Chapter 12: Nonparametric Inference

DSCC 462

Computational Introduction to Statistics

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Fall 2022

Plan for Today

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- Introduce nonparametric analogues to hypothesis tests

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- *Wilcoxon Signed-Rank Test*

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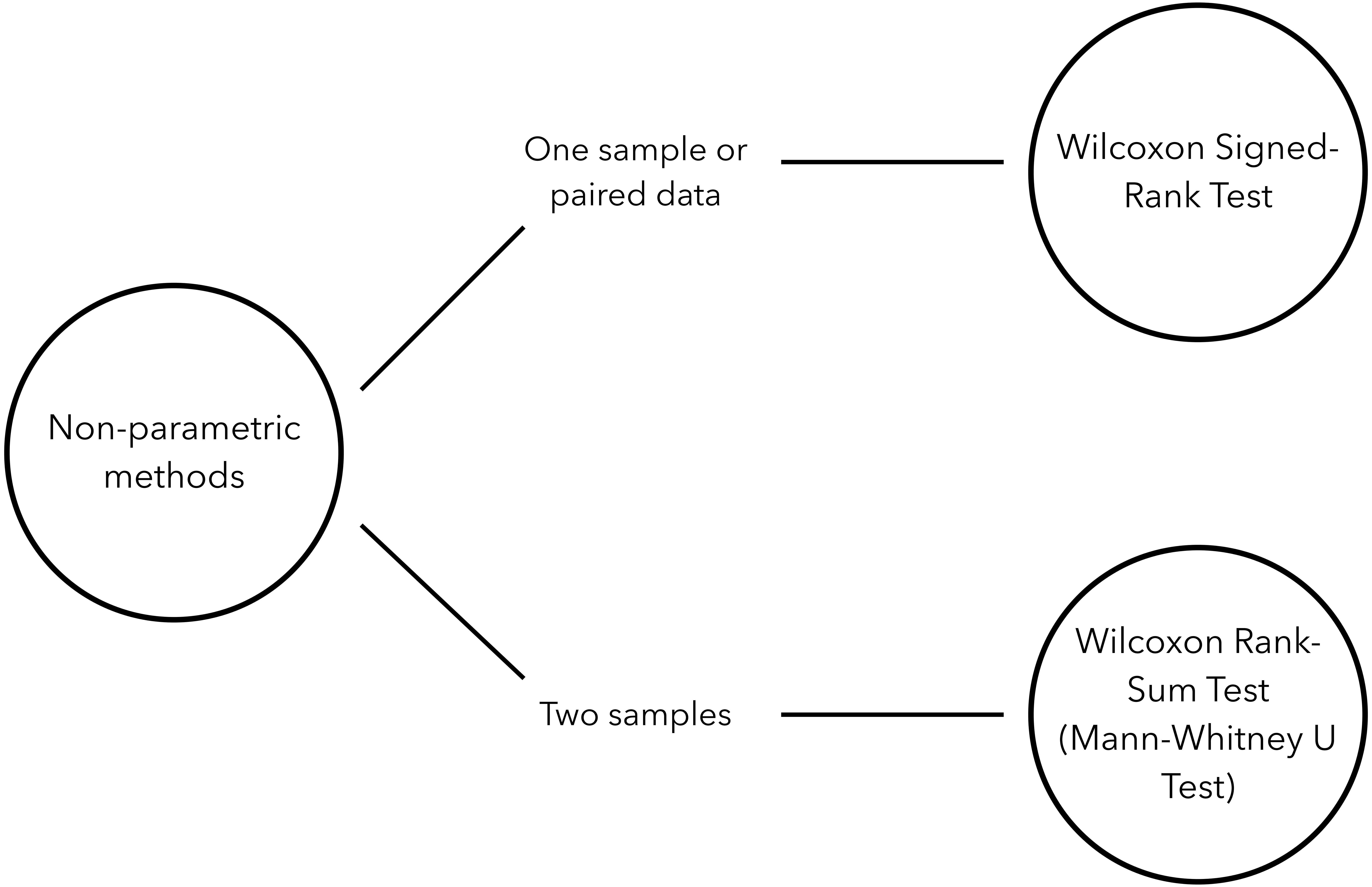
  - Nonparametric analog to the one-sample or paired t-test
- *Wilcoxon Rank-Sum Test (Mann-Whitney U Test)*
  - Nonparametric analog to the two-sample t-test

