Chapter 12: Nonparametric Inference

DSCC 462

Computational Introduction to Statistics

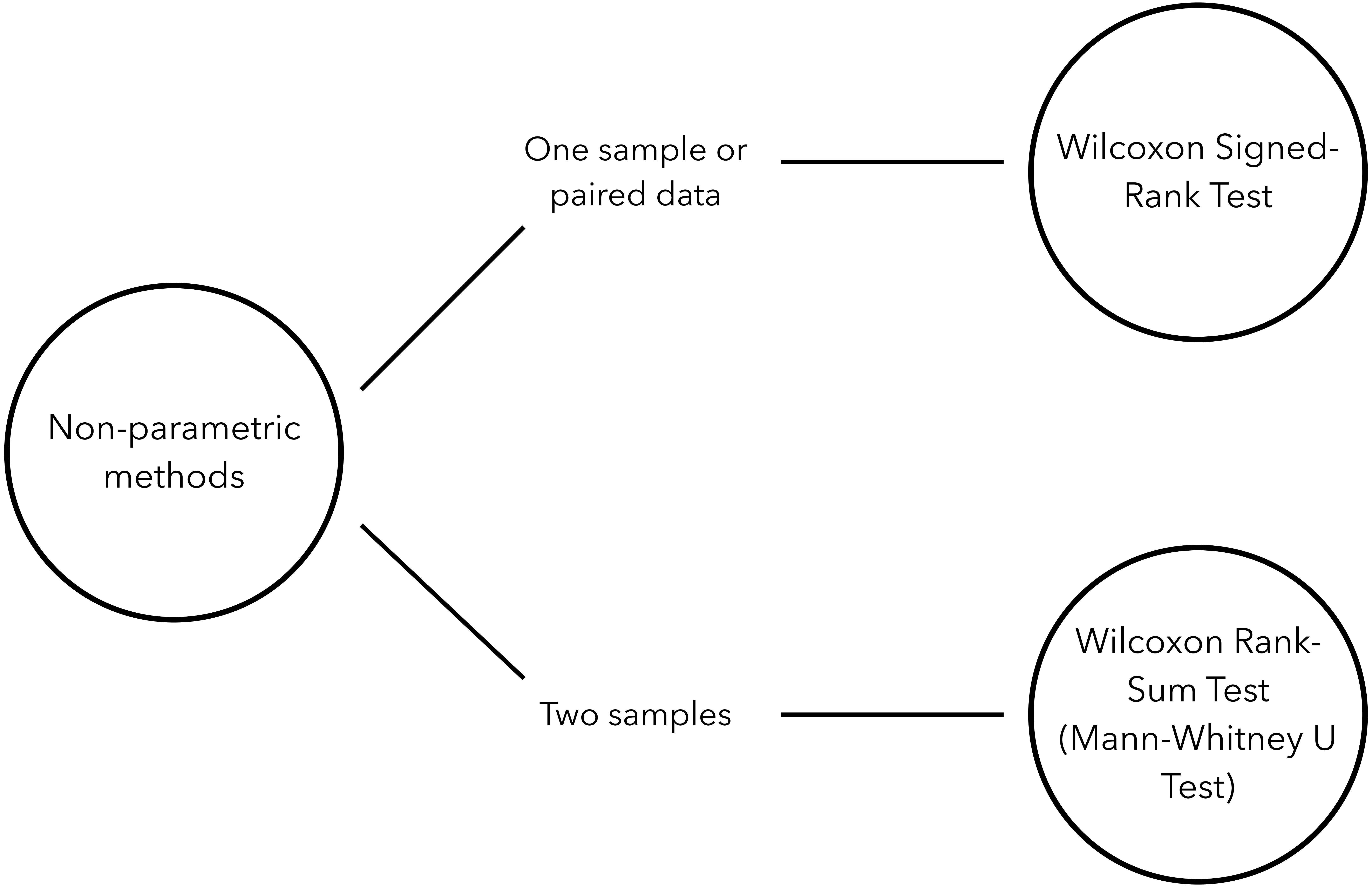
Anson Kahng

Fall 2022

Plan for Today

- Introduce nonparametric analogues to hypothesis tests
- *Wilcoxon Signed-Rank Test*
  - Nonparametric analog to the one-sample or paired t-test
- *Wilcoxon Rank-Sum Test (Mann-Whitney U Test)*
  - Nonparametric analog to the two-sample t-test

