DSGE Models
– Exercise And Solution Booklet –

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Overview

1 DSGE Models: Definition, Key Challenges, And Basic Structure[1]

1. Briefly define the term and key challenges of Dynamic Stochastic General Equilibrium (DSGE) models. What are DSGE models useful for?

   1 DSGE models use modern macroeconomic theory to explain and predict co-movements of aggregate time series. DSGE models start from what we call the micro-foundations of macroeconomics (i.e. to be consistent with the underlying behavior of economic agents), with a heart based on the rational expectation forward-looking economic behavior of agents. In reality all macro variables are related to each other, either directly or indirectly, so there is no "cetribus paribus", but a dynamic stochastic general equilibrium system.

   - General Equilibrium (GE): equations must always hold. Short-run: decisions, quantities and prices adjust such that equations are full-filled. Long-run: steady state, i.e. a condition or situation where variables do not change their value (e.g. balanced-growth path where the rate of growth is constant).
   - Stochastic (S): disturbances (or shocks) make the system deviate from its steady state, we get business cycles or, more general, a data-generating process
   - Dynamic (D): Agents are forward-looking and solve intertemporal optimization problems. When a disturbance hits the economy, macroeconomic variables do not return to equilibrium instantaneously, but change very slowly over time, producing complex reactions. Furthermore, some decisions like investment or saving only make sense in a dynamic context. We can analyze and quantify the effects after (i) a temporary shock: how does the economy return to its steady state, or (ii) a permanent shock: how does the economy move to a new steady state.

   Basic structure:

   $$E_t\left[f(y_{t+1}, y_t, y_{t-1}, u_t)\right] = 0$$

   where E_t is the expectation operator with information conditional up to and including period t, y_t is a vector of endogenous variables at time t, u_t a vector of exogenous shocks or random disturbances with proper density functions. $f(\cdot)$ is what we call economic theory. **First key challenge:** value of endogenous variables in a given period of time depend on its future expected value. We need dynamic programming techniques to find the optimality conditions which define the economic behavior of the agents.

   The solution to this system is a decision function:

   $$y_t = g(y_{t-1}, u_t)$$

   Second key challenge: DSGE models cannot be solved analytically, except for some very simple and unrealistic examples. We have to resort to numerical methods and a computer to find an approximated solution.

   Once the theoretical model and solution is at hands, the next step is the **third key challenge**: application to the data. The usual procedure consists in the calibration of the parameters of the model using previous information or matching some key ratios or moments provided by the data, or more recently, form the estimation of the parameters using maximum likelihood, Bayesian techniques, indirect inference, or general method of moments.

2. Outline the common structure of a DSGE model. How do the neoclassical, New-Classical and New-Keynesian models differ?

   2 Focus on behavior of three main types of economic agents or sectors:

[1] Based on Torres (2015, Ch. 1) and Fernández-Villaverde et al. (2016).

- Households: benefit from private consumption, leisure and possibly other things like money holdings or state services; subject to a budget constraint in which they finance their expenditures via (utility-reducing) work, renting capital and buying (government) bonds $\hookrightarrow$ maximization of utility
- Firms produce a variety of products with the help of rented equipment (capital) and labor. They (possibly) have market power over their product and are responsible for the design, manufacture and price of their products. $\hookrightarrow$ cost minimization or profit maximization
- Monetary policy follows a feedback rule, so-called Taylor rule, for instance: nominal interest rate reacts to deviations of the current (or lagged) inflation rate from its target and of current output from potential output
- Fiscal policy (the government) collects taxes from households and companies in order to finance government expenditures (possibly utility-enhancing) and government investment (possibly productivity-enhancing). In addition, the government can issue debt securities.

Also other sectors possible: financial sector, foreign sector, etc. Equilibrium results from the combination of economic decisions taken by all economic agents.

- Canonical neoclassical model (RBC model): reduce economy to the interaction of just one (representative) consumer/household and one (representative) firm. Representative household takes decisions in terms of how much to consume (save) and how much time is devoted to work (leisure). Representative firm decides how much it will produce. Equilibrium of the economy will be defined by a situation in which all decisions taken by all economic agents are compatible and feasible. One can show that business cycles can be generated by one special disturbance: total factor productivity or neutral technological shock; hence, model generates real business cycles without nominal frictions.
- Scale of DSGE models has grown over time with incorporation of a large number of features. To name a few: consumption habit formation, nominal and real rigidities, non-Ricardian agents, investment adjustment costs, investment-specific technological change, taxes, public spending, public capital, human capital, household production, imperfect competition, monetary union, steady state unemployment etc.
- New-Keynesian models have the same foundations as New-Classical general equilibrium models, but incorporate different types of rigidities in the economy. Whereas new classical DSGE models are constructed on the basis of a perfect competition environment, New-Keynesian models include additional elements to the basic model such as imperfect competitions, existence of adjustment costs in investment process, liquidity constraints or rigidities in the determination of prices and wages.

3. Comment whether or not the assumptions underlying DSGE models should be as realistic as possible. For example, a most-common assumption is that all agents live forever.

3 The degree of realism offered by an economic model is not a goal to be pursued by macroeconomists, but rather the model's usefulness in explaining macroeconomic reality. General strategy is the construction of formal structures through equations that reflect the interrelationships between the different economic variables. These simplified structures is what we call a model. The essential question is not that these theoretical constructions are realistic descriptions of the economy, but that they are able to explain the dynamics observed in the economy. Therefore, it is not possible to reject a model ex ante because it is based on assumptions that we believe not too realistic. Rather, the validations must be based on the usefulness of these models to explain reality, and whether they are more useful than other models.

Regarding the assumption that the lifetime of economic agents is assumed to be infinite: We know that in reality consumers, firms and governments have finite life. However, in our models and to be more precise, we assume that firms and governments both use the infinite time as **the**

reference period for taking economic decisions. This is not unrealistic: no government thinks it will cease to exist at some point in the future and no entrepreneur takes decisions based on the idea that the firm will go bankrupts sometime in the future. For consumers this is not so realistic, however, we may weaken this assumption, and think about families, dynasties or households rather than consumers, then the infinite time planning horizon assumption is feasible. Of course, to study the finite life cycle of an agent (school-work-retirement), the so-called Overlapping Generations (OLG) framework is more useful.

2 RBC model with leisure: steady state computations

Consider the basic Real Business Cycle (RBC) model with leisure. The representative household maximizes present as well as expected future utility

$$\max_{\{C_t,I_t,L_t,K_{t+1}\}} E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_t = \gamma \log(C_t) + \psi \log(1 - L_t)$$

is additively separable and has two arguments: consumption C_t and labor L_t. The marginal utility of consumption is positive, whereas more labor reduces utility. Accordingly, γ is the consumption utility parameter and ψ the labor disutility parameter. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $W_t L_t$ and, additionally, profits Π_t from the firm as well as revenue from lending capital K_t at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption C_t and investment I_t. In total, this defines the (real) budget constraint of the household:

$$C_t + I_t = W_t L_t + R_t K_t + \Pi_t$$

The law of motion for capital K_t at the beginning of period t is given by

$$K_{t+1} = (1-\delta)K_t + I_t$$

and δ is controlling depreciations.[2] Assume that the transversality condition is full-filled.

