Study Notes on Topology

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Preface. These notes are my own attempt to clarify some of the basic concepts of introductory topology (what is commonly known as point-set or general topology). They are rather brief, and are certainly not a comprehensive introduction, but I do hope they might be helpful to anyone wishing to learn the fundamentals of the subject. Chapter 1 gives some mathematical preliminaries, mostly to establish conventions. Chapter 2 is the main section of these notes.

Citations are not given within the text. The material is all standard, and can be found in any number of suitable textbooks. While writing these notes I relied primarily on Royden (1968), Kolmogorov and Fomin (1975), Rudin (1987) and Munkres (2013).

Chapter 1

Introduction and Mathematical Preliminaries

Some common abbrevations

It will be convenient to use the abbreviations *wrt* (with respect to), *iff* (if and only if) , as well as the following notation: $\forall$ (for all); $\exists$ (there exists); $\ni$ (such that); $\Rightarrow$ (implies).

1.1 Numbers and sets

A *set* is a collection of distinct objects of some kind. Each member of a set is referred to as an *element*, and is represented once. A set E may be *indexed*. That is, given an index set $\mathcal{T}$, each element may be assigned a unique index $t \in \mathcal{T}$, and all indices in $\mathcal{T}$ are assigned to exactly one element of E, denoted x_i. We may then write $E = \{x_i;\, t \in \mathcal{T}\}$.

The set of (finite) real numbers is denoted $\mathbb{R}$, and the set of extended real numbers is denoted $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The restriction to nonegative real numbers is written $\mathbb{R}_+ = [0, \infty)$ and $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. The strictly positive real numbers will be denoted $\mathbb{R}_{>0} = (0, \infty)$, with extension $\bar{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{\infty\}$. We use standard notation for open, closed, left closed and right closed intervals (a, b), $[a, b]$, $[a, b)$, $(a, b]$. A reference to a interval I on $\bar{\mathbb{R}}$ may be any of these types.

The set of (finite) integers will be denoted $\mathbb{I}$, while the extended integers will be $\mathbb{I}_\infty = \mathbb{I} \cup \{-\infty, \infty\}$. The set of natural numbers $\mathcal{N}$ is taken to be the set of positive integers, while $\mathcal{N}_0$ is the set of nonnegative integers. A rational number is any real number expressible as a ratio of integers. The set of all rational numbers is denoted $\mathbb{Q}$.

If $\mathcal{S}$ is a set of any type of number, $\mathcal{S}^d$, $d \in \mathcal{N}$, denotes the set of d-dimensional vectors $\tilde{s} = (s_1, \ldots, s_d)$, which are ordered collections of numbers $s_i \in \mathcal{S}$. In particular, the set of d-dimensional real vectors is written $\mathbb{R}^d$. When $0, 1 \in \mathcal{S}$, we may write the zero or one vector $\vec{0} = (0, \ldots, 0)$, $\vec{1} = (1, \ldots, 1)$, so that $c\vec{1} = (c, \ldots, c)$.

A collection of d numbers from $\mathcal{S}$ is *unordered* if no reference is made to the order (they are unlabeled). Otherwise the collection is *ordered*, that is, it is a vector. An unordered collection from $\mathcal{S}$ differs from a set in that a number $s \in \mathcal{S}$ may be represented more than once. Braces $\{\ldots\}$ enclose a set while parentheses $(\ldots)$ enclose a vector (braces will also be used to denote indexed sequences, when the context is clear).

The *cardinality* of a set E is the number of elements it contains, and is denoted $|E|$. If $|E| < \infty$ then E is a finite set. We have $|\emptyset| = 0$. If $|E| = \infty$, this statement does not suffice to characterize the cardinality of E. Two sets A, B are in a *1-1 correspondence* if a collection of pairs (a, b), $a \in A$, $b \in B$ can be constructed such that each element of A and of B is in exactly one pair. In this case, A and B are of equal cardinality. The pairing is known as a *bijection* (this term is introduced below in the context of mappings, although its meaning is the same).

If the elements of A can be placed in a 1-1 correspondence with $\mathcal{N}$ we say A is *countable* (is *denumerable*). We also adopt the convention of referring to any subset of a countable set as countable. This means all finite sets are countable. If for countable A we have $|A| = \infty$ then A is *infinitely countable*. Note that by some conventions, the term countable is reserved for infinitely countable sets. For our purposes, it is more natural to consider the finite sets as countable.

All infinitely countable sets are of equal cardinality with $\mathcal{N}$, and so are mutually of equal cardinality. Informally, a set is countable if it can be written as a list, finite or infinte. The set $\mathcal{N}^d$ is countable since, for example, $\mathcal{N}^2 = \{(1,1),(1,2),(2,1),(1,3),(2,2),(3,1),\ldots\}$. The set of rational numbers is countable, since the pairing of numerator and denominator, in any canonical representation, is a subset of $\mathcal{N}^2$.

A set A is *uncountable* (is *nondenumerable*) if $|A| = \infty$ but A is not countable. The set of real numbers, or any nonempty interval of real numbers, is uncountable.

If $A_1, \ldots, A_d$ are d sets, then $A_1 \times A_2 \times \ldots \times A_d = \times_{i=1}^d A_i$ is a product set, consisting of the set of all ordered selections of one element from each set $a_i \in A_i$. A vector is an element of a product set, but a product set is more general, since the sets A_i need not be equal, or even contain the same type of element. The definition may be extended to arbitrary forms of index sets. The dimension of a vector x is denoted $\dim(x)$.

A *tuple* (or n-*tuple*) is a finite ordered list of n mathematical objects, which may be of different types. The intention is usually that an n-tuple defines some new type of mathematical object. The term is a generalization of the sequence *single*, *double*, *triple*, *quadruple*, and so on. Any of these terms can be used, so that a 3-tuple is also a triple, or a triplet. An example of a 2-tuple, or double, is $(\mathbb{R}, +)$, which consists of the set of real numbers $\mathbb{R}$, paired with addition '+' as a binary operation. Then any pair consisting of a set and a binary operation is an example of a *group* if it additionally satisfies certain axioms (Definition 1.4).

1.1.1 Complex numbers

Let $\mathcal{C}$ denote the set of complex numbers $z = a + bi$, $a, b \in \mathbb{R}$, where the *imaginary unit* i is the solution to $i^2 = -1$ (the symbol j is sometimes used). Clearly, i is not a real number, but the arthmetic of real numbers can be extended to it in a quite useful way. If $b \in \mathbb{R}$, then bi is an *imaginary number*, which is assumed to possess the square $(bi)^2 = -b^2$. Thus, a and bi are the real and imaginary part of $z = a + bi \in \mathcal{C}$, respectively. A fact of some importance is that the real and imaginary part of a complex number are unique. Thus, a complex number $z = a + bi$ is exactly defined by a point $(a, b) \in \mathbb{R}^2$, and every distinct point $(a, b) \in \mathbb{R}^2$ defines a unique complex number.

Note that $\mathbb{R} \subset \mathcal{C}$, in the sense that $z = a + 0 \cdot i$ is a complex number with imaginary part $0 \cdot i$. In this case, $z = a$ can be taken to be a real number.

Imaginary numbers can be added and multiplied, so that $bi + b'i = b'i + bi = (b + b')i$ and $bi \times b'i = b'i \times bi = -bb'$. Imaginary and real numbers can also be multipled, giving $a \times bi = bi \times a = (ab)i$. A product of two imaginary or two real numbers is a real number, while a product of a real and imaginary number is imaginary. Addition and multiplication of complex numbers follows directly. If $z_j = a_i + b_j i$, $j = 1, 2$, then

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 = (a_1 + a_2) + (b_1 + b_2)i \\ z_1 z_2 &= z_2 z_1 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i. \end{aligned}$$

The *conjugate* of $z = a + bi \in \mathcal{C}$ is written $\bar{z} = a - bi$, so that $z\bar{z} = a^2 + b^2 \in \mathbb{R}$. Together, z and $\bar{z}$, without reference to their order, form a *conjugate pair*.

The absolute value of $a \in \mathbb{R}$ is denoted $|a| = \sqrt{a^2}$, while $|z| = (z\bar{z})^{1/2} = (a^2 + b^2)^{1/2} \in \mathbb{R}$ is also known as the magnitude or modulus of $z \in \mathcal{C}$.

Euler's formula permits exponentiation of an imaginary number:

$$e^{ix} = \cos x + i \sin x. \tag{1.1}$$

The trigonometric functions are evaluated in radians. Substituting $x = \pi$ gives Euler's identity $e^{i\pi} + 1 = 0$. This identity is remarkable for yielding a simple relationship between the transcendental numbers π, Euler's number e and the imaginary number i.

1.1.2 The supremum and infimum of a set

Consider the set $E \subset \mathbb{R}$. We say y is an *upper bound* of E if $x \leq y \ \forall \ x \in E$. Similarly, y is a *lower bound* of E if $x \geq y \ \forall \ x \in E$.

Then $y = \max E$ if y is an upper bound of E and $y \in E$. Similarly, $y = \min E$ if y is a lower bound of E and $y \in E$. The quantities $\min E$ or $\max E$ need not exist (consider $E = (0, 1)$).

The *supremum* of E, denoted $\sup E$ is the *least upper bound* of E. Similarly, The *infimum* of E, denoted $\inf E$ is the *greatest lower bound* of E.

In contrast with the min, max operations, the supremum and infimum always exist, possibly equalling $-\infty$ or ∞. For example, if $E = (0, 1)$, then $\inf E = 0$ and $\sup E = 1$. That is, $\inf E$ or $\sup E$ need not be elements of E. All numbers in $\bar{\mathbb{R}}$ are both upper and lower bounds of the empty set $\emptyset$, which means

$$\inf \emptyset = \infty \text{ and } \sup \emptyset = -\infty.$$

If $E = \{x_i;\, t \in \mathcal{T}\}$ is an indexed set we write, when possible,

$$\max E = \max_{t \in \mathcal{T}} x_i, \ \ \min E = \min_{t \in \mathcal{T}} x_i, \ \ \sup E = \sup_{t \in \mathcal{T}} x_i, \ \ \inf E = \inf_{t \in \mathcal{T}} x_i.$$

For two numbers $a, b \in \bar{\mathbb{R}}$, we may use the notations $\max\{a, b\} = x \vee y = \max(a, b)$ and $\min\{a, b\} = x \wedge y = \min(a, b)$.

1.1.3 Floor, ceiling and truncation functions

The *floor function* is defined on $\mathbb{R}$ and returns the largest integer not greater than x. This is denoted $\text{floor}(x) = \lfloor x \rfloor$. For example, $\lfloor 10.3 \rfloor = 10$, $\lfloor -7.6 \rfloor = -8$, $\lfloor 2.0 \rfloor = 2$.

The *ceiling function* is defined on $\mathbb{R}$ and returns the smallest integer not less than x. This is denoted $\text{ceiling}(x) = \lceil x \rceil$. For example, $\lceil 10.3 \rceil = 11$, $\lceil -7.6 \rceil = -7$, $\lceil 2.0 \rceil = 2$.

The *fractional part*, or sawtooth function, is defined $\text{frac}(x) = x - \lfloor x \rfloor$. For example, $\text{frac}10.3 = 0.3$, $\text{frac}{-7.6} = 0.4$, $\text{frac}2.0 = 0$.

1.2 Functions and relationships

When we write $g : \mathcal{X} \to \mathcal{Y}$, we mean g is a function, or mapping, with domain $\mathcal{X}$ and codomain $\mathcal{Y}$. Then g associates every element of $\mathcal{X}$ with a single element of $\mathcal{Y}$. The *image* of a subset $E \subset \mathcal{X}$, is written $g(E) = \{g(x) \in \mathcal{Y} : x \in E\}$, and is the set of all $y \in \mathcal{Y}$ for which $y = g(x)$ for some $x \in E$. The *preimage* of $F \subset \mathcal{Y}$ is written $g^{-1}(F) = \{x \in \mathcal{X} : g(x) \in F\}$, and is the set of all $x \in \mathcal{X}$ for which $y = g(x) \in F$. We may also refer to a *mapping* $\mathcal{X} \mapsto \mathcal{Y}$ without assigning it a symbol.

The range, or image, of g, denoted $g(\mathcal{X})$, is the set of all elements of $y \in \mathcal{Y}$ for which $y = g(x)$ for at least one $x \in \mathcal{X}$. Possibly, $g(\mathcal{X}) = \mathcal{Y}$, but it may also be a strict subset. For example, given domain $\mathcal{X} = (-\infty, \infty)$ the range of $g(x) = x^2$ is $g(\mathcal{X}) = [0, \infty)$, although it may be defined for codomain $\mathcal{Y} = (-\infty, \infty)$. The distinction between codomain and range may arise when we wish to define a family of functions on a common domain and codomain.

When the range equals the codomain, the function is *surjective*. A function is 1-1, or injective, if $g(x) \neq g(x')$ whenever $x \neq x'$. An injective function is *invertible*, that is, there exists a function $g^{-1} : g(\mathcal{X}) \to \mathcal{X}$ such that $g^{-1}(g(x)) = x$ is well defined for $x \in \mathcal{X}$ and $g(g^{-1}(y)) = y$ is well defined for $y \in g(\mathcal{X})$. Then g is *bijective* if it is both surjective and injective (note that the preimage $g^{-1}(F)$ is well defined whether or not g is injective, despite the notation).

Suppose $g : \mathcal{X} \to \mathcal{G}$, $h : \mathcal{X} \to \mathcal{H}$ are two functions on $\mathcal{X}$. We will say g is a function of h if $h(x) = h(x')$ implies $g(x) = g(x')$, which may be written $h \mapsto g$. Equivalently, there exists $g^* : \mathcal{H} \to \mathcal{G}$ such that $g(x) = g^*(h(x))$, but it is not necessary to explicitly identify g^*.

If $h \mapsto g$ and $g \mapsto h$ then g and h are equivalent, denoted $h \sim g$. The logical negation symbol $\neg$ placed in front of a statement means that that statement is not true. For example, $\neg\, h \mapsto g$ means that there exist $x \neq x'$ such that $h(x) = h(x')$ but $g(x) \neq g(x')$. Then $\neg\, h \sim g$ means that at least one of the statements $\neg\, h \mapsto g$ or $\neg\, g \mapsto h$ hold.

Example 1.1 The statements $(x, y) \mapsto x$ and $\neg\, x \mapsto (x, y)$ are both true. Note that we are actually interpreting the symbol $\mapsto$ in two ways, which are nonetheless equivalent. We can say either $(x, y) \mapsto x$ is true, or that there exists a mapping $(x, y) \mapsto x$. ///

It is important to note that $h \mapsto \vec{1}$ is always true for any h, where $\vec{1}$ is some suitably defined constant unit element. This does not mean than $h(x) = \vec{1}$ for all $x \in \mathcal{X}$. It means that if $g(x) \equiv \vec{1}$, then there exists a function g^* defined on $\mathcal{H}$ such that $g(x) = g^*(h(x))$. Thus, $\vec{1}$ can always be interpreted as a function on $\mathcal{X}$. For this reason, use of a phrase such as 'g depends on h' to describe $h \mapsto g$ must be interpreted with caution. While it means 'we may evaluate $g(x)$ if we are given the value of $h(x)$', in the special case $g(x) \equiv \vec{1}$ it is also true that 'we may evaluate $g(x)$ whether or not we are given the value of $h(x)$'.

1.3 Sequences and limits

A sequence of real numbers $a_0, a_1, a_2, \ldots$ will be written $\{a_k\}$. Depending on the context, a_0 may or may not be defined. For any sequence of real numbers, by $\lim_{k\rightarrow\infty} a_k = a$ is always meant that $\forall\epsilon > 0\ \exists K \ni k > K \Rightarrow |a - a_k| < \epsilon$. A reference to $\lim_{k\rightarrow\infty} a_k$ implies an assertion that a limit exists. This will sometimes be written $a_k \rightarrow a$ or $a_k \rightarrow_k a$ when the context makes the meaning clear.

We also take $\lim_{n\rightarrow\infty} a_n = \infty$ to mean $\forall M \in \mathbb{R}\ \exists K \ni k > K \Rightarrow a_k > M$. Then $\lim_{n\rightarrow\infty} a_n = -\infty$ *iff* $\lim_{n\rightarrow\infty} -a_n = \infty$.

Sometimes sequences are constrained to approach a limit from above. In particular, we write $a_k \downarrow a$ if $a_k \rightarrow a$ and $a_k \geq a$. Similarly, $\{a_k\}$ approaches a from below, denoted $a_k \uparrow a$, if $a_k \rightarrow a$ and $a_k \leq a$.

If $a_{k+1} \geq a_k$, the sequence must possess a limit $a_k \uparrow a$, possibly ∞. Similarly, if $a_{k+1} \leq a_k$, there exists a limit $a_k \downarrow a$, possibly $-\infty$. Then $\{a_k\}$ is an *nondecreasing* or *nonincreasing* sequence (or *increasing*, *decreasing* when the defining inequalities are strict).