Plan for Today, Visualized



Nonparametric Methods

- Think about the statistical tests we've done so far (z-test, t-test, χ^2-test, F-test, etc.)
- In all of these, we knew the distribution of the population and we only needed to perform inference on the unknown parameters
  - *Parametric methods*
  - We knew what distribution the population followed
- What if we don't know the distribution of the population?
  - *Nonparametric methods*

Nonparametric Methods

- When do we use nonparametric methods?
    - When we don’t know the underlying population distribution
    - Or the data do not meet the assumptions needed for particular parametric techniques (e.g., CLT doesn’t hold, normal approximation for proportions doesn’t hold, etc.)
- In this case, we use *nonparametric methods*, which make fewer assumptions regarding the underlying distribution
    - Also known as *distribution-free methods*

Nonparametric Methods

- Although nonparametric testing procedures make different assumptions, they still follow the same general setup as all hypothesis tests we have discussed so far
    - Make a claim, develop hypotheses, state significance level
    - Calculate a test statistic based on a random sample of data
    - Determine whether to reject or fail to reject the null hypothesis based on the test statistic and significance level

Motivating Example #1

- Suppose we want to determine whether a new drug changes tumor size
- We cannot assume that tumor sizes are normally distributed
- Let's say that we have a sample of n pairs of observations (tumor size before drug vs. tumor size after drug), where $n = 13$
- Can we apply the CLT here?
- This means that we need to use a nonparametric method
  - *Wilcoxon Signed-Rank Test*

Wilcoxon Signed-Rank Test

- Used to compare two samples from populations that are not independent
    - Nonparametric analog to the paired t-test
- Because we are considering paired data, we may look at the difference in values for each pair of observations
- Does not require populations to be normally distributed
- Takes into account both the magnitudes of the differences and their signs
- Null hypothesis: In the underlying population differences among pairs, the median difference is equal to 0
    - Note that we consider medians for nonparametric tests as opposed to means

Wilcoxon Signed-Rank Test: Back to Example #1

- Suppose we want to determine whether a new drug changes tumor size
- We cannot assume that tumor sizes are normally distributed
- Let's say that we have a sample of n pairs of observations (tumor size before drug vs. tumor size after drug), where $n = 13$
- H_0 : The median difference in tumor size equals 0
- H_1 : The median difference in tumor size is different from 0
- Test at the $\alpha = 0.05$ significance level

Wilcoxon Signed-Rank Test: Steps

- Next, take the difference for each pair of observations
- Ignoring the sign of these observations, rank their absolute values from smallest to largest
  - A difference of 0 is not ranked
  - Remove pair from data set and reduce number of pairs by 1
- Tied observations are assigned an average rank
- Finally separate the ranks by sign to either $+$ or $-$

Wilcoxon Signed-Rank Test: Data Table

Subject	Tumor Size (mm)		Difference	Rank	Signed Rank	
	Before	After			+	-
1	36.3	27.1	9.2			
2	21.7	17.4	4.3			
3	45.1	33.1	12.0			
4	27.8	32.1	-4.3			
5	5.1	8.3	-2.2			
6	23.4	22.1	1.3			
7	25.0	31.2	-6.2			
8	12.6	16.4	-3.8			
9	19.9	12.5	7.4			
10	22.1	22.1	0			
11	18.6	4.8	13.8			
12	8.9	22.6	-13.7			
13	12.7	6.4	6.3			
14	29.3	18.3	9.0			
15	26.4	21.8	4.6			