Productivity A_t is the driving force of the economy and evolves according to

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_t^A$$

where ρ_A denotes the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Real profits Π_t of the representative firm are revenues from selling output Y_t minus costs from labor $W_t L_t$ and renting capital $R_t K_t$:

$$\Pi_t = Y_t - W_t L_t - R_t K_t$$

The representative firm maximizes expected profits

$$\max_{\{L_t,K_t\}} E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} \Pi_{t+j}$$

subject to a Cobb-Douglas production function

$$f(K_t, L_t) = Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

The discount factor takes into account that firms are owned by the household, i.e. $\beta^j Q_{t+j}$ is the present value of a unit of consumption in period $t+j$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+j} = \frac{\partial U_{t+j}/\partial C_{t+j}}{\partial U_t/\partial C_t}$.

Finally, we have the non-negativity constraints $K_t \geq 0$, $C_t \geq 0$ and $0 \leq L_t \leq 1$ and clearing of the labor as well as goods market in equilibrium, i.e.

$$Y_t = C_t + I_t$$

[2]Note that capital is predetermined in period t (i.e. it was set in period $t-1$), in DYNARE we hence need to use the notation $K_t = (1-\delta)K_{t-1} + I_t$, where K_t is the capital level at the end of period t.

1. Briefly provide intuition behind the transversality condition.

   1 The transversality condition for an infinite horizon dynamic optimization problem is the boundary condition determining a solution to the problem's first-order conditions together with the initial condition. The transversality condition requires the present value of the state variables (here K_t and A_t) to converge to zero as the planning horizon recedes towards infinity. The first-order and transversality conditions are sufficient to identify an optimum in a concave optimization problem. Given an optimal path, the necessity of the transversality condition reflects the impossibility of finding an alternative feasible path for which each state variable deviates from the optimum at each time and increases discounted utility.

2. Show that the first-order conditions of the representative household are given by

$$
\begin{aligned}
U_t^C &= \beta E_t \left[U_{t+1}^C \left(1 - \delta + R_{t+1}\right) \right] \\
W_t &= -\frac{U_t^L}{U_t^C}
\end{aligned}
$$

   where $U_t^C = \gamma c_t^{-1}$ and $U_t^L = \frac{-\psi}{1-L_t}$. Interpret these equations in economic terms.

   2 Due to our assumptions , we will not have corner solutions and can neglect the non-negativity constraints. Due to the transversality condition and the concave optimization problem, we only need to focus on the first-order conditions. The Lagrangian for the household problem is

$$
\begin{aligned}
L = E_t \sum_{j=0}^{\infty} \beta^j \{ & \gamma \log(C_{t+j}) + \psi \log(1 - l_{t+j}) \\
& + \lambda_{t+j} \left(W_{t+j} L_{t+j} + R_{t+j} K_{t+j} - C_{t+j} - I_{t+j}\right) \\
& + \mu_{t+j} \left((1-\delta) K_{t+j} + I_{t+j} - K_{t+j+1}\right) \}
\end{aligned}
$$

   Note that the problem is not to choose $\{C_t, I_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ all at once in an open-loop policy, but to choose these variables sequentially given the information at time t in a closed-loop policy, i.e. at period t decision rules for $\{C_t, I_t, L_t, K_{t+1}\}$ given the information set at period t; at period $t+1$ decision rules for $\{C_{t+1}, I_{t+1}, L_{t+1}, K_{t+2}\}$ given the information set at period $t+1$.

   The first-order condition w.r.t. C_t is given by

$$
\begin{aligned}
\frac{\partial L}{\partial C_t} &= E_t \left(\gamma C_t^{-1} - \lambda_t \right) = 0 \\
\Leftrightarrow \lambda_t &= \gamma C_t^{-1} && (I)
\end{aligned}
$$

   The first-order condition w.r.t. L_t is given by

$$
\begin{aligned}
\frac{\partial L}{\partial L_t} &= E_t \left(\frac{-\psi}{1 - L_t} + \lambda_t W_t \right) = 0 \\
\Leftrightarrow \lambda_t W_t &= \frac{\psi}{1 - L_t} && (II)
\end{aligned}
$$

   The first-order condition w.r.t. I_t is given by

$$
\begin{aligned}
\frac{\partial L}{\partial I_t} &= E_t \beta^j \left(-\lambda_t + \mu_t\right) = 0 \\
\Leftrightarrow \lambda_t &= \mu_t && (III)
\end{aligned}
$$

   The first-order condition w.r.t. K_{t+1} is given by

$$
\begin{aligned}
\frac{\partial L}{\partial K_{t+1}} &= E_t(-\mu_t) + E_t \beta \left(\lambda_{t+1} R_{t+1} + \mu_{t+1}(1-\delta)\right) = 0 \\
\Leftrightarrow \mu_t &= E_t \beta(\mu_{t+1}(1-\delta) + \lambda_{t+1} R_{t+1}) && (IV)
\end{aligned}
$$

(I) and (III) in (IV) yields

$$\underbrace{\gamma C_t^{-1}}_{U_t^c} = \beta E_t \underbrace{\gamma C_{t+1}^{-1}}_{U_{t+1}^c} (1 - \delta + R_{t+1})$$

This is the Euler equation of intertemporal optimality. It reflects the trade-off between consumption and savings. If the household saves a (marginal) unit of consumption, she can consume the gross rate of return on capital, i.e. $(1 - \delta + R_{t+1})$ units, in the following period. The marginal utility of consuming today is equal to U_t^c, whereas consuming tomorrow has expected utility $E_t(U_{t+1}^c)$. Discounting expected marginal utility with β the household mist be indifferent between both choices in the optimum.

(I) in (II) yields:

$$W_t = -\frac{\frac{-\psi}{1-L_t}}{\gamma C_t^{-1}} \equiv -\frac{U_t^l}{U_t^c}$$

This equation reflects intratemporal optimality, particularly, the optimal choice for labor supply: the real wage must be equal to the marginal rate of substitution between labor and consumption.

3. Show that the first-order conditions of the representative firm are given by

$$W_t = f_L$$
$$R_t = f_K$$

where $f_L = (1-\alpha)A_t\left(\frac{K_t}{L_t}\right)^{\alpha}$ and $f_K = \alpha A_t\left(\frac{K_t}{L_t}\right)^{1-\alpha}$. Interpret these equations in economic terms.

3 First, we note that even though firms maximize expected profits it is actually a static problem. That is, the objective is to maximize profits

$$\Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R_t K_t$$

The first-order condition w.r.t. L_t is given by

$$\frac{\partial \Pi_t}{\partial L_t} = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} - W_t = 0$$
$$\Leftrightarrow W_t = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} = f_L = (1-\alpha)\frac{Y_t}{L_t}$$

The real wage must be equal to the marginal product of labor. Due to the Cobb-Douglas production function it is a constant proportion $(1-\alpha)$ of the ratio of total output and labor. This is the labor demand function.