Plan for Today, Visualized



Nonparametric Methods

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- In this case, we use *nonparametric methods*, which make fewer assumptions regarding the underlying distribution
  - Also known as *distribution-free methods*

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  - Make a claim, develop hypotheses, state significance level
  - Calculate a test statistic based on a random sample of data
  - Determine whether to reject or fail to reject the null hypothesis based on the test statistic and significance level

Motivating Example #1

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- Null hypothesis: In the underlying population differences among pairs, the median difference is equal to 0
  - Note that we consider medians for nonparametric tests as opposed to means

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- H_0 : The median difference in tumor size equals 0
- H_1 : The median difference in tumor size is different from 0
- Test at the $\alpha = 0.05$ significance level

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2 3 4 5

$d_i \rightarrow$ 0.5   1   −1   1   1   2

Ranks $\rightarrow$ 1   3   −3   3   5

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- Ignoring the sign of these observations, rank their absolute values from smallest to largest
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- Tied observations are assigned an average rank
- Finally separate the ranks by sign to either + or –

Wilcoxon Signed-Rank Test: Data Table

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5

Subject	Tumor Size (mm)		Difference	Rank	Signed Rank	
	Before	After			+	-
1	36.3	27.1	9.2			
2	21.7	17.4	4.3	4.5		
3	45.1	33.1	12.0			
4	27.8	32.1	-4.3	4.5		
5	5.1	8.3	-2.2	2		
6	23.4	22.1	1.3	1		
7	25.0	31.2	-6.2			
8	12.6	16.4	-3.8	3		
9	19.9	12.5	7.4			
10	22.1	22.1	0	—	—	—
11	18.6	4.8	13.8			
12	8.9	22.6	-13.7			
13	12.7	6.4	6.3			
14	29.3	18.3	9.0			
15	26.4	21.8	4.6			

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6	23.4	22.1	1.3	1		
7	25.0	31.2	-6.2	7		
8	12.6	16.4	-3.8	3		
9	19.9	12.5	7.4	9		
10	22.1	22.1	0	-		
11	18.6	4.8	13.8	14		
12	8.9	22.6	-13.7	13		
13	12.7	6.4	6.3	8		
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6	23.4	22.1	1.3	1	1	
7	25.0	31.2	-6.2	7		7
8	12.6	16.4	-3.8	3		3
9	19.9	12.5	7.4	9	9	
10	22.1	22.1	0	-		
11	18.6	4.8	13.8	14	14	
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$\Sigma \to T^+$ $\Sigma \to T^-$

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- Calculate $T = T^+ - T^-$
- Under the null hypothesis, the median of the underlying population differences is equal to 0

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$T^+ = \sum$ ranks all $+$

- Calculate the sum of the positive ranks, T^+, and the sum of the negative ranks, T^-

$-\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2}$

- Calculate $T = T^+ - T^-$

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- Thus, we expect approximately equal numbers of positive and negative ranks

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- Calculate the sum of the positive ranks, T^+, and the sum of the negative ranks, T^-
- Calculate $T = T^+ - T^-$

-3 -2 -1 0 1 2 3

- Under the null hypothesis, the median of the underlying population differences is equal to 0
- Thus, we expect approximately equal numbers of positive and negative ranks
- Additionally, the sum of the positive ranks should be approximately equal to the sum of the negative ranks, so T should be approximately 0

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$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$

