Chapter 13

INVERSE TRIGONOMETRIC FUNCTIONS

Introduction

We know that only a one–to–one functions will have an inverse. If a function is not one–to–one, it may be possible to restrict its domain to make it one–to–one so that its inverse can be found.

Trigonometric functions are not objective. They are periodic functions. We can find inverses of trigonometric function by restricting their domains.

Inverse trigonometric function lose its periodicity. Restricted domain trigonometric functions are called principal functions.

Inverse sine Function

is defined by

$y = \sin^{-1} x$ iff $x = \sin y$

where $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ and $-1 \le x \le 1$

Inverse cosine Function

is defined by

$y = \cos^{-1} x$ iff $x = \cos y$

where $0 \le y \le \pi$ and $-1 \le x \le 1$

Inverse tangent Function

is defined by

$y = \tan^{-1} x$ iff $x = \tan y$

where $\frac{-\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x + \infty$

Inverse cotagent Function

is defined by

$y = \cot^{-1} x$ iff $x = \cot y$

where $0 < y < \pi$ and $-\infty < x < \infty$

Inverse secant Function

is defined by

$y = \sec^{-1} x$ iff $x = \sec y$

where $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$ and $|x| \ge 1$

Inverse cosecant Function

is defined by

$y = \operatorname{cosec}^{-1} x$ iff $x = \operatorname{cosec} y$

where $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $|x| \ge 1$

EXERCISE 13.1

Q.1 Evaluate without using tables/calculator:

(i) $\sin^{-1}(1)$ (ii) $\sin^{-1}(-1)$ (iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (v) $\cos^{-1}\left(\frac{1}{2}\right)$ (vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(vii) $\cot^{-1}(-1)$ (viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ (ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

(i) $\sin^{-1}(1)$

We have to find angle whose sine is 1.

Let that angle be y

Then $y = \sin^{-1}(1)$, $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow$ $\sin y = 1 \Rightarrow y = \frac{\pi}{2}$

Thus $\sin^{-1}(1) = \frac{\pi}{2}$ Ans.

(ii) $\sin^{-1}(-1)$

We have to find angle whose sine is -1.

Let us say that angle is y.

Then $y = \sin^{-1}(-1)$, $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow$ $\sin y = -1 \Rightarrow y = \frac{-\pi}{2}$

Thus $\sin^{-1}(-1) = \frac{-\pi}{2}$ Ans.

(iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

We have to find angle whose cosine is $\frac{\sqrt{3}}{2}$

Let that angle be y

Then $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, $y \in [0, \pi]$

$\Rightarrow$ $\cos y = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $y = \frac{\pi}{6}$ thus $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ Ans.

(iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

We have to find angle whose tangent is $\frac{-1}{\sqrt{3}}$

Let that angle be y

Then $y = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\tan y = \frac{-1}{\sqrt{3}}$

$y = -\frac{\pi}{6}$ thus $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ Ans.

(v) $\cos^{-1}\left(\frac{1}{2}\right)$

We have to find angle whose cosine is $\frac{1}{2}$

Let that angle by y

Then $y = \cos^{-1}\left(\frac{1}{2}\right), \quad y \in [0, \pi]$

$\Rightarrow \quad \cos y = \frac{1}{2}$

$\Rightarrow \quad y = \frac{\pi}{3}$ thus $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ Ans.

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

We have to find angle whose tangent is $\frac{1}{\sqrt{3}}$

Let that angle be y

Then $y = \tan^{-1}\frac{1}{\sqrt{3}}, \; y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$\Rightarrow \quad \tan y = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad y = \frac{\pi}{6}$ thus $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ Ans.

(vii) $\cot^{-1}(-1)$

We have to find angle whose cotangent is -1.

Let that angle be -1.

Then $y = \cot^{-1}(-1)$

$\Rightarrow \quad \cot y = -1, \; y \in [0, \pi]$

$\Rightarrow \quad \frac{1}{\tan y} = -1$

$\Rightarrow \quad \tan y = -1$

$y = \frac{3\pi}{4}$

Thus, $\cot^{-1}(-1) = \frac{3\pi}{4}$

Alternative Solution:

$\Rightarrow \quad \cot y = -1, \; y \in [0, \pi]$

$\Rightarrow \quad y = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Thus, $\cot^{-1}(-1) = \frac{3\pi}{4}$ Ans.

