Chapter 10: Inference on Proportions

DSCC 462

Computational Introduction to Statistics

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Inference on Proportions

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- We can also extend inferential methods to cover count data
- In particular, we are often interested in the proportion of times a dichotomous (i.e., yes/no) event occurs

Sampling Distribution of a Proportion

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- Recall that the sample mean is distributed like $\hat{p} \sim N\left(p, \sqrt{\frac{(p(1-p)}{n}}\right)$, given that $np \geq 5$ and $n(1-p) \geq 5$

$$p = \frac{\text{Binom}(n, p)}{n} \xrightarrow{\text{CLT}} \frac{N\left(np, \sqrt{np(1-p)}\right)}{n}$$

$$\rightarrow N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

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$\bar{x}$

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- Thus, $Z = \dfrac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$ has a standard normal distribution

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- Draw a sample of size n and compute $\hat{p} = \dfrac{x}{n}$
- $\hat{p}$ is a point estimate of population proportion p
- We know from above that $Z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{p(1-p)}{n}}}$ is a standard normal random variable, given that n is sufficiently large (i.e., $np \geq 5$ and $n(1-p) \geq 5$)

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- Thus, $\Pr\left(-1.96 \leq \dfrac{\hat{p} - p}{\sqrt{\dfrac{p(1-p)}{n}}} \leq 1.96\right) = 0.95$

$$\Pr\left(-1.96\sqrt{\frac{p(1-p)}{n}} + \hat{p} \leq p \leq 1.96\sqrt{\frac{p(1-p)}{n}} + \hat{p}\right) = 0.95$$

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- This can be rearranged to give

$$\Pr\left(\hat{p} - 1.96\sqrt{\frac{p(1-p)}{n}} \leq p \leq \hat{p} + 1.96\sqrt{\frac{p(1-p)}{n}}\right) = 0.95$$

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- Note that this confidence interval depends on the (unknown) value of p!

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- In other words, we are 95% confident that the interval $\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ contains the true population proportion p

given Wald estimate $p = \hat{p}$.

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- In general, an approximate two-sided $(1-\alpha)\cdot 100\,\%$ confidence interval for p is given by $\left(\hat{p} - z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

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- A one-sided upper $(1-\alpha)\cdot 100\,\%$ confidence interval for p is given by $\left(0,\ \hat{p} + z_{\alpha}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

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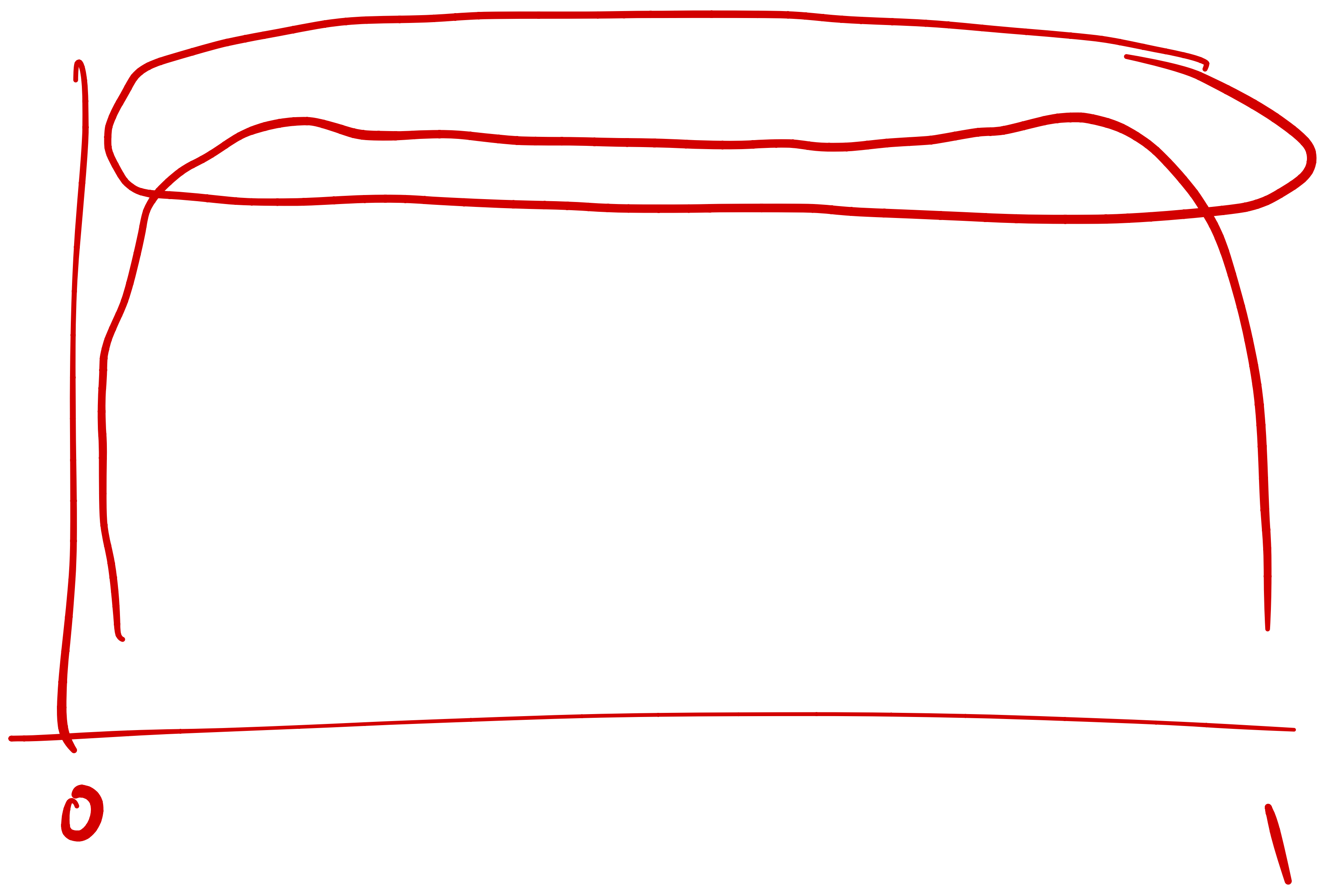

