Chapter 7: Hypothesis Testing

DSCC 462

Computational Introduction to Statistics

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Fall 2022

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- We made a confidence interval to determine a reasonable range of values that will occur with a given confidence level
- Can we now determine how probable it is to see the results in our sample given a hypothesized value of the population parameter?

Hypothesis Testing: Scenario

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- If we accept that our measuring error is normally distributed with mean 0 and standard deviation 0.4, is the observed coffee consistent with Starbucks' claims?
- What if we had measured 15.0 oz of coffee? Would we be more suspicious of Starbucks' claims?

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- The null hypothesis is that of "no change" (or status quo)

- We believe the null hypothesis to be true unless *overwhelming evidence* exists to the contrary ("innocent until proven guilty")

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- We need overwhelming evidence to conclude that H_1 is true

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- In our Starbucks example, we are interested in determining if a grande coffee is in fact 16 oz on average
- Let μ be the average amount of coffee in ounces
- $H_0 : \mu = 16$ oz – the grande coffee is 16 oz on average
- $H_1 : \mu \neq 16$ oz – the grande coffee is not 16 oz on average

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- Two-sided (true mean is not equal to the hypothesized mean):
  - $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$

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- Calculate the sample statistic and compare to the population parameter
    - Compare $\bar{x}$ to the postulated μ_0
- Is $\bar{x}$ different enough from μ_0 to conclude that H_1 is true?
    - Calculate *test statistic*

z

t

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$\Pr(\bar{x} \mid \boxed{\mu_0})$

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- Recall that if X has mean μ_0 and known variance σ^2, then by the CLT, $\overline{X} \sim N\left(\mu_0, \sigma/\sqrt{n}\right)$ for $n \geq 30$

- For a *z-test*, our test statistic is $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Significance

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$Pr\left(\bar{x} \mid H_0: \mu = \mu_0\right) <<< 1$

- If there is evidence that the sample could not have come from a population with the hypothesized parameter, we reject the null hypothesis
  - Given that H_0 is true, the probability of obtaining a sample statistic as extreme or more extreme than the observed statistic is sufficiently small
  - In this case, our data is more supportive of H_1
  - Conclude that the population parameter could not be μ_0

$Pr(\text{seeing } \bar{x} \mid H_0)$

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- Such a test is *statistically significant*
- Intuition: "If the null hypothesis were true, our observations are extremely unlikely. Therefore, this is evidence that the $\underbrace{\text{null hypothesis is not true}}_{H_1}$"

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- Intuition: "Not enough evidence to disprove the null hypothesis"

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  - Typically, let $\alpha = 0.05$: Reject H_0 when the chance that the sample could have come from a population with mean μ_0 is less than or equal to 5%

$\Pr(\bar{x} \mid \mu = \mu_0) < \alpha \rightarrow$ Rejection

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- If $p > \alpha$, we fail to reject H_0

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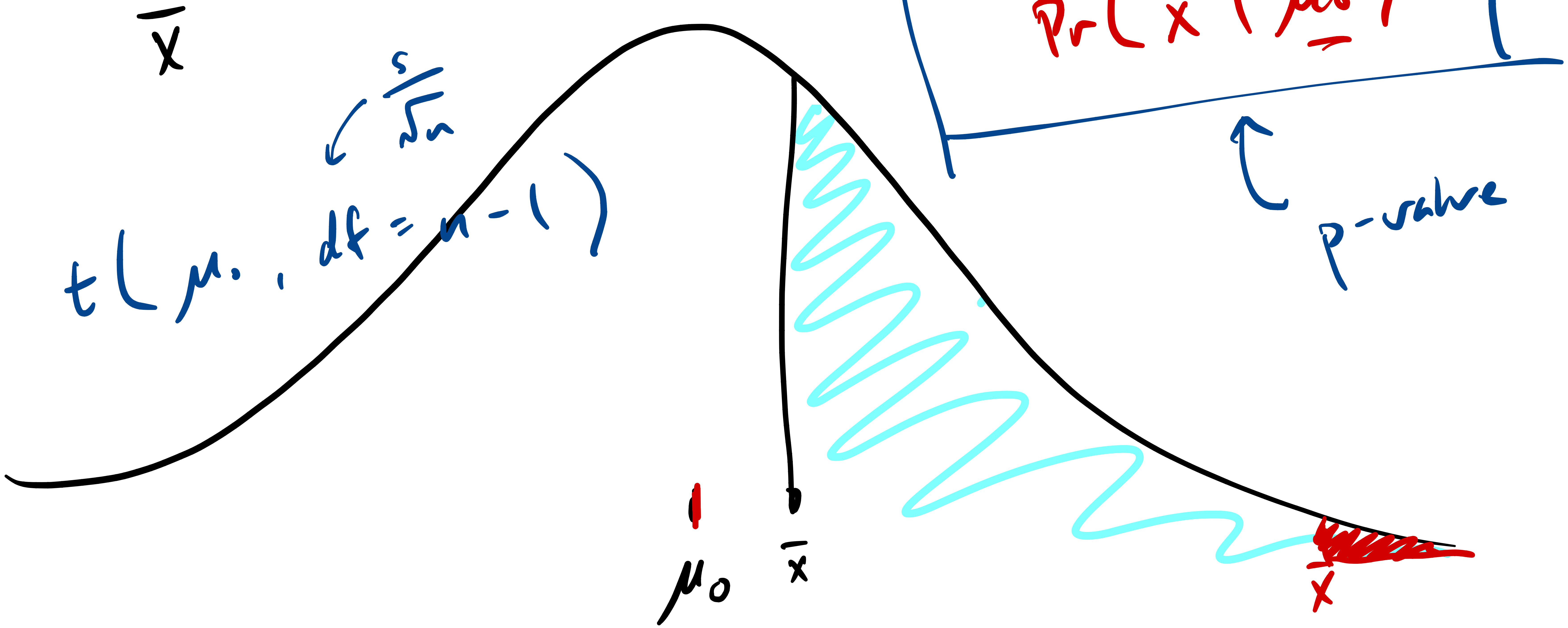
- To be less conservative, choose $\alpha = 0.1$
- Note: We must specify α **before** the test is carried out
  - Otherwise, we may do science in reverse (fit hypotheses to results)

Illustration

σ: $\boxed{\bar{x} \sim N\left(\mu_0, \frac{\sigma}{\sqrt{n}}\right)}$

$\boxed{Pr(\bar{x} \mid \mu_0)}$

p-value



s: $\bar{x} \sim t\left(\mu_0, \frac{s}{\sqrt{n}}, df = n-1\right)$

$H_0: \underline{\mu = \mu_0}$

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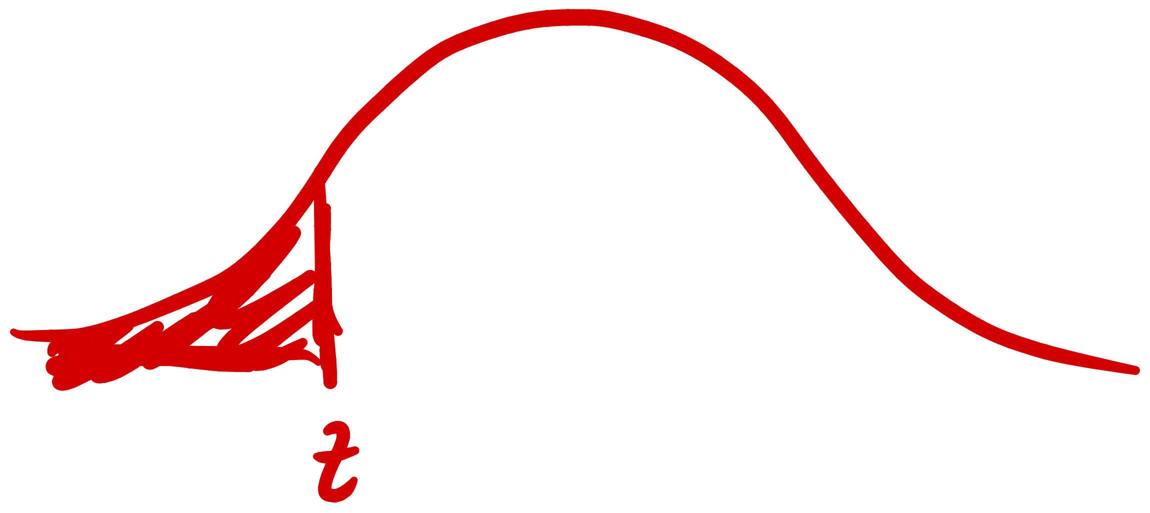

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- Two-sided hypothesis ($H_1 : \mu \neq \mu_0$):
  - If $z \leq 0$: `2*pnorm(z)`
  - If $z > 0$: `2*(1-pnorm(z))`

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$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

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- We determine the probability of seeing a test statistic t as extreme or more extreme as the one observed via a *t-test*

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z : σ
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- Two-sided hypothesis ($H_1 : \mu \neq \mu_0$):
  - If $z \leq 0$: `2*pt(t,df)`
  - If $z > 0$: `2*(1-pt(t,df))`

z test
t test $\rightarrow$ p-value vs. α

5%
10%

Conducting a Hypothesis Test

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$\bar{X}$ CLT

N

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N t

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7. Calculate the p-value or critical value

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6. Determine the proper test to use, and calculate the test statistic
7. Calculate the p-value or critical value
8. Make “reject/fail to reject” decision
9. State your conclusion in the context of the problem

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- Recall that we know that $\sigma = 0.4$ oz

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$\mu = 16$

$\mu \neq 16$

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- Let's return to our Starbucks example where we want to test whether grande coffees actually come with less than 16 oz of coffee

$n = 40$

- We buy 40 grande coffees from Starbucks and find that the average amount of coffee is 15.8 oz

- Recall that we know that $\sigma = 0.4$ oz

- Is there evidence that grande coffees are under-filled at the $\alpha = 0.05$ significance level?