(viii) $\text{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

We have to find angle whose cosecant is $\frac{-2}{\sqrt{3}}$

Let that angle be y

$y = \text{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$

$\Rightarrow \quad \text{cosec}\, y = \frac{-2}{\sqrt{3}}$

$\Rightarrow \quad \text{cosec}\, y = \frac{-2}{\sqrt{3}}$

$\Rightarrow \quad \frac{1}{\sin y} = \frac{-2}{\sqrt{3}}$

$\Rightarrow \quad \sin y = \frac{-\sqrt{3}}{2}; \; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad y = \frac{-\pi}{3}$ thus $\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \frac{-\pi}{3}$ Ans.

(ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

We have to find an angle whose sine is $-\frac{1}{\sqrt{2}}$

Let that angle be y

$y = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right); \; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \sin y = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad y = -\frac{\pi}{4}$ thus $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ Ans.

Q.2 Without using table/calculator show that

(i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

(ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$ **(Gujranwala Board 2005)**

(iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

Solution:

(i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

$\Rightarrow \quad \sin\left[\tan^{-1}\frac{5}{12}\right] = \frac{5}{13}$ .... (1)

Let $\tan^{-1}\frac{5}{12} = \theta$

$\Rightarrow \quad \tan\theta = \frac{5}{12}$ $\qquad \Rightarrow \quad \cot\theta = \frac{12}{5}$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{25}{144}} = \sqrt{\frac{144+25}{144}} = \sqrt{\frac{169}{144}}$$

$$\text{cosec}\,\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\frac{144}{25}} = \sqrt{\frac{25+144}{25}} = \sqrt{\frac{169}{25}}$$

$\sec\theta = \frac{13}{12} \Rightarrow \cos\theta = \frac{12}{13}$ $\qquad \text{cosec}\,\theta = \frac{13}{5} \Rightarrow \sin\theta = \frac{5}{13}$

Equation (1) becomes

$\sin\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{13}$

(ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

$\sin\left(2\cos^{-1}\frac{4}{5}\right) = \frac{24}{25}$ .... (1)

Let $\cos^{-1}\frac{4}{5} = \theta$

Equation (1) becomes

$\sin 2\theta = \frac{24}{25} \Rightarrow 2\sin\theta\cos\theta = \frac{24}{25}$ .... (2)

$\cos^{-1}\frac{4}{5} = \theta$

$\Rightarrow \quad \cos\theta = \frac{4}{5}$

$\Rightarrow \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}}$

$\sin\theta = \frac{3}{5}$

Put in equation (2)

$2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

$\frac{24}{25} = \frac{24}{25}$

L.H.S. = R.H.S. Hence proved

(iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

$\cot\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{3}$ .... (1)

Let $\cos^{-1}\frac{4}{5} = \theta$ Equation (1) becomes

$\cot\theta = \frac{4}{3}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4}{3}$ .... (2)

Now,

$\cos^{-1}\frac{4}{5} = \theta \Rightarrow \cos\theta = \frac{4}{5}$

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}}$

$\sin\theta = \frac{3}{5}$

Put in equation (2)

$\cot\theta = \frac{\frac{4}{5}}{\frac{3}{5}}$

$\cot\theta = \frac{4}{3}$

L.H.S. = R.H.S. Hence proved

Q.2 Find the value of each expression:

(i) $\cos\left(\sin^{-1}\dfrac{1}{\sqrt{2}}\right)$ (ii) $\sec\left(\cos^{-1}\dfrac{1}{2}\right)$

(iii) $\tan\left[\cos^{-1}\dfrac{\sqrt{3}}{2}\right]$ (iv) $\operatorname{cosec}\left[\tan^{-1}(-1)\right]$

(v) $\sec\left[\sin^{-1}\left(-\dfrac{1}{2}\right)\right]$ **(Lahore Board 2005)**

(vi) $\tan\left[\tan^{-1}(-1)\right]$

(vii) $\sin\left[\sin^{-1}\left(\dfrac{1}{2}\right)\right]$ **(Lahore Board 2007)**

