Chaper 5 - Distributions

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目录

1. General Knowledge

1.1. Expectation - the population mean

Expected value of X, denoted $E(X)$, represents a theoretical average of an infinitely large sample

for discrete variable $E(X) = \sum_{x \in S_X} x \cdot Pr(X = x)$

for continuous variable $\int_{-\infty}^{\infty} X f_X(X)\, dX$

1.2. Variance - measure the dispersion of values from the expectation(mean)

$var(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - E(X)^2$

for the case of continuous variable $\int_{-\infty}^{\infty} (X - \mu)^2 f_X(X)\, dX$

1.3. Probability Distribution

For any $E \subseteq S_X$, we can define $p_X(E) = Pr(X \in E)$, Then $\sum\limits_{x \subseteq S_X} Pr(X = x) = 1$

1.4. Covariance

$cov(X, Y) = E(XY) - E(X)E(Y)$

how to get that (hint: $\mu_X = E(X)$ and $\mu_Y = E(Y)$, and they are considered as constant):

$cov(X,Y) = E((X-\mu_X)(Y-\mu_Y))$
$= E((XY - Y\mu_X - X\mu_Y + \mu_X \cdot \mu_Y))$
$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + E((\mu_X \mu_Y))$
$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$
$= E(XY) - E(X)E(Y)$

1.5. Correlation

$corr(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY)-E(X)E(Y)}{\sigma_X \sigma_Y}$

1.6. Linear transformation

Let $Z = aX + bY$

Then the mean of Z is $\mu_Z = a\mu_X + b\mu_Y = aE(X) + bE(Y)$

The variance of Z is $\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y$

The standard deviation of Z is $\sigma_Y = \sqrt{a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y}$

1.7. General transformation

1. If $Y = g(X)$, $f(X) = p_X$ then $E(Y) = E(g(X)) = \int g(X) \cdot f(X)\, dX$
2. if $Y = g(X)$, we **don't** necessarily get $E(g(X)) = g(E(X))$

2. Theoretical Distributions

Theoretical probability distributions describe what we expect to happen based on populations on a theoretical level

2.1. The following theoretical distributions will be considered in this class (D = discrete, C = continuous):

- Bernoulli distribution (D)
- Binomial distribution (D)

- Poisson distribution (D)
- Geometric distribution (D)
- Uniform distribution (C)
- Exponential distribution (C)
- Normal distribution (C)

2.2. Bernoulli Distribution 伯努利分布

1. Let Y be a dichotomous random variable (takes one of two mutually exclusive values)
2. Successes ($= 1$) occur with probability p and failures ($= 0$) occur with probability $1 - p$, for constant $p \in [0, 1]$
3. Notation: $Y \sim Bern(p)$
4. Let Y be a dichotomous random variable representing a coin flip
   - $Y = 1$: heads, success
   - $Y = 0$: tails, fail
   - If the coin has a 60% chance to get the head/success
   - $E(Y) = 1 \cdot p + 0 \cdot (1 - p) = p$
   - $E(Y^2) = 1^2 \cdot (p) + 0^2 \cdot (1 - p) = p$
   - $var(Y) = \sigma_Y^2 = E(Y^2) - E(Y)^2 = p - p^2 = p(1 - p)$

