

# MAT232 - Lecture 3

Polar Coordinates and the Arc Length of Parametric Curves

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# Definitions and Theorems

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*Straight from the textbook — no fluff, just what we need.*

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**Quick recap before diving into the lecture.**



## Preliminary Definitions and Theorems

### Definition

#### Polar Coordinates.

Each point in the Cartesian plane can be represented in polar coordinates as an ordered pair  $(r, \theta)$ , where  $r$  is the radial coordinate (distance from the origin), and  $\theta$  is the angular coordinate (angle measured from the positive  $x$ -axis). The correspondence between Cartesian coordinates  $(x, y)$  and polar coordinates  $(r, \theta)$  is given by:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

### Theorem

#### Theorem 1.4. Converting Points Between Coordinate Systems.

Given a point  $P$  in the plane with Cartesian coordinates  $(x, y)$  and polar coordinates  $(r, \theta)$ , the following conversion formulas hold true:

$$x = r \cos \theta, \quad y = r \sin \theta,$$

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

These formulas can be used to convert between Cartesian and polar coordinates.

### Example

#### Example 1.10. Converting Between Rectangular and Polar Coordinates.

1. Convert  $(1, 1)$  to polar coordinates: Use  $x = 1$  and  $y = 1$ . Then:

$$r^2 = x^2 + y^2 = 1^2 + 1^2 = 2 \implies r = \sqrt{2}, \quad \tan \theta = \frac{y}{x} = \frac{1}{1} = 1 \implies \theta = \frac{\pi}{4}.$$

Therefore,  $(1, 1)$  can be represented as  $(\sqrt{2}, \frac{\pi}{4})$  in polar coordinates.

## Concept

**Problem-Solving Strategy: Plotting a Curve in Polar Coordinates.**

1. Create a table with two columns: one for  $\theta$  values and one for  $r$  values.
2. Calculate the corresponding  $r$  values for each  $\theta$ .
3. Plot each ordered pair  $(r, \theta)$  on the polar coordinate axes.
4. Connect the points and observe the resulting graph.

## Example

**Example 1.12. Graphing a Function in Polar Coordinates.**

Graph the curve defined by  $r = 4 \sin \theta$ .

1. Create a table of values for  $\theta$  and calculate  $r$ :

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\pi$	$2\pi$
$r = 4 \sin \theta$	0	2	$2\sqrt{2}$	4	0	0

2. Plot the points and connect them to form the curve. The result is a circle with radius 2 centered at  $(0, 2)$  in rectangular coordinates.

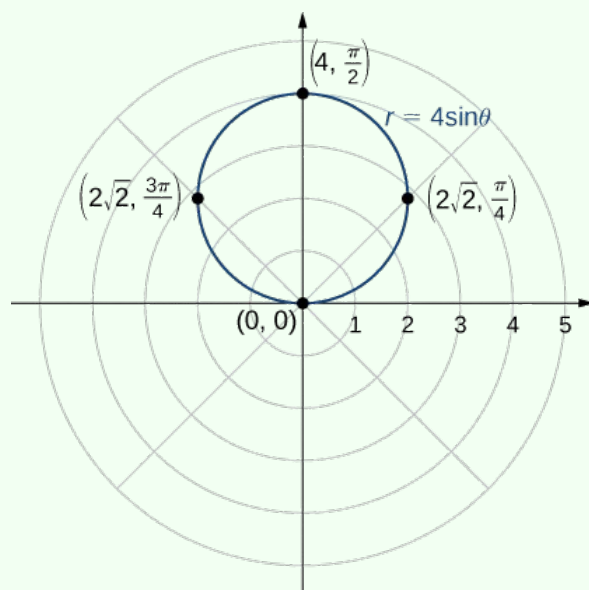


Figure 1: The graph of the function  $r = 4 \sin \theta$  is a circle.

# Let's Get Started

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*Time to dive into the lecture notes.*

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Grab your pen or pencil, and let's break this down step by step.

