Chapter 1

NUMBER SYSTEMS

The set of real numbers can be written as $R = Q \cup Q'$ where Q is the set of rational numbers and Q' is the set of irrational numbers.

RATIONAL NUMBER

Rational number is a number which can be put in the form $\frac{p}{q}$ where $p, q \in z \wedge q \neq 0$.

Thus $\sqrt{16}$, $\frac{3}{4}$, 2.7 are rational numbers.

Decimal Representation of Rational Numbers

(1) Terminating Decimals

A decimal which has only a finite number of digits in its decimal part, is called a terminating decimal. Terminating decimal represents a rational number. Thus $3.7,\ 0.0005,\ 207.9$ are rational numbers.

(2) Recurring Decimals

Recurring decimal is a decimal in which one or more digits repeat indefinitely. Every recurring decimal represents a rational number. Thus $0.333\ldots..$, $1.57575757\ldots..$ are rational numbers.

Irrational Number

Irrational number is a number which can not be put into the form $\frac{p}{q}$ where $p, q, \in z \wedge q \neq 0$. Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt{\frac{5}{6}}$ are irrational numbers.

Note: A non terminating, non recurring decimal represents an irrational number.

Binary Operation

A binary operation in a set A is a rule usually denoted by $*$ that assigns to any pair of elements of A, taken in a definite order, another element of A.

Properties of Real Numbers

1. Addition Laws:

(i) Closure Law of Addition

$\forall\ a, b \in R \quad a + b \in R$

(ii) Associative Law of Addition

$\forall\ a, b, c \in R, \ a + (b + c) = (a + b) + c$

(iii) Additive Identity

$\forall\ a \in R, \ \exists\ 0 \in R$ such that $a + 0 = 0 + a = a$

(v) Commutative Law for Addition

$\forall\ a, b \in R, \ a + b = b + a$

2. Multiplication Laws:

(vi) Closure Law of Multiplication

$\forall\ a, b \in R, \quad a\,.\,b \in R$

(vii) Associative Law for Multiplication

$\forall\ a, b, c \in R, \quad a(bc) = (ab)c$

(viii) Multiplicative Identity

$\forall\ a \in R, \ \exists\ 1 \in R$ such that $a\,.\,1 = 1\,.\,a = a$

(ix) Multiplicative Inverse

$\forall\ a (\neq 0) \in R, \ \exists\ a^{-1} \in R$ such that $a\,.\,a^{-1} = a^{-1}\,.\,a = 1$

(x) Commutative Law of Multiplication

$\forall\ a, b \in R, \quad ab = ba$

3. Multiplication – Addition Law

(xi) Distributive Property

$\forall\ a, b, c \in R$

$a(b + c) = ab + ac$ (left distributive property)

$(a + b)c = ac + bc$ (Right distributive property)

Any set possessing all the above 11 properties is called a field.

4. Properties of Equality

(i) Reflexive property: $\forall\ a \in R, \quad a = a$

(ii) Symmetric property: $\forall\ a, b \in R, \quad a = b \Rightarrow b = a$

(iii) Transitive property: $\forall\ a, b, c \in R, \quad a = b \wedge b = c \Rightarrow a = c$

(iv) Additive property: $\forall\ a, b, c, \in R, \quad a = b \Rightarrow a + c = b + c \wedge c + a = b + c$

(v) Multiplicative property: $\forall\ a, b, c, \in R, \quad a = b \Rightarrow ac = bc \wedge ca = cb$

(vi) Cancellation property w.r.t. addition: $\forall\ a, b, c \in R, \quad a + c = b + c \Rightarrow a = b$

(vii) Cancellation property w.r.t. Multiplication: $\forall\ a, b, c \in R, \ ac = bc \Rightarrow a = b, \ c \neq 0$

5. Properties of Inequalities

Trichotomy Property

$\forall\ a, b \in R$ either $a = b$ or $a > b$ or $a < b$

Transitive Property

$\forall\ a, b, c \in R$

(i) $a > b \wedge b > c \Rightarrow a > c$

(ii) $a < b \wedge b < c \Rightarrow a < c$

Additive Property

$\forall\ a, b, c \in R$

(a) (i) $a > b \Rightarrow a + c > b + c$ (ii) $a < b \Rightarrow a + c < b + c$

(b) (i) $a > b \wedge c > d \Rightarrow a + c > b + d$ (ii) $a < b \wedge c < d \Rightarrow a + c < b + d$

Multiplicative Properties

(a) $\forall\ a, b, c \in R$ and $c > 0$

(i) $a > b \Rightarrow ac > bc$ (ii) $a < b \Rightarrow ac < bc$

(b) $\forall\ a, b, c \in R$ and $c < 0$

(i) $a > b \Rightarrow ac < bc$ (ii) $a < b \Rightarrow ac > bc$

(c) $\forall\ a, b, c, d \in R$ and a, b, c, d are all positive,

(i) $a > b \wedge c > d \Rightarrow ac > bd$ (ii) $a < b \wedge c < d \Rightarrow ac < bd$

EXERCISE 1.1

Q.1 Which of the following sets have closure property w.r.t. addition and multiplication

(i) $\{0\}$

The set is closed w.r.t. addition because $0 + 0 = 0 \in \{0\}$

The set is closed w.r.t. multiplication because $0.0 = 0 \in \{0\}$

(ii) $\{1\}$

The set is not closed w.r.t. addition because $1 + 1 = 2 \notin \{1\}$

The set is closed w.r.t. multiplication because $1.1 = 1 \in \{1\}$

(iii) $\{0, -1\}$

+	0	−1
0	0	−1
−1	−1	−2

The set is not closed w.r.t. addition because $-2 \notin \{0, -1\}$

•	0	−1
0	0	0
−1	0	1

The set is not closed w.r.t. multiplication because $1 \notin \{0, -1\}$

(iv) $\{1, -1\}$

+	1	−1
1	2	0
−1	0	−2

The set is not closed w.r.t. addition because $-2, 0, 2 \notin \{-1, 1\}$

•	1	−1
1	1	−1
−1	−1	1

The set is closed w.r.t. multiplication.