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  - Get an estimate of p that is a weighted average of $\hat{p}$ and $\dfrac{1}{2}$, where the weight on $\hat{p}$ increases with n
  - Better coverage!

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- Find $\hat{p}$ (our estimate of p): $\hat{p} = \frac{53}{62} = 0.855$

- Check normality assumptions: $n\hat{p} = 53$ $\quad n(1-\hat{p}) = 9$ $\quad \alpha = 0.05$

- Apply a two-sided 95% confidence interval: $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

"by hand" $(0.767, \; 0.943)$

prop.test $\rightarrow (0.75, \; 0.95)$

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- We are using the normal distribution as an approximation for a binomial distribution
- Normal approximation (Wilson) confidence intervals can be calculated in R using `prop.test(x,n)`
- Exact binomial (Clopper-Pearson) confidence intervals can be calculated in R using `binom.test(x,n)`

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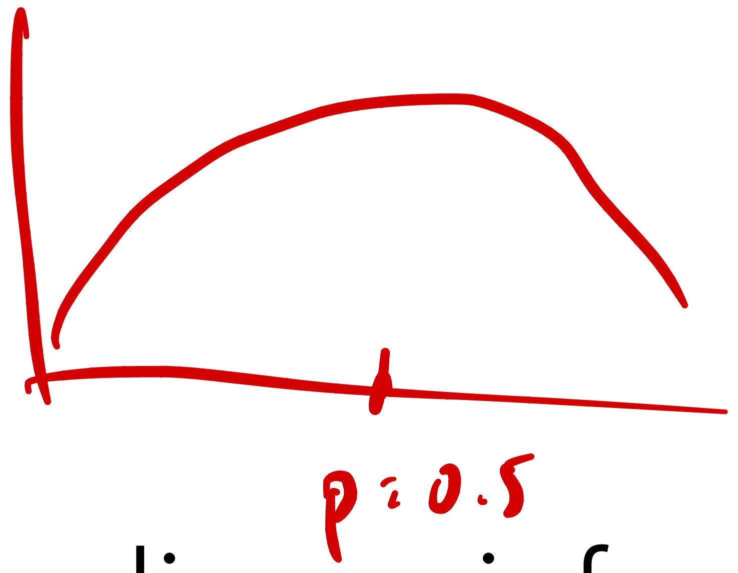

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- Otherwise, use $p = 0.5$ to get the most conservative estimate of the standard error (overestimate of the number of subjects needed)

