Chapter 13: Analysis of Variance

DSCC 462

Computational Introduction to Statistics

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Fall 2022

Some Midterm Comments

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  - Some people did **well**

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  - If you cheated, prepare to hear from me next week

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  - Groups by next Thursday, November 18

Plan for Today

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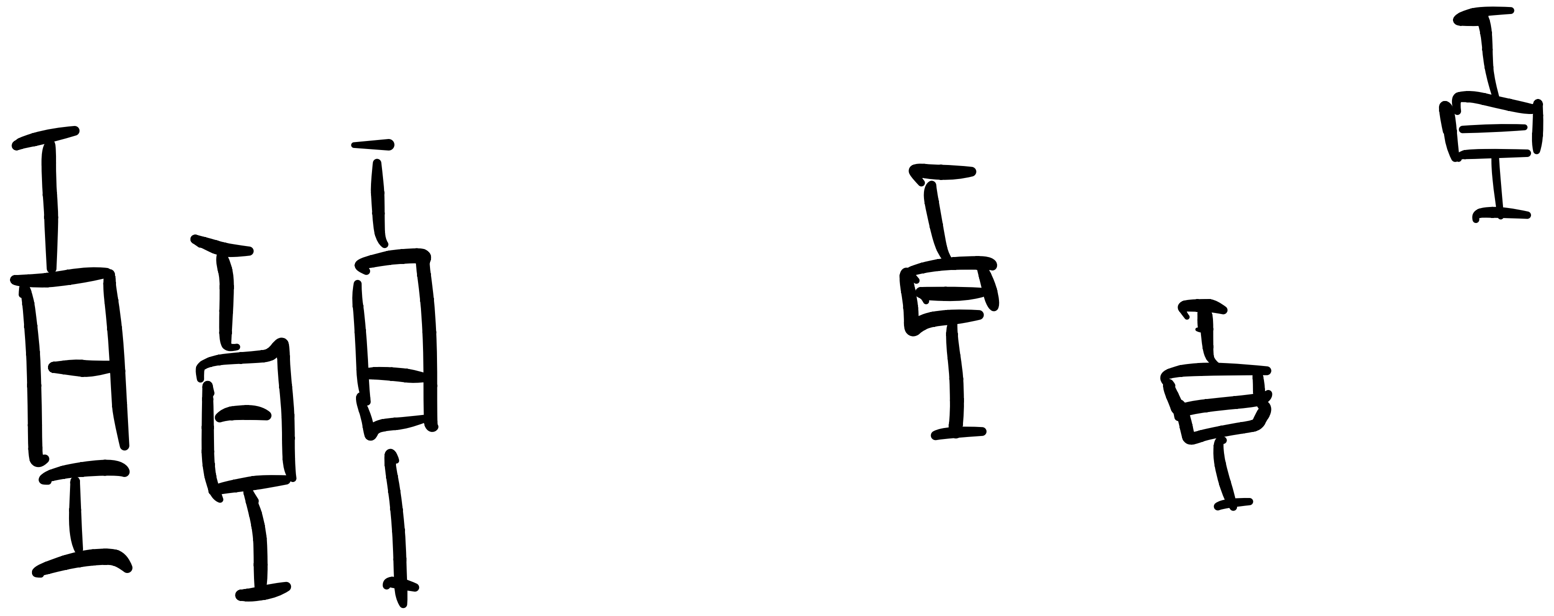
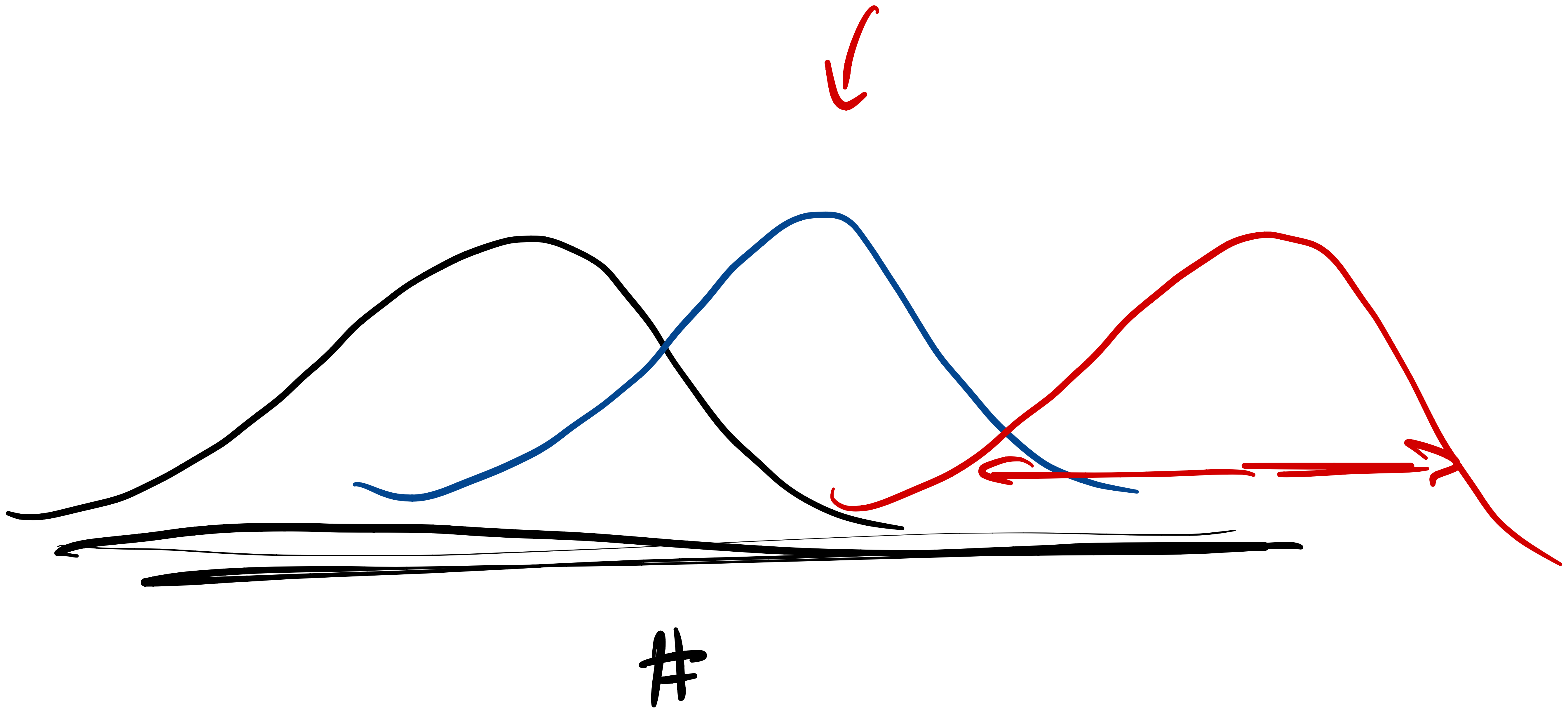
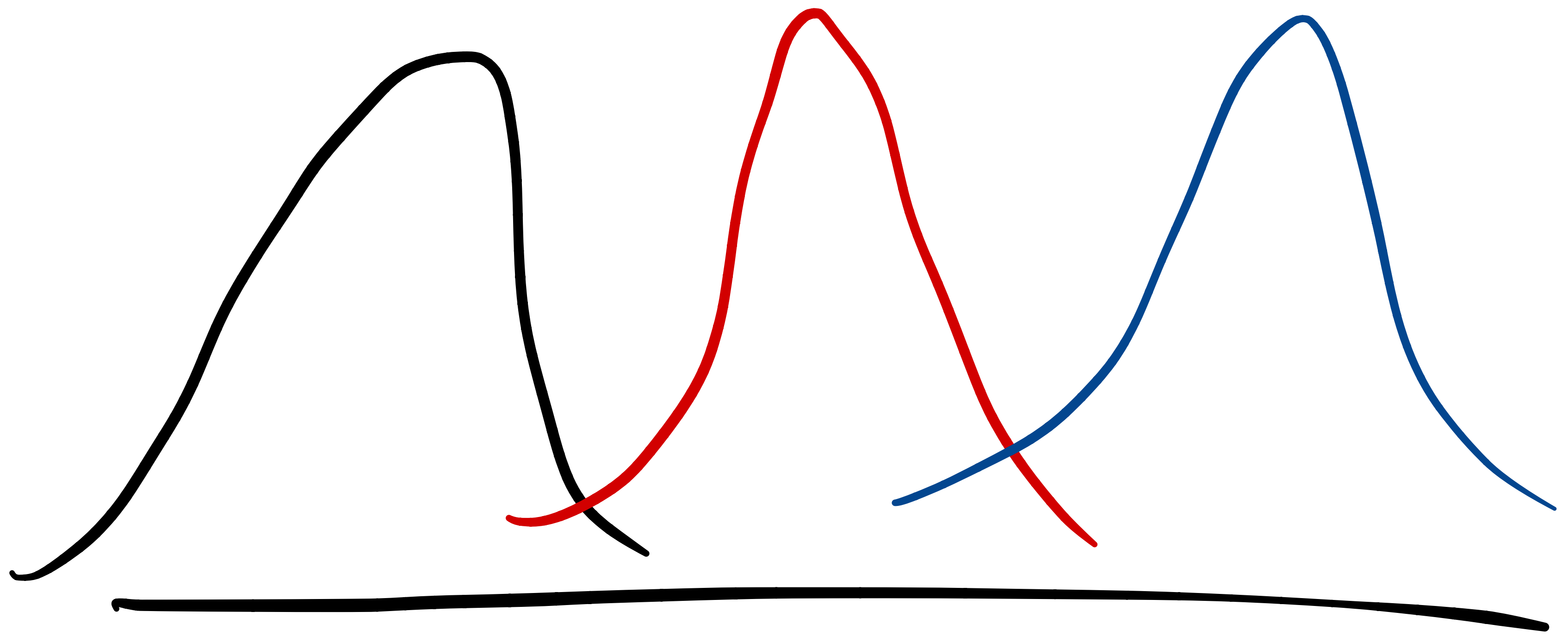
- Motivating illustration:







