Introduction to Bayesian Statistics II

Dawie van Lill

ECO872: Advanced Time Series Econometrics

July 27, 2020

What we will do today

- Marginalization
- Normal distribution with a noninformative prior (important)
- Normal distribution with a conjugate prior (important)
- Multivariate normal with known variance
- Multivariate normal with unknown variance (glance through)
- Bioassay example (very important, related to one of the exercises)

- Once again, the slides follows the work of **AG** quite closely. However, other sources were consulted to set up the slides.
- This is quite a technical session and you should try and go through the mathematics in this lecture slowly with a pen and paper.

Monte Carlo and posterior draws

- We will start the lecture with a quick discussion about **Monte Carlo methods**, since we will use this sporadically throughout.
- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization

Monte Carlo and posterior draws

- We will start the lecture with a quick discussion about **Monte Carlo methods**, since we will use this sporadically throughout.
- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization
  - to approximate expectations (integrals)

$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$$

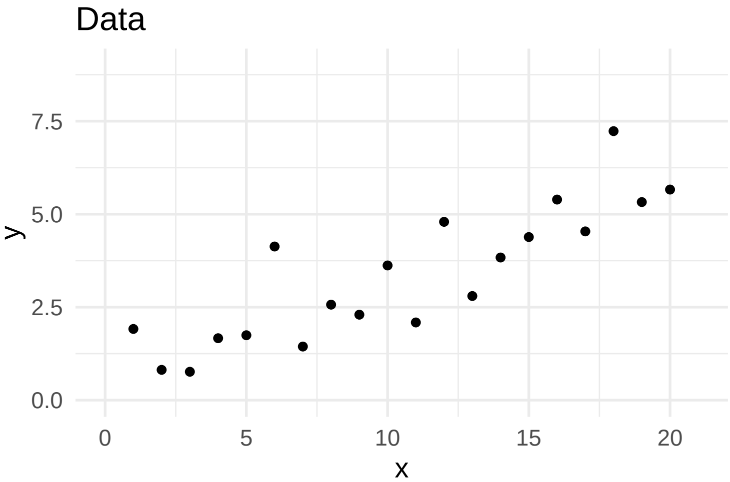
Monte Carlo and posterior draws

- We will start the lecture with a quick discussion about **Monte Carlo methods**, since we will use this sporadically throughout.
- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization
  - to approximate expectations (integrals)

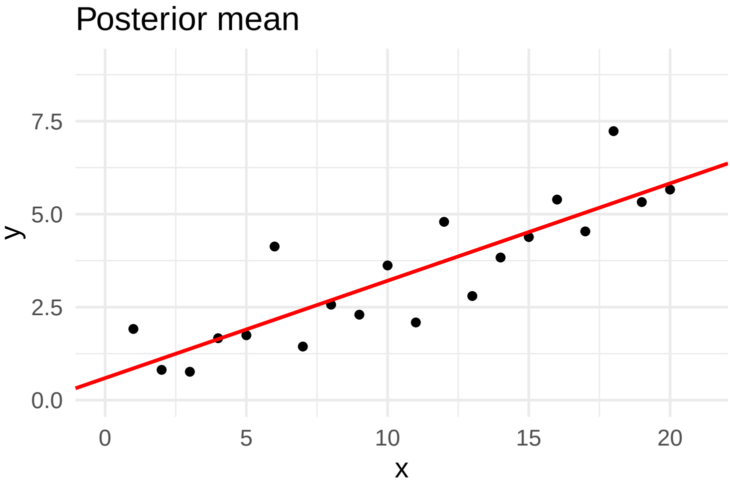
$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$$

- We now turn to an example to showcase the idea of posterior draws (visualization)

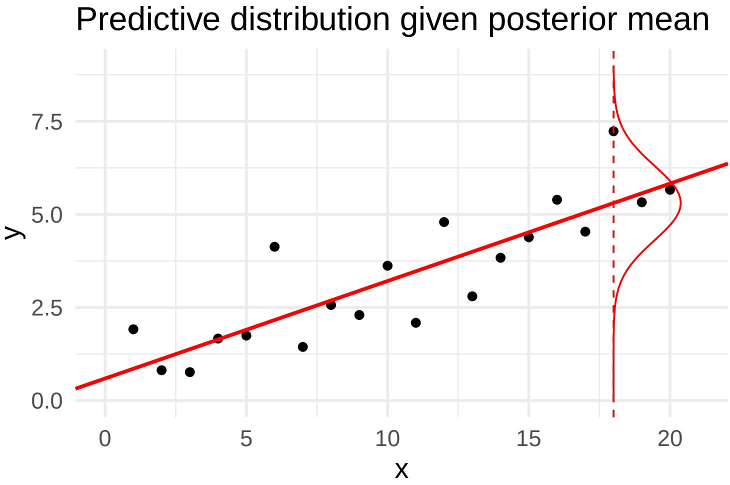
Example of uncertainty in modeling



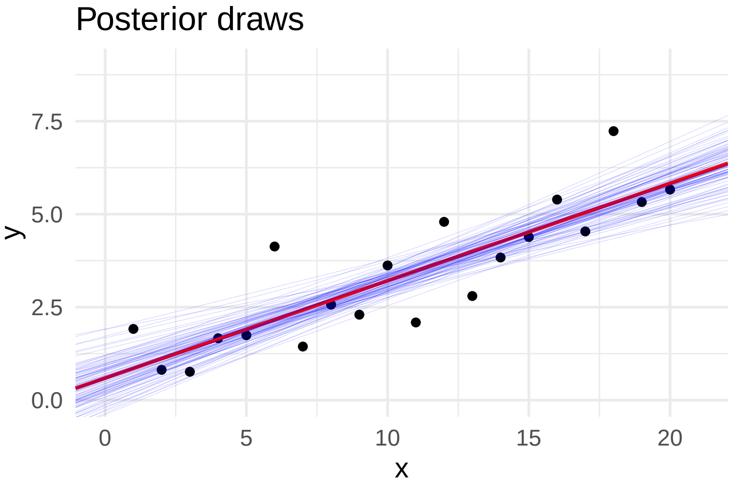
Example of uncertainty in modeling



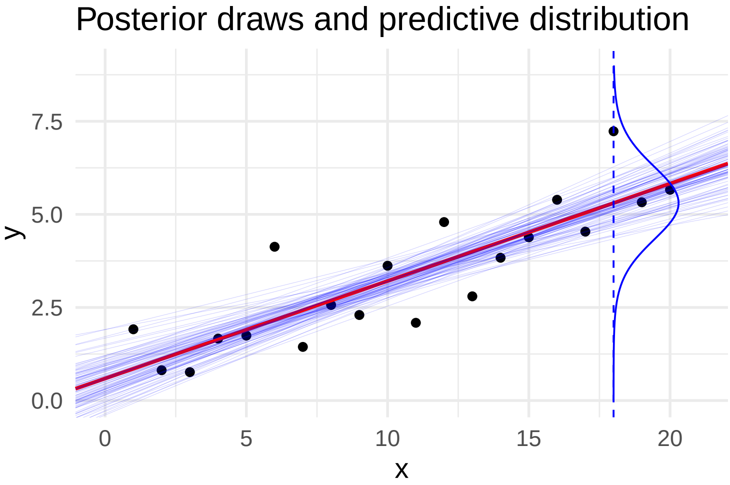
Example of uncertainty in modeling



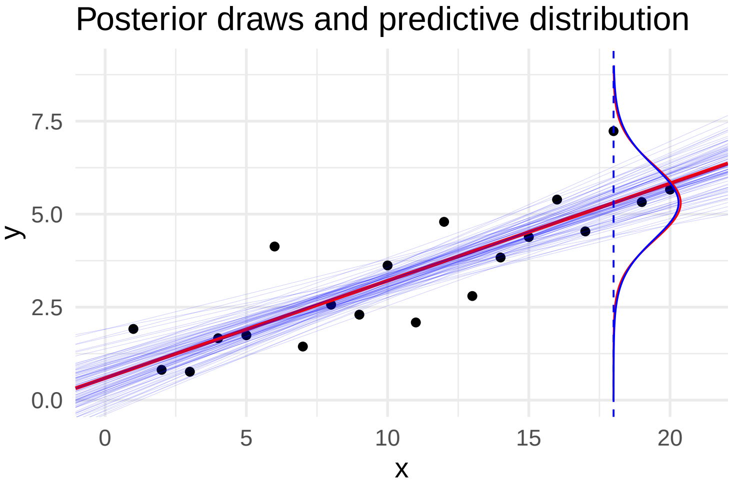
Example of uncertainty in modeling



Example of uncertainty in modeling



Example of uncertainty in modeling



Marginalization

- Joint distribution of parameters

$$p(\theta_1, \theta_2 \mid y) \propto p(y \mid \theta_1, \theta_2) p(\theta_1, \theta_2)$$

- Marginalization

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2$$

$p(\theta_1 \mid y)$ is a marginal distribution

Marginalization

- Joint distribution of parameters

$$p(\theta_1, \theta_2 \mid y) \propto p(y \mid \theta_1, \theta_2) p(\theta_1, \theta_2)$$

