Chapter 5: Distributions

DSCC 462
Computational Introduction to Statistics

Anson Kahng
Fall 2022

Random Variables

- **Random variable**: A variable that can take a number of different values and whose outcome is determined by chance
- **Discrete random variable**: A random variable whose possible outcomes are a list of discrete values (finite or countably infinite sample space)
  - Example: Coin flip (heads/tails)
- **Continuous random variable**: A random variable whose possible outcomes are any value in an interval (uncountable sample space)
  - Examples: Time required to run a mile

Notation

- Random variable: Uppercase letters (e.g., X, Y)
- Outcome of a random variable: Lowercase letters (e.g., x, y)
- Example: Let $X =$ the number of surgeries a person has had
  - $\Pr(X = 1)$: Probability of having 1 surgery
  - $\Pr(X = x)$: Probability of having x surgeries

Probability Distribution

- **Probability distribution**: List of all possible values that a random variable can take, along with their corresponding probabilities
  - Discrete: Probability mass function (PMF)
  - Continuous: Probability density function (PDF)
- Let X be a random variable defined over sample space S_X
- For any $E \subseteq S_X$, we can define $p_X(E) = \Pr(X \in E)$

Discrete Probability Distribution

- For a discrete random variable X with sample space S_X, a probability mass function (PMF) p_X maps $x \in S_X$ to a number in $[0,1]$ such that:

$$0 \leq p_X(s) = \Pr(X = x) \leq 1$$

$$\sum_{x \in S_X} p_X(x) = \sum_{x \in S_X} \Pr(X = x) = 1$$

- The support S_X consists of all x for which $p_X(x) > 0$ (all achievable outcomes)

Discrete Probability Distribution: Example

- Setup: A fair coin is flipped 3 times. Let X be a random variable that counts the number of heads observed
- Fill in the following table:
- Probability distribution tables resemble relative frequency distribution tables: probability of each outcome is the relative frequency distribution of each outcome in a large number of trials

x	Pr(X = x)
0	
1	
2	
3	

Discrete Probability Distribution: Example

- Setup: A fair coin is flipped 3 times. Let X be a random variable that counts the number of heads observed
- Fill in the following table:
- Probability distribution tables resemble relative frequency distribution tables: probability of each outcome is the relative frequency distribution of each outcome in a large number of trials

x	Pr(X = x)
0	1/8
1	3/8
2	3/8
3	1/8

Continuous Probability Distribution

- Specify continuous probability distributions through a *density function*, $f(x)$
- Properties:

$$f(x) \geq 0, \text{ for all } x \in S_X \text{ (nonnegative density)}$$

$$\int f(x)\,dx = 1 \text{ (total probability is 1)}$$

- X is continuous iff there is a density function f_X such that the following holds:

$$\Pr(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$
$$= \text{Area under } f \text{ between } a \text{ and } b$$

- The support S_X consists of all x for which $f_X(x) > 0$

Normalization

- We must ensure that probability distributions sum / integrate to 1 (i.e., total probability must equal 1)

- **Normalization**: Scalar adjustment in order to ensure that $\Pr(S_X) = 1$

- If $g(x) > 0$ for all $x \in S_X$, then

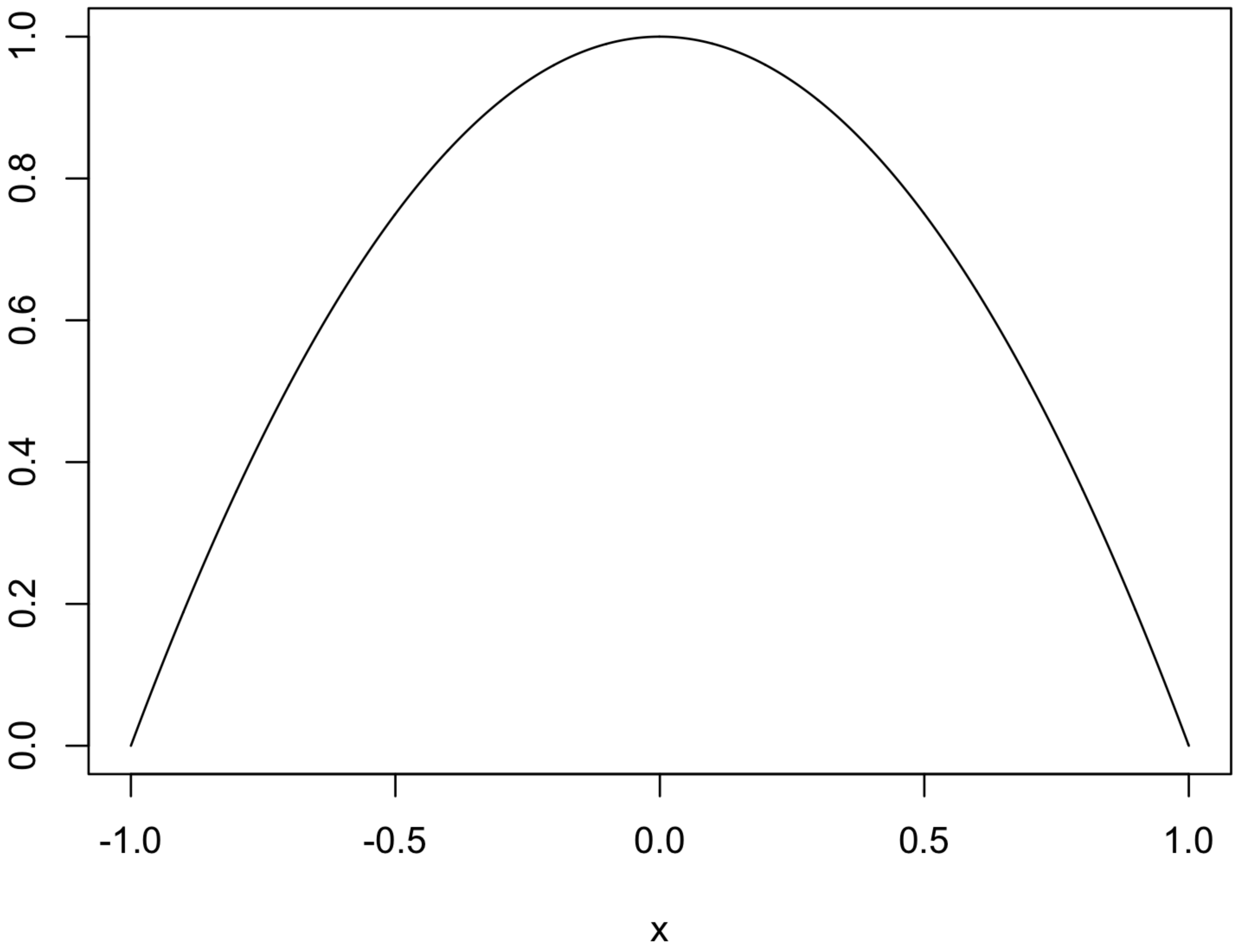
$$\text{Discrete: } p(x) = \frac{g(x)}{\sum_{x^* \in S_X} g(x^*)}$$

$$\text{Continuous: } f(x) = \frac{g(x)}{\int_{x^* \in S_X} g(x^*)\, dx^*}$$

- *Normalization constant*: 1/denominator

Normalization: Example

- Suppose that we generally know that probability is distributed according to the following curve:



Normalization: Example

- We can generally define the shape of this curve as

$$g(x) = 1 - x^2, \quad -1 \leq x \leq 1$$

- Is this a proper density?

