Confidence Intervals for Low Dimensional Parameters in High Dimensional Linear Models

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2018.12.12

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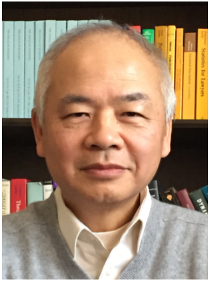

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High dimentional data
Empirical Bayes
Functional MRI
Network data
Semiparametric and nonparametric methods
Survival analysis, statistical inference
Probability theory

Outline

- Introduction
- Methodology
- Important theoretical results
- Simulations
- Discussion

Section 1

Introduction

Background

In high-dimensional statistics, much work has been made on consistency for prediction, estimation of high-dimensional objects or variable selection.

Regularized linear regression

- ℓ_1 regularized methods
- non-convex penalized methods
- greedy methods
- screening methods ...

Related work

Some related works have concerned with statistics inference:

- Knight and Fu(2000): Lasso type estimators cannot obtain a proper asymptotics distribution of unknown cofficients, even in low-dim situation.
- Leeb and Potscher(2006): Consistent estimation of the distribution of the least squares estimator after model selection is impossible.
- Berk et.al(2010); Laber and Murphy(2011): Conservative statistical inference after model selection may not yield accurate confidence regions or p-values when p is large.

Uniform signal strength condition

Existing variable selection approaches based on selection consistency theory typically requires a uniform signal strength condition:

$$\min_{\beta_j \neq 0} |\beta_j| \geq C\sigma\sqrt{(2/n)\log p},\ C > 1/2, \tag{1}$$

Advantages of OLS in low-dim

Linear model in low dimensions ($p < n$):

$$y = X\boldsymbol{\beta} + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I).$$

We get the estimator of $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'y$ with explict form of covariance structure as following

$$cov(c'\hat{\boldsymbol{\beta}}, d'\hat{\boldsymbol{\beta}}) = \sigma^2 c'(X'X)^{-}d.$$

and the confidence interval, page 129.

Section 2

Methodology

Model setting

- Considering the following linear model,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \varepsilon \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2\mathbf{I}) \tag{2}$$

where $\mathbf{y} \in \mathbb{R}^n$ is a response vector, $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_p) \in \mathbb{R}^{n\times p}$ is a design matrix with columns $\mathbf{x}_j$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \boldsymbol{\beta}_p)^T$ is a vector of unknown regression coefficients.

  - Standardize the design to $\|\mathbf{x}_j\|_2^2 = n$.
  - The design matrix $\mathbf{X}$ is assumed to be deterministic.

Least squares estimator

- In classical theory of linear models, the least squares estimator of an estimable regression coefficient β_j can be written as

$$\hat{\beta}_j^{(lse)} := (\mathrm{x}_j^{\perp})^T \mathrm{y} / (\mathrm{x}_j^{\perp})^T \mathrm{x}_j, \tag{3}$$

  where $\mathrm{x}_j^{\perp}$ is a projection of x_j to the orthogonal complement of column space of $\mathrm{X}_{-j} = (\mathrm{x}_k, k \neq j)$.
- The $\mathrm{x}_j^{\perp}$ is defined by
  - $(\mathrm{x}_j^{\perp})^T \mathrm{x}_k = 0$, for $\forall k \neq j$.

Problems

- In high dimensional situation $p > n$, $rank(\mathrm{X}_{-j}) = n$ for all j.
  - $\mathrm{x}_j^{\perp} = 0$.
  - $\hat{\beta}_j^{(lse)}$ is undefined.
- We want to preserve the properties of the least squares estimator.
  - The covariance structure of the least squares estimator:

$$\mathrm{cov}(\hat{\beta}_j^{(lse)}, \hat{\beta}_k^{(lse)}) = \sigma^2 \frac{(\mathrm{x}_j^{\perp})^T \mathrm{x}_k^{\perp}}{\|\mathrm{x}_j\|_2^2 \|\mathrm{x}_k\|_2^2} \tag{4}$$

- Motivation of LDPE:
  - Replace $\mathrm{x}_j^{\perp}$ with z_j.
  - Relaxing the constraint $\mathrm{z}_j^T \mathrm{x}_k = 0$ for $k \neq j$.