- Marginalization

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2$$

$p(\theta_1 \mid y)$ is a marginal distribution

- Monte Carlo approximation

$$p(\theta_1 \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} p(\theta_1, \theta_2^{(s)} \mid y),$$

where $\theta_2^{(s)}$ are draws from $p(\theta_2 \mid y)$

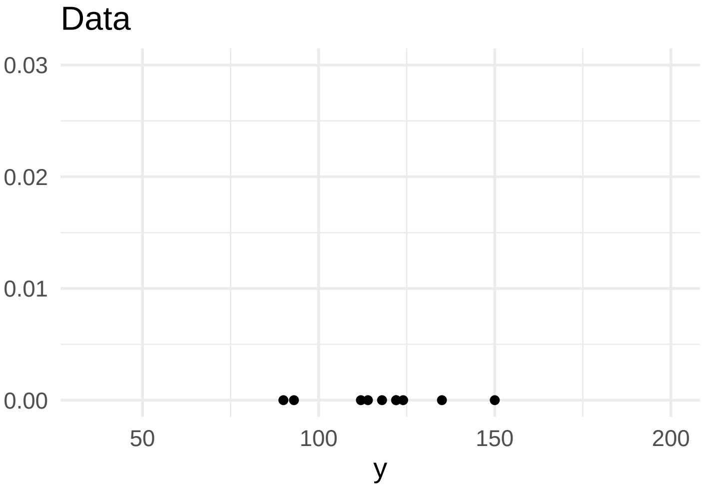
Marginalization - predictive distribution

- Marginalization over posterior distribution

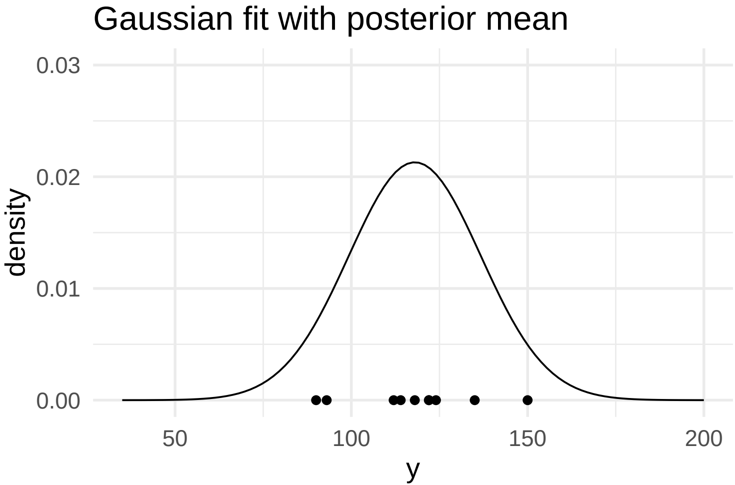
$$
\begin{aligned}
p(\tilde{y} \mid y) &= \int p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta \\
&= \int p(\tilde{y}, \theta \mid y) d\theta
\end{aligned}
$$

$p(\tilde{y} \mid y)$ is a predictive distribution

Gaussian example

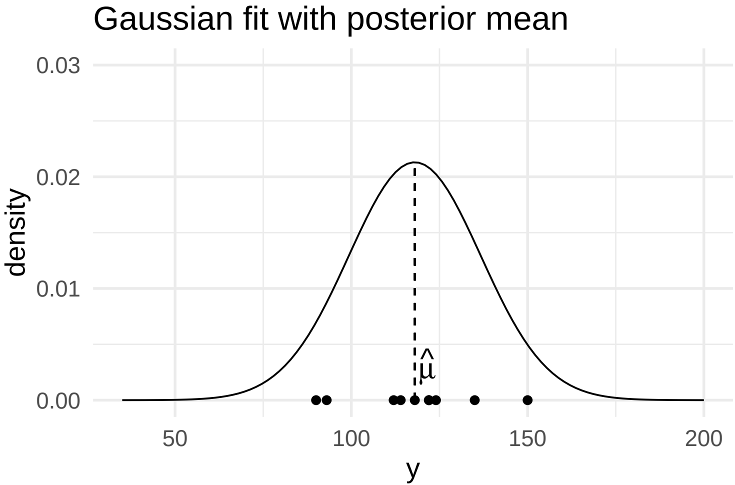


Gaussian example



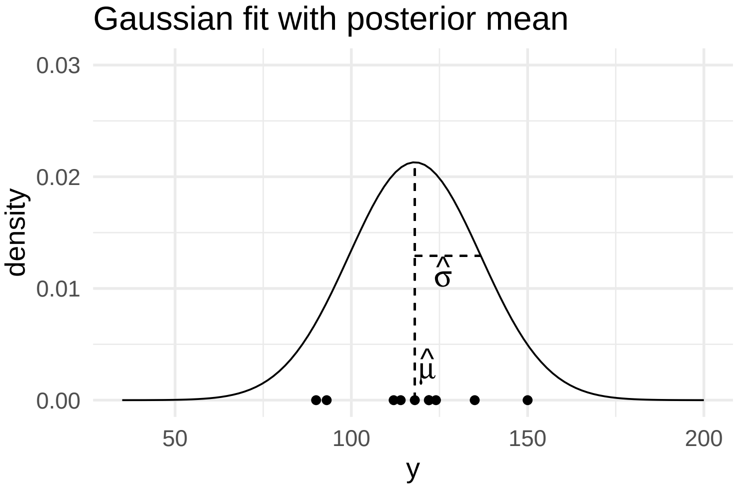
$$p(\textcolor{red}{y} \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\textcolor{red}{y} - \mu)^2\right)$$

Gaussian example



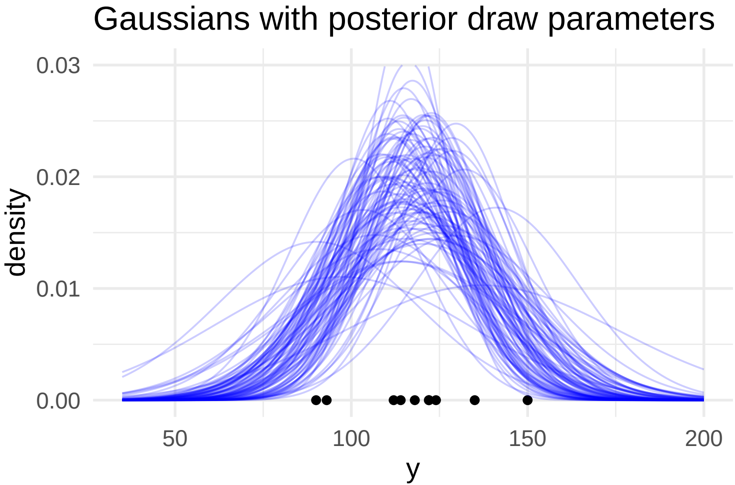
$$p(y \mid {\color{red}\mu}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - {\color{red}\mu})^2\right)$$

Gaussian example



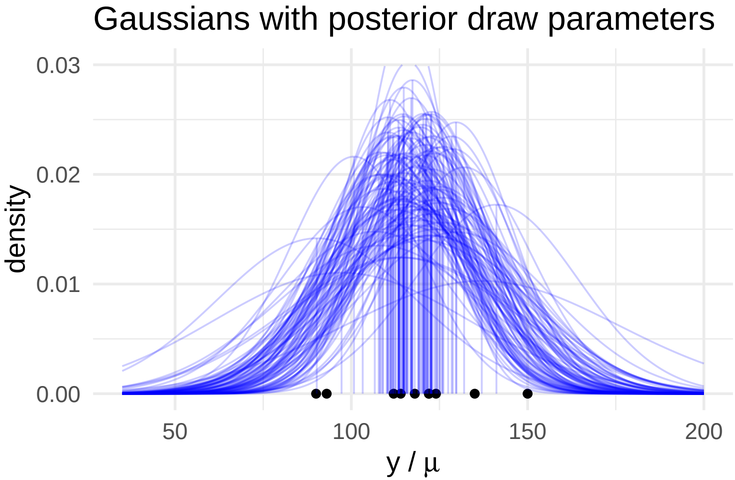
$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

Gaussian example



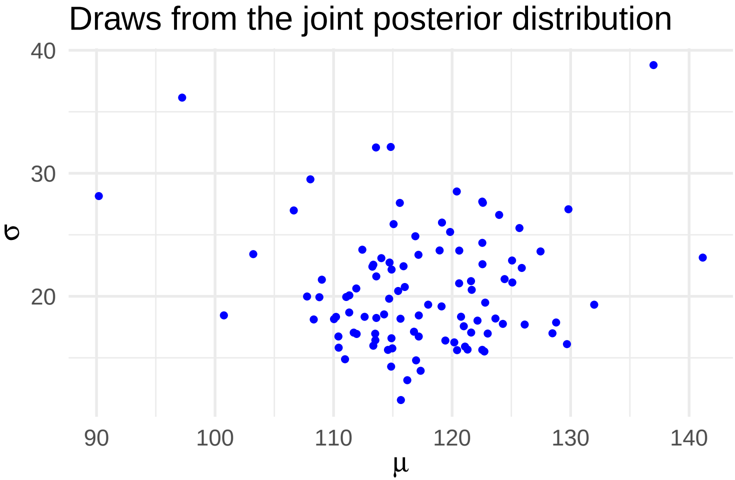
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Gaussian example



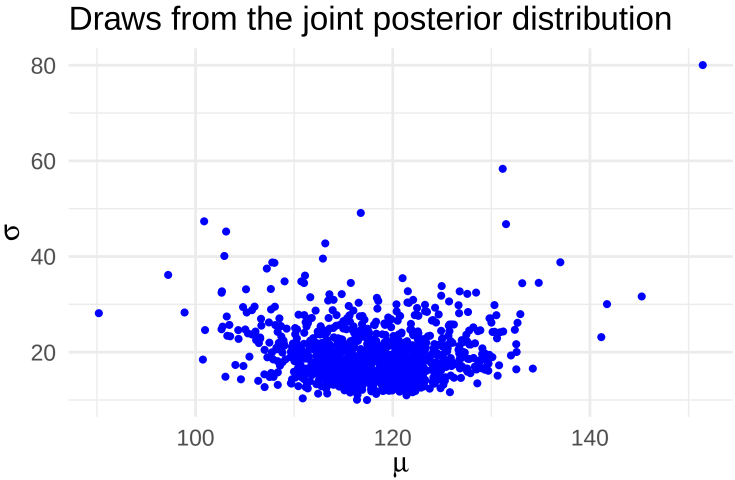
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Gaussian example