(viii) $\tan\left[\sin^{-1}\left(-\dfrac{1}{2}\right)\right]$ (ix) $\sin\left[\tan^{-1}(-1)\right]$

Solution:

(i) $\cos\left(\sin^{-1}\dfrac{1}{\sqrt{2}}\right)$

We find value of $\sin^{-1}\dfrac{1}{\sqrt{2}}$ firstly

For this we have to find angle whose sine is $\dfrac{1}{\sqrt{2}}$

Let that angle be y

Then $y = \sin^{-1}\dfrac{1}{\sqrt{2}}$, $y \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

$\Rightarrow \sin y = \dfrac{1}{\sqrt{2}}$

$\Rightarrow y = \dfrac{\pi}{4}$

Therefore $\cos\left(\sin^{-1}\dfrac{1}{\sqrt{2}}\right) = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ Ans.

(ii) $\sec\left(\cos^{-1}\dfrac{1}{2}\right)$

First we find the value of $\cos^{-1}\dfrac{1}{2}$.

For this we have to find angle whose cosine is $\dfrac{1}{2}$

Let that angle be y

Then $y = \cos^{-1}\dfrac{1}{2}$

$\Rightarrow \cos y = \dfrac{1}{2}$; $y \in [0, \pi]$

$\Rightarrow y = \dfrac{\pi}{3}$

Therefore $\sec\left(\cos^{-1}\dfrac{1}{2}\right) = \sec\dfrac{\pi}{3} = 2$ Ans.

(iii) $\tan\left[\cos^{-1}\dfrac{\sqrt{3}}{2}\right]$

First we find the value of $\cos^{-1}\dfrac{\sqrt{3}}{2}$.

For this we have to find an angle whose cosine is $\dfrac{\sqrt{3}}{2}$

Let that angle be y

$y = \cos^{-1}\dfrac{\sqrt{3}}{2}$

$\Rightarrow \cos y = \dfrac{\sqrt{3}}{2}$; $y \in [0, \pi]$

$\Rightarrow y = \dfrac{\pi}{6}$

Therefore $\tan\left(\cos^{-1}\dfrac{\sqrt{3}}{2}\right) = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$ Ans.

(iv) cosec $\left[\tan^{-1}(-1)\right]$

First we find the value of $\tan^{-1}(-1)$. For this we have to find angle whose tangent is -1.

Let that angle be y

$y = \tan^{-1}(-1)$

$\Rightarrow \tan y = -1$; $y \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$

$\Rightarrow y = \dfrac{-\pi}{4}$

Therefore $\operatorname{cosec}\left[\tan^{-1}(-1)\right] = \operatorname{cosec}\left[\dfrac{-\pi}{4}\right] = \dfrac{1}{\sin\left(\dfrac{-\pi}{4}\right)} = \dfrac{1}{\dfrac{-1}{\sqrt{2}}} = -\sqrt{2}$ Ans.

(v) $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

First we find the value of $\sin^{-1}\left(\frac{-1}{2}\right)$. For this we have to find an angle whose sin is $\frac{-1}{2}$.

Let that angle be y

$$y = \sin^{-1}\left(\frac{-1}{2}\right), \quad y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \sin y = \frac{-1}{2}$$

$$\Rightarrow \quad y = \frac{-\pi}{6}$$

Therefore, $\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \sec\left[\frac{-\pi}{6}\right] = \frac{1}{\cos\left(\frac{-\pi}{6}\right)} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$ Ans.

(vi) $\tan[\tan^{-1}(-1)]$

First we find the value of $\tan^{-1}(-1)$. For this we have to find an angle whose tan is -1.

Let that angle be y

$$y = \tan^{-1}(-1)$$

$$\Rightarrow \quad \tan y = -1 \; ; \; y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\Rightarrow \quad y = \frac{-\pi}{4}$$

Therefore $\tan[\tan^{-1}(-1)] = \tan\left(\frac{-\pi}{4}\right) = -1$ Ans.

(vii) $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$

First we find the value of $\sin^{-1}\frac{1}{2}$. For this we have to find angle whose sine is $\frac{1}{2}$.