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  - Use analysis of variance (ANOVA)

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  - H_1 : At least one of the population means differs from one of the others

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- Note: The number of observations in each sample does not need to be the same

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H_0

- Question: Are the average weights for the three age groups the same?

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    - For instance, if we have $k = 10$ groups, we need to do $\binom{10}{2} = 45$ paired t-tests

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	A	R
T_1	.95	.05
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$$\mathrm{Pr}(\overbrace{\text{Reject } H_0 \text{ given } H_0 \text{ is true}}^{R}) = 1 - \mathrm{Pr}(\text{Fail to reject in all three tests})$$

$$Pr(R) = Pr(R_1 \vee R_2 \vee R_3)$$

$$= 1 - (1 - 0.05)^3$$

$$= 1 - 0.857 = \underline{0.143} \quad > \quad 0.05$$

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- In other words, if we know that the null hypothesis is true, then this type I error rate (0.143) is much higher than the desired standard (0.05)
- With one-way ANOVA, we are able to keep the desired significance level α, unlike if we perform multiple t-tests

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    - In our example, age is the distinguishing factor

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  - Independence: Observations are not correlated

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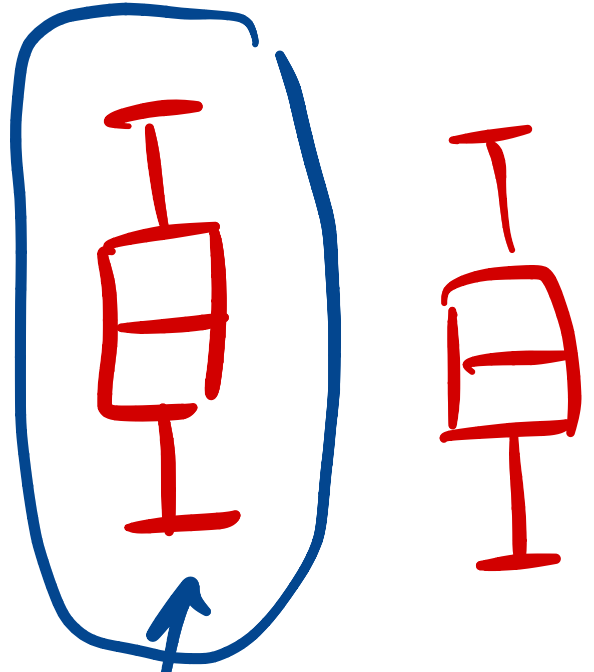

