

LECTURE 5: GLOBAL SOLUTION METHODS (CONTINUED)

Victoria Gregory

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New York University

Continue illustrating methods for solving a dynamic programming problem **globally**. Today's topics:

Howard Improvement

Policy Function Iteration

Endogenous Grid Method

Collocation

HOWARD IMPROVEMENT

For the moment, go back to value function iteration method Chase talked about last week.

- Choosing the maximizing policy is typically the most time-consuming part of these algorithms
- **Idea:** reduce the number of times we update the policy function rather than the value function
- On some iterations, use the current guess for the policy function to update the value function
- This works because the policy function typically converges faster than the value function

First need to choose a parameter H , the number of times you update the value function using the existing policy function

- Requires some experimentation
- Can increase H after each iteration
- Start using Howard Improvement only after doing VFI a few times
- Too high of an H can cause the value function to move too far away from the true one

HOWARD IMPROVEMENT

1. Choose grids on state variables: $\mathcal{A} \times \mathcal{S}$
2. Initial guess of value function: $V_0(a, s) \forall (a, s) \in \mathcal{A} \times \mathcal{S}$
3. Given V_j , for each $(a_t, s_t) \in \mathcal{A} \times \mathcal{S}$

3.1 Find $a_{t+1} \in \mathcal{A}$ such that

$$a_{t+1} \in \operatorname{argmax} u(y(s_t) + (1 + r)a_t - a_{t+1}) + \beta E [V_j(a_{t+1}, s_{t+1})]$$

3.2 Update policy function

$$a^*(a_t, s_t) = a_{t+1}$$

3.3 Update value function

$$V_{j+1}(a_t, s_t) = u(c_t) + \beta E [V_j(a_{t+1}, s_{t+1})]$$

4. If $d(V_{j+1}, V_j) < \varepsilon$ then done, otherwise return to 3

POLICY FUNCTION ITERATION

INCOME FLUCTUATION PROBLEM

- Idiosyncratic state variables: (a, y)
- Discretize the income process: y_j for $j = 1, \dots, N$
- Transition function $\pi(y'|y)$
- Household problem in recursive form:

$$V(a, y) = \max_{c, a'} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) V(a', y')$$

subject to:

$$c + a' \leq Ra + y$$

$$a' \geq -\phi$$

Euler equation, after substituting budget constraint:

$$u_c(Ra + y_j - a') - \beta R \sum_{y_i \in Y} \pi(y_i | y_j) u_c(Ra' + y_i - a'') \geq 0$$

Idea:

- Guess a policy function instead of a value function
- Iterate on the Euler equation instead of the Bellman equation
- Policy function typically converges faster than the value function

ALGORITHM

1. Construct a grid on the asset space $\{a_0, a_1, \dots, a_m\}$, where $a_0 = -\phi$.
2. Guess a policy function for a'' on the grid points: $\hat{a}_0(a_i, y_j)$.
3. For every point on the grid (a_i, y_j) , check whether the borrowing constraint binds:

$$u_c(Ra_i + y_j - a_0) - \beta R \sum_{y' \in Y} \pi(y'|y_j) u_c(Ra_0 + y' - \hat{a}_0(a_0, y_j)) > 0$$

4. **If the inequality holds**, the borrowing constraint binds.
Set $a'_0(a_i, y_j) = a_0$ and repeat step 3 for the next grid point.

5. If the inequality does not hold, there is an interior solution. Use an non-linear solver to find the solution a^* of:

$$u_c(Ra_i + y_j - a^*) - \beta R \sum_{y' \in Y} \pi(y'|y_j) u_c(Ra^* + y' - \hat{a}_0(a^*, y_j)) = 0$$

- To evaluate $\hat{a}_0(a, y')$ outside the grid points, use linear interpolation.
- For a given a^* , find the adjacent grid points $\{a_i, a_{i+1}\}$ and compute

$$\hat{a}_0(a^*, y') = \hat{a}_0(a_i, y') + (a^* - a_i) \left(\frac{\hat{a}_0(a_{i+1}, y') - \hat{a}_0(a_i, y')}{a_{i+1} - a_i} \right)$$

Set $a'_0(a_i, y_j) = a^*$ and repeat step 3 for the next grid point.

