MAT232 - Lecture 4

Polar Coordinates and Curves

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Contents

Definitions and Theorems

Straight from the textbook — no fluff, just what we need.

Quick recap before diving into the lecture.

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Polar Coordinates - Key Theorems

Converting Points between Coordinate Systems

Theorem

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

These formulas can be used to convert between rectangular and polar coordinates.

Uniqueness of Polar Coordinates

Proposition

Every point in the plane has an infinite number of representations in polar coordinates. Specifically, the polar coordinates (r, θ) of a point are not unique.

Remark

For example, the polar coordinates $(2, \pi/3)$ and $(2, 7\pi/3)$ both represent the same point in the rectangular coordinate system. Additionally, the value of r can be negative. Therefore, the point with polar coordinates $(-2, 4\pi/3)$ represents the same rectangular point as $(2, \pi/3)$.

Symmetry of Polar Curves

Theorem

Polar curves can exhibit symmetry similar to those in rectangular coordinates. The key symmetries to identify are:

- **Symmetry with respect to the polar axis:** A curve is symmetric with respect to the polar axis if replacing θ with $-\theta$ in its equation yields the same curve.
- **Symmetry with respect to the line $\theta = \frac{\pi}{2}$:** A curve is symmetric with respect to the line $\theta = \frac{\pi}{2}$ if replacing θ with $\pi - \theta$ yields the same curve.
- **Symmetry with respect to the pole (origin):** A curve is symmetric with respect to the pole if replacing r with $-r$ yields the same curve.

Let's Get Started

Time to dive into the lecture notes.

Grab your pen or pencil, and let's break this down step by step.

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Plotting Polar Coordinates

Recall the Content from Last Lecture

Note

Converting between Cartesian coordinates (x, y) and Polar coordinates (r, θ):

Algorithm

From Cartesian to Polar:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

From Polar to Cartesian:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Converting Between Degrees and Radians:

Algorithm

- **Degrees to Radians:** Multiply by $\frac{\pi}{180^\circ}$

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180^\circ}$$

- **Radians to Degrees:** Multiply by $\frac{180^\circ}{\pi}$

$$\text{Degrees} = \text{Radians} \times \frac{180^\circ}{\pi}$$

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Understanding the Convention for r in Polar Coordinates

Concept

In polar coordinates, a point is represented as (r, θ), where:

- r is the radial distance from the origin (how far the point is from the origin).
- θ is the angle, measured counterclockwise from the positive x-axis.

Note

Special Case: When r is Negative

- A negative r in $(-r, \theta)$ is interpreted as the point being reflected through the origin.
- The equivalent representation is:

$$(-r, \theta) = (r, \theta + 180^\circ)$$

or in radians:

$$(-r, \theta) = (r, \theta + \pi)$$

Intuition

- Reflecting (r, θ) through the origin is the same as rotating the point by 180° (or π radians).
- This property simplifies polar plots by offering alternate representations of the same point.

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Example: Plotting Points

Example

Let us plot the following points in polar coordinates:

$$(3, -45^\circ), \quad (3, 225^\circ), \quad (4, 330^\circ), \quad (1, -45^\circ)$$

Algorithm

Step-by-Step Process:

1. For each point, identify r and θ.
2. If θ is negative or exceeds 360°, convert it to a standard range:

$$\theta \in [0^\circ, 360^\circ)$$

   using $\theta = \theta + 360^\circ$ (for negative angles) or subtracting 360° (for angles over 360°).
3. Plot the point by measuring θ counterclockwise from the positive x-axis and placing it at a distance r from the origin.

Solution

- For $(3, -45^\circ)$: Add 360° to -45° to convert θ to 315°. Plot as $(3, 315^\circ)$.
- For $(3, 225^\circ)$: Already within the standard range, so plot directly.
- For $(4, 330^\circ)$: Angle is standard, so plot directly.
- For $(1, -45^\circ)$: Add 360° to -45°, yielding $(1, 315^\circ)$.

