Important; please read before starting:

- You must **show your work for all questions** in order to receive full credit. Showing your work will also let you receive partial credit for problems.
- For all hypothesis tests and confidence intervals, **you must demonstrate that you know how to solve the question without using any `.test()` functions in `R`**. You may use these functions to check your answers, but writing them down will not count as "showing work."

1. Let A be a random variable with a PDF of $f_A(x) = ax^2$ for $x \in [-1, 1]$.
   (a) Find a.

   (b) Find $\mathrm{E}(A)$ and $\mathrm{Var}(A)$.

   (c) Let B be an independent random variable that follows a Uniform distribution over the domain $[0, 4]$. Now, define $X = 2A + B$. What are $\mathrm{E}(X)$ and $\mathrm{Var}(X)$?

   (d) If we were to draw $n = 49$ samples according to X's distribution, what is the distribution of the sample mean?

(e) We draw a sample of size $n = 49$ according to an unknown distribution. The sample mean is $\overline{x} = 0.5$ and the sample variance is $s^2 = 4$. What is a two-sided 95% confidence interval for the true mean μ?

(f) At $\alpha = 0.1$, is the true mean greater than 0? State which test you should use, your hypotheses, test statistic, p-value, and conclusion.

(g) What is the power of the test in part (f) if we assume an alternate hypothesis of $\mu_1 = 1$?

(h) How would increasing the sample size change your answer to part (g)? Explain.

2. Probability and combinatorics.

   (a) If $\Pr(D) = 0.5$ and $\Pr(C \cap D) = 0.3$, what is $\Pr(C|D)$?

   (b) You are given three random variables A, B, and C. If you know that A and B are (pairwise) independent, A and C are (pairwise) independent, and B and C are (pairwise) independent, are A, B, and C all (mutually) independent? Explain.

   (c) If D and E are mutually exclusive events, are they independent? Explain.

   (d) Suppose candies can either be chocolate or non-chocolate. Alice finds 60% of chocolate candy tasty and 40% of non-chocolate candy tasty. If 30% of all candies are chocolate, what is the probability that a candy is chocolate given that Alice finds it tasty?

(e) (6 points) There are twenty people in a village. Five of these people own cars. Five randomly picked people will win a bicycle. Due to fairness constraints, the leader must ensure that **exactly** four of the five winners must not own a car. In how many ways can this choice be made?

(f) (4 points) Given the same setup as above, what if the leader must ensure that **at least** four of the five winners must not own a car? In how many ways can this choice be made now?

3. Starbucks is very interested in finding out what fraction of people have each season as their favorite so they can release new specialty drinks.

   (a) The CEO of Starbucks believes that 40% of people like summer the most, 30% like fall the most, 10% like winter the most, and 20% like spring the most. The marketing department takes a sample of $n = 300$ people and records their favorite season in the table below. At the $\alpha = 0.01$ significance level, use an appropriate statistical test to determine whether the CEO's belief is correct.

Summer	Fall	Winter	Spring
110	110	20	60

      i. (3 points) State which test you should use, along with your null and alternative hypotheses.

      Test:

      H_0 :

      H_1 :

      ii. (3 points) Calculate the expected number of people who have each season as their favorite:

Summer	Fall	Winter	Spring

      iii. (6 points) Calculate the appropriate test statistic. Show your work (i.e., demonstrate that you know the formula for calculating the test statistic).

      iv. (3 points) Calculate the p-value. Show your work and write the R code for calculating the p-value.

      v. (2 points) What conclusion do you draw from this test?