The first-order condition w.r.t. K_t is given by

$$\frac{\partial \Pi_t}{\partial K_t} = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0$$
$$\Leftrightarrow R_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = f_K = \alpha \frac{Y_t}{K_t}$$

The real interest rate must be equal to the marginal product of capital. Due to the Cobb-Douglas production function it is a constant proportion α of the ratio of total output and capital. This is the capital demand function.

4. Compute the steady state of the model, in the sense that there is a set of values for the endogenous variables that in equilibrium remain constant over time.

4 First, consider the steady state value of technology:

$$\log \bar{A} = \rho_A \log \bar{A} + 0 \Leftrightarrow \log \bar{A} = 0 \Leftrightarrow \bar{A} = 1$$

The Euler equation in steady state becomes:

$$\begin{aligned} \bar{U}^C &= \beta \bar{U}^C (1 - \delta + \bar{R}) \\ \Leftrightarrow \bar{R} &= \frac{1}{\beta} + \delta - 1 \end{aligned}$$

Now we will provide recursively closed-form expressions for all variables in relation to steady state labor.

- The firms demand for capital in steady state becomes

$$\begin{aligned} \bar{R} &= \alpha \bar{A} \bar{K}^{\alpha-1} \bar{L}^{1-\alpha} \\ \Leftrightarrow \frac{\bar{K}}{\bar{L}} &= \left(\frac{\alpha \bar{A}}{\bar{R}}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

- The firms demand for labor in steady state becomes:

$$W = (1-\alpha) \bar{A} \bar{K}^{\alpha} \bar{L}^{-\alpha} = (1-\alpha) \bar{A} \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}$$

- The law of motion for capital in steady state implies

$$\frac{\bar{I}}{\bar{L}} = \delta \frac{\bar{K}}{\bar{L}}$$

- The production function in steady state becomes

$$\frac{\bar{Y}}{\bar{L}} = \bar{A} \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}$$

- The clearing of the goods market in steady state implies

$$\frac{\bar{C}}{\bar{L}} = \frac{\bar{Y}}{\bar{L}} - \frac{\bar{I}}{\bar{L}}$$

Now, we need to derive steady state labor from the equilibrium on the labor market. Due to the log-utility, we can derive a closed-form expression:

$$\begin{aligned} \psi \frac{1}{1-\bar{L}} &= \gamma \bar{C}^{-1} W \\ \Leftrightarrow \psi \frac{\bar{L}}{1-\bar{L}} &= \gamma \left(\frac{\bar{C}}{\bar{L}}\right)^{-1} W \\ \Leftrightarrow \bar{L} &= (1-\bar{L}) \frac{\gamma}{\psi} \left(\frac{\bar{C}}{\bar{L}}\right)^{-1} W \\ \Leftrightarrow \bar{L} &= \frac{\frac{\gamma}{\psi}\left(\frac{\bar{C}}{\bar{L}}\right)^{-1} W}{1 + \frac{\gamma}{\psi}\left(\frac{\bar{C}}{\bar{L}}\right)^{-1} W} \end{aligned}$$

Lastly, it is straigforward to compute the remaining steady state values, i.e.

$$\bar{C} = \frac{\bar{C}}{\bar{L}} \bar{L}, \qquad \bar{I} = \frac{\bar{I}}{\bar{L}} \bar{L}, \qquad \bar{K} = \frac{\bar{K}}{\bar{L}} \bar{L}, \qquad \bar{Y} = \frac{\bar{Y}}{\bar{L}} \bar{L}$$

5. Discuss how to calibrate the following parameters α, β, δ, γ, ψ, ρ_A and σ_A.

5 General hints: construct and parameterize the model such, that it corresponds to certain properties of the true economy. One often uses steady state characteristics for choosing the parameters in accordance with observed data. For instance, long-run averages (wages, working-hours, interest rates, inflation, consumption-shares, government-spending-ratios, etc.) are used to fix steady state values of the endogenous variables, which implies values for the parameters. You can use also use micro-studies, however, one has to be careful about the aggregation!

We will focus on OECD countries and discuss one "possible" way to calibrate the model parameters (there are many other ways):

$\boldsymbol{\alpha}$ productivity parameter of capital. Due to the Cobb Douglas production function thus should be equal to the proportion of capital income to total income of economy. So, one looks inside the national accounts for OECD countries and sets α to 1 minus the share of labor income over total income. For most OECD countries this implies a range of 0.25 to 0.35.

$\boldsymbol{\beta}$ subjective intertemporal preference rate of households: it is the value of future utility in relation to present utility. Usually takes a value slightly less than unity, indicating that agents discount the future. For quarterly data, we typically set it around 0.99. A better way: fix this parameter by making use of the Euler equation in steady state: $\beta = \frac{1}{\bar{R}+1-\delta}$ where $\bar{R} = \alpha\frac{\bar{Y}}{\bar{K}}$. Looking at OECD data one usually finds that average capital productivity $\bar{K}/\bar{Y}$ is in the range of 9 to 10.

$\boldsymbol{\delta}$ depreciation rate of capital stock. For quarterly data the literature uses values in the range of 0.02 to 0.03. A better way: use steady state implication that $\delta = \frac{\bar{I}}{\bar{K}} = \frac{\bar{I}/\bar{Y}}{\bar{K}/\bar{Y}}$. For OECD data one usually finds that average ratio of investment to output, $\bar{I}/\bar{Y}$, is around 0.25.

$\boldsymbol{\gamma}$ and $\boldsymbol{\psi}$ individual's preference regarding consumption and leisure. Often a certain interpretation in terms of elasticities of substitutions is possible. Here we can make use of the First-Order-Conditions in steady state, i.e.

$$\frac{\psi}{\gamma} = \bar{W}\frac{(1-\bar{L})}{\bar{C}} = (1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}\frac{(1-\bar{L})}{\bar{C}} = (1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}\frac{\frac{1}{\bar{L}}(1-\bar{L})}{\frac{\bar{C}}{\bar{L}}}$$

and noting that $\bar{C}/\bar{L}$ as well as $\bar{K}/\bar{L}$ are given in terms of already calibrated parameters (see steady state computations). Therefore, one possible way is to normalize one of the parameters to unity (e.g. $\gamma = 1$) and calibrate the other one in terms of steady state ratios for which we would only require to calibrate steady state hours worked $\bar{L}$. Note that labor time is normalized and usually corresponds to 8 hours a day, i.e. $\bar{L} = 1/3$.