Then $\limsup_{k\rightarrow\infty} a_k = \lim_{k\rightarrow\infty} \sup_{i\geq k} a_i$. This quantity is always defined since $a'_k = \sup_{i\geq k} a_i$ defines an nonincreasing sequence. Similarly $\liminf_{k\rightarrow\infty} a_k = \lim_{k\rightarrow\infty} \inf_{i\geq k} a_i$ always exists. We always have $\liminf_{k\rightarrow\infty} a_k \leq \limsup_{k\rightarrow\infty} a_k$ and $\lim_{k\rightarrow\infty} a_k$ if and only if $a = \liminf_{k\rightarrow\infty} a_k = \limsup_{k\rightarrow\infty} a_k$ in which case $\lim_{k\rightarrow\infty} a_k = a$.

When limit operations are applied to sequences of real values functions, the limits are assumed to be evaluated pointwise. Thus, if we write $f_n \rightarrow f$, this means that $f_n(x) \rightarrow f(x)$ for all x. The same understanding holds for $f_n \uparrow f$ and $f_n \downarrow f$. Note that pointwise convergence of a function $\lim_{n\rightarrow\infty} f_n = f$ is distinct from *uniform convergence* of a sequence of functions, which is equivalent to $\lim_{n\rightarrow\infty} \sup_x |f_n(x) - f(x)| = 0$. Of course, uniform convergence implies pointwise convergence, but the converse does not hold. Unless uniform convergence is explicitly stated, pointwise convergence is intended.

A sequence $\{a_k\}$ is *of order* $\{b_k\}$ if $\limsup_k a_k/b_k < \infty$, and may be written $a_k = O(b_k)$. A sequence $\{b_k\}$ *dominates* $\{a_k\}$ if $\lim_k a_k/b_k = 0$, which may be written $a_k = o(b_k)$. A stronger condition holds if $\lambda^u\{a_k\} < \lambda^l\{b_k\}$, in which case we say $\{b_k\}$ *linearly dominates* $\{a_k\}$, which may be written $a_k = o_\ell(b_k)$. Similarly, for two real valued mappings f_t, g_t on $\mathbb{R}_{>0}$ we write $f_t = o(g_t)$ if $\lim_{t\rightarrow\infty} f_t/g_t = 0$, that is, g_t *dominates* f_t.

1.4 Classes of real valued functions

Suppose $\mathcal{X}, \mathcal{Y}$ are subsets of $\mathbb{R}$. We are given real valued function $f : \mathcal{X} \rightarrow \mathcal{Y}$. Assume $\mathcal{X}$ is open. We say f is continuous at $x_0 \in \mathcal{X}$ if

$$\lim_{n\rightarrow\infty} f(x_0 + \epsilon_n) = f(x_0), \tag{1.2}$$

for any sequence $\epsilon_n \rightarrow 0$. If (1.2) holds for sequences satisfying $\epsilon_n \downarrow 0$ ($\epsilon_n \uparrow 0$) then f is right (left) continuous. In this case we use the notation

$$\lim_{n\rightarrow\infty} f(x + \epsilon_n) = f(x_-), \text{ if } \epsilon_n \uparrow 0,$$
$$\lim_{n\rightarrow\infty} f(x + \epsilon_n) = f(x_+), \text{ if } \epsilon_n \downarrow 0.$$

Then f is continuous at x if $f(x_-) = f(x_+)$.

We say the derivative of f exists at $x_0 \in \mathcal{X}$ if there is a single finite number $f'(x_0)$ for which

$$\lim_{n\to\infty} \frac{f(x_0+\epsilon_n) - f(x_0)}{\epsilon_n} = f'(x_0), \tag{1.3}$$

for any sequence $\epsilon_n \to 0$ with $|\epsilon_n| > 0$. In this case, f must also be continuous at x_0. If (1.3) holds for sequences satisfying $\epsilon_n \downarrow 0$ ($\epsilon_n \uparrow 0$) then f is right (left) differentiable. If the left and right derivatives exist, and are equal, then f is differentiable.

We will make use of the *mean value theorem*:

Theorem 1.1 (Mean Value Theorem) Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable in (a,b). Then there exists $c \in (a,b)$ such that $f'(c) = [f(b) - f(a)]/[b-a]$. ///

Differentiation may be defined iteratively. The kth derivative of f at x is denoted $f^{(k)}(x)$, where it exists. Alternatively, an assumption of existence is implicit in a reference to $f^{(k)}(x)$. By convention $f^{(0)}(x) = f(x)$. Then $f^{(k)}(x)$ is the derivative of $f^{(k-1)}(x)$, $k \geq 1$. The class of all functions f for which $f^{(k)}(x)$ is continuous on $\mathcal{X}$ is denoted $C^k(\mathcal{X},\mathcal{Y})$. Where the context is clear, $C^k(\mathcal{X},\mathcal{Y})$ may be shortened to $C^k(\mathcal{X})$ or C^k. Then $C^0(\mathcal{X},\mathcal{Y})$ is the class of all continuous functions, and $C^\infty(\mathcal{X},\mathcal{Y})$ is the class of *infinitely differentiable functions*, or *smooth functions*. Clearly, $C^{k+1} \subset C^k$, $k \geq 0$. When interest is only in the first or second derivative, we may use the shorthand (already used above) $f'(x) = f^{(1)}(x)$, $f''(x) = f^{(2)}(x)$.

A real valued function $f : \mathcal{X} \to \bar{\mathbb{R}}$ is *lower semicontinuous* at x_0 if $x_n \to_n x_0$ implies $\liminf_n f(x_n) \geq f(x_0)$, or *upper semicontinuous* at x_0 if $x_n \to x_0$ implies $\limsup_n f(x_n) \leq f(x_0)$. We use the abbreviations *lsc* and *usc*. A function is, in general, *lsc* (*usc*) if it is *lsc* (*usc*) at all $x_0 \in \mathcal{X}$. Equivalently, f is *lsc* if $\{x \in \mathcal{X} \mid f(x) \leq \lambda\}$ is closed for all $\lambda \in \mathbb{R}$, and is *usc* if $\{x \in \mathcal{X} \mid f(x) \geq \lambda\}$ for all $\lambda \in \mathbb{R}$. A function is continuous (at x_0) if and only if it is both *lsc* and *usc* (at x_0). Note that only sequences in $\mathcal{X}$ are required for the definition, so that if f is *lsc* or *usc* on $\mathcal{X}$, it is also *lsc* or *usc* on $\mathcal{X}' \subset \mathcal{X}$.

The usual kth order partial derivatives, when they exist, are written $\partial^k f/\partial x_{i_1} \ldots \partial x_{i_k}$, and if $d = 1$ the kth total derivative is written $d^k f/dx^k = f^{(k)}(x)$. A derivative is a function on $\mathcal{X}$, unless evaluation at a specific value of $\tilde{x} \in \mathcal{X}$ is indicated, as in $d^k f/dx^k|_{x=x_0} = f^{(k)}(x_0)$. The first and second total derivative will also be written $f'(x)$ and $f''(x)$ when the context is clear.

Then $f : \mathcal{X} \to \mathcal{Y}$ is a *bounded function* if $\sup_{x\in\mathcal{X}} |f(x)| < \infty$. The class of all bounded mappings $f : \mathcal{X} \to \mathcal{Y}$ will be denoted $B(\mathcal{X},\mathcal{Y})$, or $B(\mathcal{X})$ when the context is clear.

1.5 Infinite series

Suppose we are given sequence $\{a_k\}$. The corresponding *series* (or *infinite series*) is denoted

$$\sum_{k=1}^{\infty} a_k = \sum_k a_k = a_1 + a_2 + \ldots.$$

Some care is needed in defining a sum of an infinite collection of numbers. First, define *partial sums*

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n, \;\; n \geq 1.$$

We may set $S_0 = 0$. It is natural to think of evaluating a series by sequentially adding each a_n to a cumulative total S_{n-1}. In this case, the total sum equals $\lim_n S_n$, assuming the limit exists. We say that the series (or simply, the sum) exists if the limit exists (including $-\infty$ or ∞). The series is *convergent* if the sum exists and is finite. A series is *divergent* if it is not convergent.

It is important to establish whether or not the value of the series depends on the order of the sequence. Precisely, suppose $\sigma : \mathcal{N} \rightarrow \mathcal{N}$ is a bijective mapping (essentially, an infinite permutation). If the series $\sum_k a_k$ exists, we would like to know if

$$\sum_k a_k = \sum_k a_{\sigma(k)}. \tag{1.4}$$

Since these two quantities are limits of distinct partial sums, equality need not hold. This question has a quite definite resolution. A series $\sum_k a_k$ is called *absolutely convergent* if $\sum_k |a_k|$ is convergent (so that all convergent series of nonnegative sequences are absolutely convergent). A convergent sequence is *unconditionally convergent* if (1.4) holds for all permutations σ. It may be shown that a series is absolutely convergent if and only if it is unconditionally convergent. Therefore, a convergent series may be defined as *conditionally convergent* if either it is not absolutely convergent, or if (1.4) does not hold for at least one σ. Interestingly, by the *Riemann series theorem*, if $\sum_k a_k$ is conditionally convergent then for any $L \in \bar{\mathbb{R}}$ there exists permutation σ_L for which $\sum_k a_{\sigma_L(k)} = L$.

There exist many well known tests for series convergence, which can be found in most calculus textbooks.

Finally, we make note of the following convention. We will sometimes be interested in summing over a strict subset of the index set $\mathcal{T}' \subset \mathcal{T}$. This poses no particular problem if the series $\sum_t a_t$ is well defined. If it happens that $\mathcal{T}' = \emptyset$, we will take

$$\sum_{t \in \emptyset} a_t = 0 \text{ and } \prod_{t \in \emptyset} a_t = 1. \tag{1.5}$$

1.5.1 Geometric series

We will make use of the following geometric series:

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{(i+m)!}{i!} r^i &= \frac{m!}{(1-r)^{m+1}} \text{ for } r^2 < 1, \;\; m = 0, 1, 2, \ldots \\ \sum_{i=0}^{n} r^i &= \frac{1 - r^{n+1}}{1-r} \text{ for } r \neq 1. \end{aligned} \tag{1.6}$$

1.6 Binary relations

Let $\mathcal{X}$ be a set. We sometimes define a type of relationship between two objects in $\mathcal{X}$ which does or does not hold. For example, $4 \leq 5$ holds, $4 < 4$ does not hold, $\{1, 2\} \subset \{1, 2, 3\}$ holds, and so on. While it is natural to characterize such a relationship using some analytical rule, the more abstract approach is given in the following definition:

Definition 1.1 Given a set $\mathcal{X}$, a *binary relation* $\leq$ is a set of ordered pairs $\mathcal{P} \subset \mathcal{X} \times \mathcal{X}$. We then say $x \leq y$ holds *iff* $(x, y) \in \mathcal{P}$. ///

In Definition 1.1 the symbol $\leq$ refers to *any* relation.

1.6.1 Partial orderings

Ordering is a type of binary relationship. Any two real numbers x, y can be ordered, in the sense that exactly one of the three statements $x < y$, $x > y$ or $x = y$ must hold. In set theory, the subset relation is analogous to order, but it is not always possible to order two sets in this way. Thus, it is sometimes possible to define a natural concept of ordering on some space $\mathcal{X}$, even if not all pairs $x, y \in \mathcal{X}$ can be ordered. This leads to the idea of partial ordering, conventially understood to be defined by the following axioms.

Definition 1.2 A binary relation $\leq$ on a set $\mathcal{X}$ is a *partial ordering* if it satisfies the following three axioms for any $x, y, z \in \mathcal{X}$:

> **Reflexivity:** $x \leq x$.
> **Antisymmetry:** If $x \leq y$ and $y \leq x$ then $x = y$.
> **Transitivity:** If $x \leq y$ and $y \leq z$ then $x \leq z$.

Then $\mathcal{X}$ is referred to as a *partially ordered set*, or *poset*. ///

Of course, a completely ordered set, such as $\mathbb{R}$, is a special case of a partially ordered set.

1.6.2 Equivalence relationships

Just as order can be generalized, so can equality. If we have any set $\mathcal{X}$, we can always decide whether or not $x = y$ for any pair $x, y \in \mathcal{X}$. It will sometimes be the case that $x \neq y$, but the two elements share some property which for a particular application makes them essentially the same (parallel lines, for example). The equivalence relation expresses this idea.

Definition 1.3 A binary relation $\sim$ on a set $\mathcal{X}$ is an *equivalence relation* if it satisfies the following three axioms for any $x, y, z \in \mathcal{X}$:

> **Reflexivity:** $x \sim x$.
> **Symmetry:** If $x \sim y$ then $y \sim x$.
> **Transitivity:** If $x \sim y$ and $y \sim z$ then $x \sim z$.

///

Given an equivalence relation, an *equivalence class* is any set of the form $E_x = \{y \in \mathcal{X} \mid y \sim x\}$. If $y \in E_x$ then $E_y = E_x$. Each element $x \in \mathcal{X}$ is in exactly one equivalence class, so $\sim$ induces a partition of $\mathcal{X}$ into equivalence classes.

In Euclidean space, 'is parallel to' is an equivalence relation, while 'is perpendicular to' is not.

Partitions induced by mappings

An important type of equivalence relationship involves the partition induced by a mapping. Suppose $h : \mathcal{X} \to \mathcal{Y}$ is a surjective mapping (that is, $\mathcal{Y}$ is the range of h). Define the binary relation $\sim$ on any pair $x, x' \in \mathcal{X}$ as $x \sim x'$ *iff* $h(x) = h(x')$. This is clearly an equivalence relation, which generates equivalence classes $E_y = \{x \in \mathcal{X} : h(x) = y\}$ for all $y \in \mathcal{Y}$. Then h generates a partition of $\mathcal{X}$, denoted using suitable notation:

$$\mathcal{X}_h = \{E_y : y \in \mathcal{Y}\}.$$

1.7 The most common algebraic structures

We first introduce plainly the axiomatic definitions of two of the most commonly encountered algebraic structures, the *group* and the *field*. These are of interest for their own sake, but are also used to build more complex spaces. Here, we are interested in the abstraction of arithmetic systems. The notion of real numbers can be generalized to that of a *field* $\mathcal{K}$, which is a set of *scalars* that is closed under the rules of addition and multiplication comparable to those available for real numbers $\mathbb{R}$. Both $\mathbb{R}$ and complex numbers $\mathcal{C}$ are fields.

Definition 1.4 A *group* is a pair $(G, *)$ where G is a set and $*$ is a binary operation on G satisfying the axioms:

> **Closure** If $a, b \in G$ then $a * b \in G$,
> **Associativity** For all $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$,
> **Existence of identity** There exists $e \in G$ such that for all $a \in G$ we have $a * e = e * a = a$,
> **Existence of inverse** For each $a \in G$ there exists $b \in G$ such that $a * b = b * a = e$.

///

Definition 1.5 An *abelian group* (or *commutative group*) is a group $(G, *)$ which satisfies the additional axiom

> **Commutativity** For any $a, b \in G$ we have $a * b = b * a$.

///

The group then is used to construct the field.

Definition 1.6 A *field* is a triplet $(\mathcal{K}, +, \times)$ where $\mathcal{K}$ is a set and $+$ and $\times$ are binary operations on $\mathcal{K}$ (by analogy referred to as *addition* and *multiplication*) satisfying the following axioms:

> **Group structure of addition** $(\mathcal{K}, +)$ is an abelian group,
> **Group structure of multiplication** $(\mathcal{K} - \{0\}, \times)$ is an abelian group, where 0 is the additive identity,
> **Distributivity of multiplication** For all $a, b, c \in \mathcal{K}$ we have $a \times (b+c) = (a \times b) + (a \times c)$.

///

1.8 Vector spaces

A *vector space* is then a class of objects $\mathcal{V}$ sharing the essential properties of the class of vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 1.7 Suppose we are given a field $\mathcal{K}$ of *scalars* and an abelian group $(\mathcal{V}, +)$ of *vectors* (by analogy, $+$ is referred to as vector addition). Suppose also that for each pair $a \in \mathcal{K}$ and $x \in \mathcal{V}$ there exists a unique composite product $a \circ x \in \mathcal{V}$. The collection $(\mathcal{K}, \mathcal{V}, +, \circ)$ is a *vector space* (or *linear space*) if the following additional axioms are satisfied:

> **Existence of identity for composite product** For any vector $x \in \mathcal{V}$ we have $1 \circ x = x$ where 1 is the multiplicative identity of $\mathcal{K}$,
> **Compatibility of scalar and composite product** For all $a, b \in \mathcal{K}$ and $x \in \mathcal{V}$ we have $a \circ (b \circ x) = (a \times b) \circ x$,
> **Distributivity over scalar addition** For all $a, b \in \mathcal{K}$ and $x \in \mathcal{V}$ we have $(a + b) \circ x = a \circ x + b \circ x$,
> **Distributivity over vector addition** For all $a \in \mathcal{K}$ and $x, y \in \mathcal{V}$ we have $a \circ (x + y) = (a \circ x) + (a \circ y)$.