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

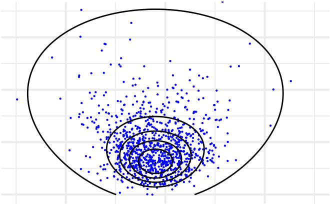
Gaussian example



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior

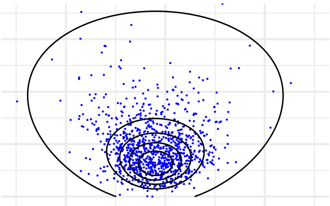
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$



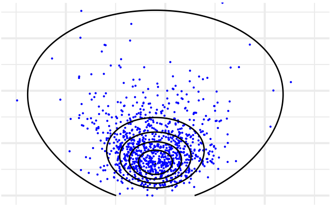
Joint posterior

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$



Joint posterior

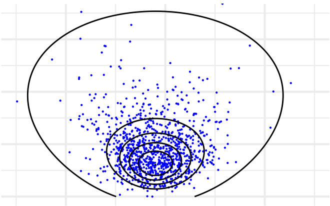


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

Joint posterior

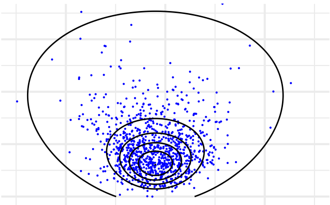


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

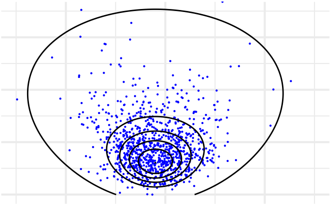
with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

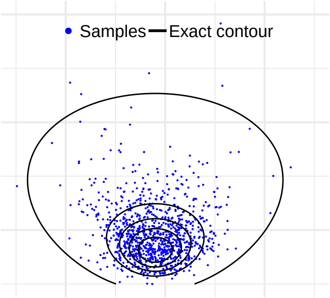
$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

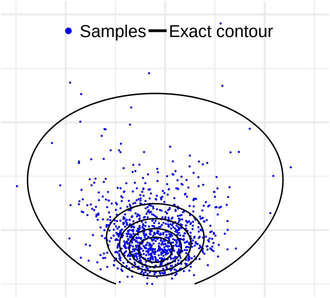
$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

Joint posterior

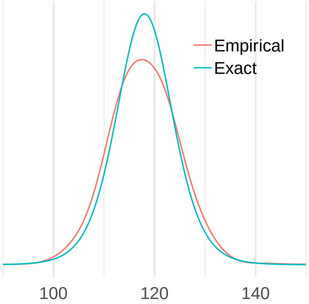


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



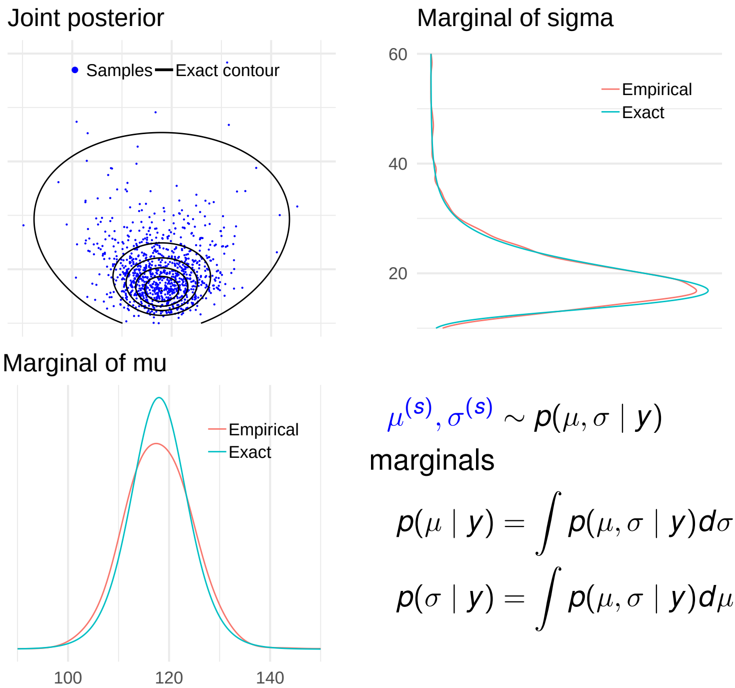
Marginal of mu



$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$



$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$p(\sigma^2 \mid y) \quad \propto \quad \int p(\mu, \sigma^2 \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$
\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n}
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\
p(\sigma^2 \mid y) &= \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)
\end{aligned}
$$

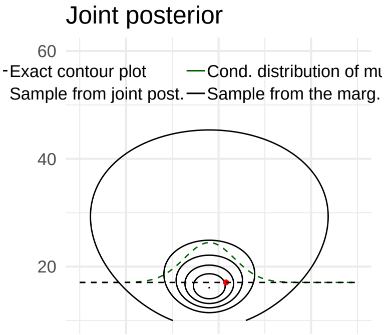
Gaussian - non-informative prior

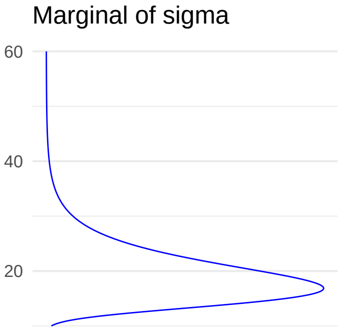
Known mean

$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, v)$$
$$\text{where} \quad v = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta)^2$$

Unknown mean

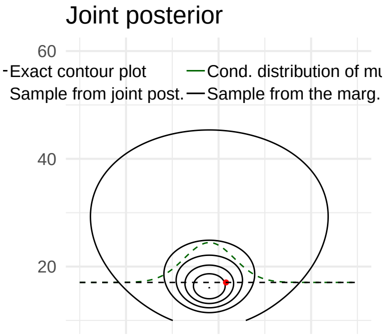
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$
$$\text{where} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

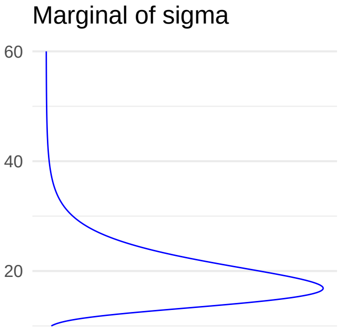




Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$



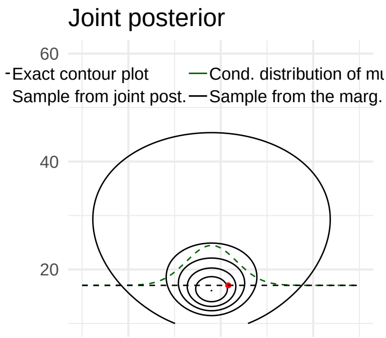


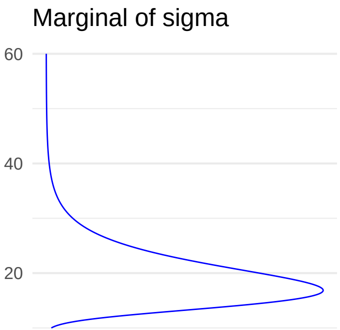
Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$





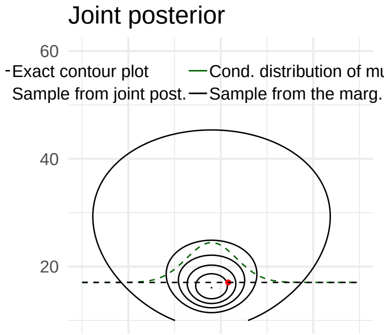
Factorization

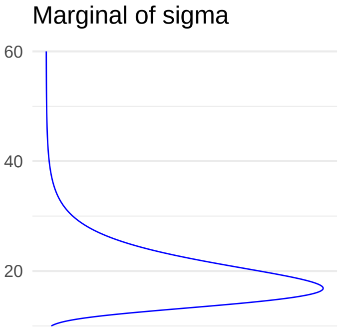
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$





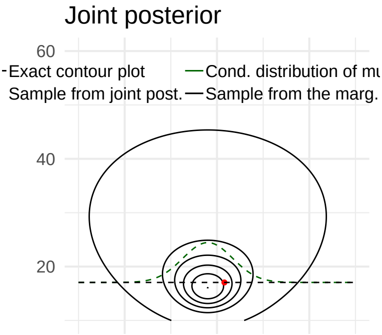
Factorization

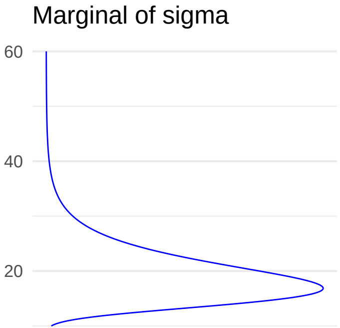
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$





Factorization

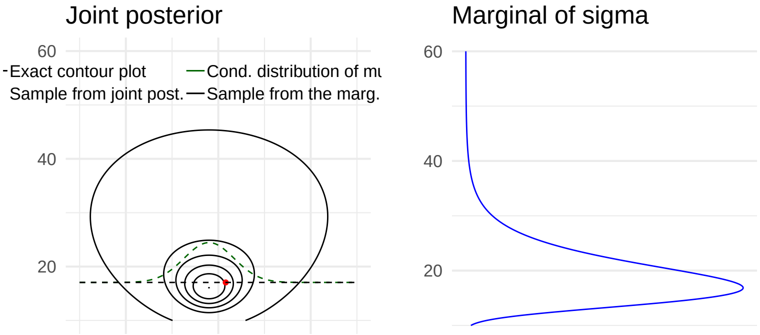
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim p(\mu \mid \sigma^2, y)$$



Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

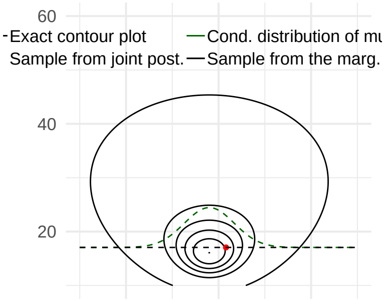
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

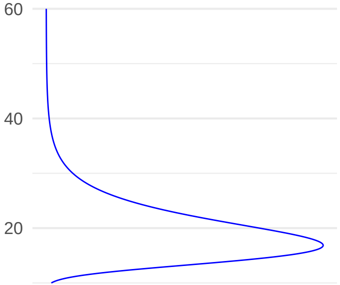
$$\mu^{(s)} \sim p(\mu \mid \sigma^2, y)$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



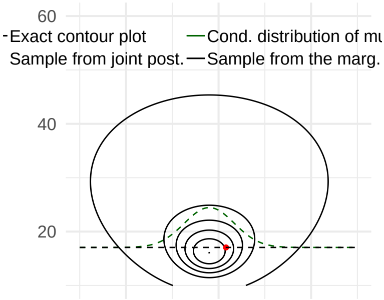
Marginal of sigma



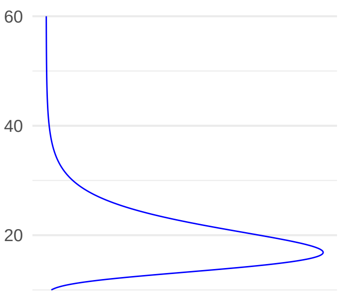
Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

Joint posterior



Marginal of sigma

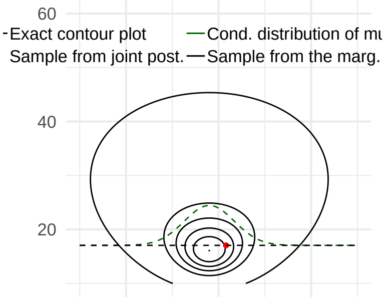


Factorization

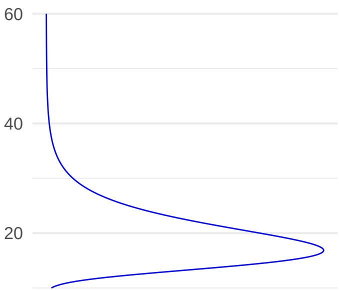
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

Joint posterior



Marginal of sigma

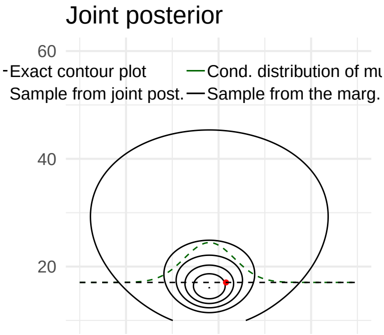


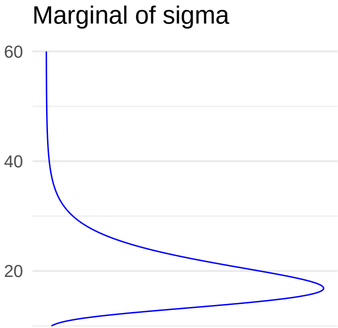
Factorization

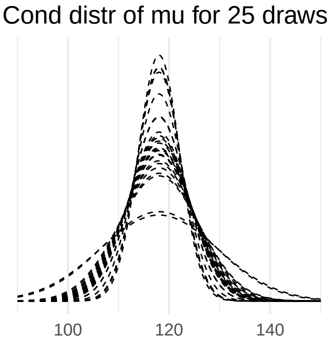
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$







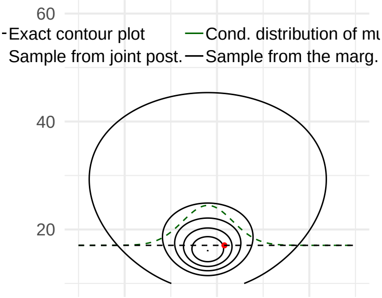
Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

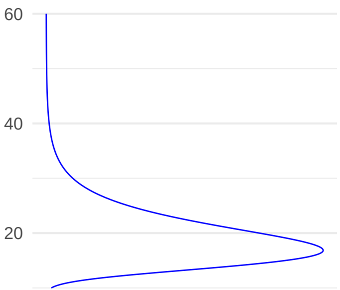
$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

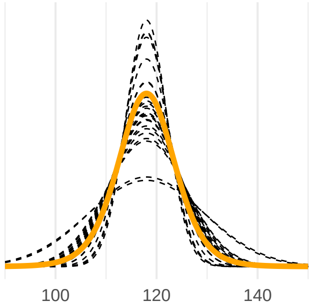
Joint posterior



Marginal of sigma



Cond distr of mu for 25 draws



Factorization

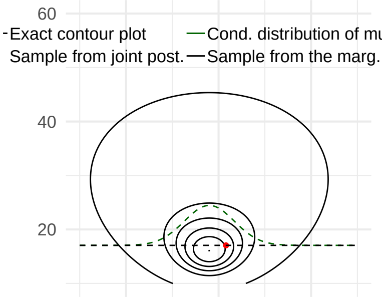
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

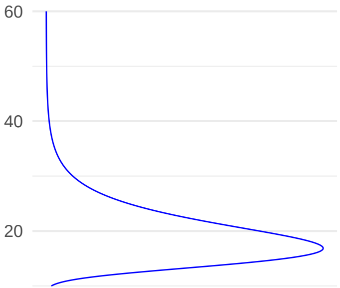
$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

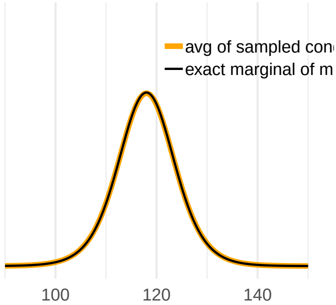
Joint posterior



Marginal of sigma



Cond. distr of mu



Factorization

$$p(\mu, \sigma^2 \mid y) = {\color{green} p(\mu \mid \sigma^2, y)}{\color{blue} p(\sigma^2 \mid y)}$$

$$(\sigma^2)^{(s)} \sim {\color{blue} p(\sigma^2 \mid y)}$$

$${\color{green} p(\mu \mid (\sigma^2)^{(s)}, y)} = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx {\color{orange} \frac{1}{S}\sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)}$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Transformation

$$
A = (n-1)s^2 + n(\mu - \bar{y})^2
$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Transformation

$$
A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}
$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Transformation

$$
A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}
$$

$$
p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz
$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2}$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

Transformation

$$
A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}
$$

$$
p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz
$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$
\begin{aligned}
&\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2} \\
&\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2}\right]^{-n/2}
\end{aligned}
$$

Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

$$\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2$$

Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

$$\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2}$$

$$\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2}\right]^{-n/2}$$

$$p(\mu \mid y) = t_{n-1}(\mu \mid \bar{y}, s^2/n) \quad \text{Student's } t$$

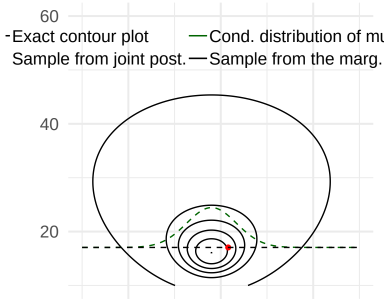
Gaussian - non-informative prior

- Marginal posterior $p(\mu \mid y)$

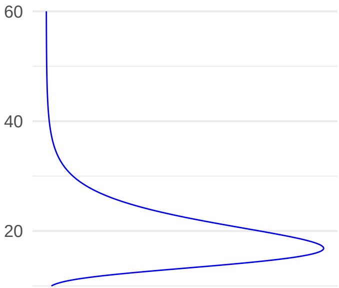
$$p(\mu \mid y) = \int_0^\infty p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y) d\sigma^2$$

- see visualization demo3_3
- marginal posterior of μ a mixture of normal distributions where mixing density is the marginal posterior of σ^2

Joint posterior

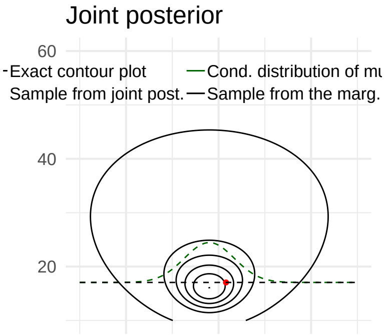


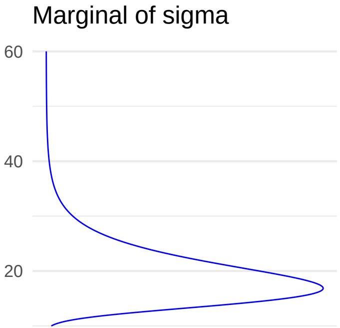
Marginal of sigma



Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$



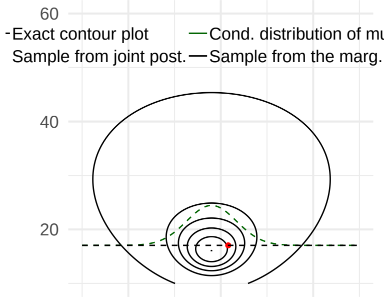


Predictive distribution for new $\tilde{y}$

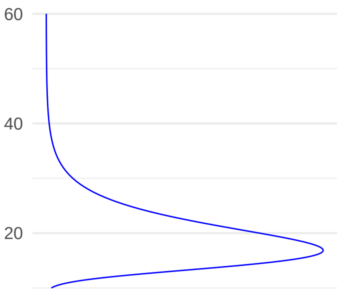
$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

$$\color{red}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

Joint posterior



Marginal of sigma



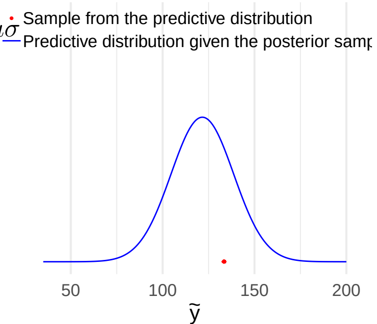
Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