$$\sqrt{\sum_{k=1}^{n} k^2}$$

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- Note that

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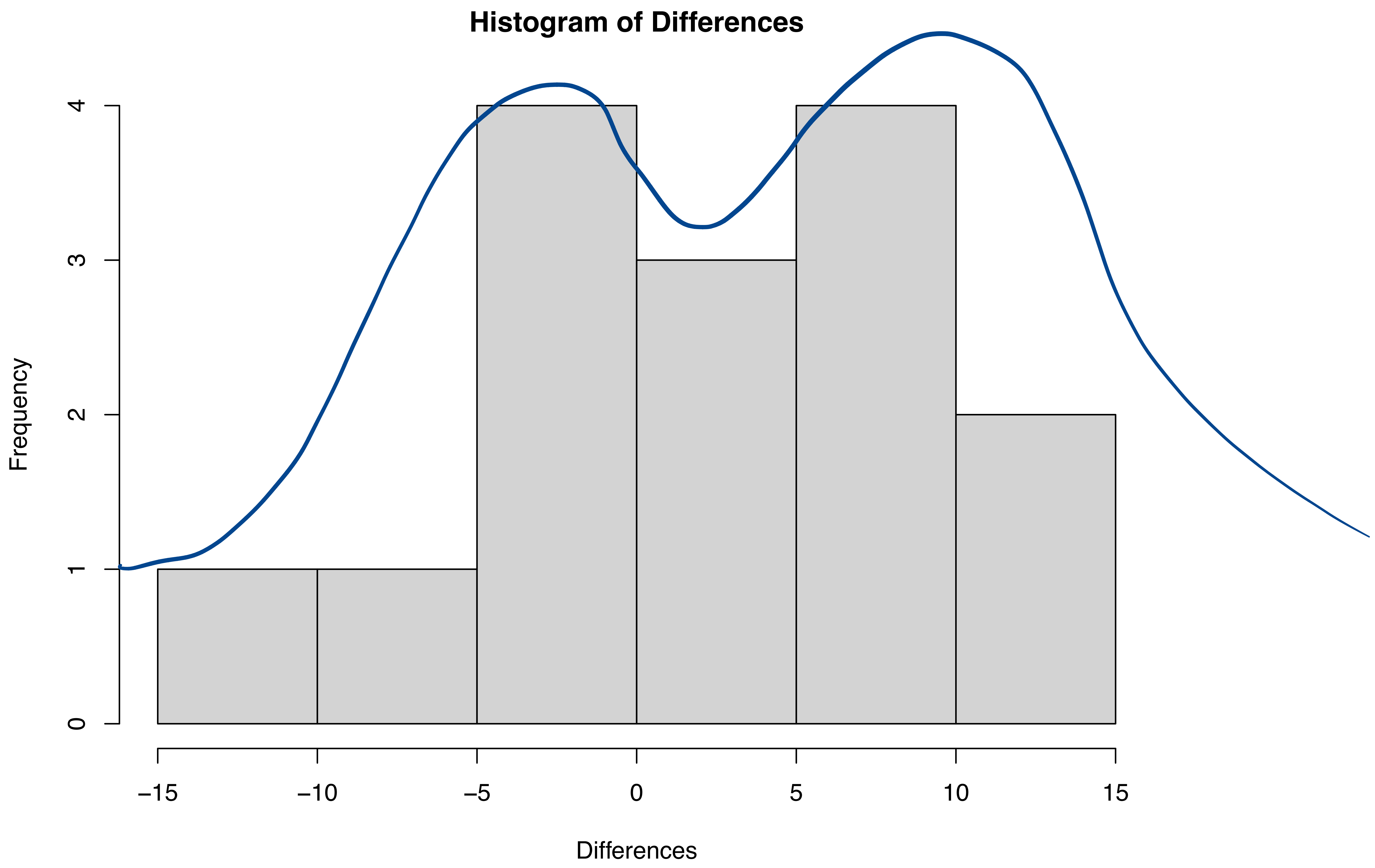
- $Z_T \sim N(0,1)$ given that n is large enough (typically $n > 12$)

Histogram of Differences



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	+	-
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5		2
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7		7
8		3
9	9	
10		
11	14	
12		13
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14	10	
15	6	

Wilcoxon Signed-Rank Test

- Based on the histogram, the difference in tumor size does not appear to be normally distributed, so we want to use the Wilcoxon signed-rank test

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- $T^+ =$

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- $T^{-} =$
- $T =$

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8		3
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10		
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Wilcoxon Signed-Rank Test

- Based on the histogram, the difference in tumor size does not appear to be normally distributed, so we want to use the Wilcoxon signed-rank test
- $T^+ =$ 75.5
- $T^- =$ 29.5
- $T =$ 46.
- $n =$ 14 > 12

Subject	Signed Rank +	Signed Rank -
1	11	
2	4.5	
3	12	
4		4.5
5		2
6	1	
7		7
8		3
9	9	
10		
11	14	
12		13
13	8	
14	10	
15	6	
	75.5	29.5

Wilcoxon Signed-Rank Test

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- Given $T = 46$, we then have the following:

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$$\mu_T = 0$$

Wilcoxon Signed-Rank Test

$n = 14$

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$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} = 31.86$$

Wilcoxon Signed-Rank Test

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$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} =$$

- Thus,

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$$\mu_T = 0$$

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} = \quad 31.86$$

- Thus,

$$z_T = \frac{T - \mu_T}{\sigma^T} = \quad \frac{46 - 0}{31.86} = 1.44$$

Wilcoxon Signed-Rank Test

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- Calculating the p-value, we have

Wilcoxon Signed-Rank Test

$$Z_T = \frac{T - \mu_T}{\sigma_T} \sim N(0,1)$$

- Calculating the p-value, we have

$$p = 2 \cdot Pr(Z > 1.44) = 2 \cdot (1 - \text{pnorm}(1.44)) = 0.149$$

- Conclusion: $p = 0.149 > \alpha = 0.05$

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- In that case, we can use `psignrank(T,n)` in R to calculate the exact p-value

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- In that case, we can use `psignrank(T,n)` in R to calculate the exact p-value
    - `2*(1-psignrank(`~~74~~ 75.5`,n=14))=`~~0.173~~ 0.135

(*) R requires $T = T^+$ for this to work correctly!

Wilcoxon Signed-Rank Test: R Code

$z \sim N(0,1)$

$d_i = x_i - y_i$ N

```
> wilcox.test(before, after, paired=T, exact=F, correct=F)

	Wilcoxon signed rank test

data:  before and after
V = 64, p-value = 0.1961
alternative hypothesis: true location shift is not equal to 0
```

pnorm(T, n)

```
> wilcox.test(before, after, paired=T, exact=T, correct=F)

	Wilcoxon signed rank test

data:  before and after
V = 64, p-value = 0.2163
alternative hypothesis: true location shift is not equal to 0
```

psignrank(T, n)

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- Take a sample of $n_2 = 15$ people who do have the disease

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- Cannot assume that body temperature is normally distributed
- Take a sample of $n_1 = 12$ people who do not have the disease
- Take a sample of $n_2 = 15$ people who do have the disease
- How can we compare the median body temperature for these two populations?

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- Cannot assume that body temperature is normally distributed
- Take a sample of $n_1 = 12$ people who do not have the disease
- Take a sample of $n_2 = 15$ people who do have the disease
- How can we compare the median body temperature for these two populations?
  - *Wilcoxon Rank-Sum Test (Mann-Whitney U Test)*

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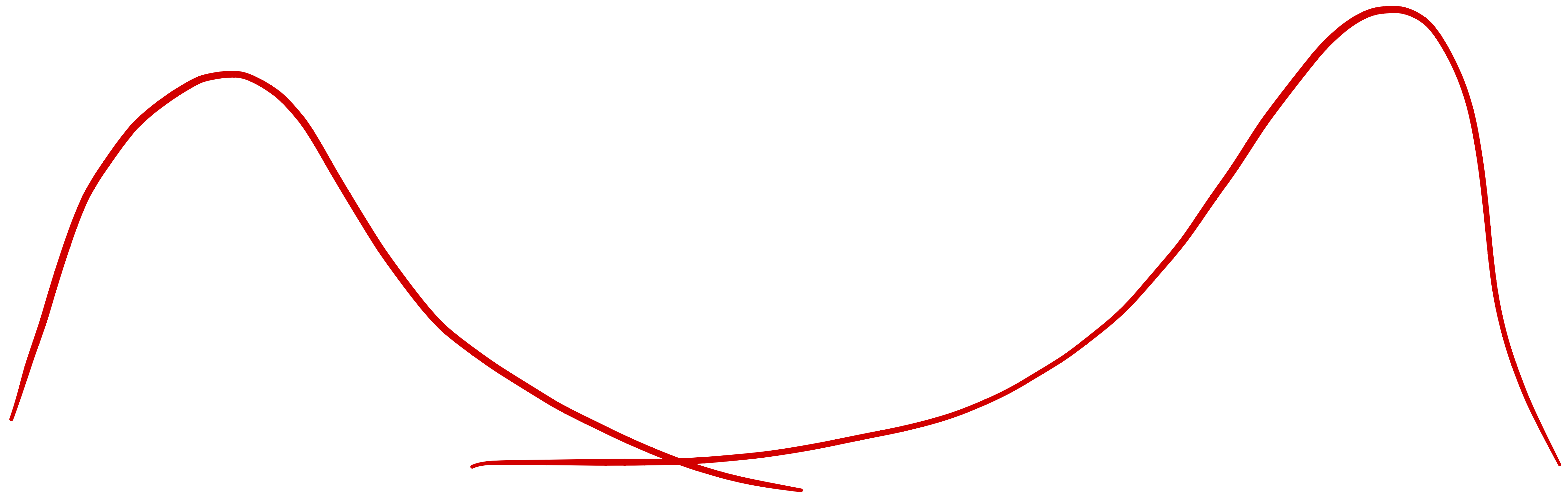