///

1.8.1 Finite dimensional vector spaces

A finite dimensional vector spaces is any subset $\mathcal{V} \subset \mathcal{K}^n$ satisfying the axioms of Definition 1.7. Elements $x_1, \ldots, x_m$ of $\mathcal{K}^n$ are *linearly independent* if $\sum_{i=1}^m a_i x_i = 0$ implies $a_i = 0$ for all i. Equivalently, no x_i is a linear combination of the remaining vectors. The *span* of a set of vectors $\tilde{x} = (x_1, \ldots, x_n)$, denoted $span(\tilde{x})$, is the set of all linear combinations of vectors in $\tilde{x}$, which must be a vector space. Suppose the vectors in $\tilde{x}$ are not linearly independent. This means that, say, x_m is a linear combination of the remaining vectors, and so any linear combination in $span(\tilde{x})$ including x_m may be replaced with one including only the remaining vectors, so that $span(\tilde{x}) = span(x_1, \ldots, x_{m-1})$. The *dimension* of a vector space $\mathcal{V}$ is the minimum number of vectors whose span equals $\mathcal{V}$. Clearly, this equals the number in any set of linearly independent vectors which span $\mathcal{V}$. Any such set of vectors forms a *basis* for $\mathcal{V}$. Any vector space has a basis.

1.8.2 Linear operators

Set domain $\mathcal{X} = \mathbb{R}^d$. A mapping $A : \mathcal{X} \rightarrow$ is a *linear operator* on $\mathcal{X}$ if for any vectors $x, y \in \mathcal{X}$ and scalars a, b we have

$$A(ax + by) = aAx + bAy.$$

When the domain is $\mathbb{R}^d$ any linear operator can be represented as a $d \times d$ matrix. For calculation purposes a vector $x \in \mathcal{X}$ is equivalent to a $d \times 1$ column vector. The image

$$\mathcal{V} = A\mathcal{X} = \{Ax : x \in \mathcal{X}\}$$

is always a vector space. In addition, the rank of A is defined as the dimension of $\mathcal{V}$. If A is invertible (or nonsingular) than it is of *full rank* d, so that $\mathcal{V} = \mathcal{X}$. Essentially, $\mathcal{V}$ is the span of the column vectors of A, which are linearly independent *iff* A is of full rank.

1.8.3 Dimension of Euclidean subsets

It will be useful to have a practical method of defining the dimension of a subset of Euclidean space. The dimension of a vector space is the size of its base. We next consider general subsets.

The dimension of a set

It will, of course, be necessary to define the dimension of subsets other than vector spaces. We first give conditions under which the dimension of $E \subset \mathbb{R}^d$ is d.

Definition 1.8 A subset $E \subset \mathbb{R}^d$ has dimension d if every point x in the interior O_E is contained in a neighborhood $B_\epsilon(x)$ also contained in O_E. ///

In general, most subsets of interest can be generated by affine transformations of the form

$$L(x) = Ax + b, \quad x \in \mathbb{R}^d \tag{1.7}$$

where A is a linear operator on $\mathbb{R}^d$, and $b \in \mathbb{R}^d$. Suppose E can be expressed

$$E = \{L(x) : x \in E^*\} \tag{1.8}$$

where $E^* \subset \mathbb{R}^d$ is a set of dimension d. Then the dimension of E is equal to the rank of A. Furthermore, a countable union $E = \cup_{i\geq 1} E_i$ of sets of dimension $d' \leq d$ is also of dimension d'.

Example 1.2 Suppose we are given a subset $E \subset \mathbb{R}^3$ defined by

$$E = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 + p_2 + p_3 = 1, \ \min_i p_i > 0\}.$$

This can be recognized as a probability simplex in $\mathbb{R}^3$. It can be visualized as a triangle with vertices $(0, 0, 1), (0, 1, 0), (0, 0, 1)$. Although it is a subset of $\mathbb{R}^3$, it has zero volume, since it is essentially a two dimensional object. But, E is neither a vector space, nor a subset of any two dimensional vector space. We can, however, represent E using (1.7) and (1.8), setting

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad b = (0, 0, 1), \quad E^* = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1 \geq 0, \ p_2 \geq 0, \ p_1 + p_2 \leq 1\}.$$

Clearly, E^* is of dimension 3, while A is of rank 2, since the first two columns are linearly independent, but all three columns are not. This verifies that E is of dimension 2. ///

1.8.4 L^p norms in Euclidean space

The concept of the *norm* is central to functional analysis, and can be thought of as a generalization of the type of norms commonly used in Euclidean analayisis. The L^p norms are an especially important class of such norms.

Definition 1.9 The L^p norm for $\boldsymbol{u} \in \mathbb{R}^m$ is defined as

$$\|\boldsymbol{u}\|_p = \left[\sum_{i=1}^{m} u_i^p\right]^{1/p},$$

for $p \in \mathbb{R}_{>0}$. In addition, the *supremum norm* (setting $p = \infty$) is defined as

$$\|\boldsymbol{u}\|_\infty = \max_{i=1,\ldots,m} |u_i|.$$

Suppose $w_i > 0$, $i = 1, \ldots, m$ are a set of *weights*. The *weighted* L^p norm, denoted L^p_w, is defined as

$$\|\boldsymbol{u}\|_{p,w} = \left[\sum_{i=1}^{m} (w_i u_i)^p\right]^{1/p},$$

for $p \in \mathbb{R}_{>0}$. In addition, the *weighted supremum norm* is defined as

$$\|\boldsymbol{u}\|_{\infty,w} = \max_{i=1,\ldots,m} |w_i u_i|.$$

///

Chapter 2

Topological Spaces

The branch of mathematics referred to as *topology* is often characterized as the study of those properties of geometric objects that are preserved under continuous transformations. However, the basic theory of *general*, or *point-set* topology concerns the axiomatization of notions such as convergence, continuity or connectedness which have natural interpretations in real analysis. In turn, these properties are defined in terms of the class of open sets. Therefore, topology is also used to create a foundation for a rigorous development of other branches of applied mathematics.

In real analysis, the conditions making a set in $\mathbb{R}^d$ open or closed are well understood, and pose no ambiguity. The notion of convergence is similarly well defined. We have no difficulty stating that the sequence $a_n = 1/n$ converges to 0 as $n \to \infty$.

On the other hand, defining convergence in function spaces admits more choices. Consider the sequence of functions $f_n(x) = I\{x > n\}$. Does f_n converge to $f_0 \equiv 0$? It converges *pointwise*, in the sense than $\lim_n f_n(x) = f_0(x)$ for each x. But it does not converge *uniformly*, since $\lim_n \sup_x |f_n(x) - f_0(x)| = 1 > 0$. On the other hand, the sequence of functions $f_n(x) = n^{-1} I\{x > n\}$ converges both pointwise and uniformly to f_0.

However, the need to precisely define convergence in non-Euclidean spaces should not obscure another important reason for understanding the axioms of topology. This is that once a coherent definition of the open set is accepted, this is all that is needed to define convergence and continuity (or even closed sets). In practice, this means that arguments within theorems might follow simply by noting that some relevant set is open or closed. The motivation is to make analysis simpler, not more challenging.

It is important to understand that while in conventional analysis openness is a property that can be tested for a single set, a topology defines openness entirely by membership in a class of subsets, with no further reference to any specific properties.

Definition 2.1 (Topological space) Let $\mathcal{T}$ be a collection of subsets of a set $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{T})$ is a *topological space* (or $\mathcal{T}$ is a *topology* on $\mathcal{X}$) if the following conditions hold:

(a) $\mathcal{X} \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;
(b) if $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$;

(c) for any collection of sets $\{A_i : i \in I\}$ in $\mathcal{T}$ (countable or uncountable) we have $\cup_{i\in I} A_i \in \mathcal{T}$.

Any element of $\mathcal{T}$ is referred to as an *open set*, and the complement of any set in $\mathcal{T}$ is a *closed set*. ///

Some remarks:

(a) To be precise, the collection of all complements of open sets is equal to the collection of all sets whose complements are open.

(b) One convention sometimes used is to reserve the symbols F and G for closed and open sets, respectively. Of course, correctness does not require consistent use of this convention, but its violation may invite scrutiny among those with whom it is familiar. While this convention will not be used exclusively in these notes, it will never be violated (that is, to the best of the author's ability, G will never be a closed set). Similarly, the symbol K is often reserved for compact sets (definition below).

(c) Note that the term "topology" will refer to a collection of open sets satisfying the axioms of Definition 2.1. We may refer to $\mathcal{X}$ as a "topological space" without reference to a specific topology $\mathcal{T}$, but this will be assumed to exist. An open subset of $\mathcal{X}$ is then an element of some topology.

(d) An indexed collection of sets is written $\{A_i : i \in I\}$, where I is an index set of any cardinality. For convenience, we will usually write only $\{A_i\}$. In this case I can be assumed to exist, and in fact may appear later within an argument. Similarly, any sequence $x_1, x_2, \ldots$ may be written $\{x_n\}$.

(e) Note also that a set can be both open and closed, in particular $\emptyset$ and Ω. This will be true of any open set that equals its own complement. We may also construct a topology in which all sets are open, and therefore also closed. A set that is both open and closed can be referred to as a *clopen set*.

The notion of a *neighborhood* is central to topology.

Definition 2.2 (Neighborhoods) If $(\mathcal{X}, \mathcal{T})$ is a topological space, then a *neighborhood* of $x \in \mathcal{X}$ is any set B containing an open set O such that $x \in O$. ///

De Morgan's Law is often stated for two sets, but the argument extends easily to any collection of sets.

Theorem 2.1 (De Morgan's Law) Let $\{A_i\}$ be any collection of sets. The following identities always hold:

$$\cup_i A_i^c = [\cap_i A_i]^c, \tag{2.1}$$

$$\cap_i A_i^c = [\cup_i A_i]^c. \tag{2.2}$$

///

Proof. First, consider Equation (2.1). Suppose $x \in \cup_i A_i^c$. Then $\exists i'$ such that $x \in A_{i'}^c$, therefore $x \notin \cap_i A_i$, so we may write

$$\cup_i A_i^c \subset [\cap_i A_i]^c. \tag{2.3}$$

Conversely, if $x \in [\cap_i A_i]^c$, then $\exists i'$ such that $x \notin A_{i'}$, therefore $x \in \cup_i A_i^c$, so we may write

$$[\cap_i A_i]^c \subset \cup_i A_i^c. \tag{2.4}$$

Then (2.3) and (2.4) imply (2.1). To prove (2.2) apply (2.1) to $\{A_i^c\}$, then take the complement of each side of the identity. □

One important theme of general topology is the equivalence of conventional ideas of real analysis and their topological counterparts (more precisely, such conventional ideas are special cases of the more general topological property). This theme gives a useful organizing principle for an introduction to general topology. Accordingly, in these notes we examine this type of equivalence for open and closed sets (Section 2.3); boundedness (Section 2.6); continuity (Section 2.7); connectedness (Section 2.9); and convergence (Sections 2.1 and 2.10).

2.1 Topological properties and convergent sequences

We can think of a *topological property* as one which can be stated by referring exclusively to the open sets of a topology. This idea will be made more precise below in Sections 2.2 and 2.8.

For now, recall our earlier comment that notions which are quite intuitive in conventional real analysis may be expressed, and in the process generalized, as topological properties. Convergence is one such property. In real analysis we say a sequence of real numbers x_n, $n \geq 1$, converges to x_0 if for all $\epsilon > 0$ there exists N such that $|x_n - x_0| < \epsilon$ for all $n > N$. No reference to open or closed sets is needed, and we could replace, for example, the condition $|x_n - x_0| < \epsilon$ with $|x_n - x_0| \leq \epsilon$, so that the open and closed properties do not seem, at least at first glance, to play an important role.

Nonetheless, the definition of convergence in a topological space is quite exact:

Definition 2.3 (Convergence in Topological Spaces) Let $\mathcal{X}$ be a topological space. A sequence $x_n \in \mathcal{X}$ converges to $x_0 \in \mathcal{X}$ if for every open subset $O \subset \mathcal{X}$ containing x_0 $\exists$ finite $N \ni x_n \in O$ for all $n > N$. Alternatively, we say the sequence x_n possesses limit x_0, or $x_n \to x_0$. ///

This gives a notion of convergence as a topological property. No attempt to measure the distance between the elements of a sequence and its limit is made, since distance has no meaning in a space characterized only by its topology. Nonetheless, the relationship between topological convergence and the conventional notion of convergence can be clearly seen. A sequence of some general type of object x_n converges to x_0 if for *all* definitions of "near" (compare to "for all $\epsilon > 0$") there is some N such that all x_n, $n > N$ are "near" x_0. In a topology, nearness of x and y is defined by common membership in an open set. So for convergence in a topology, x_0 is "near" all but a finite number of elements of the sequence, provided we may use any topological notion of "near".

What topological convergence does is to single out only the essential characteristics of convergence, yielding the most concise (but still useful) definition possible. Importantly, this may be done independently of any specific distance calculation method.

The consequence of this is that "convergence" itself is no longer simply a statement which is true or false, but a mathematical object subject to study and analysis. As will be seen in Example 2.2 below, a sequence in $\mathcal{X}$ may converge in one topological space $(\mathcal{X}, \mathcal{T})$ but not another topological space $(\mathcal{X}, \mathcal{T}')$, a fact which is an important subject of concern in topological analysis.

2.2 Strong and weak topologies

If $\mathcal{X}$ is a singleton, then $\mathcal{T} = \{\mathcal{X}, \emptyset\}$ is a topology, and is the only topology definable on $\mathcal{X}$. In fact, $\mathcal{T} = \{\emptyset\}$ also satisfies the topological axioms of Definition 2.1. However, unless $\mathcal{X}$ a singleton, or is empty, it will be possible to define more than one topology. The following definition offers a basis on which multiple topologies on $\mathcal{X}$ may be compared.

Definition 2.4 Let $\mathcal{T}$, $\mathcal{T}'$ be two topologies on a common set $\mathcal{X}$. If $\mathcal{T}' \subset \mathcal{T}$, then $\mathcal{T}$ is a *stronger* (or *finer*) topology than $\mathcal{T}'$. Alternatively, $\mathcal{T}'$ is a *weaker* (or *coarser*) topology than $\mathcal{T}$. ///

We may always identify a strongest and weakest topology.

Definition 2.5 The class of all subsets of $\mathcal{X}$ (the *power sets*), denoted $2^{\mathcal{X}}$, is the strongest possible topology, and is referred to as the *discrete topology*. Then $\{\mathcal{X}, \emptyset\}$ is the weakest possible topology, and is referred to as the *anti-discrete topology*. ///

Verification that the discrete and anti-discrete topologies are actual topologies is straightforward, and is left to the reader. Note that the discrete and anti-discrete topologies are equal *iff* $\mathcal{X}$ contains at most one element.

A topological statement may take the form:

S: There exists a collection of open sets which has property P_1,

where P_1 is either true or false in all possible topologies (for example, the statement $x \in E$ is either true or false in all topologies). Then if this statement is true in topology $\mathcal{T}$, it will also be true in any stronger topology, but will not necessarily be true in a weaker topology. We next give an example of such a property.

Example 2.1 A topological space $(\mathcal{X}, \mathcal{T})$ is a *Hausdorff space* if for any distinct $x, y \in \mathcal{X}$ there exist open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. If $(\mathcal{X}, \mathcal{T}')$ is a Hausdorff space, then any stronger topological space $(\mathcal{X}, \mathcal{T})$ will also be a Hausdorff space, since the exemplary sets satisfy $U, V \in \mathcal{T}' \subset \mathcal{T}$. We will discuss the Hausdorff space further below. ///

Alternatively, a topological statement may take the form:

S: All open sets which have property P_1 also have property P_2,

where P_1, P_2 are either true or false in all possible topologies. Then if this statement is true in topology $\mathcal{T}$, it will also be true in any weaker topology, but will not necessarily be true in a stronger topology. We next give an example of such a property.

Example 2.2 Suppose $\mathcal{T}$ is a topology on $\mathcal{X}$. According to Definition 2.3 a sequence $\{x_n\}$ converges to x in the topology $\mathcal{T}$ if any open set containing x also contains all but a finite number of elements of the sequence.

Then suppose the topology $\mathcal{T}'$ is weaker than $\mathcal{T}$, and that the sequence $\{x_n\}$ converges to x in the topology $\mathcal{T}$ Any open set of $\mathcal{T}'$ which contains x is also in $\mathcal{T}$. By assumption all such sets contain all but a finite number of elements of the sequence. Therefore $\{x_n\}$ also converges to x in the topology $\mathcal{T}'$.

At the extremes, all sequences converge to all limits in the anti-discrete topology, since $\mathcal{X}$ is the only nonempty open set. In contrast, in the discrete topology $\{x_n\}$ converges to x *iff* $x_n = x$ for all large enough n, since $\{x\}$ is an open set (Definition 2.5). ///

2.3 Open and closed sets

When ideas from topology are applied to other branches of mathematics, one surprising, and welcome, consequence is that arguments in proofs can often rely primarily on the open and closed set properties. This can make proofs more compact and elegant.

Here, we develop a characterization of the closed set. The first property we consider follows simply from the definition of a closed set as any complement of an open set, so that the axioms defining a topology may be equivalently expressed in terms of closed sets.

Theorem 2.2 The collection of all closed sets of a topological space is closed under finite union and arbitrary intersection. ///

Proof. Let A, B be two closed subsets of a topological space. Then by De Morgan's law $(A \cup B)^c = A^c \cap B^c$, which is open, therefore $A \cup B$ is closed.

