Introduction and Tools 1

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Introduction

Welcome

Thanks for being here.

Introduce yourself:

- Name
- Year of PhD
- Programming langauge you plan to use
- Answer the question you wish I asked you here

Writing Good Code

An Economics Crisis

I think we are in the middle of a simmering crisis in economics...I think the first-order issue with economists and programming languages is not (necessarily) performance, but (1) correctness and (2) software engineering...Have someone with a published numerical paper send you their code, and tell me if you can tell whether it implements the model or not...The software engineering problem in economics needs to be addressed with better training and helping people understand tools like version control, continuous integration, unit testing, the benefits of tested libraries, etc.

Intro to Software Engineering

What is “Software Engineering?”

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What is “Software Engineering?”

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What does this mean?

Write code to minimize the probability of having a bug and so that others can understand your code without excessive effort.

> *Always code as if the guy who ends up maintaining your code will be a violent pyschopath who knows where you live.*

Simple Rules to Live By

These are in no way exhaustive, but are a start

- Don’t Repeat Yourself
  - Use functions rather than writing things repeatedly
  - Writing (c^(1 — gamma) — 1) / (1 — gamma) repeatedly makes it likely you will eventually make a mistake
  - It is a pain to make sure you change all instances if you decide to use a different utility function.
- Write test cases to ensure that various functions you use do what you think they do.
- Use whitespace wisely (ALWAYS HAVE SPACE AFTER COMMA).
- Comment and document your code carefully.

Understanding the Basics

Floating Point Numbers

Computer reads numbers differently than you.

A 16 bit floating point number: $\underbrace{0}_{\text{Sign}}\ \underbrace{01101}_{\text{Exponent}}\ \underbrace{0101010101}_{\text{Mantissa}}$

$$0011010101010101 \rightarrow -1^{\text{Sign}} \left(1 + \sum_{n=1}^{10} \text{Mantissa}_n 2^{-n}\right) 2^{\text{Exponent - Bias}}$$

$$\rightarrow 1(1.3330078125)2^{13-15} = 0.33325$$

Occasionally useful to understand this fact.

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Occasionally useful to understand this fact.

For example, since arithmetic is not necessarily commutative for floating points (because of accuracy limitations) checking exact equality between floating points is a *bad* idea...

Ill Conditioned Matrices

Consider $A = \begin{bmatrix} 2 & 6 \\ 2 & 6.00001 \end{bmatrix}$, $b_1 = \begin{bmatrix} 8 \\ 8.00001 \end{bmatrix}$, and $b_2 = \begin{bmatrix} 8 \\ 8.00002 \end{bmatrix}$

- Solution to $Ax = b_1$ is $\begin{bmatrix} 1 & 1 \end{bmatrix}$
- Solution to $Ax = b_2$ is $\begin{bmatrix} 2 & -2 \end{bmatrix}$

An ill conditioned matrix is one that is close to singular. This causes, for the equation $Ax = b$, that small changes in b lead to large changes in x.

Dealing with Ill Conditioned Matrices

Can check whether a matrix is ill conditioned by looking at its condition number:

$$\kappa(A) = \sigma_{\text{max}}(A)/\sigma_{\text{min}}(A)$$

If a matrix is actually ill-conditioned ($\kappa(A)$ large), there is not much you can do except try to make sure b is calculated as closely as possible.

Column vs Row Major

Matrices (and higher dimensional) arrays are stored as sequential elements of memory. The order in which they are stored determines whether a language is column or row major.

For example, consider the following matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