- What's the normalization constant?

- What is $f(x)$?

Normalization: Example

- We can generally define the shape of this curve as

$$g(x) = 1 - x^2, \quad -1 \leq x \leq 1$$

- Is this a proper density?

$$\int_{-1}^{1} (1 - x^2)\, dx = \left(x - x^3/3\right)\Big|_{-1}^{1} = \frac{4}{3}$$

- What's the normalization constant?

Multiply both sides by $\frac{3}{4}$ in order to make it integrate to 1

- What is $f(x)$?

$$f(x) = \frac{3}{4}(1 - x^2)$$

Cumulative Distribution Functions (CDFs)

- The **cumulative distribution function (CDF)** of random variable X is

$$F_X(x) = \Pr(X \le x) \text{ for all } x \in (-\infty, \infty)$$

- If X is discrete with support S_X, then the CDF is defined as

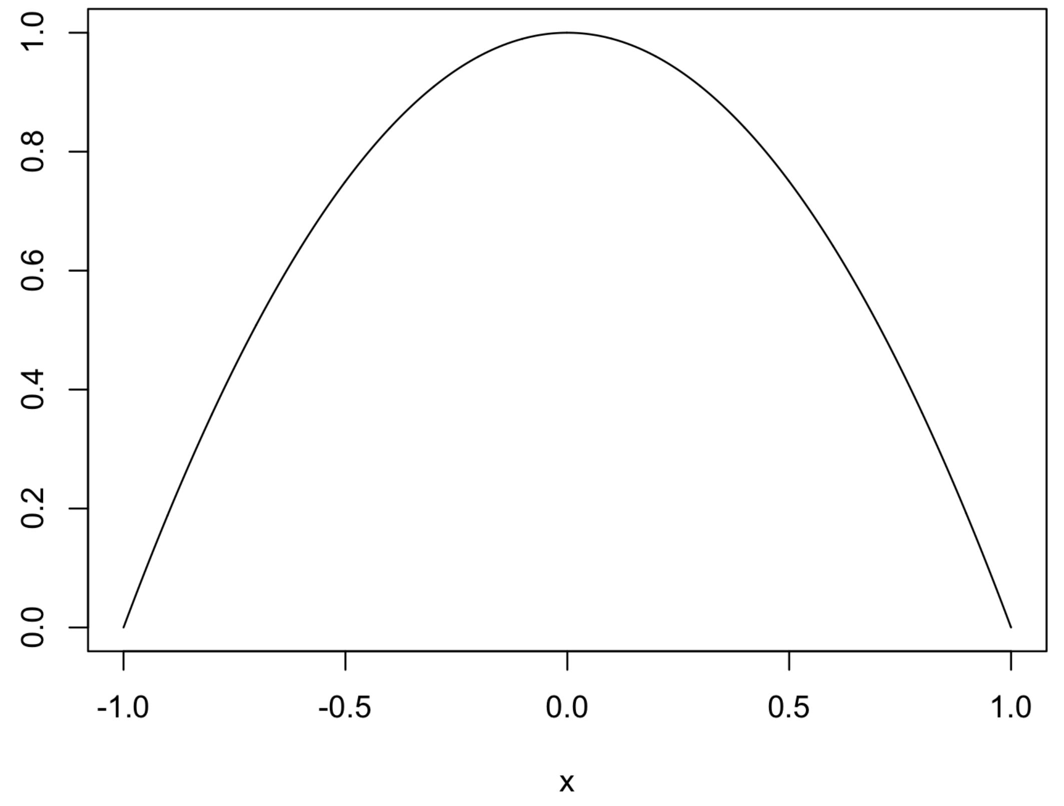
$$F_X(x) = \Pr(X \le x) = \sum_{u \in S_X : u \le x} \Pr(X = u)$$

- If X is continuous, then the CDF is defined as

$$F_X(x) = \Pr(X \le x) = \int_{-\infty}^{x} f_X(u)\, du$$

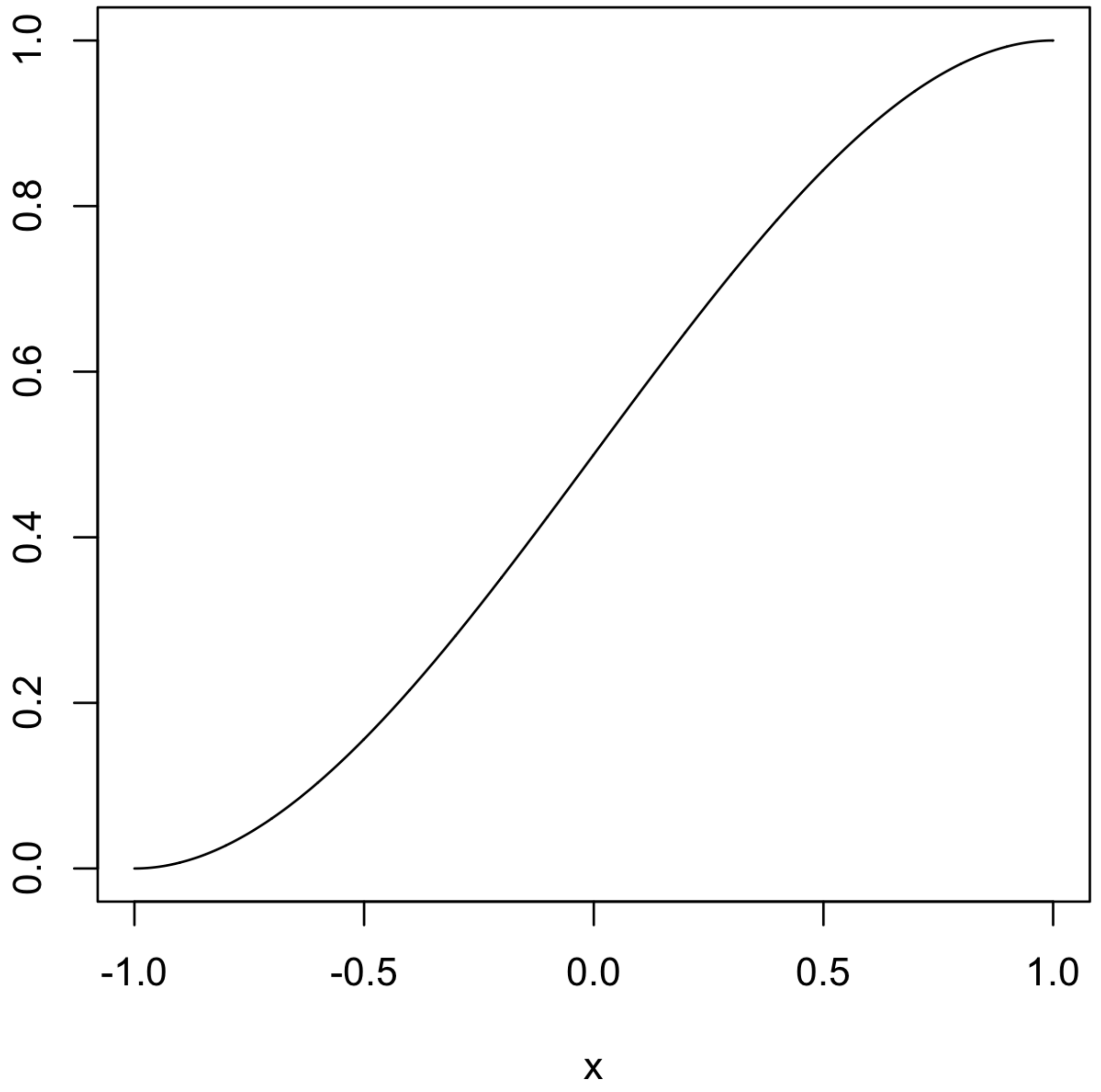
Cumulative Distribution Functions (CDFs)

