Homework 1

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Problem 1 *Let $X_1, \ldots, X_n \sim F$ and let $F_n(x)$ be the empirical distribution function, for a fixed x, find limiting distribution of $\sqrt{F_n(x)}$.*

Solution

Let $T(F) = \sqrt{F}$, we know that T is Hadamard differentiable with respect to $G = \delta_x$. Then

$$\frac{\sqrt{n}(T(F_n) - T(F))}{\hat{\tau}} \rightsquigarrow N(0,1)$$

where $\hat{\tau}^2 = \frac{1}{n}\sum_{i=1}^{n} \hat{L}^2(X_i)$ and

$$\frac{d}{dt}T(F_t)\Big|_{t=0} = \frac{d}{dt}\sqrt{F_t} = \frac{d}{dt}\sqrt{(1-t)F + t\delta_x} = -\sqrt{F}.$$

So, $\hat{\tau} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} F_n(X_i)}$.

Thus, $T(F_n) \rightsquigarrow N(\sqrt{F}, \frac{1}{n^2}\sum_{i=1}^{n} F_n(X_i))$

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Problem 2 *Let x, y be two distinct real numbers, find $Cov(F_n(x), F_n(y))$, where F_n be the empirical distribution function.*

Problem 3 *Let $X_{(1)} \leq \cdots \leq X_{(n)}$ be order statistics from continuous popolation F, prove that for any $0 < \beta < 1$*

$$P(F(X_{(n)}) - F(X_{(1)}) > \beta) = 1 - n\beta^{n-1} + (n-1)\beta^n.$$

Problem 4 *Let $X_1, \ldots, X_n$ be simple samples from $U(0,1)$, prove that sample median $\hat{\xi}_{n,1/2}$ has asymptotic distribution $N(\frac{1}{2}, \frac{1}{4n})$.*