Example: One-sided z-test

$p = Pr(\bar{x} \mid \mu_0)$

1. Check the conditions: $n \geq 30 \rightarrow$ CLT ✓
2. Parameter of interest: $\mu =$ coffee in grande
3. Significance level: $\alpha = 0.05$
4. Null hypothesis: $H_0: \mu \geq 16$ $(\mu_0 = 16)$
5. Alternative hypothesis: $H_1: \mu < 16$
6. Which test and test statistic: z b/c σ
7. p-value: $p = 2x\, Pr(Z < -3.26) = 0.0008 < 0.05$
8. Accept or reject H_0?
9. Conclusion:

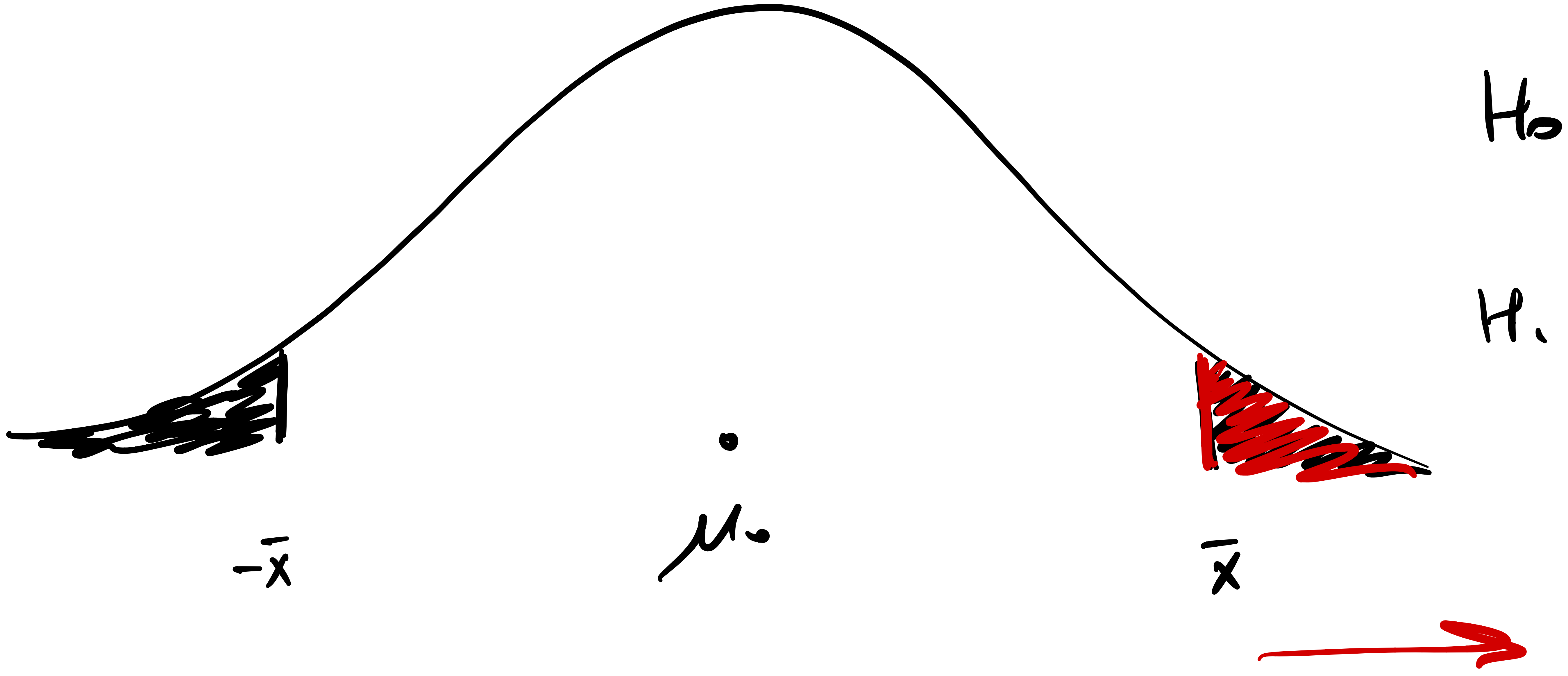
$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{15.8 - 16}{0.4/\sqrt{40}} = -3.26$$

$p = Pr(T < -3.2)$ pt(-3.2, df = 39)

Example: Two-sided z-test

$$\Pr(\text{being as extreme as } \bar{x} \text{ or more} \mid \mu_0)$$



$H_0: \mu = \mu_0$

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- The average weight for men in 1960 was 166.3 lbs
- In 2002, 30 men were sampled and their average weight was 191 lbs
- Assume $\sigma = 50$ lbs is known
- Do the data suggest that the average weight of men is significantly different in 2002 as compared to 1960 at the $\alpha = 0.05$ significance level?

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2. Parameter of interest:
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- The average heart rate for this sample is $\bar{x} = 60.9$ bpm with a sample standard deviation of $s = 34.2$ bpm

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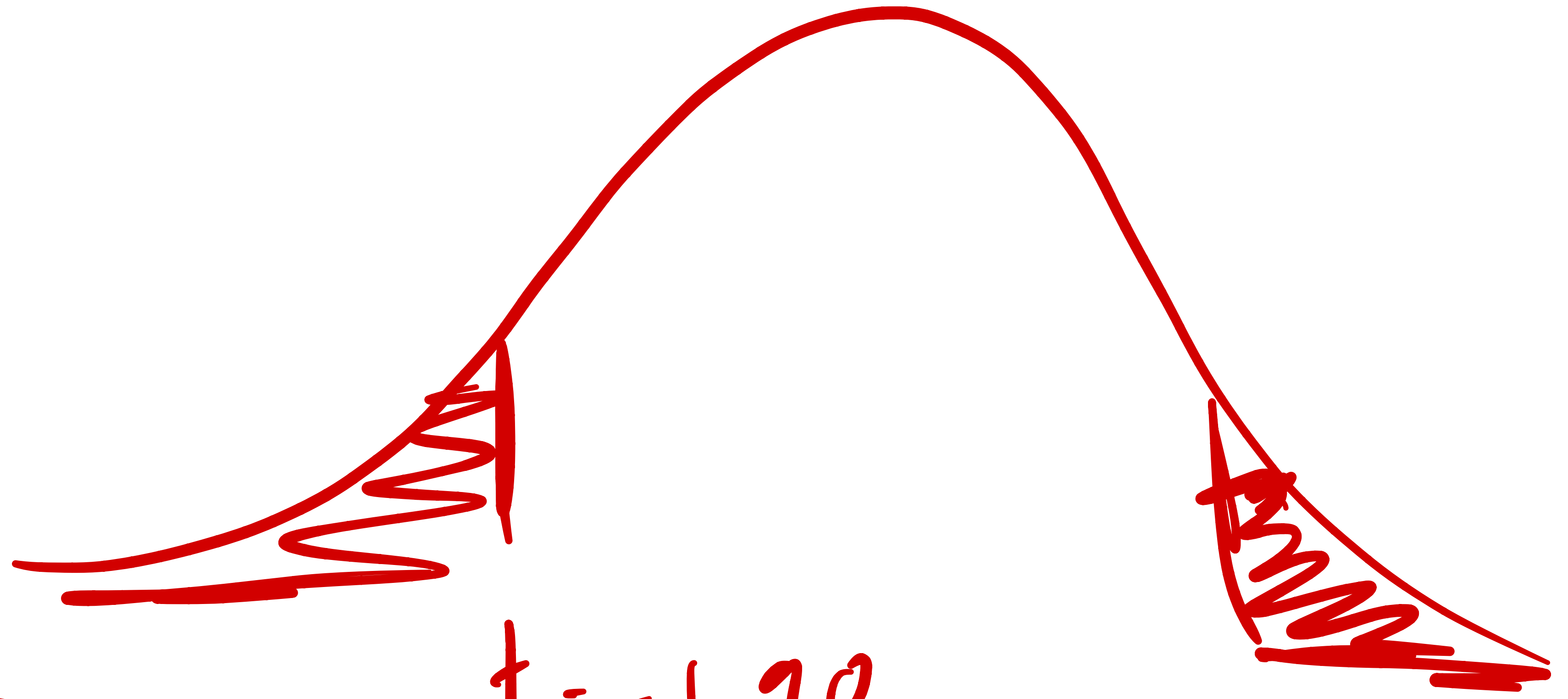
- The underlying distribution of US Olympic athlete heart rates is approximately normal with an unknown mean μ and unknown standard deviation σ
- Does the average heart rate for US Olympic athletes differ from that of the general American population at the $\alpha = 0.05$ significance level?

Example: Two-sided t-test

$\bar{X}$

1. Check the conditions: $n \geq 30 \rightarrow$ CLT   N ✓
2. Parameter of interest: heartrate for athletes
3. Significance level: $\alpha = 0.05$
4. Null hypothesis: $H_0 : \mu = 71.2$ $\boxed{H_0 : \mu \geq 71.2 \quad H_1 : \mu_0 < 71.2}$
5. Alternative hypothesis: $H_1 : \mu \neq 71.2$
6. Which test and test statistic: t : $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{60.9 - 71.2}{34.2/\sqrt{40}} = -1.90$
7. p-value: $p = \cancel{2} \cdot \Pr(T < -1.90) = 0.0658$ $0.0324 < 0.05$
8. ~~Accept~~ or reject H_0? Fail to reject H_0.
9. Conclusion:



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- The University of Rochester believes its students score better, on average, than the rest of the nation
- A sample of 52 U of R medical students is taken
- The average scores for these students is 516, with a sample standard deviation of 18
- Do U of R medical students score higher, on average, than the rest of the nation? Evaluate at the $\alpha = 0.05$ significance level