- Consider the parabolic density

""

$$f(x) = \begin{cases} 0 & x \in (-\infty, -1) \\ \frac{3}{4}(1 - x^2) & x \in [-1,1] \\ 0 & x \in (1,\infty) \end{cases}$$

- Over the range $x \in (-\infty, -1)$, we have $F(x) = 0$



- Over the range $x \in [-1,1]$, we have

$$F(x) = \int_{-1}^{x} \frac{3}{4}(1 - u^2)\, du = -x^3/4 + 3x/4 + 1/2$$

- Over the range $x \in (1,\infty)$, we have $F(x) = 1$

Quantiles and Percentiles

- Suppose that a student with an 85 on an exam scored higher than 72% of their classmates

- Then, $\Pr(X \leq 85) = 0.72$

- We say that $q = 85$ is the $p = 0.72$ quantile of this distribution (also called the 72nd percentile)

Quantiles and Percentiles

- More generally: For a random variable X, q is the p-quantile of X if

$$\Pr(X < q) \leq p \text{ and } \Pr(X > q) \leq 1 - p$$

- The quantile function of X is then defined as

$$Q(p) = \min\{x \in S_X : \Pr(X \leq x) \geq p\}$$

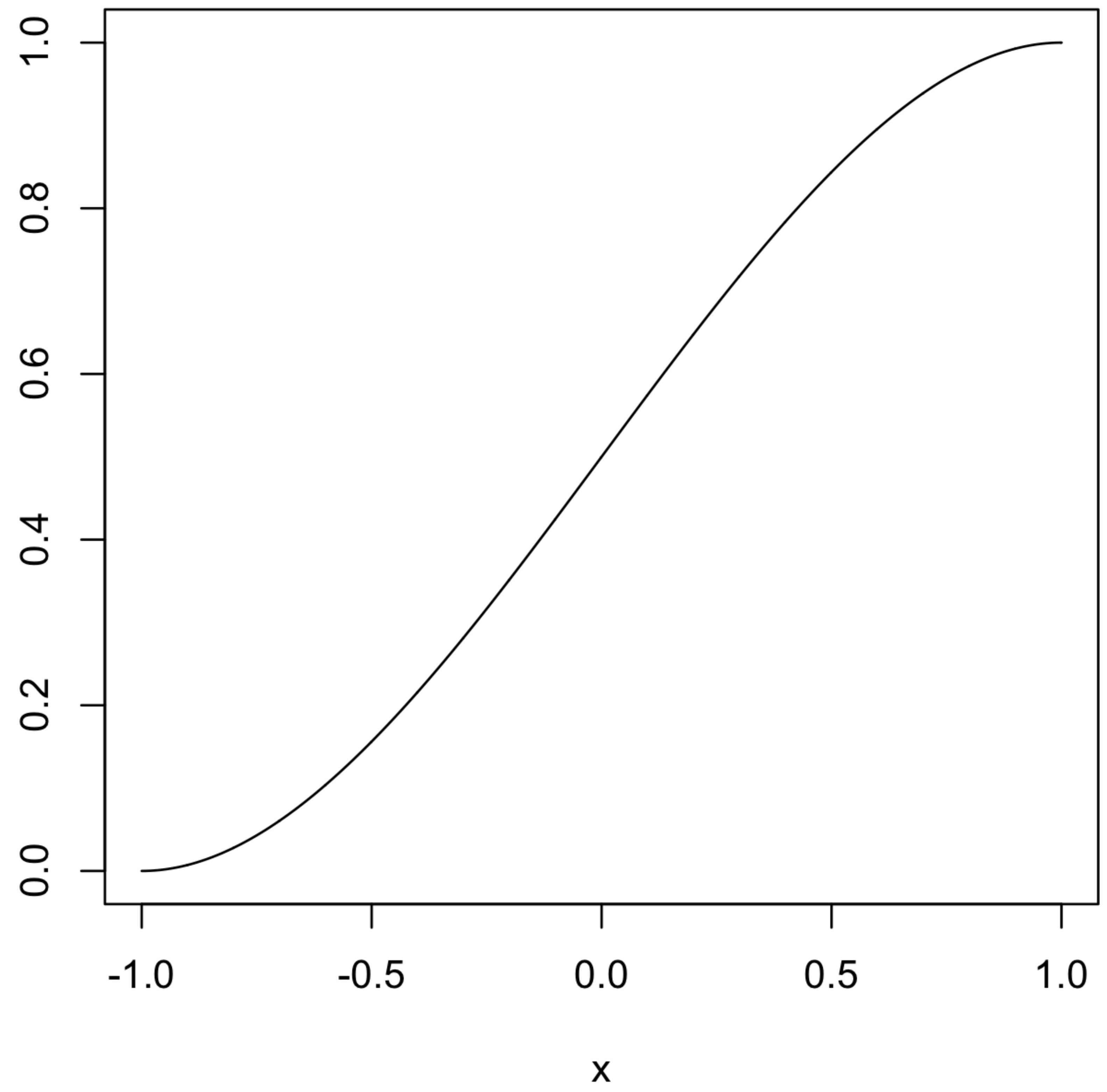
- If the CDF $F_X(x)$ is continuous and strictly increasing on S_X, then

$$Q(p) = F_x^{-1}(p)$$

- Although $Q(p)$ is uniquely defined, the p-quantile may not be unique

Quantiles and Percentiles: Example

- Consider the parabolic density, $f(x) = \dfrac{3}{4}(1 - x^2)$
- What is the 0.25-quantile?