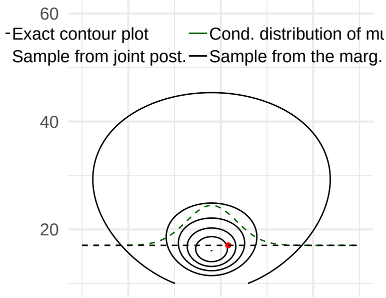
$$\color{red}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

$$\color{red}{\tilde{y}^{(s)}} \sim \color{blue}{p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})}$$

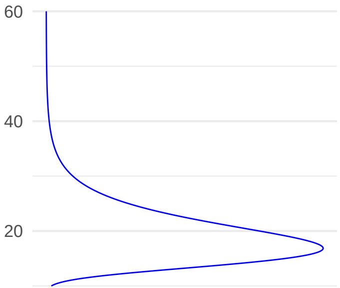
Posterior predictive distribution



Joint posterior



Marginal of sigma



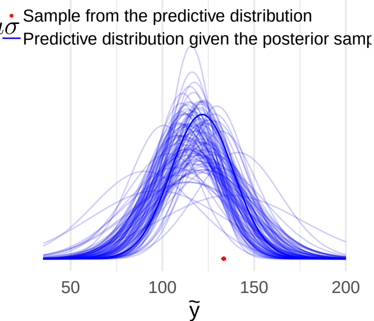
Predictive distribution for new $\tilde{y}$

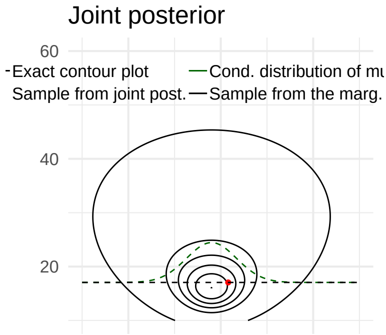
$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

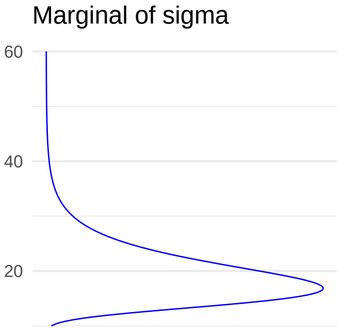
$$\color{red}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

$$\color{red}{\tilde{y}^{(s)}} \sim \color{blue}{p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})}$$

Posterior predictive distribution





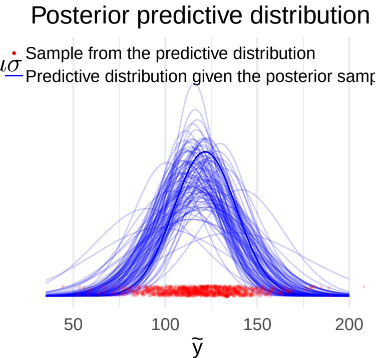


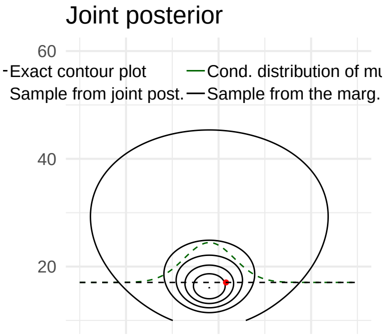
Predictive distribution for new $\tilde{y}$

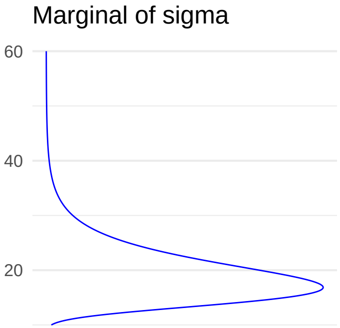
$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})$$





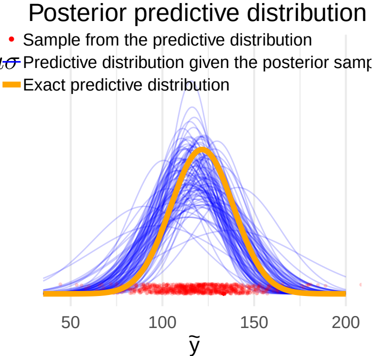


Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})$$



Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$p(\tilde{y} \mid \sigma^2, y) = \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \end{aligned}$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\ &= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2) \end{aligned}$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\ &= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2) \end{aligned}$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\ &= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2) \end{aligned}$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

$$p(\tilde{y} \mid y) = t_{n-1}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n}) s^2)$$

Gaussian - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)

Gaussian - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)
- Handy parametrization

$$\mu \mid \sigma^2 \sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0)$$
$$\sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)$$

which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

Gaussian - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)
- Handy parametrization

$$\begin{aligned} \mu \mid \sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

- μ and σ^2 are a priori dependent
  - if σ^2 is large, then μ has wide prior

Gaussian - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}$$

Gaussian - conjugate prior

- Conditional $p(\mu \mid \sigma^2, y)$

$$\begin{aligned}\mu \mid \sigma^2, y &\sim \mathrm{N}(\mu_n, \sigma^2/\kappa_n)\\ &= \mathrm{N}\left(\frac{\frac{\kappa_0}{\sigma^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{\kappa_0}{\sigma^2} + \frac{n}{\sigma^2}}, \frac{1}{\frac{\kappa_0}{\sigma^2} + \frac{n}{\sigma^2}}\right)\end{aligned}$$

- Marginal $p(\sigma^2 \mid y)$

$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2)$$

- Marginal $p(\mu \mid y)$

$$\mu \mid y \sim t_{\nu_n}(\mu \mid \mu_n, \sigma_n^2/\kappa_n)$$

Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^{k} \theta_j^{y_j},$$

- BDA3 p. 69–

Multivariate Gaussian

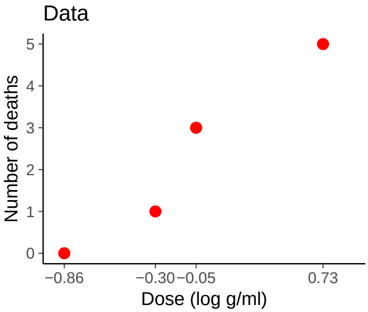
- Observation model

$$p(y \mid \mu, \Sigma) \propto \mid \Sigma \mid^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)^T \Sigma^{-1}(y-\mu)\right),$$

- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

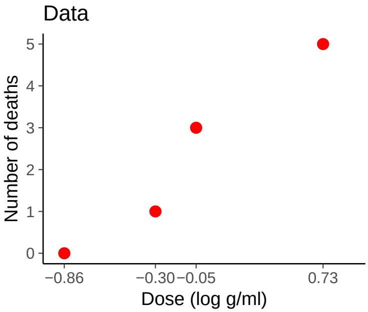
Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5

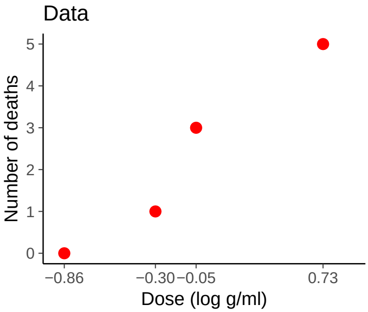


Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



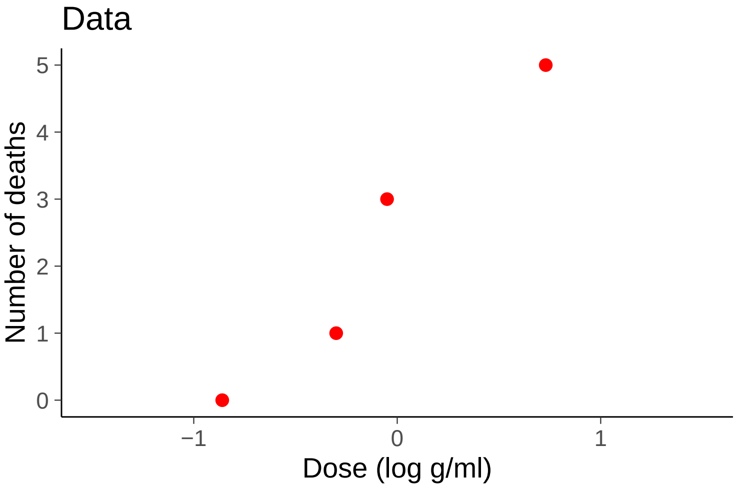
Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