## Recall: 1st Year Calculus

### Definition

The **definite integral** of a function  $y = f(x)$ , where  $f(x) \geq 0$ , represents the area under the curve from  $x = a$  to  $x = b$ :

$$\text{Area} = \int_{x=a}^{x=b} f(x) dx$$

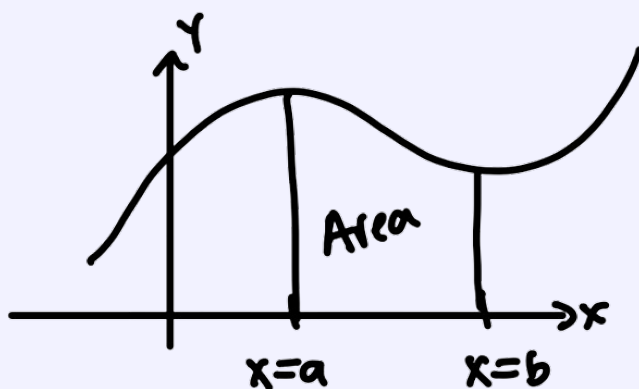


Figure 2: Illustration of the area under  $y = f(x)$ .



## Section 1.2: Area Enclosed by a Parametric Curve

### Definition

A **parametric curve** is defined by:

$$x = f(t), \quad y = g(t), \quad \alpha \leq t \leq \beta$$

with the following properties:

- The curve lies above the  $x$ -axis.
- The curve does not self-intersect.

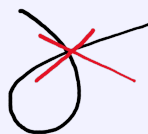


Figure 3: A curve that self-intersects.

### Theorem

The area enclosed by the curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t)f'(t) dt$$

or equivalently:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} f(t)g'(t) dt$$

## Alternative Forms for Area

### Note

In specific cases, the area can also be calculated using:

$$\text{Area} = \int_{y=c}^{y=d} x(y) dy$$

or:

$$\text{Area} = \int_{x=a}^{x=b} y(x) dx$$



## Example: Standard Parametric Curve

### Example

Calculate the area enclosed by the parametric curve:

$$x = \cos(t), \quad y = \sin(t), \quad 0 \leq t \leq \pi$$

### Solution

The area is calculated as:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t) dt,$$

where  $x = f(t)$  and  $y = g(t)$ . Here,  $f(t) = \cos(t)$ ,  $g(t) = \sin(t)$ , and  $f'(t) = -\sin(t)$ .

Substituting:

$$\text{Area} = \int_{t=0}^{t=\pi} \sin(t)(-\sin(t)) dt = \int_{t=0}^{t=\pi} -\sin^2(t) dt.$$

Using  $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$ , we get:

$$\text{Area} = - \int_{t=0}^{t=\pi} \frac{1}{2}(1 - \cos(2t)) dt = -\frac{1}{2} \left[ \int_{t=0}^{t=\pi} 1 dt - \int_{t=0}^{t=\pi} \cos(2t) dt \right].$$

Evaluate the integrals:

$$\int_{t=0}^{t=\pi} 1 dt = \pi, \quad \int_{t=0}^{t=\pi} \cos(2t) dt = \left[ \frac{\sin(2t)}{2} \right]_0^\pi = 0.$$

Thus:

$$\text{Area} = -\frac{1}{2}(\pi - 0) = -\frac{\pi}{2}.$$

Taking the absolute value (since area is positive):

$$\text{Area} = \frac{\pi}{2}.$$

## Example: Cycloid

### Example

**Example:** Find the area under the cycloid defined by:

$$x = t - \sin(t), \quad y = 1 - \cos(t), \quad 0 \leq t \leq 2\pi.$$

### Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t) dt, \quad f'(t) = \frac{dx}{dt}.$$

#### Step 1: Substitution

From  $x = t - \sin(t)$  and  $y = 1 - \cos(t)$ :

$$f'(t) = 1 - \cos(t), \quad g(t) = 1 - \cos(t).$$

Substitute into the formula:

$$\text{Area} = \int_0^{2\pi} (1 - \cos(t))^2 dt.$$

#### Step 2: Expand and Separate Terms

Expand  $(1 - \cos(t))^2$ :

$$\text{Area} = \int_0^{2\pi} [1 - 2\cos(t) + \cos^2(t)] dt.$$

Split the integral:

$$\text{Area} = \int_0^{2\pi} 1 dt - 2 \int_0^{2\pi} \cos(t) dt + \int_0^{2\pi} \cos^2(t) dt.$$

...cont'd...