Quantiles and Percentiles: Example

- Consider the parabolic density, $f(x) = \frac{3}{4}(1 - x^2)$
- What is the 0.25-quantile?

$$Q(p) = \min\{x \in S_X : \Pr(X \leq x) \geq p\}$$

Want to find x such that $\Pr(X \leq x) = 0.25$

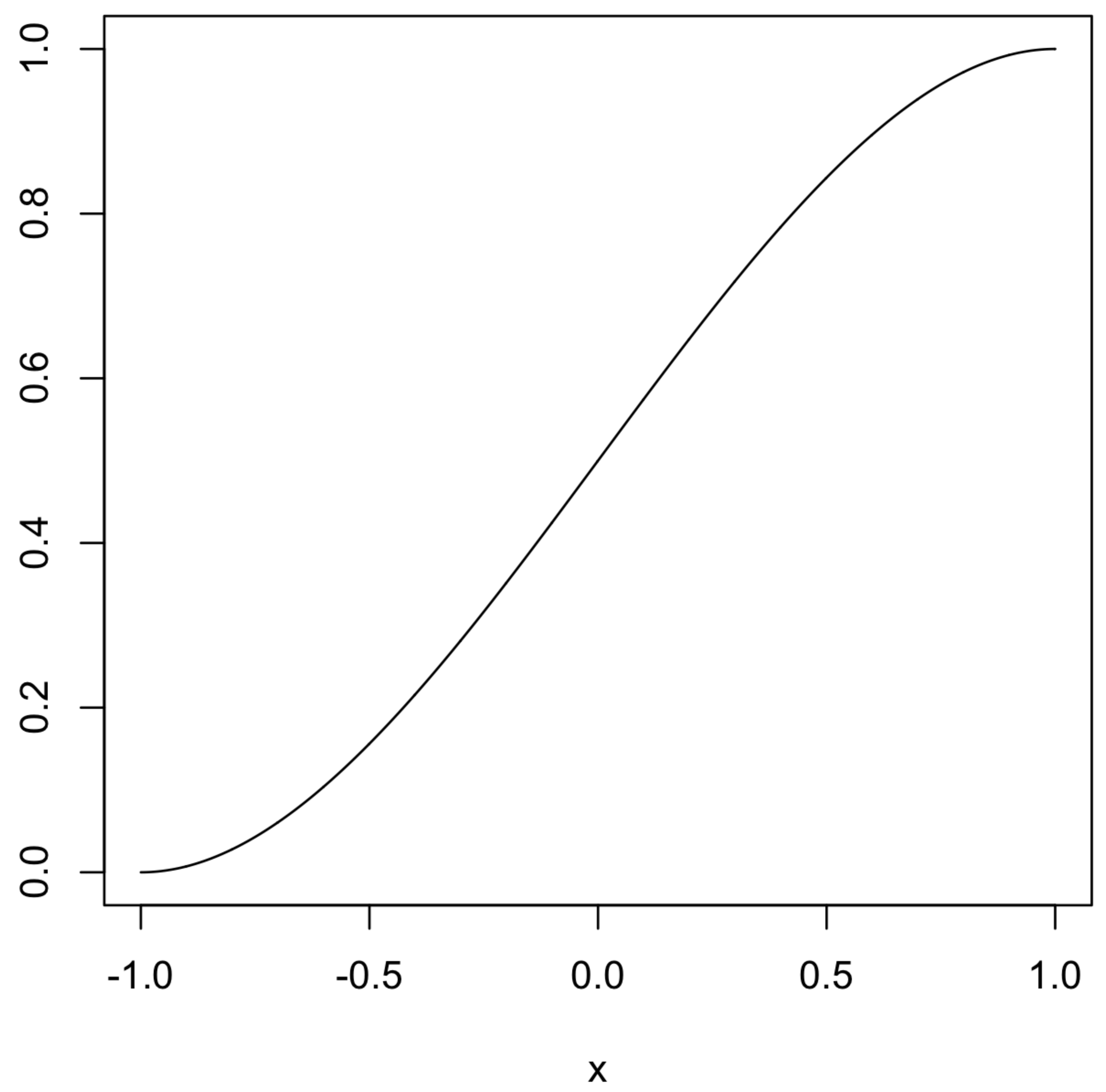
Solve for x:

$$-x^3/4 + 3x/4 + 1/2 = 1/4$$

$$x^3 - 3x - 1 = 0$$

roots: $x = \{-1.53, -0.35, 1.879\}$

Because we know that $x \in [-1,1]$, we have that the 0.25-quantile occurs at $x = -0.35$



Summarizing Probability Distributions

- Many random variables can take a large number of values, so an explicit probability distribution may be quite complicated
- We can describe a probability distribution with measures of central tendency and dispersion
- *Population mean*: Average value that a random variable takes
- *Population variance*: Dispersion of the values relative to the population mean
- *Population standard deviation*: The square root of the population variance

Expected Value

- **Expected value** of X, denoted $E(X)$, represents a theoretical average of an infinitely large sample
    - $E(X)$ is what we "expect" X to equal; the population mean of X
- We use the notation $\mu = \mu_X = E(X)$

Expected Value

- If X is a discrete random variable:

$$\mu_X = E(X) = \sum_{x \in S_X} x \cdot \Pr(X = x)$$

- If X is a continuous random variable:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

- If c is a constant, then

$$E(c) = c$$

Linearity of Expectation

- For any random variables X and Y:

$$E(X+Y) = E(X) + E(Y)$$

- This holds even if X and Y are *not* independent

- In general, for random variables $X_1, \ldots, X_n$:

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

Mean of a Random Variable: Example

- Q1: What is the expected number of heads when flipping a fair coin (1/2 H, 1/2 T)?

- Q2: What is the expected number of heads when flipping an unfair coin (2/3 H, 1/3 T)?

- Q3: What is the expected number of heads when flipping three fair coins (1/2 H, 1/2 T) and two unfair coins (2/3 H, 1/3 T)?