Let that angle be y

$$y = \sin^{-1}\frac{1}{2}$$

$$\Rightarrow \quad \sin y = \frac{1}{2}; \quad y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \frac{\pi}{6}$$

Therefore $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\frac{\pi}{6} = \frac{1}{2}$ Ans.

(viii) $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

First we find the value of $\sin^{-1}\left(\frac{-1}{2}\right)$. For this we have to find angle whose sin is $\frac{-1}{2}$.

Let that angle be y

$$y = \sin^{-1}\left(\frac{-1}{2}\right)$$

$$\Rightarrow \quad \sin y = \frac{-1}{2}; \quad y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \frac{-\pi}{6}$$

Therefore $\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \tan\left(\frac{-\pi}{6}\right) = \frac{-1}{\sqrt{3}}$ Ans.

(ix) $\sin[\tan^{-1}(-1)]$

First we find the value of $\tan^{-1}(-1)$. For this we have to find angle whose tan is -1.

Let that angle be y

$$y = \tan^{-1}(-1)$$

$$\Rightarrow \quad \tan y = -1 \; ; \; y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\Rightarrow \quad y = \frac{-\pi}{4}$$

Therefore $\sin[\tan^{-1}(-1)] = \sin\left(\frac{-\pi}{4}\right) = \frac{-1}{\sqrt{2}}$ Ans.

EXERCISE 13.2

Q.1 Prove that $\sin^{-1}\frac{15}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$

Solution:

$$\sin^{-1}\frac{15}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$$

$$\cos\left(\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}\right)=\frac{253}{325} \quad \text{......... (1)}$$

Let $\sin^{-1}\frac{5}{13}=\alpha,\quad \sin^{-1}\frac{17}{25}=\beta$

Equation (1) becomes

$$\cos(\alpha+\beta)=\frac{253}{325}$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\frac{253}{325} \quad \text{......... (2)}$$

$$\sin\alpha=\frac{5}{13}$$
$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$
$$\cos\alpha=\sqrt{1-\frac{25}{169}}$$
$$\cos\alpha=\sqrt{\frac{169-25}{169}}$$
$$\cos\alpha=\sqrt{\frac{144}{169}}$$
$$\cos\alpha=\frac{12}{13}$$

$$\sin\beta=\frac{7}{25}$$
$$\cos\beta=\sqrt{1-\sin^2\beta}$$
$$\cos\beta=\sqrt{1-\frac{49}{625}}$$
$$\cos\beta=\sqrt{\frac{625-49}{625}}$$
$$\cos\beta=\sqrt{\frac{576}{625}}$$
$$\cos\beta=\frac{24}{25}$$

Substitute values in equation (2)

$$\cos(\alpha+\beta)=\frac{12}{13}\times\frac{24}{25}-\frac{5}{13}\times\frac{7}{25}$$
$$=\frac{288}{325}-\frac{35}{325}=\frac{288-35}{325}$$
$$\cos(\alpha+\beta)=\frac{253}{325}$$

Hence proved.

Q.2 $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{9}{19}$ **(Gujranwala Board 2007)**

Formula $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\frac{A+B}{1-AB}$

Solution:

$$\text{L.H.S.}=\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}$$
$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{4}\cdot\frac{1}{5}}\right)=\tan^{-1}\left(\frac{\frac{5+4}{20}}{\frac{20-1}{20}}\right)$$
$$=\tan^{-1}\frac{9}{19}=\text{R.H.S.} \quad \text{Hence proved}$$

Q.3 $2\tan^{-1}\frac{2}{3}=\sin^{-1}\frac{12}{13}$

Solution:

$$2\tan^{-1}\frac{2}{3}=\sin^{-1}\frac{12}{13}$$
$$\sin\left(2\tan^{-1}\frac{2}{3}\right)=\frac{12}{13} \quad \text{......... (1)}$$

Let $\tan^{-1}\frac{2}{3}=\theta$

Equation (1) becomes

$$\sin 2\theta=\frac{12}{13} \quad \text{......... (2)}$$
$$\tan\theta=\frac{2}{3}$$
$$\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\frac{4}{9}}=\sqrt{\frac{9+4}{9}}=\frac{\sqrt{13}}{3}$$
$$\cos\theta=\frac{3}{\sqrt{13}}$$
$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{9}{13}}=\sqrt{\frac{13-9}{13}}=\sqrt{\frac{4}{13}}$$
$$\sin\theta=\frac{2}{\sqrt{13}}$$
$$\sin 2\theta=2\sin\theta\cos\theta=2\times\frac{2}{\sqrt{13}}\times\frac{3}{\sqrt{13}}=\frac{12}{13} \quad \text{Hence proved}$$