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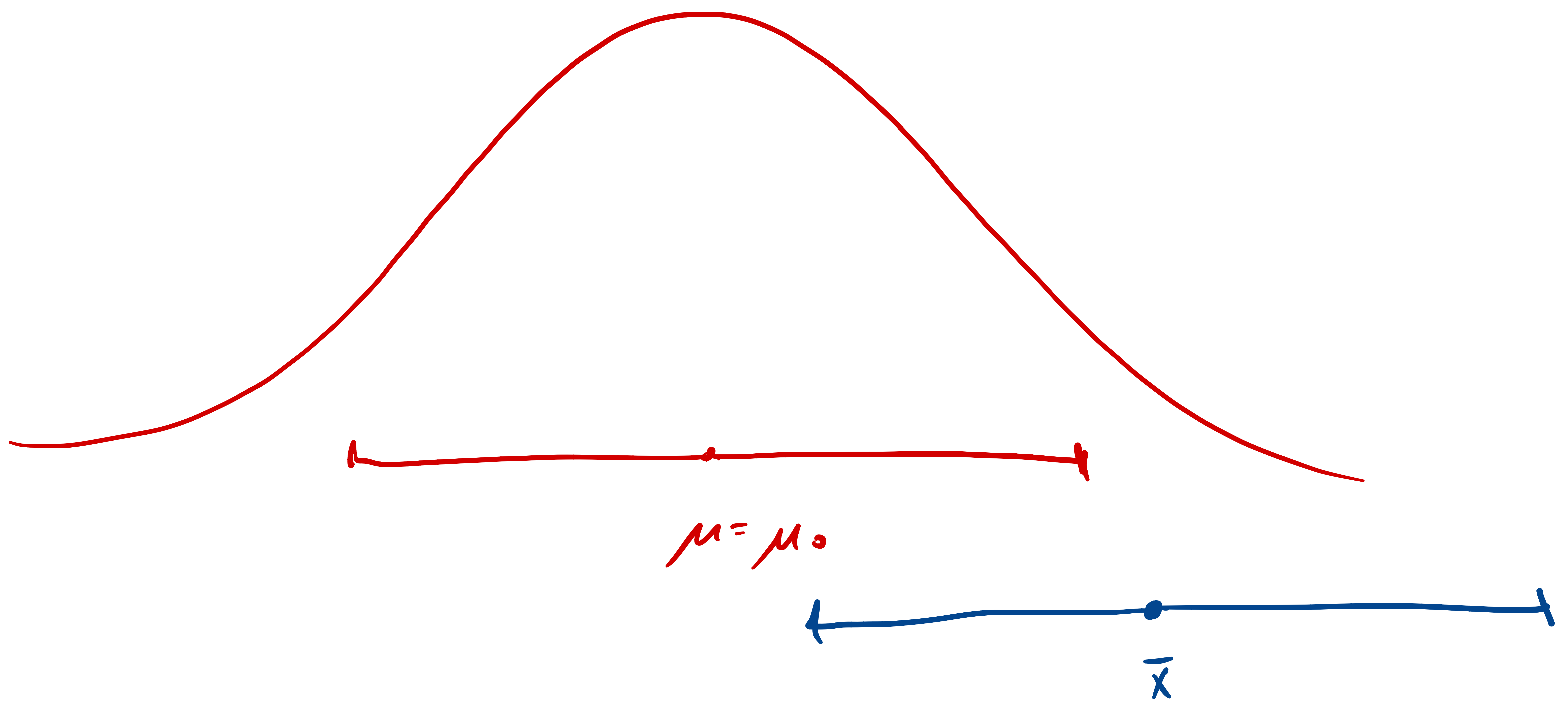
9. Conclusion:

Hypothesis Tests vs. Confidence Intervals



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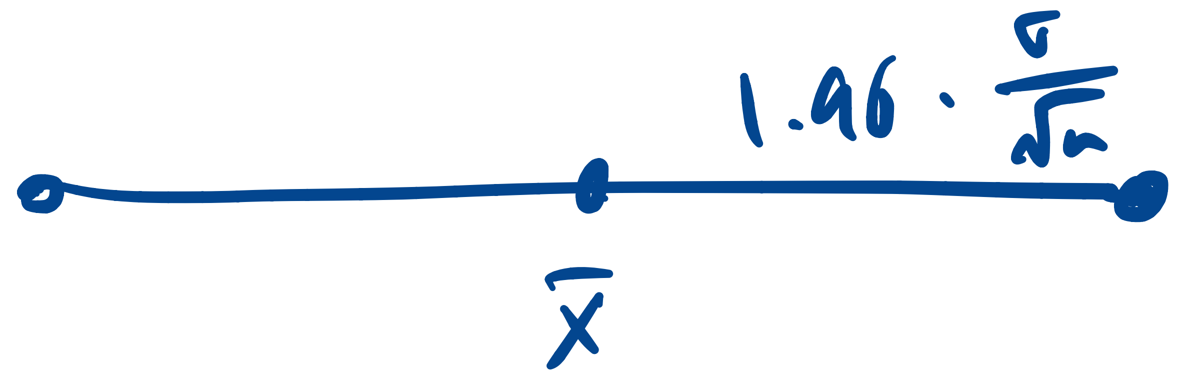

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- We say that -1.96 and 1.96 are the *critical values* of the test statistic at the $\alpha = 0.05$ significance level (for 2-sided z-tests)
- Conversely, we fail to reject a null hypothesis at $\alpha = 0.05$ if μ_0 falls within the 95% confidence interval for μ
  - We reject a null hypothesis at $\alpha = 0.05$ if μ_0 lies outside the 95% confidence interval for μ

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- Therefore, the two-sided 95% confidence interval is

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\begin{aligned}
CI &= 191 \pm 1.96 \frac{50}{\sqrt{30}} \\
&= 191 \pm 17.89 \\
&= (173.11, 208.89)
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- Since 166.3 falls outside this interval, we reject the null hypothesis and conclude that the average weight of men is not equal to 166.3 lbs

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- Hypothesis tests are centered around a null hypothesis that we are interested in gathering evidence against in order to reject it in favor of our alternative supposition

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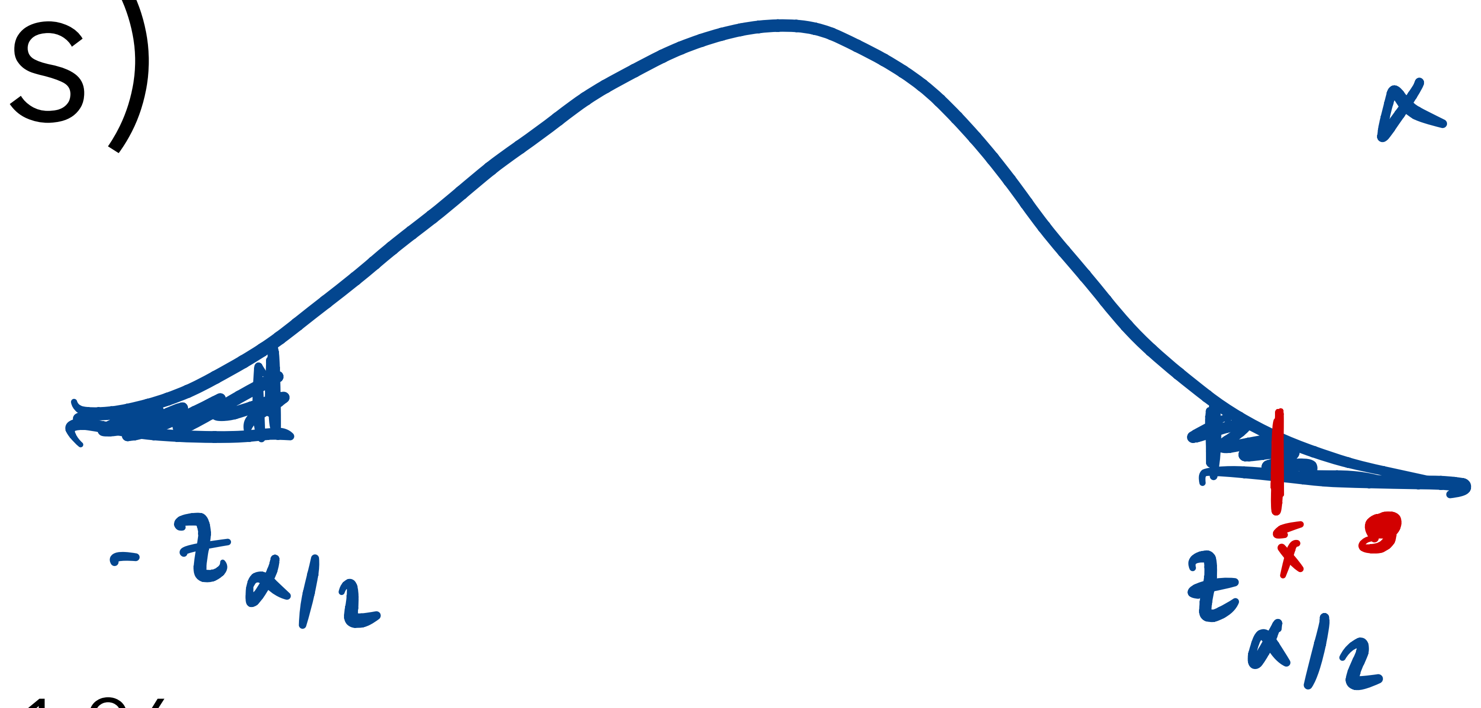

- Assume σ^2 is known (i.e., z-test)
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- For a two-sided z-test with $\alpha = 0.05$, our critical value is $z_{\alpha/2}$ = `qnorm(0.975)` = 1.96
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    - If $|z| < z_{\alpha/2}$, fail to reject H_0

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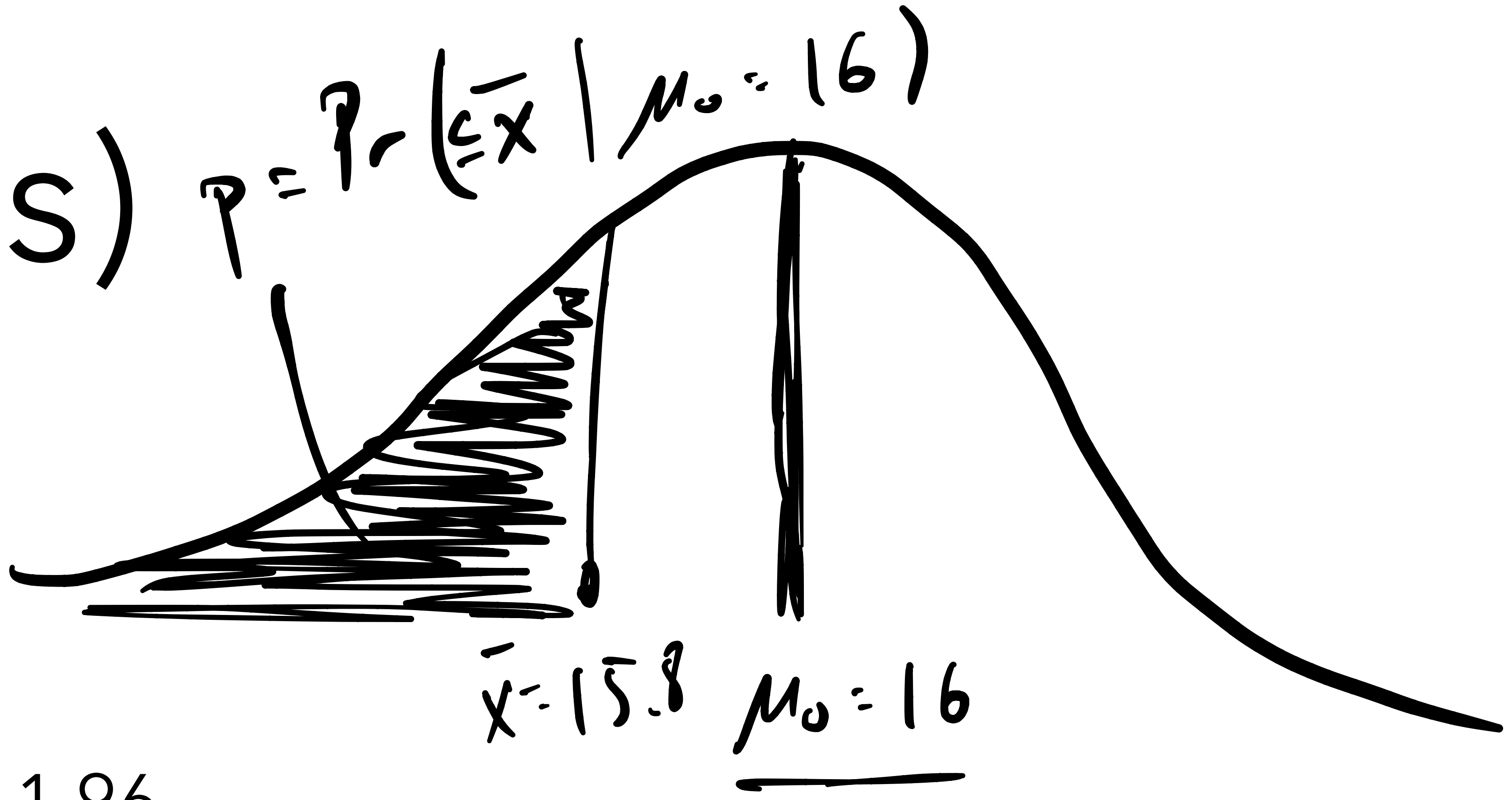