Let $\{F_i\}$ be any collection of closed sets. Then $\{F_i^c\}$ is a collection of open sets, therefore $\cup_i F_i^c$ is open. By De Morgan's law $\cup_i F_i^c = (\cap_i F_i)^c$, therefore $\cap_i F_i$ is closed. □

2.3.1 Construction of closed sets by closure

Although the closedness property follows from the definition of a topology, it will be useful to offer alternative constructive definitions of a closed set, the technical problem being to verify that these conform to the topological definition.

Essentially, we start with any set E, then construct a union of E with all other points which are in some sense on the *border* of E (which may or may not be already included in E). These are referred to as *points of closure*.

Definition 2.6 (Point of closure) If $(\mathcal{X}, \mathcal{T})$ is a topological space, then $x \in \mathcal{X}$ is a *point of closure* of a subset E if every open neighborhood of x also contains an element $y \in E$. Note that if $x \in E$, it follows that x is a point of closure of E.

In addition, x is an *accumulation point* if it is a point of closure of the subset $E - \{x\}$. A point of closure that is not an accumulation point is referred to as an *isolated point*.

Note that points of closure are also referred to as *adherent points* or *contact points*. Accumulation points are also referred to as a *limit points*. ///

Example 2.3 Consider the subset $E = [0, 1) \cup \{2\}$, and accept the conventional definitions of the open and closed set on $\mathbb{R}$. All elements of E are points of closure. All elements of $[0, 1)$ are accumulation points. Then $x = 1$ is both a point of closure and an accumulation point. Finally, $x = 2$ is an isolated point but not an accumulation point. ///

We will make use of the following simple theorem.

Theorem 2.3 Suppose we are given a topological space $(\mathcal{X}, \mathcal{T})$. If $A \subset B$, and x is a point of closure of A, then it is also a point of closure of B. ///

Proof. If every open neighborhood of x contains an element of A, it also contains an element of B. □

The open and closed properties lead to the notions of the *interior* and *closure* of a set, respectively.

Definition 2.7 (Interiors, closures and boundaries) Suppose we are given a topological space $(\mathcal{X}, \mathcal{T})$ and subset $E \subset \mathcal{X}$. The *interior* of E is the largest open set E^o contained in E. Equivalently, E^o is the union of all open subsets of E. The *closure* of E is the smallest closed set $\bar{E}$ containing E. Equivalently, $\bar{E}$ is the intersection of all closed sets containing E.

The *boundary* of E is then $\partial E = \bar{E} - E^o$.

If $E \subset \mathcal{X}$, then D is *dense* in E (or is a *dense* subset of E) if $\bar{D} = E$. ///

Thus, any set E can be assigned an envelope $E^o \subset E \subset \bar{E}$.

Suppose for any subset E we provisionally define $\hat{E}$ to be the set of all points of closure of E. We next show that $\bar{E} = \hat{E}$. First, we characterize $\hat{E}$.

Theorem 2.4 Suppose we are given a topological space $(\mathcal{X}, \mathcal{T})$. Any subset $E \subset \mathcal{X}$ is closed if and only if $E = \hat{E}$. ///

Proof. First, suppose $E = \hat{E}$. Then any $y \in E^c$ is not a point of closure of E. This means $\exists$ open set O_y with $y \in O_y$ and $O_y \subset E^c$. Then $E^c = \cup_{y \in E^c} O_y$, and is thus open. Therefore E is closed.

Next, suppose E is closed, and that $y \in E^c$. Since E^c is open, y has an open neighborhood contained in E^c, and can therefore not be a point of closure of E. Therefore $E = \hat{E}$. □

We can now verify that $\bar{E} = \hat{E}$.

Theorem 2.5 Suppose we are given a topological space $(\mathcal{X}, \mathcal{T})$. Then for any subset $E \subset \mathcal{X}$, $\bar{E} = \hat{E}$. ///

Proof. Let E^* be any closed set containing E. By Theorems 2.3 and 2.4, $\hat{E} \subset \hat{E}^* = E^*$. Then $\hat{E}$ is a closed set contained in all closed sets containing E, and therefore satisfies the definition of $\bar{E}$ (Definition 2.7). □

2.4 Construction of topologies

There are various methods of generating topologies or developing constructive characterizations of topologies. For example, topologies are easily constructed from any collection of subsets.

Theorem 2.6 Let $\mathcal{S}$ be any collection of subsets of $\mathcal{X}$ such that $\mathcal{X} = \cup_{S \in \mathcal{S}} S$. Let $\hat{\mathcal{S}}$ be the collection of all finite intersections of sets in $\mathcal{S}$, and $\mathcal{T}$ be the set of all arbitrary unions of sets in $\hat{\mathcal{S}}$ (including the empty union). Then $\mathcal{T}$ is a topology. ///

Proof. By assumption, $\mathcal{X} \in \mathcal{T}$, then we take $\emptyset$ to be an empty union of sets from $\hat{\mathcal{S}}$, therefore $\emptyset \in \mathcal{T}$. That $\mathcal{T}$ is closed under arbitrary union follows from its construction. Then let $A = \cup_{i \in I} A_i$ and $B = \cup_{j \in J} B_j$ be two unions of sets from $\hat{\mathcal{S}}$. Then

$$A \cap B = \cup_{(i,j) \in I \times J} A_i \cap B_j$$

which is a union of finite intersections of sets in $\mathcal{S}$, and is therefore included in $\mathcal{T}$. All axioms defining a topology are therefore satisfied. □

2.4.1 Subspaces

Given a topological space $(\mathcal{X}, \mathcal{T})$, a *subspace topology* can be induced on any subset $S \subset \mathcal{X}$ by taking all intersections $O \cap S$, $O \in \mathcal{T}$. Note that S need not be open in $\mathcal{X}$ (that is, it is not required that $S \in \mathcal{T}$).

Definition 2.8 Given a topological space $(\mathcal{X}, \mathcal{T})$ and any subset $S \subset \mathcal{X}$, define the class of sets

$$\mathcal{T}_S = \{S \cap O : O \in \mathcal{T}\}.$$

Then $(S, \mathcal{T}_S)$ is a *topological subspace* (or simply *subspace*) of $(\mathcal{X}, \mathcal{T})$. ///

Clearly, the next task is to verify that a topological subspace is an actual topological space. This is not hard to do, but it must be remembered that an open set in $(S, \mathcal{T}_S)$ need not be an open set in $(\mathcal{X}, \mathcal{T})$ (this comment holds for S itself, as well).

Theorem 2.7 Suppose $(\mathcal{X}, \mathcal{T})$ is a topological space, and let $S \subset \mathcal{X}$ be any subset. Then the topological subspace $(S, \mathcal{T}_S)$ of Definition 2.8 is a topological space. ///

Proof. By definition, $S = S \cap \mathcal{X}$ and $\emptyset = S \cap \emptyset$ are included in $\mathcal{T}_S$. If $A, B \in \mathcal{T}_S$, there exists $U, V \in \mathcal{T}$ such that $A = S \cap U$, $B = S \cap V$, so that

$$A \cap B = (S \cap U) \cap (S \cap V) = S \cap (U \cap V).$$

but $U \cap V \in \mathcal{T}$, therefore $A \cap B = S \cap (U \cap V) \in \mathcal{T}_S$.

Then suppose $\{A_i : i \in I\}$ is a collection of sets in $\mathcal{T}_S$. Then there exists a collection of sets $\{U_i : i \in I\}$ such that $A_i = S \cap U_i$ for each index $i \in I$. Then

$$\cup_{i \in I} A_i = \cup_{i \in I} S \cap U_i = S \cap [\cup_{i \in I} U_i].$$

But $\cup_{i \in I} U_i \in \mathcal{T}$, therefore $\cup_{i \in I} A_i \in \mathcal{T}_S$. □

2.4.2 Topological bases

A *base* (or *basis*) for a topology is any collection of subsets which is representative in the following sense:

Definition 2.9 (Base of a Topological Space) Let $(\mathcal{X}, \mathcal{T})$ be a topological space. Then a collection of open sets $\mathcal{B} \subset \mathcal{T}$ is a base for topology $\mathcal{T}$ if every $O \in \mathcal{T}$ is a union of some collection of sets in $\mathcal{B}$. ///

Usually, a base is a simpler object than the topology, which admits some convenient analytical representation. The first problem to consider is whether or not a given collection $\mathcal{B}$ is equivalent to a given topology $\mathcal{T}$ in the sense of Definition 2.9. One set of conditions guaranteeing this equivalence is given in the following theorem.

Theorem 2.8 Let $(\mathcal{X}, \mathcal{T})$ be a topological space. Then a collection of open sets $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ *iff* for every $O \in \mathcal{T}$ and $x \in O$ there exists $B \in \mathcal{B}$ such that $x \in B \subset O$. ///

Proof. First, assume $\mathcal{B}$ is a base, then select any $O \in \mathcal{T}$, and $x \in O$. By assumption, there exists a collection $\{B_i : i \in I\} \subset \mathcal{B}$ such that $O = \cup_{i \in I} B_i$. Then, there must exist some $B_x \in \{B_i : i \in I\}$ for which $x \in B_x \subset O$.

To prove the converse, select any $O \in \mathcal{T}$. By assumption, for each $x \in O$, there exists $B_x \in \mathcal{B}$ for which $x \in B_x \subset O$. The proof is completed by noting $O = \cup_{x \in O} B_x$. □

Following Theorem 2.8, Definition 2.9 can be restricted to a single element of a topology.

Definition 2.10 Let $(\mathcal{X}, \mathcal{T})$ be a topological space. Then a collection of open sets $\mathcal{B}_x \subset \mathcal{T}$ is a *local* (or *neighborhood*) *base* at $x \in \mathcal{X}$ if for every $O \in \mathcal{T}$ containing x there exists $B \in \mathcal{B}_x$ such that $x \in B \subset O$. ///

We next consider a somewhat different question. Rather than resolve the equivalency of a specific base and a specific topology, we ask if a collection of subsets $\mathcal{B}$ can function as a base. The topology is not specified. Rather, if $\mathcal{B}$ can function of a base, a topology is induced in a manner consistent with Definition 2.9. It is important to note that we do not begin with a specific topology, so that the sets in $\mathcal{B}$ are not assumed to be open. Rather, if $\mathcal{B}$ is conceivably a base for *some* topology, then those sets are taken to be open, which then induces the topology.

Theorem 2.9 A collection $\mathcal{B}$ of subsets of a set $\mathcal{X}$ is a base for a topology on $\mathcal{X}$ *iff* (i) for each $x \in \mathcal{X}$ there exists $B \in \mathcal{B}$ such that $x \in B$; and if (ii) whenever $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, there exists a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. ///

Proof. First, assume $\mathcal{B}$ is a base for topology $\mathcal{T}$ on $\mathcal{X}$. Fix $x \in \mathcal{X}$. By Theorem 2.8 there exists $B_x \in \mathcal{B}$ for which $x \in B_x$. Then, suppose $x \in B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$. Then $B_1 \cap B_2$ is open (since all base sets are open). By Theorem 2.8, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

To prove the converse, following Definition 2.9, we need to verify that the set of all arbitrary unions of sets in $\mathcal{B}$, say $\mathcal{T}^*$, is a topology, assuming conditions (i) and (ii) hold. Since each $x \in \mathcal{X}$ is contained in at least one set $B_x \in \mathcal{B}$, we have $\mathcal{X} = \cup_{x \in \mathcal{X}} B_x \in \mathcal{T}^*$. Then $\emptyset$ an empty union of sets in $\mathcal{B}$, so that $\emptyset \in \mathcal{T}^*$. Next, note that by construction $\mathcal{T}^*$ is closed under arbitrary union. Finally, suppose $O_1, O_2 \in \mathcal{T}^*$. We may write $O_1 = \cup_{i \in I} B_i$, $O_2 = \cup_{j \in J} B_j$ as unions of sets in $\mathcal{B}$. Then

$$O_1 \cap O_2 = [\cup_{i \in I} B_i] \cap [\cup_{j \in J} B_j] = \cup_{(i,j) \in I \times J} B_i \cap B_j.$$

By assumption, for each $x \in B_i \cap B_j$ there exists $B' \in \mathcal{B}$ such that $x \in B' \subset B_i \cap B_j$. But this implies that $B_i \cap B_j$, and therefore $O_1 \cap O_2$, is a union of sets from $\mathcal{B}$. Therefore, $\mathcal{T}^*$ is closed under finite intersection, which completes the proof. □

2.4.3 First and second countability axioms

Every topology has a base, in particular, the topology of open sets itself (a consequence of the closure of the open sets under finite intersection). Furthermore, this base can be made strictly smaller than the topology by removing the empty set. One important problem in topology is to determine a base which is in some sense minimal. The degree to which this is possible is an important property of a topological space. An important example of this is given in the following definition.

Definition 2.11 A topological space $\mathcal{X}$ satisfies the *first axiom of countability* (is *first countable*) if a countable local base exists at every $x \in \mathcal{X}$ (Definition 2.10).

Then $\mathcal{X}$ satisfies the *second axiom of countability* (is *second countable*) if it possesses a countable base. ///

Example 2.4 The collection of all bounded, nonempty open intervals (a, b) defines a base $\mathcal{B}$ for a topology $\mathcal{T}$ on $\mathcal{X} = \mathbb{R}$ (conditions (i) and (ii) of Theorem 2.9 are easily verified).

Next, suppose $\mathcal{B}_{\mathbb{Q}}$ is the collection of all bounded, nonempty intervals (a, b) with rational endpoints $a, b \in \mathbb{Q}$. Theorem 2.9 may be similarly used to verify that $\mathcal{B}_{\mathbb{Q}}$ is a base for some topology $\mathcal{T}_{\mathbb{Q}}$. Since $\mathcal{B}_{\mathbb{Q}} \subset \mathcal{B}$ it follows from the definition of a base that $\mathcal{T}_{\mathbb{Q}} \subset \mathcal{T}$ (Definition 2.9). On the other hand, it is easily verified that any open interval in $\mathcal{B}$ is a union of intervals in $\mathcal{B}_{\mathbb{Q}}$, so $\mathcal{T} \subset \mathcal{T}_{\mathbb{Q}}$, and so $\mathcal{T} = \mathcal{T}_{\mathbb{Q}}$. Thus, $\mathcal{B}_{\mathbb{Q}}$ is a base for the topology $\mathcal{T}$. But $\mathcal{B}_{\mathbb{Q}}$ is countable, so that the topological space $(\mathcal{X}, \mathcal{T})$ is second countable. ///

2.5 Separable spaces

Some properties of topological spaces follow from the notion of dense subsets (Definition 2.7). An interesting problem is to determine the simplest dense subset of $\mathcal{X}$. For example, a dense subset of $\mathcal{X}$ may have strictly smaller cardinality.

Definition 2.12 A topological space $\mathcal{X}$ is *separable* if it possesses a countable dense subset. ///

Here we will introduce a number of topologies on $\mathcal{X} = \mathbb{R}$, and determine whether or not the induced topological spaces are separable. We adopt the convention that $\mathcal{T}_{\mathcal{B}}$ is a topology induced by a base $\mathcal{B}$.

Also note, following the discussion of Section 2.2, that any sequence that converges in a topological space also converges in a weaker topological space. To say that D is a dense subset of E is to say that for each x there is a sequence $\{x_n\} \subset D$ such that $x_n \to x$. Therefore, if D in a dense subset in one topological space, it will also be dense in a weaker topological space (Example 2.2).

Example 2.5 (Euclidean topology on $\mathbb{R}$) The Euclidean topology on $\mathbb{R}$ is induced by the base $\mathcal{B}_E = \{(a, b) : a < b; a, b \in \mathbb{R}\}$ and unless otherwise specified, when $\mathbb{R}$ is described as a topological space, this is the one intended (we also use this convention in these notes). That $\mathcal{B}_E$ is a base may be verified by Theorem 2.9. This topology defines convergence in the conventional manner, that is $x_n \to x$ *iff* for any $\epsilon\ \exists N \ni n > N \implies |x_n - x| < \epsilon$.

Clearly, the set of rational number $\mathbb{Q}$ is dense in $(\mathcal{X}, \mathcal{T}_{\mathcal{B}_E})$, since any real number is a limit of a rational sequence. We conclude that $(\mathcal{X}, \mathcal{T}_{\mathcal{B}_E})$ is separable. ///

Example 2.6 (Discrete topology on $\mathbb{R}$) For the discrete topology all sets are open, and therefore also closed. By definition, the closure $\bar{E}$ of a subset E is the smallest closed set containing E. But all sets are closed, therefore $\bar{E} = E$, and so the only dense subset of E is E itself. Therefore $\mathcal{X} = \mathbb{R}$ cannot be separable. ///

Example 2.7 (Anti-discrete topology on $\mathbb{R}$) The anti-discrete topological is weaker than the Euclidean topological, which is separable. Therefore the anti-discrete topology is separable. ///

Example 2.8 (Zariski topology on $\mathbb{R}$) The Zariski topology on $\mathbb{R}$ defines as open sets

$$\mathcal{T}_Z = \{U \subset \mathbb{R} : U = \emptyset \text{ or } U = \mathbb{R} - S,\ |S| < \infty\}.$$

That $\mathcal{T}_Z$ is a topology is easily verified. Then the closed sets are those of finite cardinality, and $\mathbb{R}$. Let $E \subset \mathbb{R}$ be any countably infinite subset. The smallest closed set containing E is $\mathbb{R}$, meaning $\bar{E} = \mathbb{R}$, so that E is a countably dense subset of $\mathbb{R}$. Thus, the Zariski topological space is separable.