Q.4 **Prove that** $\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$

Solution:

$$2\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{120}{119}$$

$$\tan\left(2\cos^{-1}\frac{12}{13}\right) = \frac{120}{119} \qquad \text{.........(1)}$$

Let $\cos^{-1}\frac{12}{13} = \theta \Rightarrow \cos\theta = \frac{12}{13}$

Equation (1) becomes

$$\tan 2\theta = \frac{120}{119}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{119} \qquad \text{.........(2)}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{114}{169}} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}}$$

$$\sin\theta = \frac{5}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{5}{13}\times\frac{12}{13} = \frac{120}{169}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{144}{169} - \frac{25}{169} = \frac{144-25}{169} = \frac{119}{169}$$

$$\tan 2\theta = \frac{\frac{120}{169}}{\frac{119}{169}}$$

$$\tan 2\theta = \frac{120}{119} \quad \text{Hence proved.}$$

Q.5 $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$

Solution:

Let $\sin^{-1}\frac{1}{\sqrt{5}} = \alpha$, $\cot^{-1}3 = \beta$

$$\sin\alpha = \frac{1}{\sqrt{5}} \qquad \cot\beta = 3$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} \qquad \Rightarrow \tan\beta = \frac{1}{3}$$

$$\cos\alpha = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{5-1}{5}} = \sqrt{\frac{4}{5}} \qquad \sec\beta = \sqrt{1+\tan^2\beta}$$

$$\cos\alpha = \frac{2}{\sqrt{5}} \qquad \sec\beta = \sqrt{1+\frac{1}{9}} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$$

$$\cos\beta = \frac{3}{\sqrt{10}} \Rightarrow \sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{9}{10}} = \sqrt{\frac{10-9}{10}} = \frac{1}{\sqrt{10}}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \frac{1}{\sqrt{5}}\times\frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{10}} = \frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}} = \frac{5}{\sqrt{50}}$$

$$\sin(\alpha+\beta) = \frac{5}{5\sqrt{2}}$$

$$\alpha+\beta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4} \quad \text{Hence proved}$$

Q.6 $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$ **(Lahore Board 2006, Gujranwala Board 2007)**

Solution: $\boxed{\textbf{Formula } \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)}$

$$\text{L.H.S.} = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{64}{289}} + \frac{8}{17}\sqrt{1-\frac{9}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{289-64}{289}} + \frac{8}{17}\sqrt{\frac{25-9}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{225}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right)$$

$$= \sin^{-1}\left(\frac{45}{85} + \frac{32}{85}\right) = \sin^{-1}\frac{77}{85}$$

= R.H.S. Hence proved.

Q.7 $\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$ **(Lahore Board 2009, 2010)**

Solution:

$$\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$$

$$\cos\left(\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5}\right) = \frac{15}{17} \qquad \ldots\ldots\ldots (1)$$

Let $\sin^{-1}\frac{77}{85} = \alpha$ $\qquad$ $\sin^{-1}\frac{3}{5} = \beta$

$$\sin\alpha = \frac{77}{85}$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{5929}{7225}} = \sqrt{\frac{7225-5929}{7225}} = \sqrt{\frac{1296}{7225}} = \frac{36}{85}$$

$$\sin\beta = \frac{3}{5}$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \frac{36}{85}\times\frac{4}{5} + \frac{77}{85}\times\frac{3}{5}$$

$$= \frac{144}{425} + \frac{231}{425} = \frac{144+231}{425} = \frac{375}{425} = \frac{15}{17}$$

$\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\left(\frac{15}{17}\right)$ Hence proved.

