Chapter 15: Linear Regression I

DSCC 462
Computational Introduction to Statistics

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Announcements

- Midterm grades are out
- Pickup tomorrow from 3-4 pm in my office (Wegmans 2401) or at office hours next Tuesday
- Project groups are due tomorrow! Datasets will be released tomorrow (no project proposal necessary)

Plan For Today

- Explain how one variable affects another: simple linear regression!
- Basics of regression
- Inference for parameters
- Confidence intervals for true values

Simple Linear Regression

- Simple linear regression allows us to explore the relationship between two continuous random variables (think scatterplot)
- Unlike correlation analyses, we can *directly* model how a change in one variable affects another variable
  - *Explanatory variable* affects the *response variable*
- Goal: Estimate the value of the response variable that is associated with a given value of the explanatory variable
  - Example: If a child is 9 years old, how tall do we expect them to be?

Simple Linear Regression

- In *linear* regression, we estimate the relationship between x (explanatory variable) and y (response variable) by a line
- Regression model: $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ for $i = 1,\ldots,n$ and $\epsilon_i \sim N(0,\sigma^2)$
  - Expressed another way: $y_i \mid x_i \sim N(\beta_0 + \beta_1 x_i,\ \sigma^2)$
  - β_0 is the y-intercept and β_1 is the slope for the population
- **Goal**: Estimate β_0 and β_1 based on a sample in order to model the relationship between y and x

Simple Linear Regression

- Assumptions:
  - Given x, the y's are independent
  - There is a linear relationship between y and x (i.e., $E(\epsilon) = 0$)
  - The variance σ^2 is constant across all values of x (i.e., $Var(\epsilon) = \sigma^2$), known as homoscedasticity
  - For a specified value of x, y is normally distributed
  - x are fixed, known quantities
- When the regression assumptions are met, the use of linear regression is appropriate for describing the relationship between y and x

Simple Linear Regression

- Once we have estimates $\hat{\beta}_0$ and $\hat{\beta}_1$, we can estimate what y_i would be for a given x_i, under the model
- $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$
- But how do we fit a linear regression model?

Simple Linear Regression

- Use the *method of least squares* to fit a straight line to a set of points (x_i, y_i)
- If we look at how much each predicted $\hat{y}_i$ deviates from the true observed value y_i, we have the residual $e_i = y_i - \hat{y}_i$
    - R = A – P (residual = actual – predicted)
- Residuals of $e_i = 0$ indicate that the observed point lies directly on the regression line
- Ideally, we would want every point to lie directly on the line

