MAT232 - Lecture 4

Polar Coordinates and Curves

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Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

Polar Coordinates - Key Theorems

Converting Points between Coordinate Systems

Theorem

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between rectangular and polar coordinates.

Uniqueness of Polar Coordinates

Proposition

Every point in the plane has an infinite number of representations in polar coordinates. Specifically, the polar coordinates (r, θ) of a point are not unique.

Remark

For example, the polar coordinates $(2, \pi/3)$ and $(2, 7\pi/3)$ both represent the same point in the rectangular coordinate system. Additionally, the value of r can be negative. Therefore, the point with polar coordinates $(-2, 4\pi/3)$ represents the same rectangular point as $(2, \pi/3)$.

Symmetry of Polar Curves

Theorem

Polar curves can exhibit symmetry similar to those in rectangular coordinates. The key symmetries to identify are:

- **Symmetry with respect to the polar axis:** A curve is symmetric with respect to the polar axis if replacing θ with $-\theta$ in its equation yields the same curve.
- **Symmetry with respect to the line $\theta = \frac{\pi}{2}$:** A curve is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if replacing θ with $\pi - \theta$ yields the same curve.
- **Symmetry with respect to the pole (origin):** A curve is symmetric with respect to the pole if replacing r with $-r$ yields the same curve.

Let’s Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let’s break this down step by step.

Plotting Polar Coordinates

Recall the Content from Last Lecture

Note

Converting between Cartesian coordinates (x, y) and Polar coordinates (r, θ):

Algorithm

From Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

From Polar to Cartesian:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Converting Between Degrees and Radians:

Algorithm

- **Degrees to Radians:** Multiply by $\frac{\pi}{180^\circ}$

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180^\circ}$$

- **Radians to Degrees:** Multiply by $\frac{180^\circ}{\pi}$

$$\text{Degrees} = \text{Radians} \times \frac{180^\circ}{\pi}$$

Understanding the Convention for r in Polar Coordinates

Concept

In polar coordinates, a point is represented as (r, θ), where:

- r is the radial distance from the origin (how far the point is from the origin).
- θ is the angle, measured counterclockwise from the positive x-axis.

Note

Special Case: When r is Negative

- A negative r in $(-r, \theta)$ is interpreted as the point being reflected through the origin.
- The equivalent representation is:

$$(-r, \theta) = (r, \theta + 180^\circ)$$

or in radians:

$$(-r, \theta) = (r, \theta + \pi)$$

Intuition

- Reflecting (r, θ) through the origin is the same as rotating the point by 180° (or π radians).
- This property simplifies polar plots by offering alternate representations of the same point.

Example: Plotting Points

Example

Let us plot the following points in polar coordinates:

$$(3, -45^\circ), \quad (3, 225^\circ), \quad (4, 330^\circ), \quad (1, -45^\circ)$$

Algorithm

Step-by-Step Process:

1. For each point, identify r and θ.
2. If θ is negative or exceeds 360°, convert it to a standard range:

$$\theta \in [0^\circ, 360^\circ)$$

   using $\theta = \theta + 360^\circ$ (for negative angles) or subtracting 360° (for angles over 360°).
3. Plot the point by measuring θ counterclockwise from the positive x-axis and placing it at a distance r from the origin.

Solution

- For $(3, -45^\circ)$: Add 360° to -45° to convert θ to 315°. Plot as $(3, 315^\circ)$.
- For $(3, 225^\circ)$: Already within the standard range, so plot directly.
- For $(4, 330^\circ)$: Angle is standard, so plot directly.
- For $(1, -45^\circ)$: Add 360° to -45°, yielding $(1, 315^\circ)$.