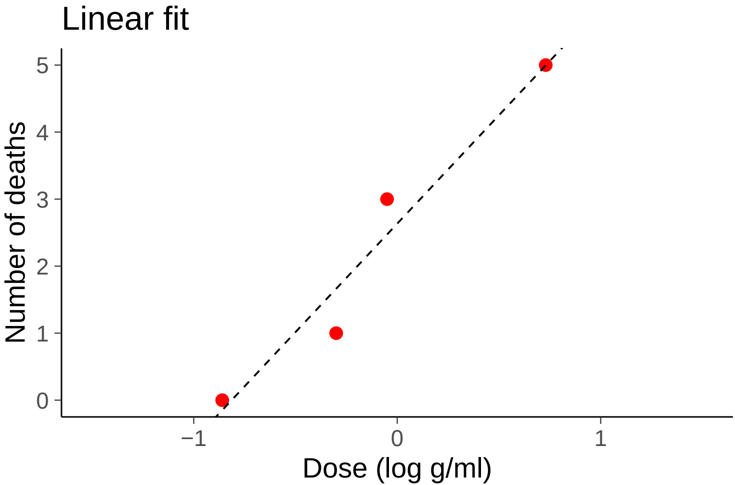
Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

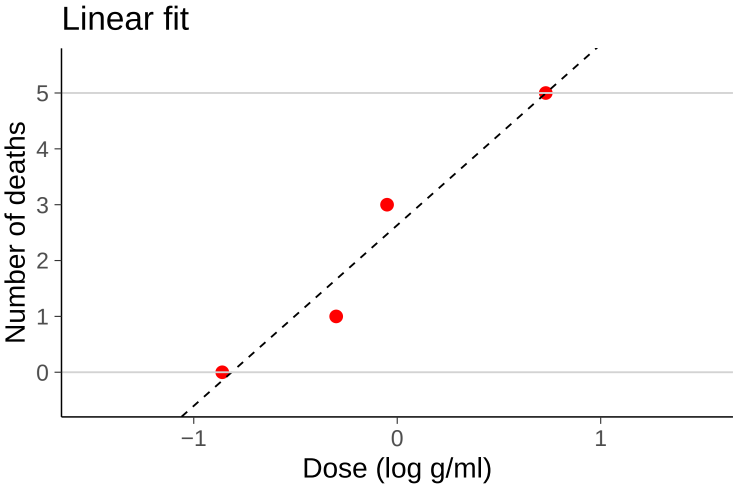
Bioassay



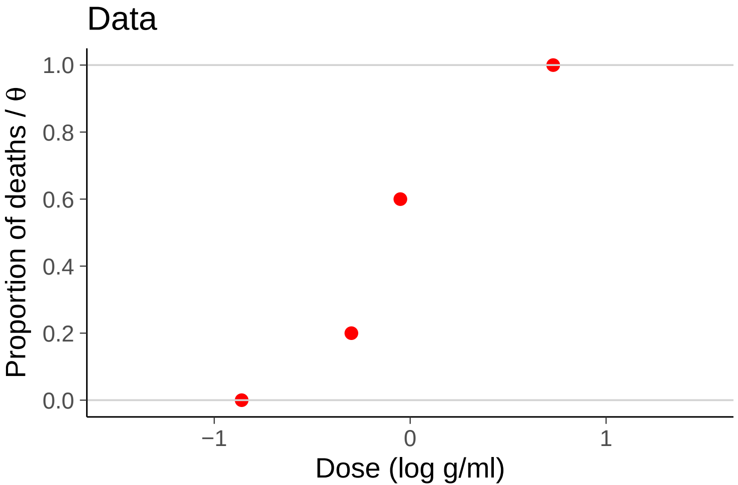
Bioassay



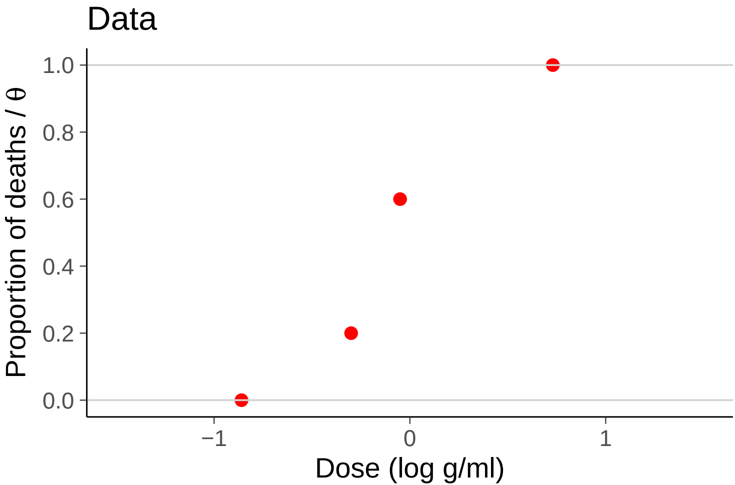
Bioassay



Bioassay



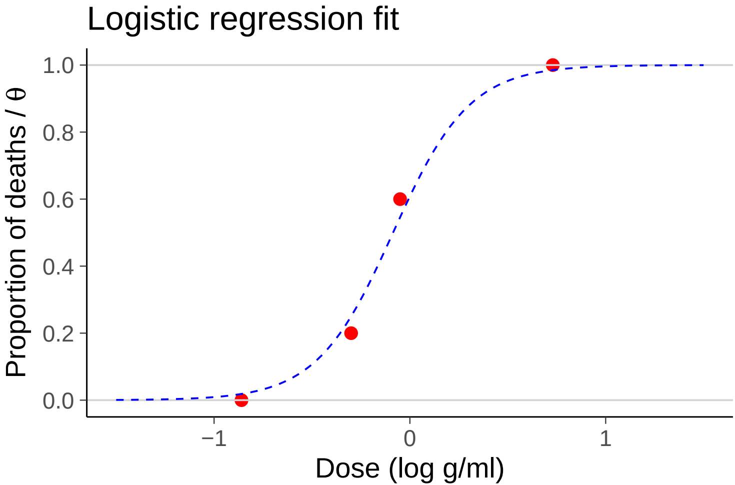
Bioassay



Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Bioassay



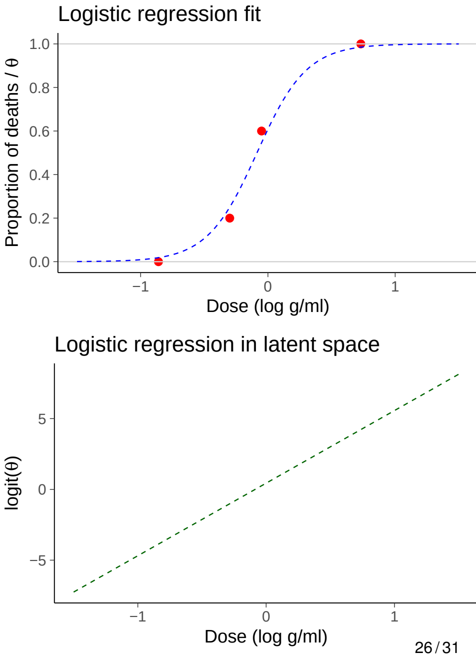
Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i), \quad \mathrm{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

Bioassay

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

$$\begin{aligned}\text{logit}(\theta_i) &= \log\left(\frac{\theta_i}{1-\theta_i}\right)\\ &= \alpha + \beta x_i\end{aligned}$$

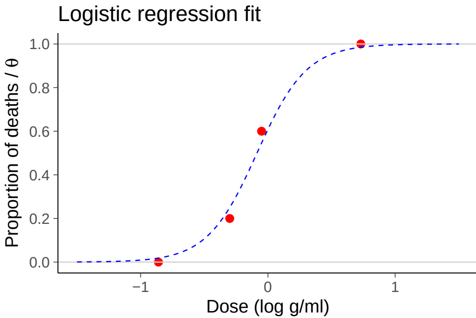


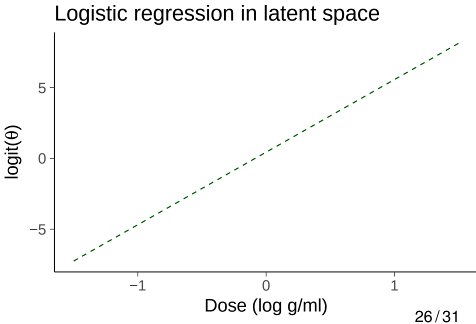
Bioassay

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

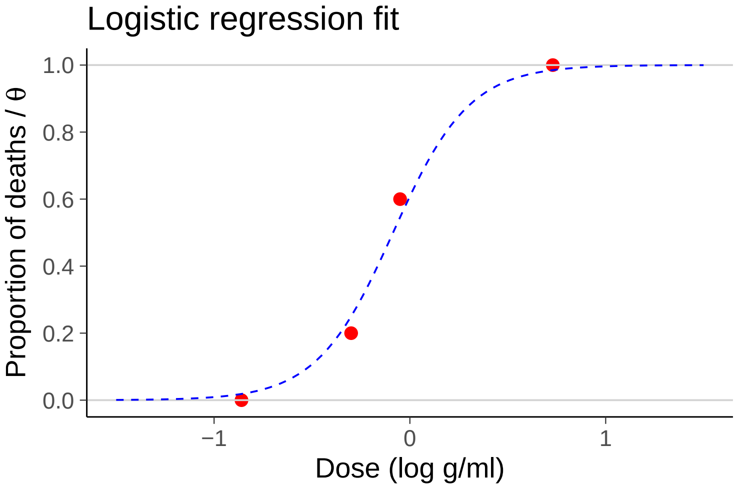
$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