Wilcoxon Signed-Rank Test: Data Table

Subject	Tumor Size (mm)		Difference	Rank	Signed Rank	
	Before	After			+	-
1	36.3	27.1	9.2	11		
2	21.7	17.4	4.3	4.5		
3	45.1	33.1	12.0	12		
4	27.8	32.1	-4.3	4.5		
5	5.1	8.3	-2.2	2		
6	23.4	22.1	1.3	1		
7	25.0	31.2	-6.2	7		
8	12.6	16.4	-3.8	3		
9	19.9	12.5	7.4	9		
10	22.1	22.1	0	-		
11	18.6	4.8	13.8	14		
12	8.9	22.6	-13.7	13		
13	12.7	6.4	6.3	8		
14	29.3	18.3	9.0	10		
15	26.4	21.8	4.6	6		

Wilcoxon Signed-Rank Test: Data Table

Subject	Tumor Size (mm)		Difference	Rank	Signed Rank	
	Before	After			+	-
1	36.3	27.1	9.2	11	11	
2	21.7	17.4	4.3	4.5	4.5	
3	45.1	33.1	12.0	12	12	
4	27.8	32.1	-4.3	4.5		4.5
5	5.1	8.3	-2.2	2		2
6	23.4	22.1	1.3	1	1	
7	25.0	31.2	-6.2	7		7
8	12.6	16.4	-3.8	3		3
9	19.9	12.5	7.4	9	9	
10	22.1	22.1	0	-		
11	18.6	4.8	13.8	14	14	
12	8.9	22.6	-13.7	13		13
13	12.7	6.4	6.3	8	8	
14	29.3	18.3	9.0	10	10	
15	26.4	21.8	4.6	6	6	

Wilcoxon Signed-Rank Test

- Calculate the sum of the positive ranks, T^+, and the sum of the negative ranks, T^-
- Calculate $T = T^+ - T^-$
- Under the null hypothesis, the median of the underlying population differences is equal to 0
- Thus, we expect approximately equal numbers of positive and negative ranks
- Additionally, the sum of the positive ranks should be approximately equal to the sum of the negative ranks, so T should be approximately 0

Wilcoxon Signed-Rank Test

- Evaluate the null hypothesis using the test statistic:

$$z_T = \frac{T - \mu_T}{\sigma_T}$$

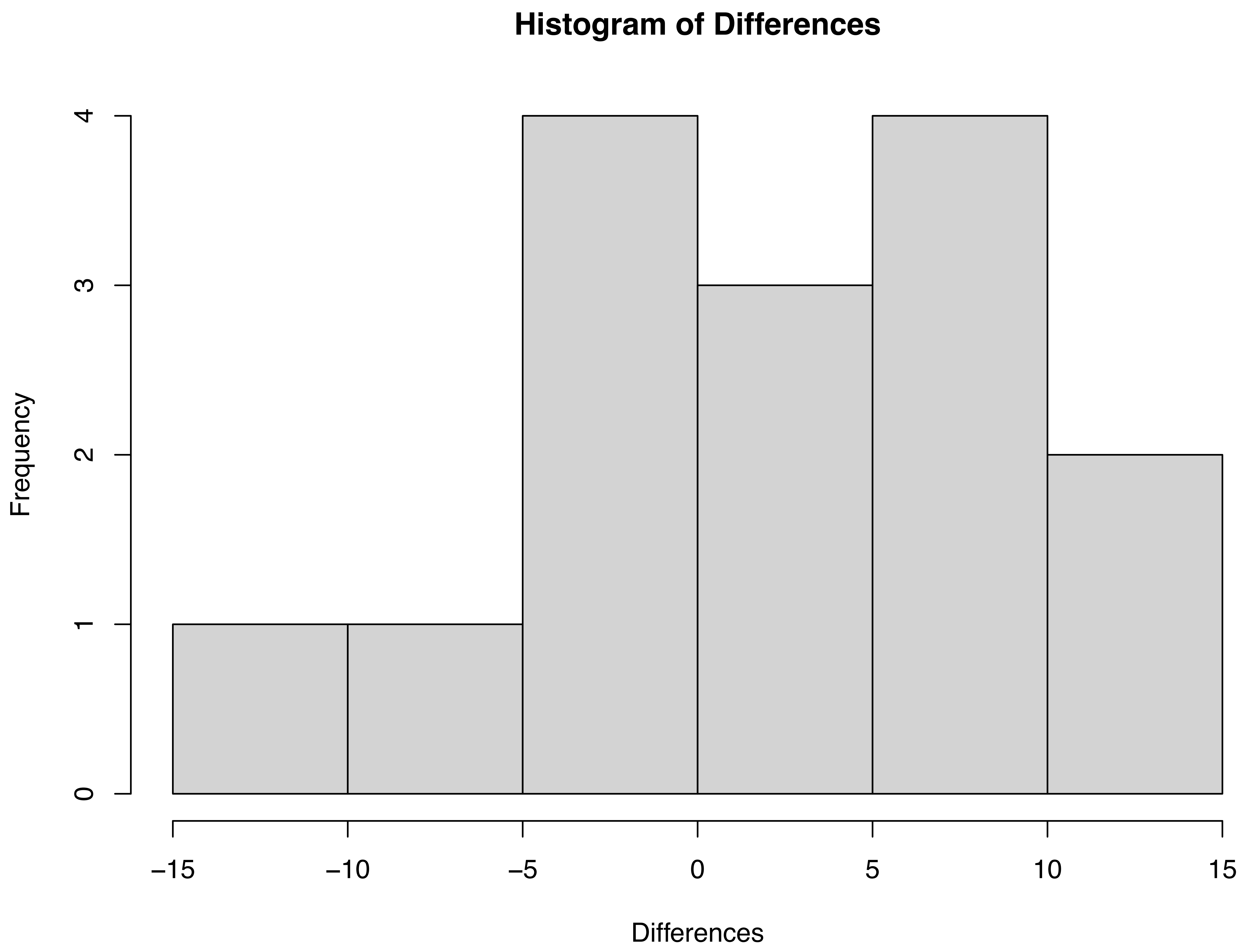
- Note that

$$\mu_T = 0$$

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$

- $Z_T \sim N(0,1)$ given that n is large enough (typically $n > 12$)

Histogram of Differences



Wilcoxon Signed-Rank Test

- Based on the histogram, the difference in tumor size does not appear to be normally distributed, so we want to use the Wilcoxon signed-rank test
- $T^{+} =$
- $T^{-} =$
- $T =$
- $n =$

Subject	**Signed Rank**	
	+	**-**
1	11	
2	4.5	
3	12	
4		4.5
5		2
6	1	
7		7
8		3
9	9	
10		
11	14	
12		13
13	8	
14	10	
15	6	

Wilcoxon Signed-Rank Test

- Based on the histogram, the difference in tumor size does not appear to be normally distributed, so we want to use the Wilcoxon signed-rank test

- $T^+ = 11 + 4.5 + 12 + 1 + 9 + 14 + 8 + 10 + 6 = 75.5$

- $T^- = 4.5 + 2 + 7 + 3 + 13 = 29.5$

- $T = 46$

- $n = 14$

Subject	**Signed Rank**	
	+	**-**
1	11	
2	4.5	
3	12	
4		4.5
5		2
6	1	
7		7
8		3
9	9	
10		
11	14	
12		13
13	8	
14	10	
15	6	

Wilcoxon Signed-Rank Test

- Given $T = 46$, we then have the following:

$$\mu_T = 0$$

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} =$$

- Thus,

$$z_T = \frac{T - \mu_T}{\sigma^T} =$$

Wilcoxon Signed-Rank Test

- Given $T = 46$, we then have the following:

$$\mu_T = 0$$

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} = 31.86$$

- Thus,

$$z_T = \frac{T - \mu_T}{\sigma^T} = 1.44$$

Wilcoxon Signed-Rank Test

- Calculating the p-value, we have

- Conclusion:

Wilcoxon Signed-Rank Test

- Calculating the p-value, we have
  $p = 2 \cdot \Pr(Z > 1.44) =$ `2*(1-pnorm(1.44))` $= 0.149$

