Chapter 4: Probability and Combinatorics

DSCC 462

Computational Introduction to Statistics

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Probability

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  - Draw a card
  - Roll a die
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- Probability is the mathematics of random occurrences

Events

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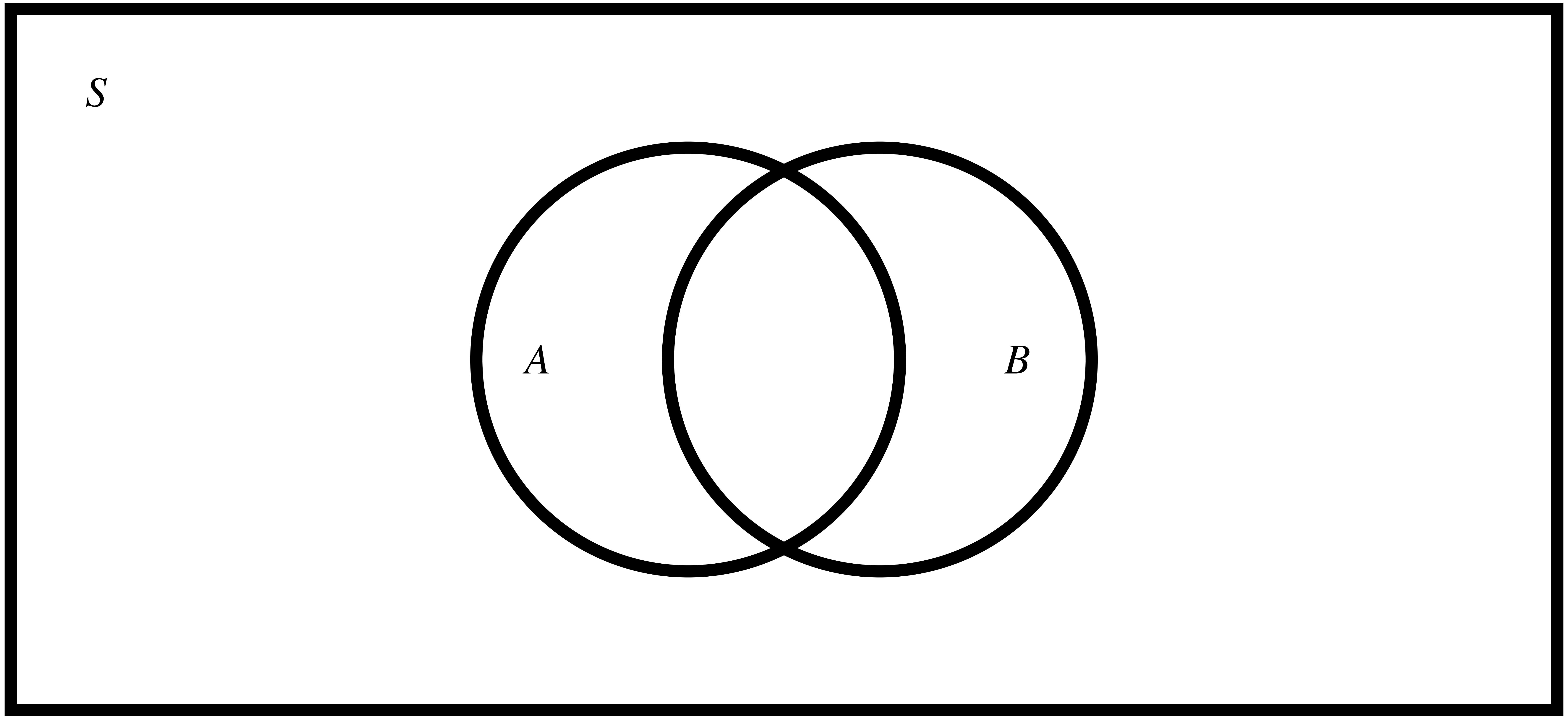
- Example: $A = \{\text{roll an even number on a six-sided die}\} = \{2,4,6\}$