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- For a one-sided (lower) z-test with $\alpha = 0.05$, our critical value is z_α = `qnorm(0.05)` = -1.645
  - If $z \leq z_\alpha$, reject H_0
  - If $z > z_\alpha$, fail to reject H_0

$H_1: \mu < 16$

$H_0: \mu \geq 16$

$\boxed{\mu > 16}$

$H_0 = \mu \geq 16$

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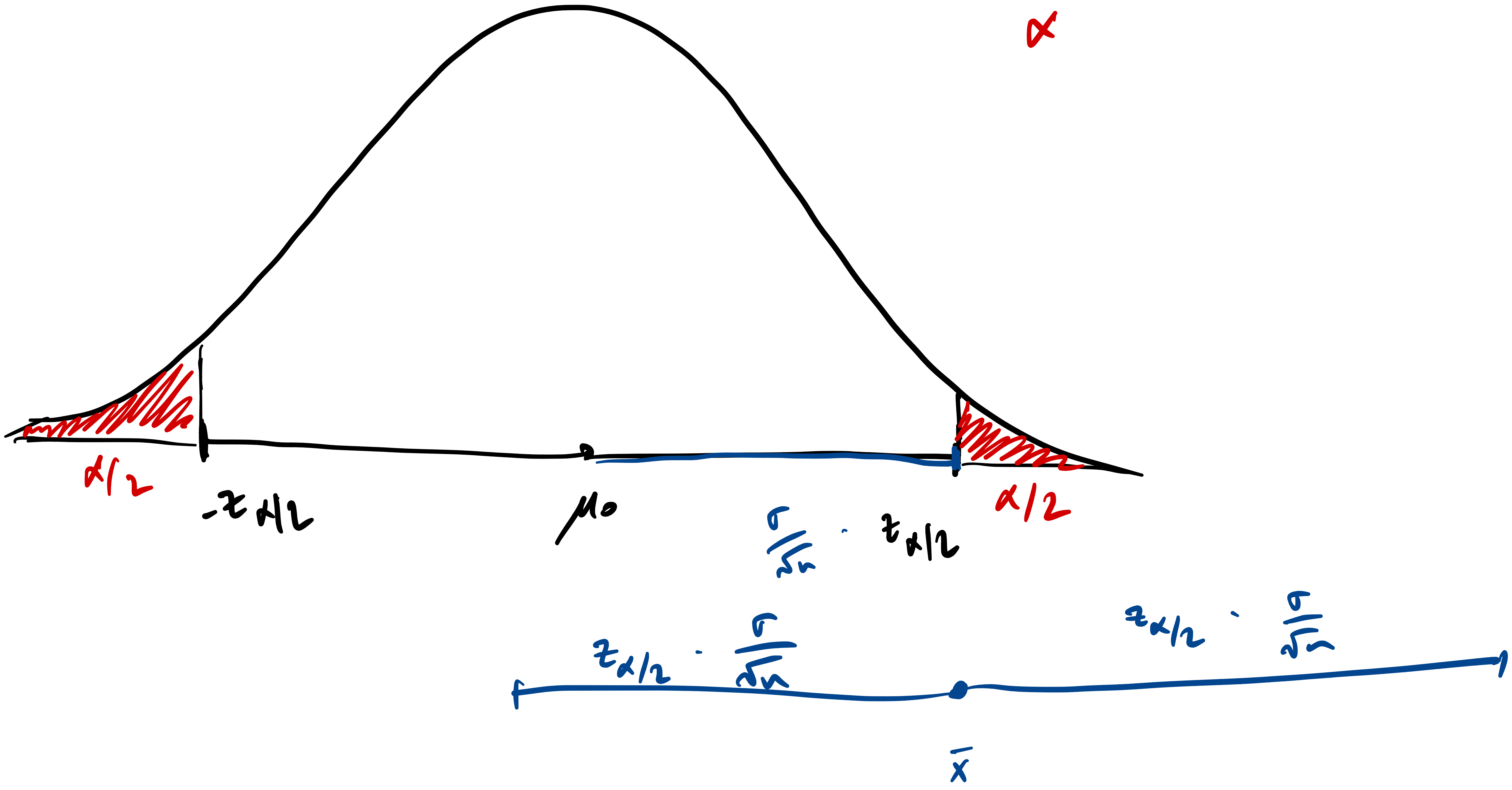
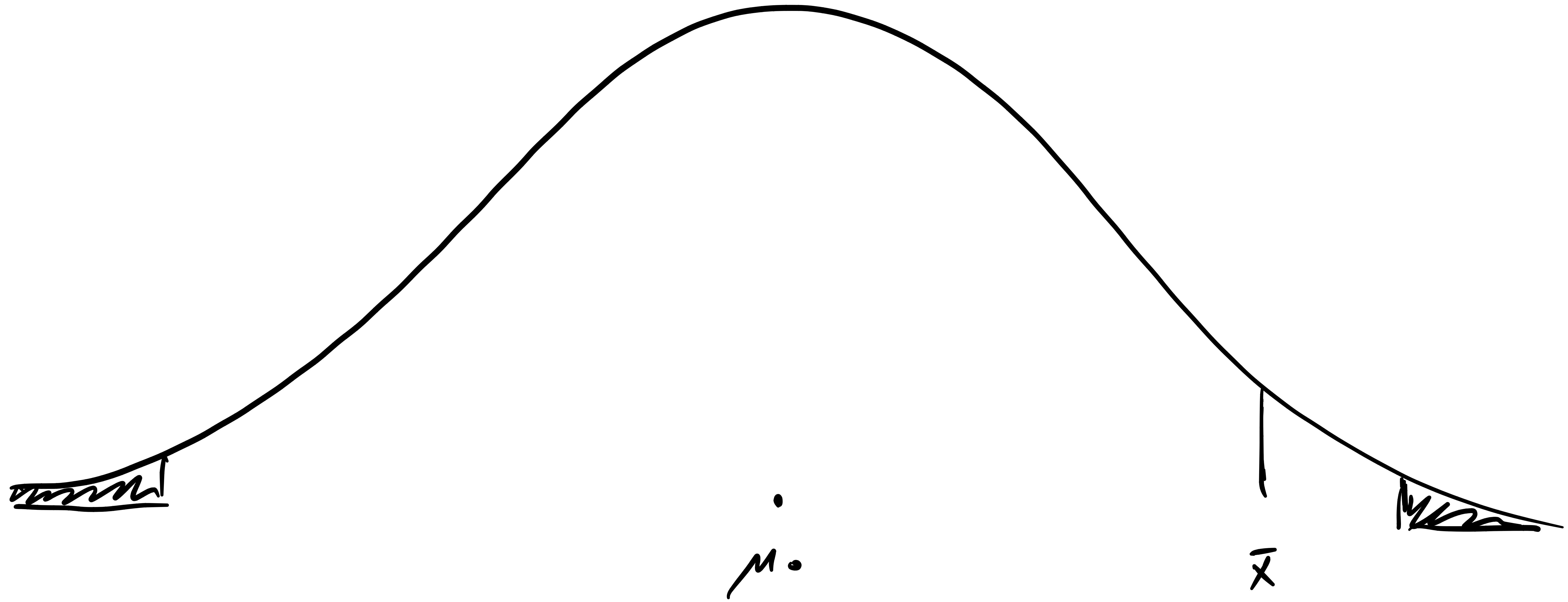
Rejection Region Example: One-sided z-test (lower)



Rejection Region Example: One-sided z-test (lower)

- Back to Starbuck's example: $n = 40$, $\bar{x} = 15.8$, $\sigma = 40$, $\mu_0 = 16$



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- Critical value: z_α = `qnorm(0.05)` = -1.645
- Since $-3.16 \leq -1.645$, reject H_0
- There is sufficient evidence to conclude that less than 16 oz of coffee is being poured into grande cups

Rejection Region Example: Two-sided z-test

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- Back to men's weight example: $n = 30$, $\bar{x} = 191$, $\sigma = 50$, $\mu_0 = 166.3$

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- Back to men's weight example: $n = 30,\ \bar{x} = 191,\ \sigma = 50,\ \mu_0 = 166.3$
- $H_0 : \mu = 166.3$ lbs, $H_1 : \mu \neq 166.3$ lbs

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- Back to men's weight example: $n = 30,\ \bar{x} = 191,\ \sigma = 50,\ \mu_0 = 166.3$
- $H_0 : \mu = 166.3$ lbs, $H_1 : \mu \neq 166.3$ lbs
- σ is known, so we use a z-test: $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{191 - 166.3}{50/\sqrt{30}} = 2.706$

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- Critical value: $z_{\alpha/2}$ = `qnorm(0.975)` = 1.96

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- Since $2.706 \geq 1.96$, reject H_0

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- Critical value: $z_{\alpha/2}$ = `qnorm(0.975)` = 1.96
- Since $2.706 \geq 1.96$, reject H_0
- There is sufficient evidence to conclude that the average weight of men has significantly changed between 1960 and 2002

Rejection Region Example: Two-sided t-test

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- Back to US Olympic athletes' heart rates example:
  $n = 40,\ \bar{x} = 60.9,\ s = 34.2,\ \mu_0 = 71.2$

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- σ is unknown, so we use a t-test: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{60.9 - 71.2}{34.2/\sqrt{40}} = -1.90$

-

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- Critical value: $t_{\alpha/2}$ = `qt(0.975,df=39)` = 2.023
- Since $|-1.90| < 2.023$, fail to reject H_0
- There is insufficient evidence to conclude that the average heart rate of US Olympic athletes is significantly different from the average heart rate of all Americans

Rejection Region Example: One-sided t-test (upper)