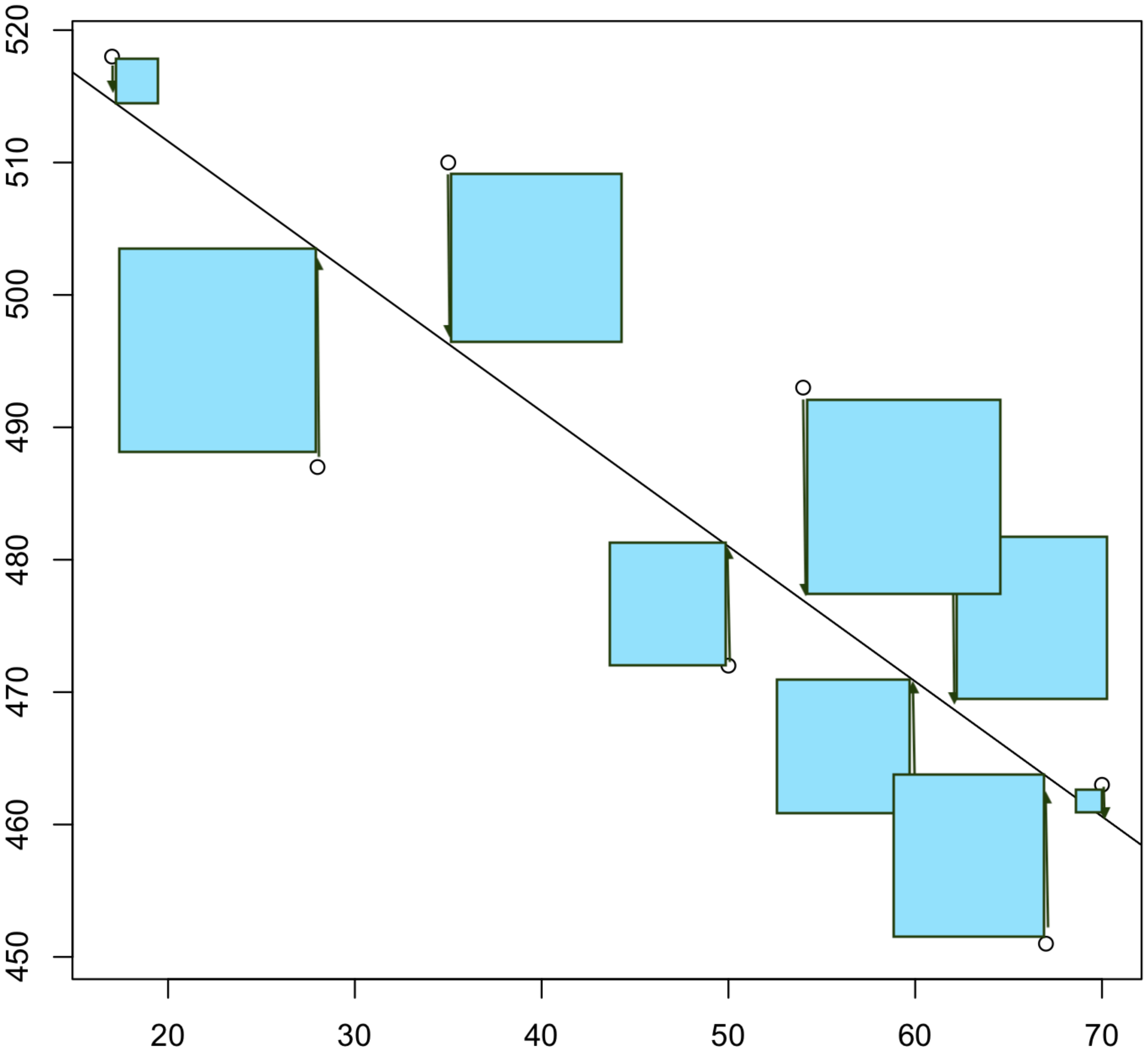
Simple Linear Regression

- Since points do not all lie on the regression line, we must determine the best criterion for fitting the line in such a way as to make these residuals as small as possible

- *Residual sum of squares*:

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Simple Linear Regression



Simple Linear Regression

- Plugging in $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$, we get:

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

- Using this, we get our parameter estimates as follows:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} = r\frac{s_y}{s_x}$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$$

- Note, r is Pearson's correlation coefficient, s_y is the standard deviation of y, and s_x is the standard deviation of x

Simple Linear Regression

```
> set.seed(223542)
> dat1 <- rmvnorm(10,c(11,10), sigma=matrix(c(1,.5, .5, 1),2,2))
> colnames(dat1) <- c("X","Y")
> x <- dat1[,1]
> y <- dat1[,2]
> model1 <- lm(y~x)
> plot(x,y, xlab="X", ylab="Y")
> abline(model1)
> model1

Call:
lm(formula = y ~ x)

Coefficients:
(Intercept)            x
     0.3236       0.8578
```