## Example

## Solution

...cont'd...

**Step 3: Evaluate Each Term**

1. First Term:

$$\int_0^{2\pi} 1 \, dt = 2\pi.$$

2. Second Term:

$$\int_0^{2\pi} \cos(t) \, dt = [\sin(t)]_0^{2\pi} = 0.$$

3. Third Term:

Using  $\cos^2(t) = \frac{1 + \cos(2t)}{2}$ :

$$\int_0^{2\pi} \cos^2(t) \, dt = \frac{1}{2} \int_0^{2\pi} 1 \, dt + \frac{1}{2} \int_0^{2\pi} \cos(2t) \, dt.$$

Evaluate:

$$\frac{1}{2} \int_0^{2\pi} 1 \, dt = \pi, \quad \frac{1}{2} \int_0^{2\pi} \cos(2t) \, dt = 0.$$

Thus:

$$\int_0^{2\pi} \cos^2(t) \, dt = \pi.$$

**Step 4: Combine Results**

$$\text{Area} = 2\pi - 0 + \pi = 3\pi.$$

## Answer

$$\text{Area} = 3\pi$$



## Homework Practice Question: Area Under a Parametric Curve

### Exercise

Find the area under the curve defined by

$$x = 3 \cos(t) + \cos(3t), \quad y = 3 \sin(t) - \sin(3t), \quad 0 \leq t \leq \pi.$$

Hint: Recall that  $\sin^2(x) + \cos^2(x) = 1$ .

### Solution

The area under a parametric curve is given by:

$$\text{Area} = \int_{t=\alpha}^{t=\beta} g(t) f'(t) dt,$$

where  $x = f(t)$ ,  $y = g(t)$ , and  $f'(t) = \frac{dx}{dt}$ .

#### Step 1: Differentiate $x(t)$

Given  $x = 3 \cos(t) + \cos(3t)$ , compute:

$$f'(t) = \frac{d}{dt}[3 \cos(t) + \cos(3t)] = -3 \sin(t) - 3 \sin(3t).$$

#### Step 2: Substitute into the Formula

The parametric area formula becomes:

$$\text{Area} = \int_0^\pi [3 \sin(t) - \sin(3t)] [-3 \sin(t) - 3 \sin(3t)] dt.$$

#### Step 3: Simplify the Expression

Expand the product:

$$[3 \sin(t) - \sin(3t)][-3 \sin(t) - 3 \sin(3t)] = -9 \sin^2(t) - 9 \sin(t) \sin(3t) + 3 \sin(3t) \sin(t) + 3 \sin^2(3t).$$

Combine terms:

$$-9 \sin^2(t) + 3 \sin^2(3t) - 6 \sin(t) \sin(3t).$$

...cont'd...

## Exercise

## Solution

...cont'd..

Using the product-to-sum identity for  $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ :

$$\sin(t)\sin(3t) = \frac{1}{2}[\cos(2t) - \cos(4t)].$$

Substitute this back:

$$\text{Area} = \int_0^\pi [-9\sin^2(t) + 3\sin^2(3t) - 3[\cos(2t) - \cos(4t)]] dt.$$

**Step 4: Break the Integral into Separate Terms**

Split the integral:

$$\text{Area} = -9 \int_0^\pi \sin^2(t) dt + 3 \int_0^\pi \sin^2(3t) dt - 3 \int_0^\pi \cos(2t) dt + 3 \int_0^\pi \cos(4t) dt.$$

**Step 5: Evaluate Each Integral**

1. First Term ( $-9 \int_0^\pi \sin^2(t) dt$ ):

Use the identity  $\sin^2(t) = \frac{1 - \cos(2t)}{2}$ :

$$\int_0^\pi \sin^2(t) dt = \int_0^\pi \frac{1 - \cos(2t)}{2} dt = \frac{1}{2} \int_0^\pi 1 dt - \frac{1}{2} \int_0^\pi \cos(2t) dt.$$

Evaluate:

$$\frac{1}{2} \int_0^\pi 1 dt = \frac{\pi}{2}, \quad \frac{1}{2} \int_0^\pi \cos(2t) dt = \frac{1}{2}[0] = 0.$$

So:

$$\int_0^\pi \sin^2(t) dt = \frac{\pi}{2}.$$

Multiply by  $-9$ :

$$-9 \int_0^\pi \sin^2(t) dt = -9 \cdot \frac{\pi}{2} = -\frac{9\pi}{2}.$$

...cont'd...