Q.2 Name the properties used in the following equations (letters, where used, represents real numbers)

Solution:

(i) $4 + 9 = 9 + 4$ Commutative property w.r.t. '+'

(ii) $(a + 1) + \frac{3}{4} = a + \left(1 + \frac{3}{4}\right)$ Associative property w.r.t. '+'

(iii) $(\sqrt{3} + \sqrt{5}) + \sqrt{7} = \sqrt{3} + (\sqrt{5} + \sqrt{7})$ Associative property w.r.t. '+'

(iv) $100 + 0 = 100$ Additive Identity

(v) $100 \times 1 = 100$ Multiplicative Identity

(vi) $4.1 + (-4.1) = 0$ Additive Inverse

(vii) $a - a = 0$ Additive Inverse.

(viii) $\sqrt{2} \times \sqrt{5} = \sqrt{5} \times \sqrt{2}$ Commutative property w.r.t. '.'

(ix) $a(b - c) = ab - ac$ Left distributive property.

(x) $(x - y)z = xz - yz$ Right distributive property.

(xi) $4 \times (5 \times 8) = (4 \times 5) \times 8$ Associative property w.r.t. '.'

(xii) $a(b + c - d) = ab + ac - ad$ Left distributive property

Q.3 Name the properties used in the following inequalities.

Solution:

(i) $-3 < -2 \Rightarrow 0 < 1$ Additive property.

(ii) $-5 < -4 \Rightarrow 20 > 16$ Multiplication property.

(iii) $1 > -1 \Rightarrow -3 > -5$ Additive property.

(iv) $a < 0 \Rightarrow -a > 0$ Multiplicative property.

(v) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ Multiplicative property.

(vi) $a > b \Rightarrow -a < -b$ Multiplicative property.

Q.4 Prove the following Rules of Addition

(i) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

Solution:

L.H.S

$= \frac{a}{c} + \frac{b}{c}$

$= a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$ $\because \frac{a}{b} = a \cdot \frac{1}{b}$

$= (a + b) \cdot \frac{1}{c}$ Distributive property

$= \frac{a+b}{c}$ $\because a \cdot \frac{1}{c} = \frac{a}{c}$

(ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

Solution:

L.H.S. $=\frac{a}{b}+\frac{c}{d}$

$=\frac{a}{b}\cdot 1+\frac{c}{d}\cdot 1$ Multiplicative Identity

$=\frac{a}{b}\cdot\left(d\cdot\frac{1}{d}\right)+\frac{c}{d}\left(b\cdot\frac{1}{b}\right)$ Multiplicative Inverse

$=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}$ $\because d\cdot\frac{1}{d}=\frac{d}{d},\ b\cdot\frac{1}{b}=\frac{b}{b}$

$=\frac{ad}{bd}+\frac{cb}{db}$

$=\frac{ad}{bd}+\frac{bc}{bd}$ Commutative Property w.r.t. '.'

$=ad\cdot\frac{1}{bd}+bc\cdot\frac{1}{bd}$ $\because \frac{a}{b}=a\cdot\frac{1}{b}$

$=(ad+bc)\cdot\frac{1}{bd}$ Distributive Property

$=\frac{ad+bc}{bd}$ $\because a\cdot\frac{1}{b}=\frac{a}{b}$

$=$ R.H.S.

Q.5 Prove that $-\frac{7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

Solution:

L.H.S. $=-\frac{7}{12}-\frac{5}{18}$

$=-\frac{7}{12}\cdot 1-\frac{5}{18}\cdot 1$ Multiplicative Identity

$=-\frac{7}{12}\left(3\cdot\frac{1}{3}\right)-\frac{5}{18}\cdot\left(2\cdot\frac{1}{2}\right)$ Multiplicative Inverse

$=-\frac{7}{12}\cdot\frac{3}{3}-\frac{5}{18}\cdot\frac{2}{2}$ $\because a\cdot\frac{1}{b}=\frac{a}{b}$

$=-\frac{21}{36}-\frac{10}{36}$

$=-21\cdot\frac{1}{36}-10\cdot\frac{1}{36}$ $\because \frac{a}{b}=a\cdot\frac{1}{b}$

$=(-21-10)\cdot\frac{1}{36}$ Distributive Property

$=\frac{-21-10}{36}$

$=$ R.H.S.

Q.6 Simplify by justifying each step:

(i) $\frac{4+16x}{4}$

Solution:

$\frac{4+16x}{4}$

$=\frac{1}{4}\cdot(4+16x)$ $\because \frac{a}{b}=\frac{1}{b}\cdot a$

$=\frac{1}{4}\cdot 4+\frac{1}{4}\cdot 16\,x$ Distributive Property

$=\frac{1}{4}\cdot 4\cdot 4x$ Multiplicative Inverse

$=1+1\cdot 4x$ Multiplicative Inverse

$=1+4x$ Multiplicative Identity

(ii) $\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

Solution:

$\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

$=\left(\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}\right)\cdot 1$ Multiplicative Identity

$$= \left(\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}\right).20.\frac{1}{20}$$ Multiplicative Inverse

$$= \frac{\left(\frac{1}{4}+\frac{1}{5}\right).20}{\left(\frac{1}{4}-\frac{1}{5}\right).20}$$ $\because\ 20.\frac{1}{20} = \frac{20}{20}$

$$= \frac{\frac{1}{4}.20+\frac{1}{5}.20}{\frac{1}{4}.20+\frac{1}{5}.20}$$ Distributive Property

$$= \frac{\left(\frac{1}{4}.4\right)5+\left(\frac{1}{5}.5\right)4}{\frac{1}{4}.4.5+\frac{1}{5}.5.4} = \frac{1.5+1.4}{1.5-1.4}$$ Multiplicative Inverse

$$= \frac{5+4}{5-4}$$ Multiplicative Identity

$$= \frac{9}{1} = 9$$

(iii) $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$$

$$= \left(\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}\right).1$$ Multiplicative Identity

$$= \left(\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}\right).bd.\frac{1}{bd}$$ Multiplicative Inverse

$$= \frac{\left(\frac{a}{b}+\frac{c}{d}\right).bd}{\left(\frac{a}{b}-\frac{c}{d}\right).bd}$$ $\because\ bd.\frac{1}{bd} = \frac{bd}{bd}$

$$= \frac{\frac{a}{b}.bd+\frac{c}{d}.bd}{\frac{a}{b}.bd-\frac{c}{d}.bd}$$ Distributive Property

$$= \frac{a.\frac{1}{b}.b.d+c.\frac{1}{d}.b.d}{a\frac{1}{b}.b.d-c\frac{1}{d}.b.d}$$ $\because\ \frac{a}{b} = a.\frac{1}{b},\ \frac{c}{d} = c.\frac{1}{d}$

$$= \frac{a\left(\frac{1}{b}.b\right)d+c.\left(\frac{1}{d}.d\right)b}{a\left(\frac{1}{b}.b\right)d-c\left(\frac{1}{d}.d\right)b}$$ Commutative Property

$$= \frac{a.1.d+c.1.b}{a.1.d-c.1.b}$$ Multiplicative Inverse

$$= \frac{ad+cb}{ad-cb}$$ Multiplicative Identity

(iv) $$\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}$$

$$= \left(\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}\right).1$$ Multiplicative Identity

$$= \left(\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}.\frac{1}{b}}\right).ab.\frac{1}{ab}$$ Multiplicative Inverse

$$= \frac{\left(\frac{1}{a}-\frac{1}{b}\right).ab}{\left(1-\frac{1}{a}.\frac{1}{b}\right).ab}$$ $\because\ ab.\frac{1}{ab} = \frac{ab}{ab}$

$$= \frac{\frac{1}{a}.ab-\frac{1}{b}.ab}{ab-\frac{1}{a}.\frac{1}{b}.ab}$$ Distributive Property

$$= \frac{\left(\frac{1}{a}.a\right)b-\left(\frac{1}{b}.b\right)a}{ab-\left(\frac{1}{a}.a\right)\left(\frac{1}{b}.b\right)}$$ Commutative Property

$$= \frac{1.b-1.a}{ab-1.1}$$ Multiplicative Inverse

$$= \frac{b-a}{ab-1}$$ Multiplicative Identity

Complex Numbers

The numbers of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, are called complex numbers, here 'x' is called real part and 'y' is called imaginary part of the complex number. For example $3 + 4i$, $2 - \frac{5}{7}i$ etc. are complex numbers.

A complex number can be written in the form of an ordered pair i.e. $x + iy = (x, y)$. the set 'C' of complex numbers does not satisfy the order axioms. Infact there is no sense in saying that one complex number is greater or less than another.

EXERCISE 1.2

Q.1 Verify the addition properties of complex numbers.

Solution:

Addition properties of complex numbers are:

(i) Closure property

$\forall\ (a, b),\ (c, d) \in C$

$(a, b) + (c, d) = a + ib + c + id = a + c + i(b + d) = (a + c, b + d) \in C$

(ii) Associative Property

$\forall\ (a, b),\ (c, d),\ (e, f) \in C$

$[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$

L.H.S.

$$\begin{aligned}[(a, b) + (c, d)] + (e, f) &= [a + ib + c + id] + (e + if)\\ &= a + ib + c + id + e + if\\ &= a + c + e + i(b + d + f)\\ &= (a + c + e\ ,\ b + d + f)\end{aligned}$$

R.H.S.

$(a, b) + [(c, d) + (e, f)]$

$= (a + ib) + [c + id + e + if]$

$= a + ib + c + id + e + if$

$= a + c + e + i(b + d + f)$

$= (a + c + e\ ,\ b + d + f)$

$\Rightarrow$ L.H.S. = R.H.S.

(iii) Additive Identity

$\forall\ (a, b) \in C \quad \exists\ (0, 0) \in C$

such that $(a, b) + (0, 0) = (0, 0) + (a, b) = (a, b)$

(iv) Additive Inverse

$\forall\ (a, b) \in C \quad \exists\ (-a, -b) \in C$

such that $(a, b) + (-a, -b) = (0, 0) = (-a, -b) + (a, b)$

(v) Commutative Property

$\forall\ (a, b),\ (c, d) \in C$

$$\begin{aligned}(a, b) + (c, d) &= a + ib + c + id\\ &= a + c + i(b + d)\\ &= c + a + i(d + b) = (c + id) + (a + ib) = (c, d) + (a, b)\end{aligned}$$

Q.2 Verify the multiplication properties of complex numbers.