Mean of a Random Variable: Example

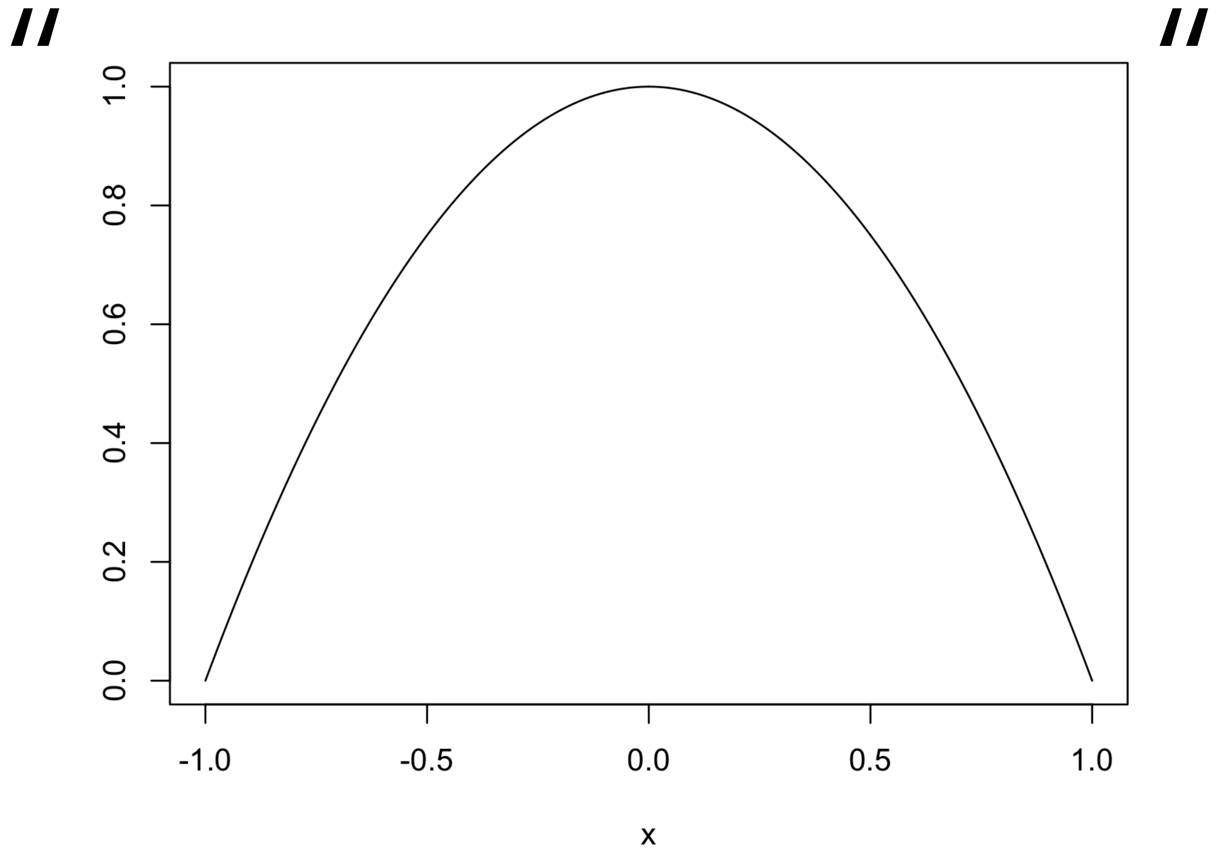
- Q1: What is the expected number of heads when flipping a fair coin (1/2 H, 1/2 T)?
  Let X be the random variable representing the number of heads
  $E(X) = (1/2) \cdot 1 + (1/2) \cdot 0 = 1/2$

- Q2: What is the expected number of heads when flipping an unfair coin (2/3 H, 1/3 T)?
  $E(X) = (2/3) \cdot 1 + (1/2) \cdot 0 = 2/3$

- Q3: What is the expected number of heads when flipping three fair coins (1/2 H, 1/2 T) and two unfair coins (2/3 H, 1/3 T)?
  $E(X) = 3 \cdot (1/2) + 2 \cdot (2/3) = 3/2 + 4/3 = 17/6$ by linearity of expectation

Mean of a Random Variable: Example

- Consider the parabolic density, $f(x) = \dfrac{3}{4}(1 - x^2)$



- What is $E(X)$?

- Intuition: By symmetry, $E(X) = 0$

Mean of a Random Variable: Example

- Consider the parabolic density, $f(x) = \frac{3}{4}(1 - x^2)$

- What is $E(X)$?

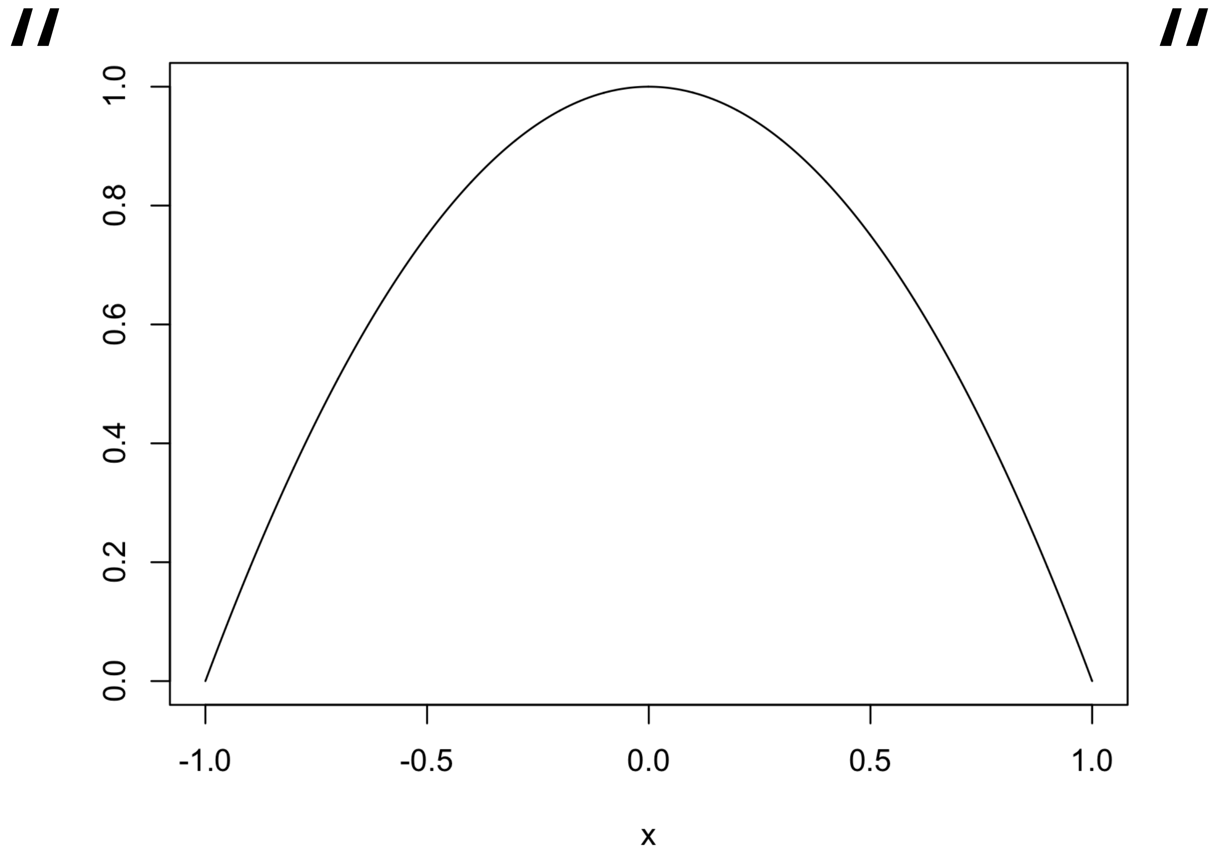
$$E(X) = \int_{-1}^{1} (x)\frac{3}{4}(1 - x^2)dx = \int_{-1}^{1} \frac{3x}{4} - \frac{3x^3}{4}dx$$

$$= \frac{3x^2}{8} - \frac{3x^4}{16}\Big|_{-1}^{1}$$

$$= \frac{3}{8} - \frac{3}{16} - \left(\frac{3}{8} - \frac{3}{16}\right)$$

$$= 0$$

Intuition: By symmetry, $E(X) = 0$



Variance

- The variance of X, denoted $var(X)$, measures the tendency of X to deviate from $E(X)$ and is defined as follows

$$
\begin{aligned}
var(X) &= E\left(\left(X - E(X)\right)^2\right) \\
&= E(X^2) - E(X)^2
\end{aligned}
$$