Bias-corrected linear estimators

- For any $\mathbf{z}_j$ that is not orthogonal to $\mathbf{x}_j$, the corresponding univariate linear regression estimator satisfies

$$\hat{\beta}_j^{(lin)} = \frac{\mathbf{z}_j^T \mathbf{y}}{\mathbf{z}_j^T \mathbf{x}_j} = \beta_j + \frac{\mathbf{z}_j^T \varepsilon}{\mathbf{z}_j^T \mathbf{x}_j} + \sum_{k \neq j} \frac{\mathbf{z}_j^T \mathbf{x}_k \beta_k}{\mathbf{z}_j^T \mathbf{x}_j}.$$

  - Here, $\hat{\beta}_j^{(lin)}$ has the same covariance structure with $\hat{\beta}_j^{(lse)}$.

- Note that the bias of $\hat{\beta}_j^{(lin)}$ is linear in β_k, which is unbounded. It is impossible to have $\mathbf{z}_j^T \mathbf{x}_k = 0$ for all $k \neq j \quad (\mathbf{z}_j \neq 0)$.

Low dimensional projection estimator

- Bias correction with a non-linear initial estimator $\hat{\beta}^{(init)}$:

$$\hat{\beta}_j = \hat{\beta}_j^{(lin)} - \sum_{k \neq j} \frac{z_j^T x_k \hat{\beta}_k^{(init)}}{z_j^T x_j} = \frac{z_j^T y}{z_j^T x_j} - \sum_{k \neq j} \frac{z_j^T x_k \hat{\beta}_k^{(init)}}{z_j^T x_j}. \quad (5)$$

  - The estimation error of $\hat{\beta}_j$:

$$\hat{\beta}_j - \beta_j = \frac{z_j^T \varepsilon}{z_j^T x_j} + \frac{\sum_{k \neq j} z_j^T x_k (\beta_k - \hat{\beta}_k^{(init)})}{z_j^T x_j} \triangleq A + B. \quad (6)$$

  - It can be viewed as a sum of noise term and bias term.

Error analysis of LDPE(1)

- The approximation error of the LDPE (Term B) can be controlled:

$$\Big|\sum_{k\neq j} z_j^T x_k(\beta_k - \widehat{\beta}_k^{(init)})\Big| \leq \Big(\max_{k\neq j} |z_j^T x_k|\Big)\|\widehat{\boldsymbol{\beta}}^{(init)} - \boldsymbol{\beta}\|_1. \quad (7)$$

- For z_j, define

$$\eta_j = \max_{k\neq j} |z_j^T x_k|/\|z_j\|_2, \qquad \tau_j = \|z_j\|_2/|z_j^T x_j|. \quad (8)$$

  - Bias factor η_j: $\eta_j\|\hat{\boldsymbol{\beta}}^{(init)} - \boldsymbol{\beta}\|_1$ controls the approximation error.
  - Noise factor τ_j: $\tau_j\sigma$ is the standard deviation of noise term.

Error analysis of LDPE (2)

- Since $z_j^T \varepsilon \sim N(0, \sigma^2 \|z_j\|_2^2)$, equation (5) yields

$$\eta_j \|\hat{\beta}^{(init)} - \beta\|_1/\sigma = o(1) \Rightarrow \tau_j^{-1}(\hat{\beta}_j - \beta_j) \approx N(0, \sigma^2). \quad (9)$$

- Confidence intervals can be constructed by condition (8) and a consistent estimator of σ.
- Need to solve:
  - Choose proper z_j.
  - Choose $\hat{\beta}^{(init)}$.

How can we choose z_j ?