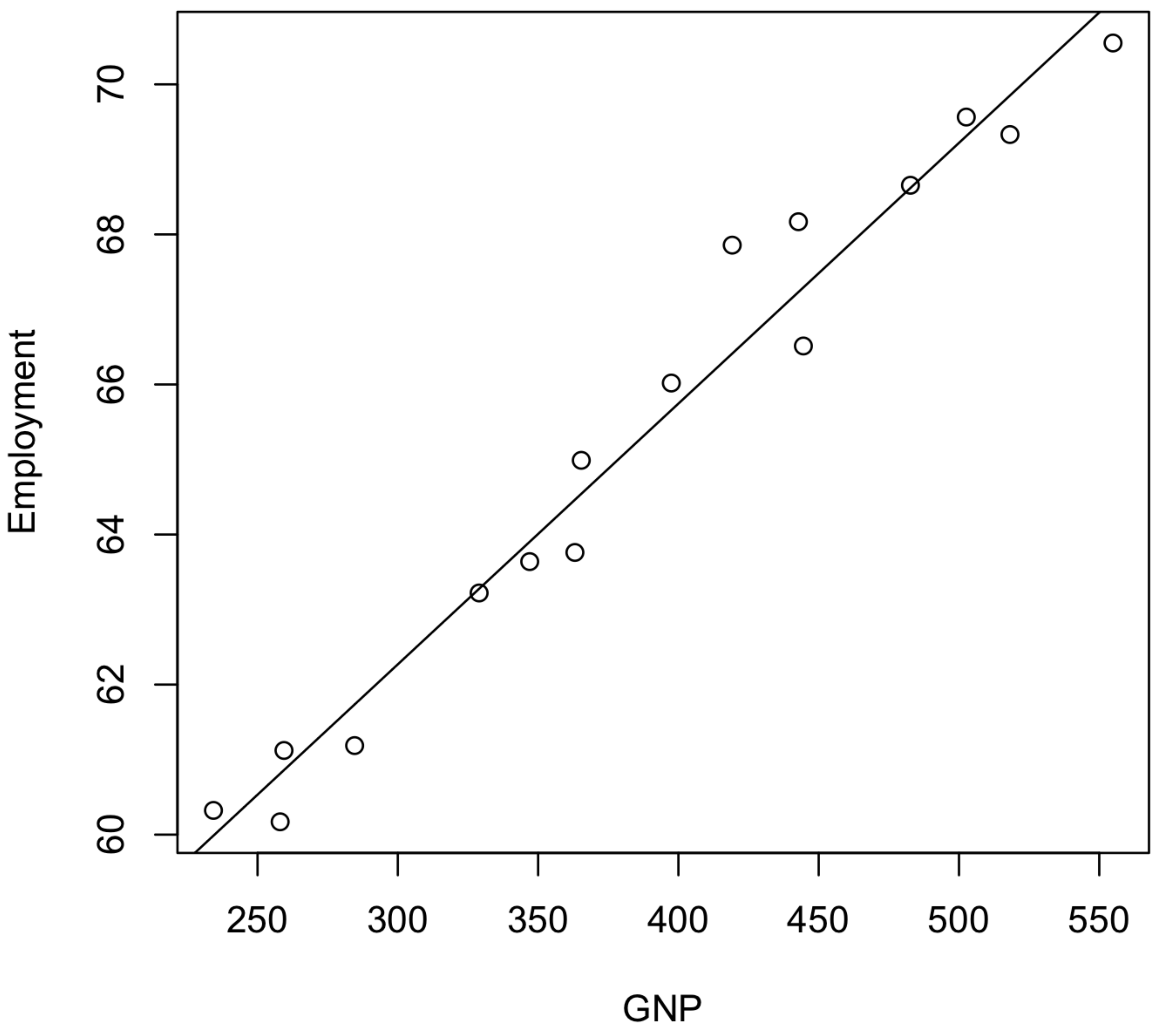
```
> summary(model1)

Call:
lm(formula = y ~ x)

Residuals:
     Min       1Q   Median       3Q      Max
-1.27555 -0.34855 -0.09534  0.52797  1.28676

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.3236     2.3820   0.136  0.89530
x             0.8578     0.2209   3.884  0.00465 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8009 on 8 degrees of freedom
Multiple R-squared:  0.6534,Adjusted R-squared:  0.6101
F-statistic: 15.08 on 1 and 8 DF,  p-value: 0.004651
```