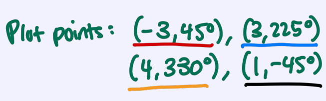

Figure 1: Colour Legend

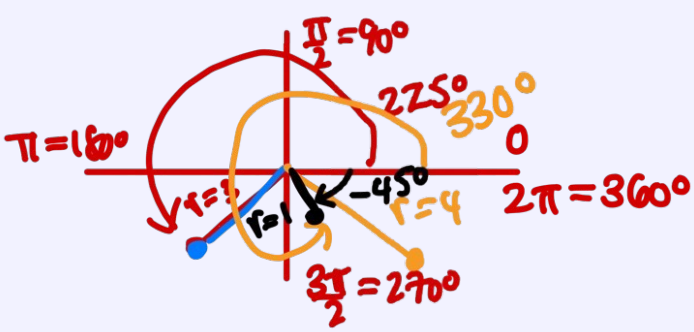

Figure 2: Polar Coordinates Plot and "Trajectories"

Tip

Ensure to label points clearly on the polar grid, and verify angle conversions and reflections for accuracy.

Example: Converting from Polar Coords to Cartesian Coords

Example

Find the **rectangular coordinates** (or Cartesian coordinates) of the point p whose polar coordinates are $(6, \frac{\pi}{3})$.

Solution

To convert from polar to Cartesian coordinates, we use the following formulas:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Substitute the given values for $r = 6$ and $\theta = \dfrac{\pi}{3}$:

- For x:
$$x = 6\cos\left(\frac{\pi}{3}\right) = 6\left(\frac{1}{2}\right) = 3$$
- For y:
$$y = 6\sin\left(\frac{\pi}{3}\right) = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

Thus, $(x, y) = (3, 3\sqrt{3})$.

Answer

The cartesian coordinates of the point are $(x, y) = (3, 3\sqrt{3})$.

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Converting from Cartesian Coordinates to Polar Coordinates

Example

Find the polar coordinate of the point p whose rectangular coordinates are $-2, 2\sqrt{3}$.

Solution

Recall that (the circle equation):

$$x^2 + y^2 = r^2$$

It follows that:

$$(-2)^2 + (2\sqrt{3})^2 = r^2$$

$$4 + 4 \cdot 3 = r^2$$

$$16 = r^2$$

$$\pm\sqrt{16} = r$$

$$r = \pm 4$$

Note that the radius is positive. Thus:

$$r = 4.$$

Recall that:

$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\theta) = \frac{2\sqrt{3}}{-2}$$

$$\tan(\theta) = -\sqrt{3}$$

Note

Note that:

$$\arctan\left(\frac{y}{x}\right) = \theta, \quad \frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Tip

Manually determining θ from $\tan(\theta) = \sqrt{3}$.
Note the special angles (in radians):

- 0
- $\frac{\pi}{6}$
- $\frac{\pi}{4}$
- $\frac{\pi}{3}$
- $\frac{\pi}{2}$

Check: At $\theta = \frac{\pi}{6}$,

$$LHS = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}$$

Tip

When practicing for this course, you are encouraged to leverage any available graphing websites and/or software.

Ideally, you want to know how to draw lines and circles.

Polar Curves

Example

Consider $r = f(\theta)$.

Sketch the following functions:
(a) $r = 1$
(b) $\theta = \frac{\pi}{4}$
(c) $r = \theta, \quad \theta \geqslant 0$
(d) $r = \sin(\theta)$
(e) $r = \cos(2\theta)$

(a)

Solution

Here, $r = 1$ and θ is an arbitrary angle.

Converting from a polar-coordinate curve to a cartesian-coordinate equation:

$$x^2 + y^2 = r^2 = 1^2 = 1$$

Clearly, we are working with the unit cirle.
self-note: actually show the illustration as andie drew on the lecture notes

(b)

Solution

[fill this in]

(c) $r = \theta, \quad \theta \geqslant 0$

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Solution

As $r \to \infty$, θ increases.

$$r = f(\theta)$$

$$\pi \doteq 3.14$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$\doteq 1$	$\doteq 1.2$	$\doteq 1.4$	$\doteq 1.5$

Check out the illustration: [add-illustration-here]

