Chapter 14: Inference for Correlation

DSCC 462

Computational Introduction to Statistics

Anson Kahng

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Final Project Announcements

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- Final project description is up on Blackboard
  - Setup: Ad dataset, company wants to understand their data…
  - Some open-ended questions, some less so (room for interpretation!)
- Datasets will be released once teams are formed (Friday)

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- Introduce tools that allow us to go past our previous assumptions of independence when comparing random variables
- In particular, learn how to infer whether or not a linear relationship exists between two variables, X and Y

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- We estimate ρ with Pearson's coefficient of correlation, r (Chapter 3)

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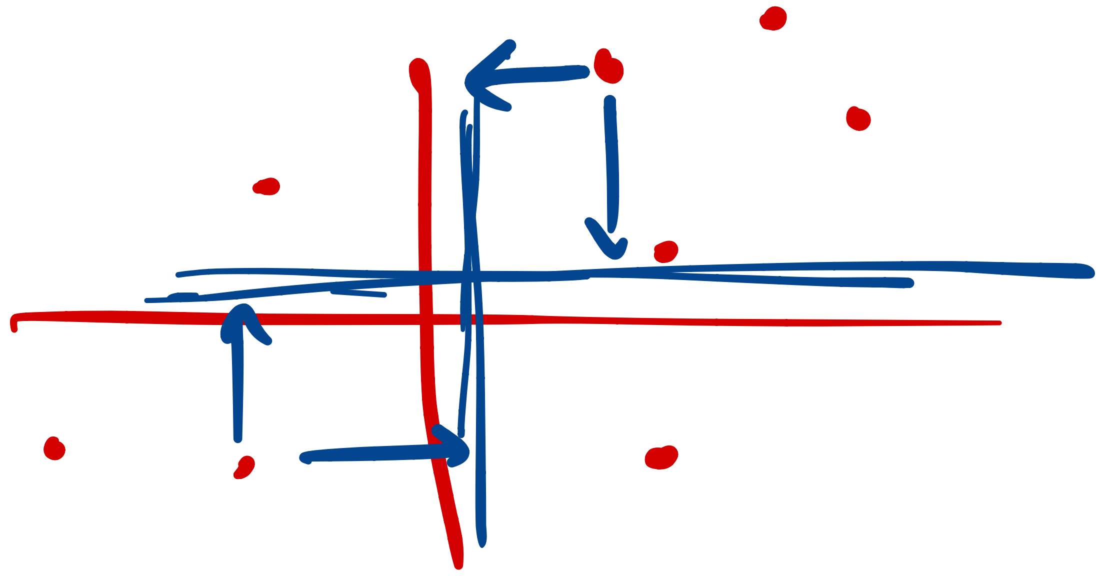

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$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]\left[\sum_{i=1}^{n}(y_i - \bar{y})^2\right]}} = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

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$s_x \cdot (n-1)$

- Note that $-1 \leq r \leq 1$

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- In other words, we're investigating whether a linear relationship exists between X and Y (age and weight)
- Use the sample correlation r for our statistic in hypothesis testing

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$z = \frac{\bar{x} - \mu}{SE}$

$\sigma/\sqrt{n}$

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- If pairs (x_i, y_i) come from normally distributed X and Y, then if we standardize r, we get a statistic that has a t distribution with $n - 2$ degrees of freedom
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- Test statistic: $t = \dfrac{r - \rho}{SE(r)}$, where $SE(r) = \sqrt{\dfrac{1 - r^2}{n - 2}}$
- We thus get $t = \dfrac{r - \rho}{SE(r)} = r\sqrt{\dfrac{n - 2}{1 - r^2}}$

$r \in [-1, 1]$

$$\frac{r - \rho}{SE(r)} = \frac{r}{\sqrt{\frac{1-r^2}{n-2}}} =$$

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- We know that the correlation between weight and age for this sample is $r = 0.865$

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- Test this null hypothesis at the $\alpha = 0.05$ significance level

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- p-value:

Correlation: Inference Example

- $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$, test at $\underline{\alpha = 0.05}$ $df = 7 - 2 = 5$

- The correlation between weight and age for this sample is $r = 0.865$

- Test statistic:

$$t = r \cdot \sqrt{\frac{n-2}{1-r^2}} = 0.865 \cdot \sqrt{\frac{7-2}{1-(0.865)^2}} = 3.856$$

- p-value:

$$p = 2 \cdot Pr(T \geq 3.856) = 2 \cdot (1 - pt(3.856, df = 5)) = 0.012$$

- Conclusion:

Reject H_0, some correlation.