Solution:

Multiplication Properties of complex numbers are:

Closure property

$\forall\ (a, b),\ (c, d) \in C$

$$\begin{aligned}(a, b) \cdot (c, d) &= (a + ib)(c + id) = ac + aid + ibc + i^2bd\\ &= ac - bd + i(ad + bc) = (ac - bd, ad + bc) \in C\end{aligned}$$

Associative Property

$\forall\ (a, b),\ (c, d)\ (e, f) \in C$

$[(a, b) \cdot (c, d)] \cdot (e, f) = (a, b)\,[(c, d) \cdot (e, f)]$

L.H.S.

$$\begin{aligned}[(a, b) \cdot (c, d)] \cdot (e, f) &= [(a + ib)(c + id)] \cdot (e + if)\\ &= [ac + iad + ibc + i^2bd] \cdot (e + if)\\ &= [ac - bd + iad + ibc] \cdot (e + if)\\ &= ace + iacf - bde - ibdf + iade + i^2adf + ibce + i^2bcf\\ &= ace - adf - bcf - bde + i(acf - bdf + ade + bce)\end{aligned}$$

Now R.H.S.

$$\begin{aligned}(a, b)\,[(c, d) \cdot (e, f)] &= (a + ib)\,[(c + id)(e + if)]\\ &= (a + ib)\,[ce + icf + ide + i^2df]\\ &= (a + ib)\,[ce - df + icf + ide]\\ &= ace - adf + iacf + iade + ibce - ibdf + i^2bcf + i^2bde\\ &= ace - adf - bcf - bde + i(acf - bdf + ade + bce)\end{aligned}$$

$\Rightarrow$ L.H.S. = R.H.S

Multiplicative Identity

$\forall\ (a, b) \in C \quad \exists\ (1, 0) \in C$ such that

$(a, b) . (1, 0) = (a + ib) . (1 + 0\,i) = a + 0i + ib + 0 = a + ib = (a, b)$

and

$(1, 0)\,(a, b) = (1 + 0i) . (a + ib) = a + ib + 0i + 0 = a + ib = (a, b)$

$\Rightarrow$ (1,0) is multiplicative identity in C.

Multiplicative Inverse

$\forall\ (a + ib) \in C \quad \exists\ (a + ib)^{-1} \in C$

where

$$(a + ib)^{-1} = \frac{1}{a + ib} = \frac{1}{a + ib} \cdot \frac{a - ib}{a - ib}$$

$$= \frac{a - ib}{(a)^2 - (ib)^2} = \frac{a - ib}{a^2 - i^2b^2} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

$$= \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

Commutative Property

$\forall\ (a, b), (c, d) \in C$

$(a, b) . (c, d) = (c, d) . (a, b)$

L.H.S.

$(a, b) . (c, d) = (a + ib)(c + id) = ac + iad + ibc + i^2bd$

$= (ac - bd) + i\,(ad + bc)$

R.H.S.

$(c, d) . (a, b) = (c + id)(a + ib) = ca + icb + ida + i^2db$

$= (ac - bd) + i\,(bc + ad)$

$\Rightarrow$ L.H.S. = R.H.S

Q.3 Verify the distributive law of complex numbers.

$(a, b)\,[(c, d) + (e, f)] = (a, b)\,(c, d) + (a, b)\,(e, f)$

Solution:

To show

$(a, b)\,[(c, d) + e, f)] = (a, b)\,(c, d) + (a, b)\,(e, f)$

L.H.S.

$(a, b)\,[(c, d) + (e, f)]$

$= (a + ib)\,[c + id + e + if] = ac + iad + ae + iaf + ibc + i^2bd + ibe + i^2bf \qquad \because i^2 = -1$

$= ac + ae - bd - bf + i\,(ad + af + bc + be)$

R.H.S.

$(a, b)\,(c, d) + (a, b)\,(e, f)$

$= (a + ib)\,(c + id) + (a + ib)\,(e + if)$

$= ac + iad + ibc + i^2bd + ae + iaf + ibe + i^2bf \qquad \because i^2 = -1$

$= ac + ae - bd - bf + i\,(ad + bc + af + be)$

L. H. S. = R.H.S.

Hence Proved.

Q.4 Simplify the following:

(i) i^9

Solution:

$i^9 = i^8 . i = (i^2)^4 . i = (-1)^4 . i = (1)(i) = i \qquad \because i^2 = -1$

(ii) i^{14}

Solution:

$i^{14} = (i^2)^7 = (-1)^7 = -1 \qquad$ ¤ $i^2 = -1$

(iii) $(-i)^{19}$ **(Lahore Board 2004)**

Solution:

$(-i)^{19} = (-1)\,i^{19} = -(1)\,i^{18} . i = -(i^2)^9 . i = -(-1)^9\,i = (-1) . i = i \qquad$ ¤ $i^2 = -1$

(iv) $(-1)^{-21/2}$ **(Lahore Board 2007)**

Solution:

$$(-1)^{-21/2} = [(-1)^{1/2}]^{-21} = [(i^2)^{1/2}]^{-21} = i^{-21}$$

$$= \frac{1}{i^{21}} = \frac{1}{i^{20} . i} = \frac{1}{(i^2)^{10} . i}$$

$$= \frac{1}{(-1)^{10}\,i} = \frac{1}{i} . \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i \qquad \because i^2 = -1$$

Q.5 Write in terms of i

Solution:

(i) $\sqrt{-1}\,b$

Solution:

$\sqrt{-1}\,b = ib \qquad$ ¤ $\sqrt{-1} = i$

(ii) $\sqrt{-5}$

Solution:

$\sqrt{-5} = \sqrt{-1} . \sqrt{5} = i\sqrt{5} \qquad \because \sqrt{-1} = i$

(iii) $\sqrt{\dfrac{-16}{25}}$

Solution:

$$\sqrt{\frac{-16}{25}} = \sqrt{-1}\sqrt{\frac{16}{25}} = i\left(\frac{4}{5}\right) = \frac{4}{5}i \qquad \because \sqrt{-1} = i$$

(iv) $\sqrt{\dfrac{1}{-4}}$

Solution:

$$\sqrt{\frac{1}{-4}} = \frac{1}{\sqrt{-1}}\sqrt{\frac{1}{4}} = \frac{1}{i}\cdot\frac{1}{2} = \frac{i}{i\,.\,i}\frac{1}{2} = \frac{i}{i^2\,.\,2} = \frac{+i}{-2} = \frac{i}{2} \qquad \because i^2 = -1$$