- Conclusion: Since the p-value of 0.149 is greater than $\alpha = 0.05$, we fail to reject the null hypothesis and have insufficient evidence to conclude that the median difference is not equal to 0

- There is not enough evidence to conclude that the new drug significantly changes median tumor size

Wilcoxon Signed-Rank Test

- If the sample size is $n \leq 12$, we cannot use the normal approximation
- In that case, we can use `psignrank(T,n)` in R to calculate the exact p-value
    - `2*(1-psignrank(75.5,n=14))=0.135`

- R requires $T = T^{+}$ for this to work correctly!

Wilcoxon Signed-Rank Test: R Code

```
> wilcox.test(before, after, paired=T, exact=F,correct=F)

	Wilcoxon signed rank test

data:  before and after
V = 64, p-value = 0.1961
alternative hypothesis: true location shift is not equal to 0
```

```
> wilcox.test(before, after, paired=T, exact=T,correct=F)

	Wilcoxon signed rank test

data:  before and after
V = 64, p-value = 0.2163
alternative hypothesis: true location shift is not equal to 0
```

Motivating Example #2

- Suppose that we want to determine whether having a certain disease is associated with a raised body temperature
- Cannot assume that body temperature is normally distributed
- Take a sample of $n_1 = 12$ people who do not have the disease
- Take a sample of $n_2 = 15$ people who do have the disease
- How can we compare the median body temperature for these two populations?
    - *Wilcoxon Rank-Sum Test (Mann-Whitney U Test)*

Wilcoxon Rank-Sum Test

- Used to compare samples from independent populations
  - Nonparametric analog to the two-sample t-test
- Does not require populations to be normally distributed
- Requires the two populations to have the same general shape
- H_0 : The medians of the two populations are identical

Wilcoxon Rank-Sum Test: Back to Example #2

- Suppose that we want to determine whether having a certain disease is associated with a raised body temperature
- Cannot assume that body temperature is normally distributed
- Take a sample of $n_1 = 12$ people who do not have the disease
- Take a sample of $n_2 = 15$ people who do have the disease
- H_0 : The median body temperature for those without the disease is greater than or equal to those with the disease
- H_1 : The median body temperature for those without the disease is less than those with the disease
- Test at the $\alpha = 0.05$ significance level

Wilcoxon Rank-Sum Test: Steps

- Combine all data from the two samples and rank the observations from smallest to largest
- If ranks are tied, we assign the average rank to those values
- We then find the sum of ranks for each of the two original samples, denoted W_1 and W_2, and then let $W = \min(W_1, W_2)$
- Under H_0, the underlying populations have the same median, so we would expect ranks to be randomly distributed between the two groups
- Thus, the average ranks for the two samples (i.e., W_1/n_1 and W_2/n_2) should be approximately equal

Data Table

No Disease		Disease	
Temp	**Rank**	**Temp**	**Rank**
98.1		99.3	
98.5		99.4	
98.6		99.4	
98.8		99.5	
98.9		99.5	
99.0		99.6	
99.2		99.7	
99.5		99.7	
99.6		100.0	
99.7		100.0	
100.5		100.1	
101.0		100.1	
		100.1	
		101.1	
		101.9	