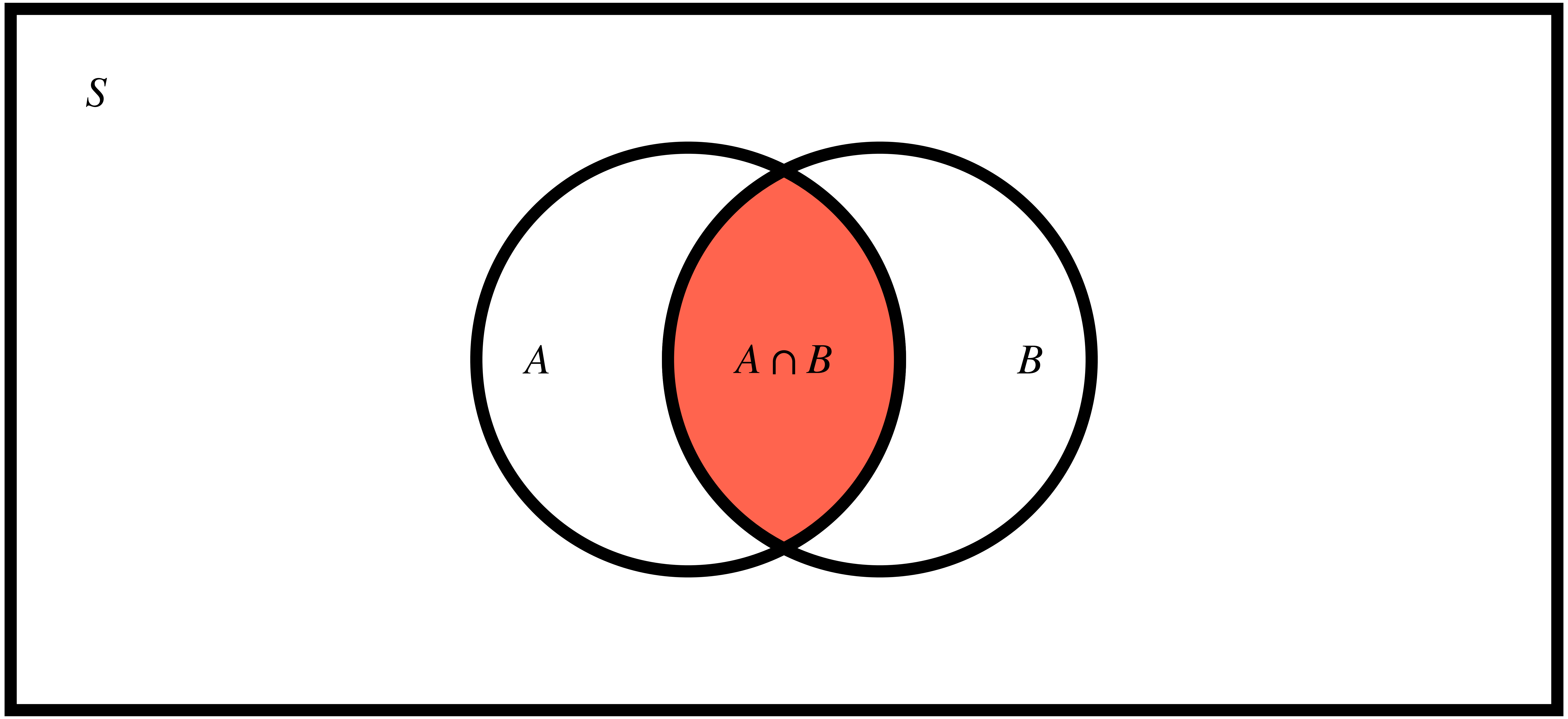
Operations on Events

- Let A and B be events, or subsets of S, where $A \subset S$ and $B \subset S$



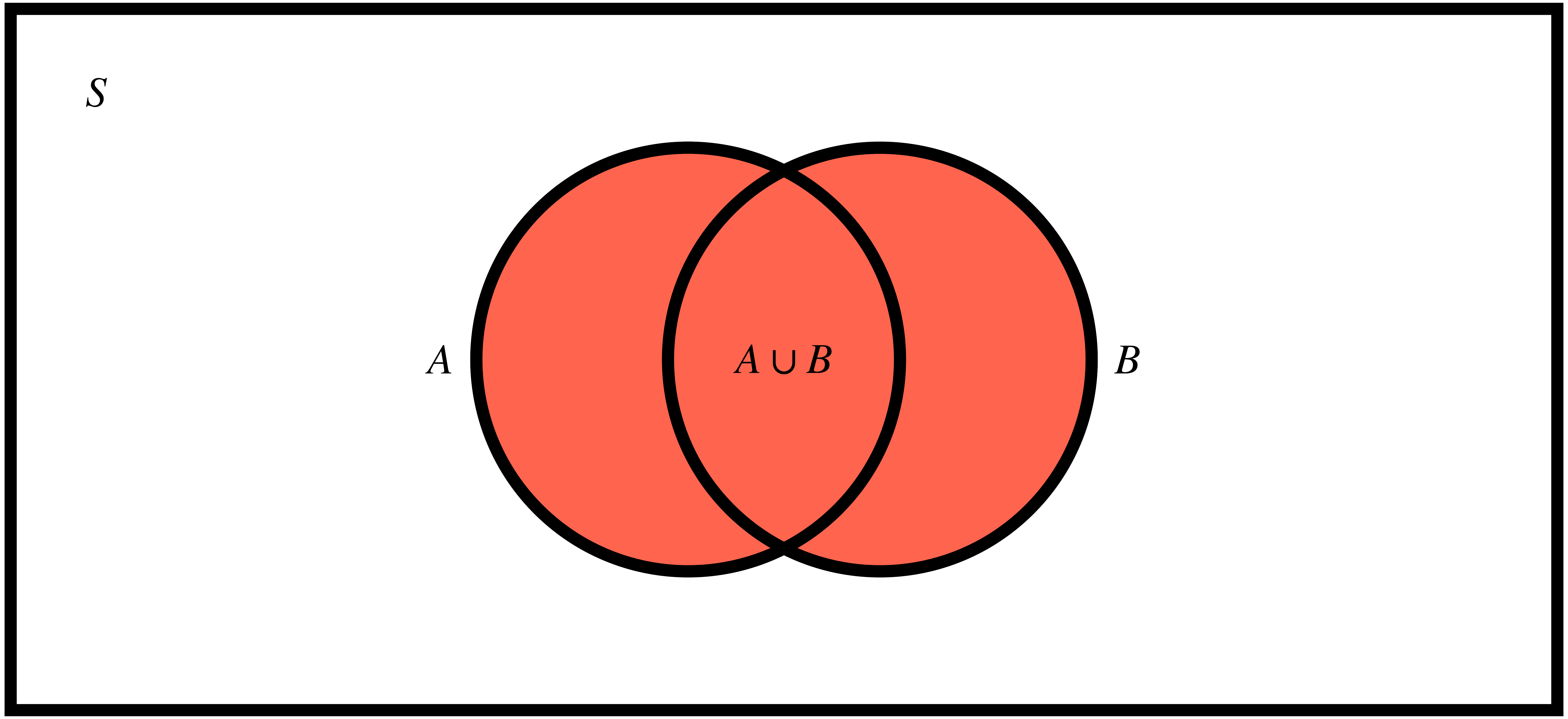
Intersection

- Intersection ($A \cap B$): The event "both A and B", or all elements in S in both A and B