Q.6 Simplify the following:
$(7, 9) + (3, -5)$

Solution:

$$(7, 9) + (3, -5) = 7 + 9i + 3 - 5i = 10 + 4i = (10, 4)$$

Q.7 $(8, -5) - (-7, 4)$

Solution:

$$(8, -5) - (-7, 4) = 8 - 5i - (-7 + 4i) = 8 - 5i + 7 - 4i = 15 - 9i = (15, -9)$$

Q.8 $(2, 6)\,(3, 7)$

Solution:

$$(2, 6)\,(3, 7) = (2 + 6i)\,.\,(3 + 7i) = 6 + 14i + 18i + 42\,i^2$$
$$= 6 + 32i - 42 = -36 + 32i = (-36, 32) \qquad \because i^2 = -1$$

Q.9 $(5, -4)\,(-3, -2)$

Solution:

$$(5, -4)\,(-3, -2) = (5 - 4i)\,(-3 - 2i)$$
$$= -15 - 10i + 12i + 8i^2 = -15 + 2i - 8 = -23 + 2i = (-23, 2)$$

Q.10 $(0, 3)\,(0, 5)$

Solution:

$$(0, 3)\,(0, 5) = (0 + 3i)\,(0 + 5i)$$
$$= 0 + 0 + 0 + 15i^2 = 0 + 15(-1) = -15 = -15 + 0i = (-15, 0)$$

Q.11 $(2, 6) \div (3, 7)$

Solution:

$$(2, 6) \div (3, 7) = (2 + 6i) \div (3 + 7i) = \frac{(2 + 6i)}{(3 + 7i)} \times \frac{(3 - 7i)}{(3 - 7i)} \qquad \text{Rationalizing}$$
$$= \frac{(2 + 6i)(3 - 7i)}{(3 + 7i)(3 - 7i)} = \frac{6 - 14i + 18i - 42\,i^2}{(3)^2 - (7i)^2} \qquad \because i^2 = -1$$
$$= \frac{6 + 4i + 42}{9 - 49i^2} = \frac{48 + 4i}{9 + 49} = \frac{48 + 4i}{58}$$
$$= \left(\frac{48}{58}, \frac{4i}{58}\right) = \left(\frac{24}{29}, \frac{2i}{29}\right)$$

Q.12 $(5, -4) \div (-3, -8)$

Solution:

$$(5, -4) \div (-3, -8) = (5 - 4i) \div (-3 - 8i)$$
$$= \frac{5 - 4i}{-3 - 8i} \times \frac{-3 + 8i}{-3 + 8i} \qquad \text{Rationalizing}$$
$$= \frac{(5 - 4i)(-3 + 8i)}{(-3 - 8i)(-3 + 8i)} = \frac{-15 + 40i + 12i - 32\,i^2}{(-3)^2 - (8i)^2}$$
$$= \frac{-15 + 52i + 32}{9 - 64i^2} = \frac{17 + 52i}{9 + 64} = \frac{17 + 52i}{73}$$
$$= \frac{17}{73} + \frac{52}{73}i = \left(\frac{17}{73}, \frac{52}{73}\right)$$

Q.13 Prove that the sum as well as product of any two conjugate complex numbers is a real number.

Solution:

Let $z = a + bi$ is a complex number then its conjugate is $\bar{z} = \overline{a + ib} = a - ib$

Sum $= z + \bar{z} = a + ib + a - ib = 2a$ (real number)

Product $= z\,.\,\bar{z} = (a + ib)\,(a - ib) = (a)^2 - (ib)^2 = a^2 - i^2 b^2 \qquad \because i^2 = -1$
$= a^2 + b^2$ (real number)

Hence the sum as well as the product of any two conjugate complex numbers is a real number.

Q.14 Find the multiplicative inverse of each of the following numbers.
(i) $(-4, 7)$ **(Lahore Board 2007)**

Solution:

(i) $(-4, 7)$

As multiplicative inverse of $(a, b) = \left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(-4, 7) = \left(\dfrac{-4}{(-4)^2 + (7)^2}, \dfrac{-7}{(-4)^2 + (7)^2}\right)$

$$= \left(\frac{-4}{16 + 49}, \frac{-7}{16 + 49}\right) = \left(\frac{-4}{65}, \frac{-7}{65}\right)$$

(ii) $(\sqrt{2}, -\sqrt{5})$

Solution:

As multiplicative inverse of $(a, b) = \left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(\sqrt{2}, -\sqrt{5}) = \left(\dfrac{\sqrt{2}}{(\sqrt{2})^2 + (-\sqrt{5})^2}, \dfrac{+\sqrt{5}}{(\sqrt{2})^2 + (-\sqrt{5})^2}\right)$

$$= \left(\frac{\sqrt{2}}{2 + 5}, \frac{\sqrt{5}}{2 + 5}\right) = \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$$

(iii) $(1, 0)$

As multiplicative inverse of $(a, b) = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

So multiplicative inverse of $(1, 0) = \left(\dfrac{1}{(1)^2+(0)^2}, \dfrac{-0}{(1)^2+(0)^2}\right)$

$$= \left(\frac{1}{1+0}, 0\right) = (1, 0)$$

Q.15 Factorize the following:

(i) $a^2 + 4b^2$

Solution:

$a^2 + 4b^2$

$= a^2 - (-4b^2) = (a^2) - (i^2 4b^2) \quad \because i^2 = -1$

$= (a)^2 - (2bi)^2 = (a + 2bi)(a - 2bi)$

(ii) $9a^2 + 16b^2$ **(Lahore Board 2006)**

Solution:

$9a^2 + 16b^2 = 9a^2 - (-16b^2)$

$= 9a^2 - (i^2 16b^2) \quad \because i^2 = -1$

$= (3a)^2 - (4bi)^2 = (3a + 4bi)(3a - 4bi)$

(iii) $3x^2 + 3y^2$ **(Gujranwala Board 2007)**

Solution:

$3x^2 + 3y^2 = 3[x^2 - (-y)^2] = 3[x^2 - (i^2y^2)]$

$= 3[(x)^2 - (iy)^2] = 3(x + iy)(x - iy)$

Q.16 Separate into real and imaginary parts (write as a simple complex number)

(i) $\dfrac{2-7i}{4+5i}$ **(Lahore Board 2011)**

Solution:

$\dfrac{2-7i}{4+5i} = \dfrac{2-7i}{4+5i} \times \dfrac{4-5i}{4-5i}$ Rationalizing

$= \dfrac{(2-7i)(4-5i)}{(4+5i)(4-5i)} = \dfrac{8-10i-28i+35i^2}{(4)^2-(5i)^2} \quad \because i^2 = -1$

$= \dfrac{8-38i-35}{16-25i^2} = \dfrac{-27-38i}{16+25}$

$= \dfrac{-27-38i}{41} = \dfrac{-27}{41} - \dfrac{38}{41}i$

(ii) $\dfrac{(-2+3i)^2}{(1+i)}$ **(Lahore Board 2003)**

Solution:

$\dfrac{(-2+3i)^2}{(1+i)} = \dfrac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$

$= \dfrac{4+9i^2-12i}{1+i} = \dfrac{4-9-12i}{1+i} = \dfrac{-5-12i}{1+i}$

$= \dfrac{-5-12i}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{(-5-12i)(1-i)}{(1+i)(1-i)} = \dfrac{-5+5i-12i+12i^2}{(1)^2-(i)^2}$

$= \dfrac{-5+5i-12i-12}{1-i^2} = \dfrac{-5-7i-12}{1-(-1)}$

$= \dfrac{-17-7i}{2} = -\dfrac{17}{2} - \dfrac{7}{2}i$

(iii) $\dfrac{i}{1+i}$ **(Gujranwala Board 2007)**

Solution:

$\dfrac{i}{1+i}$

$= \dfrac{i}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{i(1-i)}{(1+i)(1-i)} = \dfrac{i-i^2}{(1)^2-i^2} = \dfrac{i-(-1)}{1-(-1)}$

$= \dfrac{i+1}{1+1} = \dfrac{i+1}{2}$

$= \dfrac{i}{2} + \dfrac{1}{2} = \dfrac{1}{2} + \dfrac{1}{2}i$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

The Complex Plane:

We can represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of the coordinate plane.

In this representation the x-axis is called the real axis and the y-axis is called the imaginary axis. The figure representing one or more complex numbers on the complex plane is called an Argand diagram.

EXERCISE 1.3

Q.1 Graph the following numbers on the complex plane:

(i) $2 + 3i = (2, 3)$

(ii) $2 - 3i = (2, -3)$

(iii) $-2 - 3i = (-2, -3)$

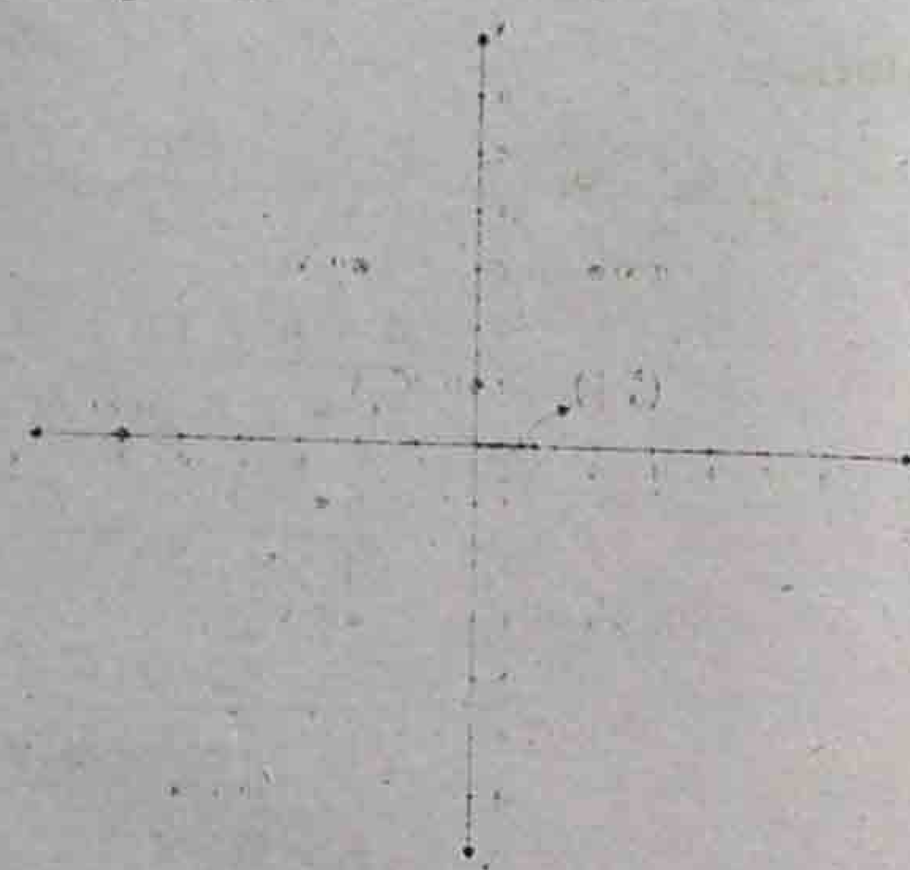
(iv) $-2 + 3i = (-2, 3)$

(v) $-6 = (-6, 0)$

(vi) $i = (0, 1)$

(vii) $\frac{3}{5} - \frac{4}{5}i = \left(\frac{3}{5}, -\frac{4}{5}\right) = (0.6, -0.8)$