Simple Linear Regression

- Consider the economic data presented in the longley dataset
- Include economic variables from mid-1900s America
- GNP: in billions of US dollars
- Employment: in millions of people

$\hat{y}$ vs. y

$$\hat{y} = 51.844 + 0.03475x$$



Simple Linear Regression

- cor(x,y) = $r = 0.98355$
- sd(x) = $s_x = 99.395$
- sd(y) = $s_y = 3.512$
- mean(x) = $\bar{x} = 387.699$
- mean(y) = $\bar{y} = 65.317$
- $\hat{\beta}_1 = 0.98355\left(\frac{3.512}{99.395}\right) = 0.03475$
- $\hat{\beta}_0 = 65.317 - 0.03475(387.699) = 51.844$

Interpretation of Regression Estimates

- $\hat{\beta}_0$ is the y-intercept
  - When $x = 0$ we expect y to be equal to $\hat{\beta}_0$
  - Only makes sense if $x = 0$ is within the range of your data and has contextual meaning
  - E.g., 51.844 million people are expected to be employed with the GNP is $0
- $\hat{\beta}_1$ is the slope
  - For each 1 unit increase in x, we expect y to increase by $\hat{\beta}_1$ according to the model
  - E.g., for each $1 billion increase in GNP, we expect the number of people employed to increase by 0.03475 million

Prediction

- We can use our regression line to make predictions
- Suppose we want to predict employment numbers when GNP is $350 billion
- $\hat{y} = 51.844 + 0.03475x = 51.844 + 0.03475 \cdot 350 = 64.0065$ billion USD