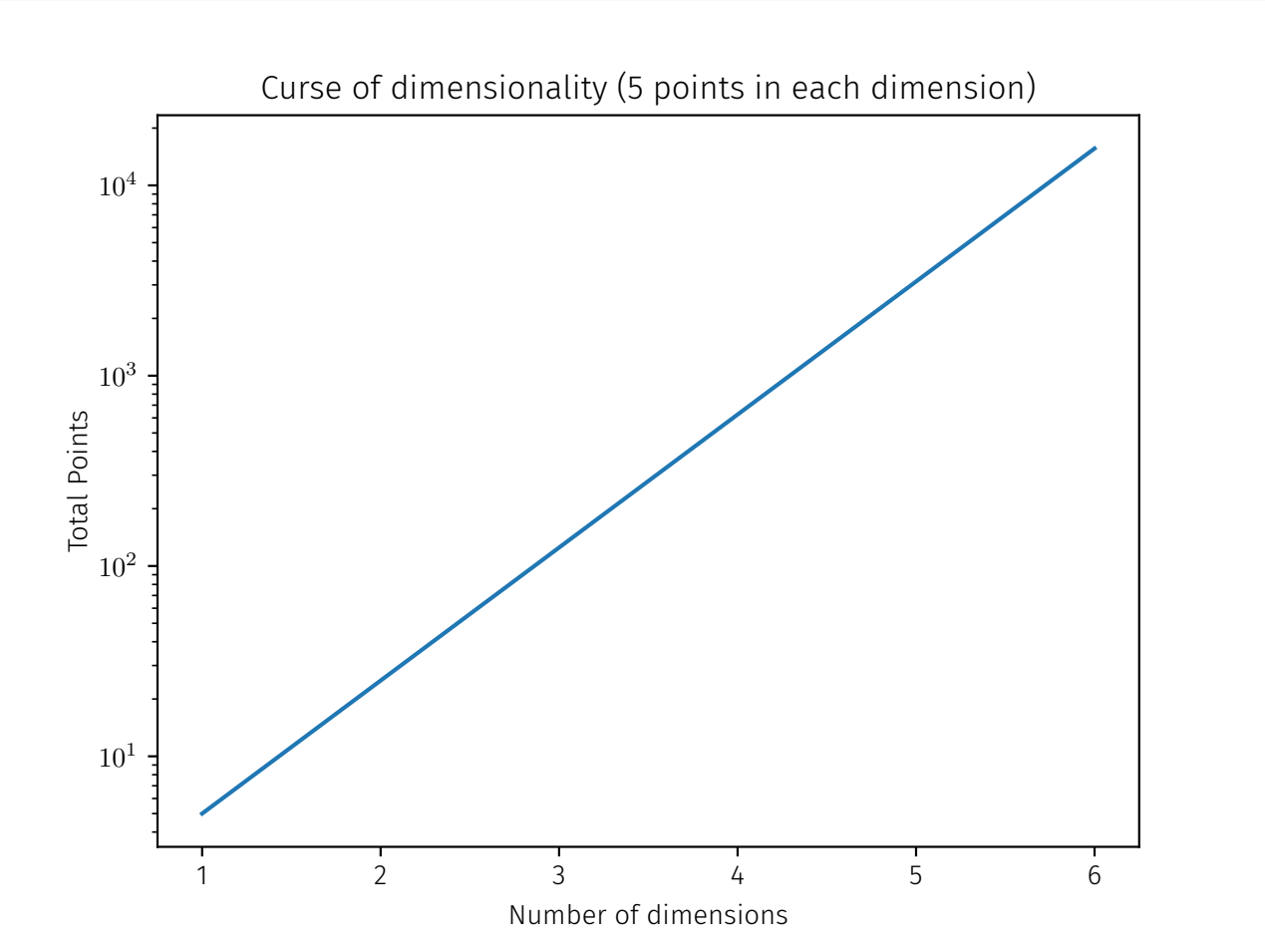
- A row major language would store this as: 1, 2, 3, 4, 5, 6
- A column major language would store this as: 1, 4, 2, 5, 3, 6

This has performance implications (More info here)

