Homework 18

Liu Huihang
SA18017026
QQ: 184050473
MAIL: huihang@mail.ustc.edu.cn

Problem 1 (18.1) *Find the the asymptotic distributions of the local linear estimators (36) and (37) in Slide 18.*

Solution

Equation (36):

$$\hat{\beta}_{ll} = \left[(X - SX)'(X - SX)\right]^{-1} (X - SX)'(Y - SY) \tag{1}$$

Equation (37):

$$\hat{\alpha}_{ll}(z) = \alpha_{\hat{\beta}}(z) = S(z)(Y - X\hat{\beta}_{ll}) \tag{2}$$

A semiparametric partially linear model is given by

$$Y_i = X_i'\beta_0 + g(Z_i) + \mu_i, \quad i = 1, \ldots, n.$$

Equation (36) and (37) can be achieved by using local linear approximation to improve the result of local constant approximation with smoother operator

$$S(z) = \left[\tilde{Z}_z' W_z \tilde{Z}_z\right]^{-1} \tilde{Z}_z' W_z,$$

and

$$\tilde{Z}_z = \begin{pmatrix} 1 & (Z_1 - z)' \\ 1 & (Z_2 - z)' \\ & \vdots \\ 1 & (Z_n - z)' \end{pmatrix}$$

and $W_z = diag\left(K_h(Z_1 - z), K_h(Z_2 - z), \ldots, K_h(Z_n - z)\right)$.

$$\begin{aligned}
\hat{\beta}_{ll} - \beta_0 &= \left[(X - SX)'(X - SX)\right]^{-1} (X - SX)'[Y - SY - (X - SX)\beta_0] \\
&= \left[X'(I - S)'(I - S)X\right]^{-1} X'(I - S)'(I - S)(Y - X\beta_0) \\
&= \left[X'(I - S)'(I - S)X\right]^{-1} X'(I - S)'(I - S)\left[g(Z) + \mu\right]
\end{aligned}$$

Let $A = (I - S)'(I - S)$, then

$$\begin{aligned}
\hat{\beta}_{ll} - \beta_0 &= \left[X'AX\right]^{-1} X'A\left[g(Z) + \mu\right] \\
&= \underbrace{\left[X'AX\right]^{-1} X'Ag(Z)}_{Bias} + \underbrace{\left[X'AX\right]^{-1} X'A\mu}_{Variance}
\end{aligned}$$

$$E\left[\hat{\beta}_{ll} - \beta_0\right] = \left[X'AX\right]^{-1} X'Ag(Z) \triangleq Bias$$

and

$$Var\left[\hat{\beta}_{ll} - \beta_0\right] = \left[X'AX\right]^{-1} X'A\Sigma A'X \left[X'AX\right]^{-1} \triangleq Var$$

where $\Sigma = \mathrm{diag}(\sigma^2(X_1, Z_1), \ldots, \sigma^2(X_n, Z_n))$.

Thus we have

$$\hat{\beta}_{ll} - \beta_0 \rightsquigarrow N\left(Bias, Var\right).$$

According to $\alpha_0(z) = S(z)\left(Y - X\beta_0\right)$, we have

$$\begin{aligned}\hat{\alpha}_{ll}(z) - \alpha_0(z) &= -S(z)(\hat{\beta}_{ll} - \beta_0) \\ &\rightsquigarrow N\left(-S(z)Bias, S(z)VarS(z)'\right).\end{aligned}$$

■