Alternatively, we may simply note that $\mathcal{T}_Z \subset \mathcal{T}_{\mathcal{B}_E}$. In other words, the Zariski topology is weaker than the Euclidean topology, therefore the Zariski topological space is separable. ///

Example 2.9 (Arrow topology on $\mathbb{R}$) That $\mathcal{B}_A = \{[a,b) : a < b; a, b \in \mathbb{R}\}$ is a base on $\mathcal{X} = \mathbb{R}$ may be verified by Theorem 2.9. This is known as the *arrow topology* $\mathcal{T}_{\mathcal{B}_A}$. It is easily shown that (a,b) may be written as a union of left closed intervals, and so is open in the arrow topology, so we have $\mathcal{T}_{\mathcal{B}_E} \subset \mathcal{T}_{\mathcal{B}_A}$, that is, any set that is open in the Euclidean topology is open in the arrow topology. The converse does not hold, since $[a,b) \in \mathcal{T}_{\mathcal{B}_A}$ but $[a,b) \notin \mathcal{T}_{\mathcal{B}_E}$. Thus $\mathcal{T}_{\mathcal{B}_A}$ is a stronger topology than $\mathcal{T}_{\mathcal{B}_E}$.

This has some interesting implications. Suppose $\{x_n\}$ converges to x in the Euclidean topological space, but that $x_n < x$ for all n (clearly, this is possible). However, $[x, x+1)$ is open in the arrow topological space, contains x, but contains no elements of $\{x_n\}$. This means $\{x_n\}$ is not convergent in the arrow topological space. On the other hand, if $x_n > x$, and $x_n \to x$ in the Euclidean topological space, then $x_n \to x$ in the arrow topological space. For this reason, it is still possible to assert that $\mathbb{Q}$ is a dense subset of $\mathcal{X}$, so that the arrow topological space is separable ///

2.5.1 Second countable spaces are separable

Recall that a topological space is second countable if it has a countable base (Definition 2.11). Any such space is also separable.

Theorem 2.10 Any second countable topological space is separable. ///

Proof. Suppose $\mathcal{X}$ has a countable base $\mathcal{B} = \{B_n : n \geq 1\}$. Select any $x_n \in B_n$, and define the set $D = \{x_n : n \geq 1\}$. Then any $x \in \mathcal{X}$ is a point of closure of D, since any open set containing x must contain an element of D (recall that all open sets are unions of sets in $\mathcal{B}$). Therefore $\bar{D} = \mathcal{X}$, D is countable, and so $\mathcal{X}$ is separable. $\square$

2.6 Compactness

The *compactness property* is analogous to boundedness in Euclidean space.

Definition 2.13 Suppose $(\mathcal{X}, \mathcal{T})$ is a topological space. An open covering $\mathcal{U}$ of $\mathcal{X}$ is any collection of open sets whose union equals $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{T})$ is *compact* if any open covering of $\mathcal{X}$ contains a finite subcovering of $\mathcal{X}$, that is, any open covering $\mathcal{U}$ includes a finite number of sets $\{O_1, \ldots, O_m\} \subset \mathcal{U}$ for which $\cup_{i=1}^m O_i = \mathcal{X}$. A subset $K \subset \mathcal{X}$ is compact if its induced topological space is compact. A topological space $(\mathcal{X}, \mathcal{T})$ is *locally compact* if for every point x there exists open set O such that $\bar{O}$ is compact. ///

It is possible to test the compactness of a subset K without reference to a subspace topology, if we take an open covering of K *in the topology* $\mathcal{X}$ to be a superset of K.

Theorem 2.11 Suppose $(\mathcal{X}, \mathcal{T})$ is a topological space. Then K is a compact subset *iff* any open covering $\mathcal{U} \subset \mathcal{T}$ $(K \subset \cup_{\mathcal{U}} O)$ includes a finite number of sets $\{O_1, \ldots, O_m\} \subset \mathcal{U}$ for which $K \subset \cup_{i=1}^m O_i$. ///

Proof. First assume K is compact. The topology of the subspace on K is equivalent to all sets of the form $K \cap O$, $O \in \mathcal{T}$. Then, if $\{O_i\}$ is an open covering of K in $\mathcal{X}$, $\{K \cap O_i\}$ is an open covering of the subspace K. Since K is compact $\exists$ a finite subcover $\{K \cap O_1, \ldots, K \cap O_m\}$ of K. Then $\{O_1, \ldots, O_m\}$ is a finite subcover of K in $\mathcal{X}$.

Conversely, assume that every open cover of K in $\mathcal{X}$ contains a finite subcover of K. Let $\{K \cap O_i\}$ be an open covering of the subspace K. Then $\{O_i\}$ is an open covering of K in $\mathcal{X}$. By assumption, $\exists$ a finite subcover $\{O_1, \ldots, O_m\}$ of K, then $\{K \cap O_1, \ldots, K \cap O_m\}$ is a finite subcover of the subspace K. $\square$

A closed subset of a compact set is compact.

Theorem 2.12 Suppose $(\mathcal{X}, \mathcal{T})$ is a compact topological space. Then any closed subset $F \subset \mathcal{X}$ is also compact. ///

Proof. An open covering of the subspace F may be written $\{K \cap O_i\}$, where $\{O_i\} \subset \mathcal{T}$. But $\{F^c\} \cup \{O_i\}$ is an open covering of $\mathcal{T}$, which is by assumption is compact. A finite subcover of the subspace F must therefore exist. $\square$

In Euclidean space, a compact set is closed and bounded.

Theorem 2.13 (Heine - Borel Theorem) A subset $F \subset \mathbb{R}^d$ is compact *iff* it is closed and bounded. ///

Proof. See, for example, Royden (1968). $\square$

2.7 Continuity

We next define continuity for a mapping between topological spaces $\mathcal{X}$ and $\mathcal{Y}$.

Definition 2.14 (Continuity in Topological Spaces) Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping between topological spaces $\mathcal{X}$ and $\mathcal{Y}$. Then f is *continuous* if for any open set $G \subset \mathcal{Y}$, the preimage $f^{-1}(G)$ is an open subset of $\mathcal{X}$. ///

In the definition of continuity, the open and closed properties are interchangeable.

Theorem 2.14 Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping between topological spaces $\mathcal{X}$ and $\mathcal{Y}$. Then f is continuous *iff* for any closed set $F \subset \mathcal{Y}$, the preimage $f^{-1}(F)$ is an closed subset of $\mathcal{X}$. ///

Proof. Assume f is continuous, and $F \subset \mathcal{Y}$ is closed. Then $f^{-1}(F) = f^{-1}(F^c)^c$. But F^c is open, and by continuity of f, $f^{-1}(F^c)$ is open, hence $f^{-1}(F)$ is closed.

Conversely, assume that $f^{-1}(F)$ is closed whenever F is closed. Then $f^{-1}(F^c)$ is open. The proof is completed by writing any open subset $G \subset \mathcal{Y}$ as $G = F^c$. $\square$

2.7.1 Preservation of continuity

We next enumerate a number of conditions under which continuity is preserved.

Continuity is preserved under domain restriction

We will sometimes need to restrict a function to a strict subset of a domain. In this case, continuity is preserved.

Definition 2.15 Suppose we are given mapping $f : \mathcal{X} \to \mathcal{Y}$, and subset $E \subset \mathcal{X}$. Then f_E is the unique mapping $f_E : E \to f(E)$ between topological subspaces E and $f(E)$ for which $f_E(x) = f(x)\ \forall x \in E$. ///

Theorem 2.15 Following Definition 2.15, if $f : \mathcal{X} \to \mathcal{Y}$ is continuous then so is f_E. ///

Proof. Suppose V is an open subset of the subspace topology $f(E)$. Then there exists an open subset $V' \subset \mathcal{Y}$ such that $V = f(E) \cap V'$. Since f is continuous, the set $f^{-1}(V')$ is open, and $E \cap f^{-1}(V')$ is an open subset of the subspace E. The proof is completed by noting the identity $f^{-1}(V) = E \cap f^{-1}(V')$. □

Continuity is preserved under composition

Theorem 2.16 Given topological spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ suppose $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are continuous mappings. Then the composition $g \circ f : \mathcal{X} \to \mathcal{Z}$ is also continuous. ///

Proof. Let $G \subset \mathcal{Z}$ be open. Then by assumption $g^{-1}(G) \subset \mathcal{Y}$ is open, and so is $f^{-1}(g^{-1}(G))$. But this is the preimage $(g \circ f)^{-1}(G)$. □

2.7.2 Properties preserved by continuity

We next enumerate a number of properties which are preserved under continuity.

Compactness is preserved under continuous mappings

For a continuous mapping, an image of a compact set is compact.

Theorem 2.17 Given topologies, $\mathcal{X}, \mathcal{Y}$ suppose $f : \mathcal{X} \to \mathcal{Y}$ is a continuous mapping. Then if $K \subset \mathcal{X}$ is compact, so is $f(K)$. ///

Proof. We will make use of Theorem 2.11. Suppose $\{V_i\}$ is an open covering of $f(K)$ in the topological space $\mathcal{Y}$. By the continuity of f, $\{f^{-1}(V_i)\}$ is an open covering of K in the topological space $\mathcal{X}$. Since K is assumed compact, there is a finite subcovering $\{f^{-1}(V_1), \ldots, f^{-1}(V_m)\}$ of K. Then $\{V_1, \ldots, V_m\}$ is a finite subcovering of $f(K)$, which completes the proof. □

Convergence is preserved under continuous mappings

In real spaces a continuous function f is conventionally taken to be one for which $f(x_n) \to f(x)$ whenever $x_n \to x$. In contrast, we say f is a continuous mapping between topological spaces if the pre-image of any open set is open. As it happens the two properties are not equivalent for general topologies. We can, however, say that for any topology the implication works in at least one direction. We will later show that the implications work in both directions for metric spaces.

Theorem 2.18 Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping between topological spaces $\mathcal{X}$ and $\mathcal{Y}$. If $\{x_n\} \subset \mathcal{X}$ is a sequence with limit x_0, then $f(x_n) \to f(x_0)$ in $\mathcal{Y}$. ///

Proof. Suppose $f(x_0) \in G \subset \mathcal{Y}$, and that G is open. Then $U = f^{-1}(G)$ is an open subset of $\mathcal{X}$ containing x_0. By assumption, there exists finite N such that $x_n \in U$ for all $n \geq N$. This implies $f(x_n) \in G$ for all $n \geq N$, which completes the proof. □

2.8 Homeomorphisms

The homeomorphic property of a mapping $f : X \to \mathcal{Y}$ is in a sense a stronger version of continuity.

Definition 2.16 (Homeomorphisms) A mapping $f : \mathcal{X} \to \mathcal{Y}$ between topological spaces is a *homeomorphism* if f is bijective, and if f and f^{-1} are both continuous. Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homeomorphic* if a homeomorphism between them exists. This is denoted $\mathcal{X} \sim \mathcal{Y}$. ///

Example 2.10 The identity function $I : \mathcal{X} \to \mathcal{X}$ is continuous, bijective, and equals its own inverse, and is therefore a homeomorphism. ///

Example 2.11 If $f : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then so is f^{-1}, since Definition 2.16 is perfectly symmetric *wrt* $\mathcal{X}$ and $\mathcal{Y}$. ///

Example 2.12 If $f : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then f_E is a homeomorphism between subspaces E and $f(E)$ (see Definition 2.15). To see this, note that if f is a bijection, then so is f_E. By Theorem 2.15, if f is continuous then so is f_E. Then f_E^{-1} is the restriction of $f^{-1} : \mathcal{Y} \to \mathcal{X}$ to domain $f(E)$, and is therefore also continuous. ///

It is important to note that the homeomorphism property defines a binary relation between topological spaces, and is, in fact, an equivalence relation (Section 1.6). This is verified in the following theorem.

Theorem 2.19 Given topological spaces $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ suppose $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are homeomorphisms. Then the composition $g \circ f : \mathcal{X} \to \mathcal{Z}$ is also a homeomorphism. As a consequence, homeomorphisms define an equivalence relation. ///

Proof. If f and g are bijections, then so is $g \circ f$. Then, by assumption, f, g, f^{-1}, g^{-1} are continuous. By Theorem 2.16 $g \circ f$ and $(g \circ f)^{-1}$ are also continuous. Therefore, $(g \circ f)$ is a homeomorphism.

The homeomorphism property defines an equivalence relation (Definition 1.3). The reflexive property ($\mathcal{X} \sim \mathcal{X}$) is verified in Example 2.10; the symmetric property ($\mathcal{X} \sim \mathcal{Y}$ *iff* $\mathcal{Y} \sim \mathcal{X}$) is verified in Example 2.11; and the transitive property ($\mathcal{X} \sim \mathcal{Y}$ and $\mathcal{Y} \sim \mathcal{Z}$ implies $\mathcal{X} \sim \mathcal{Z}$) has been verified above. □

Example 2.13 Any two nonempty open intervals $(a, b), (c, d)$ can be shown to be homeomorphic by constructing a linear transformation between them. Therefore, since homeomorphism is an equivalence relation, to show that $\mathbb{R}$ is homeomorphic to any open interval, we need only show, for example, $\mathbb{R} \sim (-1, 1)$. Then $f(x) = x/(1 + |x|)$ serves as the required homeomorphism $f : \mathbb{R} \to (-1, 1)$. ///

2.8.1 Topological invariants

A fundamental problem of topology is to establish whether of not two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic. At a minimum, there must be a bijection between them, so that they must be of the same cardinality. It immediately follows, for example, that the set of real numbers and the set of rational numbers are not homeomorphic.

However, identical cardinality is not sufficient for homeomorphic equivalence, and considerable effort may be needed to resolve the existence or nonexistence of the homeomorphic property for two topologies of equal cardinality. The effort is motivated by the notion of the *topological invariant*.

Definition 2.17 Suppose $\mathcal{X} \sim \mathcal{Y}$. A property of a topological space is a *topological invariant* if it holds for both $\mathcal{X}$ and $\mathcal{Y}$ or for neither. Because the homeomorphic property is an equivalence relation, all topologies in any equivalence class either do or do not possess a topologically invariant property. ///

Compactness is a topological invariant

The topological invariance of the compactness property follows from the fact that compactness is preserved under continuous mappings (Theorem 2.17).

Theorem 2.20 Compactness is a topological invariant. ///

Proof. Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, and that $\mathcal{X}$ is compact. Since f is surjective we have $f(\mathcal{X}) = \mathcal{Y}$. By Theorem 2.17 $\mathcal{Y}$ is compact. That the compactness of $\mathcal{Y}$ implies the compactness of $\mathcal{X}$ then follows from the fact that f^{-1} is a continuous surjective mapping from $\mathcal{Y}$ to $\mathcal{X}$. □

2.8.2 Homeomorphisms and Euclidean space

Here, we will set up what might be considered an archetype problem in topology: are the topologies $\mathbb{R}^m$ and $\mathbb{R}^n$ homeomorphic if $m \neq n$? The first question is whether or not they are bijective. We start with a simple case.

Example 2.14 We will construct a bijective mapping $f : (0,1] \times (0,1] \to (0,1]$. Suppose $x, y \in (0,1]$. An obvious idea is to simply interleave the digits of x, y to yield $z = f(x,y)$. Then $(x,y) = f^{-1}(z)$ can be evaluated by separating the alternating digits of z. That is, if

$$\begin{aligned} x &= 0.a_1a_2a_3\ldots, \text{ and} \\ y &= 0.b_1b_2b_3\ldots, \end{aligned}$$

then

$$f(x,y) = 0.a_1b_1a_2b_2a_3b_3\ldots,$$

and if

$$z = 0.c_1c_2c_3\ldots,$$

then

$$f^{-1}(z) = (0.c_1c_3c_5\ldots, 0.c_2c_4c_6\ldots).$$

However, this idea fails for a number of reasons. First, it does not take into account the nonuniqueness of a decimal representation, for example, $1/2 = 0.500\ldots = 0.499\ldots$. Unless we are more precise, f will not be surjective, noting that if $1/2$ is represented by $0.500\ldots$, it will not be in the codomain of f using the interleaving scheme.

So, we will accept as the canonical decimal representation of a real number sequences which end in repeated nines in place of those ending in repeated zeros (this is where the choice lies). However, the interleaving scheme will still not be surjective, given counterexample $z = 0.202020\ldots$.

To solve this problem, rather than decompose a real number into a sequence of digits $0.a_1a_2a_3\ldots$ the elements a_i can instead be either a single nonzero digit, or a finite sequence of zeros followed by a nonzero digit. For example, $0.3200604\ldots$ is represented by the sequence $a_1 = 3$, $a_2 = 2$, $a_3 = 006$, $a_4 = 04$, and so on. No canonical representation contains an infinite sequence of zeros, so this scheme suffices to define the required bijection. ///

The argument of Example 2.14 can be extended to show that any two topologies $\mathbb{R}^m$ and $\mathbb{R}^n$ are bijective for any m, n. To show that they are not homeomorphic when $m \neq n$ we will rely on some ideas introduced below.