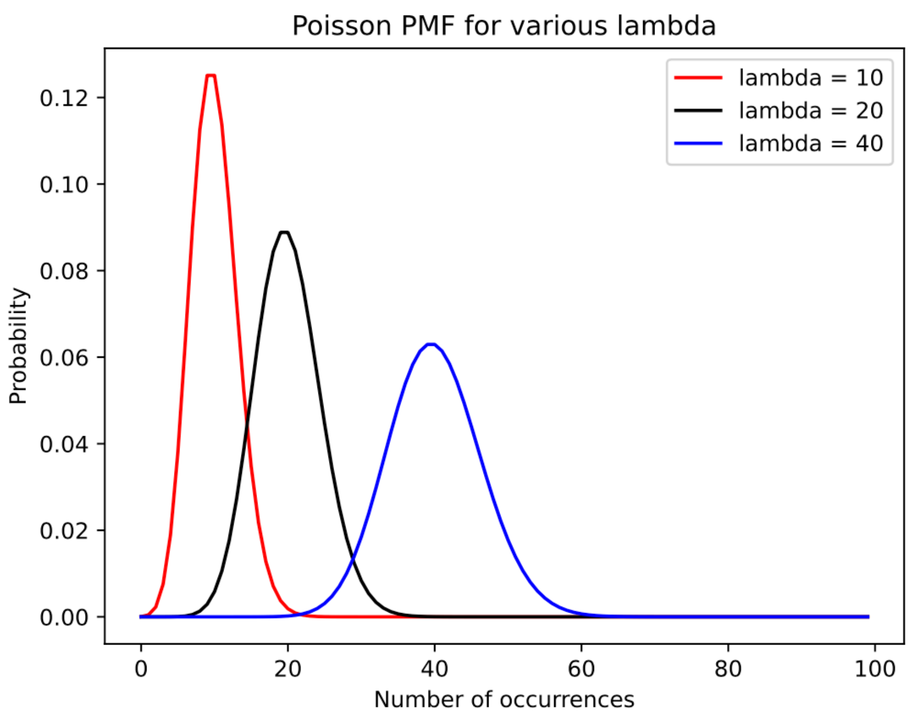
2.3. Binomial Distribution 二项分布

1. Definition: If we have a sequence of n Bernoulli variables, each with a probability of success p, then the total number of successes is a binomial random variable.
   - Assumptions: fixed number of trials, independent, constant p
2. Notation: $X \sim Bin(n, p)$
3. Note for *Combination* and *Permutation*
   1. Combination: $C(n, k)$ or $\binom{n}{k}$
   2. Permutation: $P(n, k)$

4. Probability Mass Function:
   1. $Pr(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$
   2. $Pr(X = x) = C(n, k) \cdot p^x \cdot (1-p)^{n-x}$
5. Then if you flip coin for 100 times, $n = 100$, the probability to get head for k times is $Pr(X = x) = C(100, k) \cdot p^k (1-p)^{100-k}$
6. How do you calculate it in ***R***?
   1. Calculate the probability of x successes $Pr(X = x)$ using dbinom(x, n, p)
   2. Calculate $Pr(X \leq x)$ using pbinom(x, n, p)
   3. Calculate $Pr(X \geq x)$ using 1 - pbinom(x - 1, n, p)
7. Summary measures
   1. Expection $E(X) = np$
   2. Variance $var(X) = \sigma_X^2 = np(1-p)$
   3. Stdev $\sigma_X = \sqrt{np(1-p)}$
8. How do you get those above:
   1. Consider Binomial Distribution as the sum of n times of Bernoulli Experiments
   2. When $X \sim Bern(p)$
      1. $E(X) = p$
      2. $\sigma_X^2 = p(1-p)$
   3. Then let $Y \sim Bin(n, p)$
      1. $E(Y) = np$
      2. $\sigma_Y^2 = n\sigma_X^2 = np(1-p)$
9. Main take-away points from the binomial distribution:
   1. Fixed number of independent Bernoulli trials, n
   2. Constant probability of success, p (Bernoulli parameter)
   3. Interested in the total number of successes in n trials (not order)
   4. Mean: $\mu_X = np$
   5. Variance: $\sigma^2 = np(1-p)$