- Choose z_j as the residual of lasso:

$$z_j = x_j - X_{-j}\hat{\gamma}_j,\ \hat{\gamma}_j = \arg\min_{b}\left\{\frac{\|x_j - X_{-j}b\|_2^2}{2n} + \lambda_j\|b\|_1\right\}. \quad (10)$$

- Karush-Kuhn-Tucker conditions for equation (9)

$\Rightarrow \quad |x_k^T z_j/n| \le \lambda_j$ for all $k \neq j$

$\Rightarrow \quad \eta_j \le n\lambda_j/\|z_j\|_2.$

How can we pick initial estimator of β?

- The scaled lasso is a joint convex minimization method

$$\{\hat{\beta}^{(init)}, \sigma\} = \arg\min_{\mathrm{b},\sigma} \left\{ \frac{\|\mathrm{y} - \mathrm{Xb}\|_2^2}{2\sigma n} + \frac{\sigma}{2} + \lambda_0 \|\mathrm{b}\|_1 \right\}. \quad (11)$$

- The scaled lasso is biased, an alternative method scaled lasso-LSE can be applied:

$$\{\hat{\beta}^{(init)}, \sigma\} = \arg\min_{\mathrm{b},\sigma} \{ \frac{\|\mathrm{y} - \mathrm{Xb}\|_2^2}{2\sigma max(n - |\hat{S}^{(scl)}|, 1)} + \frac{\sigma}{2} \} \quad (12)$$

where $\hat{S}^{(scl)}$ is the set of non-zero estimated coefficients produce by scaled lasso.

Procedure of computing $\mathbf{z}_j$

Along the Lasso path for regressing $\mathbf{x}_j$ against $\mathbf{X}_{-j}$, let

$$\begin{aligned} \gamma_j(\lambda) &= \arg\min_{\mathbf{b}} \{\|\mathbf{x}_j - \mathbf{X}_{-j}\mathbf{b}\|_2^2/(2n) + \lambda\|\mathbf{b}\|_1\}, \\ \mathbf{z}_j(\lambda) &= \mathbf{x}_j - \mathbf{X}_{-j}\gamma_j(\lambda), \\ \eta_j(\lambda) &= \max_{k\neq j}|\mathbf{x}_k^T\mathbf{z}_j(\lambda)|/\|\mathbf{z}_j(\lambda)\|_2, \\ \tau_j(\lambda) &= \|\mathbf{z}_j(\lambda)\|_2/|\mathbf{x}_j^T\mathbf{z}_j(\lambda)|, \end{aligned} \tag{13}$$

be the coefficient estimator γ_j, residual $\mathbf{z}_j$, the bias factor η_j, and the noise factor τ_j, as functions of λ.

Proposition 1

(a) In the Lasso path, $\|\mathrm{z}_j(\lambda)\|_2$, $\eta_j(\lambda)$, and $\sigma_j(\lambda)$ are nondecreasing functions of λ, and $\tau_j(\lambda) \leq 1/\|\mathrm{z}_j(\lambda)\|_2$. Moreover, $\gamma_j(\lambda) \neq 0$ implies $\eta_j(\lambda) = \lambda n/\|\mathrm{z}_j(\lambda)\|_2$.
(b) Let $\lambda_{univ} = \sqrt{(2/n)\log p}$. Then,

$$\sigma_j(C\lambda_{univ}) > 0 \text{ iff } \{\lambda > 0 : \eta_j(\lambda) \leq C\sqrt{2\log p}\} \neq \emptyset, \tag{14}$$

and in this case, the algorithm in Table 2 provides

$$\eta_j \leq \eta_j^* \leq (1 \vee C)\sqrt{2\log p}, \tag{15}$$

$$\tau_j \leq n^{-1/2}(1+\kappa_0)/\hat{\sigma}_j(C\lambda_{univ}). \tag{16}$$

Moreover, when $\mathrm{z}_j(0) = \mathrm{x}_j^{\perp} = 0$, $\eta_j(0+)\inf\{\|\gamma_j\|_1 : \mathrm{X}_{-j}\gamma_j = \mathrm{x}_j\} = \sqrt{n}$.
(c) Let $0 < a_0 < 1 \leq C_0 < \infty$. Suppose that for $s = a_0 n/\log p$

