Homework 13

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Problem 1 (HW 13.1) *Assume that bandwidth selection method in N-W estimator $\hat{m}$ without leave-one-out method is given by*

$$h_0 = \arg\min \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{m}(X_i))^2.$$

Prove $h_0 = 0$.

Solution

N-W estimator at x is given by

$$\hat{m}(x) = \frac{n^{-1}\sum_{i=1}^{n}\mathcal{K}_h\left(x - X_i\right)Y_i}{n^{-1}\sum_{i=1}^{n}\mathcal{K}_h\left(x - X_i\right)}.$$

We can rewrite the N-W estimator

$$\begin{aligned}
\hat{m}(x) &= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathcal{K}_h\left(x - X_i\right)}{n^{-1}\sum_{i=1}^{n}\mathcal{K}_h\left(x - X_i\right)}\right)Y_i \\
&= \frac{1}{n}\sum_{i=1}^{n}W_{hi}(x)Y_i.
\end{aligned}$$

$$\begin{aligned}
h_0 &= \arg\min \frac{1}{n}\sum_{i=1}^{n}\left((Y_i - \hat{m}(X_i)\right)^2 \\
&= \arg\min \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \frac{1}{n}\sum_{j=1}^{n}W_{hj}(x_i)Y_j\right)^2
\end{aligned}$$

Notice that $\left(Y_i - \frac{1}{n}\sum_{j=1}^{n}W_{hj}(x_i)Y_j\right)^2$ reaches its mininal 0, if $Y_i = \frac{1}{n}\sum_{j=1}^{n}W_{hj}(x_i)Y_j$. And the equation holds, as long as $h = 0$ for every $i = 1, 2, \ldots, n$.

Thus $h_0 = 0$. ∎

Problem 2 (HW 13.2) *Prove that N-W estimator $\hat{m}(x)$ with bandwidth selected by leave-one-out method satisfies*

$$CV(h) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{m}_{-i}(X_i))^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i - \hat{m}(X_i)}{1 - W_i(X_i)}\right)^2,$$

where $W_i(x) = \mathcal{K}_h(X_i - x)/\sum_{j=1}^{n}\mathcal{K}_h(X_j - x)$.

Solution

We have N-W estimator $\hat{m}_{-i}(X_i)$ at X_i with bandwidth selected by leave-one-out method as

$$\begin{aligned}
\hat{m}_{-i}(X_i) &= \frac{\sum_{j=1,j\neq i}^{n} \mathcal{K}_h(X_i - X_j)\, Y_j}{\sum_{j=1,j\neq i}^{n} \mathcal{K}_h(X_i - X_j)} \\
&= \frac{\sum_{j=1,j\neq i}^{n} \mathcal{K}_h(X_i - X_j)\, Y_j}{\sum_{j=1,j\neq i}^{n} \mathcal{K}_h(X_i - X_j) + \mathcal{K}(0) - \mathcal{K}(0)} \\
&= \frac{\sum_{j=1,j\neq i}^{n} \mathcal{K}_h(X_i - X_j)\, Y_j}{\sum_{j=1}^{n} \mathcal{K}_h(X_i - X_j) - \mathcal{K}(0)} \\
&= \frac{\sum_{j=1,j\neq i}^{n} W_i(X_j) Y_j}{1 - W_i(X_i)}.
\end{aligned}$$

The last equation is proved by simultaneously dividing numerator and denominator by $\sum_{j=1}^{n} \mathcal{K}_h(X_j - x)$.

Then we can rewrite the $CV(h)$ by replacing $\hat{m}_{-i}(X_i)$ with $\frac{\sum_{j=1,j\neq i}^{n} W_i(X_j)Y_j}{1-W_i(X_i)}$ as following.

$$\begin{aligned}
CV(h) &= \frac{1}{n}\sum_{i=1}^{n} (Y_i - \hat{m}_{-i}(X_i))^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(Y_i - \frac{\sum_{j\neq i} W_i(X_j) Y_j}{1 - W_i(X_i)} \right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{(1 - W_i(X_i))Y_i}{1 - W_i(X_i)} - \frac{\sum_{j\neq i} W_i(X_j) Y_j}{1 - W_i(X_i)} \right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{Y_i - W_i(X_i)Y_i - \sum_{j\neq i} W_i(X_j) Y_j}{1 - W_i(X_i)} \right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\frac{Y_i - \hat{m}(X_i}{1 - W_i(X_i)} \right)^2.
\end{aligned}$$

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