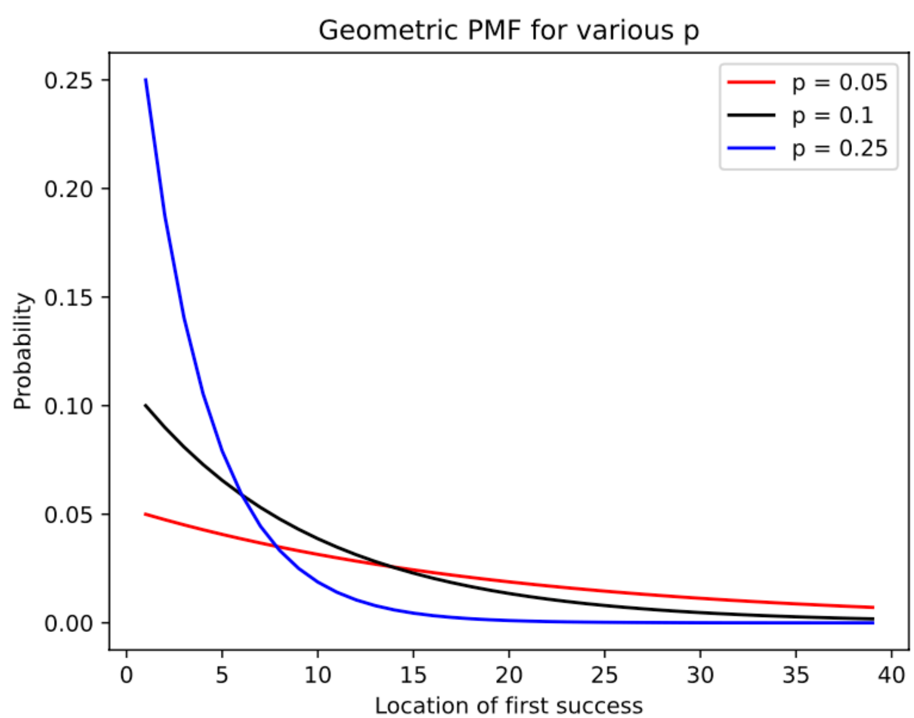
2.4. Poisson Distribution 泊松分布

1. Probability function is given by $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$
2. If $X \sim Pois(\lambda)$, then $\mu_X = \sigma_x^2 = \lambda$
3. Example problem in class slides
   - setup: on average, 1.95 people develop the disease per year
   - Q1: probability of no one developing the disease in the next year
     - $\lambda = 1.95 = \mu_X = \sigma_X^2$
     - $x = 0$
     - $p = \frac{e^{-\lambda}\lambda^x}{x!} = (e^{-1.95} * (1.95)^0/0!) = e^{-1.95}$
     - in R: exp(-1.95) = 0.1422741
   - Q2: probability of one person developing the disease in the next year
     - $x = 1$
     - $p = \frac{e^{-\lambda}\lambda^x}{x!} = (e^{-1.95} \cdot (1.95)^1/1!) = e^{-1.95} \cdot (1.95)$
     - in R: exp(-1.95) * (1.95) = 0.2774344



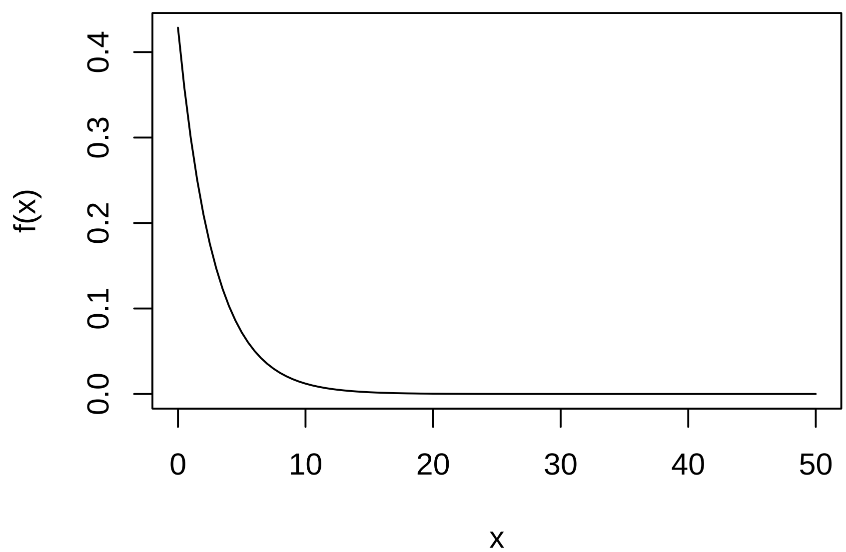
2.5. Geometric Distribution 几何分布

1. Suppose $Y_1, Y_2, \ldots$ is an infinite sequence of independent Bernoulli random variables with parameter p
2. Let X be the first index i for which $Y_i = 1$ (location of first success)
3. PMF: $P(X = x) = p(1-p)^{x-1}$
4. plain English: what is the probability to take x times to get the first success, given that the Bernoulli parameter is p, or the success rate is p.
5. Notation: $X \sim Geom(p)$