- Used to compare samples from independent populations
  - Nonparametric analog to the two-sample t-test
- Does not require populations to be normally distributed
- Requires the two populations to have the same general shape

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  - Nonparametric analog to the two-sample t-test
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- Requires the two populations to have the same general shape
- H_0 : The medians of the two populations are identical

Wilcoxon Rank-Sum Test: Back to Example #2

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- Take a sample of $n_2 = 15$ people who do have the disease
- H_0 : The median body temperature for those without the disease is greater than or equal to those with the disease

Wilcoxon Rank-Sum Test: Back to Example #2

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- Take a sample of $n_2 = 15$ people who do have the disease
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- Test at the $\alpha = 0.05$ significance level

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- Thus, the average ranks for the two samples (i.e., W_1/n_1 and W_2/n_2) should be approximately equal

Data Table

No Disease		Disease	
Temp	**Rank**	**Temp**	**Rank**
98.1	1	99.3	8
98.5	2	99.4	9.5
98.6	3	99.4	9.5
98.8	4	99.5	
98.9	5	99.5	
99.0	6	99.6	
99.2	7	99.7	
99.5		99.7	
99.6		100.0	
99.7		100.0	
100.5		100.1	
101.0		100.1	
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99.5	12	99.7	17
99.6	14.5	100.0	19.5
99.7	17	100.0	19.5
100.5	24	100.1	22
101.0	25	100.1	22
		100.1	22
		101.1	26
		101.9	27

$w_1 =$

$w_2 =$

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$W - \binom{n_1}{2}$

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- $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2}$ and $\sigma_W = \sqrt{\dfrac{n_1 n_2(n_1 + n_2 + 1)}{12}}$