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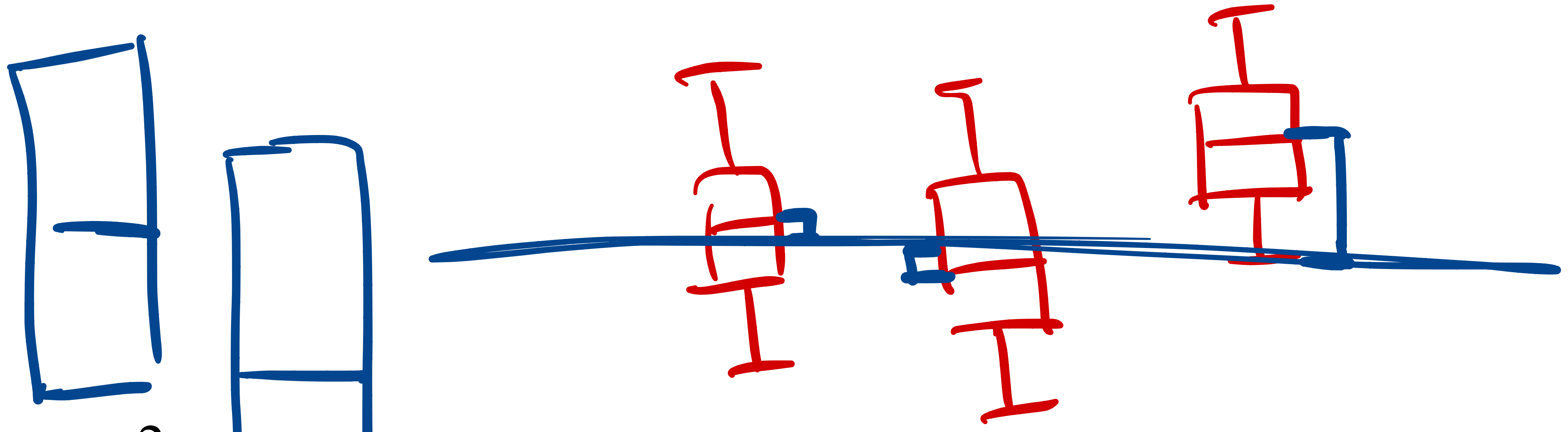

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  - Intuition: tightly clustered and separated from each other

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  - Thinking back to variances, s_b^2/s_w^2 follows an F distribution