$\boldsymbol{\rho_A}$ and $\boldsymbol{\sigma_A}$ parameters of process for total factor productivity. These can be estimated based on a regression of the Solow Residual, i.e. production function residuals. ρ_A is mostly set above 0.9 to reflect persistence of the technological process and σ_A around 0.6 in the simple RBC model. Another way would be to try different values for σ_A and then try to match the shape of impulse-response-functions of corresponding (S)VAR models.

6. Write a DYNARE mod file for this RBC model with a feasible calibration for an OECD country and compute the steady state of the model by using
   a) a `steady_state_model` block .
   b) an `initval` block with initial values.

6 In the mod file, set `logutility = 1` and for (a) `steadystatemethod = 2` and for (b) `steadystatemethod = 1`. For (b) make sure that `RBC_leisure_steadystate.m` is not in the same folder otherwise initval won't be effective.

7. Now assume a contemporaneous utility function of the CRRA (constant Relative Risk Aversion) type:[3]

$$U_t = \gamma \frac{C_t^{1-\eta_C} - 1}{1-\eta_C} + \psi \frac{(1-L_t)^{1-\eta_L} - 1}{1-\eta_L}$$

   a) Derive the model equations and steady state analytically.
   b) Write a DYNARE mod file with a feasible calibration for an OECD country and compute the steady state for this model by using an external Matlab file.

7 For the first-order conditions of the household we know use

$$\begin{aligned} U_t^C &= \gamma C_t^{-\eta_C} \\ U_t^L &= -\psi(1-L_t)^{-\eta_L} \end{aligned}$$

The steady state for labor changes to

$$\begin{aligned} W\gamma C^{-\eta_C} &= \psi(1-L)^{-\eta_L} \\ W\left(\frac{C}{L}\right)^{-\eta_C} &= \psi(1-L)^{-\eta_L} L^{\eta_C} \end{aligned}$$

This cannot be solved for L in closed-form. Rather, we need to condition on the values of the parameters and use an numerical optimizer to solve for L as is done in the file `RBC_leisure_steadystate.m`: Copy `RBC_leisure_steadystate.m` into the same folder as `RBC_leisure.mod`. Set `logutility = 0` and `steadystatemethod = 0`. Run the mod file with DYNARE to compute the steady state. Note that in the case of log utility the external function uses the closed-form expression to output steady state labor. So this file is most general.

[3] Note that due to L'Hopital's rule $\eta_C = \eta_L = 1$ implies the original specification, $U_t = \gamma \log C_t + \psi \log(1-L_t)$.

3 Brockman and Mirman Model

Consider the simple RBC model with log-utility and full depreciation. The objective is to maximize

$$\max_{\{C_t\}} E_t \sum_{j=0}^{\infty} \beta^j \log(C_{t+j})$$

subject to the law of motion for capital K_t at the beginning of period t

$$K_{t+1} = A_t K_t^{\alpha} - C_t$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. Productivity A_t is the driving force of the economy and evolves according to

$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_t^A$$

where ρ_A denotes the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Finally, we assume that the transversality condition is full-filled and the following non-negativity constraints $K_t \geq 0$ and $C_t \geq 0$.

1. Show that the first-order condition is given by

$$C_t^{-1} = \alpha\beta E_t C_{t+1}^{-1} A_{t+1} K_{t+1}^{\alpha-1}$$

1 Due to our assumptions , we will not have corner solutions and can neglect the non-negativity constraints. Due to the transversality condition and the concave optimization problem, we only need to focus on the first-order conditions. The Lagrangian for the household problem is

$$L = E_t \sum_{j=0}^{\infty} \beta^j \left\{ \log(C_{t+j}) + \lambda_{t+j} \left(A_{t+j} K_{t+j}^{\alpha} - C_t - K_{t+j+1} \right) \right\}$$

Note that the problem is not to choose $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ all at once in an open-loop policy, but to choose these variables sequentially given the information at time t in a closed-loop policy.

The first-order condition w.r.t. C_t is given by

$$\frac{\partial L}{\partial C_t} = E_t \left(C_t^{-1} - \lambda_t \right) = 0$$
$$\Leftrightarrow \lambda_t = C_t^{-1} \qquad (I)$$

The first-order condition w.r.t. K_{t+1} is given by

$$\frac{\partial L}{\partial K_{t+1}} = E_t(-\lambda_t) + E_t\beta \left(\lambda_{t+1} \alpha A_{t+1} K_{t+1}^{\alpha-1} \right) = 0$$
$$\Leftrightarrow \lambda_t = \alpha\beta E_t \lambda_{t+1} A_{t+1} K_{t+1}^{\alpha-1} \qquad (II)$$

(I) and (II) yields

$$C_t^{-1} = \alpha\beta E_t C_{t+1}^{-1} A_{t+1} K_{t+1}^{\alpha-1}$$

2. Compute the steady state of the model, in the sense that there is a set of values for the endogenous variables that in equilibrium remain constant over time.

2 First, consider the steady value of technology:

$$\log \bar{A} = \rho_A \log \bar{A} + 0 \Leftrightarrow \log \bar{A} = 0 \Leftrightarrow \bar{A} = 1$$

The Euler equation in steady state becomes:

$$\bar{K} = (\alpha\beta\bar{A})^{\frac{1}{1-\alpha}}$$

3. The solution to a DSGE model is characterized by decision rules g and h, so-called policy functions, for the control variable C_t and the state variable K_{t+1}:

$$\begin{aligned} C_t &= g(A_t, K_t) \\ K_{t+1} &= h(A_t, K_t) \\ \log A_{t+1} &= \rho_A \log A_t + \varepsilon_{t+1} \end{aligned}$$

Guess that the functions g and h are linear in $A_t K_t^\alpha$:

$$\begin{aligned} C_t &= g_C A_t K_t^\alpha \\ K_{t+1} &= h_K A_t K_t^\alpha \end{aligned}$$

Derive the scalar values g_C and h_C in terms of model parameters α and β.

3 Inserting the guessed policy function for C_t inside the the capital accumulation equation yields:

$$K_{t+1} = A_t K_t^\alpha - g_C A_t K_t^\alpha = (1 - g_C) A_t K_t^\alpha$$

Therefore, $h_K = (1 - g_C)$. Once we derive g_C, we get h_K.
Inserting the guessed policy function for C_t inside the Euler equation yields

$$\begin{aligned} \frac{1}{C_t} &= \alpha\beta E_t \frac{1}{C_{t+1}} A_{t+1} K_{t+1}^{\alpha-1} \\ \frac{1}{g_C A_t K_t^\alpha} &= \alpha\beta E_t \frac{1}{g_C A_{t+1} K_{t+1}^\alpha} A_{t+1} K_{t+1}^{\alpha-1} \\ A_t K_t^\alpha &= \frac{1}{\alpha\beta} E_t K_{t+1} \end{aligned}$$

Inserting the decision rule for capital:

$$\begin{aligned} A_t K_t^\alpha &= \frac{1}{\alpha\beta}(1 - g_C) A_t K_t^\alpha \\ &\Leftrightarrow g_C = (1 - \alpha\beta) \end{aligned}$$

Thus the policy function for C_t is

$$C_t = (1 - \alpha\beta) A_t K_t^\alpha$$

and for K_{t+1}:

$$K_{t+1} = \alpha\beta A_t K_t^\alpha$$

In summary we have found analytically the policy functions. This will not be possible for other DSGE models and we have to rely on numerical methods to approximate the highly nonlinear functions g and h.