Now, converting from polar coordinates to cartesian coordinates:

$$x^2 + y^2 = r^2$$

$$\sqrt{x^2 + y^2} = r$$

[and also add the other equation]

(d) $r = \sin\theta$

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Solution

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Just use the table to directly plot the points for the graph!

add-illustration-here We now need an equation that will help us get rid of $r = \sin\theta$. Consider the possibilities:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $\frac{y}{r} = \sin\theta$

$$r = \sin\theta$$
$$r = \frac{y}{r}$$
$$r^2 = y$$

$x^2 + y^2 = r^2$ So,

$$x^2 + y^2 = y.$$

Proposition

Recall how to complete the square from Grade 10 math: **self-note: add that here to reference**

Proceed to complete the square:

$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$$

step no. 1: $-\frac{1}{2}$
step no. 2: $(-\dfrac{1}{2})^2 = \frac{1}{4}$. Recall that:

$$(y+a)^2 = (y+a)(y+a)$$
$$= y^2 + 2ay + a^2$$

This would represent the $y^2 - y + \frac{1}{4}$ part.
Note that the $(y-a)^2$ represents the $-\frac{1}{2}$
result:

$$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

Centre: $(0, \frac{1}{2})$
Radius: $= \frac{1}{2}$

Exercise

Try:

$$r = \cos\theta$$

under the same context as denoted for the above questions.

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The Derivative of a Polar Curve

Tangents to Polar Curves

Definition

Note

Recall that polar curves are defined by:

$$r = f(\theta)$$

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

Intuition

The goal is to have everything on x depend on **one** parameter.

Do the exact same thing on y.

So,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{\frac{dx}{d}}{\theta}}.$$

We want require $\frac{dx}{d\theta} \neq 0$.
So...

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{df(\theta)}{d\theta}\sin\theta + \cos\theta f(\theta)}{\frac{df(\theta)}{d\theta}\cos\theta - \sin\theta f(\theta)}$$

So,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(subbed r in for $f(\theta)$). Conclusion:

- Horizontal Tangents: $\frac{dy}{d\theta} = 0, \quad \frac{dx}{d\theta} \neq 0$
- Vertical Tangents: $\frac{dx}{d\theta} = 0, \quad \frac{dy}{d\theta} \neq 0$
- Singular Points (discard; we will not be doing further analysis for this case in MAT232): $\frac{dy}{d\theta} = \frac{dx}{d\theta} = 0$

Examples

Example

Find the **vertical tangent** angles of the polar curve $r = 1 - \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$.

Solution

Recall that $\frac{dr}{d\theta} = \sin\theta$.
Obtain the first derivative:

$$\frac{dy}{dx} = \ldots$$

self-note: prof is going way too fast; finish the notes according to your camera roll later! the good thing is that you didn't actually miss any sections! fulfilling incomplete sections is just a matter of reviewing and comparing to the pictures taken of the prof's projected live notes!

Answer

The vertical tangents are located at $x = \{\frac{\pi}{3}, \pi\}$.

Figure 3: Sample image illustrating the concept.

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Next Week: Vector Week

Theorem

- Circle: $x^2 + y^2 = r^2$.
- Generic form for a circle centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

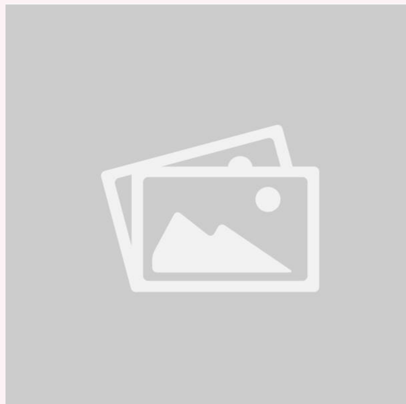

Figure 4: Graphical representation of the theorem.

Example

Sketch $x^2 + y^2 - 2x = 10$.

Solution

Recall how to complete the square:

$$x^2 - 2x + 1 + y^2 = 10 + 1$$

Step #1: $-\frac{2}{2} = -1$;
Step #2: $(-1)^2 = 1$ **self-note: complete this below**

Additional Notes

Note

Always check the domain of the parameter t when solving problems involving parametric equations.