6. if $p = 0.3$, draw PMF for $x \in [0, 40]$

```
p = 0.3
f <- function(x) {
    return(p * (1 - p)^(x - 1))
}
curve(f, from = 0, to = 50)
```



7. Mean $E(X) = \frac{1}{p}$
8. Variance $\sigma^2 = \frac{1-p}{p^2}$
9. **Why??** CDF $P(X \leq x) = 1 - (1-p)^x$ (1 minus the probability that the first x trials all failed?)

2.6. Uniform Distribution (Continuous)

1. PDF:
$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$
2. Why $f(x) = \frac{1}{b-a}$? Because only by that $\int_a^b f(x)dx = 1$
3. Notation: $X \sim Unif(a, b)$
4. $\mu = \frac{a+b}{2}$, $\sigma = \frac{(b-a)^2}{12}$

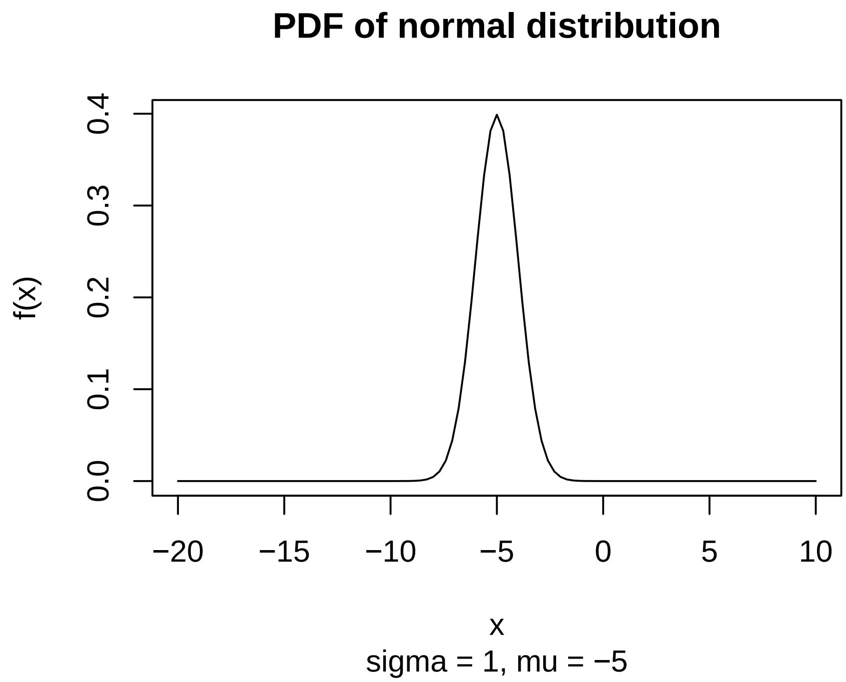
2.7. Exponential Distribution (Continuous)

1. PDF: $f_X(x) = \lambda e^{-\lambda x}$, $\lambda > 0$
2. Notation: $X \sim Exp(\lambda)$
3. $\mu = 1/\lambda$, $\sigma^2 = 1/\lambda^2$
4. CDF: $F_X(x) = 1 - e^{-\lambda x}$

