Ph21 Assignment 2

Yuchen Tang

Part 1

1. The definition of the fourier transform is consistent

   Insert Eq.(3) into Eq.(2):

$$h(x) = \sum_{k=-\infty}^{\infty} \tilde{h_k} e^{-2\pi i f_k x},\ f_k = \frac{k}{L}$$

$$\begin{aligned} h(x) &= \sum_{k=-\infty}^{\infty} (\frac{1}{L}\int_0^L h(x) e^{2\pi i f_k x} dx) e^{-2\pi i f_k x} \\ &= \sum_{k=-\infty}^{\infty} (\frac{1}{L}\int_0^L (\sum_{j=-\infty}^{\infty} \tilde{h_j} e^{-2\pi i f_j x}) e^{2\pi i f_k x} dx) e^{-2\pi i f_k x} \end{aligned}$$

   Since $\tilde{h_j}$ is not dependent on x, we can treat it as a constant in the integral. We have

$$\int_0^L e^{-2\pi i f_k x} e^{2\pi i f_j x} dx = \begin{cases} 0, & \text{if } k \neq j \\ L, & \text{if } k = j \end{cases} \tag{1}$$

$$h(x) = \sum_{k=-\infty}^{\infty} \frac{1}{L} L \tilde{h_k} e^{-2\pi i f_k x} = \sum_{k=-\infty}^{\infty} \tilde{h_k} e^{-2\pi i f_k x}$$

2. Linear combination for sine function

$$\begin{aligned} A\sin(2\pi x/L + \phi) &= \frac{A}{2i}(e^{2\pi i x + \phi} - e^{-2\pi i x - \phi}) \\ &= \frac{A}{2i} e^{\phi} e^{2\pi i x} - \frac{A}{2i} e^{-\phi} e^{-2\pi i x} \end{aligned}$$

3. $\tilde{h}_{-k} = \tilde{h}_k^*$

   By the definition of $\tilde{h}_k$:

$$\tilde{h}_{-k} = \frac{1}{L}\int_0^L h(x) e^{2\pi i(-k)x/L} dx = \frac{1}{L}\int_0^L h(x) e^{-2\pi i k x/L} dx$$

We assume that the $x \in \mathbb{R}$. Find the complex conjugate $\tilde{h}_k^*$:

$$\begin{aligned}
\tilde{h}_k^* &= \overline{\frac{1}{L}\int_0^L h(x)e^{2\pi ikx/L}dx} \\
&= \overline{\frac{1}{L}\int_0^L h(x)(\cos(2\pi kx/L) + i\sin(2\pi kx/L))dx} \\
&= \frac{1}{L}\int_0^L h(x)\overline{(\cos(2\pi kx/L) + i\sin(2\pi kx/L))}dx \\
&= \frac{1}{L}\int_0^L h(x)(\cos(2\pi kx/L) - i\sin(2\pi kx/L))dx \\
&= \frac{1}{L}\int_0^L h(x)e^{-2\pi ikx/L}dx \\
&= \tilde{h}_{-k}
\end{aligned}$$

4. Convolution theorem

$$\begin{aligned}
H(x) &= \sum_{k=-\infty}^{\infty} \tilde{H}_k e^{-2\pi i f_k x} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{k'=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \tilde{h}_{k'}^{(2)} \right) e^{-2\pi i f_k x} \\
&= \sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} \sum_{k=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} e^{-2\pi ikx/L} \\
&= \sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} \sum_{k=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} e^{-2\pi i(k-k')x/L} e^{-2\pi ik'x/L}
\end{aligned}$$

Since we are summing from $k = -\infty$ to $k = \infty$, we get the same result by subtracting k'.

$$\begin{aligned}
&= \sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} e^{-2\pi ik'x/L} \sum_{k=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} e^{-2\pi i(k-k')x/L} \\
&= h(x)^{(1)} * h(x)^{(2)}
\end{aligned}$$

Thus, $\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \tilde{h}_{k'}^{(2)}$.

5. Fourier transform of the cosine and gaussian functions
Fourier transform of $f(t) = A\cos(f_k t + \phi) + C = A\cos(\frac{2\pi nt}{L} + \phi) + C$:
$f(t)$ is periodic: $f(0) = f(L), t \in [0, L]$.