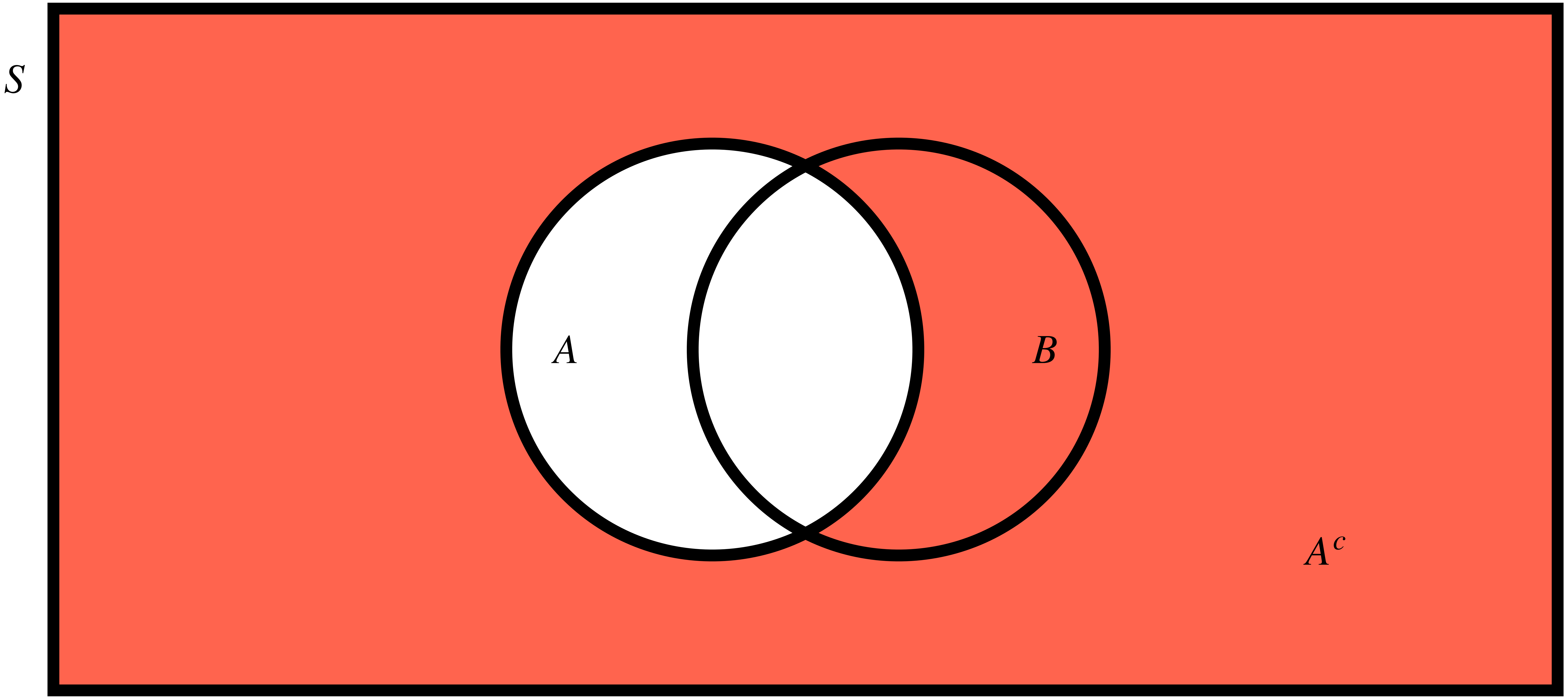
Union

- Union ($A \cup B$): The event "either A or B", or all elements in S in either A or B



Complement

- Complement (A^c, $\overline{A}$, or A'): The event "not A", or all elements in S not in A



Operations Example

Operations Example

- Suppose we have the following, where $A \subset S, B \subset S,$ and $C \subset S$:

$$S = \{1,2,3,4,5,6,7,8\}$$

$$A = \{1,2,3,4\}$$

$$B = \{2,4,6,8\}$$

$$C = \{7,8\}$$

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- Evaluate the following expressions:

$$A \cap B = \{2,4\}$$

$$(A \cup C) \cap B = \{2,4,8\}$$

$$A^c \cap C = \{7,8\}$$

$$(A \cap B^c) \cup C = \{1,3,7,8\}$$

Operations on Events: De Morgan’s Laws

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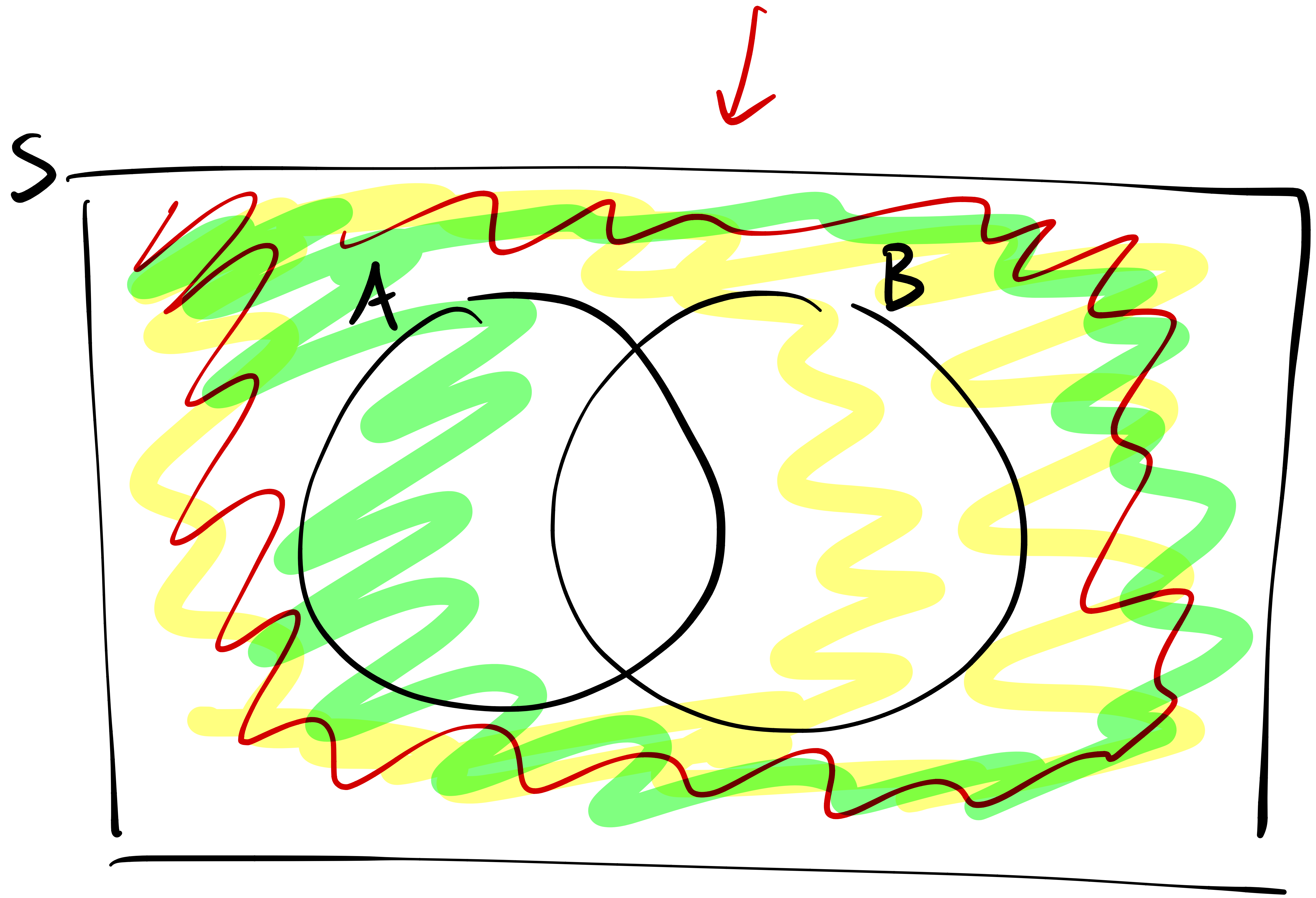