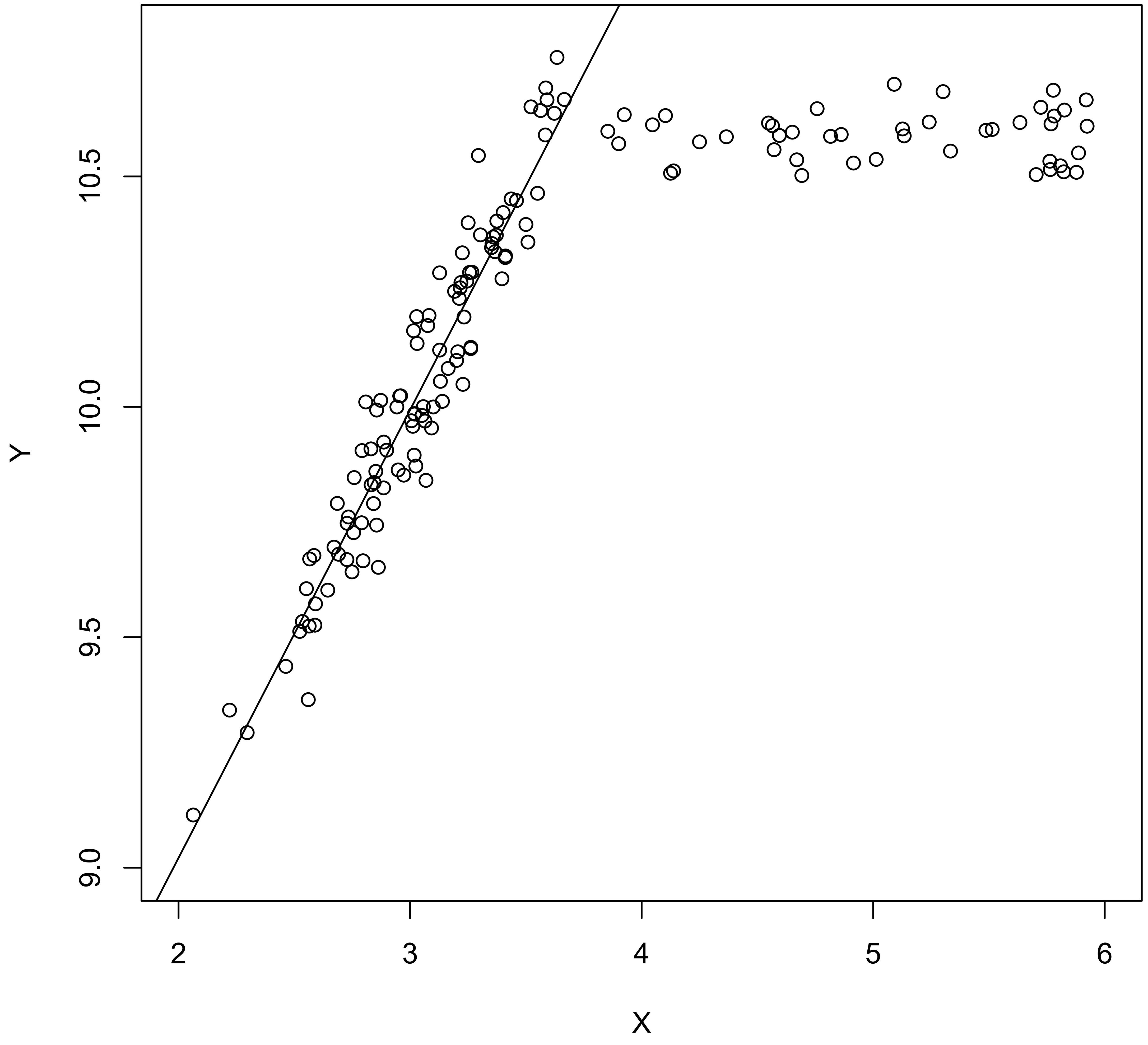
Extrapolation

- We can only use our regression line to make predictions over the set of values for which we have observations

- The regression line should not be extended outside the range for which we have data

- Intuition: Our model was created only for our range of data, and we do not know what happens outside of this range

Extrapolation



Inference for Regression Coefficients

- Regression line is based on a sample, but we want to make a conclusion about a population
- Create confidence intervals for $\hat{\beta}_0$ and $\hat{\beta}_1$
- Perform hypothesis tests
  - If $\beta_1 = 0$, this implies that a change in x has no impact on y
  - Hypothesis tests for β_0 are often unimportant and have little meaning, so we will focus only on β_1

Inference for Regression Coefficients

- Since different samples will lead to different values of $\hat{\beta}_0$ and $\hat{\beta}_1$, this implies that there is variability to these estimates
- $Var(\hat{\beta}_1) = \dfrac{\sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$
- $Var(\hat{\beta}_0) = \sigma^2 \left(\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right)$
- Note that σ^2 is the variance of the residuals around the predicted regression line
- Estimate σ^2 with sample $s^2 = Var(e_i) = \dfrac{1}{n-2} \sum_{i=1}^{n} e_i^2 = \dfrac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

Inference for Regression Coefficients

- Using this estimate s^2 for unknown σ^2, we get the following
- $Var(\hat{\beta}_1) = \dfrac{s^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$
- $Var(\hat{\beta}_0) = s^2\left(\dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)$

Inference for Regression Coefficients

- We typically are interested only in hypothesis tests regarding $\hat{\beta}_1$
- The slope tells us of the relationship between x and y
- Test the null hypothesis $H_0 : \beta_1 = \beta_1^*$ vs. $H_1 : \beta_1 \neq \beta_1^*$, where β_1^* is some population slope value, at the α significance level
  - Generally interested in the case of $\beta_1^* = 0$
  - No relationship vs. some relationship

Inference for Regression Coefficients

- Consider the test statistic $t = \dfrac{\hat{\beta}_1 - \beta_1^*}{SE(\hat{\beta}_1)}$

  - $SE(\hat{\beta}_1) = \sqrt{\dfrac{s^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$

- Under null hypothesis H_0, t follows a t-distribution with $n - 2$ degrees of freedom