- We use the notation $\sigma^2 = \sigma_X^2 = var(X)$

- The standard deviation is the square root of the variance: $\sigma = \sigma_X = \sqrt{var(X)}$

Variance

- Recall: $var(X) = E\left(\left(X - E(X)\right)^2\right)$
- Let X be a discrete random variable with mean μ_X:

$$\sigma_X^2 = \sum_{S_X} (x - \mu_X)^2 \Pr(X = x)$$

- Let X be a continuous random variable with mean μ_X

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx$$

Variance: Example

- Setup: Flip two fair coins; let X be the number of heads
- Q: What is $var(X)$?
- Q: What is the standard deviation of X?

Variance: Example

- Setup: Flip two fair coins; let X be the number of heads
- Q: What is $var(X)$?

  We have that $\sigma_X^2 = \sum_{S_X} (x - \mu_X)^2 \Pr(X = x)$

  We also know that $\mu_X = 1$, $P(X = 0) = P(X = 2) = 1/4$, and $P(X = 1) = 1/2$

  Therefore, $\sigma_X^2 = (0 - 1)^2 \cdot (1/4) + (1 - 1)^2 \cdot (1/2) + (2 - 1)^2 \cdot (1/4) = 1/2$

- Q: What is the standard deviation of X?

  $$\sigma_X = \sqrt{\sigma_X^2} = \frac{1}{\sqrt{2}}$$

Functions of Random Variables

- Take random variable X and function $g(\,.\,)$
- We can obtain a new random variable: $Y = g(X)$
- This is what is called a *transformation of variables*
- In general, to get the distribution of Y, we have that for any event $E \subseteq S_Y$, we have $p_Y(E) = p_X(g^{-1}(E))$

Linear Transformations: Mean and Variance

- Let g be a linear function of the form $g(X) = aX + b$
- Let X be a random variable with mean μ_X and variance σ_X^2
- Define a new random variable $Y = g(X) = aX + b$
- Finding the mean of Y:

$$\mu_Y = E(Y) = E(aX + b) = aE(X) + b = a\mu_X + b$$

- Finding the variance of Y:

$$\begin{aligned}
\sigma_Y^2 &= var(Y) = E((aX + b - E(aX + b))^2) \\
&= E((aX + b - aE(X) - b)^2) = E((aX - aE(X))^2) \\
&= E(a^2(X - E(X))^2) = a^2E((X - E(X))^2) \\
&= a^2 \cdot var(X) = a^2 \cdot \sigma_X^2
\end{aligned}$$

General Transformations: Mean

- If we have $Y = g(X)$ for general $g(X)$, then we have:

$$\mu_Y = E(Y) = E(g(X))$$

- We do **not** necessarily have that:

$$E(g(X)) = g(E(X))$$

- Example: Consider X = the outcome of rolling a fair six-sided die, and let $g(X) = X^2$

$$E(g(X)) = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6} \approx 15.17$$

$$g(E(X)) = \left(\frac{1 + 2 + 3 + 4 + 5 + 6}{6}\right)^2 = (3.5)^2 = 12.25$$

Independence

- Two random variables X_1 and X_2 are independent if the following holds, for any two events E_1 and E_2:

$$P(X_1 \in E_1 \cap X_2 \in E_2) = P(X_1 \in E_1) \cdot P(X_2 \in E_2)$$

- Notation:

$$X_1 \perp X_2 \text{ means } X_1 \text{ and } X_2 \text{ are independent}$$

- If a collection of random variables $X_1, X_2, \ldots, X_n$ are all independent and have the same distribution, we say that they are i.i.d. (independent and identically distributed)
  - Example: Roll two dice, or flip three fair coins

Covariance

- If two variables are not independent, we measure their dependency through their **covariance**
- Let X and Y be two random variables with means μ_X and μ_Y, respectively
- The covariance of X and Y is defined as follows:

$$cov(X, Y) = \sigma_{XY} = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - E(X)E(Y)$$

- Correlation (essentially normalized covariance):

$$corr(X, Y) = \rho = \rho_{XY} = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

Properties of Covariance

- Given random variables X and Y, the following hold:
  - If either X or Y is a constant, then $cov(X, Y) = 0$ and $corr(X, Y)$ is undefined
  - If $X \perp Y$, then $cov(X, Y) = corr(X, Y) = 0$
  - $cov(X, X) = var(X)$
  - $cov(X, Y) = cov(Y, X)$

Linear Combinations

- Suppose you have random variables X and Y with means μ_X, μ_Y and variances σ_X^2, σ_Y^2
- Let $Z = aX + bY$
- The mean of Z is

$$\mu_Z = E(Z) = E(aX + bY) = E(aX) + E(bY) = a\mu_X + b\mu_Y$$

- The variance of Z is

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

- The standard deviation of Z is

$$\sigma_Z = \sqrt{a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}}$$

Theoretical Distributions

- Theoretical probability distributions describe what we expect to happen based on populations on a theoretical level
- We will consider the following theoretical distributions (D = discrete, C = continuous):
  - Bernoulli distribution (D)
  - Binomial distribution (D)
  - Poisson distribution (D)
  - Geometric distribution (D)
  - Uniform distribution (C)
  - Exponential distribution (C)
  - Normal distribution (C)

Bernoulli Distribution

- Let Y be a dichotomous random variable (takes one of two mutually exclusive values)
  - Classic example: Coin flip
- Successes (= 1) occur with probability p and failures (= 0) occur with probability $1 - p$, for constant $p \in [0,1]$
- Notation: $Y \sim Bern(p)$

Bernoulli Distribution: Example

- Let Y be a dichotomous random variable representing a coin flip
  - $Y = 1$: heads
  - $Y = 0$: tails
- If the coin is fair, then $p =$ and $1 - p =$
- If the coin has a 60% chance of landing heads, then $p =$ and $1 - p =$
- What is $E(Y)$ in terms of p?
- What is $var(Y)$ in terms of p?