2.9 Connectedness

Topology is often associated with the study of geometric properties which remain invariant under continuous transformations (the well known equivalence of a donut and coffee mug, for example). One of these is *connectedness*.

Definition 2.18 A topological space $\mathcal{X}$ is *connected* if there do not exist two disjoint and nonempty open sets whose union equals $\mathcal{X}$. If there exists nonempty open sets U, V such that $U \cap V = \emptyset$ and $U \cup V = \mathcal{X}$, then we say $\{U, V\}$ is a *separation* of $\mathcal{X}$. ///

Connectedness is preserved under continuous mappings.

Theorem 2.21 Given topologies, $\mathcal{X}, \mathcal{Y}$ suppose there exists a continuous surjective mapping $f : \mathcal{X} \to \mathcal{Y}$. Then if $\mathcal{X}$ is connected so is $\mathcal{Y}$.

In addition, if f is continuous, but not necessarily surjective, then if $\mathcal{X}$ is connected so is the subspace $f(\mathcal{X})$. ///

Proof. Suppose U, V is a separation of $\mathcal{Y}$. Since f is continuous and surjective $f^{-1}(U), f^{-1}(V)$ must be a separation of $\mathcal{X}$, leading to a contradiction.

The rest of the proof follows by restricting the codomain of f to the subspace $f(\mathcal{X})$. □

We then have a new topological invariant.

Theorem 2.22 Connectedness is a topological invariant. ///

Proof. Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. The theorem follows from Theorem 2.21, after noting that both f and f^{-1} are continuous and surjective. □

2.9.1 Connectedness and intervals in $\mathbb{R}$

Since connectedness is a topological invariant, it may be used to show that two topologies are *not* homeomorphic. For example, are the nonempty intervals (a, b) and $[a, b]$ homeomorphic? First, we show that intervals of any kind are connected.

Theorem 2.23 Suppose $a < b$. Then (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ are connected topological spaces. ///

Proof. First consider $[a, b]$. Suppose U, V are open subsets of $[a, b]$ such that $U \cup V = [a, b]$. We will show that U, V cannot be a separation. If a or b are included in both U, V this is obviously true, so we may assume without loss of generality that $a \in U$ and $b \in V$. Then let $x_0 = \inf V$. Either $x_0 \notin U$ or $x_0 \in U$.

If $x_0 \notin U$, then $x_0 \neq a$, therefore $a < x_0 \in V$. But V is open, and so contains $(x_0 - \epsilon, x_0 + \epsilon)$ for some small enough $\epsilon > 0$. This in turn implies that there exists $x < x_0 = \inf V$ such that $x \in V$, which leads to a contradiction.

Therefore, we must have $x_0 \in U$. Since U is open, it follows that $[x_0, x_0 + \epsilon') \subset U$ for some small enough $\epsilon' > 0$. The proof is completed by noting that if $[x_0, x_0 + \epsilon') \cap V = \emptyset$ we must have $\inf V > x_0$, which leads to a contradiction.

Now let I be any of the intervals (a, b), $[a, b]$, $(a, b]$ or $[a, b)$. Let U, V be nonempty open subsets of I for which $U \cup V = I$. We may choose $c < d$ such that $c \in U$ and $d \in V$. Then let $U' = U \cap [c, d]$, $V' = V \cap [c, d]$. U', V' are open subsets of $[c, d]$ for which $U' \cup V' = [c, d]$. But we have just shown that $U' \cap V'$ is nonempty, therefore so is $U \cap V$. $\square$

Example 2.15 From Example 2.13 $(a, b) \sim \mathbb{R}$, assuming $a < b$. By Theorem 2.23 (a, b) is connected, and by Theorem 2.25 connectedness is a topological invariant. It follows that $\mathbb{R}$ is connected. ///

We may also shown that (a, b) and $[a, b]$ are not homeomorphic.

Example 2.16 Suppose $f : [a, b] \to (a, b)$ is a homeomorphism. Let $E = (a, b]$ be a subset of $[a, b]$. By Example 2.12 $f_E : (a, b] \to (a, b) - f(a)$ is a homeomorphism. However, by Theorem 2.23 $(a, b]$ is connected, whereas $(a, b) - f(a)$ possesses separation $\{(a, f(a)), (f(a), b)\}$, and is therefore not connected. Therefore, no homeomorphism between $[a, b]$ and (a, b) can exist, that is, $[a, b]$ and (a, b) are not homeomorphic.

A similar argument can be used to show that $[a, b]$ is not homeomorphic to $(a, b]$ or $[a, b)$, and that neither $(a, b]$ or $[a, b)$ are homeomorphic to (a, b). However, a linear, and therefore homeomorphic, mapping between $(a, b]$ and $[a, b)$ can be constructed. ///

2.9.2 Path connectedness

We next consider a stronger version of connectedness.

Definition 2.19 A *path* in a topological space $\mathcal{X}$ is a continuous function $\omega : [0, 1] \to \mathcal{X}$. We say ω *joins* x_0 and x_1 if $\omega(0) = x_0$ and $\omega(1) = x_1$. Then we say $\mathcal{X}$ is *path connected* if there exists a path which joins any two $x_0, x_1 \in \mathcal{X}$. ///

That path connectedness is a stronger property than connectedness is verified in the following theorem.

Theorem 2.24 Any path connected topological space is connected. ///

Proof. Suppose $\mathcal{X}$ is a path connected topological space but is not connected. Then there exists a separation U, V of $\mathcal{X}$. We may select $x_0 \in U$ and $x_1 \in V$. Since $\mathcal{X}$ is path connected, there exists a path ω which joins x_0 and x_1. Then ω is a continuous mapping from $[0,1]$ to the topological subspace $\omega([0,1])$. Then $U' = \omega([0,1]) \cap U$ and $V' = \omega([0,1]) \cap V$ are open subsets of $\omega([0,1])$. Neither is empty, since $\omega(0) \in U$ and $\omega(1) \in V$. By assumption, $U \cup V = \mathcal{X}$, therefore $U' \cup V' = \omega([0,1])$. Finally, U, V are disjoint, therefore so are U', V'. This implies $\omega([0,1])$ is not connected. However, by Theorem 2.21, since $[0,1]$ is connected and ω is continuous, $\omega([0,1])$ must also be connected, leading to a contradiction. Therefore $\mathcal{X}$ must be connected. □

We can begin our resolution of the problem posed in Section 2.8.2.

Example 2.17 The topologies $\mathbb{R}$ and $\mathbb{R}^n$, $n > 1$ are not homeomorphic. Suppose, in contradiction, that $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is a homeomorphism. Define subspace $E = \mathbb{R} - \{0\}$. Then by Example 2.12 $f_E : E \rightarrow \mathbb{R}^n - \{f(0)\}$ is also a homeomorphism. Then note that E is not connected. On the other hand, the topology produced by deleting a single element from $\mathbb{R}^n$ is clearly path connected, and by Theorem 2.24 also connected. But connectedness is a topological invariant, therefore f_E cannot be a homeomorphism, therefore neither can f. We conclude that $\mathbb{R}$ and $\mathbb{R}^n$, $n > 1$ are not homeomorphic. ///

We then note that path connectedness is a topological invariant.

Theorem 2.25 Path connectedness is a topological invariant. ///

Proof. Suppose $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a homeomorphism, and that $\mathcal{X}$ is path connected. Suppose $y_0, y_1 \in \mathcal{Y}$, and that $x_0 = f^{-1}(y_0)$, $x_1 = f^{-1}(y_1)$. Then there exists a path ω in $\mathcal{X}$ joining x_0, x_1. It follows that the composition $f \circ \omega$ is a path joining y_0, y_1 (since continuity is preserved by composition). □

2.10 Metric spaces

The *metric* can be thought of as a generalization of the notion of Euclidean distance, allowing more flexible notions of distance, while retaining the most important properties.

Definition 2.20 Suppose we have a set of objects $\mathcal{X}$ and a real-valued mapping $d : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ operating on two observations $\boldsymbol{u}, \boldsymbol{v}$. Then d is a *metric* if

(i) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{u} = \boldsymbol{v}$ (identifiabilty);
(ii) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$ (symmetry);
(iii) $d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v})$ for any $\boldsymbol{w} \in \mathcal{X}$ (triangle inequality). The pair $(\mathcal{X}, d)$ is referred to as a *metric space*.

///

Non-negativity ($d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$) is often included as an axiom, but this is not needed.

Theorem 2.26 A metric (Definition 2.20) is non-negative. ///

Proof. Making use of all three axioms, we may write

$$0 = d(\boldsymbol{u}, \boldsymbol{u}) \leq d(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, \boldsymbol{u}) = 2d(\boldsymbol{u}, \boldsymbol{v}).$$

□

Example 2.18 Definition 2.20 can applied to quite general objects. Suppose $\mathcal{X}$ is the set of all bounded functions on the interval $[0,1]$. For any two functions $f, g \in \mathcal{X}$ set

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

It can be verifies that $d(f,g)$ satifies Definition 2.20. That it satisfies the triangle inequality follows from:

$$\begin{aligned} d(f,g) &= \sup_{x\in[0,1]} |f(x) - g(x)| \\ &\leq \sup_{x\in[0,1]} |f(x) - h(x)| + |h(x) - g(x)| \\ &\leq \sup_{x\in[0,1]} |f(x) - h(x)| + \sup_{x\in[0,1]} |h(x) - g(x)| \\ &= d(f,h) + d(h,g). \end{aligned}$$

///

Convergence on a metric space is defined by the metric. However, the definition of convergence need not refer to an actual limit.

Definition 2.21 Given a metric space $(\mathcal{X}, d)$ of Definition 2.20, a sequence $x_n \in \mathcal{X}$, $n \geq 1$ possesses limit $x_0 \in \mathcal{X}$ if $\lim_n d(x_n, x_0) = 0$. The sequence is a *Cauchy sequence* if for all $\epsilon > 0$ $\exists$ finite $N \ni d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. If all Cauchy sequences possess limits in $\mathcal{X}$ then $(\mathcal{X}, d)$ is a *complete metric space*. ///

In general, a metric space can be *completed* to form a complete metric space by incorporating (or defining, if needed) all limits of Cauchy sequences.

Example 2.19 The sequence $x_n = 1/n$, is a Cauchy sequence in $\mathcal{X} = (0, \infty)$, but does not possess a limit in $\mathcal{X}$, and the characterization of convergence does not refer to one. ///

Every point in an open interval (a,b) is contained in a strictly smaller open interval contained in (a,b). The limit of every sequence in a closed interval $[a,b]$ is contained in $[a,b]$. These notions can be expressed in general metric spaces.

Definition 2.22 We are given a metric space $(\mathcal{X}, d)$. An (*open*) *ball* is any subset of the form

$$B_\epsilon(x) = \{y \in \mathcal{X} : d(x,y) < \epsilon\}.$$

A subset $E \subset \mathcal{X}$ is *open* if for each $x \in E$ $\exists$ $\epsilon > 0 \ni B_\epsilon(x) \subset E$.

An element $x \in E \subset \mathcal{X}$ is a *point of closure* of E if $\forall$ $\epsilon > 0$ $\exists$ $y \in E \ni y \in B_\epsilon(x)$. The set of all points of closure of E (the *closure* of E) is denoted $\bar{E}$. Then a set E is *closed* if $E = \bar{E}$.

A set D is a *dense subset* of E if $\bar{D} = E$.

An element $x \in \mathcal{X}$ is an *accumulation point* of $E \subset \mathcal{X}$ if it is a point of closure of $E - \{x\}$. ///

Note that the definition of an open ball on a space $\mathcal{X}$ forces the inclusion $B_\epsilon(x) \subset \mathcal{X}$. So, if $\mathcal{X} = [0,1]$, then $B_{1/2}(0) = [0, 1/2)$, and is an open subset of $\mathcal{X}$.

We present a few properties of Cauchy sequences.

Theorem 2.27 Any sequence possessing a limit in a metric space $(\mathcal{X}, d)$ is a Cauchy sequence. ///

Proof. Suppose $x_n \to x$ in $\mathcal{X}$. For any $\epsilon > 0 \ \exists\ N < \infty$ such that $d(x_n, x) < \epsilon/2$ for all $n > N$. Then for any $m, n > N$, by the triangle inequality we have $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon$. □

Theorem 2.28 Any Cauchy sequence in a metric space $(\mathcal{X}, d)$ is bounded, that is, any a Cauchy sequence is contained in some open ball $B_\epsilon(x) \subset \mathcal{X}$. ///

Proof. For any $\epsilon' > 0$ we can select N such that $m, n > N$ implies $d(x_m, x_n) < \epsilon'$. Fix some $n' > N$. Then $d(x_n, x_{n'}) < \epsilon'$ for all $n \geq n'$. We can then select some finite

$$\epsilon > \max\{\epsilon', d(x_1, x_{n'}), d(x_2, x_{n'}), \ldots, d(x_{n'-1}, x_{n'})\},$$

in which case the Cauchy sequence is contained in $B_\epsilon(x_{n'})$. □

Definition 2.22 contains separate definitions for open and closed sets. It turns out this is not necessary, since a set if closed *iff* it is the complement of an open set. In fact, we will show below that the notions of an open and closed set in a metric space that we have just explicitly defined are topological.

Theorem 2.29 The complement of a closed (open) set in metric space $(\mathcal{X}, d)$ is open (closed). ///

Proof. If $x \in O$, and O is open, then there is an open ball $B_\epsilon(x) \subset O$. Then there is no $y \in O^c$ also contained in $B_\epsilon(x)$. Thus, no $x \in O$ is a point of closure of O^c, which must therefore be closed.

Then suppose F is closed, and $x \in F^c$. Then x is not a point of closure of F. There must therefore be an open ball $B_\epsilon(x)$ containing no point in F. This implies $B_\epsilon(x) \subset F^c$, therefore F^c is closed. □

Trivially, any $x \in E$ is a point of closure of E, therefore $E \subset \bar{E}$. Thus, if $E = (a, b)$, $\bar{E} = [a, b]$. Note also that a point of closure need not be an accumulation point. For example, if $E = [0, 1] \cup \{2\}$, then $x = 2$ is a point of closure but not an accumulation point.

The closure property in a metric can be conveniently characterized in the following way.

Theorem 2.30 A subset F in a metric space $(\mathcal{X}, d)$ is closed *iff* whenever $\{x_n\} \subset F$ and $x_n \to x$ we also have $x \in F$. ///

Proof. First assume F is closed. Then suppose $\{x_n\} \subset F$ and $x_n \to x$. Then for any $\epsilon > 0 \ \exists$ index n' such that $x_{n'} \in B_\epsilon(x)$. By assumption $x_{n'} \in F$, so that x is a point of closure. Then $x \in F$, since a closed set includes all of its points of closure.

Then suppose $\{x_n\} \subset F$ and $x_n \to x$ implies $x \in F$. Suppose y is a point of closure of F. Then for each $n \geq 1$ there exists $y_n \in F$ such that $y_n \in B_{1/n}(y)$. This means $y_n \to y$, so by assumption $y \in F$. Equivalently, F contains all of its points of closure, and so is closed. □

The continuity of a mapping between two metric spaces is defined in terms of convergence.

Definition 2.23 Suppose $(\mathcal{X}, d)$ and $(\mathcal{Y}, d')$ are two metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is *continuous* if for any sequence $x_n \in \mathcal{X}$ with limit $x_0 \in \mathcal{X}$, the sequence $f(x_n)$ converges in $\mathcal{Y}$ to limit $f(x_0)$, with convergence defined in Definition 2.21. Equivalently, for each $x \in \mathcal{X}$ and $\epsilon > 0 \; \exists \; \delta > 0$ such that $d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$. ///

Example 2.20 (The set $\mathbb{R}$ is complete) The metric space $(\mathbb{R}, d)$, where d is Euclidean distance, is complete. To see this, suppose $\{x_n\}$ is a Cauchy sequence. By Theorem 2.28 $\{x_n\}$ is bounded. Therefore $\liminf_n x_n = x_0 \in \mathbb{R}$, and there exists a subsequence $\{x_{n_k}\}$ which converges to x_0. Then, for any $\epsilon > 0$ we may select finite N such that $d(x_{n_k}, x_0) < \epsilon/2$ and $d(x_m, x_n) < \epsilon/2$ whenever $n_k, m, n > N$. In particular, by the triangle inequality

$$d(x_n, x_0) \le d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon.$$

But there exists some $n_k > N$ for any N, so we conclude that $x_n \to x_0$, and therefore $\mathbb{R}$ is complete. ///

Example 2.21 (The completeness property for metric spaces is not a topological invariant) From Example 2.13 $\mathbb{R}$ is homeomorphic to any bounded open interval (a, b). Note that $\mathbb{R}$ is a complete metric space (Example 2.20) but (a, b) is not, since there exists a Cauchy sequence in (a, b) with limit $a \notin (a, b)$. Therefore completeness cannot be a topological invariant. ///

2.10.1 A metric space is a topological space

In our definition of a topological space, it was stressed that once the class of open sets $\mathcal{T}$ is defined, properties such as convergence and continuity may be defined entirely in terms of this class. An important question is whether or not this holds for a metric space, and the definition of an open set associated with it. Suppose we are given metric space $(\mathcal{X}, d)$, and we wish to define a topology on $\mathcal{X}$. The first step is to construct a base $\mathcal{B}$. The obvious choice is the class of all open balls $B_\epsilon(x)$, $x \in \mathcal{X}$, $\epsilon > 0$. It is not too difficult to then verify that $\mathcal{B}$ satisfies the hypothesis of Theorem 2.9. Then the topology $\mathcal{T}$ on $\mathcal{X}$ is the class of all unions of sets in $\mathcal{B}$, as well as $\emptyset$.

