Closed-Form Finite Recurrences for SU(2) 3nj Symbols

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Abstract

Building on the universal closed-form hypergeometric representation of SU(2) 3nj symbols [1], we present a finite set of algebraic three-term recurrence relations in the spin labels j_k, together with minimal boundary data, which uniquely determine any SU(2) $3nj$ recoupling coefficient. These recurrences allow computation of all $3nj$ symbols in closed form without nested sums of $6j$ symbols.

1 General Recurrence System

For each edge k in the coupling tree with Schwinger variable x_k, define

$$\Delta_0^{(k)} = \det\bigl(I - K(\mathbf{x})\bigr)\Big|_{x_k=0}, \quad \Delta_1^{(k)} = \det\bigl(I - K(\mathbf{x})\bigr)\Big|_{\substack{\text{rows/cols of edge } k\\ \text{removed}}},$$

and let

$$G(\mathbf{x}) = \sum_{\{j\}} T(j_1, \ldots, j_E) \prod_{\ell=1}^{E} x_\ell^{2j_\ell}.$$

The closed three-term recurrence in the label j_k follows from

$$\Delta_0^{(k)}(\mathbf{x}_{\neq k})\, G(\mathbf{x}) = x_k^2\, \Delta_1^{(k)}(\mathbf{x}_{\neq k})\, G(\mathbf{x}).$$

Equating coefficients of $\prod_\ell x_\ell^{2j_\ell}$ yields

$$\sum_\mu a_\mu\, T\bigl(j_k, \mathbf{j}_{\neq k} - \mu\bigr) \;=\; \sum_\nu b_\nu\, T\bigl(j_k - 1, \mathbf{j}_{\neq k} - \nu\bigr),$$

where a_μ, b_ν are the expansion coefficients of $\Delta_0^{(k)}$ and $\Delta_1^{(k)}$ in the other variables.

2 Seed Data

For each k, the seed values

$$T(0, \mathbf{j}_{\neq k})$$

are given by the SU(2) $3(n-1)j$ symbol on the tree with edge k removed, bottoming out at the Racah $6j$ case. These seeds, together with the recurrences, uniquely determine every $3nj$ coefficient.

References

[1] Arcticoder, *Universal Closed-Form Hypergeometric Representation of SU(2) 3nj Symbols*, `https://arcticoder.github.io/su2-3nj-uniform-closed-form/`, 2025.