- Our p-value is thus `2*pt(-abs(t), df=n-2)`

- If the p-value $p \leq \alpha$, we reject the null hypothesis

Inference for Regression Coefficients

- In this simple regression case, if $\beta_1 = 0$, then $\rho = 0$
- The test of the null hypothesis $H_0 : \beta_1 = 0$ is the same as the test of $H_0 : \rho = 0$
  - Both hypotheses claim that y does not change as x increases

Inference for Regression Coefficients

- Consider our employment example
- Test the hypotheses $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$ at the $\alpha = 0.05$ significance level
- We calculated $\hat{\beta}_1 = 0.03475$
- $s = 0.6566$
- $SE(\hat{\beta}_1) = \dfrac{0.6566}{\sqrt{\sum_{i=1}^{n} (x_i - \overline{x})^2}} = 0.001706$
- $t = \dfrac{0.03475}{0.001706} = 20.37$
- $p =$ 2*pt(-20.37, df=14) $= 8.4 \times 10^{-12}$
- Since the p-value is less than 0.05, we reject the null hypothesis and conclude that there is a significant linear relationship between GNP and employment

Confidence Intervals for Regression Coefficients

- We can also create confidence intervals for regression coefficients
- A $(1-\alpha) \times 100\,\%$ confidence interval for $\hat{\beta}_1$ is given as $\left(\hat{\beta}_1 - t_{\alpha/2} SE(\hat{\beta}_1), \hat{\beta}_1 + t_{\alpha/2} SE(\hat{\beta}_1)\right)$

Confidence Intervals for Regression Coefficients

- Applied to our example:

$$\begin{aligned} CI &= 0.03475 \pm \text{qt}(0.975, 14)(0.001706) \\ &= 0.03475 \pm 2.145(0.001706) \\ &= (0.03109,\ 0.03841) \end{aligned}$$

- We are 95% confident that the interval $(0.03109,\ 0.03841)$ contains the true population slope

Confidence Intervals for Regression Coefficients

```
> confint(lm1)
                 2.5 %   97.5 %
(Intercept) -5.1694143 5.816569
dat1[, 1]    0.3484624 1.367178
```

ANOVA Approach to Regression

- We can decompose the variability in a regression model in a way similar to what we did with ANOVA
- The sum of squared errors (SSE) is defined as follows:

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

- This is also known as the residual sum of squares and describes the random variability about the regression line
- The mean squared error is then

$$MSE = \frac{SSE}{n-2}$$

- Since we have two unknown parameters (β_0 and β_1), this is analogous to the $n - k$ degrees of freedom with the MSE in ANOVA

ANOVA Approach to Regression

- The treatment sum of squares (SST) gets redefined as the regression sum of squares (the explained variability):

$$SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

- This describes variability that is explained due to the regression line

- The total sum of squares is then $SSTo = SSR + SSE$

ANOVA Approach to Regression

- We can compile this information into an ANOVA table

Source	SS	df	MS	F
Regression	SSR	1	$MSR = \frac{SSR}{1}$	$F = \frac{MSR}{MSE}$
Error	SSE	n-2	$MSE = \frac{SSE}{n-2}$	
Total	SSTo	n-1		

- F can be used to test the hypothesis $H_0 : \beta_1 = 0$
- F has an F distribution with 1 and $n-2$ degrees of freedom

Inference for Mean Response

- Regression parameter estimates have sampling distributions
  - If we selected a different sample, the regression line would be slightly different than the line we got from our sample
- We can use the regression line to estimate the mean value of y corresponding to a particular value of $x = x*$
  - If we took many samples for a particular $x*$ and found their average response y, it would be equal to the estimated response from our regression

Inference for Mean Response

- Recall that we predicted, on average, employment to be 64.0065 million given GNP was 350 billion USD
- In actuality, when the GNP is 350 billion USD, the employment will not necessarily equal exactly 64.0065 million, but instead that will be the average response