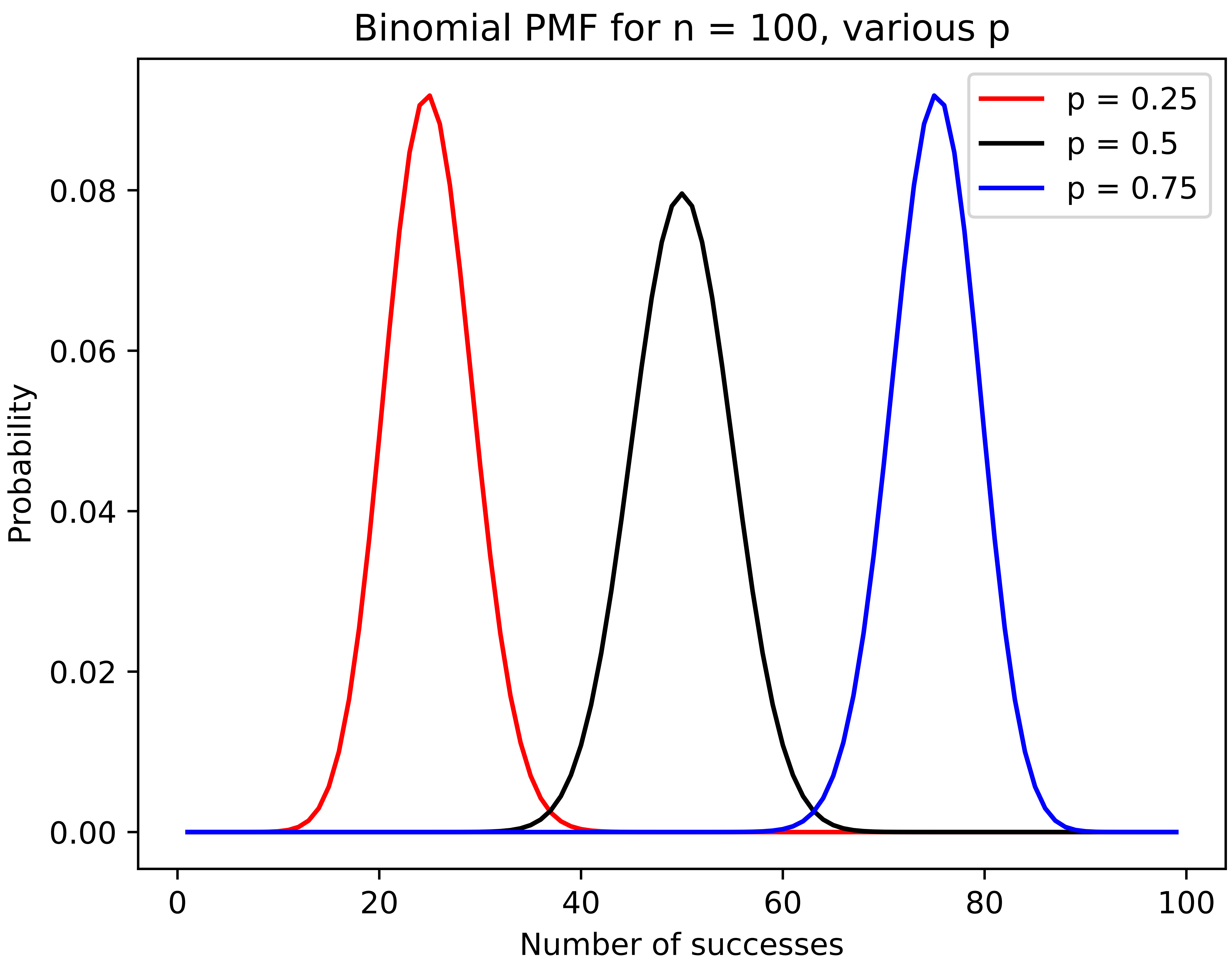
Bernoulli Distribution: Example

- Let Y be a dichotomous random variable representing a coin flip
  - $Y = 1$: heads
  - $Y = 0$: tails
- If the coin is fair, then $p = 1/2$ and $1 - p = 1/2$
- If the coin has a 60% chance of landing heads, then $p = 0.6$ and $1 - p = 0.4$
- What is $E(Y)$ in terms of p? $E(Y) = p \cdot 1 + (1 - p) \cdot 0 = p$
- What is $var(Y)$ in terms of p? $var(Y) = E(Y^2) - E(Y)^2 = p - p^2 = p(1 - p)$

Binomial Distribution

- Suppose we flip n i.i.d. coins instead of just one coin
- Let $X = \sum_{i=1}^{n} X_i$ be the number of heads we observe
- X is binomially distributed
- **Binomial distribution**: If we have a sequence of n Bernoulli random variables, each with a probability of success p, then the total number of successes is a binomial random variable
  - Assumptions: fixed number of trials, independent, constant p
- Notation: $X \sim Bin(n, p)$

Binomial Distribution



Binomial Coefficients

- Let $X = \sum_{i=1}^{n} X_i$ be the number of heads we observe when we flip n i.i.d. coins
- Each coin has probability of heads p, and flips are independent
- Q: What is the probability of getting exactly x out of n successes?
  - Choose which x trials succeed:
  - Probability that these x trials succeed:
  - Probability that the other $n - x$ trials fail:
- In general,

Binomial Coefficients

- Let $X = \sum_{i=1}^{n} X_i$ be the number of heads we observe when we flip n i.i.d. coins
- Each coin has probability of heads p, and flips are independent
- Q: What is the probability of getting exactly x out of n successes?
  - Choose which x trials succeed: $\binom{n}{x}$
  - Probability that these x trials succeed: p^x
  - Probability that the other $n - x$ trials fail: $(1 - p)^{n-x}$
- In general, $\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$

Binomial Probabilities in R

- Calculate probabilities in R:
  - Calculate $\Pr(X = x)$ using `dbinom(x,n,p)`
  - Calculate $\Pr(X \leq x)$ using `pbinom(x,n,p)`
  - Calculate $\Pr(X \geq x)$ using `1-pbinom(x-1,n,p)`

Binomial Distribution: Summary Measures

- Note that a binomial distribution with parameters n and p is the sum of n independent Bernoulli distributions with parameter p

$$E(X) = \mu_X = np$$

$$var(X) = \sigma_X^2 = np(1-p)$$

$$stdev(X) = \sigma_X = \sqrt{np(1-p)}$$

- Q: How does $var(X)$ change with $p \in [0,1]$?