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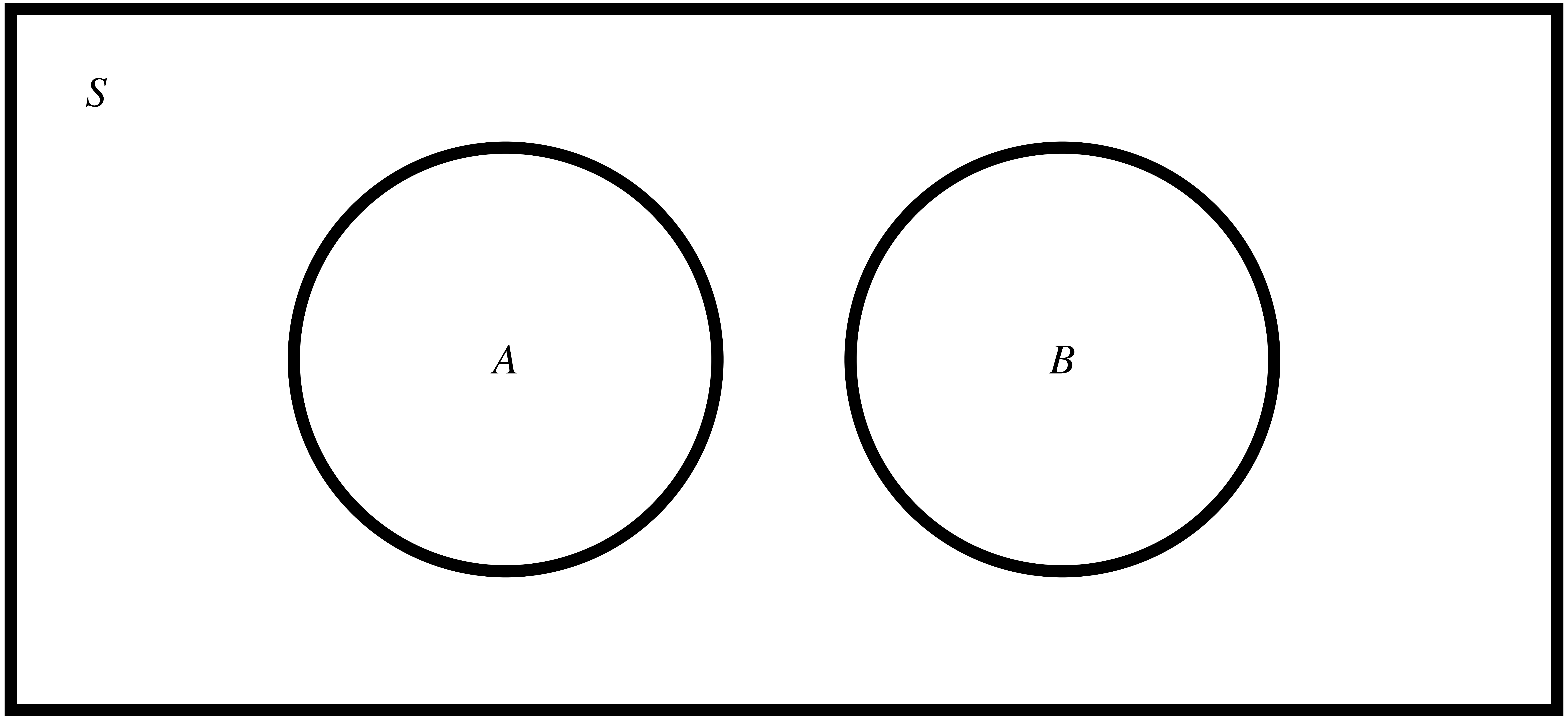
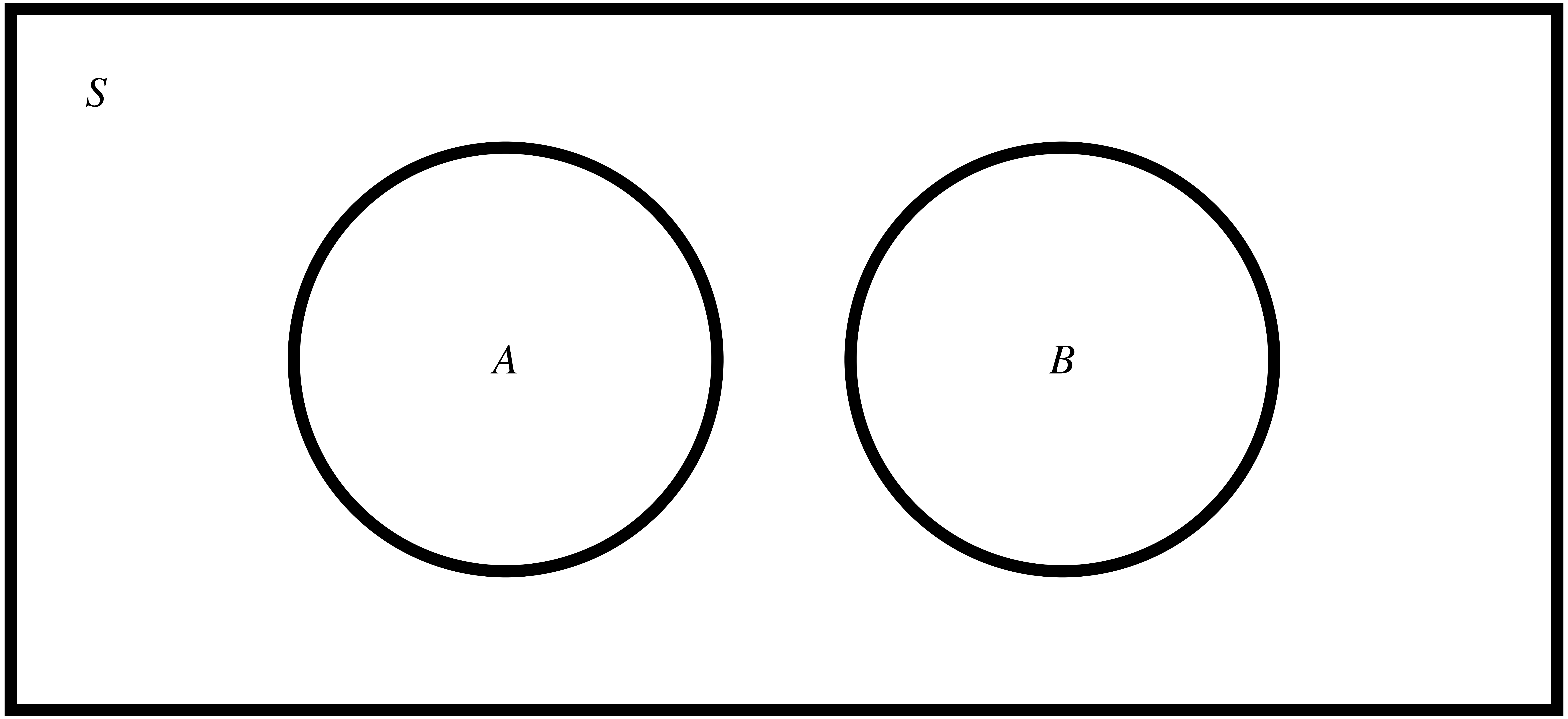