Theorem 2.31 Given a metric space $(\mathcal{X}, d)$, the set of all open balls $\mathcal{B}$ is a base for a topology on $\mathcal{X}$. ///

Proof. We apply Theorerm 2.9. Fix $x \in \mathcal{X}$. Condition (i) is satisfied by any $B = B_\epsilon(x)$. Then suppose $x \in B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$ for some $x_1, x_2 \in \mathcal{X}$, $\epsilon_1, \epsilon_2 > 0$. Then

$$\begin{aligned} d(x, x_1) &= \epsilon_1' < \epsilon_1, \\ d(x, x_2) &= \epsilon_2' < \epsilon_2. \end{aligned}$$

Select $\epsilon_3 < \min(\epsilon_1 - \epsilon_1', \epsilon_2 - \epsilon_2')$, and suppose $y \in B_{\epsilon_3}(x)$. Then, using the triangle inequality,

$$\begin{aligned} d(y, x_1) &\le d(y, x) + d(x, x_1) \le \epsilon_3 + \epsilon_1' < \epsilon_1 - \epsilon_1' + \epsilon_1' = \epsilon_1, \\ d(y, x_2) &\le d(y, x) + d(x, x_2) \le \epsilon_3 + \epsilon_2' < \epsilon_2 - \epsilon_2' + \epsilon_2' = \epsilon_2, \end{aligned}$$

so that $y \in B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$, therefore $B_{\epsilon_3}(x) \subset B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$. Hypothesis (ii) follows by equating $B_1 = B_{\epsilon_1}(x_1)$, $B_2 = B_{\epsilon_2}(x_2)$, $B_3 = B_{\epsilon_3}(x)$. $\square$

Suppose we regard $\mathcal{X}$ as both a metric space, induced by metric d (denoted $\mathcal{X}_d$), and a topological space, induced by the base $\mathcal{B}$ consisting of all open balls defined by the metric d (denoted $\mathcal{X}_\mathcal{B}$). That $\mathcal{X}_\mathcal{B}$ is always defined follows from Theorem 2.31. With respect to a metric space, we have explicitly defined the properties of convergence; openness and closedness; and continuity (Definitions 2.21, 2.22, 2.23). In contrast, for a topological space these properties are defined exclusively by the topology of open sets, in a manner entirely independent of the properties of the set $\mathcal{X}$. So it is worth asking if these properties are identical for $\mathcal{X}$ as a metric space and $\mathcal{X}$ as a topological space. We resolve the question in the following series of theorems.

Theorem 2.32 Closedness and openness are equivalent in metric spaces and topological spaces. ///

Proof. First, by Theorem 2.8, G is open in $\mathcal{X}_d$ *iff* G is open in $\mathcal{X}_\mathcal{B}$. In a topology, a closed set is simply any complement of an open set. But by Theorem 2.29, this relation also exists between open and closed sets in $\mathcal{X}_d$. So the closed sets of $\mathcal{X}_d$ and $\mathcal{X}_\mathcal{B}$ are also identical. $\square$

Theorem 2.33 Convergence is equivalent in metric spaces and topological spaces. ///

Proof. Suppose $\{x_n\}$ converges to x in $\mathcal{X}_\mathcal{B}$. Then for each $\epsilon > 0$, all but a finite number of elements of $\{x_n\}$ are contained in $B_\epsilon(x)$, equivalently all but a finite number of elements of $\{x_n\}$ satisfy $d(x_n, x) < \epsilon$. We conclude that $x_n \to x$ in $\mathcal{X}_d$.

Conversely, suppose $\{x_n\}$ converges to x in $\mathcal{X}_d$. Let G be any open subset of $\mathcal{X}_\mathcal{B}$ containing x. Then by Theorem 2.32 G is also an open subset of $\mathcal{X}_d$, from which is follows that $B_\epsilon(x) \subset G$ for some small enough $\epsilon > 0$. Since $\{x_n\}$ converges to x in $\mathcal{X}_d$, all but a finite number of elements of $\{x_n\}$ are contained in $B_\epsilon(x)$ and therefore also in G. We conclude that $x_n \to x$ in $\mathcal{X}_\mathcal{B}$. $\square$

Theorem 2.34 Continuity is equivalent in metric spaces and topological spaces. ///

Proof. Let $\mathcal{X}_d = (\mathcal{X}, d)$, $\mathcal{Y}_{d'} = (\mathcal{Y}, d')$ be two metric spaces, and let $\mathcal{X}_\mathcal{B}$, $\mathcal{Y}_{\mathcal{B}'}$ be the topological spaces on $\mathcal{X}, \mathcal{Y}$ induced by the open neighborhoods of the respective metric spaces. We will rely on the fact that that by Theorems 2.32 and 2.33 closedness and convergence are equivalent in metric spaces and topological spaces. We may therefore conclude directly by Theorem 2.18 that if $f : \mathcal{X}_\mathcal{B} \to \mathcal{Y}_{\mathcal{B}'}$ is a continuous mapping, then so is $f : \mathcal{X}_d \to \mathcal{Y}_{d'}$.

Conversely, suppose $f : \mathcal{X}_d \to \mathcal{Y}_{d'}$ is continuous. Let A be a closed subset of $\mathcal{Y}_{\mathcal{B}'}$, and suppose we have sequence $\{x_n\} \subset f^{-1}(A)$ with limit $x \in \mathcal{X}$. By assumption $f(x_n) \to f(x)$. Since A is closed, by Theorem 2.30 $f(x) \in A$. This implies $x \in f^{-1}(A)$, so by a second application of Theorem 2.30 we conclude that $f^{-1}(A)$ is closed, and therefore $f : \mathcal{X}_\mathcal{B} \to \mathcal{Y}_{\mathcal{B}'}$ is a continuous mapping. $\square$

2.10.2 Metrizable spaces

All metric spaces have topological properties, but not all topologies have metric properties. In fact, the problem of whether or not a topological space can be induced by a metric space is a fundamental concern of topology. We then have the following definition.

Definition 2.24 A topological space $\mathcal{X}$ is *metrizable* if there exists a metric d such that the open balls $B_\epsilon(x) = \{y : d(x,y) < \epsilon\}$ define a base. ///

There are various methods of determining whether or not a space is metrizable. The following theorem provides a characterization of metric spaces which may be used for this purpose.

Theorem 2.35 If $(\mathcal{X}, d)$ is a metric space, the induced topological space is separable *iff* it is second countable. ///

Proof. First note that any second countable space is separable (Theorem 2.10).

Then, suppose D is countable and dense in $\mathcal{X}$. Define the collection of open balls $\mathcal{B} = \{B_{1/n}(x) : x \in D; n = 1, 2, \ldots\}$. The proof is completed by verifying that $\mathcal{B}$ forms a base for the metric space $\mathcal{X}$, and noting that $\mathcal{B}$ is countable. □

Example 2.22 (The arrow topological space on $\mathbb{R}$ is not metrizable) Recall the arrow topology on $\mathbb{R}$, induced by the base $\mathcal{B}_A = \{[a,b) : a < b; a, b \in \mathbb{R}\}$ (Example 2.9). This topology is stronger than the Euclidean topology, but retains sufficient convergence properties to be separable. This means that if the arrow topological space is not second countable, then by Theorem 2.35 it is not metrizable.

We show this is the case by contradiction. Suppose a countable base $\mathcal{B}'_A = \{B_n : n \geq 1\}$ does exist. Let $b_n = \inf B_n$ for each $n \geq 1$. Select $b' \notin \{b_n\}$. Then $G = [b', b'+1)$ is open in the arrow topology, and must be a union of sets in $\mathcal{B}'_A$, say $G' = \cup_{m \in M} B_m$. This union must include at least one set B_k for which $b_k < b'$ since $b' \in G$, but $b' \notin B_n$ whenever $b_n > b'$, and $b_n \neq b'$ for any n. However, this implies $\inf G' \leq b_k < b' = \inf G$, therefore $G \neq G'$. Thus, the arrow topological space cannot have a countable base, and so by Theorem 2.35 cannot be metrizable. ///

2.10.3 Compact metric spaces

The compactness property is of some consequence for metric spaces.

Definition 2.25 A topological space is *sequentially compact* if every sequence $\{x_n\}$ contains a convergent subsequence. ///

We will show that for metric spaces sequential compactness is equivalent to compactness.

Lemma 2.1 Let $\mathcal{X}$ be a metric space. Suppose we are given a sequence $\{x_n\}$ which does not have any convergent subsequence. Then

(i) For each $x \in \mathcal{X}$ there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains no element of $\{x_n\}$ that is not equal to x.

(ii) The set $\{x_n\}$ is closed. ///

Proof. (i) If there exists no convergent subsequence of $\{x_n\}$, then for any $x \in \mathcal{X}$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap \{x_n\}$ has finite cardinality. In this case there also exists small enough $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains no elements of $\{x_n\}$ that are not equal to x.

(ii) Suppose $x \notin \{x_n\}$. By part (i) there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains no elements of $\{x_n\}$. Then

$$\{x_n\}^c = \cup_{x \notin \{x_n\}} B_{\epsilon_x}(x),$$

which is an open set. □

The first part of the equivalence of compactness and sequential compactness for metric spaces follows from Lemma 2.1.

Theorem 2.36 A compact metric space is sequentially compact. ///

Proof. We argue by contradiction. Suppose $\mathcal{X}$ is a metric space. Let $\{x_n\}$ be a sequence which contains no convergent subsequence. By Lemma 2.1, for each $k \geq 1 \ \exists \ \epsilon_k > 0$ such that $B_{\epsilon_k}(x_k) \cap \{x_n\} = x_k$. Also by Lemma 2.1 $\{x_n\}^c$ is open, so that $\mathcal{U} = \{\{x_n\}^c, B_{\epsilon_1}(x_1), B_{\epsilon_2}(x_2), \ldots\}$ is an open cover of $\mathcal{X}$. But $\mathcal{U}$ contains no finite subcover of $\mathcal{X}$, which is therefore not compact. □

To prove the converse of Theorem 2.36 we will need a few new constructions.

Definition 2.26 Let $\mathcal{U} = \{U_i\}$ be an open cover of a metric space $\mathcal{X}$. A *Lebesgue number* for $\mathcal{U}$ is any number $\lambda_{\mathcal{U}} > 0$ such that for each $x \in \mathcal{X} \ \exists \ U_i \in \mathcal{U}$ such that $B_{\lambda_{\mathcal{U}}}(x) \subset U_i$. ///

Definition 2.27 Suppose we are given a metric space $(\mathcal{X}, d)$. An *ϵ-net* is a set of points $\{x_i : i \in I\} \subset \mathcal{X}$ for which $\mathcal{X} = \cup_{i \in I} B_\epsilon(x_i)$. Alternatively, $\{x_i\} \subset \mathcal{X}$ is an ϵ-net *iff* for every $x \in \mathcal{X} \ \exists \ i \in I \ni d(x, x_i) < \epsilon$. ///

If $\mathcal{X}$ is a sequentially compact metric space then any open cover possesses a Lebesgue number, and $\mathcal{X}$ possesses a finite ϵ-net for any $\epsilon > 0$.

Theorem 2.37 Any open cover of a sequentially compact metric space possesses a Lebesgue number. ///

Proof. We argue by contradiction. Suppose $\mathcal{X}$ is a sequentially compact metric space for which $\mathcal{U} = \{U_i\}$ is an open cover. If $\mathcal{U}$ does not possess a Lebesgue number, then for each $n \geq 1 \ \exists \ x_n \in \mathcal{X} \ni B_{1/n}(x_n)$ is not contained in any $U_i \in \mathcal{U}$. But $\mathcal{X}$ is sequentially compact, so $\{x_n\}$ contains a subsequence, say $\{x_{n_k} : k \geq 1\}$, which possesses some limit $x_0 \in \mathcal{X}$. Since $\mathcal{U}$ is a cover of $\mathcal{X}$ there must be some $U_0 \in \mathcal{U}$ which contains x_0, and some small enough $\epsilon > 0$ such that $B_\epsilon(x_0) \subset U_0$. We may also select large enough k such that $1/n_k < \epsilon/2$ and $d(x_{n_k}, x_0) < \epsilon$. Suppose $y \in B_{1/n_k}(x_{n_k})$. By the triangle inequality

$$d(y, x_0) \leq d(y, x_{n_k}) + d(x_{n_k}, x_0) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This implies

$$B_{1/n_k}(x_{n_k}) \subset B_\epsilon(x_0)),$$

which contradicts the choice of $\{x_n\}$. Therefore a Lebesgue number must exist. □

Theorem 2.38 A sequentially compact metric space possesses a finite ϵ-net for all $\epsilon > 0$. ///

Proof. Suppose $\mathcal{X}$ is a sequentially compact metric space, but that no finite ϵ-net exists. Then let $x_1, \ldots, x_n$ be any finite sequence in $\mathcal{X}$. Since no finite ϵ-net exists, we must have $\cup_{k=1}^n B_\epsilon(x_k) \neq \mathcal{X}$, so that there exists $x_{n+1} \in \mathcal{X}$ for which $x_{n+1} \notin cup_{k=1}^n B_\epsilon(x_k)$. This means $d(x_{n+1}, x_k) \geq \epsilon$ for each $k = 1, \ldots, n$. Thus, starting with any $x_1 \in \mathcal{X}$, we may construct a sequence $\{x_n\}$ for which $d(x_n, x_m) \geq \epsilon$ for all $n \neq m$. This means that no subsequence of $\{x_n\}$ is a Cauchy sequence, and by Theorem 2.27 no subsequence of $\{x_n\}$ is convergent, which contradicts the hypothesis that $\mathcal{X}$ is sequentially compact. □

We can now prove the converse of Theorem 2.36.

Theorem 2.39 A sequentially compact metric space is compact. ///

Proof. Suppose $\mathcal{X}$ is a sequentially compact metric space for which $\mathcal{U} = \{U_i\}$ is an open cover. By Theorem there exists a Lebesgue number $\lambda_{\mathcal{U}} > 0$ for $\mathcal{U}$, and by Theorem 2.38 there exists a finite $\lambda_{\mathcal{U}}$-net $\{x_1, \ldots, x_n\}$ for $\mathcal{X}$. For each $k = 1, \ldots, n$ there exists U_{i_k} such that $B_{\lambda_{\mathcal{U}}}(x_k) \subset U_{i_k}$. Then

$$\mathcal{X} = \cup_{k=1}^{n} B_{\lambda_{\mathcal{U}}}(x_k) \subset \cup_{k=1}^{n} U_{i_k},$$

which gives us the required finite subcover. □

We then have:

Theorem 2.40 A metric space is compact *iff* it is sequentially compact. ///

Proof. The theorem follows from Theorems 2.36 and 2.39. □

The completeness property of a metric space is not a topological invariant (Example 2.21), but may still be characterized using topological properties.

Theorem 2.41 A compact metric space is complete. ///

Proof. Let $\{x_n\}$ be a Cauchy sequence in a compact metric space $\mathcal{X}$. Since a compact space is sequentially compact (Theorem 2.40) $\{x_n\}$ possesses a convergent subsequence $\{x_{n_k}\}$ with some limit $x_0 \in \omega$. For any $\epsilon > 0$ we may select N such that $d(x_{n_k}, x_0) \leq \epsilon/2$, and $d(x_m, x_n) < \epsilon/2$ whenever $n_k, m, n > N$. Then by the triangle inequality,

$$d(x_n, x_0) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon.$$

We may then conclude that the sequence $\{x_n\}$ converges to $x_0 \in \mathcal{X}$, so that $\mathcal{X}$ is complete. □

The converse is clearly not true, since $\mathbb{R}$ is complete but not compact. However, a subset of complete metric spaces can be shown to be compact.

Theorem 2.42 If a metric space $\mathcal{X}$ is complete, and for every $\epsilon > 0$ there exists a finite ϵ-net, then $\mathcal{X}$ is compact. ///

Proof. We proceed by showing that if a finite ϵ-net exists for any ϵ, any sequence $X = \{x_n\}$ must possess a subsequence which is a Cauchy sequence. Suppose we have selected indices $n_1 < n_2 < \ldots < n_k$, so that we have finite subsequence $\{x_{n_1}, \ldots, x_{n_k}\}$. There exists a finite $(1/(k+1))$-net, so there must be some open ball $B_{1/(k+1)}(y)$ which contains an infinite number of elements of $\{x_n\}$, say X_{k+1}. Let n_{k+1} be any index represented in X_{k+1} that is larger than n_k. Then $x_{n_{k+1}}$ is appended to the finite sequence. We can then argue that there exists some open ball $B_{1/(k+2)}(y')$ that contains an infinite number of elements of X_{k+1}, say X_{k+2}, then select n_{k+2} in the same manner.