$$h(t) = A\cos(f_n t + \phi) + C = \sum_{k=-\infty}^{\infty} \tilde{h_k} e^{-2\pi i f_k x}$$

$$
\begin{aligned}
\tilde{h_k} &= \frac{1}{L}\int_0^L (A\cos(\frac{2\pi nt}{L}+\phi)+C)e^{\frac{2\pi ikt}{L}}dt \\
&= \frac{A}{2L}\int_0^L (e^{\frac{i2\pi nt}{L}+i\phi}+e^{-\frac{i2\pi nt}{L}-i\phi})e^{\frac{2\pi ikt}{L}}dt + \frac{C}{L}\int_0^L e^{\frac{2\pi ikt}{L}}dt \\
&= \frac{A}{2L}(e^{i\phi}\delta_{-nk}+e^{-i\phi}\delta_{nk})+C\delta_{0k}
\end{aligned}
$$

$$
\delta_{ij} = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j. \end{cases}
$$

Fourier transform of $f(t) = Ae^{-B(t-\frac{L}{2})^2}$

Normalized form of a gaussian function: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-a)^2}{2\sigma^2}}$

If B is large enough, $f(t)$ is periodic and $f(0) = f(L)$.

$$
h(t) = Ae^{-B(t-\frac{L}{2})^2} = \sum_{k=-\infty}^{\infty}\tilde{h_k}e^{-2\pi if_kx}
$$

For convenience, we shift the center of the gaussian peak to $x = 0$. Then $f(t)$ is periodic with $f(-\frac{L}{2}) = f(\frac{L}{2})$.

$$
h(t) = Ae^{-Bt^2} = \sum_{k=-\infty}^{\infty}\tilde{h_k}e^{-2\pi if_kx}
$$

$$
\begin{aligned}
\tilde{h_k} &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} Ae^{-Bt^2}e^{\frac{2\pi ikt}{L}}dt \\
&= \frac{A}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-B(t^2-\frac{2\pi ikt}{BL})}dt \\
&= \frac{A}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-B(t^2-2t\frac{i\pi k}{BL}+(\frac{i\pi k}{BL})^2-(\frac{i\pi k}{BL})^2)}dt \\
&= \frac{A}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-B(t-\frac{i\pi k}{BL})^2} * e^{-\frac{\pi^2k^2}{B^2L^2}}dt \\
&= \frac{A}{L}e^{-\frac{\pi^2k^2}{B^2L^2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-B(t-\frac{i\pi k}{BL})^2}dt
\end{aligned}
$$

In the complex plane, the gaussian function is shifted upwards (along the imaginary) axis by $\frac{\pi k}{BL}$ as shown in 1. We can integrate the function along the contour C. Since there are no singularities in the C, by Cauchy's integral theorem, the integral along C is 0. Since we require the B is large, the vertical parts of C is negligible and thus the top horizontal part is equal to the bottom horizontal part.

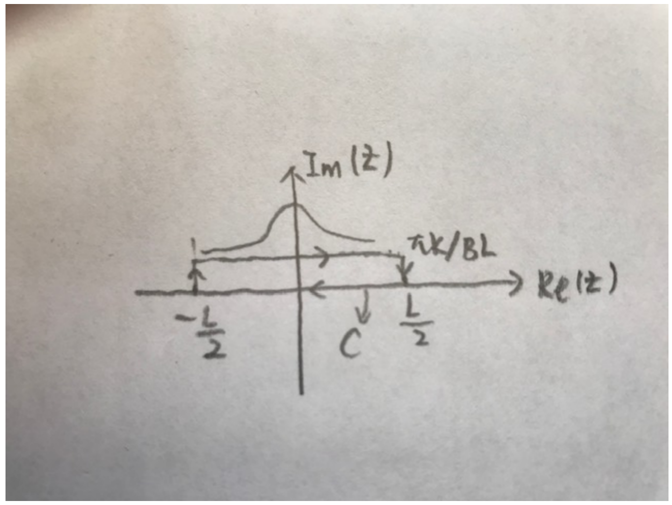

Figure 1: The gaussian function is shifted along the imaginary axis.

Assume that B is large and the gaussian function decays to a negligible value before reaching $t = -\frac{L}{2}$ and $t = \frac{L}{2}$.

$$\begin{aligned}\tilde{h_k} &= \frac{A}{L} e^{\frac{-\pi^2 k^2}{B^2 L^2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-Bt^2} dt \\ &= \frac{A}{L}\sqrt{\frac{\pi}{B}} e^{-\frac{\pi^2 k^2}{B^2 L^2}}\end{aligned}$$

The inverse ffts of the cosine and gaussian functions return the original functions. The maximum difference between inverse fft and the original function is much smaller than 1. The input cosine function is $10\cos\left(\frac{2\pi 1000t}{2\pi}\right)$. The amplitude of the cosine fft is ploted with respect to frequency. The peaks are at $f = 1000$ rad/s and $f = -1000\ rad/s = 9000\ rad/s$ as expected. The input gaussian function is $e^{-10000(t-2*\pi/2)^2}$. The gaussian fft is a gaussian peak centered at zero.

The fourier transforms give expected results as shown in Figure 2.