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- Q1: How large of a sample should you take?

qnorm(0.975)

1.96

0.5   0.5

$m = 0.08$

$p = 0.5$

$$n = \left\lceil \frac{z_{\alpha/2}^2 \cdot p(1-p)}{m^2} \right\rceil = \underline{151}$$

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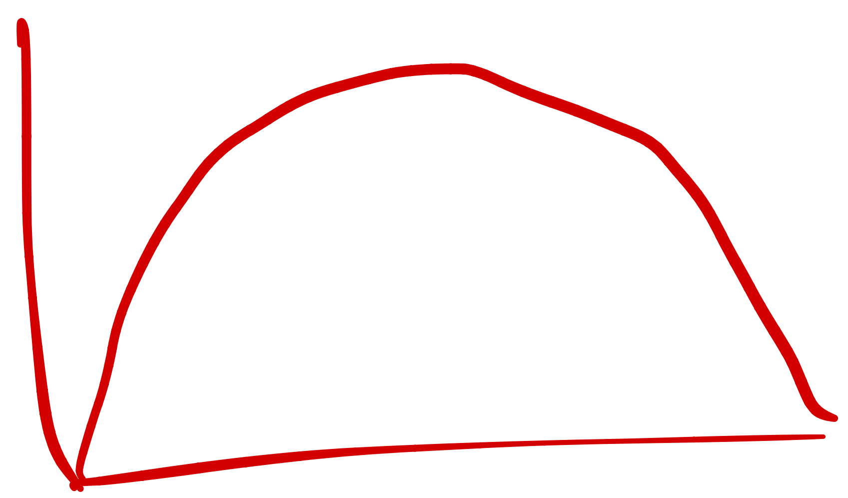

- Q2: A national study determined that 38% of all Americans own iPhones. Now, how large of a sample should you take?

$m = 0.08$

$p = 0.38$

$1-p = 0.62$

$$n = \left\lceil \frac{1.96^2 \cdot 0.38 \cdot 0.62}{0.08^2} \right\rceil = \cancel{142}$$

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- Draw a random sample of size n observations from the underlying population (each observation is a dichotomous yes/no)

$\hat{p} = \frac{x}{n}$

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- Draw a random sample of size n observations from the underlying population (each observation is a dichotomous yes/no)

- Calculate a z-statistic: $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1 - p_0)}{n}}}$

CI: $z = \dfrac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})}}$

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- If $p \leq 0.05$, we reject the null hypothesis and conclude that $p \neq p_0$
- If $p > 0.05$, we fail to reject the null hypothesis and conclude that there is not significant evidence to say that $p \neq p_0$

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- Test at the $\alpha = 0.05$ level whether the true population proportion of right-handed people is equal to 0.9

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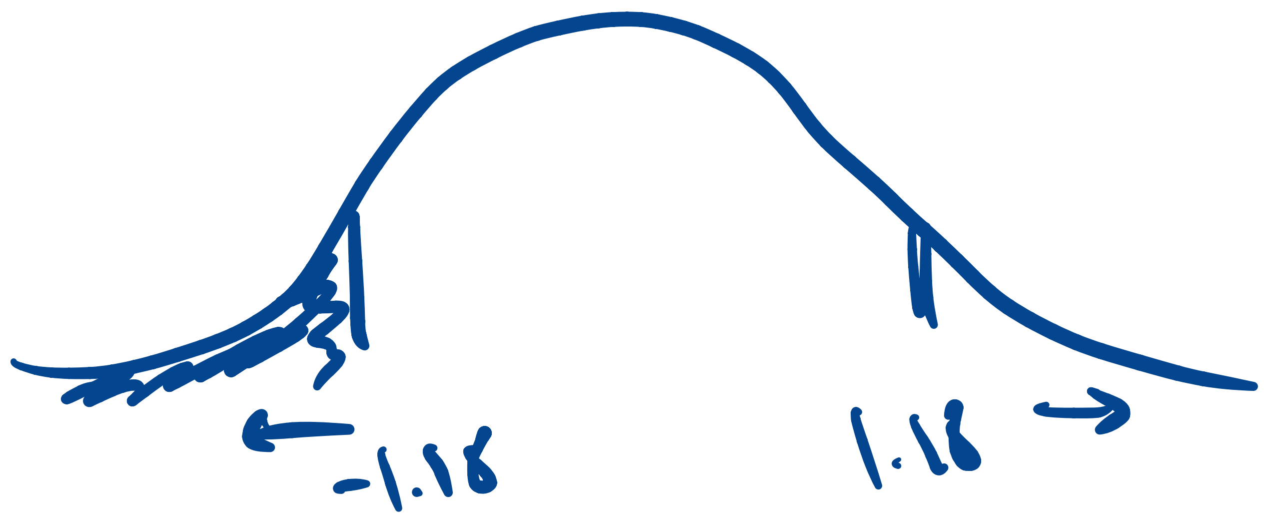
- $H_0 : p = 0.9$ vs. $H_1 : p \neq 0.9$