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- Back to MCAT example: $n = 52,\ \bar{x} = 516,\ s = 18,\ \mu_0 = 500$

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- Back to MCAT example: $n = 52,\ \bar{x} = 516,\ s = 18,\ \mu_0 = 500$
- $H_0 : \mu \leq 500$, $H_1 : \mu > 500$

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- Back to MCAT example: $n = 52,\ \bar{x} = 516,\ s = 18,\ \mu_0 = 500$
- $H_0 : \mu \leq 500, H_1 : \mu > 500$
- σ is unknown, so we use a t-test: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{516 - 500}{18/\sqrt{52}} = 6.41$

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- σ is unknown, so we use a t-test: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{516 - 500}{18/\sqrt{52}} = 6.41$
- Critical value: t_α = `qt(0.95,df=51)` = 1.675

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- Critical value: t_α = `qt(0.95,df=51)` = 1.675
- Since $6.41 \geq 1.675$, reject H_0

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- Back to MCAT example: $n = 52,\ \bar{x} = 516,\ s = 18,\ \mu_0 = 500$
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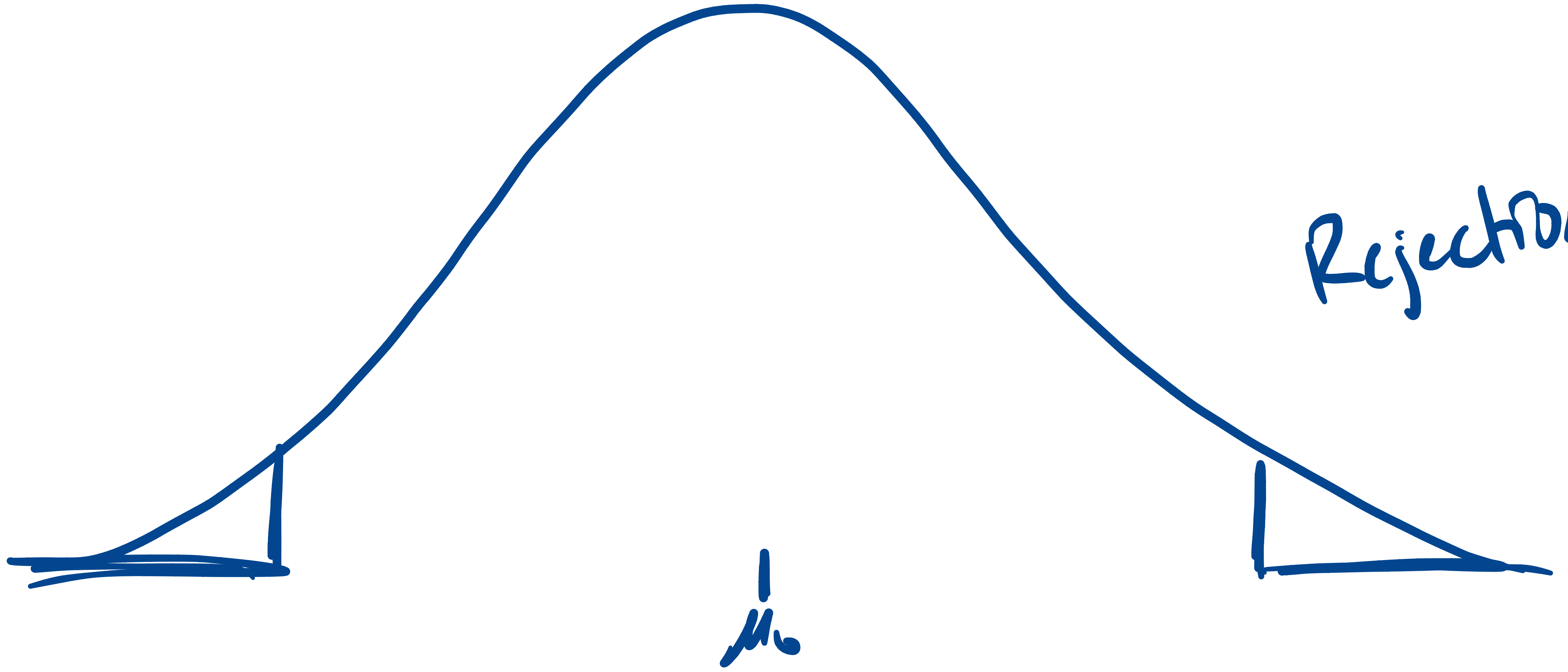
- Critical value: t_α = `qt(0.95,df=51)` = 1.675
- Since $6.41 \geq 1.675$, reject H_0
- There is sufficient evidence to conclude that U of R medical students score significantly higher on the MCAT than the nationwide average

Types of Error



Rejection regions.

$H_0 : \mu = \mu_0$

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H₀ false

	$\mu = \mu_0$	$\mu \neq \mu_0$
Fail to reject	Correct	Incorrect (Type II)
Reject	Incorrect (Type I)	Correct

Example for two-sided test

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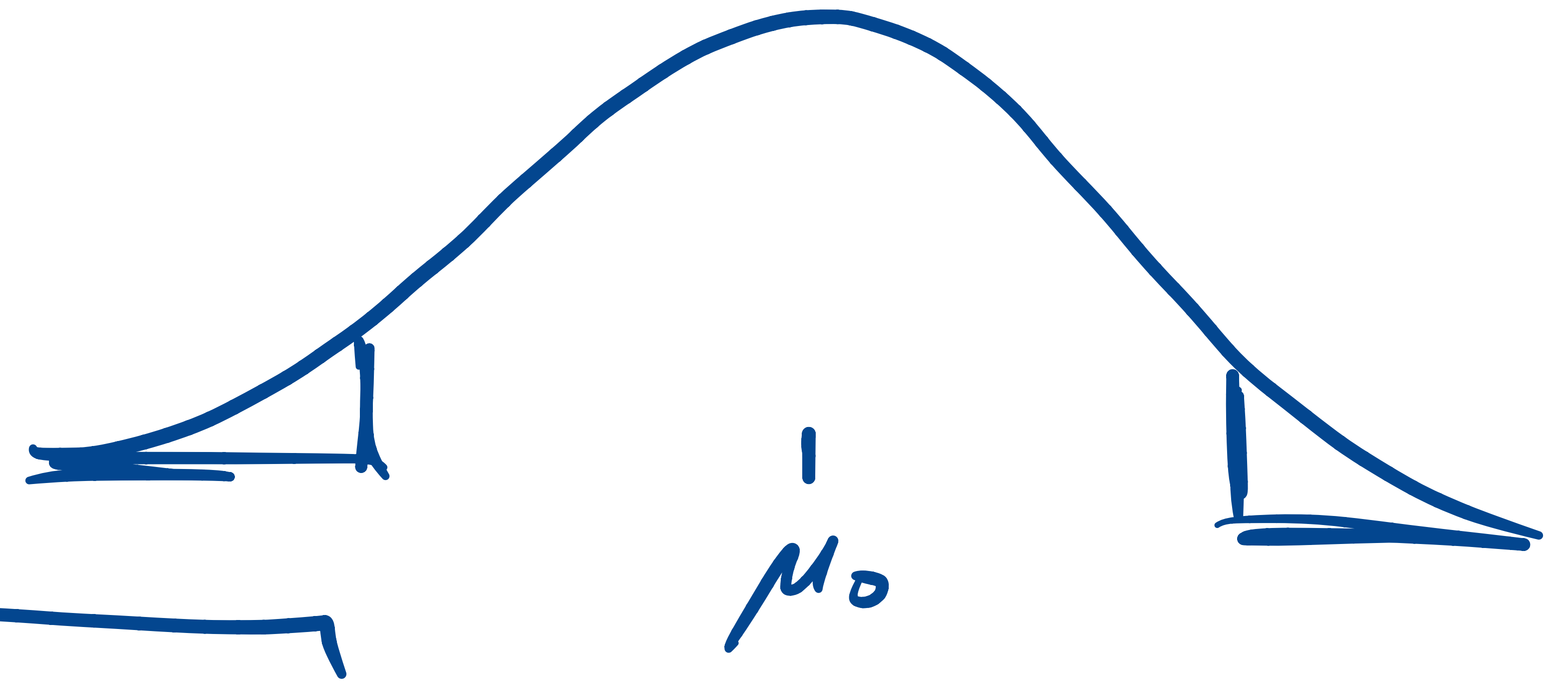
- The chance that this happens is $\Pr(\text{reject } H_0 \,|\, H_0 \text{ is true})$



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- However, recall that $\alpha = \Pr(\text{reject } H_0 \,|\, H_0 \text{ is true})$
- Thus, the significance level α is the probability of making a type I error
- We decide what α is for our test

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- Typically, we want β to be around 0.10 (or less)
- Holding all else constant, the smaller we make α, the larger $\underline{\beta}$ becomes
- Exact type II error depends on the *particular* alternative population mean μ_1