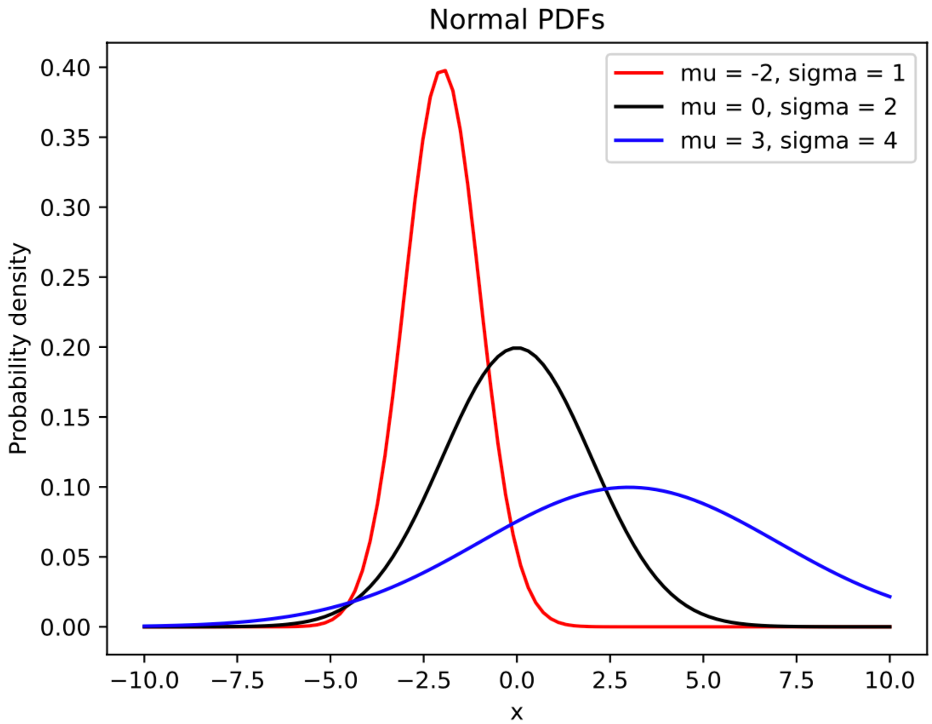
2.8. Normal Distribution (Continuous)

1. The most common continuous distribution is the normal distribution (also called a Gaussian distribution or bell-shaped curve)
   - Shape of the binomial distribution when p is constant but $n \to \infty$
   - Shape of the Poisson distribution when $\lambda \to \infty$
2. **PDF**: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
3. Notation: $X \sim N(\mu, \sigma^2)$, note that in R, use stdev instead of variance
4. Mean = median = mode = μ, variance = σ^2, standard deviation = σ

```
sigma = 1
mu = -5
f <- function(x) {
    return(1/(sqrt(2 * pi) * sigma) * exp(-0.5 * ((x - mu)/sigma)^2))
}
curve(f, from = -20, to = 10)
title(main = "PDF of normal distribution", sub = "sigma = 1, mu = -5")
```

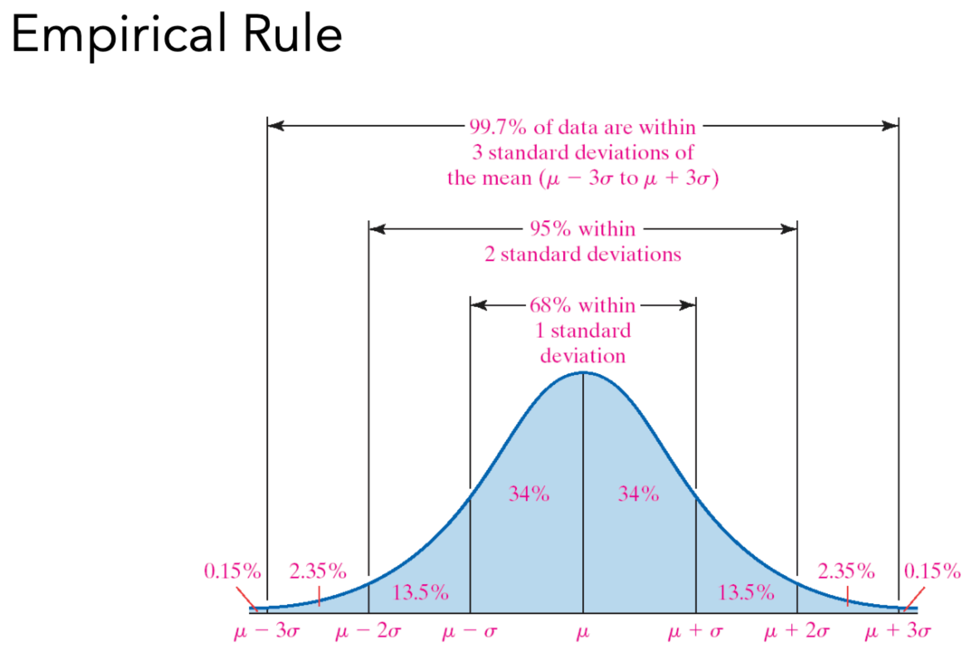


5. When $\mu = 0$ and $\sigma^2 = 1$, we have the standard normal distribution.



6.