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```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y)

	Pearson's product-moment correlation

data:  x and y
t = 3.856, df = 5, p-value = 0.01193
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.3213733 0.9798243
sample estimates:
      cor
0.8650727
```

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- Correlation can be very sensitive to outliers and can thus give misleading results when outliers are present

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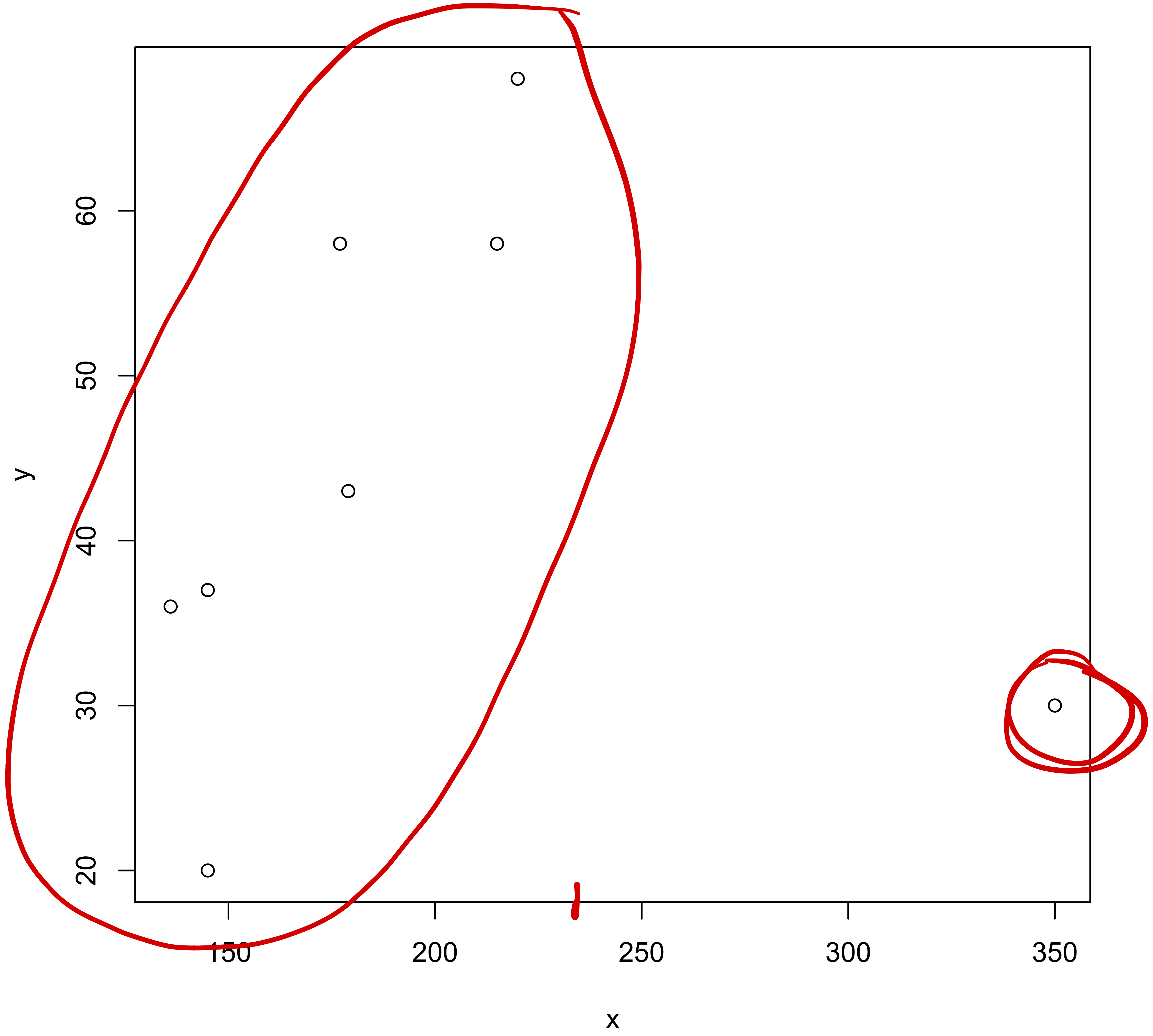

Correlation Limitations