$H_0 : \mu = \mu_0$

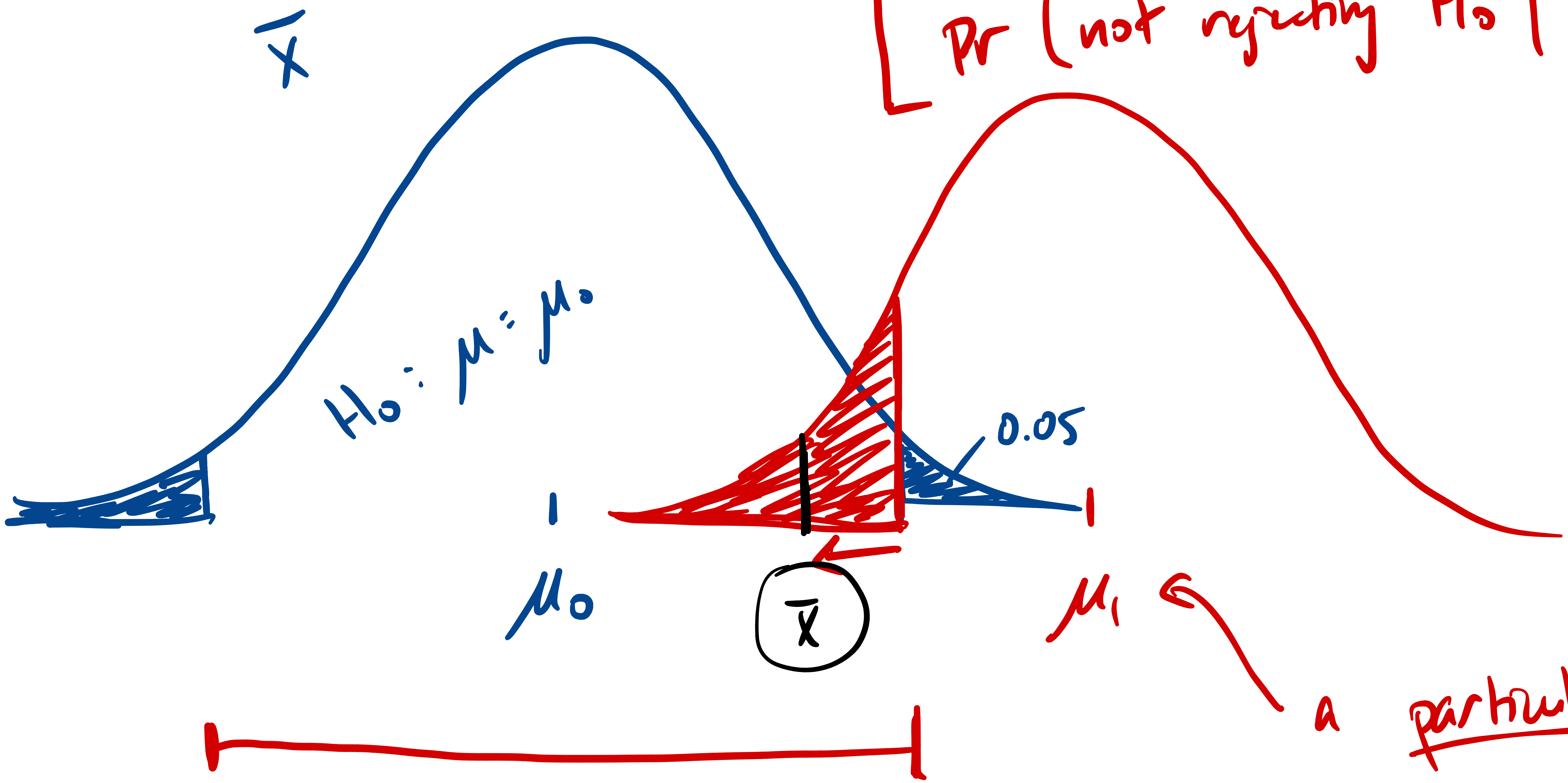
$H_1 : \mu \neq \mu_0$

$\mu = \mu_1$

Type I and Type II Error, Illustrated

$\Pr(\text{reject } H_0 \mid H_0 \text{ is true})$ α

$\Pr(\text{not rejecting } H_0 \mid H_1 \text{ is true})$ β



$N(\mu, \sigma)$

a particular alt hypothesis

$H_1: \mu = \mu_1$

Types of Error: Example

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- Setup: A pharmaceutical company has developed a cancer treatment and wants to know if it is effective
- H_0: The treatment is not effective
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- What are type I and type II errors in this context?

<u>Type I</u>: Reject H_0 | H_0 true

<u>Type II</u>: Fail to reject H_0 | H_0 false

Power

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- Example: Convict the criminal who committed the crime
- Power must be computed for a particular alternative population mean μ_1

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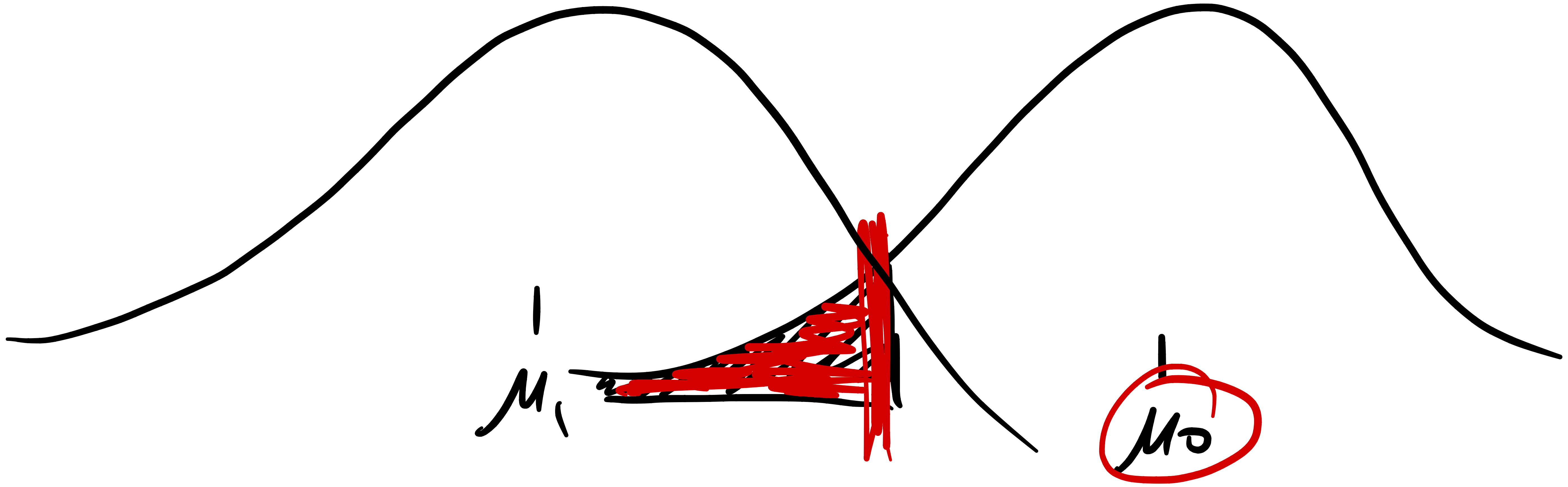
- What is the value of β associated with a test of the null hypothesis $H_0 : \mu \geq 16$ oz?



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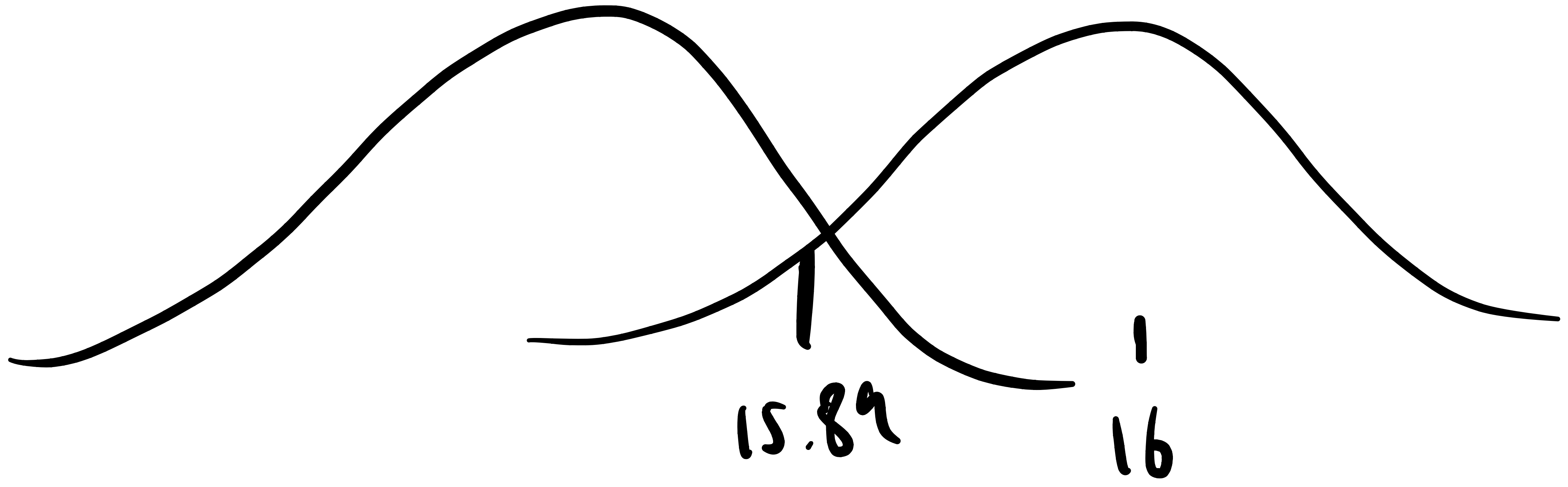

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- Interpretation: the null hypothesis will be rejected if our sample has a mean $\bar{x}$ that is less than or equal to 15.89 oz

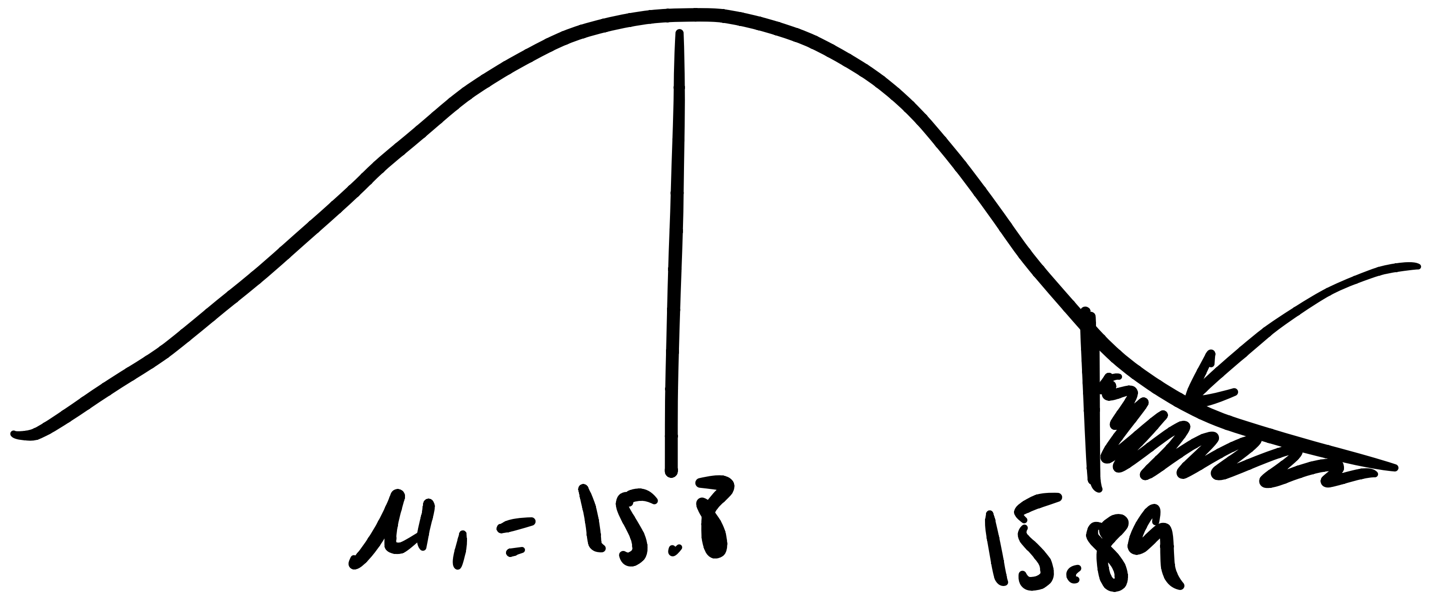
    - If the sample mean is larger, we lack sufficient evidence to reject H_0

Power: Example

Power: Example

- Now, given this cutoff for rejecting the null hypothesis (15.89 oz), what is β?

