2(3t)\, dt - 3\int_0^\pi \cos(2t)\, dt + 3\int_0^\pi \cos(4t)\, dt.$$

Step 5: Evaluate Each Integral

1. First Term ($-9\int_0^\pi \sin^2(t)\, dt$):

Use the identity $\sin^2(t) = \dfrac{1-\cos(2t)}{2}$:

$$\int_0^\pi \sin^2(t)\, dt = \int_0^\pi \frac{1-\cos(2t)}{2}\, dt = \frac{1}{2}\int_0^\pi 1\, dt - \frac{1}{2}\int_0^\pi \cos(2t)\, dt.$$

Evaluate:

$$\frac{1}{2}\int_0^\pi 1\, dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^\pi \cos(2t)\, dt = \frac{1}{2}[0] = 0.$$

So:

$$\int_0^\pi \sin^2(t)\, dt = \frac{\pi}{2}.$$

Multiply by -9:

$$-9\int_0^\pi \sin^2(t)\, dt = -9 \cdot \frac{\pi}{2} = -\frac{9\pi}{2}.$$

...cont'd...

Exercise

Solution

...cont'd...

2. Second Term ($3\int_0^\pi \sin^2(3t)\,dt$):

Similarly, $\sin^2(3t) = \dfrac{1-\cos(6t)}{2}$:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{1}{2}\int_0^\pi 1\,dt - \frac{1}{2}\int_0^\pi \cos(6t)\,dt.$$

Evaluate:

$$\frac{1}{2}\int_0^\pi 1\,dt = \frac{\pi}{2}, \quad \frac{1}{2}\int_0^\pi \cos(6t)\,dt = 0.$$

So:

$$\int_0^\pi \sin^2(3t)\,dt = \frac{\pi}{2}.$$

Multiply by 3:

$$3\int_0^\pi \sin^2(3t)\,dt = 3\cdot\frac{\pi}{2} = \frac{3\pi}{2}.$$

3. Third Term ($-3\int_0^\pi \cos(2t)\,dt$):

Since $\int_0^\pi \cos(2t)\,dt = 0$:

$$-3\int_0^\pi \cos(2t)\,dt = 0.$$

4. Fourth Term ($3\int_0^\pi \cos(4t)\,dt$):

Similarly, $\int_0^\pi \cos(4t)\,dt = 0$:

$$3\int_0^\pi \cos(4t)\,dt = 0.$$

Step 6: Combine Results

Add the evaluated terms:

$$\text{Area} = -\frac{9\pi}{2} + \frac{3\pi}{2} + 0 + 0 = -\frac{6\pi}{2} = -3\pi.$$

However, the area is always positive, so:

$$\text{Area} = 3\pi.$$

Exercise

Solution

...cont'd...

Answer

Area $= 3\pi$

The Arc Length of a Parametric Curve

Theorem

Theorem: self-note: grab the actual theorem from the textbook lol

- (x_1, y_1) and (x_2, y_2) are points
- $\Delta x = x_1 - x_2$, Δ = Delta

The distance between two points is denoted by D as follows:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Substitute Δx and Δy as follows:

$$D = \sqrt{\Delta x^2 + \Delta y^2}$$

It follows that...

$$D = \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2}\Delta t$$

Now, notice the similarity to Riemann sums from MAT136. As $\Delta x \to 0$:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

L is the (arc) length of a curve. This is confirmed to be included on term test 1, and will be on the formula sheet.

Figure 4: Graphical representation of the theorem.

Example

Example

Find the arc length of the curve defined by

$$x = 3\cos(t), \quad y = 3\sin(t), \quad t \in [0, 2\pi].$$

The arc length is denoted by L. Evaluate as follows:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2} dt \\ &= \textbf{self-note: finish this using the notes in the camera roll} \end{aligned}$$

Homework Practice Problem

Note

Find the arc length of the curve defined by

$$x = 3t^2, \quad y = 2t^3, \quad 1 \leqslant t \leqslant 3.$$

self-note: do the solution to this

Section 1.3: Polar Coordinates

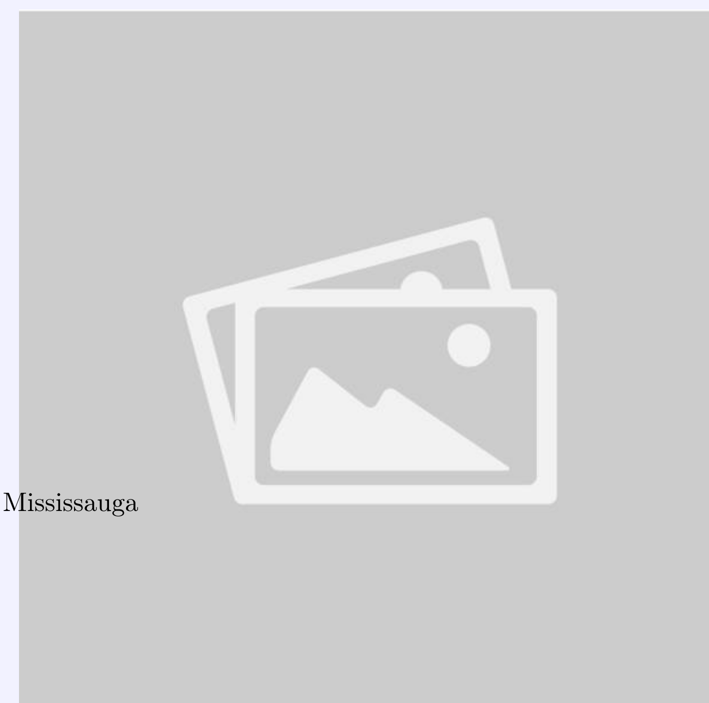
Definition

give the actual definition here from the textbook lol

Cartesian Coordniates:

Figure 5: Graphical representation of the theorem.

Polar Coordinates:



Additional Notes

Note

Always check the domain of the parameter t when solving problems involving parametric equations.

Further Visualization

Figure 7: Additional visualization for parametric curves.