4. Write a DYNARE mod file for this RBC model with a feasible calibration for an OECD country. Compute the steady state and approximate the policy and transition functions with a first-order perturbation method (`stoch_simul(order=1,irf=0,periods=0,nomoments)`).

4

```
% =========================================================================
% Stochastic growth model of Brock and Mirman (1972) with technology shock
% Computes simulated data and impulse response functions based on the true
% decision functions and compares these with the corresponding objects
% based on Dynare's approximated model solutions
% =========================================================================
% Willi Mutschler, January 2018
% willi@mutschler.eu
% =========================================================================

% =========================================================================
% Set options for Dynare preprocessor
% =========================================================================
% comparison: 1 for comparing simulated data and IRFs based on true
%             solution vs. Dynare's approximated one
@#define comparision = 0

% =========================================================================
% Declare variables, shocks and parameters
% =========================================================================
var
    C         ${C}$ (long_name='consumption')
    K         ${K}$ (long_name='capital')
    A         ${Z}$ (long_name='total factor productivity')
;

varexo
    eps_A     ${\varepsilon_A}$ (long_name='TFP shock')
;

parameters
    alph    ${\alpha}$   (long_name='capital share')
    betta   ${\beta}$    (long_name='discount factor')
    rhoA    ${\rho_A}$   (long_name='persistence TFP')
    sigA    ${\sigma_A}$ (long_name='standard deviation TFP shock')
;

% =========================================================================
% Calibrate parameter values based on OECD data
% =========================================================================
alph  = 0.35;
betta = 0.99;
rhoA  = 0.9;
sigA  = 0.6;

% =========================================================================
% Model equations
% =========================================================================
model;
[name='Euler equation']
C^(-1)=alph*betta*C(+1)^(-1)*A(+1)*K^(alph-1);
[name='capital law of motion']
K=A*K(-1)^alph-C;
[name='exogenous TFP process']
log(A)=rhoA*log(A(-1))+sigA*eps_A;
end;

% =========================================================================
% Define shock covariance matrix
% =========================================================================
shocks;
    var eps_A = 1;
end;
```

```
% ==========================================================================
% Define steady state computation
% ==========================================================================
% Analytical steady state using steady_state_model
steady_state_model;
    A = 1;                              % technology level
    K = (alph*betta*A)^(1/(1-alph)); % capital level
    C = A*K^alph-K;                     % consumption level
end;

% ==========================================================================
% Computations
% ==========================================================================
steady; % compute steady state given the starting values
resid;  % check the starting values for the steady state
check;  % check Blanchard & Khan rank condition
@# if comparision == 0
    stoch_simul(order=1);
@# else
    stoch_simul(order=1,irf=40,periods=200);
    % order=1:     first-order approximation of solution
    % irf=40 :     compute IRFs for 40 periods for one-standard deviation shock
    %              from approximated solution
    % periods=200: draw 200 shocks and simulate 200 observations
    %              from approximated solution

    %% Compare Trajectories
    % This is the function used by DYNARE to simulate 200 observations from the
    % approximated solution, note that these are already saved in C K A
    epsA = oo_.exo_simul;
    ybar = oo_.steady_state;
    yapprox=simult_(ybar,oo_.dr,epsA,options_.order);
    isequal(yapprox(:,2:end),[C K A]')

    % Simulate the model using the exact policy functions given the same draws
    % of exogenous shocks as was used by DYNARE
    ytrue = zeros(size(ybar,1),options_.periods+1);
    ytrue(:,1) = ybar;
    for i = 2:(options_.periods+1)
        ytrue(3,i) = ytrue(3,i-1)^rhoA * exp(epsA(i-1));            % technology
        ytrue(2,i) = alph*betta*ytrue(3,i)*ytrue(2,i-1)^alph;       % capital
        ytrue(1,i) = (1-alph*betta)*ytrue(3,i)*ytrue(2,i-1)^alph;   % consumption
    end

    % Plot trajectories of exact and approximated solution
    figure('Name','Trajectories of Simulated Data')
    subplot(3,1,1);
        plot((ytrue(1,2:end)));
        hold on
        title('Consumption');
        plot(C-ybar(1),'--')
        hlineC = refline([0 ybar(3)]);
        hlineC.Color = 'black';
        hlineC.LineStyle = ':';
        xlim([1 options_.periods]);
        hold off
        legend('Exact Solution','Approximated Solution','Steady-State')
    subplot(3,1,2);
        plot(log(ytrue(2,2:end)));
        hold on
        title('Capital');
        plot(K-ybar(2),'--')
        hlineK = refline([0 ybar(3)]);
        hlineK.Color = 'black';
```

```
        hlineK.LineStyle = ':';
        xlim([1 options_.periods]);
        hold off
        legend('Exact Solution','Approximated Solution','Steady-State')
    subplot(3,1,3);
        plot(log(ytrue(3,2:end)));
        hold on
        title('Technology');
        plot(A-ybar(3),'--')
        hlineA = refline([0 ybar(3)]);
        hlineA.Color = 'black';
        hlineA.LineStyle = ':';
        xlim([1 options_.periods]);
        hold off
        legend('Exact Solution','Approximated Solution','Steady-State')


    %% Compare Impulse Response Functions
    % This is the function used by DYNARE to compute IRFs from the approximated
    % solution, note that these are already saved in C_eps_A, K_eps_A A_epsA
    onestddev = chol(M_.Sigma_e+1e-14*eye(M_.exo_nbr));
    irfapprox=irf(oo_.dr,onestddev, options_.irf, options_.drop, options_.replic, options_.
    isequal(irfapprox,[C_eps_A K_eps_A A_eps_A]')

    % Compute IRFs using the exact policy functions
    irftrue = zeros(size(ybar,1),options_.periods+1);
    epsA2 = zeros(options_.irf,M_.exo_nbr);
    epsA2(1,:) = onestddev;
    irftrue = zeros(size(ybar,1),options_.irf+1);
    irftrue(:,1) = ybar;
    for i = 2:(options_.irf+1)
        irftrue(3,i) = irftrue(3,i-1)^rhoA * exp(epsA2(i-1));     % technology
        irftrue(2,i) = alph*betta*irftrue(3,i)*irftrue(2,i-1)^alph;       % capital
        irftrue(1,i) = (1-alph*betta)*irftrue(3,i)*irftrue(2,i-1)^alph;  % consumption
    end

    % Plot IRFs of exact and approximated solution
    figure('Name','Impulse Response Functions');
    subplot(3,1,1);
        plot(log(irftrue(1,2:end))-log(ybar(1)));
        hold on
        title('Consumption');
        plot(C_eps_A,'--')
        hlineC = refline([0 0]);
        hlineC.Color = 'black';
        hlineC.LineStyle = ':';
        xlim([1 options_.irf]);
        hold off
        legend('Exact IRF','Approximated IRF','Steady-State')
    subplot(3,1,2);
        plot(log(irftrue(2,2:end))-log(ybar(2)));
        hold on
        title('Capital');
        plot(K_eps_A,'--')
        hlineK = refline([0 0]);
        hlineK.Color = 'black';
        hlineK.LineStyle = ':';
        xlim([1 options_.irf]);
        hold off
        legend('Exact IRF','Approximated IRF','Steady-State')
    subplot(3,1,3);
        plot(log(irftrue(3,2:end))-log(ybar(3)));
        hold on
        title('Technology');
        plot(A_eps_A,'--')
```

```
        hlineA = refline([0 0]);
        hlineA.Color = 'black';
        hlineA.LineStyle = ':';
        xlim([1 options_.irf]);
        hold off
        legend('Exact IRF','Approximated IRF','Steady-State')
@# endif
```