$$\boxed{\mu = p_0 , \quad \sigma = \frac{p_0(1-p_0)}{n}}$$

- Check normality assumptions based on p: $np = 62 \cdot 0.9 > 5 \qquad n(1-p) = 62 \cdot 0.1 = 6.2$
- Calculate z-score: $z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = \dfrac{0.855 - 0.9}{\sqrt{\dfrac{0.9 \times 0.1}{62}}} = -1.18$
- Calculate p-value:



$2 \cdot \text{pnorm}(-1.18) = 0.238$

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- For proportion (Wald) confidence intervals, we calculate the standard error based on $\hat{p}$ as $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ (i.e., our frame of reference is centered at our observed sample proportion)

$H_0: p = 0.3 \qquad H_1: p \neq 0.3$

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- As with confidence intervals, we can perform an exact test based on the binomial distribution instead of using the normal approximation: `binom.test(x,n,p=p₀)`

$\Pr(\hat{x} \mid p_0, n)$

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- For two-sided tests, we are concerned with the area in both tails of the distribution

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- However, we are sometimes interested in deviations only in one direction
  - $H_1 : p > p_0$
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- Analyses follow directly as they did for one-sided tests for sample means

Comparison of Two Proportions

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Comparison of Two Proportions

$\hat{p}_1 \sim N\left(p_1, \sqrt{\frac{p_1(1-p_1)}{n_1}}\right)$

$\hat{p}_2 \sim N\left(p_2, \sqrt{\frac{p_2(1-p_2)}{n_2}}\right)$

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  - $\hat{p} = \dfrac{x_1 + x_2}{n_1 + n_2} = \dfrac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$

    $\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \right.$ var: $\left(\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$

  - $\hat{p}$ is the weighted average of the two sample proportions (or total successes over total trials)

    $\sigma = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{(p_2(1-p_2))}{n_2}}$

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  - Standard error: $\sqrt{\hat{p}(1-\hat{p})\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}$
  - Similar to the "pooled" estimate for sample means

Comparison of Two Proportions

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2,\ \sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right)$$

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Wald

$p_1 \overset{?}{=} p_2$

$p(1-p)$

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- If n_1 and n_2 are sufficiently large, this z statistic is approximately a standard normal (mean 0, standard deviation 1)
- Typically, we want $n_1\hat{p}_1,\ n_1(1 - \hat{p}_1),\ n_2\hat{p}_2,$ and $n_2(1 - \hat{p}_2)$ to all be greater than 5 (this is a conservative standard)

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- If $p \leq \alpha$, we reject the null hypothesis
- If $p > \alpha$, we fail to reject the null hypothesis

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    - $H_0 : p_A = p_B$ vs. $H_1 : p_A \neq p_B$

- We take samples of $n_A = 54$ and $n_B = 62$

- We observe $x_A = 48$ and $x_B = 60$ subjects being right handed

Comparison of Two Proportions

Comparison of Two Proportions

- Calculate proportions:

$$\hat{p}_A = \frac{48}{54} = 0.889$$

$$\hat{p}_B = \frac{60}{62} = 0.968$$

Comparison of Two Proportions

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0.889

0.968

- Is this difference too large to be attributed to chance?

Comparison of Two Proportions

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$\hat{p}_A = .889$

$\hat{p}_B = 0.968$

- Is this difference too large to be attributed to chance?