Power: Example



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- Interpretation: The probability of failing to reject $H_0 = 16$ oz when the true population mean is $\mu_1 = 15.8$ oz is 0.0885
- Hence, the power of the test is $1 - \beta = 0.9115$

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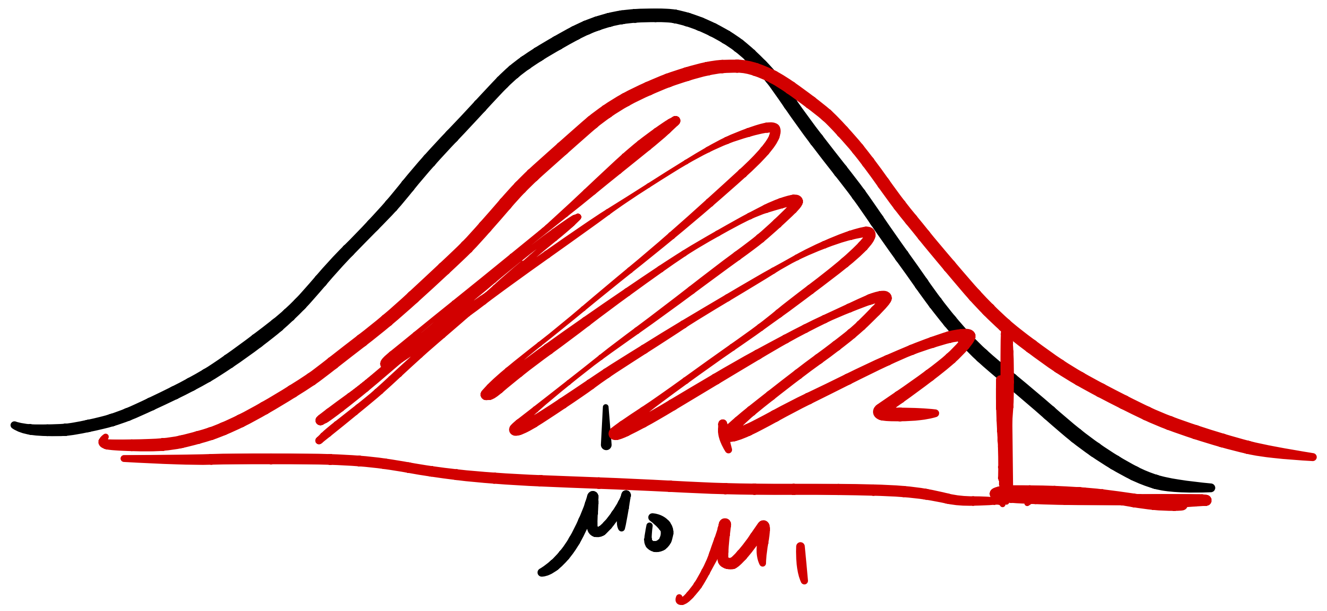

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- But, type II error is calculated for a single value of μ_1 that falls under the alternative hypothesis

- Different choices of the "truth" will lead to different values of β

- In general, the closer μ_1 is to μ_0, the harder it will be to reject H_0

Power Revisited

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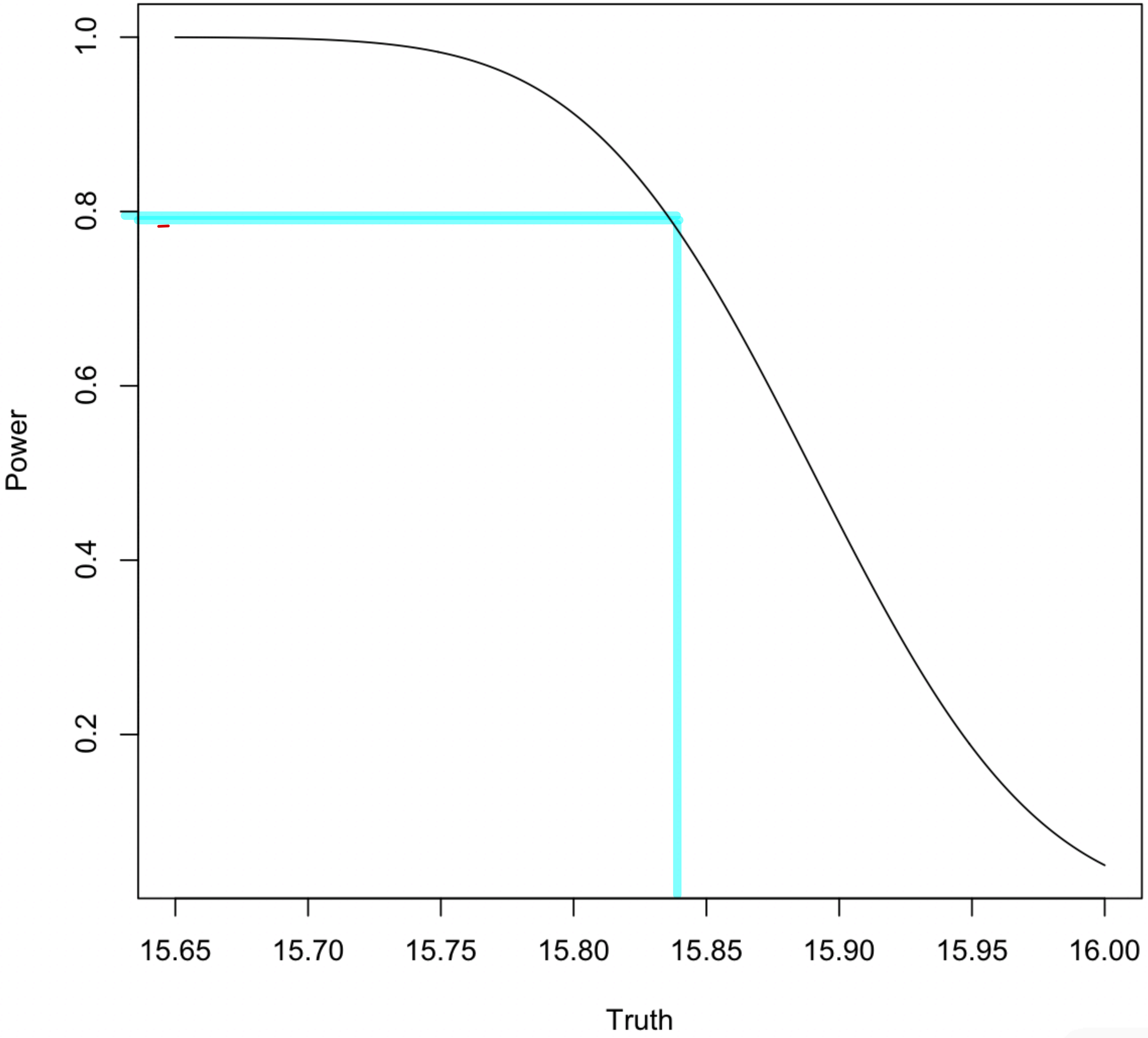
- In the Starbucks example ($n = 36$, $\alpha = 0.05$, $\sigma = 0.4$, $\mu_0 = 16$, $\mu_1 = 15.8$, one-sided lower z-test), we can more directly calculate power as follows:

$$
\begin{aligned}
\text{Power} &= \Pr(\text{reject } H_0 \,|\, \mu = 15.8) \\
&= \Pr(\boxed{\text{reject } \mu \geq 16} \,|\, \mu = 15.8) \\
&= \Pr(\bar{X} \leq 15.89 \,|\, \mu = 15.8) \\
&= \Pr(Z \leq 1.35) \\
&= 0.9115
\end{aligned}
$$

15.89

Power Curve



Power
1.0
0.8
0.6
0.4
0.2
15.65
15.70
15.75
15.80
15.85
15.90
15.95
16.00
Truth

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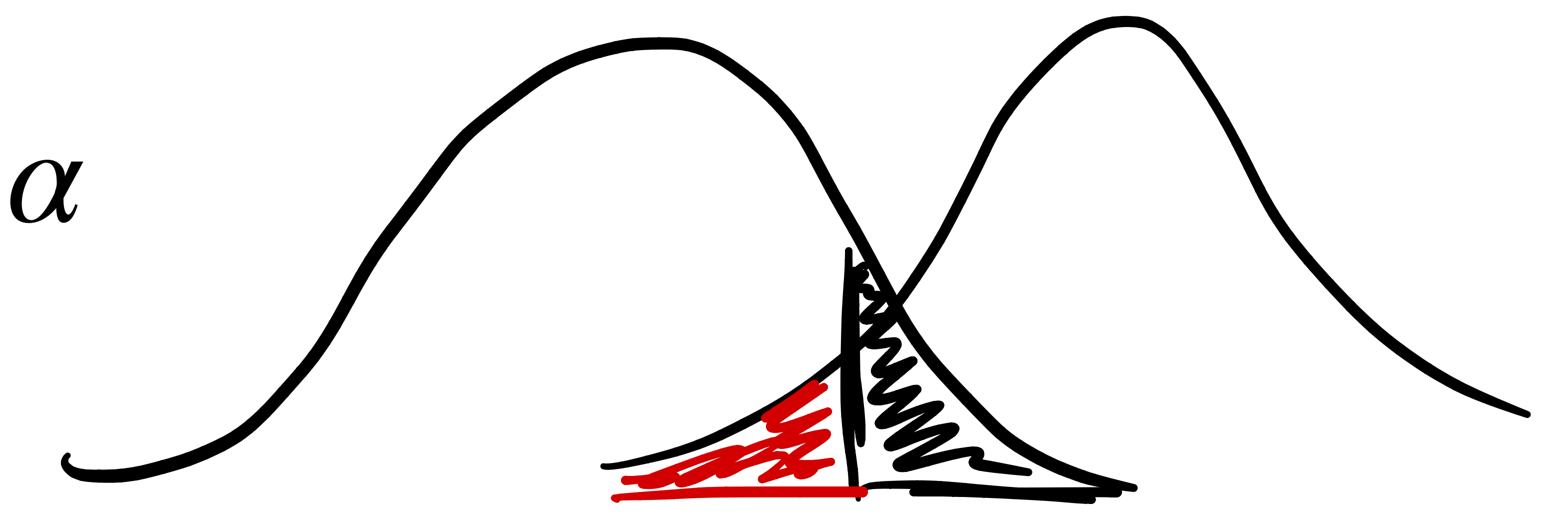
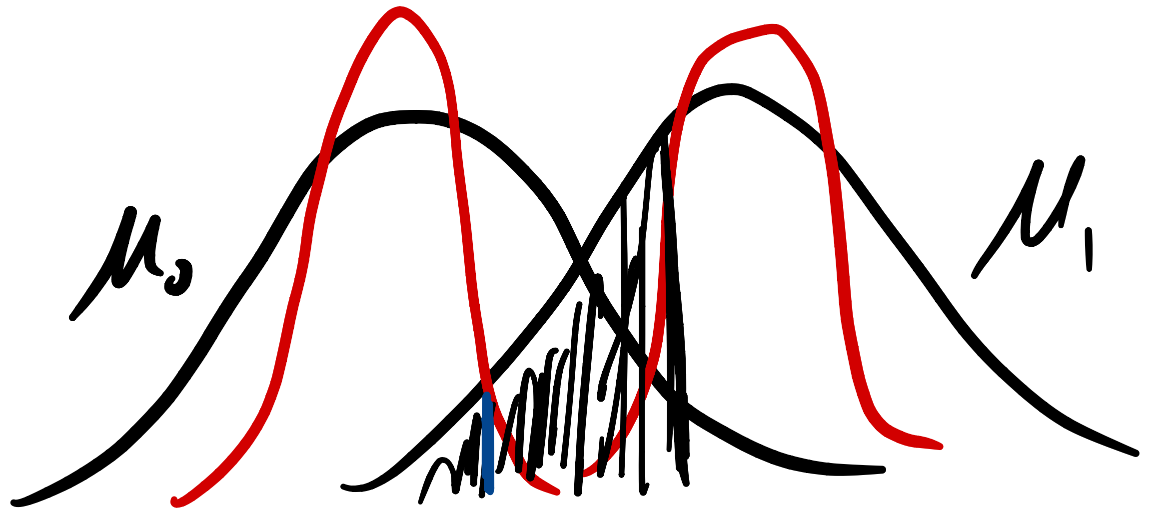