6. Check convergence by comparing $a'_0(a_i, y_j)$ to $\hat{a}_0(a_i, y_j)$.
Declare convergence on iteration n if:

$$\max_{i,j} \{|a'_n(a_i, y_j) - \hat{a}_n(a_i, y_j)|\} < \varepsilon$$

7. Stop if convergence is achieved, otherwise go back to step 3 with new guess $\hat{a}_1(a_i, y_j) = a'_0(a_i, y_j)$.

ENDOGENOUS GRID METHOD

Euler equation:

$$u_c(c(a, y)) \geq \beta R \sum_{y' \in Y} \pi(y'|y) u_c(c(a', y'))$$

(with equality if $a' > \phi$)

Idea:

- Instead of building a grid over a , build a grid over a' , next period's asset holdings
- Iterate over the consumption policy $c(a, y)$

ALGORITHM

1. Construct a grid for (a', y) . The bottom point on the grid for a' should be the borrowing limit, $-\phi$.
2. Guess a consumption policy, $\hat{c}_0(a_i, y_j)$. Reasonable initial guess: $\hat{c}_0(a_i, y_j) = ra_i + y_j$
3. For each pair $\{a'_i, y_j\}$ construct the right-hand side of the Euler equation using the current guess for consumption \hat{c}_0 :

$$B(a'_i, y_j) = \beta R \sum_{y' \in Y} \pi(y'|y_j) u_c(\hat{c}_0(a'_i, y'))$$

(RHS of the Euler equation with a'_i assets tomorrow when your income today is y_j)

4. **Key step:** now we can use the Euler equation to find $\tilde{c}(a'_i, y_j)$ that satisfies

$$u_c(\tilde{c}(a'_i, y_j)) = B(a'_i, y_j)$$

This can be done analytically: if $u_c(c) = c^{-\gamma}$, then $\tilde{c}(a'_i, y_j) = [B(a'_i, y_j)]^{-\frac{1}{\gamma}}$.

Advantages:

- Don't need to use a nonlinear solver!
- Consequently, only need to compute the expectation in step 3 once.

Note that this requires u_c to be invertible.

ALGORITHM (CONT'D)

5. Now that we have consumption, can solve for assets today that would lead the consumer to have a'_i assets tomorrow if his current income shock is y_j : $a^*(a'_i, y_j)$. We can back this out of the budget constraint:

$$\tilde{c}(a'_i, y_j) + a'_i = Ra_i^* + y_j$$

Now you have $c(a_i^*, y_j) = \tilde{c}(a'_i, y_j)$, which is **not** defined on the original grid – this is the **endogenous grid**.

6. Let a_0^* be the value of assets that induces the borrowing constraint to bind next period (the value of a^* at the bottom grid point). You'll have one for each possible income state.

ALGORITHM (CONT'D)

7. To update the guess for consumption on the original grid, $\hat{c}_1(a_i, y_j)$:

- On grid points $a_i > a_0^*$, find a_n^*, a_{n+1}^* such that $a_n^* < a_i < a_{n+1}^*$ and use linear interpolation between $c(a_n^*, y_j)$ and $c(a_{n+1}^*, y_j)$ to get $\hat{c}_1(a_i, y_j)$.
- On grid points $a_i < a_0^*$, use the budget constraint:

$$\hat{c}_1(a_i, y_j) = Ra_i + y_j - a'_0$$

since we know that borrowing constraint is going to be binding next period, so the Euler equation won't hold with equality.

8. Check convergence:

$$\max_{i,j} \{|c_{n+1}(a_i, y_j) - c_n(a_i, y_j)|\} < \varepsilon$$

COLLOCATION

- **Main idea**
 - Approximate the value function with a linear combination of basis functions
 - Need to solve for the coefficients on these basis functions
 - Family of basis functions is chosen by the economist
- Algorithm based on the one on Simon's website
- Uses tools from Laszlo's lecture on function approximation (now, the function we need to approximate is unknown)

ADVANTAGES

- Very fast, stable
- Many objects only need to be computed once, ahead of time
- End up with a linear system on each iteration
- Once you've solved for the coefficients, you can easily evaluate the value and policy function on any grid
- Extends to other settings
- Julia package `BasisMatrices.jl` very convenient for this setup
- If you're using MATLAB, everything is vectorized

$$V(a, y) = \max_{a' \in B(a, y)} u(Ra + y - a') + \beta \sum_{y' \in Y} \pi(y, y') V(a', y')$$

where $B(a, y) = [-\phi, Ra + y]$.