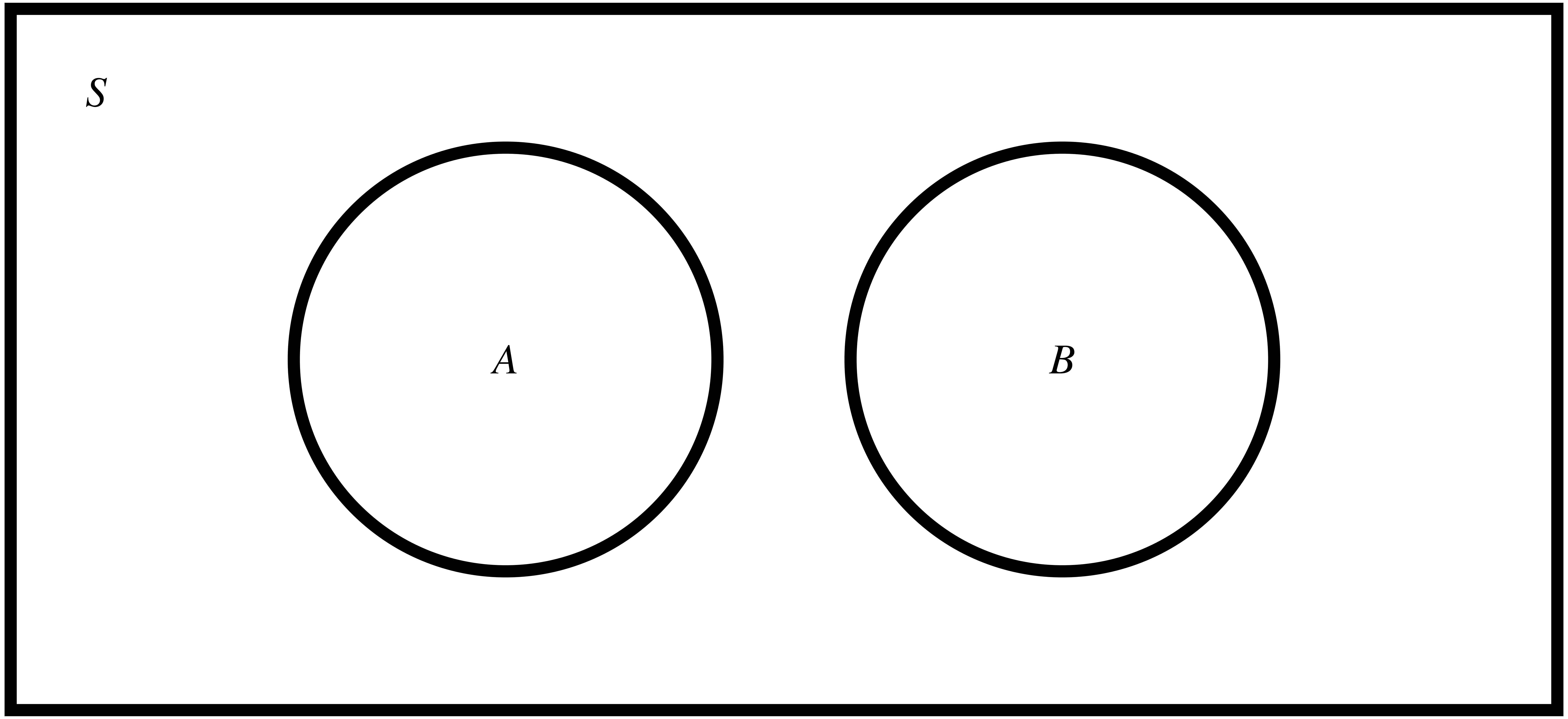
Events

- Null events are events that can never occur, represented as $\varnothing$



Events

- Null events are events that can never occur, represented as $\varnothing$
- Disjoint or mutually exclusive events are events that cannot occur simultaneously; A and B are disjoint if and only if $A \cap B = \varnothing$



Cardinality

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- Three types of cardinality:
  - Finite: $|A| < \infty$
  - Countable: $|A| = \infty$ but elements can be listed as $x_1, x_2, \ldots$
  - Uncountable: $|A| = \infty$ and elements cannot be listed as $x_1, x_2, \ldots$

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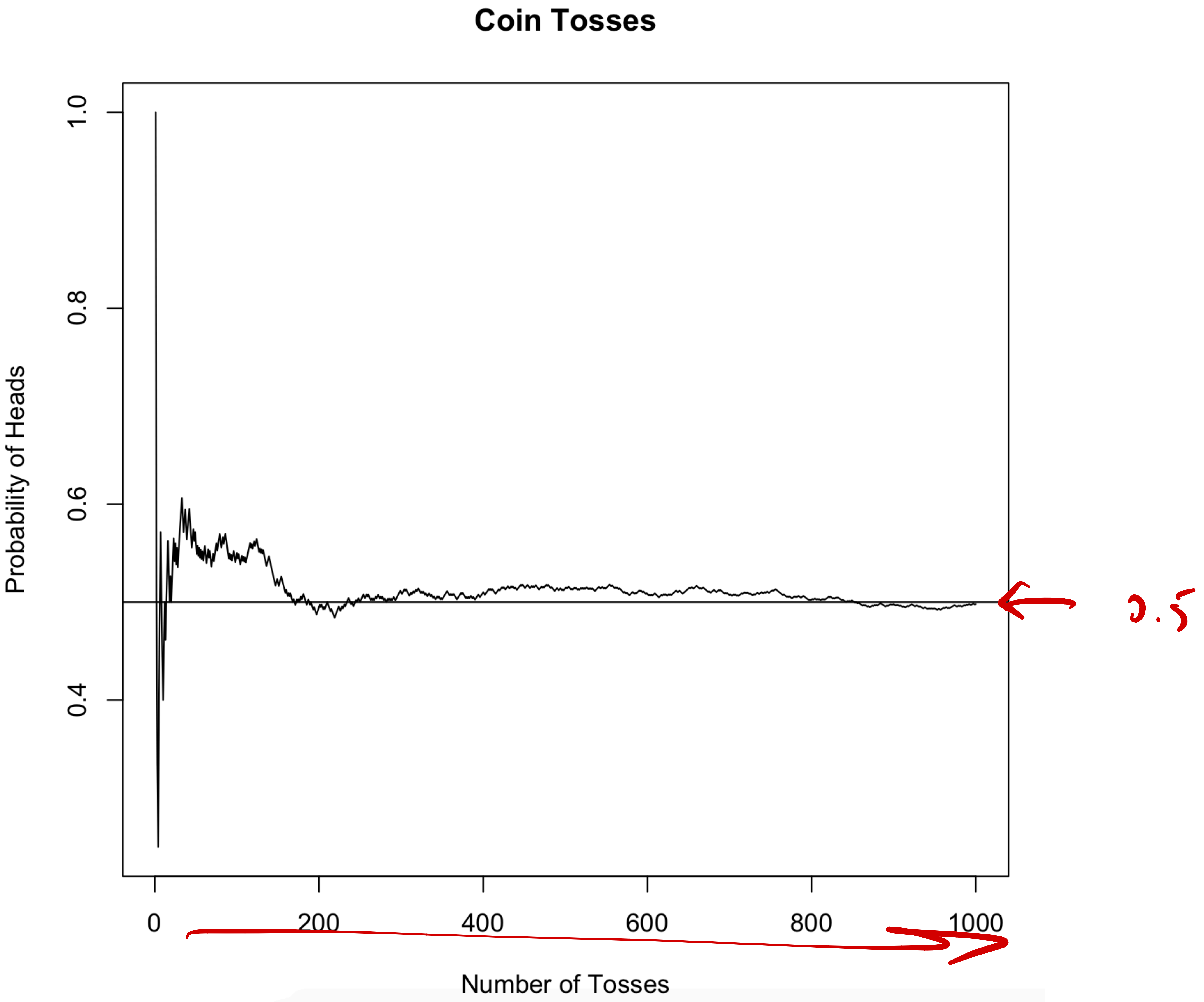
- Relative frequency of occurrence of an event when repeated many times
- $\Pr(A) = \dfrac{\text{\# of times } A \text{ occurs}}{\text{total \# of trials}}$