## Exercise

## Solution

...cont'd...

2. Second Term ( $3 \int_0^\pi \sin^2(3t) dt$ ):

Similarly,  $\sin^2(3t) = \frac{1 - \cos(6t)}{2}$ :

$$\int_0^\pi \sin^2(3t) dt = \frac{1}{2} \int_0^\pi 1 dt - \frac{1}{2} \int_0^\pi \cos(6t) dt.$$

Evaluate:

$$\frac{1}{2} \int_0^\pi 1 dt = \frac{\pi}{2}, \quad \frac{1}{2} \int_0^\pi \cos(6t) dt = 0.$$

So:

$$\int_0^\pi \sin^2(3t) dt = \frac{\pi}{2}.$$

Multiply by 3:

$$3 \int_0^\pi \sin^2(3t) dt = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}.$$

3. Third Term ( $-3 \int_0^\pi \cos(2t) dt$ ):

Since  $\int_0^\pi \cos(2t) dt = 0$ :

$$-3 \int_0^\pi \cos(2t) dt = 0.$$

4. Fourth Term ( $3 \int_0^\pi \cos(4t) dt$ ):

Similarly,  $\int_0^\pi \cos(4t) dt = 0$ :

$$3 \int_0^\pi \cos(4t) dt = 0.$$

### Step 6: Combine Results

Add the evaluated terms:

$$\text{Area} = -\frac{9\pi}{2} + \frac{3\pi}{2} + 0 + 0 = -\frac{6\pi}{2} = -3\pi.$$

However, the area is always positive, so:

$$\text{Area} = 3\pi.$$



## Exercise

## Solution

*...cont'd...*

## Answer

$$\text{Area} = 3\pi$$

## The Arc Length of a Parametric Curve

### Theorem

Let a curve be parameterized by  $t$ , such that:

$$x = x(t) \quad \text{and} \quad y = y(t), \quad \text{for } t \in [\alpha, \beta].$$

The **arc length**  $L$  of the curve between  $t = \alpha$  and  $t = \beta$  is given by:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### Concept

Consider two points  $(x_1, y_1)$  and  $(x_2, y_2)$  on a curve. The differences in their coordinates are defined as:

$$\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1$$

The distance  $D$  between the two points is given by the Pythagorean theorem:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substituting  $\Delta x$  and  $\Delta y$ :

$$D = \sqrt{\Delta x^2 + \Delta y^2}$$

Now, consider the curve parameterized by  $t$ , where  $x = x(t)$  and  $y = y(t)$ . Dividing  $\Delta x$  and  $\Delta y$  by the parameter  $\Delta t$ :

$$D \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t$$

Taking the limit as  $\Delta t \rightarrow 0$ , this becomes a Riemann sum. Therefore, the arc length  $L$  of the curve is:

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### Note

$L$  represents the total length of the curve between  $t = \alpha$  and  $t = \beta$ . This formula is provided on the formula sheet for term test 1.

## Illustration

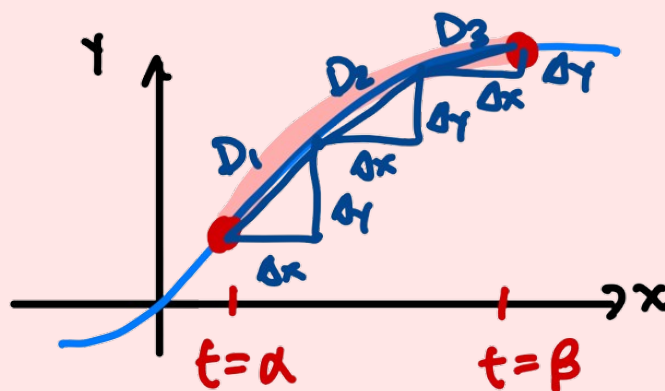


Figure 4: Visualizing the arc length theorem — the connection between Riemann sums and the integral for a curve's length.