$$\inf_{\delta}\sup_{\boldsymbol{\beta}}\left\{\|\delta(\mathrm{X},\mathrm{y})-\boldsymbol{\beta}\|_2^2 : \mathrm{y} = \mathrm{X}\boldsymbol{\beta}, \sum_{j=1}^{p}\min(|\beta_j|/\lambda_{univ}, 1) \leq s+1\right\} \leq 2C_0 s(\log p)/n.$$

Computation of z_j

Figure 1: Computation of z_j from the Lasso (12)

Input:	$\eta_j^* = \sqrt{2\log p}$
	$\kappa_0 = 0.25$
Step 1:	Compute z_j for $\lambda \geq \lambda_*$,
	Compute η_j and τ_j for $\lambda \geq \lambda_*$ η_j^*
Step 2:	If $\eta_j(\lambda_*) \geq \eta_j^*$, return $z_j \leftarrow z_j(\lambda_*)$;
	otherwise
	$\tau_j^* \leftarrow (1+\kappa_0)\min\{\tau_j(\lambda) : \eta_j(\lambda) \geq \eta_j^*\}$
	$\lambda \leftarrow \arg\min\{\eta_j(\lambda) : \tau_j(\lambda) \geq \tau_j^*\}$
	$z_j \leftarrow z_j(\lambda)$

[1]

[1] λ_* is the smallest non-zero penalty level in lasso path.

Restricted LDPE

- The reason for using restricted lasso relaxation for $\mathbf{z}_j$.
  - The summands with larger absolute correlation $|\mathbf{x}_j^T \mathbf{x}_k / n|$ are likely to have a greater contribution to the bias due to initial estimation error $|\hat{\beta}_k^{(init)} - \beta_k|$.
- How to implement restricted LDPE(RLDPE)?
  - Force smaller $|\mathbf{z}_j^T \mathbf{x}_k / n|$ for large $|\mathbf{x}_j^T \mathbf{x}_k / n|$ with a weighted relaxation:

$$\mathbf{z}_j = \mathbf{x}_j - \mathbf{X}_{-j}\gamma_j, \quad \gamma_j = \arg\min_{\mathbf{b}} \left\{ \frac{\|\mathbf{x}_j - \mathbf{X}_{-j}\mathbf{b}\|_2^2}{2n} + \lambda_j \sum_{k \neq j} w_k |b_k| \right\}, \qquad (17)$$

- Simply set $w_k = 0$ for large $|\mathbf{x}_j^T \mathbf{x}_k / n|$ and $w_k = 1$ for other k in the RLDPE.

Confidence interval

- The covariance of the noise component in (5) is proportional to

$$V = (V_{jk})_{p\times p}, \quad V_{jk} = \frac{z_j^T z_k}{|z_j^T x_j||z_k^T x_k|} = \sigma^{-2} cov\left(\frac{z_j^T \varepsilon}{z_j^T x_j}, \frac{z_k^T \varepsilon}{z_k^T x_k}\right). \tag{18}$$

- For sparse vectors a with bounded $||a||_0$, an approximate $(1-\alpha)100\%$ confidence interval is

$$\left|a^T \hat{\boldsymbol{\beta}} - a^T \boldsymbol{\beta}\right| \le \hat{\sigma}\Phi^{-1}(1-\alpha/2)(a^T V a)^{1/2}, \tag{19}$$

where $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \ldots, \hat{\beta}_p)^T$ is the vector of LDPEs $\hat{\beta}_j$ in (4), Φ is the standard normal distribution function.