We can separate the expected value function to get the following system:

$$V(a, y) = \max_{a' \in B(a, y)} u(Ra + y - a') + \beta V_e(a', y)$$

$$V_e(a, y) = \sum_{y' \in Y} \pi(y, y') V(a, y')$$

Idea: if we directly approximate the expected value function, we can avoid computing the expectation each time we solve for a' .

REWRITING THE SYSTEM

- Let \mathbf{a} be a length N_a grid for assets
- Let \mathbf{y} be a length N_y grid for income, with transition matrix Π
- Stack each combination of these to get a set of collocation nodes \mathbf{s} , which is $N \times 2$ where $N = N_a N_y$ – `gridmake` will do this for any number of grids
- Let s_i be an asset, income pair, or a row in \mathbf{s}

In this notation, the system becomes:

$$V(s_i) = \max_{a' \in B(s_i)} u(Rs_{i1} + s_{i2} - a') + \beta V_e([a', s_{i2}])$$
$$V_e(s_i) = \sum_{y' \in Y} \pi(s_{i2}, y') V([s_{i1}, y'])$$

INTERPOLANTS

Let's replace the value functions with interpolants, consisting of a basis function ϕ and sets of coefficients $\{c_j\}_{j=1}^N$.

$$V(s_i) = \sum_{j=1}^N \phi(s_i) c_j$$

$$V^e(s_i) = \sum_{j=1}^N \phi(s_i) c_j^e$$

- Approximating functions can be splines, Chebyshev, linear, etc.
- N coefficients and N points on the state space

Substitute these back into our system:

$$\sum_{j=1}^N \phi(s_i) c_j = \max_{a' \in B(s_i)} u(Rs_{i1} + s_{i2} - a') + \beta \sum_{j=1}^N \phi([a', s_{i2}]) c_j^e$$

$$\sum_{j=1}^N \phi(s_i) c_j^e = \sum_{y' \in Y} \pi(s_{i2}, y') \sum_{j=1}^N \phi([s_{i1}, y']) c_j$$

Linear system in $2N$ equations and $2N$ unknowns.

- $V(s_i) = \sum_{j=1}^N \phi(s_i) c_j$ becomes $\mathbf{V}(s) = \Phi(s)c$
 - $\Phi(s)$ is the basis matrix of the interpolant evaluated at s
 - $c = (c_1, \dots, c_N)'$ is the vector of coefficients
- So the first equation becomes

$$\Phi(s)c = \max_{a' \in \mathbf{B}(s)} u(s, a') + \beta \Phi([a', s_2])c^e$$

- The second equation becomes

$$\Phi(s)c^e = (\Pi \otimes I_{N_a})\Phi(s)c$$

$$\Phi(s)c = \max_{a' \in \mathbf{B}(s)} \mathbf{u}(s, a') + \beta \Phi([a', s_2])c^e$$

$$\Phi(s)c^e = (\Pi \otimes I_{N_a})\Phi(s)c$$

How to implement this computationally?

1. Compute most of the basis matrices in advance
2. Given a guess for $\{c, c^e\}$, compute the max
3. Iterate on the equations, update coefficients until convergence

- Create a `Basis` object to set up the type of interpolant, collocation nodes, etc.:

```
basis = Basis(SplineParams(agrid0, 0, order),  
              LinParams(ygrid0, 0))  
s, (agrid, ygrid) = nodes(basis)
```

- To compute the basis matrix, $\Phi(s)$:

```
bs = BasisMatrix(basis, Direct(), s, [0 0])  
 $\Phi$  = convert(Expanded, bs).vals[1]
```

What else can be stored in advance?