## Example: Parametric Curve

### Example

Find the arc length of the curve defined by:

$$x = 3 \cos(t), \quad y = 3 \sin(t), \quad t \in [0, 2\pi].$$

### Solution

The arc length is denoted by  $L$ . We evaluate it as follows:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sqrt{(-3 \sin(t))^2 + (3 \cos(t))^2} dt \\ &= \int_0^{2\pi} \sqrt{9 \sin^2(t) + 9 \cos^2(t)} dt. \end{aligned}$$

Using the Pythagorean identity  $\sin^2(t) + \cos^2(t) = 1$ , simplify the integrand:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9 \cdot (\sin^2(t) + \cos^2(t))} dt \\ &= \int_0^{2\pi} \sqrt{9 \cdot 1} dt \\ &= \int_0^{2\pi} 3 dt. \end{aligned}$$

The integral simplifies to:

$$\begin{aligned} L &= 3 \int_0^{2\pi} 1 dt \\ &= 3 [t]_0^{2\pi} \\ &= 3 \cdot (2\pi - 0) \\ &= 6\pi. \end{aligned}$$

### Answer

The arc length of the curve is  $L = 6\pi$ .

## Homework Practice Problem: Parametric Curve

### Note

Find the arc length of the curve defined by:

$$x = 3t^2, \quad y = 2t^3, \quad 1 \leq t \leq 3.$$

### Solution

The arc length is denoted by  $L$ , which is evaluated as follows:

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Compute the derivatives:

$$\frac{dx}{dt} = \frac{d}{dt}(3t^2) = 6t, \quad \frac{dy}{dt} = \frac{d}{dt}(2t^3) = 6t^2.$$

Substitute these into the arc length formula:

$$\begin{aligned} L &= \int_1^3 \sqrt{(6t)^2 + (6t^2)^2} dt \\ &= \int_1^3 \sqrt{36t^2 + 36t^4} dt \\ &= \int_1^3 \sqrt{36t^2(1 + t^2)} dt \\ &= \int_1^3 6t\sqrt{1 + t^2} dt. \end{aligned}$$

Use substitution to simplify the integral. Let:

$$u = 1 + t^2, \quad \frac{du}{dt} = 2t, \quad dt = \frac{du}{2t}.$$

Change the limits of integration:

$$\text{When } t = 1, u = 1 + 1^2 = 2; \quad \text{When } t = 3, u = 1 + 3^2 = 10.$$

Substitute into the integral:

$$\begin{aligned} L &= \int_2^{10} 6t \cdot \sqrt{u} \cdot \frac{du}{2t} \\ &= \int_2^{10} 3\sqrt{u} du. \end{aligned}$$

## Note

## Solution

Evaluate the integral:

$$\begin{aligned} L &= 3 \int_2^{10} u^{1/2} du \\ &= 3 \left[ \frac{2}{3} u^{3/2} \right]_2^{10} \\ &= 2 \left[ u^{3/2} \right]_2^{10} \\ &= 2 \left[ (10)^{3/2} - (2)^{3/2} \right]. \end{aligned}$$

Simplify the result:

$$L = 2 \left[ 10\sqrt{10} - 2\sqrt{2} \right].$$

## Answer

The arc length of the curve is  $L = 2 (10\sqrt{10} - 2\sqrt{2})$ .



## Section 1.3: Polar Coordinates

### Definition

**Polar coordinates** represent a point in the plane by specifying its distance from the origin ( $r$ ) and the angle ( $\theta$ ) it makes with the positive x-axis, measured counterclockwise.

### Concept

**Cartesian Coordinates:**

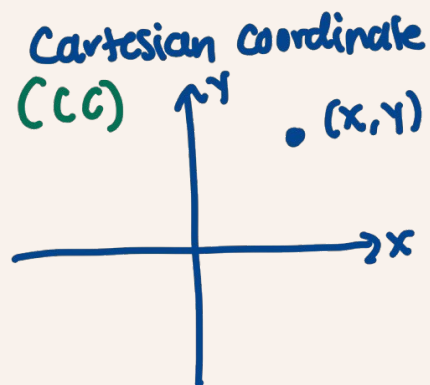


Figure 5: Graphical representation of Cartesian coordinates.