Section 3

Important theoretical results

Conditions

Let $\lambda_{univ} = \sqrt{(2/n)\log p}$. Suppose the model (1) holds with a vector β satisfying the following capped-ℓ_1 sparsity condition:

$$\sum_{j=1}^{p} \min\left\{|\beta_j|/(\sigma\lambda_{univ}), 1\right\} \le s. \tag{20}$$

This condition holds if β is ℓ_0 sparse with $\|\beta\|_0 \le s$ or ℓ_q sparse with $\|\beta\|_q^q/(\sigma\lambda_{univ})^q \le s$, $0 < q \le 1$. Let $\sigma^* = \|\varepsilon\|_2/\sqrt{n}$. A generic condition we impose on the initial estimator is

$$P\left\{\|\hat{\beta}^{(init)} - \beta\|_1 \ge C_1 s\sigma^* \sqrt{(2/n)\log(p/\epsilon)}\right\} \le \epsilon \tag{21}$$

for a certain fixed constant C_1 and all $\alpha_0/p^2 \le \epsilon \le 1$, where $\alpha_0 \in (0,1)$ is a preassigned constant. We also impose a similar generic condition on an estimator $\hat{\sigma}$ for the noise level:

$$P\left\{|\hat{\sigma}/\sigma^* - 1| \ge C_2 s(2/n)\log(p/\epsilon)\right\} \le \epsilon, \forall \alpha_0/p^2 \le \epsilon \le 1, \tag{22}$$

with a fixed C_2.

Theorem 1

Let $\hat{\beta}_j$ be the LDPE with an initial estimator $\hat{\boldsymbol{\beta}}^{(init)}$. Let η_j and τ_j be the bias and noise factors in (7), $\sigma^* = \|\varepsilon\|_2/\sqrt{n}$, $\max(\epsilon'_n, \epsilon''_n) \to 0$, and $\eta^* > 0$. Suppose (20) holds with $\eta^* C_1 s\sqrt{(2/n)\log(p/\epsilon)} \le \epsilon'_n$. If $\eta_j \le \eta^*$, then

$$P\left\{\left|\tau_j^{-1}(\hat{\beta}_j - \beta_j) - \mathrm{z}_j^T\varepsilon/\|\mathrm{z}_j\|_2\right| > \sigma^*\epsilon'_n\right\} \le \epsilon. \tag{23}$$

If in addition (21) holds with $C_2 s(2/n)\log(p/\epsilon) \le \epsilon''_n$, then for all $t \ge (1+\epsilon'_n)/(1-\epsilon''_n)$,

$$P\left\{|\hat{\beta}_j - \beta_j| \ge \tau_j\hat{\sigma}t\right\} \le 2\Phi_{n-1}(-(1-\epsilon''_n)t + \epsilon'_n) + 2\epsilon, \tag{24}$$

where $\Phi_n(t)$ is the student-t distribution function with n degrees of freedom. Moreover, for the covariance matrix V in and all fixed m,

$$\lim_{n\to\infty} \inf_{\mathrm{a}\in\mathscr{A}_{n,p,m}} P\left\{\left|\mathrm{a}^T\hat{\boldsymbol{\beta}} - \mathrm{a}^T\boldsymbol{\beta}\right| \le \hat{\sigma}\Phi^{-1}(1-\alpha/2)(\mathrm{a}^T\mathrm{V}\mathrm{a})^{1/2}\right\} = 1-\alpha, \tag{25}$$

where $\Phi(t) = P\{N(0,1) \le t\}$ and $\mathscr{A}_{n,p,m} = \{\mathrm{a} : \|\mathrm{a}\|_0 \le m, \max_{j\le p}|a_j|\eta_j \le \eta^*\}$.

Remark 1

- Condition (22) establishes the joint asymptotic normality of the LDPE under condition (20) This allows us to write the LDPE as an approximate Gaussian sequence.

$$\hat{\beta}_j = \beta_j + N(0, \tau_j^2\sigma^2) + o_P(\tau_j\sigma). \tag{26}$$

- Condition (23) and (24) justify the approximate coverage probability of the resulting confidence interval.
- The uniform signal strength condition is not required for condition (20) and (21).