5. Compare simulated data for the endogenous variables as well as impulse response functions of a one-standard-deviation technology shock based on the true solution with DYNARE's approximated one.

   5 Set `comparison = 1` and then run the above mod file with Dynare. You see that the first-order approximation of the true solution is quite accurate only when we are in the vicinity of the steady state. For technology the first-order approximation is exactly the true decision function, because the first-order approximation is equal to a log-linearization. In the IRFs we understate the effect of the technology shock.

4 Klein's linear approximation of rational expectation models

Consider a general DSGE model given in the framework of SGU

1. Define the dimensions of the framework.
2. Ouline the method by Klein to get approximated policy functions with perturbation.
3. Show for the Brockman and Mirman model that the approximated policy function computed by DYNARE and Klein's algorithm are numerically equivalent.

5 RBC model with leisure and habit formation

Consider the basic Real Business Cycle (RBC) model with leisure and habit formation. The representative household maximizes present as well as expected future utility

$$\max_{\{C_t,I_t,L_t,K_t\}} E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_t = \gamma \ln(C_t - \phi C_{t-1}) + (1-\gamma)\ln(1-L_t)$$

has three arguments: consumption C_t and labor L_t of the current period and consumption C_{t-1} of the previous period. The marginal utility of consumption is positive, whereas more labor reduces utility. Accordingly, γ is the elasticity of substitution between consumption and labor and ϕ a coefficient of persistence in habits. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $W_t L_t$ and, additionally, profits Π_t from the firm as well as revenue from lending capital in the previous period K_{t-1} at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption C_t and investment I_t. In total, this defines the (real) budget constraint of the household:

$$C_t + I_t = W_t L_t + R_t K_{t-1} + \Pi_t$$

The law of motion for capital K_t at the end of period t is given by

$$K_t = (1-\delta)K_{t-1} + I_t$$

where δ is the depreciation rate. Assume that the transversality condition is full-filled.

Productivity A_t is the driving force of the economy and evolves according to

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

where ρ_A denotes the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Real profits Π_t of the representative firm are revenues from selling output Y_t minus costs from labor $W_t L_t$ and renting capital $R_t K_{t-1}$:

$$\Pi_t = Y_t - W_t L_t - R_t K_{t-1}$$

The representative firm maximizes expected profits

$$\max_{\{L_t,K_{t-1}\}} E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} \Pi_{t+j}$$

subject to a Cobb-Douglas production function

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

The discount factor takes into account that firms are owned by the household, i.e. $\beta^j Q_{t+j}$ is the present value of a unit of consumption in period $t+j$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+j} = \frac{\partial U_{t+j}/\partial C_{t+j}}{\partial U_t/\partial C_t}$.

Finally, we have the non-negativity constraints $K_t \geq 0$, $C_t \geq 0$ and $0 \leq L_t \leq 1$ and clearing of the labor as well as goods market in equilibrium, i.e.

$$Y_t = C_t + I_t$$

1. Briefly provide intuition behind the consumption habit formation behavior.

2. Show that the first-order conditions of the representative household are given by

$$E_t\left[\frac{\frac{1}{C_t-\phi C_{t-1}}-\beta\phi\frac{1}{C_{t+1}-\phi C_t}}{\frac{1}{C_{t+1}-\phi C_t}-\beta\phi\frac{1}{C_{t+2}-\phi C_{t+1}}}\right]=E_t\left[\beta(R_{t+1}+1-\delta)\right]$$

$$E_t\left[\gamma\frac{1}{C_t-\phi C_{t-1}}-\beta\gamma\phi\frac{1}{C_{t+1}-\phi C_t}\right]W_t=(1-\gamma)\frac{1}{1-L_t}$$

   Interpret these equations in economic terms.

3. Show that the first-order conditions of the representative firm are given by

$$W_t=(1-\alpha)A_t\left(\frac{K_{t-1}}{L_t}\right)^{\alpha}$$

$$R_t=\alpha A_t\left(\frac{L_t}{K_{t-1}}\right)^{1-\alpha}$$

   Interpret these equations in economic terms.

4. Discuss how to calibrate the parameter ϕ.

5. Write a DYNARE mod file for this model with a feasible calibration and compute the steady state of the model either analytically or numerically.

6. Study the effects of a positive aggregate productivity shock using an impulse response analysis. What are the main differences relative to the same shock in the basic RBC model without consumption habit, i.e. $\phi=0$?

7. Simulate data for consumption growth for 200 periods. Estimate three parameters (of your choosing) with

   (i) maximum likelihood methods

   (ii) Bayesian methods

   Provide feasible upper and lower bounds and discuss the intuition behind your priors.

8. Explain whether or not you are satisfied with your estimation results?

6 RBC model with leisure and non-Ricardian agents

Consider the basic Real Business Cycle (RBC) model with leisure and non-Ricardian agents. Assume that there is a continuum of consumers given on the interval $[0,1]$. A proportion of the population, ω, are Ricardian agents who have access to financial markets and are indexed by $i \in [0,\omega)$. The other part of the population, $1-\omega$, is composed of non-Ricardian agents who do not have access to financial markets and are indexed by $j \in (\omega,1]$.

A Ricardian household maximizes present as well as expected future utility

$$\max_{\{C_{i,t},I_{i,t},L_{i,t},K_{i,t}\}} E_t \sum_{s=0}^{\infty} \beta^s U_{i,t+s}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_{i,t} = \gamma \ln C_{i,t} + (1-\gamma) \ln(1-L_{i,t})$$

has two arguments: consumption $C_{i,t}$ and labor $L_{i,t}$. The marginal utility of consumption is positive, whereas more labor reduces utility. Accordingly, γ is the elasticity of substitution between consumption and labor. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $W_t L_{i,t}$ and, additionally, profits $\Pi_{i,t}$ from the firm as well as revenue from lending capital in the previous period $K_{i,t-1}$ at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the Ricardian households. Income and wealth are used to finance consumption $C_{i,t}$ and investment $I_{i,t}$. In total, this defines the (real) budget constraint of the Ricardian agent:

$$C_{i,t} + I_{i,t} = W_t L_{i,t} + R_t K_{i,t-1} + \Pi_{i,t}$$

The law of motion for capital $K_{i,t}$ at the end of period t is given by

$$K_{i,t} = (1-\delta)K_{i,t-1} + I_{i,t}$$

where δ is the depreciation rate. Assume that the transversality condition is full-filled.