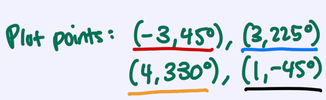

Figure 1: Colour Legend

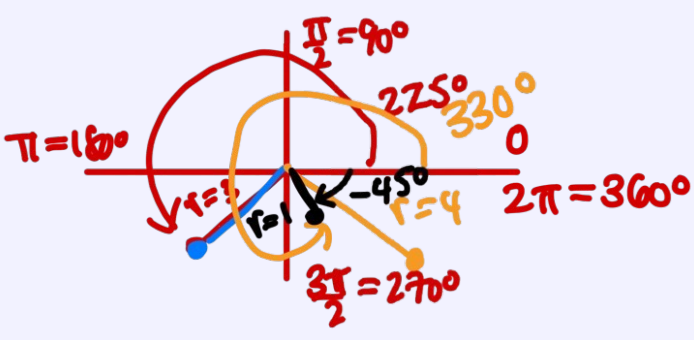

Figure 2: Polar Coordinates Plot and “Trajectories”

Tip

Ensure to label points clearly on the polar grid, and verify angle conversions and reflections for accuracy.

Example: Converting from Polar Coordinates to Cartesian Coordinates

Example

Find the **rectangular coordinates** of the point p with polar coordinates $(6, \frac{\pi}{3})$.

Solution

To convert from polar to Cartesian coordinates, use:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Substitute $r = 6$ and $\theta = \frac{\pi}{3}$:

$$x = 6\cos\left(\frac{\pi}{3}\right) = 6 \cdot \frac{1}{2} = 3, \quad y = 6\sin\left(\frac{\pi}{3}\right) = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}.$$

Thus, the Cartesian coordinates are:

$$(x, y) = (3, 3\sqrt{3}).$$

Answer

The rectangular coordinates are $(3, 3\sqrt{3})$.

Converting from Cartesian Coordinates to Polar Coordinates

Example

Find the **polar coordinates** of the point p with rectangular coordinates $(-2, 2\sqrt{3})$.

Solution

To find the polar coordinates, use:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}.$$

Step 1: Solve for r:

$$r^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16 \implies r = 4.$$

Step 2: Solve for θ:

$$\tan(\theta) = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}.$$

The point $(-2, 2\sqrt{3})$ lies in Quadrant II. The reference angle for $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$. Thus:

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Tip

Alternatively, for a negative angle:

$$\theta = -\frac{\pi}{3}, \quad \text{adjust to Quadrant II: } -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Thus, $(r, \theta) = (4, \frac{2\pi}{3})$.

Answer

The polar coordinates are $(4, \dfrac{2\pi}{3})$ or $(4, 120^\circ)$.

Note

To enhance material understanding, make use of graphing websites (e.g. Desmos, Geogebra) or software whenever possible. Focus especially on mastering how to plot lines and circles, as these are fundamental for MAT232.

Polar Curves

Example

Consider $r = f(\theta)$. Sketch the following functions:

(a) $r = 1$

(b) $\theta = \frac{\pi}{4}$

(c) $r = \theta, \quad \theta \geqslant 0$

(d) $r = \sin(\theta)$

(e) $r = \cos(2\theta)$

(a) $r = 1$

Solution

Here, $r = 1$, and θ can take any value.

This means the point is always at a distance of 1 from the origin, regardless of the angle θ. Hence, the graph is a **circle** with radius 1, centred at the origin.

Remark

Cartesian Conversion

From the polar equation:

$$x^2 + y^2 = r^2 = 1$$

This confirms the equation of a unit circle in Cartesian coordinates.

Figure 3: Sample image illustrating the concept.

(b) $\theta = \dfrac{\pi}{4}$

Solution

Here, $\theta = \dfrac{\pi}{4}$, and r can take any value.
This represents all points that lie along the line passing through the origin at an angle of $\frac{\pi}{4}$ (or $45°$) with the positive x-axis. The graph is a straight line through the origin.

Remark

Cartesian Conversion
In polar coordinates:

$$\tan(\theta) = \frac{y}{x}$$

Substituting $\theta = \frac{\pi}{4}$, we get:

$$\tan\left(\frac{\pi}{4}\right) = 1 \quad \Rightarrow \quad \frac{\pi}{4} = \tan^{-1}(1) = \tan^{-1}\left(\frac{y}{x}\right) \quad \Rightarrow \quad 1 = \frac{y}{x} \quad \Rightarrow \quad y = x$$

Thus, the Cartesian equation is $y = x$.

Figure 4: Sample image illustrating the concept.

(c) $r = \theta, \quad \theta \geqslant 0$

Solution

Here, r increases as θ increases. This creates a spiral that starts at the origin and winds outward as θ grows.