Curse of Dimensionality

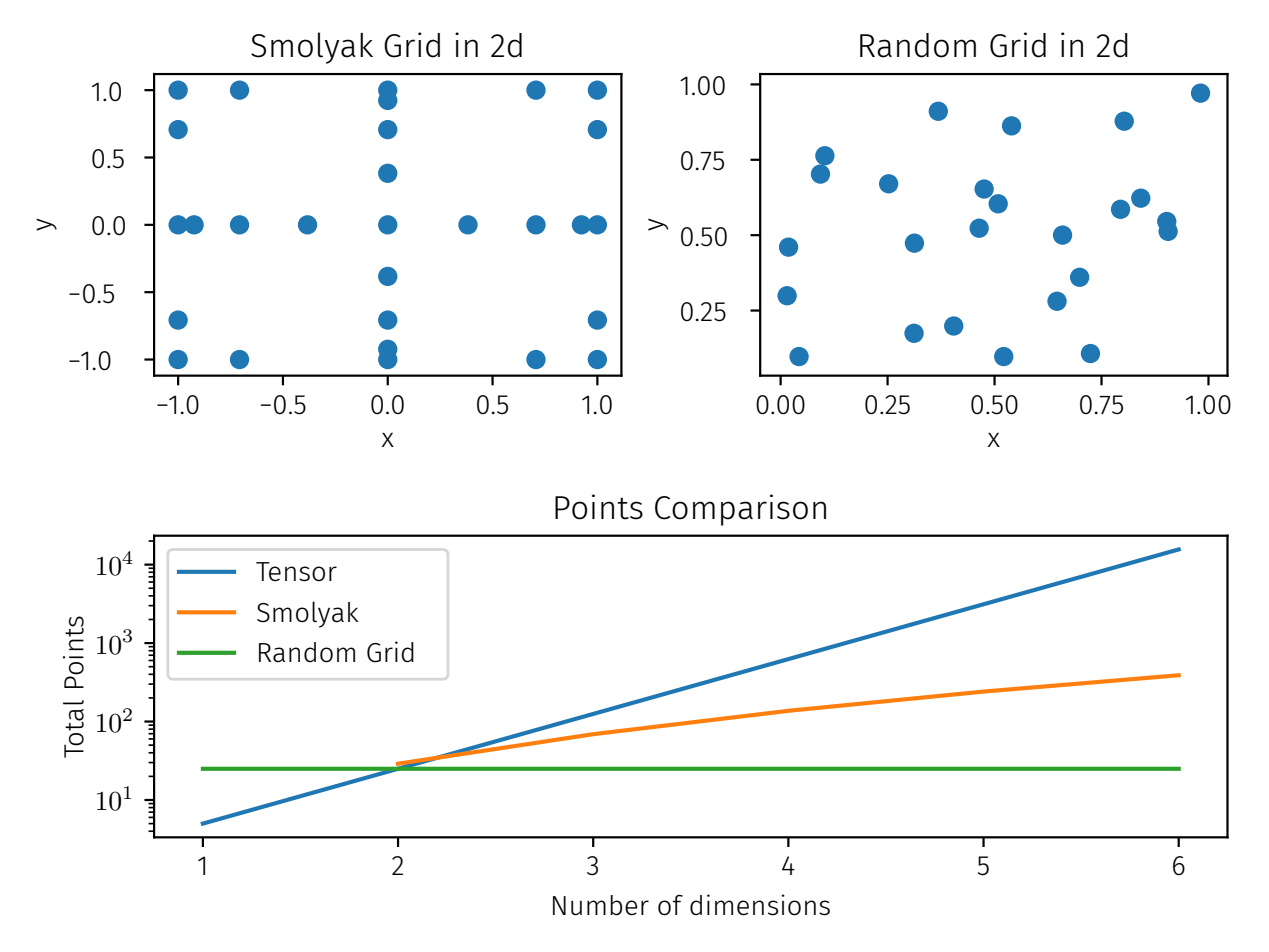
Often need to approximate functions $f : \mathbb{R}^d \to \mathbb{R}$

If want to use n points to approximate function in each dimension using a tensor grid of points then have a total of n^d points.



Curse of Dimensionality

Most obvious way to slow down the curse of dimensionality is to move away from tensor grids!



Vectorization

Sometimes useful to recognize that loops can be avoided by using vectorized operations.

For example, imagine a Markov chain, (S_t, P, π_0), for endowments.

$$V(s_t) = u(s_t) + \beta E\left[V(s_{t+1})|s_t\right]$$

Recognize that $\beta E[V(s_{t+1})|s_t]$ is just $\beta P\vec{V}$ thus $\vec{V} = (I - \beta P)u(\vec{s})$

Root Finding and Optimization

Goal of Root Finding

Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that $f(x) = 0$

Whenever we search for prices in a general equilibrium model, we are seeking to clear markets — i.e. find prices such that the excess demand functions are zero.

In algorithms presented, the stopping point will be determined by how close x_{k+1} and x_k are, but the algorithm is only a success if ultimately $f(x_{k+1}) \approx 0$. For space constraints, I have left this function convergence check out of the algorithms.

Bisection: Simplest Root Finding Algorithm

Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $f(a)f(b) < 0$. Then Intermediate Value Theorem states, $\exists c \in (a, b)$ such that $f(c) = 0$.

Require: $f(a)f(b) < 0$
  $fc \leftarrow 10$
  while $|b - a| > tol$ **do**
    $c \leftarrow (a + b)/2$
    $fc \leftarrow f(c)$
    if $fa * fc < 0$ **then**
      $b \leftarrow c$
    else
      $a \leftarrow c$
    end if
  end while

Bisection

Simplest root finding algorithm

- Pros: Relatively fast, simple, guaranteed to find a solution given certain conditions
- Cons: Uses little info about function, not natural to extend to higher than 1 dimension

Newton's Method

Natural follow up to Bisection is Newton's Method which uses information about derivatives by linearizing and find the zero of the linearized version.