$$\theta_i = \frac{1}{1+\exp(-(\alpha+\beta x_i))}$$

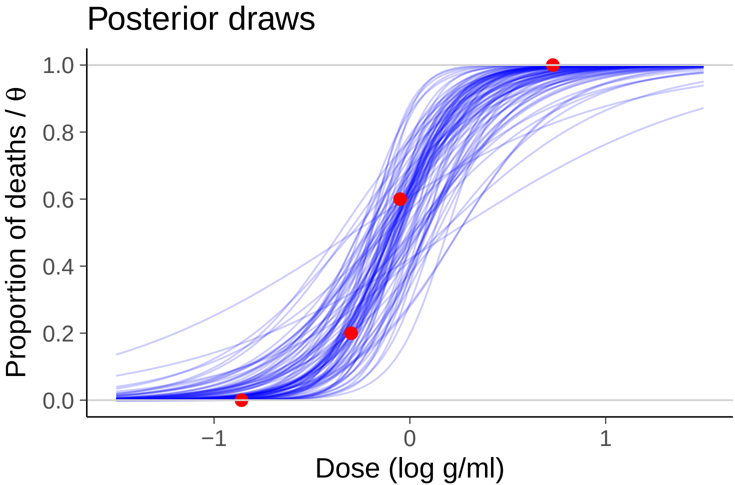




Bioassay

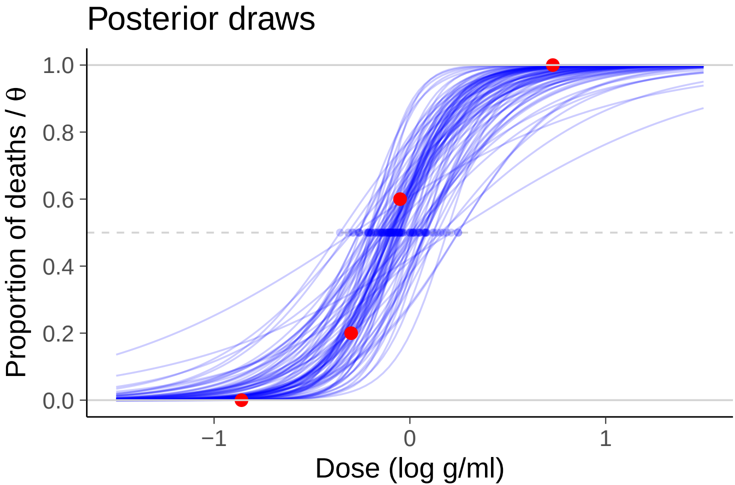

Logistic regression fit
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



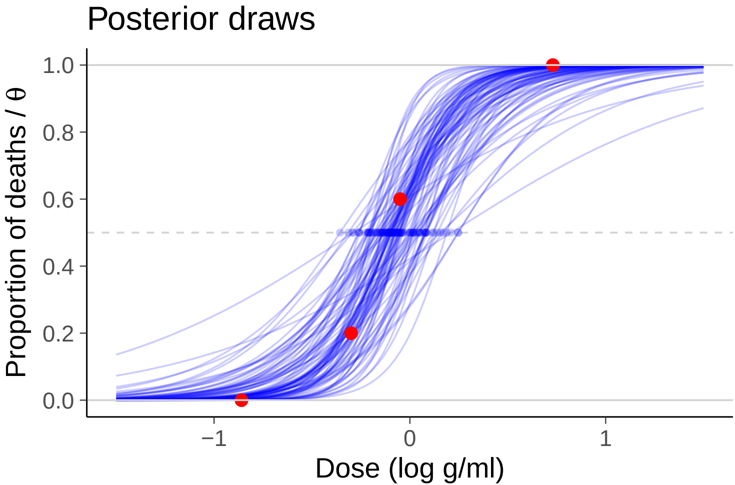
Posterior draws
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



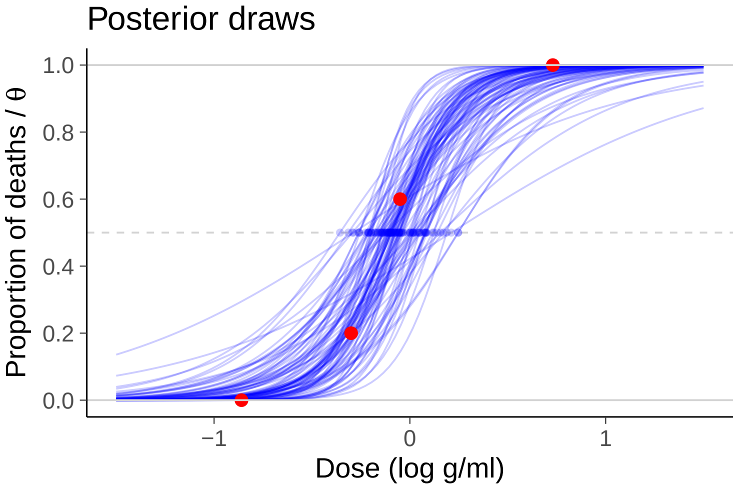
LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5$

Bioassay



LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

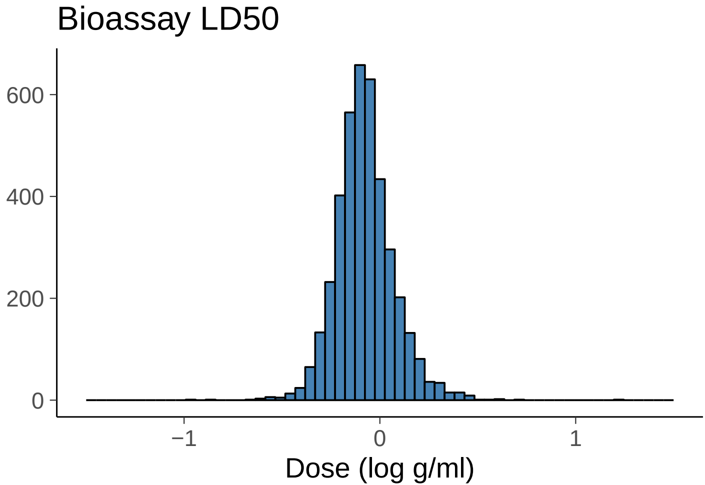
Bioassay



LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

$$x_{\mathrm{LD50}}^{(s)} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay



LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

$$x^{(s)}_{\mathrm{LD50}} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i} [1 - \theta_i]^{n_i - y_i}$$

Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha + \beta x_i)]^{y_i}[1 - \mathrm{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

Likelihood

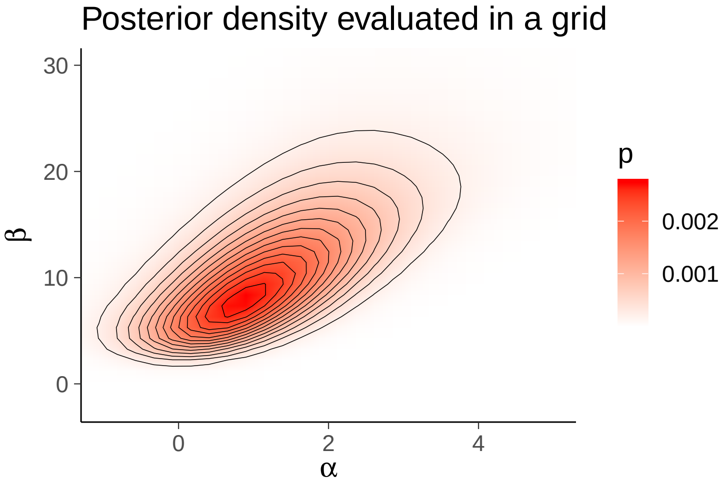
$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha + \beta x_i)]^{y_i}[1 - \mathrm{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

Posterior (with uniform prior on α, β)

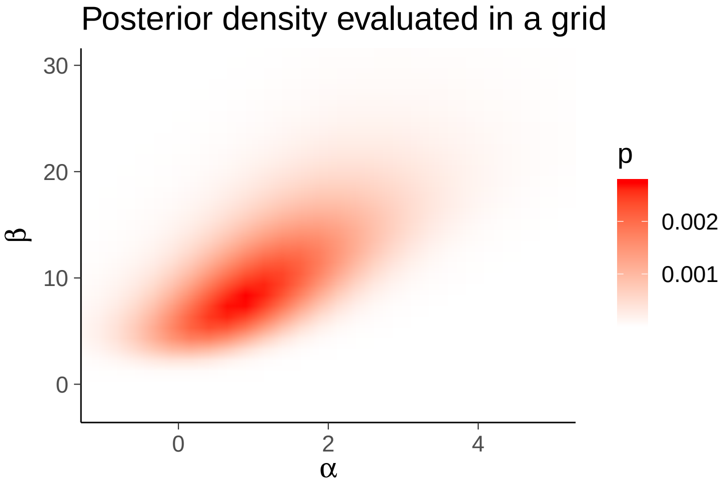
$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i)$$

Bioassay



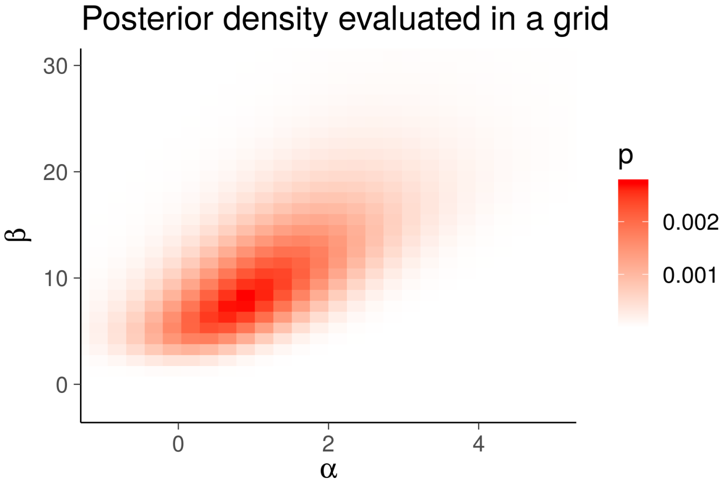
Posterior density evaluated in a grid
30
20
10
0
β
0
2
4
α
p
0.002
0.001