- Under H_0, $p_A = p_B$, so we can estimate their common value p

$$p = \frac{X_A + X_B}{n_A + n_B} = \frac{48+60}{54+62} = \frac{108}{116} = 0.931$$

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Comparison of Two Proportions

- Checking normality assumptions:
  - $n_A p_A = 48 > 5$
  - $n_A(1 - p_A) = 6 > 5$
  - $n_B p_B = 60 > 5$
  - $n_B(1 - p_B) = 2 < 5 \implies$ proceed with caution

Comparison of Two Proportions

Comparison of Two Proportions

- Calculate z-statistic:

Comparison of Two Proportions

$H_0: p_A - p_B = 0$

- Calculate z-statistic:

$$z = \frac{(\hat{p}_A - \hat{p}_B) - \cancel{(p_A - p_B)}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}} = \frac{0.889 - 0.963}{\sqrt{0.931(0.069)\left(\frac{1}{54} + \frac{1}{62}\right)}}$$

$\hat{p}$: 0.931

$= -1.675$

$p = 2 \cdot Pr(Z < -1.675) = \underline{0.09}$

- Conclusion:

$\alpha = 0.01$

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- As in the one-sample case, the standard error is not the same for the confidence interval and hypothesis test

- For a two-sided confidence interval, we are $(1-\alpha)\cdot 100\,\%$ confident that the interval $\left(\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{n_2}}\right)$ contains the true population difference, $p_1 - p_2$

$\sqrt{p(1-p)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$

(Wald)

Miettinen - Nurminen (MN)

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$$\left(\underset{\hat{p}_A}{0.889} - \underset{\hat{p}_B}{0.968}\right) \pm 1.96\sqrt{\frac{0.889 \cdot 0.111}{54} + \frac{0.968 \cdot 0.032}{62}}$$

$$= (-\underline{0.173}, \underline{0.016})$$

$$\sqrt{\frac{p_A(1-p_A)}{n_1} + \frac{p_B(1-p_B)}{n_2}}$$

Comparison of Two Proportions

- Continuing with our dominant hand by group example, we can construct a two-sided 95% confidence interval for $p_A - p_B$ as follows:

$$(0.889 - 0.968) \pm 1.96\sqrt{\frac{0.889 \cdot 0.111}{54} + \frac{0.968 \cdot 0.032}{62}}$$

$$= (-0.173, 0.016)$$

- We are 95% confident that the interval (-0.173, 0.016) contains the true difference in the proportion of members of Group A and Group B who are right-handed

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- One tail, lower bound:

Comparison of Two Proportions

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$$\left(\hat{p}_1 - \hat{p}_2 - z_\alpha \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}},\ 1\right)$$

Comparison of Two Proportions

- One tail, lower bound:

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- One tail, upper bound:

Comparison of Two Proportions

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- One tail, upper bound:

$$\left(-1,\ \hat{p}_1 - \hat{p}_2 + z_\alpha \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}\right)$$

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$$z_{1-\alpha} = \frac{\bar{x} - \mu}{\sqrt{\sigma/n}}$$

$$z_{1-\alpha} \cdot \sqrt{\sigma/n} + \mu_0 = \bar{x}$$

$$z_\beta = \frac{\bar{x} - \mu_1}{\sqrt{\sigma/n}}$$

$$z_\beta \cdot \sqrt{\frac{\sigma}{n}} = \mu_0 + z_{1-\alpha} \cdot \sqrt{\sigma/n} - \mu_1$$