Binomial Distribution: Summary Measures

- Note that a binomial distribution with parameters n and p is the sum of n independent Bernoulli distributions with parameter p

$$E(X) = \mu_X = np$$

$$var(X) = \sigma_X^2 = np(1-p)$$

$$stdev(X) = \sigma_X = \sqrt{np(1-p)}$$

- Q: How does $var(X)$ change with $p \in [0,1]$?
  Highest when $p = 0.5$, lowest (= 0) when $p = 0$ or 1

Binomial Distribution: Summary

- Main take-away points from the binomial distribution:
  - Fixed number of independent Bernoulli trials, n
  - Constant probability of success, p (Bernoulli parameter)
  - Interested in the total number of successes in n trials (not order)
  - Mean: $\mu_X = np$
  - Variance: $\sigma_X^2 = np(1-p)$
- Examples:
  - Number of heads in 15 flips of a fair coin

Poisson Distribution

- **Poisson distribution**: Probability of observing a certain number of discrete events within a known interval
  - Models discrete events that occur infrequently in time or space
- Example:
  - The number of babies born at Strong Memorial Hospital between 10 am and 2 pm today
  - The number of students who enter River Campus today

Poisson Distribution

- Let $X \in [0,\infty)$ be the number of occurrences of some event over a given interval

- Let $\lambda > 0$ be the average number of occurrences of the event over the specified interval

- In this case, we say that $X \sim Pois(\lambda)$

- The probability function is given by $P(X = x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

- If $X \sim Pois(\lambda)$, then $\mu_X = \sigma_X^2 = \lambda$
  - For a Poisson distribution, both the mean and the variance are equal to λ

Poisson Distribution

- Poisson distribution assumptions:
  - The probability of an event occurring is proportional to the length of the interval
  - Within an interval, an infinite number of events is theoretically possible
  - Events occur independently
  - The number of events that occur must be non-negative

Poisson Distribution: Example

- Setup: We want to examine the probability of certain numbers of people developing a rare disease in the next year. On average, 1.95 people develop the disease per year

- Q: What is the probability of no one developing the disease in the next year?

- Q: What is the probability of one person developing the disease in the next year?

Poisson Distribution: Example

- Setup: We want to examine the probability of certain numbers of people developing a rare disease in the next year. On average, 1.95 people develop the disease per year

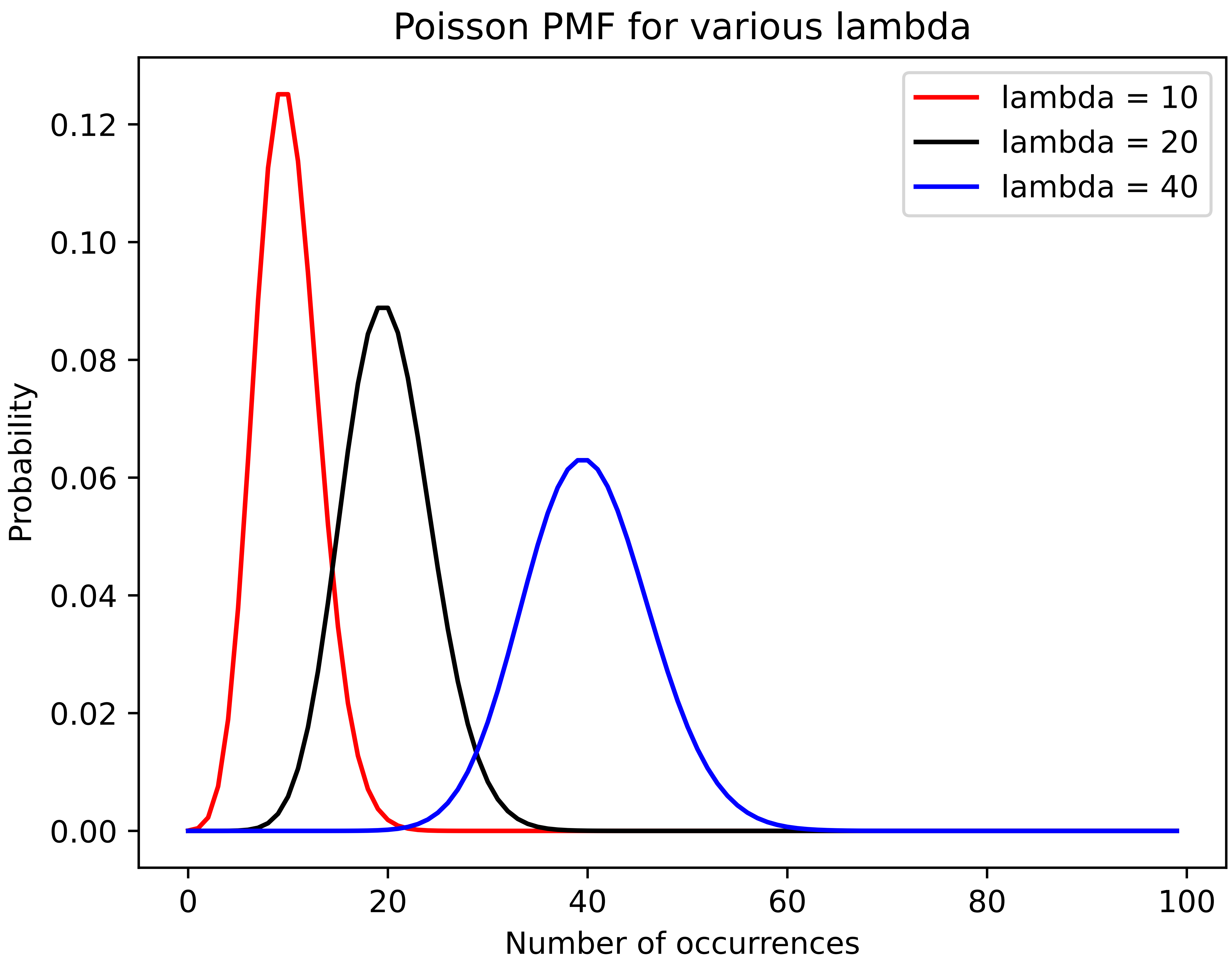
- Q: What is the probability of no one developing the disease in the next year?

$$P(X = 0) = \frac{e^{-1.95}1.95^0}{0!} = e^{-1.95} \approx 0.142$$

- Q: What is the probability of one person developing the disease in the next year?

$$P(X = 1) = \frac{e^{-1.95}1.95^1}{1!} = 1.95 \cdot e^{-1.95} \approx 0.277$$

Poisson Distribution: Visualized



Poisson Probabilities in R

- Calculate probabilities in R:
  - Calculate $\Pr(X = x)$ using `dpois(x,lambda)`
  - Calculate $\Pr(X \leq x)$ using `ppois(x,lambda)`
  - Calculate $\Pr(X \geq x)$ using `1-ppois(x-1,lamda)`

Poisson Distribution: Summary

- Main take-away points from the Poisson distribution:
  - Fixed interval, independent events, interested in number of events in interval
  - Unlimited number of events is theoretically possible
  - Mean: $\mu_X = \lambda$
  - Variance: $\sigma_X^2 = \lambda$
- Examples:
  - Number of calculators the book store sells this year
  - Number of babies born today

Geometric Distribution

- Suppose $Y_1, Y_2, \ldots$ is an *infinite* sequence of independent Bernoulli random variables with parameter p
- Let X be the first index i for which $Y_i = 1$ (location of first success)
- The probability mass function (PMF) is given by

$$P(X = x) = p(1-p)^{x-1}$$

- Notation: $X \sim Geom(p)$
- Mean and variance:

$$E(X) = \mu_X = \frac{1}{p}$$

$$var(X) = \sigma_X^2 = \frac{1-p}{p^2}$$

- CDF: $P(X \leq x) = 1 - (1-p)^x$