Q.8 $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$

Solution:

$$\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$$

$$\therefore \quad 2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{2\left(\frac{1}{5}\right)}{1-\frac{1}{25}} = \tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}$$

$$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$$

Given equation becomes

$$\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{3}{5}$$

$$\Rightarrow \quad \sin\left(\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{5}{12}\right) = \frac{3}{5} \qquad \ldots\ldots\ldots (1)$$

Let $\cos^{-1}\frac{63}{65} = \alpha$ $\qquad$ $\tan^{-1}\frac{5}{12} = \beta$

Equation (1):

$$\cos\alpha = \frac{63}{65}, \quad \tan\beta = \frac{5}{12}$$

$$\sin(\alpha+\beta) = \frac{3}{5} \qquad \ldots\ldots\ldots (2)$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{3969}{4225}} = \sqrt{\frac{4225-3969}{4225}} = \sqrt{\frac{256}{4225}}$$

$$\sin\alpha = \frac{16}{65}$$

$$\sec\beta = \sqrt{1+\tan^2\beta} = \sqrt{1+\frac{25}{144}}$$

$$\sec\beta = \sqrt{\frac{144+25}{144}} = \sqrt{\frac{169}{144}} = \frac{13}{12}$$

$$\cos\beta = \frac{12}{13}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}}$$

$$\sin\beta = \sqrt{\frac{25}{169}}$$

$$\sin\beta = \frac{5}{13}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{16}{65}\times\frac{12}{13} + \frac{63}{65}\times\frac{5}{13}$$

$$= \frac{192}{845} + \frac{315}{845} = \frac{507}{845} = \frac{3}{5}$$

$$\alpha+\beta = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{3}{5}$$ Hence proved.

Q.9 $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

Solution:

$$\text{L.H.S.} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right] - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{\frac{15+12}{20}}{\frac{20-9}{20}} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{513-88}{11\times19}}{\frac{209+216}{11\times19}}\right]$$

$$= \tan^{-1}\frac{425}{425}$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

= R.H.S. Hence proved.

Q.10 $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Solution:

$$\text{L.H.S.} = \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{16}{25}} + \frac{4}{5}\sqrt{1-\frac{25}{169}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{\frac{25-16}{25}} + \frac{4}{5}\sqrt{\frac{169-25}{169}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{5}{13}\times\frac{3}{5} + \frac{4}{5}\times\frac{12}{13}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{3}{13} + \frac{48}{65}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{16}{65}\sqrt{1-\frac{3969}{4225}} + \frac{63}{65}\sqrt{1-\frac{256}{4225}}\right]$$

$$= \sin^{-1}\left[\frac{16}{65}\times\frac{16}{65} + \frac{63}{65}\times\frac{63}{65}\right]$$

$$= \sin^{-1}\left[\frac{256}{4225} + \frac{3969}{4225}\right]$$

$$= \sin^{-1}\left[\frac{256+3969}{4225}\right]$$

$$= \sin^{-1}\left(\frac{4225}{4225}\right)$$

$$= \sin^{-1}(1)$$

$$= \frac{\pi}{2}$$ = R.H.S. Hence proved.

Q.11 $\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$ **(Lahore Board 2011)**

Solution:

$$\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$$

$$\tan^{-1}\left(\frac{\frac{1}{11}+\frac{5}{6}}{1-\frac{1}{11}\times\frac{5}{6}}\right)=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right)$$

$$\tan^{-1}\left[\frac{\frac{6+55}{66}}{\frac{66-5}{66}}\right]=\tan^{-1}\left[\frac{\frac{2+3}{6}}{\frac{6-1}{6}}\right]$$

$$\tan^{-1}\left(\frac{61}{61}\right)=\tan^{-1}\left(\frac{5}{5}\right)$$

$$\tan^{-1}(1)=\tan^{-1}(1)$$

$$\frac{\pi}{4}=\frac{\pi}{4}$$ Hence proved.

Q.12 $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

(Gujranwala Board 2005, 2006) (Lahore Board 2006, 2007, 2008)

Solution:

$$\text{L.H.S.}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)$$

$$=\tan^{-1}\left[\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right]$$

$$=\tan^{-1}\frac{25}{25}=\tan^{-1}(1)=\frac{\pi}{4}$$

= R.H.S. Hence proved.