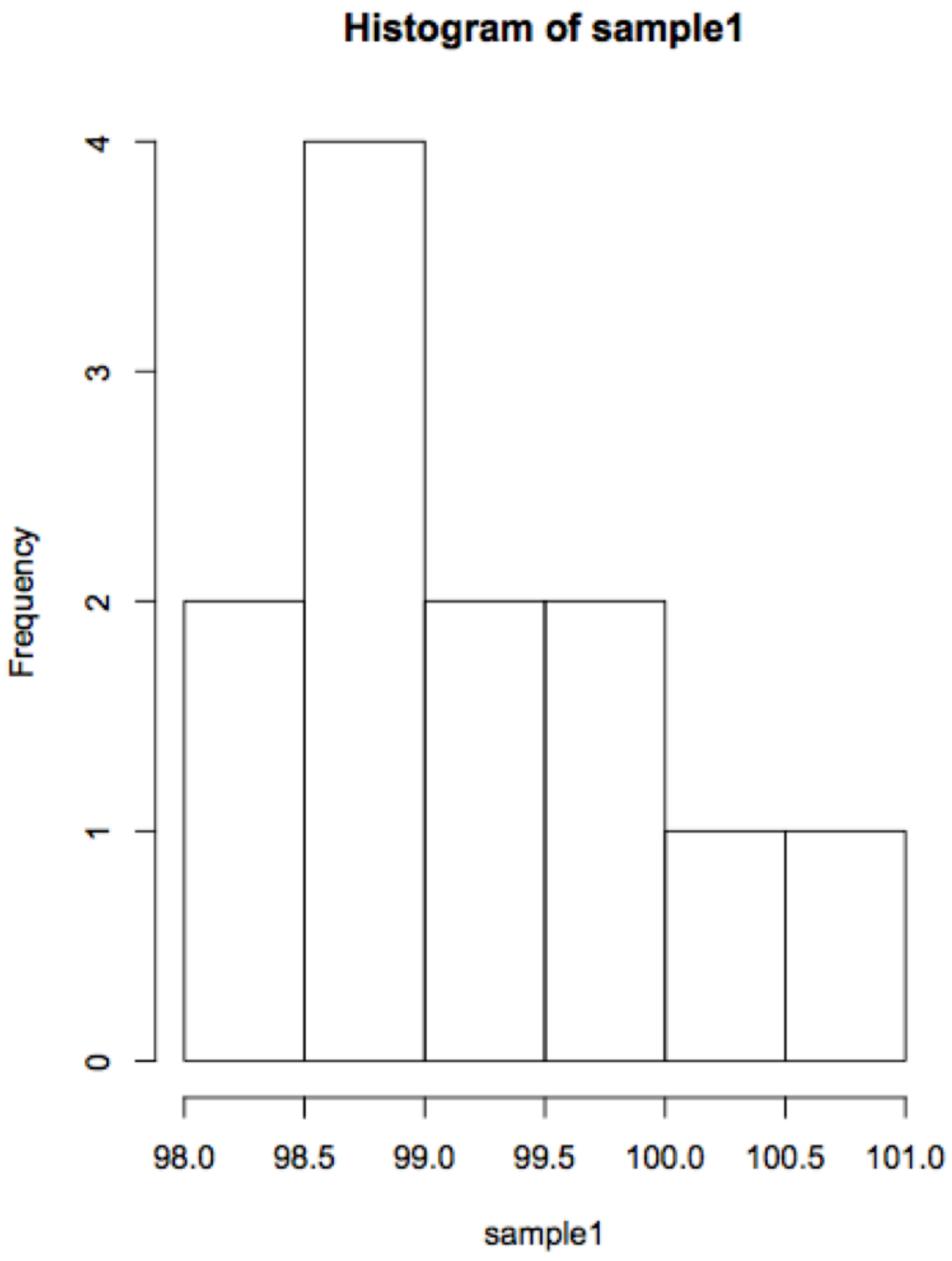
Data Table

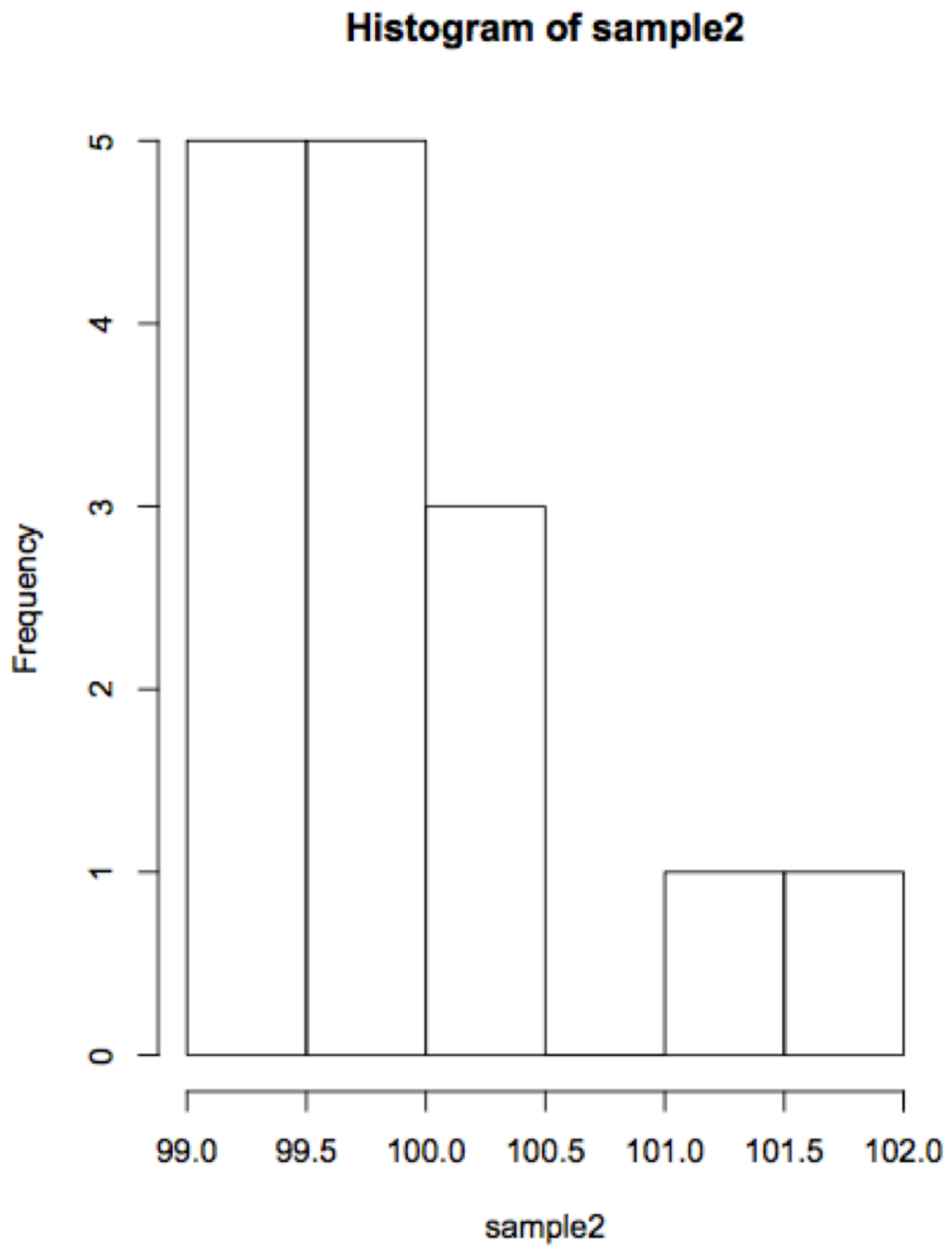
No Disease		Disease	
Temp	**Rank**	**Temp**	**Rank**
98.1	1	99.3	8
98.5	2	99.4	9.5
98.6	3	99.4	9.5
98.8	4	99.5	12
98.9	5	99.5	12
99.0	6	99.6	14.5
99.2	7	99.7	17
99.5	12	99.7	17
99.6	14.5	100.0	19.5
99.7	17	100.0	19.5
100.5	24	100.1	22
101.0	25	100.1	22
		100.1	22
		101.1	26
		101.9	27

Wilcoxon Rank-Sum Test: Steps

- Evaluate the null hypothesis using the test statistic $z_W = \dfrac{W - \mu_W}{\sigma_W}$
- Let n_1 be the number of observations in the sample with the smaller sum of ranks
- Let n_2 be the number of observations in the sample with the larger sum of ranks
- Then,
  - $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2}$ and $\sigma_W = \sqrt{\dfrac{n_1 n_2 (n_1 + n_2 + 1)}{12}}$
  - $z_W \sim N(0,1)$ when n_1 and n_2 are large enough $(n_1, n_2 > 10)$