Probability



Probability Rules

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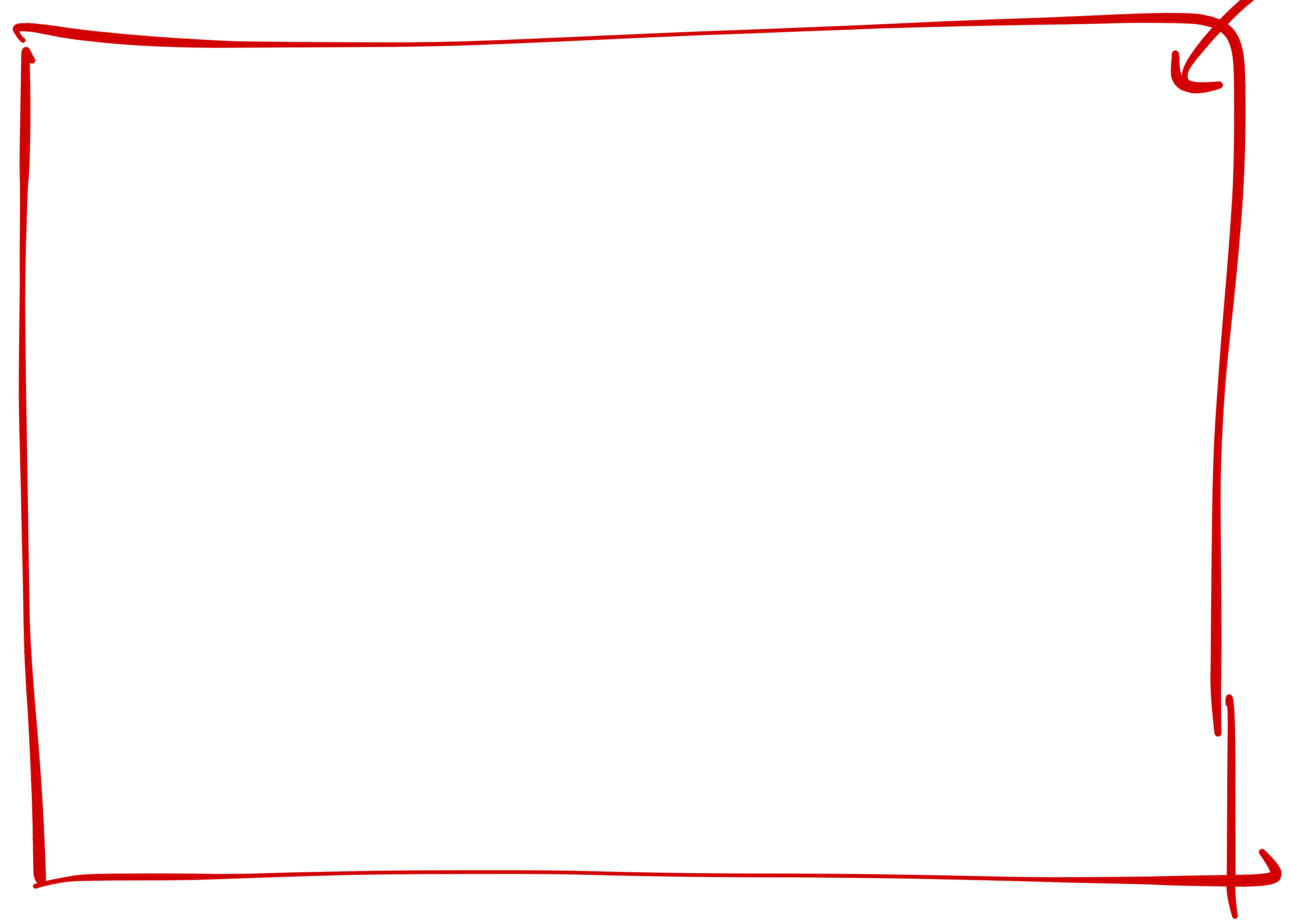
- If $A \subset B$, then $\Pr(A) \leq \Pr(B)$



outcome

$S = \{TTTT, TTTH, \ldots, HHHH\}$

$A = \{TTHH, THTH, \ldots, HHTT\}$

event

Mutual Exclusivity and Exhaustiveness

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- When the probabilities of mutually exclusive events sum to 1, the events are *exhaustive* (i.e., no other possible outcomes)

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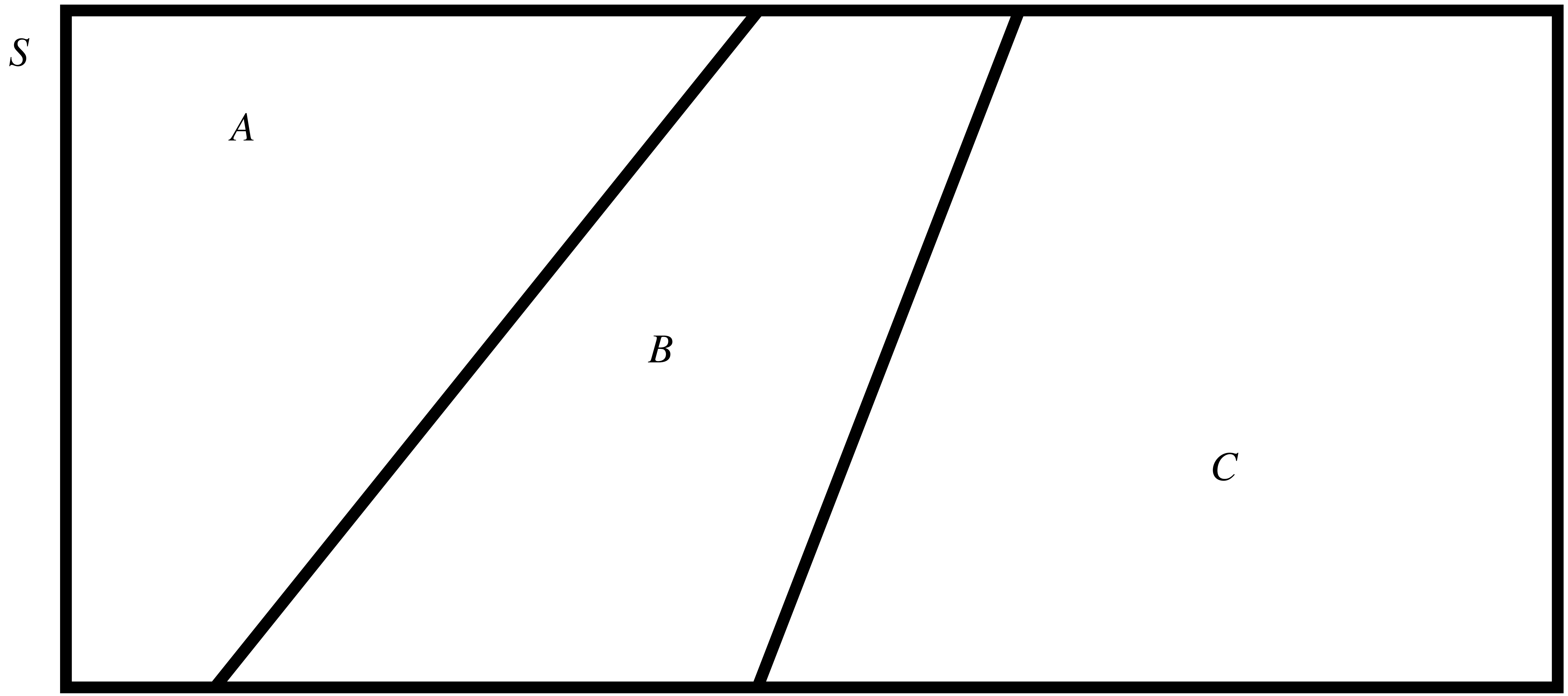