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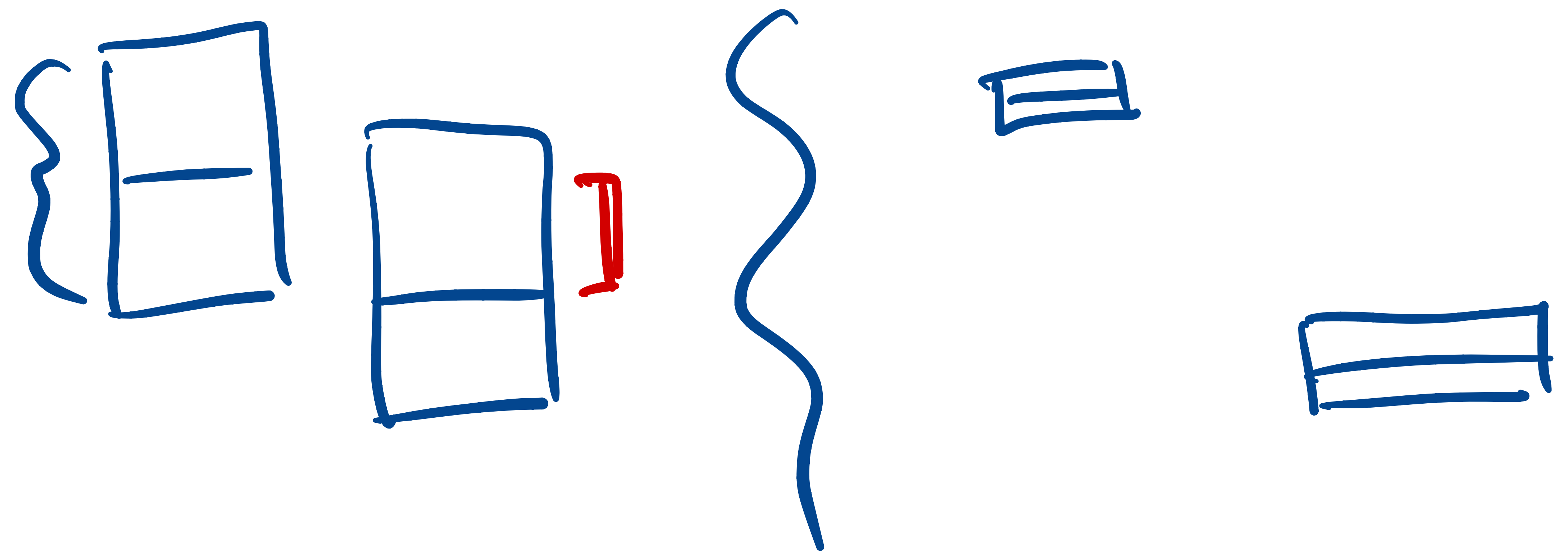

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- If a difference exists between population means, $s_b^2 > s_w^2$ and thus F will be larger than 1
  - If $s_b^2 \leq s_w^2$, then there is no difference between population means

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$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = \overbrace{n_1 + n_2 + \cdots + n_k}^{n} - k$$

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- Under H_0, $F = s_b^2/s_w^2$ has an F distribution with $k-1$ degrees of freedom in the numerator and $n-k$ degrees of freedom in the denominator

- If $k=2$, this F-test reduces to a two-sample t-test

t^2

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  - Reject H_0 if $p \leq \alpha$
- To get p-values in R (recall that we are only interested in the upper tail probability): `1-pf(F,df1,df2)`
- Or, we can compare our test statistic to the critical value that cuts of the upper $\alpha \cdot 100\,\%$ of the F distribution with degrees of freedom df1 and df2

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- We are interested in comparing the mean weight for three age groups: 18-30 years old, 31-50 years old, and 51+ years old
- At the $\alpha = 0.05$ significance level, we want to test $H_0 : \mu_1 = \mu_2 = \mu_3$ against H_1 : at least one of the age groups has an average weight that is different from at least one of the other age groups

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- Recall our sample summary statistics:

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- Using this information, we can calculate s_w^2, s_b^2, and our F-statistic

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- Given $n_1 = 26$, $\bar{x}_1 = 151$, $s_1 = 8.9$, $n_2 = 31$, $\bar{x}_2 = 174$, $s_2 = 11.4$, and $n_3 = 44$, $\bar{x}_3 = 162$, $s_3 = 9.9$

"within"

- $s_w^2 = \dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2 + (n_3-1)s_3^2}{n_1+n_2+n_3-3} = \dfrac{(26-1)\times 8.9^2 + (31-1)\times 11.4^2 + (44-1)\times 9.9^2}{26+31+44-3}$

  $= \boxed{103.}$

- $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1+n_2+n_3} = \underline{162.85}$

- $s_b^2 = \dfrac{n_1(\bar{x}_1-\bar{x})^2 + \cdots + n_3(\bar{x}_3-\bar{x})^2}{k-1 = 3-1 = 2} = \boxed{7521}$