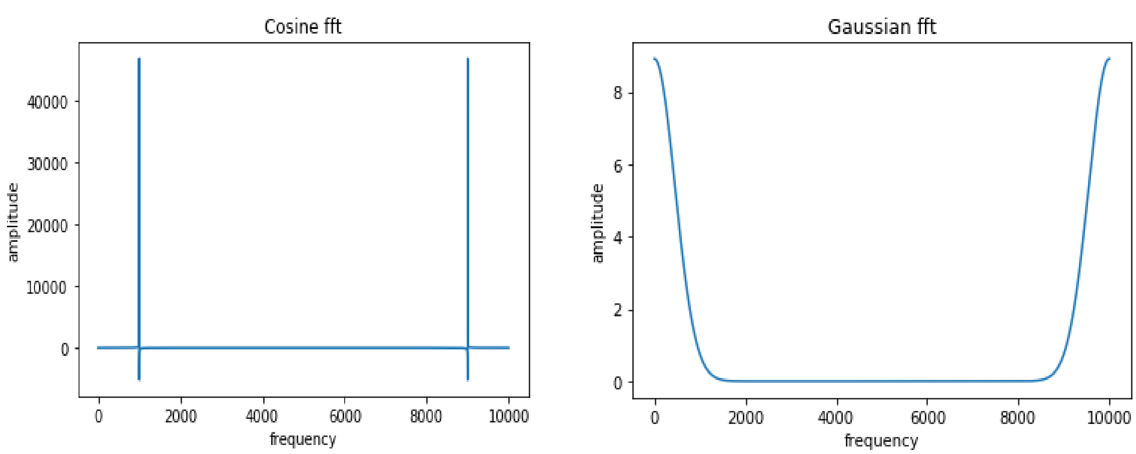

Figure 2: Plots of cosine and gaussian ffts.

Part 2

1. Plotting the function with respect to time reveals no obvious features as in Figure 3.

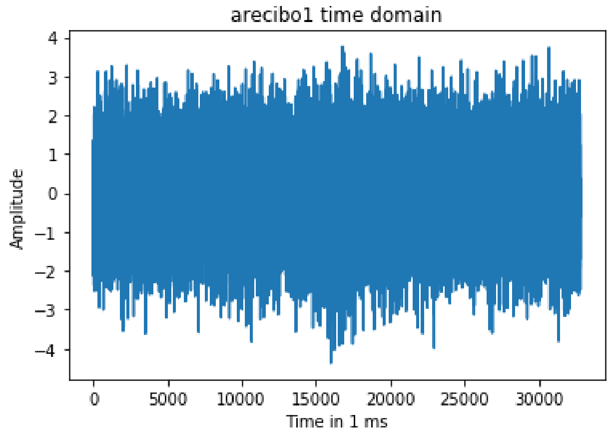

Figure 3: There is no obvious feature in the time domain of the arecibo1 data set.

After taking the fast fourier transform of the data set and plotting it with respect to frequency index, the peaks can be clearly seen (Figure 4). Since we assumed that our time domain function is proportional to $\cos(\frac{2\pi nt}{L}) * e^{-\frac{(t-L/2)^2}{\Delta t^2}}$, we found that $n = 4489, 28279$ at the peaks. So the frequency is $\frac{4489}{32768*10^{-3}s} = 137\text{Hz}$. The peak at $n = 28279$ can be seen as $n = 28279 - 32768 = -4489$.

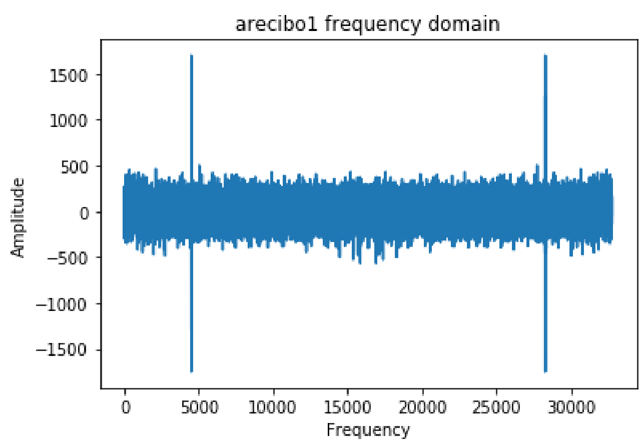

Figure 4: The indices of the two peaks are 4489 and 22279.

2. We modeled the data set as a product of an gaussian envelope and a cosine function. We zoomed in to the $n = 4489$ peak and fitted with various values of Δt and amplitude. The final fit result is in Figure 5. The fit function is $0.95 * \cos(\frac{2\pi 4489t}{32768}) * e^{-0.23*10^{-6}(t-32768/2)^2}$. So, $\Delta t = \sqrt{\frac{1}{0.23*10^{-6}}} = 2085.64$ ms.