A non-Ricardian household maximizes present as well as expected future utility

$$\max_{\{C_{j,t},L_{j,t}\}} E_t \sum_{s=0}^{\infty} \beta^s U_{j,t+s}$$

The contemporaneous utility function is the same as for non-Ricardian households, i.e.

$$U_{j,t} = \gamma \ln C_{j,t} + (1-\gamma) \ln(1-L_{j,t})$$

As non-Ricardian agents do not have access to the credit market, their (real) budget constraint is given by:

$$C_{j,t} = W_t L_{j,t}$$

It is assumed that all agents, independently the group they belong to, are identical. Therefore, aggregate values (in per capita terms) are given by:

$$C_t = \omega C_{i,t} + (1-\omega) C_{j,t}, \qquad L_t = \omega L_{i,t} + (1-\omega) L_{j,t}, \qquad K_t = \omega K_{i,t} \qquad I_t = \omega I_{i,t}$$

where the right two expressions are due to the fact that only Ricardian agents invest in physical capital.

Productivity A_t is the driving force of the economy and evolves according to

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

where ρ_A denotes the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Real profits $\Pi_t = \omega \Pi_{i,t}$ of the representative firm are revenues from selling output Y_t minus costs from labor $W_t L_t$ and renting capital $R_t K_{t-1}$:

$$\Pi_t = Y_t - W_t L_t - R_t K_{t-1}$$

The representative firm maximizes expected profits

$$\max_{\{L_t, K_{t-1}\}} E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} \Pi_{t+j}$$

subject to a Cobb-Douglas production function

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

The discount factor takes into account that firms are owned by the Ricardian households, i.e. $\beta^s Q_{t+s}$ is the present value of a unit of consumption in period $t+s$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+s} = \frac{\partial U_{i,t+s}/\partial C_{i,t+s}}{\partial U_{i,t}/\partial C_{i,t}}$.

Finally, we have the non-negativity constraints $K_{i,t} \geq 0$, $C_{i,t} \geq 0$, $C_{j,t} \geq 0$, $0 \leq L_{i,t} \leq 1$ and $0 \leq L_{j,t} \leq 1$. Furthermore, clearing of the labor as well as goods market in equilibrium implies

$$Y_t = C_t + I_t$$

1. Briefly provide intuition behind the introduction of non-Ricardian households.
2. Show that the first-order conditions of the agents are given by

$$E_t \left[\frac{C_{i,t+1}}{C_{i,t}} \right] = \beta E_t \left[1 - \delta + R_{t+1} \right], \qquad W_t = \frac{1-\gamma}{\gamma} \frac{C_{i,t}}{1 - L_{i,t}},$$

$$C_{j,t} = W_t L_{j,t}, \qquad W_t = \frac{1-\gamma}{\gamma} \frac{C_{j,t}}{1 - L_{j,t}}.$$

   Interpret these equations in economic terms.
3. Show that the first-order conditions of the representative firm are given by

$$W_t = (1-\alpha) A_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha}, \qquad R_t = \alpha A_t \left(\frac{L_t}{K_{t-1}} \right)^{1-\alpha}$$

   Interpret these equations in economic terms.
4. Discuss how to calibrate the parameter ω.
5. Write a DYNARE mod file for this model with a feasible calibration and compute the steady state of the model either analytically or numerically.
6. Study the effects of a positive aggregate productivity shock using an impulse response analysis for each group as well as on aggregate variables. What are the main differences relative to the same shock in the basic RBC model without non-Ricardian agents, i.e. $\omega = 1$?
7. Simulate data for consumption growth for 200 periods. Estimate three parameters (of your choosing) with
   (i) maximum likelihood methods
   (ii) Bayesian methods

   Provide feasible upper and lower bounds and discuss the intuition behind your priors.
8. Explain whether or not you are satisfied with your estimation results?

7 RBC model with leisure and investment-specific technological change

Consider the basic Real Business Cycle (RBC) model with leisure and investment-specific technological change. The representative household maximizes present as well as expected future utility

$$\max_{\{C_t,I_t,L_t,K_t\}} E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_t = \gamma \ln(C_t) + (1-\gamma) \ln(1-L_t)$$

has two arguments: consumption C_t and labor L_t. The marginal utility of consumption is positive, whereas more labor reduces utility. Accordingly, γ is the elasticity of substitution between consumption and labor. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $W_t L_t$ and, additionally, profits Π_t from the firm as well as revenue from lending capital K_{t-1} in the previous period at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption C_t and investment I_t. In total, this defines the (real) budget constraint of the household:

$$C_t + I_t = W_t L_t + R_t K_{t-1} + \Pi_t$$

The law of motion for capital K_t at the end of period t is given by

$$K_t = (1-\delta)K_{t-1} + Z_t I_t$$

where δ is the depreciation rate and Z_t investment-specific technological change. Assume that the transversality condition is full-filled.

The model includes two driving forces of the economy, a neutral technological progress A_t and a technological progress specific to investment Z_t. The laws of motion for these processes are given by:

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$
$$\ln Z_t = \rho_Z \ln Z_{t-1} + \varepsilon_t^Z$$

where ρ_A and ρ_Z denote the persistence parameters and ε_t^A and ε_t^Z are assumed to be independently normally distributed with zero means and variances equal to σ_A^2 and σ_Z^2, respectively.

Real profits Π_t of the representative firm are revenues from selling output Y_t minus costs from labor $W_t L_t$ and renting capital $R_t K_{t-1}$:

$$\Pi_t = Y_t - W_t L_t - R_t K_{t-1}$$

The representative firm maximizes expected profits

$$\max_{\{L_t,K_{t-1}\}} E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} \Pi_{t+j}$$

subject to a Cobb-Douglas production function

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

The discount factor takes into account that firms are owned by the household, i.e. $\beta^j Q_{t+j}$ is the present value of a unit of consumption in period $t+j$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+j} = \frac{\partial U_{t+j}/\partial C_{t+j}}{\partial U_t/\partial C_t}$.