```
> x<-c(220,215,179,145,145,177,136, 350)
> y<-c(68,58,43,37,20,58,36, 30)
> plot(x,y)
> cor(x,y)
[1] 0.06260467
```

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- This is *Spearman's correlation coefficient* r_s

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$$r_s = \frac{\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)(y_{ri} - \bar{y}_r)}{\sqrt{\left[\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)^2\right]\left[\sum_{i=1}^{n} (y_{ri} - \bar{y}_r)^2\right]}}$$

$X \rightarrow X_r$

$Y \rightarrow Y_r$

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- Note that $-1 \leq r_s \leq 1$
- Same interpretations of association between X and Y

Spearman's Rank Correlation Coefficient

Patient	Weight	Rank	Age	Rank
1	220	7	68	7
2	215	6	~~58~~	5.5
3	179	5	~~43~~	4
4	145	2.5	37	3
5	145	2.5	- 20	1
6	- 177	4	58	5.5
7	- 136	1	36	2

Spearman's Rank Correlation Coefficient

- Consider the following age and weight measurements for the 7 subjects

Patient	Weight	Rank	Age	Rank
1	220		68	
2	215		58	
3	179		43	
4	145		37	
5	145		20	
6	177		58	
7	136		36	

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  - $\bar{y}_r =$
  - $r_s =$

Spearman's Rank Correlation Coefficient

1 2 3 4 5 6 7 (x, y)

- Using this information, we can calculate r_s as follows:
  - $\bar{x}_r =$ 4
  - $\bar{y}_r =$ 4
  - $r_s =$ $\dfrac{\sum_{i=1}^{n} (x_{r_i} - 4)(y_{r_i} - 4)}{\sqrt{\left[\sum (x_{r_i} - 4)^2\right]\left[\sum (y_{r_i} - 4)^2\right]}} = \underline{0.873}$
- Recall that Pearson's correlation coefficient was given as 0.865. How does Spearman's rank correlation coefficient compare?

Spearman’s Rank Correlation Coefficient: R

```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor(x,y, method="spearman")
[1] 0.8727273
>
> x1 <- rank(x)
> y1 <- rank(y)
> x1; y1
[1] 7.0 6.0 5.0 2.5 2.5 4.0 1.0
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Spearman's Rank Correlation Coefficient: R

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shortcut

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long:

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- If there is no linear correspondence between the ranks, then $r_s = 0$

Spearman’s Rank Correlation Coefficient: Outliers

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- If the sample size is large enough (generally $n \geq 10$) and if we can assume that pairs of ranks (x_{ri}, y_{ri}) are chosen randomly, then we can test the null hypothesis $H_0 : \rho = 0$ vs. the alternative hypothesis $H_1 : \rho \neq 0$

Pearson $\quad t_s = \dfrac{r_s - \rho}{\sqrt{\dfrac{1 - r_s^2}{n-2}}}$

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- Use a similar test statistic: $t_s = r_s \cdot \sqrt{\dfrac{n-2}{1-r_s^2}}$

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- Compare t_s to a t distribution with $n-2$ degrees of freedom

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- Calculating the p-value:

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- Calculating our t-statistic: $r_s = 0.873$

$$t_s = r_s \sqrt{\frac{n-2}{1-r_s^2}} = 0.873 \sqrt{\frac{7-2}{1-0.873^2}} = 3.997$$

- Calculating the p-value:

$$p = 2 \cdot Pr(T > 3.997) = \dots = 0.0103$$

- Conclusion:

$p < \alpha \to$ Reject H_0

Spearman's Rank Correlation Coefficient: R

"n ≥ 10"

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> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y,method="spearman",exact=FALSE)

	Spearman's rank correlation rho

data:  x and y
S = 7.1273, p-value = 0.01035
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.8727273
```

$r_s \rightarrow$