**Polar Coordinates:**

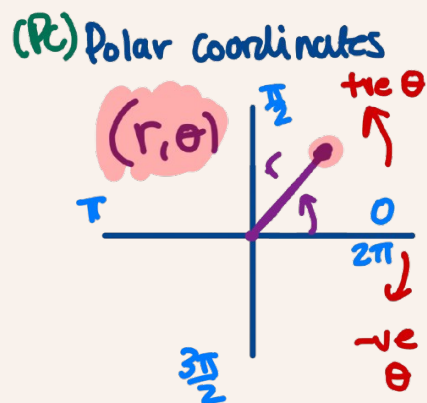


Figure 6: Graphical representation of Polar coordinates.

## How to Work with Polar Coordinates

### Tip

1. Start from the origin and move radially outward by a distance equal to  $r$ .
2. From this position, rotate the point counterclockwise by an angle  $\theta$  (in radians or degrees).
3. The resultant position is the point represented by the polar coordinate  $(r, \theta)$ .

## Converting from Cartesian to Polar Coordinates

### Algorithm

1. Given a point  $(x, y)$  in Cartesian coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

2.  $r$  represents the radial distance from the origin.
3.  $\theta$  is the angle measured from the positive x-axis to the line segment joining the origin to the point.
4. Adjust  $\theta$  based on the quadrant of the point to ensure the correct angle.

## Converting from Polar to Cartesian Coordinates

### Algorithm

1. Given a point  $(r, \theta)$  in polar coordinates:

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

2. Compute  $x$  and  $y$  using trigonometric functions with  $r$  and  $\theta$ .
3. The result  $(x, y)$  represents the Cartesian coordinates of the point.

## Examples: Converting Between Rectangular and Polar Coordinates

### Example

To convert a point from rectangular coordinates  $(x, y)$  to polar coordinates  $(r, \theta)$ , we use the following formulas:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \left( \frac{y}{x} \right)$$

Now, let's convert the following points:

### Exercise

1. For the point  $(1, 1)$ :

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad \theta = \tan^{-1} \left( \frac{1}{1} \right) = \frac{\pi}{4}$$

So, the polar coordinates are  $(\sqrt{2}, \frac{\pi}{4})$ .

### Exercise

2. For the point  $(-3, 4)$ :

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5, \quad \theta = \tan^{-1} \left( \frac{4}{-3} \right) = \tan^{-1} \left( -\frac{4}{3} \right)$$

Since the point is in the second quadrant, we add  $\pi$  to the angle:

$$\theta = \pi - \tan^{-1} \left( \frac{4}{3} \right)$$

The polar coordinates are approximately  $(5, \pi - \tan^{-1}(\frac{4}{3}))$ .

### Exercise

3. For the point  $(0, 3)$ :

$$r = \sqrt{0^2 + 3^2} = 3, \quad \theta = \frac{\pi}{2}$$

So, the polar coordinates are  $(3, \frac{\pi}{2})$ .

## Examples: Converting Polar to Rectangular Coordinates

### Example

To convert a point from polar coordinates  $(r, \theta)$  to rectangular coordinates  $(x, y)$ , we use the following formulas:

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Now, let's convert the following points:

### Exercise

1. For the point  $(3, \frac{\pi}{3})$ :

$$x = 3 \cos\left(\frac{\pi}{3}\right) = 3 \times \frac{1}{2} = \frac{3}{2}, \quad y = 3 \sin\left(\frac{\pi}{3}\right) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

So, the rectangular coordinates are  $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ .

### Exercise

2. For the point  $(2, \frac{3\pi}{2})$ :

$$x = 2 \cos\left(\frac{3\pi}{2}\right) = 2 \times 0 = 0, \quad y = 2 \sin\left(\frac{3\pi}{2}\right) = 2 \times (-1) = -2$$

So, the rectangular coordinates are  $(0, -2)$ .

### Exercise

3. For the point  $(6, -\frac{5\pi}{6})$ :

$$x = 6 \cos\left(-\frac{5\pi}{6}\right) = 6 \times \left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}, \quad y = 6 \sin\left(-\frac{5\pi}{6}\right) = 6 \times \left(-\frac{1}{2}\right) = -3$$

So, the rectangular coordinates are  $(-3\sqrt{3}, -3)$ .