Finally, we have the non-negativity constraints $K_t \geq 0$, $C_t \geq 0$ and $0 \leq L_t \leq 1$ and clearing of the labor as well as goods market in equilibrium, i.e.

$$Y_t = C_t + I_t$$

1. Briefly provide intuition behind the introduction of investment-specific technological change.
2. Show that the first-order conditions of the agents are given by

$$E_t\left[\frac{C_{t+1}}{C_t}\right] = \beta E_t\left[\frac{Z_t}{Z_{t+1}}\left(1-\delta+Z_{t+1}R_{t+1}\right)\right],$$
$$W_t = \frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t},$$

   Interpret these equations in economic terms.
3. Show that the first-order conditions of the representative firm are given by

$$W_t = (1-\alpha)A_t\left(\frac{K_{t-1}}{L_t}\right)^{\alpha},$$
$$R_t = \alpha A_t\left(\frac{L_t}{K_{t-1}}\right)^{1-\alpha}$$

   Interpret these equations in economic terms.
4. Discuss how to calibrate the parameters ρ_Z and σ_Z^2.
5. Write a DYNARE mod file for this model with a feasible calibration and compute the steady state of the model either analytically or numerically.
6. Study the effects of both a positive neutral productivity shock and a positive investment-specific productivity shock using an impulse response analysis. How would you design a short-run identification scheme for a SVAR model based on your DSGE model to disentangle both technological shocks? In other words, which variable(s) behave differently in the short-run?
7. Simulate data for investment and consumption growth for 200 periods. Estimate three parameters (of your choosing) with
   (i) maximum likelihood methods
   (ii) Bayesian methods

   Provide feasible upper and lower bounds and discuss the intuition behind your priors.
8. Explain whether or not you are satisfied with your estimation results?

8 RBC model with leisure and home production

Consider the basic Real Business Cycle (RBC) model with leisure and home production, that is time devoted, for instance, to maintain the house, parenting or nursing care of the elderly. The representative household maximizes present as well as expected future utility

$$\max_{\{C_{m,t},C_{h,t},I_t,L_{m,t},L_{h,t},K_t\}} E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. Total consumption C_t is composed by the consumption of market goods $C_{m,t}$ and of home production services $C_{h,t}$. The aggregation follows a CES-type function:

$$C_t = \left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta}\right]^{1/\eta}$$

where η is the parameter measuring the willingness of agents to substitute between the two goods and ω is the proportion of each good in total consumption.

The production function for home activities is labor-intensive and given by

$$C_{h,t} = B_t L_{h,t}^{\theta}$$

where $0 < \theta < 1$ implies decreasing returns. Productivity B_t is the driving force of the home production sector and evolves according to

$$\ln B_t = \rho_B \ln B_{t-1} + \varepsilon_t^B$$

where ρ_B denotes the persistence parameter and ε_t^B is assumed to be normally distributed with mean zero and variance σ_B^2.

Non-leisure time $L_t = L_{m,t} + L_{h,t}$ is devoted either to working in the good market production ($L_{m,t}$) or providing home production services ($L_{h,t}$). Hence, the contemporaneous utility function is given by

$$U_t = \gamma \ln(C_t) + (1-\gamma)\ln(1 - L_{m,t} - L_{h,t})$$

The marginal utility of consumption is positive, whereas more labor in either sector reduces utility. Accordingly, γ is the elasticity of substitution between consumption and labor. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm in the good production sector. In return, she receives real labor income in the amount of $W_t L_{m,t}$ and, additionally, profits Π_t from the firm as well as revenue from lending capital K_{t-1} in the previous period at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption of market goods $C_{m,t}$ and investment I_t. In total, this defines the (real) budget constraint of the household:

$$C_{t,m} + I_t = W_t L_{t,m} + R_t K_{t-1} + \Pi_t$$

The law of motion for capital K_t at the end of period t is given by

$$K_t = (1-\delta) K_{t-1} + I_t$$

where δ is the depreciation rate. Assume that the transversality condition is full-filled.

Real profits Π_t of the representative firm in the goods market sector are revenues from selling output Y_t minus costs from labor $W_t L_{m,t}$ and renting capital $R_t K_{t-1}$:

$$\Pi_t = Y_t - W_t L_{m,t} - R_t K_{t-1}$$

The representative firm maximizes expected profits

$$\max_{\{L_{m,t},K_{t-1}\}} E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} \Pi_{t+j}$$

subject to a Cobb-Douglas production function

$$Y_t = A_t K_{t-1}^{\alpha} L_{m,t}^{1-\alpha}$$

The discount factor takes into account that firms are owned by the household, i.e. $\beta^j Q_{t+j}$ is the present value of a unit of consumption in period $t+j$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+j} = \frac{\partial U_{t+j}/\partial C_{t+j}}{\partial U_t/\partial C_t}$. Productivity A_t is the driving force of the goods market sector and evolves according to

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

where $\bar{A}$ denotes the technology level in steady state, ρ_A the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Finally, we have the non-negativity constraints $K_t \geq 0$, $C_t \geq 0$, $0 \leq L_{h,t} \leq 1$ and $0 \leq L_{m,t} \leq 1$ and clearing of the labor as well as goods market in equilibrium, i.e.

$$Y_t = C_{m,t} + I_t$$

1. Briefly provide intuition behind the introduction of the home production sector.
2. Show that the first-order conditions of the household are given by

$$E_t \left[\beta \frac{C_{m,t+1}^{\eta-1}}{C_{t+1}} (R_{t+1} + 1 - \delta) \right] = \frac{C_{m,t}^{\eta-1}}{C_t}$$

$$\frac{1-\gamma}{1 - L_{m,t} - L_{h,t}} = \gamma\omega \frac{C_{m,t}^{\eta-1}}{C_t} W_t$$

$$\frac{1-\gamma}{1 - L_{m,t} - L_{h,t}} = \gamma(1-\omega)\theta \frac{C_{h,t}^{\eta}}{C_t L_{h,t}}$$

   Interpret these equations in economic terms.
3. Show that the first-order conditions of the representative firm in the goods market sector are given by

$$W_t = (1-\alpha) A_t \left(\frac{K_{t-1}}{L_{m,t}} \right)^{\alpha},$$

$$R_t = \alpha A_t \left(\frac{L_{m,t}}{K_{t-1}} \right)^{1-\alpha}$$

   Interpret these equations in economic terms.
4. Discuss how to calibrate the parameters η, ω, θ, ρ_B and σ_B^2.
5. Write a DYNARE mod file for this model with a feasible calibration and compute the steady state of the model either analytically or numerically.
6. Study the effects of a positive aggregate technology shock in the goods market sector using an impulse response analysis. Compare this to a model without a production sector, i.e. $\omega = 1$.
7. Simulate data for labor and consumption growth in the goods market for 200 periods. Estimate three parameters (of your choosing) with
   (i) maximum likelihood methods
   (ii) Bayesian methods

   Provide feasible upper and lower bounds and discuss the intuition behind your priors.
8. Explain whether or not you are satisfied with your estimation results?

References

Fernández-Villaverde, J., J. F. Rubio-Ramírez, and F. Schorfheide (2016). Solution and Estimation Methods for DSGE Models. In J. B. Taylor and H. Uhlig (Eds.), *Handbook of Macroeconomics Volume 2 (forthcoming)*. North-Holland.

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