$\dfrac{s_b^2}{s_w^2} \approx 70$

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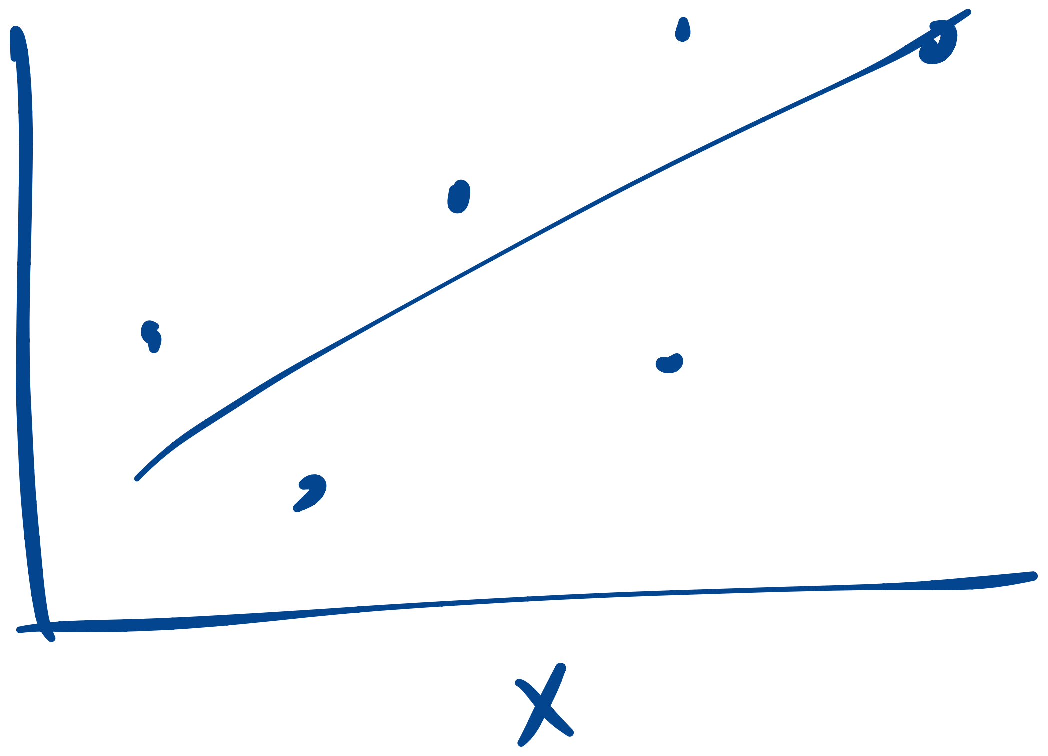
- Therefore, our test statistic is $F = s_b^2/s_w^2 =$ 73
- For an F distribution with $k - 1 =$ 2 and $n - k =$ 98 degrees of freedom, we get a p-value of $p =$ 1- pf(73, 2, 98) = 0
- Conclusion: $p <<< \alpha$