Remark

Illustration

Create a table of values to plot key points:

θ	r
0	0
$\frac{\pi}{6}$	$\frac{\pi}{6} \approx 0.52$
$\frac{\pi}{4}$	$\frac{\pi}{4} \approx 0.79$
$\frac{\pi}{3}$	$\frac{\pi}{3} \approx 1.05$
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$

Note

Using $x = r\cos\theta$ and $y = r\sin\theta$, compute x and y for each point in the table above to visualize the curve in Cartesian coordinates.

Figure 5: Sample image illustrating the concept.

(d) $r = \sin(\theta)$

Solution

Here, $r = \sin(\theta)$. Since $\sin(\theta)$ oscillates between 0 and 1, the graph forms a **cardioid**.

Table of Values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
r	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0

Cartesian Conversion

To express the equation in Cartesian form:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Eliminating r, substitute $r = \sin\theta$:

$$r = \frac{y}{r} \quad \Rightarrow \quad r^2 = y$$

Using $r^2 = x^2 + y^2$, we get:

$$x^2 + y^2 = y$$

Complete the square for y:

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

This represents a circle centred at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$.

Figure 6: The graph of $r = \sin(\theta)$.

The graph is a cardioid, touching the origin and looping around the point $\left(0, \frac{1}{2}\right)$.

Exercise

Try sketching the curve:

$$r = \cos\theta$$

under the same context as the previous questions.

Solution

Here, $r = \cos\theta$. Since $\cos\theta$ oscillates between -1 and 1, the graph will form a **limacon** with an inner loop.

Table of Values To visualize the curve, create a table of key points:

θ	r
0	1
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
π	-1

Note

Plot the points in the table and connect them smoothly to form the graph. Note that for negative r, the points are plotted in the opposite direction from the origin.

Cartesian Conversion Substitute $r = \cos\theta$ into the Cartesian equations:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Substituting $r = \sqrt{x^2 + y^2}$ and eliminating r, we derive:

$$x^2 + y^2 = x$$

Complete the square for x:

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

This represents a circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

Figure 7: Sample image illustrating the concept.

The Derivative of a Polar Curve

Tangents to Polar Curves

Definition

Note

Recall that polar curves are defined by:

$$r = f(\theta)$$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Intuition

The goal is to have everything on x depend on **one** parameter.

Do the exact same thing on y.

So,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{\frac{dx}{d}}{\theta}}.$$

We want require $\frac{dx}{d\theta} \neq 0$.
So...

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{df(\theta)}{d\theta}\sin\theta + \cos\theta f(\theta)}{\frac{df(\theta)}{d\theta}\cos\theta - \sin\theta f(\theta)}$$

So,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(subbed r in for $f(\theta)$). Conclusion:

- Horizontal Tangents: $\frac{dy}{d\theta} = 0, \quad \frac{dx}{d\theta} \neq 0$
- Vertical Tangents: $\frac{dx}{d\theta} = 0, \quad \frac{dy}{d\theta} \neq 0$
- Singular Points (discard; we will not be doing further analysis for this case in MAT232): $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$

Examples

Example

Find the **vertical tangent** angles of the polar curve $r = 1 - \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$.

Solution

Recall that $\frac{dr}{d\theta} = \sin\theta$.
Obtain the first derivative:

$$\frac{dy}{dx} = \ldots$$

self-note: prof is going way too fast; finish the notes according to your camera roll later! the good thing is that you didn't actually miss any sections! fulfilling incomplete sections is just a matter of reviewing and comparing to the pictures taken of the prof's projected live notes!

Answer

The vertical tangents are located at $x = \{\frac{\pi}{3}, \pi\}$.

Figure 8: Sample image illustrating the concept.

Next Week: Vector Week

Theorem

- Circle: $x^2 + y^2 = r^2$.
- Generic form for a circle centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

Figure 9: Graphical representation of the theorem.

Example

Sketch $x^2 + y^2 - 2x = 10$.

Solution

Recall how to complete the square:

$$x^2 - 2x + 1 + y^2 = 10 + 1$$

Step #1: $-\frac{2}{2} = -1$;
Step #2: $(-1)^2 = 1$ **self-note: complete this below**