(viii) $-5 - 6i = (-5, -6)$



Q.2 Find the multiplicative inverse of the following numbers:

(i) $-3i$

Solution:

$-3i = (0 - 3)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(0, -3) = \left(\frac{0}{(0)^2 + (-3)^2}, \frac{3}{(0)^2 + (-3)^2}\right)$

$= \left(0, \frac{3}{9}\right) = \left(0, \frac{1}{3}\right) = 0 + \frac{1}{3}i = \frac{1}{3}i$

(ii) $1 - 2i$ (Lahore Board 2006)

Solution:

$1 - 2i = (1, -2)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(1, -2) = \left(\frac{1}{(1)^2 + (-2)^2}, \frac{-(-2)}{(1)^2 + (-2)^2}\right)$

$= \left(\frac{1}{1 + 4}, \frac{2}{1 + 4}\right)$

$= \left(\frac{1}{5}, \frac{2}{5}\right) = \frac{1}{5} + \frac{2}{5}i$

(iii) $-3 - 5i$

Solution: (Gujranwala Board 2004)

$-3 - 5i = (-3, -5)$

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(-3, -5) = \left(\frac{-3}{(-3)^2 + (-5)^2}, \frac{-(-5)}{(-3)^2 + (-5)^2}\right)$

$= \left(\frac{-3}{9 + 25}, \frac{5}{9 + 25}\right)$

$= \left(\frac{-3}{34}, \frac{5}{34}\right) = \frac{-3}{34} + \frac{5}{34}i$

(iv) $(1, 2)$

Solution:

As multiplicative inverse of $(a, b) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

So multiplicative inverse of $(1, 2) = \left(\frac{1}{(1)^2 + (2)^2}, \frac{-2}{(1)^2 + (+2)^2}\right)$

$= \left(\frac{1}{1 + 4}, \frac{-2}{1 + 4}\right)$

$= \left(\frac{1}{5}, \frac{-2}{5}\right)$

Q.3 Simplify

(i) i^{101} (Gujranwala Board 2007)

Solution:

$i^{101} = i^{100} . i = (i^2)^{50} . i = (-1)^{50} . i = i \qquad \because i^2 = -1$

(ii) $(-ai)^4, a \in R$

Solution:

$(-ai)^4 = (-1)^4 . a^4 i^4 = a^4 (i^2)^2 = a^4 (-1)^2 = a^4 \qquad \because i^2 = -1$

(iii) i^{-3}

Solution:

$i^{-3} = \frac{1}{i^3} = \frac{1}{i^2 . i} = \frac{1}{-i} . \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i$

(iv) i^{-10}

Solution:

$i^{-10} = \frac{1}{i^{10}} = \frac{1}{(i^2)^5} = \frac{1}{(-1)^5} = \frac{1}{-1} = -1$

Q.4 Prove that $\overline{z} = z$ iff z is real. **(Lahore Board 2009)**

Solution:

Let $z = a + ib$ then $\overline{z} = a - ib$

Suppose $z = \overline{z}$

$\Rightarrow \quad a + ib = a - ib$

$\Rightarrow \quad a + ib - a + ib = 0$

$\Rightarrow \quad 2ib = 0 \quad \Rightarrow \quad b = 0 \qquad \because 2i \neq 0$

$\Rightarrow \quad z = a + i0$

$\Rightarrow \quad z = a$

$\Rightarrow \quad z$ is real.

Now conversely;

Suppose that z is real

Such that $z = a$

then $\quad \overline{z} = a$

$\Rightarrow \quad z = \overline{z}$

Hence proved.

Q.5 Simplify by expressing in form $a + bi$

(i) $5 + 2\sqrt{-4}$

Solution:

$$5 + 2\sqrt{-4} = 5 + 2\sqrt{-1}.\sqrt{4} = 5 + 2i(2) = 5 + 4i \qquad \because \sqrt{-1} = 1$$

(ii) $(2 + \sqrt{-3})(3 + \sqrt{-3})$

Solution:

$$\begin{aligned}(2+\sqrt{-3})(3+\sqrt{-3}) &= 6 + 2\sqrt{-3} + 3\sqrt{-3} + (\sqrt{-3})^2 \\ &= 6 + 2\sqrt{-1}.\sqrt{3} + 3\sqrt{-1}.\sqrt{3} + (-3) \\ &= 6 + 2i\sqrt{3} + 3i\sqrt{3} - 3 \\ &= 3 + 5\sqrt{3}\,i\end{aligned}$$

(iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$

$$\begin{aligned}\frac{2}{\sqrt{5}+\sqrt{-8}} &= \frac{2}{\sqrt{5}+\sqrt{-8}}.\frac{\sqrt{5}-\sqrt{-8}}{\sqrt{5}-\sqrt{-8}} \qquad \text{Rationalizing} \\ &= \frac{2(\sqrt{5}-\sqrt{-8})}{(\sqrt{5})^2-(\sqrt{-8})^2} = \frac{2\sqrt{5}-2\sqrt{-8}}{5-(-8)} \\ &= \frac{2\sqrt{5}-2\sqrt{-1}.\sqrt{8}}{5+8} = \frac{2\sqrt{5}-2i\sqrt{8}}{13} \\ &= \frac{2\sqrt{5}}{13} - \frac{4i\sqrt{2}}{13}\end{aligned}$$

(iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

Solution:

$$\begin{aligned}\frac{3}{\sqrt{6}-\sqrt{-12}} &= \frac{3}{\sqrt{6}-\sqrt{12}} \times \frac{\sqrt{6}+\sqrt{-12}}{\sqrt{6}+\sqrt{-12}} \qquad \text{Rationalizing} \\ &= \frac{3(\sqrt{6}+\sqrt{12})}{(\sqrt{6})^2-(\sqrt{-12})^2} \\ &= \frac{3\sqrt{6}+3\sqrt{-12}}{6-(-12)} = \frac{3\sqrt{6}+3\sqrt{-1}.\sqrt{12}}{6+12} \\ &= \frac{3\sqrt{6}+3i\sqrt{12}}{18} = \frac{3\sqrt{6}}{18} + \frac{3i.2\sqrt{3}}{18} \\ &= \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{3}i = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}i\end{aligned}$$

Q.6 Show that $\forall z \in C$

(i) $z^2 + \overline{z}^2$ **is real number**

Solution:

Let $\quad z = a + bi$

Then $\quad \overline{z} = a - bi$

Now

$$\begin{aligned}z^2 + \overline{z}^2 &= (a+bi)^2 + (a-bi)^2 \\ &= (a)^2 + (ib)^2 + 2abi + a^2 + (ib)^2 - 2abi \\ &= a^2 + i^2b^2 + a^2 + i^2b^2 \qquad \because i^2 = -1 \\ &= a^2 - b^2 + a^2 - b^2 = 2a^2 - 2b^2 = 2(a^2 - b^2)\end{aligned}$$

which is real number

Hence proved.

(ii) $(z - \overline{z})^2$ **is real number** **(Lahore Board 2009)**

Solution:

Let $\quad z = a + bi$

then $\quad \overline{z} = a - ib$

$$\begin{aligned}\Rightarrow \quad (z-\overline{z})^2 &= [(a+ib)-(a-ib)]^2 \qquad \because i^2 = -1 \\ &= [a + ib - a + ib]^2 = (2ib)^2 = 4i^2b^2 = -4b^2\end{aligned}$$

which is a real number.

Hence proved.

Q.7 Simplify the followings

(i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$ **(Gujranwala Board 2003, 2005)**

Solution:

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 \qquad \because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$$

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{(-1+\sqrt{3i})^3}{(2)^3}$$

$$= \frac{1}{8}[(-1)^3+(\sqrt{3i})^3+3(-1)^2(\sqrt{3i})+3(-1)(\sqrt{3i})^2]$$

$$= \frac{1}{8}[-1+(\sqrt{3})^3 i^3+3(1)\sqrt{3}i-3(3i^2)]$$

$$= \frac{1}{8}[-1+\sqrt{3^3}i^2.i+3\sqrt{3}i-3(3)(-1)]$$

$$= \frac{1}{8}[-1+\sqrt{27}(-1)i+3\sqrt{3}i+9]$$

$$= \frac{1}{8}[-1-3\sqrt{3}i+3\sqrt{3}i+9]$$

$$= \frac{1}{8}(8) = 1$$

(ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

Solution:

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{-1-\sqrt{3}i}{2}\right)^3 = \left(\frac{(-1)(1+\sqrt{3}i)}{2}\right)^3 = \frac{(-1)^3(1+\sqrt{3}i)^3}{2^3}$$

$$= -\frac{1}{8}[(1)^3+(\sqrt{3i})^3+3(1)^2(\sqrt{3i})+3(1)(\sqrt{3i})^2]$$

$$= -\frac{1}{8}[1+(\sqrt{3})^3 i^3+3\sqrt{3}i+3(3i^2)]$$

$$= -\frac{1}{8}[1+\sqrt{3^3}i^2.i+3\sqrt{3}i+9(-1)]$$

$$= -\frac{1}{8}[1+\sqrt{27}(-1)i+3\sqrt{3}i-9]$$

$$= -\frac{1}{8}[1-3\sqrt{3}i+3\sqrt{3}i-9]$$

$$= -\frac{1}{8}[-8] = 1$$

(iii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

Solution:

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-1}$$

$$= \left(\frac{-1-\sqrt{3i}}{2}\right)^{-1}$$

$$= \frac{2}{-1-\sqrt{3i}}\cdot\frac{-1+\sqrt{3i}}{-1+\sqrt{3i}} \qquad \text{Rationalizing}$$

$$= \frac{2(-1+\sqrt{3}i)}{(-1)^2-(\sqrt{3}i)^2} = \frac{-2+2\sqrt{3}i}{1-3i^2} = \frac{-2+2\sqrt{3}i}{1+3} \qquad \because i^2 =$$

1

$$= \frac{-2+2\sqrt{3}i}{4} = \frac{-2}{4}+\frac{2\sqrt{3}i}{4} = -\frac{1}{2}+\frac{\sqrt{3}i}{2}$$

(iv) $(a+bi)^2$

Solution:

$$(a+bi)^2 = (a)^2+2(a)(bi)+(bi)^2 = a^2+b^2i^2+2abi$$

(v) $(a+bi)^2$

Solution:

$$(a+bi)^2 = \frac{1}{(a+bi)^2} = \left(\frac{1}{a+bi}\cdot\frac{a-bi}{a-bi}\right)^2 \qquad \text{Rationalizing}$$

$$= \left(\frac{a-bi}{a^2-b^2i^2}\right)^2 = \left(\frac{a-bi}{a^2+b^2}\right)^2 \qquad \because i^2 = -1$$

$$= \frac{(a-bi)^2}{(a^2+b^2)^2}$$

$$= \frac{a^2+b^2i^2-2abi}{(a^2+b^2)^2}$$

$$= \frac{a^2-b^2-2abi}{(a^2+b^2)^2} \qquad \because i^2 = -1$$

$$= \frac{a^2-b^2}{(a^2+b^2)^2}-\frac{2abi}{(a^2+b^2)^2}$$