Q.13 Show that $\cos(\sin^{-1}x)=\sqrt{1-x^2}$ (Gujranwala Board 2007)

Solution:

$$\text{L.H.S.}=\cos(\sin^{-1}x)$$

Let $\sin^{-1}x=\alpha \Rightarrow \sin\alpha = x \quad \alpha\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-x^2}$$

$$\cos(\sin^{-1}x)=\sqrt{1-x^2}=\text{R.H.S.}$$

Hence proved.

Q.14 Show that $\sin(2\cos^{-1}x)=2x\sqrt{1-x^2}$

Solution:

$$\text{L.H.S.}=\sin(2\cos^{-1}x) \quad \text{.........(1)}$$

Let $\cos^{-1}x=\alpha \Rightarrow x=\cos\alpha \quad \alpha\in[0,\pi]$

Equation (1) becomes

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \quad \text{.........(2)}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-x^2}$$

Put values in equation (2)

$$\sin 2\alpha = 2x\sqrt{1-x^2}$$

$\sin 2(\cos^{-1}x)=2x\sqrt{1-x^2}$ R.H.S. Hence proved.

Q.15 Show that $\cos(2\sin^{-1}x)=1-2x^2$

Solution:

$$\text{L.H.S.}=\cos(2\sin^{-1}x) \quad \text{.........(1)}$$

Let $\sin^{-1}x=\alpha \Rightarrow \sin\alpha=x, \quad \alpha\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Equation (1) becomes

$$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha \quad \text{.........(2)}$$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-x^2}$$

Put in equation (2)

$$\cos 2\alpha = (\sqrt{1-x^2})^2-x^2$$
$$=1-x^2-x^2$$
$$\cos 2(\sin^{-1}x)=1-2x^2$$

= R.H.S. Hence proved.

Q.16 Show that $\tan^{-1}(-x) = -\tan^{-1} x$

Solution:

$$\tan^{-1}(-x) + \tan^{-1} x = 0$$

$$\text{L.H.S.} = \tan^{-1}\left[\frac{-x+x}{1-(-x)(x)}\right]$$

$$= \tan^{-1}\left[\frac{0}{1+x^2}\right]$$

$$= \tan^{-1} 0$$

$$= 0 = \text{R.H.S. Hence proved}$$

Q.17 Show that $\sin^{-1}(-x) = -\sin^{-1} x$

Solution:

$$\sin^{-1}(-x) + \sin^{-1} x = 0$$

$$\text{L.H.S.} = \sin^{-1}(-x) + \sin^{-1} x$$

$$= \sin^{-1}\left[(-x)\sqrt{1-x^2} + x\sqrt{1-x^2}\right]$$

$$= \sin^{-1}(0)$$

$$= 0 = \text{R.H.S. Hence proved.}$$

Q.18 Show that $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Solution:

$$\cos^{-1}(-x) + \cos^{-1} x = \pi$$

$$\text{L.H.S.} = \cos^{-1}(-x) + \cos^{-1} x$$

$$\boxed{\text{Formula } \cos^{-1} A + \cos^{-1} B = \cos^{-1}\left[AB - \sqrt{(1-A^2)(1-B^2)}\right]}$$

$$\cos^{-1}(-x) + \cos^{-1} x = \cos^{-1}\left[-x \times x - \sqrt{(1-x^2)(1-x^2)}\right]$$

$$= \cos^{-1}\left[-x^2 - \sqrt{(1-x^2)^2}\right]$$

$$= \cos^{-1}(-1)$$

$$= \pi = \text{R.H.S. Hence proved.}$$

Q.19 Show that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$ (Lahore Board 2008)

Solution:

$$\tan(\sin^{-1} x) \quad \text{.......... (1)}$$

Let $\sin^{-1} x = \alpha$

Equation (1) becomes

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \quad \text{.......... (2)}$$

$$x = \sin\alpha$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$$

Put in equation (2)

$$\tan\alpha = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \quad \text{Hence proved.}$$

Q.20 Given that $x = \sin^{-1}\frac{1}{2}$, find the values of following trigonometric functions sin x, cos x, tan x, cosec x, sec x, cot x.

Solution:

$$x = \sin^{-1}\frac{1}{2}$$

$$\Rightarrow \quad \sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{4-1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\cot x = \sqrt{3}$$

$$\sec x = \frac{1}{\cos x} = \frac{2}{\sqrt{3}}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = 2 \quad \text{Ans.}$$