Simultaneous confidence interval

Theorem 2

Suppose (20) holds with $\eta^* C_1 s\sqrt{(2/n)\log(p/\epsilon)} \le \epsilon'_n$. Then,

$$P\left\{\max_{\eta_j \le \eta^*} \left|\tau_j^{-1}(\hat{\beta}_j - \beta_j) - \mathrm{z}_j^T \epsilon / \|\mathrm{z}_j\|_2\right| > \sigma^* \epsilon'_n\right\} \le \epsilon. \qquad (27)$$

If (21) also holds with $C_2 s(2/n)\log(p/\epsilon) \le \epsilon''_n$, then

$$P\left\{\max_{\eta_j \le \eta^*} |\hat{\beta}_j - \beta_j|/(\tau_j \hat{\sigma}) > t\right\} \le 2\Phi_n(-(1-\epsilon''_n)t + \epsilon'_n)\#\{j : \eta_j \le \eta^*\} + 2\epsilon. \qquad (28)$$

If, in addition to (20) and (21), $\max_{j \le p} \eta_j \le \eta^*$ and $\max(\epsilon'_n, \epsilon) \to 0$ as $\min(n,p) \to \infty$, then for fixed $\alpha \in (0,1)$ and $c_0 > 0$,

$$\liminf_{n\to\infty} P\left\{\max_{j \le p}\left|\frac{\hat{\beta}_j - \beta_j}{\tau_j(\hat{\sigma} \wedge \sigma)}\right| \le c_0 + \sqrt{2\log(p/\alpha)}\right\} \ge 1 - \alpha. \qquad (29)$$

Thresholded LDPE

- From (25), the $\hat{\beta}_j$ can be viewed as an approximate Gaussian sequence.
- The approximate Gaussian sequence is not sparse but can be thresholded. Using either the hard or the soft thresholding method:

$$\hat{\beta}_j^{(thr)} = \begin{cases} \hat{\beta}_j I(|\hat{\beta}_j| > \hat{t}_j), \\ sgn(\hat{\beta}_j)(|\hat{\beta}_j| - \hat{t}_j)^+, \end{cases} \tag{30}$$

with

$$\hat{S}^{(thr)} = \{j : |\hat{\beta}_j| > \hat{t}_j\}$$

where $\hat{t}_j \approx \hat{\sigma}\tau_j \Phi^{-1}(1 - \alpha/(2p))$ with $\alpha > 0$.

Theorem 3

Let $L_0 = \Phi^{-1}(1-\alpha/(2p))$, $\tilde{t}_j = \tau_j \sigma L_0$, and $\hat{t}_j = (1+c_n)\hat{\sigma}\tau_j L_0$ with positive constants α and c_n. Suppose condition (queshao) holds with $\eta^* C_1 s/\sqrt{n} \le \epsilon_n'$, $\max_{j\le p}\eta_j \le \eta^*$, and

$$P\left\{ \frac{(\hat{\sigma}/\sigma)\vee(\sigma/\hat{\sigma}) - 1 + \epsilon_n'\sigma^*/(\hat{\sigma}\wedge\sigma)}{1-(\hat{\sigma}/\sigma-1)_+} > c_n \right\} \le 2\epsilon. \tag{31}$$

Let $\boldsymbol{\beta}^{(thr)} = (\beta_1^{(thr)}, \ldots, \beta_p^{(thr)})^T$ be the soft thresholded LDPE with these $\hat{t}_j$. Then, there is an event Ω_n with $P\{\Omega_n^c\} \le 3\epsilon$ such where $L_n = 4/L_0^3 + 4c_n/L_0 + 12c_n^2 L_0$. Moreover, with at least probability $1-\alpha-3\epsilon$,

$$\{j : |\beta_j| > (2+2c_n)\tilde{t}_j\} \subseteq \hat{S}^{(thr)} \subseteq \{j : \beta_j \neq 0\}. \tag{32}$$

Section 4

Simulation

Setting

Set $n = 200$, $p = 3000$, and run several simulation experiments with 100 replications in each setting.