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$$

while $|x_{k+1} - x_k| > tol_x$ **do**
  $x_{k+1} \to x_k - f(x_k)/f'(x_k)$
end while

Newton's Method

- Pros: Even faster than bisection, simple, uses information about derivative
- Cons: Does not always find a solution, requires ability to compute derivative

Secant Method

Secand method is related to Newton's Method, but, rather than use the analytical derivative, it approximates $f'(x)$ by taking secant line between $f(x_{k-1})$ and $f(x_k)$

while $|x_{k+1} - x_k| > tol_x$ **do**
$x_{k+1} \rightarrow x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$
end while

Secant Method

- Pros: Similar in speed to Newton's method — The relative speed depends on how much time evaluating the derivative takes vs possibly slower convergence because not using actual derivative, sometimes useful not to need exact derivative
- Cons: Not guaranteed to converge, doesn't use analytical derivative

Multi-dimesional Newton

Use Jacobian rather than derivative

while $|x_{k+1} - x_k| > tol_x$ **do**
  $A_k = J(x^k)$
  Solve $A_k s_k = -f(x^k)$ for s_k
  $x_{k+1} \to x_k + s_k$
end while

Multi-dimesional Secant (aka Broyden)

Must determine how to approximate Jacobian?

One way[1] : $A_{k+1} = A_k + \frac{(y_k - A_k s_k) s_k^T}{s_k^T s_k}$

$A_0 = I$
while $|x_{k+1} - x_k| > tol_x$ **do**
  Solve $A_k s_k = -f(x^k)$ for s_k
  $x_{k+1} \to x_k + s_k$
  $y_k = f(x^{k+1}) - f(x^k)$
  $A_{k+1} = A_k + \frac{(y_k - A_k s_k) s_k^T}{s_k^T s_k}$
end while

[1] This ensure that $A_{k+1} q = A_k q$ whenever $q^T s_k = 0$ (predicted change in directions orthogonal to s_k with new Jacobian matrix are equal to predicted change in that direction with old Jacobian matrix)

Gauss-Jacobi and Gauss-Seidel

Both of these algorithms simplify a n-dimensional problem to a 1-dimensional problem. Solve for x_i such that $f(x_i, x_{-i}) = 0$. However, the approach is slightly different.

- **Gauss-Jacobi**: Find x^{k+1} such that its elements satisfy following equations

$$f^1(x_1^{k+1}, x_2^k, \ldots, x_n^k) = 0$$
$$f^2(x_1^k, x_2^{k+1}, \ldots, x_n^k) = 0$$
$$\ldots$$

- **Gauss-Seidel**: Find x^{k+1} such that its elements satisfy following equations

$$f^1(x_1^{k+1}, x_2^k, \ldots, x_n^k) = 0$$
$$f^2(x_1^{k+1}, x_2^{k+1}, \ldots, x_n^k) = 0$$
$$\ldots$$

Goal of Optimization

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, find $x^* \in \mathbb{R}^n$ such that
$x^* \in \arg\max_x f(x)$

Economic models typically have various agents who are each solving a maximization problem — For example, consumers choose consumption and savings in order to maximize their value today (flow + continuation).

Grid Search

“Premature optimization is the root of all evil” —Donald Knuth

A very simple way to get an idea of where the maximum of a function is, is to simply compare on a discrete number of points.

This approach is often more effective than people give it credit for.

Golden-Section Search

Nelder-Mead (Simplex Method)

Newton's Method (Optimization)

Like finding the root of $f'(x)$...

while $|x_{k+1} - x_k| > tol_x$ **do**
  Compute gradient, $\nabla f(x^k)$
  Compute Hessian, $H(x^k)$
  Solve for s^k in $H(x^k)s^k = -(\nabla f(x^k))^T$
  $x^{k+1} \leftarrow x^k + s^k$
end while

Hill Climbing

BFGS Method

Constrained: Penalty Function Methods

Constrained: Sequential Quadratic

Column vs Row Major: Supplement

Computers have different layers of memory storage. In order to apply operations, data must be moved to a small piece of memory called L1 memory. L1 memory lives directly on the processor and is very fast.

As computers have evolved, they have gotten better at “guessing” what data you will need next. Instead of moving just a single element of an array, the processor will retrieve a small block of consecutive elements.

As you iterate along an array, this means the computer will need to spend less time passing memory between RAM and L1 and can spend more time doing actual operations.

Additional info: online

Back to Column vs Row Major

References

- Numerical Methods in Economics by Kenneth L Judd