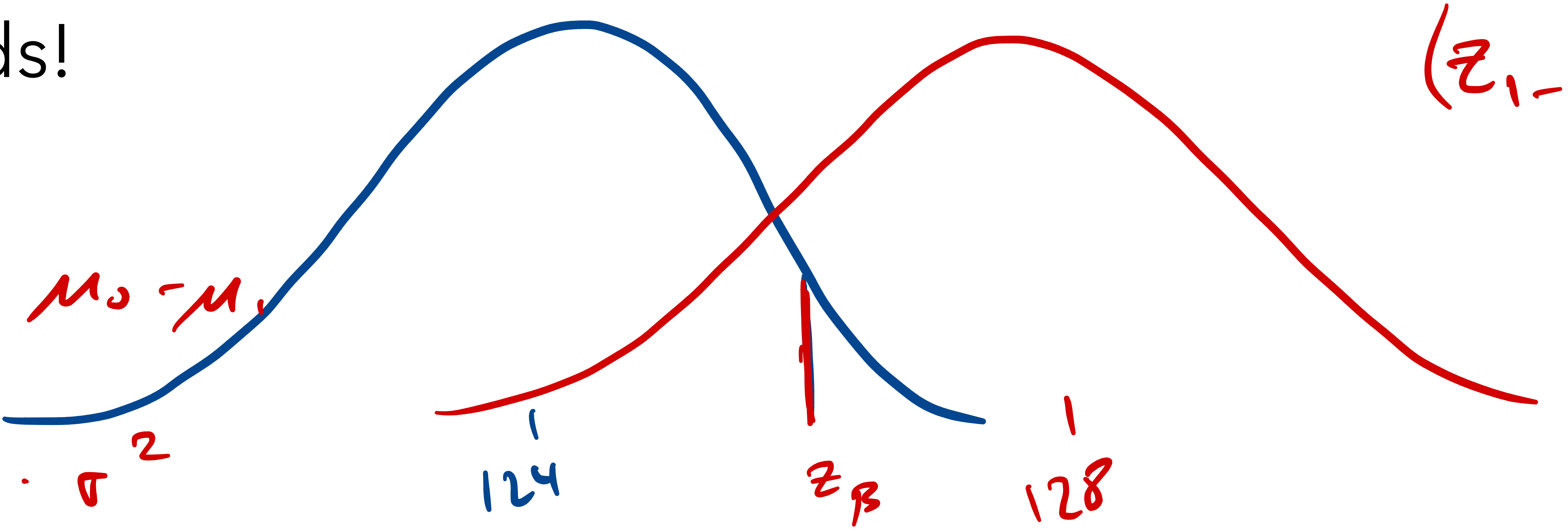
$$(z_\beta - z_{1-\alpha}) \sqrt{\frac{\sigma}{n}} = \mu_0 - \mu_1$$

- https://forms.gle/Zt3Qzrb7S7UXFXY28
- If you fill it out: +2.5% on midterm
- If at least 90% of the class fills it out: +2.5% on each person's midterm
  - Tell your friends!

$$(z_{1-\alpha} + z_{1-\beta})$$

$$(z_\beta + z_\alpha) \sqrt{\frac{\sigma}{n}} = \mu_0 - \mu_1$$

$$n = \left(\frac{z_\alpha + z_\beta}{\mu_0 - \mu_1}\right)^2 \cdot \sigma^2$$



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$\bar{X} \leftarrow \alpha$

plot(mu1, some function of mu1, ...)

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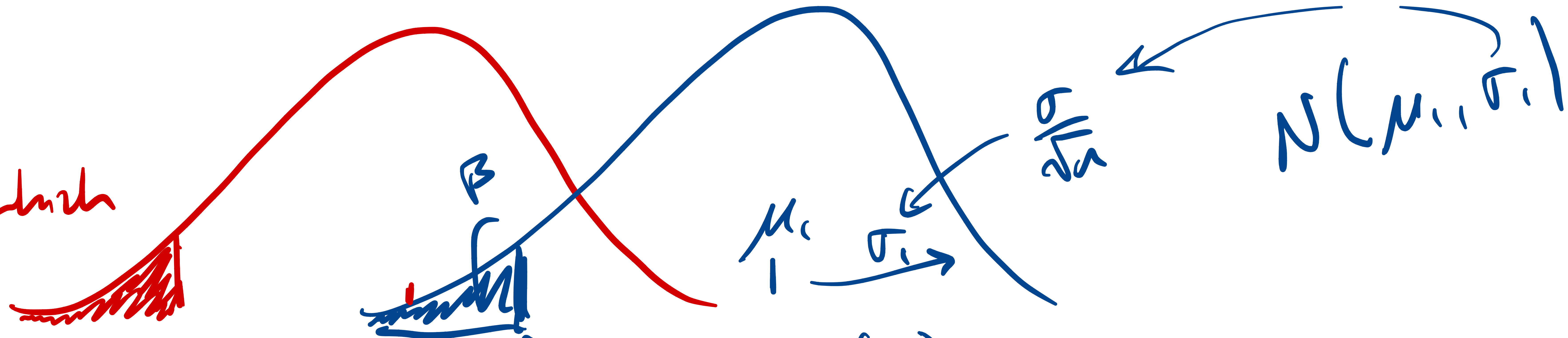

z_α is value for cutoff

$Pr(Z < z_\alpha) = \alpha$

z_α = qnorm(α)

z_β = qnorm(β)

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