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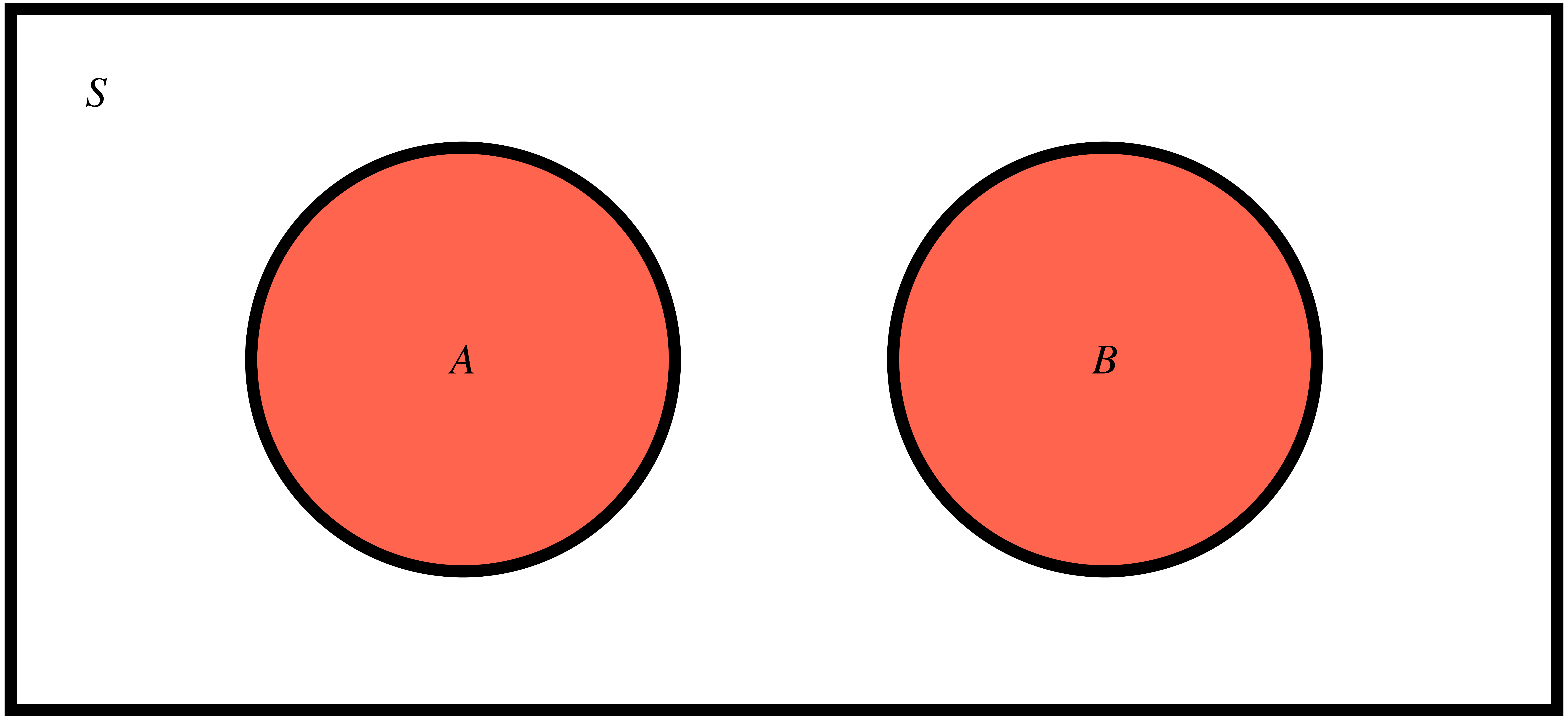

Addition Rule: General

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- In general, we have $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