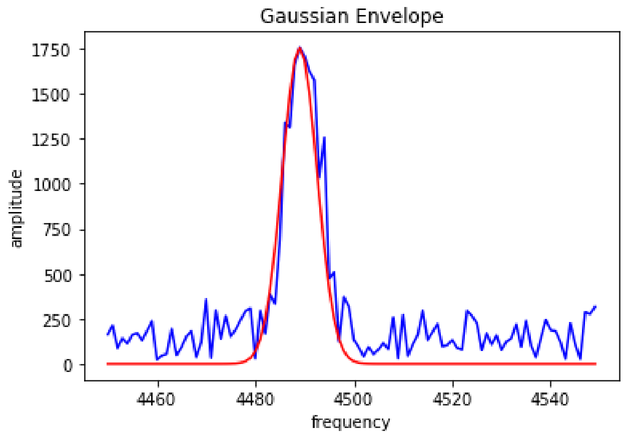

Figure 5: The original signal is in blue. The fit function is in red.

Part 3

1. We used the Lomb-Scargle algorithm from scipy.signal.
2. We used the Lomb-Scargle algorithm to transform the gaussian function in part 1.5. The frequency in Figure 6 is the same as what we found in part 1.5.

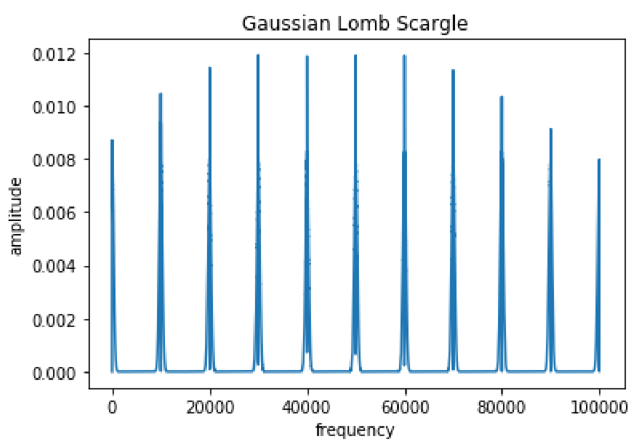

Figure 6: The gaussian distribution is analyzed with Lomb-Scargle algorithm.

The Lomb-Scargle algorithm is used to analyze the arecibo1 data set. The fourier transform plot is in Figure 7. The peak is found at about the same frequency.

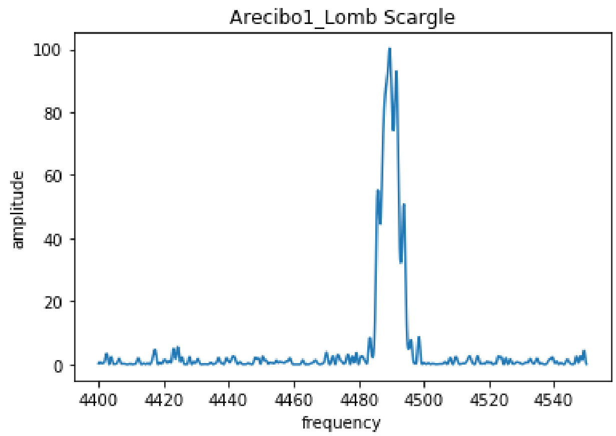

Figure 7: Plot of the transform of the arecibo1 data set.

3. We analyzed the Her X-1 data set from assignment 1 with Lomb-Scargle. The highest peak is at 3.696 rad/day. The period of Her X-1 is $\frac{2\pi}{3.696} = 1.7$ days as expected.

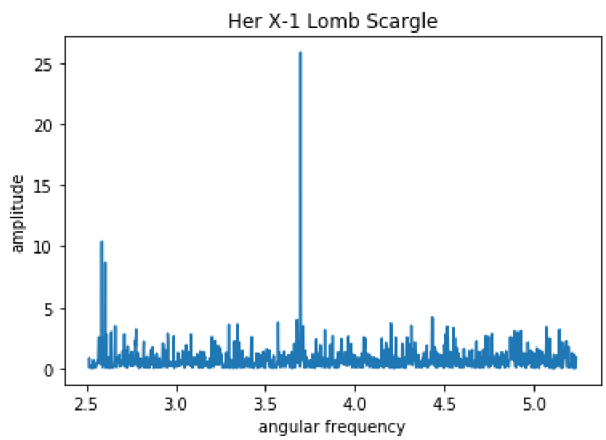

Figure 8: There is a peak at 3.696 rad/day.

There might be significant frequencies from nearby bodies.