Histograms of Samples





Wilcoxon Rank-Sum Test

- Based on the histograms, the two samples do not appear to be coming from normally distributed populations, so we want to use the Wilcoxan rank sum test
- Wilcoxon rank sum test is appropriate since both populations have similar shapes
- The sum of ranks for sample 1 is 120.5
- The sum of ranks for sample 2 is 257.5
- Thus, $W = 120.5$, $n_1 = 12$, and $n_2 = 15$

Wilcoxon Rank-Sum Test

- Given $W = 120.5$, $n_1 = 12$, and $n_2 = 15$, we then have
- $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2} =$
- $\sigma_W = \sqrt{\dfrac{n_1 n_2(n_1 + n_2 + 1)}{12}} =$
- Thus, we have $z_W = \dfrac{W - \mu_W}{\sigma_W} =$

Wilcoxon Rank-Sum Test

- Given $W = 120.5$, $n_1 = 12$, and $n_2 = 15$, we then have
- $\mu_W = \dfrac{n_1(n_1 + n_2 + 1)}{2} = 168$
- $\sigma_W = \sqrt{\dfrac{n_1 n_2 (n_1 + n_2 + 1)}{12}} = 20.49$
- Thus, we have $z_W = \dfrac{W - \mu_W}{\sigma_W} = -2.318$

Wilcoxon Rank-Sum Test

- Calculating the p-value:

- Conclusion:

Wilcoxon Rank-Sum Test

- Calculating the p-value:
  $\Pr(Z < -2.318) =$ `pnorm(-2.318)` $= 0.010$

accept H1 that summed rank is significantly lower than expected rank sum. Since lower rank == lower value, we conclude that group 1 has lower value, which is the group that without disease

- Conclusion: Since the p-value of 0.01 is less than $\alpha = 0.05$, we reject the null hypothesis and conclude that the median body temperature for those with the disease is greater than the median body temperature for those without the disease

Wilcoxon Rank-Sum Test

- If n_1 and n_2 are very small (i.e., either is less than or equal to 10), we cannot use the normal approximation

- When sample sizes are small, we can use the exact distribution to calculate p-values

- In R, we use `pwilcox(`$W_{\texttt{obs}}, n_1, n_2$`)`

  - In this case, $W_{\text{obs}} = W - \dfrac{n_1(n_1+1)}{2}$

  - `pwilcox(120.5-78,12,15)=0.0093`

Wilcoxon Rank-Sum Test: R Code

```
> wilcox.test(sample1,sample2, exact=F, correct=F, alt="less")

	Wilcoxon rank sum test

data:  sample1 and sample2
W = 42.5, p-value = 0.01009
alternative hypothesis: true location shift is less than 0
```

Nonparametric Methods: Pros and Cons

- Advantages:
  - Do not impose restrictive assumptions
  - Do not require normally distributed populations
  - Are sometimes easier to compute by hand
  - Ranks are less sensitive to measurement error
  - Permits the use of ordinal data
- Disadvantages:
  - If a parametric test can be used, it is more powerful than its nonparametric counterpart
  - Hypotheses tend to be less specific for nonparametric tests
  - Variances are typically overestimated

Nonparametric Methods: Summary

- Sometimes we want to run tests on variables but do not know their distributions
- Nonparametric tests are a flexible but sometimes underpowered way of doing so
- Nonparametric analog to the one-sample or paired t-test: Wilcoxon Signed-Rank Test
- Nonparametric analog to the two-sample t-test: Wilcoxon Rank-Sum Test (Mann-Whitney U Test)

Review Sessions Next Week

- TA review sessions (Wegmans 1201):
  - Tuesday, November 1, from 2 - 3 pm (Lucinda's normal OH)
  - Wednesday, November 2, from 6 - 8 pm
- Instructor: Tuesday and Thursday during class
  - Submit specific requests via Google Form