$\mu = n_1 \left(\underbrace{\dfrac{n_1 + n_2}{2}}_{\text{avg rank}} \right)$

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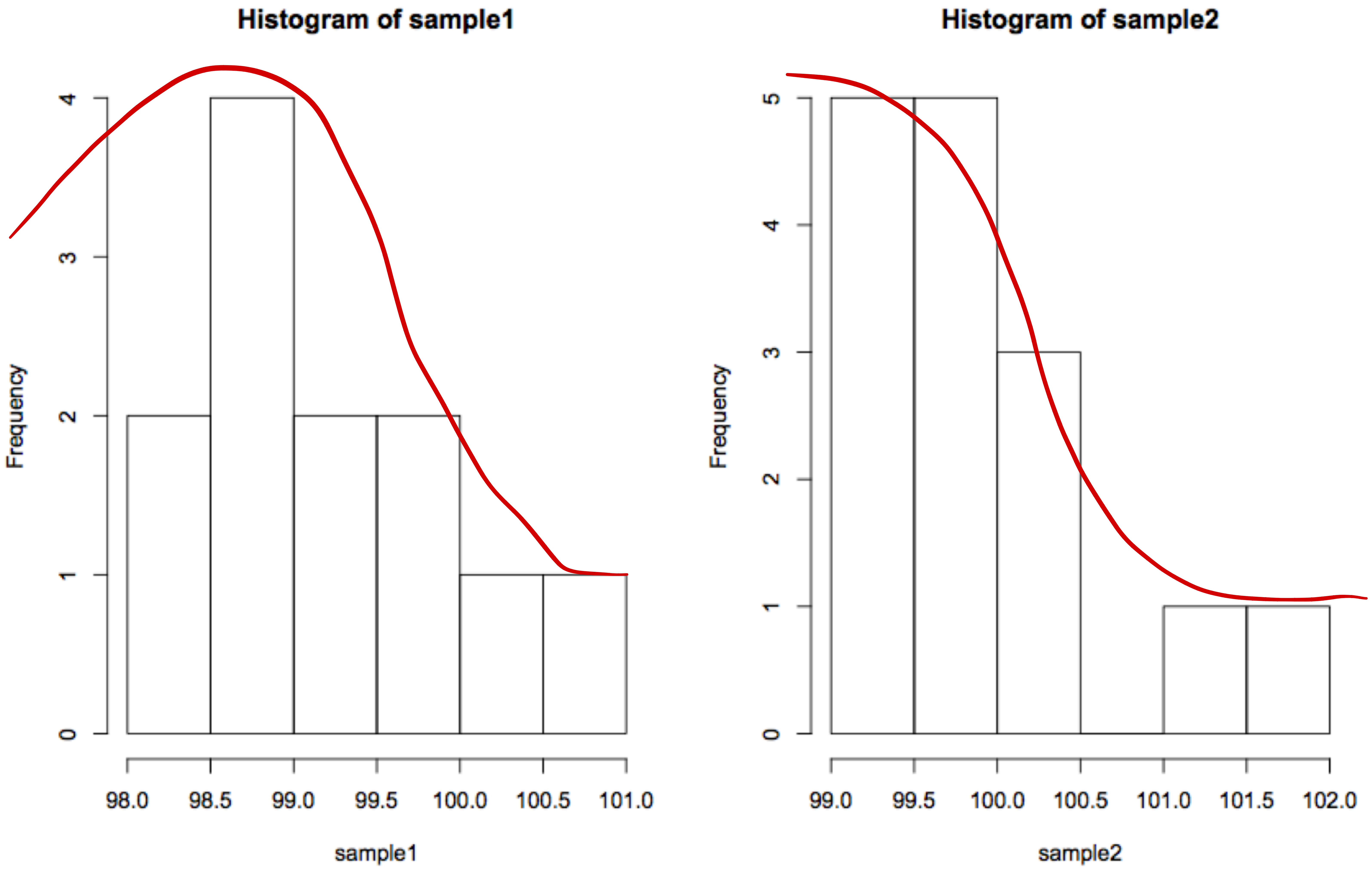
- Then,
  - $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2}$ and $\sigma_W = \sqrt{\dfrac{n_1 n_2(n_1 + n_2 + 1)}{12}}$
  - $z_W \sim N(0,1)$ when n_1 and n_2 are large enough $(n_1, n_2 > 10)$

Histograms of Samples



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- The sum of ranks for sample 1 is 120.5 $= w_1$
- The sum of ranks for sample 2 is 257.5 $= w_2$

$$w = \min(w_1, w_2) = \underline{120.5}$$

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- The sum of ranks for sample 2 is 257.5

- Thus, $W = 120.5$, $n_1 = 12$, and $n_2 = 15$

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- $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2} =$

Wilcoxon Rank-Sum Test

$$\frac{1}{n_1+n_2}\cdot 1 + \frac{1}{n_1+n_2}\cdot 2 + \cdots + \frac{1}{n_1+n_2}\cdot (n_1+n_2)$$