ANOVA Table

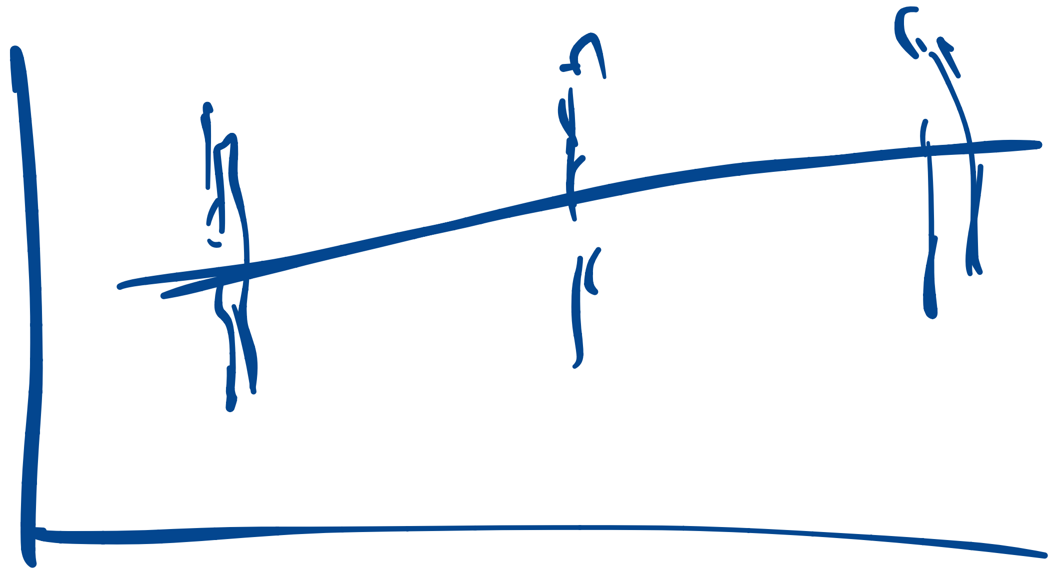
Source of Variation	Sum of Squares (SS)	df	Mean Squares (MS)	F	P-value
Between (treatment)	$SSB = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})^2$	$k-1$	$s_b^2 = \frac{SSB}{k-1}$	$\frac{s_b^2}{s_w^2}$	p^{***}
Within (error)	$SSE = \sum_{i=1}^{k} (n_i - 1)s_i^2$	$n-k$	$s_w^2 = \frac{SSE}{n-k}$		
Total	$\boxed{SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2}$	$n-1$			

$1 - pf(F, k-1, n-k)$

ANOVA Table for Example



Source of Variation	Sum of Squares (SS)	df	Mean Squares (MS)	F	P-value
explain Between (treatment)	15041.754	$k-1$ 2	$s_b^2 = 7500$	≈ 75	0
unexplained Within (error)	10093.51	$n-k$ 98	$s_w^2 = 100$		
Total					



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"post hoc"

- We need to conduct additional tests to find where the differences are
- In this case, we are performing *multiple comparisons*

Multiple Comparisons

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- Make it more difficult to reject each individual comparison so that the overall significance level remains at α
    - We call this the *familywise type I error*, or α_{FWE}

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- $\alpha* = \dfrac{\alpha}{\binom{k}{2}}$ is the significance level for an individual comparison
- This is called the *Bonferroni correction*
- For instance, if we want $\alpha = 0.05$ significance for $k = 5$ populations, then each pairwise test should have significance level $\alpha* = 0.05/10 = \underline{0.005}$

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- Calculate the p-value, p, based on a t distribution with $n - k$ degrees of freedom
- Reject H_0 if $p \leq \alpha^*$ $= \alpha / \binom{k}{2}$

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- Total of $\binom{k}{2} = \binom{3}{2} = 3$ comparisons
- Overall desired significance $\alpha = 0.05$

$s_w^2 = 100$

$\alpha^* = \frac{\alpha}{\binom{k}{2}} = 0.0167$

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- $t_{12} =$

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Multiple Comparisons: Example

- $t_{12} = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_w^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \dfrac{151 - 174}{\sqrt{100\left(\frac{1}{26} + \frac{1}{31}\right)}} = \underline{-8.52}$

- $t_{13} = \underline{-4.38}$

- $t_{23} = \underline{5.04}$

$df = n-k = 98$

Multiple Comparisons: Example

Multiple Comparisons: Example

- From previous slide, we have $t_{12} =$ -8.5, $t_{13} =$ -4.4, and $t_{23} =$ 5

$$\alpha^* = \frac{\alpha}{\binom{4}{2}} = \boxed{0.0167}$$

$$df = 101 - 3 = \underline{98}.$$

$$p_{12} = 2 \times pt(-8.5, df = 98) = 0.000000000000000195$$

$$p_{13}, \quad p_{23} < 1 \times 10^{-4}$$

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- Conclusions from ANOVA and multiple comparisons may contradict each other!
  - Significant ANOVA and no significant pairwise comparisons: Overly conservative pairwise comparisons test
  - Non-significant ANOVA but significant pairwise comparisons: Generally consider pairwise comparisons result valid

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- Some others: Tukey, Newman-Keuls, Scheffee, Dunnett, etc.

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  - `anova(model1)`