Hypothesis Testing: α and Power



- Ideally, we would like a test with small α and high power $(1 - \beta)$
- In practice, power less than 80% is considered insufficient and the test is deemed not worthwhile
- We can increase power (decrease β) by increasing α
- With larger sample sizes, we are less likely to commit either type of error (and thus, have higher power)

  $\frac{\sigma}{\sqrt{n}}$

  - Intuition: Larger sample sizes = more sharply peaked distributions = less overlap in the two normal distributions = increased power

Sample Size Estimation

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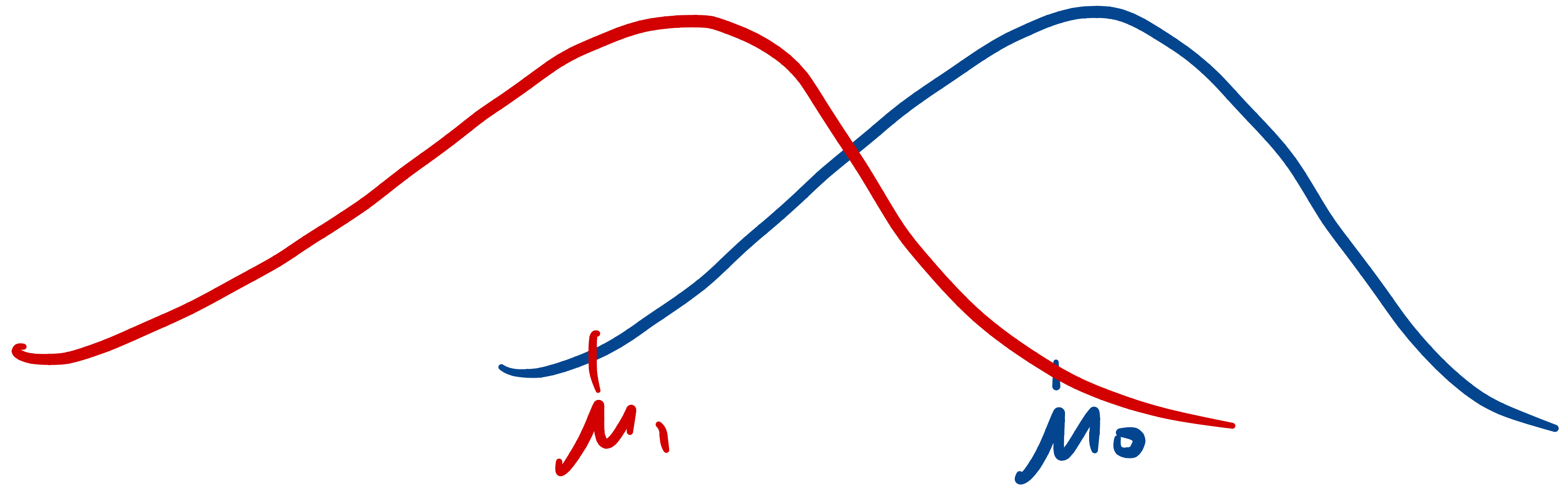
- However, sample size (n) can impact the power of a test for a given significance level (α)
- Thus, if we want to achieve a certain power at a given significance level, we can calculate the sample size necessary to do so

Sample Size Estimation

Sample Size Estimation

- Back to Starbucks (one-sided z-test): $\sigma = 0.4$, $\alpha = \beta = 0.05$, $\mu_1 = 15.8$ oz, $H_0 : \mu \geq 16$ oz, $H_1 : \mu < 16$ oz
    - In other words, we want $\alpha = 0.05$ and power = 0.95 (so $\beta = 0.05$).

Sample Size Estimation



- Back to Starbucks (one-sided z-test): $\sigma = 0.4$, $\alpha = \beta = 0.05$, $\mu_1 = 15.8$ oz, $H_0 : \mu \geq 16$ oz, $H_1 : \mu < 16$ oz
  - In other words, we want $\alpha = 0.05$ and power = 0.95 (so $\beta = 0.05$).
- What size sample (n) do we need?
  - First, find cutoff where we reject the null hypothesis: $z = \text{qnorm}(0.05) = -1.645$

$$z = -1.645 = \frac{\bar{x} - 16}{0.4/\sqrt{n}} \rightarrow \bar{X} = -1.645(.4/\sqrt{n}) + 16$$

  - Recall that we want a power of 0.95 ($\beta = 0.05$)
  - If the true mean were actually $\mu_1 = 15.8$, we want to reject the null hypothesis with probability 0.95
  - For $\beta = 0.05$, we have a z-score of $\text{qnorm}(0.95) = 1.645$

Sample Size Estimation

Sample Size Estimation

- We would like the cutoff for $\alpha = 0.05$, which is $\bar{x} = 16 - 1.645 \cdot \frac{.4}{\sqrt{n}}$, to correspond with the z-score for $\beta = 0.05$ of 1.645
- In other words:

$$z = 1.645 = \frac{\bar{x} - \mu_1}{0.4/\sqrt{n}} = \frac{\left(16 - 1.645 \cdot \frac{0.4}{\sqrt{n}}\right) - 15.8}{0.4/\sqrt{n}}$$

$$\therefore \underbrace{16 - 15.8}_{\mu_0 - \mu_1} = \underbrace{1.645\left(\frac{0.4}{\sqrt{n}}\right)}_{z_\beta} + \underbrace{1.645\left(\frac{0.4}{\sqrt{n}}\right)}_{z_\alpha}$$

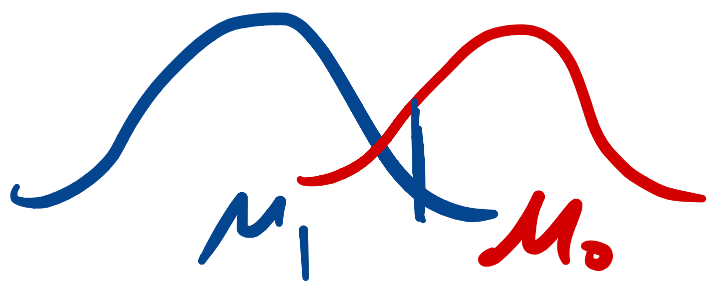
$$\rightarrow n = 43.3 \rightarrow n = 44$$

Sample Size Estimation

- We would like the cutoff for $\alpha = 0.05$, which is , to correspond with the z-score for $\beta = 0.05$ of 1.645
- In other words:

- Thus, we need to sample $n =$ coffees

Sample Size Estimation: One-sided z-test



$\sigma, \alpha, \beta, \mu_0, \mu_1$: find n to get α and β

cutoff

$$-z_\alpha = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \bar{x} = -z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) + \mu_0$$

want:

$$z_\beta = \frac{\bar{x}-\mu_1}{\sigma/\sqrt{n}} \rightarrow z_\beta = \frac{-z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) + \mu_0 - \mu_1}{\sigma/\sqrt{n}}$$

$$z_\beta(\sigma/\sqrt{n}) + z_\alpha(\sigma/\sqrt{n}) = \mu_0 - \mu_1$$

$$\sqrt{n} = \frac{\sigma(z_\beta + z_\alpha)}{\mu_0 - \mu_1}$$

Sample Size Estimation: One-sided z-test

- We can write this sample size calculation formula more generally for any one-sided hypothesis test
- Let z_α be the value that cuts off an area of α in the upper tail of the standard normal distribution
- Let z_β be the value of z that corresponds to a type II error probability of β
- Consider either set of one-sided hypotheses:
  - $H_0 : \mu \leq \mu_0$ and $H_1 : \mu > \mu_0$
  - $H_0 : \mu \geq \mu_0$ and $H_1 : \mu < \mu_0$
- If we want to achieve a power of $1 - \beta$ while keeping a significance level of α, our sample size formula is

$$\boxed{n = \left\lceil \left(\frac{\sigma \cdot (z_\alpha + z_\beta)}{\mu_1 - \mu_0} \right)^2 \right\rceil}$$

Sample Size Estimation: Two-sided z-test

Sample Size Estimation: Two-sided z-test

- For a two-sided hypothesis test, instead of having α in the upper tail, we need $\alpha/2$ in the upper tail
- Let $z_{\alpha/2}$ be the value that cuts off an area of $\alpha/2$ in the upper tail of the standard normal distribution
- Let z_β be the value of z that corresponds to a type II error probability of β
- Consider the two-sided hypothesis:
  - $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$
- If we want to achieve a power of $1 - \beta$ while keeping a significance level of α, our sample size formula is

$$n = \left[\left(\frac{\sigma \cdot (z_{\alpha/2} + z_\beta)}{\mu_1 - \mu_0}\right)^2\right]$$