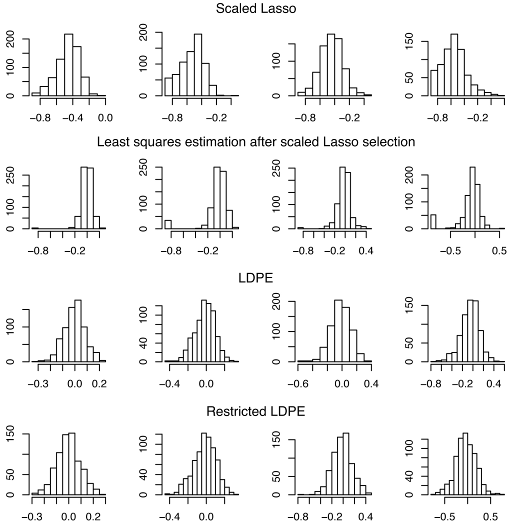
- Generate independent copy of (X, y):
  Given a particular $\rho \in (-1, 1)$, $\tilde{X} = (\tilde{x}_{ij})_{n \times p}$ has iid $N(0, \Sigma)$ rows with $\Sigma = (\rho^{|j-k|})_{p \times p}$, $x_j = \tilde{x}_j \sqrt{n}/|\tilde{x}_j|_2$, and (X, y) is as in (1) with $\sigma = 1$.
- Generate $\boldsymbol{\beta}$:
  Given a particular $\alpha \geq 1$, $\beta_j = 3\lambda_{univ}$ for $j = 1500, 1800, 2100, \ldots, 3000$, and $\beta_j = 3\lambda_{univ}/j^{\alpha}$ for all other j, where $\lambda_{univ} = \sqrt{(2/n) \log p}$.
- Set α and ρ:
  This simulation example includes four cases, labeled (A), (B), (C), and (D), respectively: $(\alpha, \rho) = (2, \frac{1}{5})$, $(1, \frac{1}{5})$, $(2, \frac{4}{5})$, and $(1, \frac{4}{5})$.

Comparison between different methods

Table 3. Summary statistics for various estimates of the maximal $\beta_j = |\beta|_\infty$: the lasso, the scaled lasso, the scaled lasso–LSE method, the oracle estimator, the LDPE and the RLDPE

Setting	*Statistic*	*Lasso*	*Scaled lasso*	*Scaled lasso–LSE*	*Oracle*	*LDPE*	*RLDPE*
A	Bias	−0.2965	−0.4605	−0.0064	−0.0045	−0.0038	−0.0028
	Standard deviation	0.0936	0.1360	0.1004	0.0730	0.0860	0.0960
	Median absolute error	0.2948	0.4519	0.0549	0.0507	0.0531	0.0627
B	Bias	−0.2998	−0.5341	−0.0476	0.0049	−0.0160	−0.0167
	Standard deviation	0.1082	0.1590	0.2032	0.0722	0.1111	0.1213
	Median absolute error	0.2994	0.5150	0.0693	0.0500	0.0705	0.0799
C	Bias	−0.3007	−0.4423	−0.0266	−0.0049	−0.0194	−0.0181
	Standard deviation	0.1207	0.1520	0.1338	0.1485	0.1358	0.1750
	Median absolute error	0.3000	0.4356	0.0657	0.0994	0.0902	0.1150
D	Bias	−0.3258	−0.5548	−0.1074	−0.0007	−0.0510	−0.0405
	Standard deviation	0.1367	0.1844	0.2442	0.1455	0.1768	0.2198
	Median absolute error	0.3319	0.5620	0.0857	0.0955	0.1112	0.1411