Inference for Mean Response

- We can create a confidence interval for this mean value
- The $(1-\alpha)\,\%$ confidence interval for the true mean $\bar{y}$ for the regression line at a particular point $x*$ is given as $\left(\hat{y} - t_{\alpha/2}SE(\hat{y}), \hat{y} + t_{\alpha/2}SE(\hat{y})\right)$
    - Here, $SE(\hat{y}) = s\sqrt{\left[\frac{1}{n} + \frac{(x* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]}$
- The standard error depends on the particular estimate of $x*$
- The closer $x*$ is to $\bar{x}$, the smaller the variability around the line
    - At the mean, there is the most information and thus this is the best estimated point

Inference for Mean Response

- Create a confidence interval for $\overline{y}$ at $x^* = 350$
- Recall that $s = 0.6566,\ \bar{x} = 387.698,\ s_x = 99.395$, and $n = 16$
- $\hat{y} = 64.0065$
- $SE(\hat{y}) = s\sqrt{\left[\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]} = 0.6566\sqrt{\left[\frac{1}{16} + \frac{(350 - 387.698)^2}{15 * 99.395^2}\right]} = 0.17629$
- `qt(0.975, df=14)` = 2.145
- CI: $64.0065 \pm 2.145 \cdot 0.17629$
- Conclusion: I am 95% confident that the interval (63.628, 64.385) million contains the true mean employment number when the GNP is 350 billion USD

Inference for Predicted Response

- Instead of considering the mean response of y for a given value of x, perhaps we are interested in the response of a single observation of $x*$
    - "You find the GNP of a given area to be 350. What do we expect the area's employment numbers to be?"
- The best estimate we have is still the predicted value from the regression line
    - $y* = \hat{\beta}_0 + \hat{\beta}_1 x* = \hat{y}$
- We are less certain in this estimate; we know that on average it is good, but for one point, it is probably going to be a bit off

Inference for Predicted Response

- In creating the regression line, we have variability based on our sample
  - The variability of the regression line around the mean response at $x = x^*$ is what we calculated as $Var(\hat{y})$
- Now, we have added variability of y^* around the regression line
  - The variability of y^* around the regression line is given as σ^2, since the outcomes of y are assumed to be normally distributed at a given value of $x = x^*$ with variance σ^2
- This means that the total standard error of y^* can be written as $SE(y^*) = s\sqrt{\left[1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]}$
- The $(1 - \alpha)\,\%$ confidence interval for the true observed y^* (not the mean $\bar{y}$!) for the regression line at a particular point x^* is given as $\left(\hat{y} - t_{\alpha/2}SE(y^*), \hat{y} + t_{\alpha/2}SE(y^*)\right)$

Inference for Predicted Response

- Create a confidence interval for y^* at $x^* = 350$
- Recall that $s = 0.6566,\ \bar{x} = 387.698,\ s_x = 99.395$, and $n = 16$
- $\hat{y} = 64.0065$
- $SE(y^*) = s\sqrt{\left[1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]} = 0.6566\sqrt{\left[1 + \frac{1}{16} + \frac{(350 - 387.698)^2}{15 * 99.395^2}\right]} = 0.6799$
- `qt(0.975, df=14)` = 2.145
- CI: $64.0065 \pm 2.145 \cdot 0.6799$
- Conclusion: I am 95% confident that the interval (62.548, 65.465) million contains the true employment number when the GNP is 350 billion USD

Inference for Mean and Predicted Response

```
> set.seed(223542)
> x <- rnorm(10, 5, 2)
> y <- rnorm(10, 13, 3)
> predict(lm(y~x), data.frame(x=6), conf.level=0.95, interval="confidence")
       fit      lwr      upr
1 10.41062 7.617309 13.20392
> predict(lm(y~x), data.frame(x=6), conf.level=0.95, interval="prediction")
       fit     lwr      upr
1 10.41062 2.83408 17.98715
```