Suppose we start with $k = 1$, and set $n_1 = 1$, constructing a subsequence in the manner just described. Then for fixed k the elements of the resulting subsequence $x_{n_{k'}}$ for which $k' \geq k$ are contained in X_k. This in turn implies that $d(x_{n_{k'}}, x_{n_{k''}}) < 2/k$ for all $k', k'' \geq k$. In other words, the resulting subsequence is a Cauchy sequence, which converges to some $x_0 \in \mathcal{X}$, since $\mathcal{X}$ is assumed complete. Thus, $\mathcal{X}$ is sequentially compact, and hence compact by Theorem 2.40. □

We then have the following equivalence:

Theorem 2.43 Let $\mathcal{X}$ be a metric space. Then the following statements are equivalent.
(i) $\mathcal{X}$ is compact.
(ii) $\mathcal{X}$ is complete and possesses a finite ϵ-net for all $\epsilon > 0$. ///

Proof. The Theorem follows from Theorems 2.41, 2.36, 2.38 and 2.42 $\square$

Example 2.23 In Example 2.21 it was shown that completeness is not a topological invariant. But compactness is, so by Theorem 2.43 if two homeomorphic metric spaces are complete, then either both, or neither, possess a finite ϵ-net for all $\epsilon > 0$. ///

2.10.4 Semimetrics and the completion of metric spaces

We begin with the following definition.

Definition 2.28 Suppose we have a set of objects $\mathcal{X}$ and a real-valued mapping $d : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ operating on two observations $\boldsymbol{u}, \boldsymbol{v}$. Then d is a *semimetric* (or *pseudometric*) if

(i) $d(\boldsymbol{u}, \boldsymbol{u}) = 0$;
(ii) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$ (symmetry);
(iii) $d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v})$ for any $\boldsymbol{w} \in \mathcal{X}$ (triangle inequality). The pair $(\mathcal{X}, d)$ is referred to as a *metric space*.

///

As for a metric, the axioms for a semimetric imply non-negativity.

Theorem 2.44 A semimetric (Definition 2.28) is non-negative. ///

Proof. The proof is identical to that of Theorem 2.26. $\square$

The only difference between a semimetric and a metric is that the latter admits the possibility that $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ when $\boldsymbol{u} \neq \boldsymbol{v}$. However, it turns out this induces an interesting mathematical structure.

Theorem 2.45 Suppose d is a semimetric on $\mathcal{X}$ (Definition 2.28). Then the binary relation

$$x \sim y \iff d(x, y) = 0 \tag{2.5}$$

is an equivalence relation. ///

Proof. Given Definition 1.3, reflexivity and symmetry follow immediately from Definition 2.44. Then suppose $d(x, y) = d(y, z) = 0$, so that $x \sim y$ and $y \sim z$. The the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) = 0,$$

so that $x \sim z$, and transitivity also holds. $\square$

The set of all equivalence classes forms a partition of $\mathcal{X}$. It is sometimes useful to consider the set of all equivalence sets as a space, which may be of interest on its own. This appears in various branches of mathematics, so conventions may differ, but this type of space is generally known as a *quotient space* $\mathcal{X}_Q$ generated by some equivalence relation $\sim$ on $\mathcal{X}$. Elements of $\mathcal{X}_Q$ may be represented as *cosets*:

$$[x] = \{y \in \mathcal{X} : x \sim y\}, \ \ x \in \mathcal{X}.$$

Any element of $\mathcal{X}$ is in exactly one coset, and $[y] = [x]$ *iff* $x \sim y$.

We now consider the main problem of this section. The definition of a *complete* metric space is given in Definition 2.21. Nothing in the definition of a metric space forces the completeness property. Clearly, $\mathcal{X} = (0, 1)$ endowed with the Euclidean metric is a metric space, but is not complete.

We earlier made the claim that a metric space can always be completed. In the example just given, this is easily done by extending the metric to $\mathcal{X} \cup \{0, 1\}$. However, this cannot serve as a general method. The definition of a metric space does not require specific properties of the set $\mathcal{X}$, and so we cannot assume, in general, that $\mathcal{X}$ is a subset of some strictly larger set that contains limits of all Cauchy sequences.

Let $\mathcal{C}$ be the class of all Cauchy sequences in $\mathcal{X}$. It turns out that a metric space can always be completed by defining limits of Cauchy sequences, or rather, constructing a new space $\mathcal{X}_Q$ for which exists a surjective mapping $\mathcal{C} \mapsto \mathcal{X}_Q$, and for which $\mathcal{X}_Q$ can reasonably be interpreted as the set of all elements of $\mathcal{X}$, to which are appended elements interpretable as limits of Cauchy sequences.

As the notation suggests, $\mathcal{X}_Q$ will be a quotient space, but a quotient space on $\mathcal{C}$. The argument then has two parts. The first is the construction of the quotient space, the second is the interpretation of $\mathcal{X}_Q$ as the completion of $(\mathcal{X}, d)$.

First, construct a distance function

$$\Delta(X, Y) = \lim_{n\to\infty} d(x_n, y_n) \tag{2.6}$$

for any $X = \{x_n\}$, $Y = \{y_n\}$ in $\mathcal{C}$. We need to verify that this is always defined.

Lemma 2.2 The distance function defined in Equation 2.6 is well defined for any two Cauchy sequences ///

Proof. We proceed by showing that $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$. For any $\epsilon > 0$ we may select N large enough that $m, n > N$ implies both $d(x_m, x_n) < \epsilon$ and $d(y_m, y_n) < \epsilon$. Then by the triangle inequality

$$\begin{aligned} d(x_m, y_m) &\leq d(x_m, y_n) + d(y_n, y_m) \\ &\leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m). \end{aligned}$$

A similar argument yields

$$d(x_n, y_n) \leq d(x_m, x_n) + d(x_m, y_m) + d(y_n, y_m),$$

and the two inequalities imply

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_m, x_n) + d(y_m, y_n) < 2\epsilon$$

for $m, n > N$. We may conclude that $\{d(x_n, y_n) : n \geq 1\}$ is a Cauchy sequence in $\mathbb{R}$, which is a complete metric space (Example 2.20). Therefore $d(x_n, y_n)$ possesses a limit, and the distance function defined in Equation 2.6 is well defined. □

At this point, we provide an outline of the remaining argument. We first verify that $\Delta(X, Y)$ is a semimetric on $\mathcal{C}$. This generates the space of equivalence classes on $\mathcal{C}$ defined by Δ, say $\mathcal{C}_\Delta$. We next note that $\Delta(X, Y)$ may be used to define a *metric* Δ^* on $\mathcal{C}_\Delta$. This is because if $[X], [Y]$ are any two elements of $\mathcal{C}_\Delta$, expressed as cosets, we may define

$$\Delta^*([X], [Y]) = \Delta(X, Y).$$

This is well defined, since for any other $X' \in [X]$, $Y' \in [Y]$,

$$\begin{aligned} \Delta(X', Y') &\leq \Delta(X, Y) + \Delta(X, X') + \Delta(Y, Y') \\ &= \Delta(X, Y). \end{aligned}$$

since $\Delta(X, X') = \Delta(Y, Y') = 0$. An identical argument $\Delta(X, Y) \leq \Delta(X', Y')$, so that $\Delta(X, Y) = \Delta(X', Y')$, and we may calculate $\Delta^*([X], [Y])$ using any element of the respective cosets. It may then be verified that Δ^* is a metric, therefore $(\mathcal{C}_\Delta, \Delta^*)$ is a metric space.

It turns out that despite its rather unintuitive construction, the metric space $(\mathcal{C}_\Delta, \Delta^*)$ is the completion we wish to construct. For any element $[X] \in \mathcal{C}_\Delta$, either all of its Cauchy sequence converge to some common element in $\mathcal{X}$, or none do. And any element $x \in \mathcal{X}$ is the limit of at least one Cauchy sequence (for example, $x_n = x, n \geq 1$). Thus, $\mathcal{C}_\Delta$ can be partitioned into two subsets, one of which is in a bijective relationship with $\mathcal{X}$. The remaining subset is then interpretable as the limit of a Cauchy sequence, and all Cauchy sequences in $\mathcal{X}$ are represented in this way. This idea is made rigorous by formally proving that the metric space $(\mathcal{C}_\Delta, \Delta^*)$ is, in fact, itself complete.

2.11 Separation axioms

Separation axioms define topological invariants which characterize the degree to which distinct points can be isolated from each other. Conventionally, they consist of a hierarchical collection of four axioms, labelled T_1, T_2, T_3, T_4. They serve the important function of characterizing the degree to which a topological space behaves like a metric space.

Definition 2.29 (Separation axiom T_1) A topological space $\mathcal{X}$ satisfies T_1 if for all points $x \neq y$ $\exists$ open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. ///

Definition 2.30 (Separation axiom T_2) A topological space $\mathcal{X}$ satisfies T_2 if for all points $x \neq y$ $\exists$ open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. A space satisfying T_2 is referred to as a *Hausdorff space*. ///

Definition 2.31 (Separation axiom T_3) A topological space $\mathcal{X}$ satisfies T_3 if it satisfies T_1, and if for all points x and closed sets F for which $x \notin F$ $\exists$ open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. A space satisfying T_3 is referred to as a *regular space*. ///

Definition 2.32 (Separation axiom T_4) A topological space $\mathcal{X}$ satisfies T_4 if it satisfies T_1, and if for any disjoint closed sets A, B $\exists$ open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. A space satisfying T_4 is referred to as a *normal space*. ///

Verification of the hierarchical relationship of the separation axioms will require the following theorem.

Theorem 2.46 A topological space $\mathcal{X}$ satisfies T_1 *iff* the subset $\{x\}$ is closed for each $x \in \mathcal{X}$. ///

Proof. First assume $\mathcal{X}$ satisfies T_1. Set $E = \{x\}$. We will show that x is the only point of closure of E (Definition 2.6). Suppose $y \neq x$. By the T_1 axiom $\exists$ open set V such that $y \in V$ but $x \notin V$. Therefore y cannot be a point of closure of E, hence by Theorem 2.4 E is closed.

Conversely, suppose any set $\{x\}$ is closed. Then $y \neq x$ cannot be a point of closure of $\{x\}$, so there must exist open set V containing y but not x. By applying the same argument to $\{y\}$ we conclude that there must exist open set U containing x but not y. Therefore axiom T_1 must hold. □

We then have our hierarchy:

Theorem 2.47 For any topology $\mathcal{X}$ the following sequence of implications hold.

$$T_4 \implies T_3 \implies T_2 \implies T_1.$$

///

Proof. First, $T_2 \implies T_1$ is verified by noting that the sets U, V described in Definition 2.30 also satisfy the conditions stated in Definition 2.29. We next note that by definition T_3 and T_4 also satisfy T_1, from which by Theorem 2.46 it follows that any set $\{x\}$ is closed. Then $T_3 \implies T_2$ is verified by setting $F = \{y\}$ for any $y \neq x$ in Definition 2.31. The resulting statement is equivalent to Definition 2.30. Finally, $T_4 \implies T_3$ is similarly verified by setting $A = \{x\}$, $B = \{y\}$ for any $x \neq y$ in Definition 2.32. □

2.11.1 Metric spaces are normal spaces

The following theorem provides a useful sufficient condition for a topological space to be normal. It is worth recalling throughout the conclusion of Section 2.10.1, that openness, closedness, continuity and convergence are equivalent for metric spaces and their induced topological spaces.

Theorem 2.48 Suppose $\mathcal{X}$ is a topological space satisfying T_1. Then suppose for any disjoint closed subsets A, B $\exists$ a continuous mapping $f : \mathcal{X} \to [0,1]$ such that $A \subset f^{-1}(\{0\})$ and $B \subset f^{-1}(\{1\})$. Then $\mathcal{X}$ is normal. ///

Proof. Let A, B be the closed sets, and f the mapping described in the hypothesis. Fix $\epsilon \in (0, 1/2)$. Let $U = f^{-1}([0,\epsilon))$ and $V = f^{-1}((1-\epsilon, 1])$. Since $[0,\epsilon)$ and $(1-\epsilon, 1]$ are disjoint open subsets of $[0,1]$, and f is continuous, U and V are disjoint open subsets of $\mathcal{X}$. The proof is completed by noting that $A \subset U$ and $B \subset V$. □

A metric space can then be shown to be normal by constructing the family of mappings described in the hypothesis of Theorem 2.48.

Definition 2.33 Suppose $(\mathcal{X}, d)$ is a metric space. The distance between a point $x \in \mathcal{X}$ and a subset $A \subset \mathcal{X}$ may be defined by the quantity

$$d(x, A) = \inf\{d(x, y) : y \in A\}$$

///

We will need to verify the following properties of the quantity $d(x, A)$ given in Definition 2.33.

Theorem 2.49 Suppose $(\mathcal{X}, d)$ is a metric space, and $d(x, A)$ is the distance between a point $x \in \mathcal{X}$ and subset $A \subset \mathcal{X}$ given in Definition 2.33. Then
(i) If A is closed, then $d(x, A) = 0$ *iff* $x \in A$.
(ii) For any fixed A, the function $\phi(x) = d(x, A)$ is continuous. ///

Proof. (i) If $x \in A$ then $d(x, A) \leq d(x, x) = 0$. Conversely, suppose $d(x, A) = 0$. Then for each $n \geq 1$ $\exists$ $y_n \in A$ such that $d(x, y_n) < 1/n$. This means that $y_n \rightarrow x$. But A is closed, so by Theorem 2.30 we must have $x \in A$.
(ii) Select $x, y \in \mathcal{X}$ and $a \in A$. Then

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

Since this holds for any $a \in A$ we may conclude

$$d(x, A) \leq d(x, y) + d(y, A).$$

After exchanging x and y we have

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

The continuity of $\phi(x)$ follows directly. □

We can now finish our argument.

Theorem 2.50 Metric spaces are normal. ///

Proof. Suppose $(\mathcal{X}, d)$ is a metric space. First note that a metric space satisfies axiom T_1. To verify this, for any $x \neq y$ set $U = B_\epsilon(x)$, $V = B_{\epsilon'}(y)$ in Definition 2.29, then make ϵ, ϵ' small enough. We then proceed by constructing an example of the mapping $f : \mathcal{X} \rightarrow [0, 1]$ described in the hypothesis of Theorem 2.48.
Suppose A, B are disjoint closed subsets of $\mathcal{X}$. Set

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

By Theorem 2.49 (i), since A, B are closed and disjoint we must have $d(x, A) + d(x, B) > 0$ for all $x \in \mathcal{X}$, so $f(x)$ is well defined. By Theorem 2.49 (ii), $d(x, A)$ and $d(x, B)$ are continuous in x, therefore so is $f(x)$. The hypothesis of Theorem 2.48 is satisfied by this construction. □

We now have a powerful method of proving that a topological space is not metrizable.

Theorem 2.51 If a topological space does not satisfy any of the separation axioms T_1, T_2, T_3, T_4 it is not metrizable. ///

Proof. The theorem is an immediate consequence of Theorems 2.47 and 2.50. □

Example 2.24 (The Zariski topological space is not metrizable) Consider the Zariski topological space of Example 2.8. No two nonempty open sets are disjoint, so separation axiom T_2 cannot hold. By Theorem 2.51 the Zariski topological space is not metrizable. ///

Theorem 2.52 (Urysohn Metrization Theorem) A normal topological space satisfying the second axiom of countability is metrizable. ///

2.12 Product Spaces

We have defined the product of m sets $A_1, \ldots, A_m$, denoted $A_1 \times A_2 \times \ldots \times A_m = \times_{i=1}^m A_i$, consisting of the set of all ordered selections of one element from each set $a_i \in A_i$. A subset of $\times_{i=1}^m A_i$ need not itself be a product of sets.

It will often be necessary to construct product spaces of various kinds, and it will be important to verify when that product space retains the various properties of its components. The process is usually straightforward when m is finite.

Product metric spaces

Let (X_i, d_i), $i = 1, \ldots, m$, be m metric spaces. Define the product $X = \times_{i=1}^m X_i$. For any $x, y \in \mathcal{X}$, define the vector $\vec{d}(x, y) = (d_1(x_1, y_1), \ldots, d_m(x_m, y_m))$. It may be verfied that the distance function

$$d(x, y) = \|\vec{d}(x, y)\|_p$$

is a true metric on $\mathcal{X}$ for any L^p norm, therefore (X, d) is a metric space. The metric spaces for distinct L^p norms are equivalent (that is, the topological properties do not depend on p).

Product topologies

Let $(\mathcal{X}_i, \mathcal{O}_i)$, $i = 1, \ldots, m$, be m topological spaces. Define the product $\mathcal{X} = \times_{i=1}^m \mathcal{X}_i$. Define the class of sets $\mathcal{B} = \{\times_{i=1}^m O_i : O_i \in \mathcal{O}_i\}$. It may be verfied that $\mathcal{B}$ is a topological base for $\mathcal{X}$, which induces the *product topology*. Furthermore, if $\mathcal{X}_i$ are metric spaces, the product topology is equivalent to the topology induced on $\mathcal{X}$ by the product metric.

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Bibliography

Kolmogorov, A. and Fomin, S. (1975). *Introductory Real Analysis*. Courier Corporation.

Munkres, J. (2013). *Topology: Pearson New International Edition*. Pearson.

Royden, H. L. (1968). *Real Analysis*. MacMillan Publishing, New York, NY, second edition.

Rudin, W. (1987). *Real and complex analysis*. Tata McGraw-Hill Education.