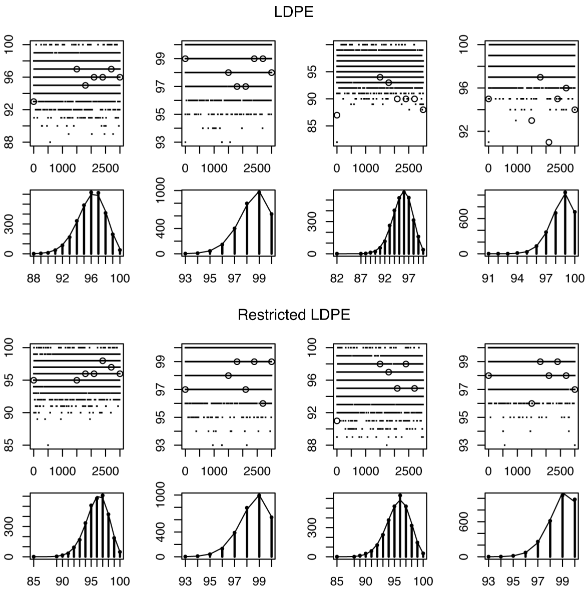
Histogram of error



Result of mean coverage probability

Figure 2: Mean coverage probability of LDPE and R-LDPE.

		A	B	C	D
all β_j	LDPE	0.9597	0.9845	0.9556	0.9855
	R-LDPE	0.9595	0.9848	0.9557	0.9885
maximal β_j	LDPE	0.9571	0.9814	0.9029	0.9443
	R-LDPE	0.9614	0.9786	0.9414	0.9786


LDPE
Restricted LDPE

Simulation 1: Summary statistics for different estimators

We set $p = 300$, $n = 50$, replication$= 100$, let Σ, $\alpha = 2$, $\rho = 1/5$ the same as Setting A.

Figure 3: Summary statistics for different estimators of the maximal $\beta_j = \|\beta\|_\infty$

	Lasso	Scaled lasso	LDPE
Bias	2.070	2.110	0.580
Standard deviation	0.046	0.042	0.181

Simulation 2: Mean coverage probability

We set $p = 300$, $n = 50$, replication$= 100$. Generating design matrix X from different Σ to compare the mean coverage probability by LDPE.

- Setting A: $\Sigma = (\rho^{|j-k|})_{p\times p}, \rho = 0.2$
- Setting B: $\Sigma = (\rho^{|j-k|})_{p\times p}, \rho = 0.8$
- Setting C: $\Sigma = (\rho)_{p\times p}, j \neq k, \rho = 0.2$
- Setting D: $\Sigma = (\rho)_{p\times p}, j \neq k, \rho = 0.8$

Figure 4: Mean coverage probability of LDPE.

	A	B	C	D
all β_j	0.919	0.934	0.910	0.913
maximal β_j	0.940	0.980	0.950	0.960

Simulation 3: Choosing initial estimator of β by ℓ_0 and scaled lasso

We set $p = 300$, $n = 50$, replication=100. $\beta_j = 3\lambda_{univ}$ for $j = 15, 18, \ldots, 30$, and $\beta_j = 3\lambda_{univ}/j^\alpha$ for other j.

- Setting A: $\Sigma = (\rho^{|j-k|})_{p\times p}$, $\rho = 0.2$
- Setting B: $\Sigma = (\rho^{|j-k|})_{p\times p}$, $\rho = 0.6$

Figure 5: Mean coverage probability of ℓ_0 and Scaled lasso.

		A	B
all β_j	ℓ_0	0.915	0.942
	Scaled lasso	0.907	0.953
maximal β_j	ℓ_0	0.930	0.980
	Scaled lasso	0.920	0.970

Summary of LDPE

Advantages

- Construct confidence intervals for regression coefficients
- Without requirement of the uniform signal strength condition
- Handle the situation with high-corelated design matrix.

Thank you !

Discussion

- Why lasso to obtain z_j? What about other methods?
- Conditions.
- Limits of Thresholded-LDPE in variable selection.