7. Z score of X when $X \sim N(\mu, \sigma)$
   - definition of Z score: $z = \frac{x-\mu}{\sigma}$
   - When X follows Normal distribution, always $Z \sim N(0, 1)$
   - Usage example: when μ and σ are known, how do we know the probability that $x \leq a$
     - $z = (a - \mu)/\sigma$, $Z \sim N(0, 1)$
     - $P = pnorm((a - \mu)/\sigma)$

8. 

9. Does empirical rule work well for Z score?

- $\Pr(-1 \leq Z \leq 1) = 0.683$
- $\Pr(-2 \leq Z \leq 2) = 0.954$
- $\Pr(-3 \leq Z \leq 3) = 0.997$

10.

Normal Distribution: Example

- Setup: Let X be a random variable that represents weights of patients in American hospital EDs; X is normally distributed with $\mu = 160$ and $\sigma = 15$
- Q1: Find the probability that a randomly selected patient in the ED weighs between 140 pounds and 210 pounds

  Find z-scores: $z = \dfrac{x-\mu}{\sigma}$, so $z_1 = \dfrac{140-160}{15} = -4/3$ and $z_1 = \dfrac{190-160}{15} = 2$

  `pnorm(2) - pnorm(-4/3) = 0.886`

- Q2: Find the value that cuts off the upper 10% of the curve in American ED patient weights

  Find z-score: $z_{0.9} =$ `qnorm(0.9) = 1.282` $= \dfrac{x-160}{15}$

  $x = 160 + 1.282 \cdot 15 = 179.2$

  pnorm(): give z score or value, calculate probability
  qnorm(): give percentile, calculate the corresponding z score (if you did not give it mean and sd)

11.

2.9. Central Limit Theorem(CLT) and Sampling Distribution

1. **Sampling distribution**: If $X \sim N(\mu, \sigma)$, then $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$
2. Central Limit Theorem(CLT): If the population we are sampling from is not normal, then the shape of the distribution of $\overline{X}$ will be normal as long as n is sufficiently large (typically $n \geq 30$ suffices).
3. Therefore, when n is large enough, even X does not follow normal distribution, $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$
4. Then the Z score of sampling mean is $Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$, also, $Z \sim N(0,1)$.

2.10 Sampling Distribution of a Proportion

1. Suppose we are interested in the proportion of the time that an event occurs
2. If we take a sample of size n and observe x successes, then we could estimate the population proportion p by $\hat{p} = x/n$.

3. When $np \geq 5$ or $n(1-p) \geq 5$, it is considered that $\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$.

Sampling Distribution of a Proportion: Example

- Setup: Suppose 20% of Americans favor Advil as a pain reducer. A polling organization takes a sample of 100 Americans and asks if they prefer Advil or some other pain relief medicine.
- Q1: What is the mean of this sample proportion?
  $\mu = 0.20$
- Q2: What is the standard error of this sample proportion?
  $\sqrt{\frac{0.2(1-0.2)}{100}} = 0.04$
- Q3: What distribution does the sample proportion follow?
  $np = 20 > 5$, and $n(1-p) = 80 > 5$, so by CLT, $\hat{p} \sim N(0.2, 0.04)$
- Q4: What is the probability that the sample proportion is less than 18%?
  $\Pr(\hat{p} < 0.18) = \Pr(Z < (0.18 - 0.2)/0.04) = \Pr(Z < -0.5) \approx 0.31$
- Q5: What is the 20th percentile of the distribution of the sample proportion?
  $z_{0.20} = \frac{x - \mu}{\sigma} \rightarrow x = 0.2 + (-0.84) \cdot 0.04 \approx 0.167$

4.