Bioassay



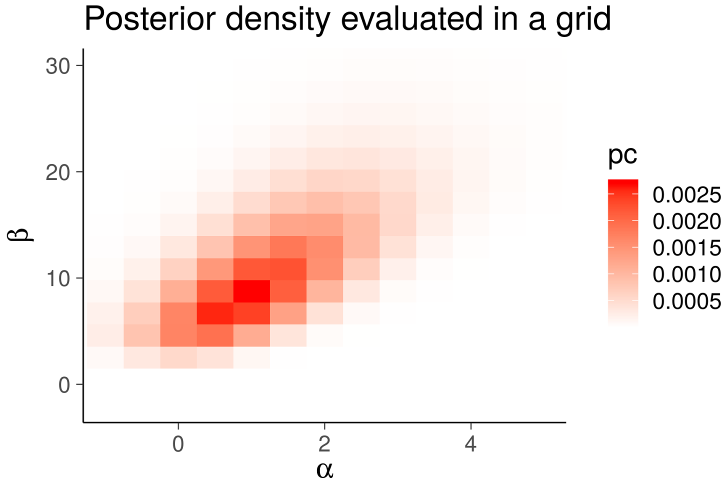
Density evaluated in grid, but plotted using interpolation

Bioassay



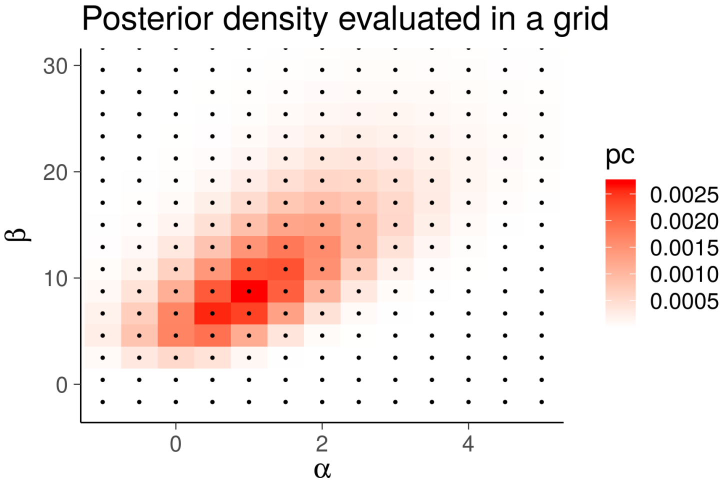
Density evaluated in grid, and plotted without interpolation

Bioassay



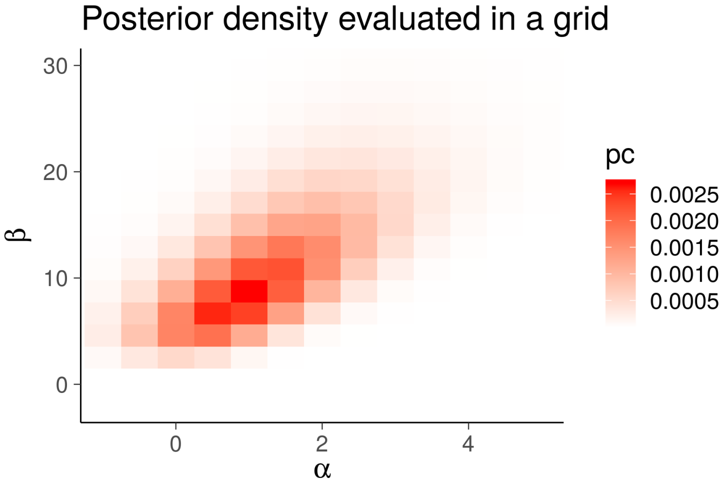
Density evaluated in a coarser grid

Bioassay



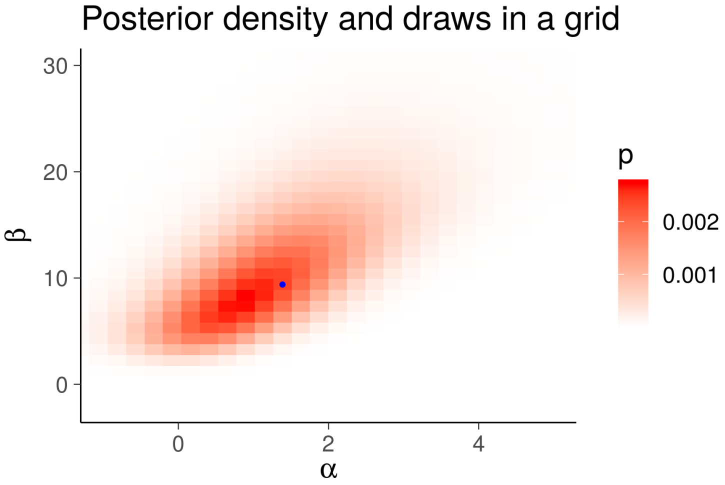
- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell

Bioassay



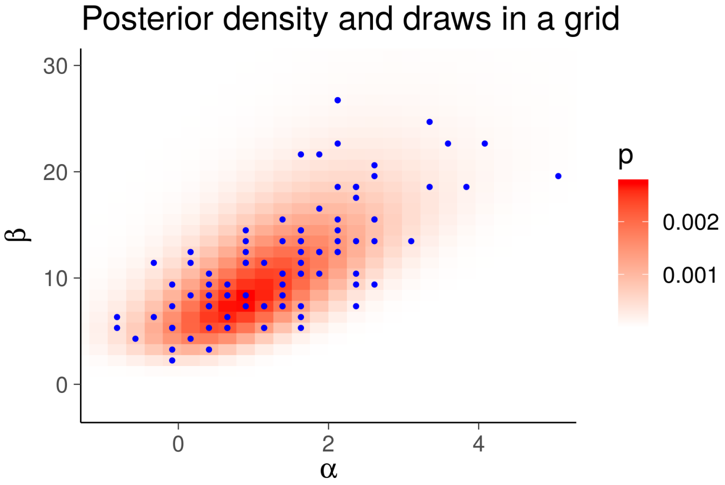
- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

Bioassay



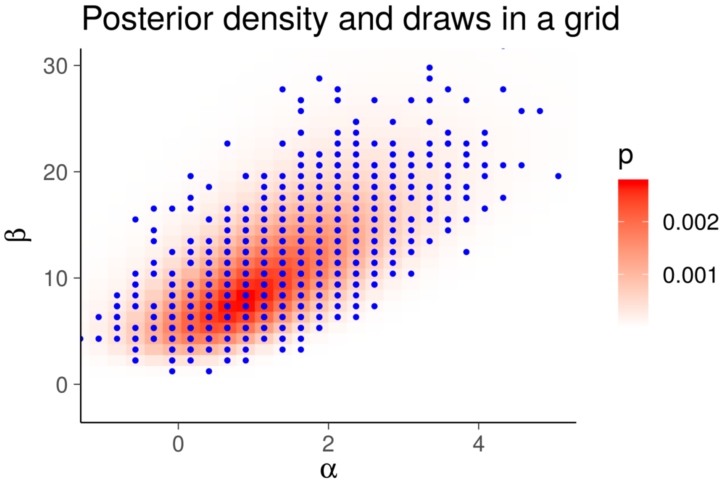
- Sample according to grid cell probabilities

Bioassay



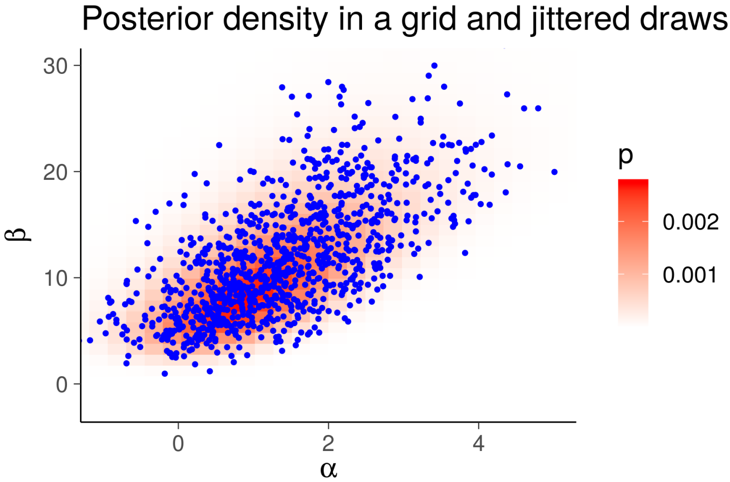
- Sample according to grid cell probabilities

Bioassay



- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

Bioassay



- Jitter can be added to improve visualization

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S}\frac{\alpha^{(s)}}{\beta^{(s)}}$$

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S}\frac{\alpha^{(s)}}{\beta^{(s)}}$$

- Instead of sampling, grid could be used to evaluate functions directly, for example

$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} w_{\mathrm{cell}}^{(t)}\frac{\alpha^{(t)}}{\beta^{(t)}},$$

where $w_{\mathrm{cell}}^{(t)}$ is the normalized probability of a grid cell t, and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S}\frac{\alpha^{(s)}}{\beta^{(s)}}$$

- Instead of sampling, grid could be used to evaluate functions directly, for example

$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} w_{\mathrm{cell}}^{(t)}\frac{\alpha^{(t)}}{\beta^{(t)}},$$

  where $w_{\mathrm{cell}}^{(t)}$ is the normalized probability of a grid cell t, and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

- Grid sampling gets computationally too expensive in high dimensions