- $(\Pi \otimes I_{N_a})\Phi(s)$

`Emat = kron(Π , speye(N_a))* Φ`

- Part of $\Phi([a', s_2])$

- This is really a tensor product: $\Phi_y(y)\Phi_{a'}(a')$
- Since the values of y we want to evaluate at never change, we can compute just that part of the interpolation in advance, and then later on create the basis matrix for a' .

```
 $\Phi_y$  = bs.vals[2]           # store this
```

```
...
```

```
bs_a = BasisMatrix(h.basis[1], Direct(), ap, 0)
```

```
 $\Phi_a$  = bs_a.vals[1]
```

```
 $\Phi_{apy}$  = row_kron(h. $\Phi_y$ ,  $\Phi_a$ )
```

- Less basis matrices need to be computed by the solver

$$\Phi(s)c = \max_{a' \in B(s)} u(s, a') + \beta \Phi([a', s_2])c^e$$

$$\Phi(s)c^e = (\Pi \otimes I_{N_a})\Phi(s)c$$

- Now that we know how to compute basis matrices, it's straightforward to compute the objective function given a guess for the coefficients
- Golden search works well for computing the max (at least in one dimension)

```
lower_bound = zeros(size(s, 1), )
upper_bound = (1 + h.r).*s[:, 1] + s[:, 2]
f(ap::Vector{Float64}) = value(h, s, ap)
ap, v1 = golden_method(f, lower_bound, upper_bound)
```

How to update the guesses for c, c^e ?

1. Bellman iteration
2. Newton iteration (linearity of the system allows us to write down the Jacobian)

Compute the right-hand side and invert $\Phi(s)$ to get the new coefficient guesses:

$$c_{k+1} = \Phi(s)^{-1} [\mathbf{u}(s, a'(s)) + \beta \Phi([a'(s), s_2])c_k^e]$$

$$c_{k+1}^e = \Phi(s)^{-1} [(\Pi \otimes I_{N_a})\Phi(s)c_k]$$

- Do this a few times to get a decent initial guess for the Newton iterations
- Sometimes useful to let this run for a while to make sure you have a contraction mapping

View this as a root-finding problem

$$\mathbf{g}_1(c, c^e) = \Phi(s)c - [\mathbf{u}(s, a'(s)) + \beta\Phi([a'(s), s_2])c^e]$$

$$\mathbf{g}_2(c, c^e) = \Phi(s)c^e - [(\Pi \otimes I_{N_a})\Phi(s)c]$$

Because the system is linear in the coefficients, we can easily write down the Jacobian:

$$\mathbf{D}(c, c^e) = \begin{bmatrix} \Phi(s) & -\beta\Phi([a'(s), s_2]) \\ -(\Pi \otimes I_{N_a})\Phi(s) & \Phi(s) \end{bmatrix}$$

Updating:

$$\begin{bmatrix} c_{k+1} \\ c_{k+1}^e \end{bmatrix} = \begin{bmatrix} c_k \\ c_k^e \end{bmatrix} - \mathbf{D}(c_k, c_k^e)^{-1} \begin{bmatrix} \mathbf{g}_1(c_k, c_k^e) \\ \mathbf{g}_2(c_k, c_k^e) \end{bmatrix}$$

These converge **much** faster than the Bellman iterations!

HOMEWORK

Solve the following problem of firm investment:

$$V(k, z) = \max_{i, n} z k^{\alpha} n^{\nu} - i - \eta(k' - k)^2 - wn + \beta \mathbb{E} [V(k', z')]$$

where $k' = (1 - \delta)k + i$.

- Eliminate n by solving static labor choice problem
- Parameters: $w = 0.99$; $w = 1.26$; $\alpha = 0.26$; $\nu = 0.64$;
 $\delta = 0.07$; $\eta = 0.5$
- z follows a Markov chain with values
[0.7578 0.9117 1.0969 1.3196]' and transition matrix:

$$\begin{bmatrix} 0.9269 & 0.0713 & 0.0018 & 0 \\ 0.0238 & 0.9281 & 0.0475 & 0.0001 \\ 0.0001 & 0.0475 & 0.9281 & 0.0238 \\ 0 & 0.0018 & 0.0713 & 0.9269 \end{bmatrix}$$