- Given $W = 120.5$, $n_1 = 12$, and $n_2 = 15$, we then have
- $\mu_W = \dfrac{n_1(n_1+n_2+1)}{2} =$
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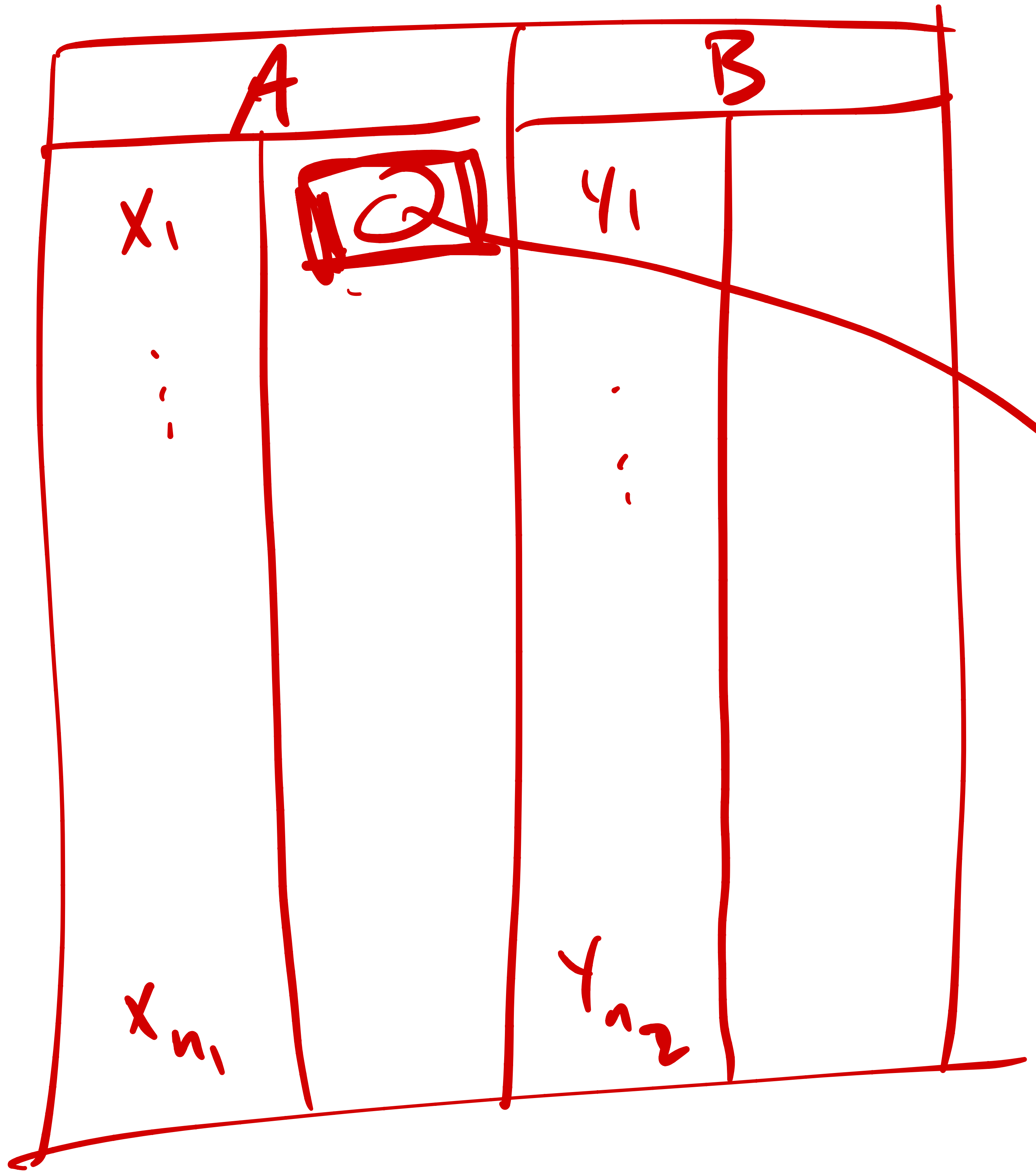

Ranks

1

⋮

$n_1 + n_2$

$$\frac{(n_1+n_2)+1}{2}$$

Wilcoxon Rank-Sum Test

- Given $W = 120.5$, $n_1 = 12$, and $n_2 = 15$, we then have
- $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2} =$ $\dfrac{12(12+15+1)}{2} = 168$
- $\sigma_W = \sqrt{\dfrac{n_1 n_2 (n_1 + n_2 + 1)}{12}} =$ 20.49
- Thus, we have $z_W = \dfrac{W - \mu_W}{\sigma_W} =$ $\dfrac{120.5 - 168}{20.49} = \underline{-2.318}$

Wilcoxon Rank-Sum Test

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- Calculating the p-value:

Wilcoxon Rank-Sum Test

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$p = Pr(Z < -2.318) = \text{pnorm}(-2.318) = 0.01$

- Conclusion:

$p < \alpha \rightarrow$ Reject H_0.

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  - `pwilcox(120.5-78,12,15)=0.0093`

Wilcoxon Rank-Sum Test: R Code

F → N(0,1)

T → wilcox distr.

```
> wilcox.test(sample1,sample2, exact=F, correct=F, alt="less")

	Wilcoxon rank sum test

data:  sample1 and sample2
W = 42.5, p-value = 0.01009
alternative hypothesis: true location shift is less than 0
```

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