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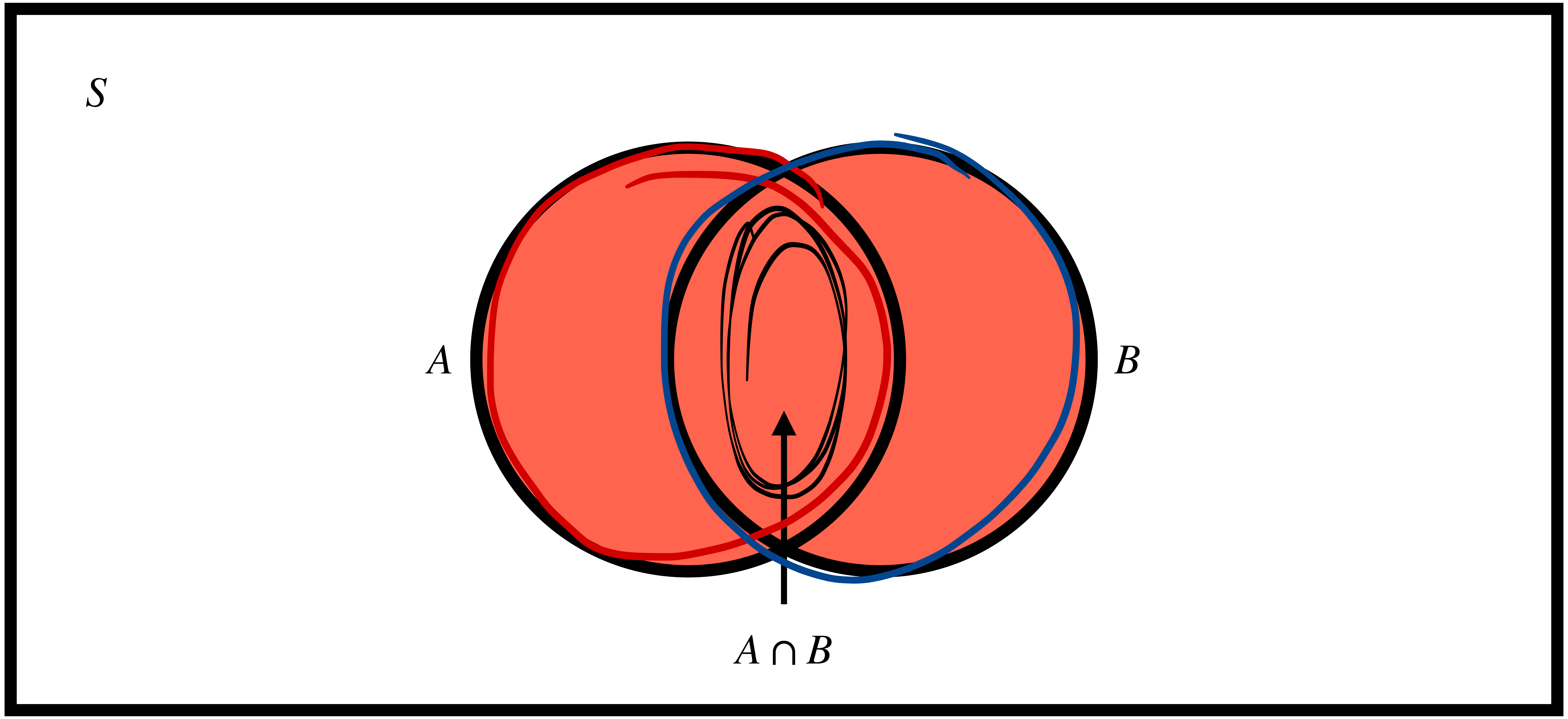

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- Suppose that 55% of cancer patients are female, 20% of cancer patients have previously undergone chemotherapy, and 15% of cancer patients are both female and have undergone chemotherapy

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A B

- Suppose that 55% of cancer patients are female, 20% of cancer patients have previously undergone chemotherapy, and 15% of cancer patients are both female and have undergone chemotherapy
- What is the probability that a patient is female or has undergone chemotherapy?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$55 + 20 - 15 = \underline{60\%}$$

Conditional Probability

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- **Conditional Probability**: The probability that event A will occur given that we already know the outcome of event B
- $\Pr(A \mid B) =$ probability of A given B

Multiplicative Rule

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- The *multiplicative rule of probability* tells us the following:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B \mid A)$$

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- Rearranging yields *conditional probability expressions*:

$$\underline{\Pr(B \mid A)} = \frac{\Pr(A \cap B)}{\Pr(A)}$$

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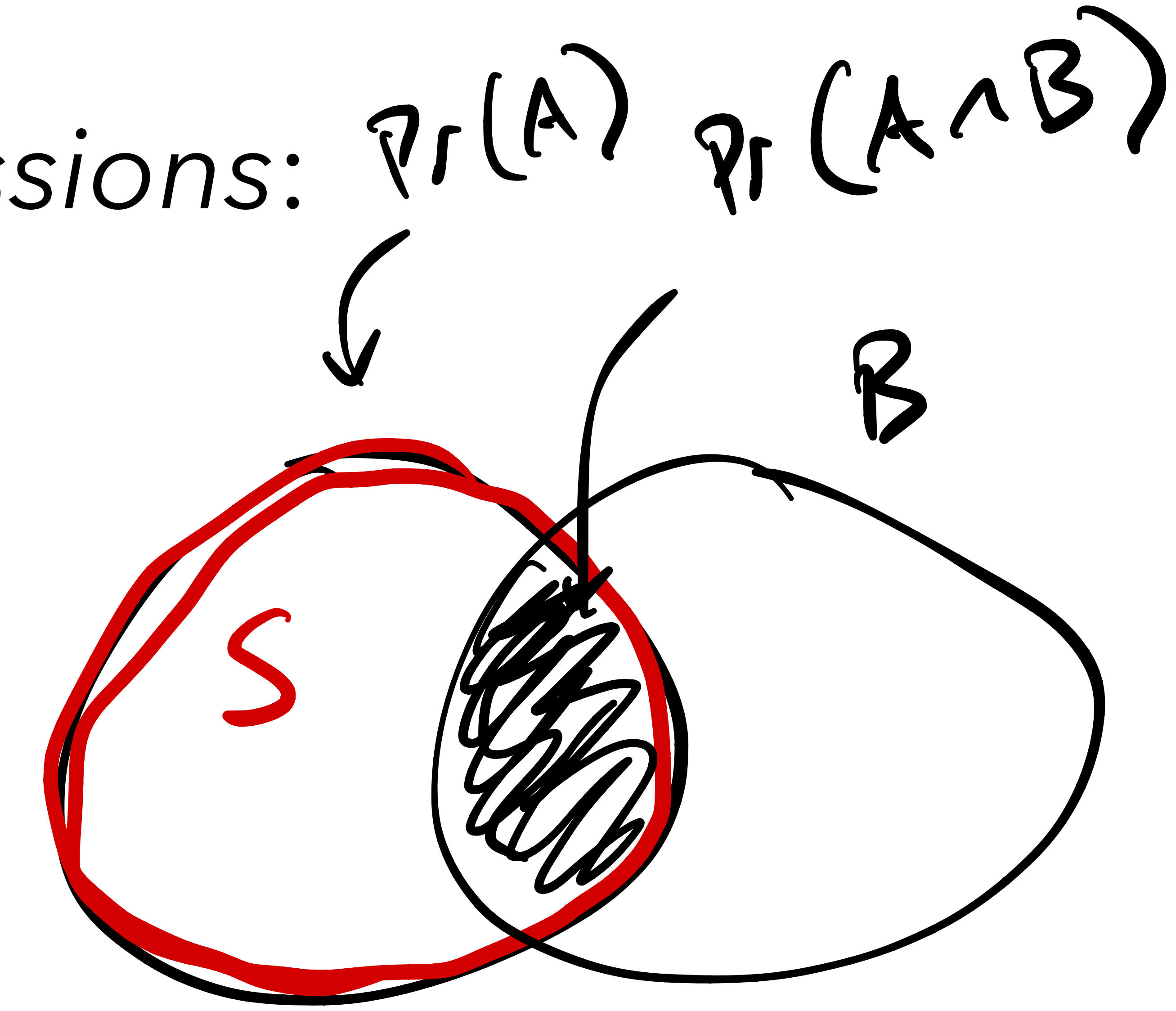

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  - 450 students changed majors
  - 300 students who changed majors were males
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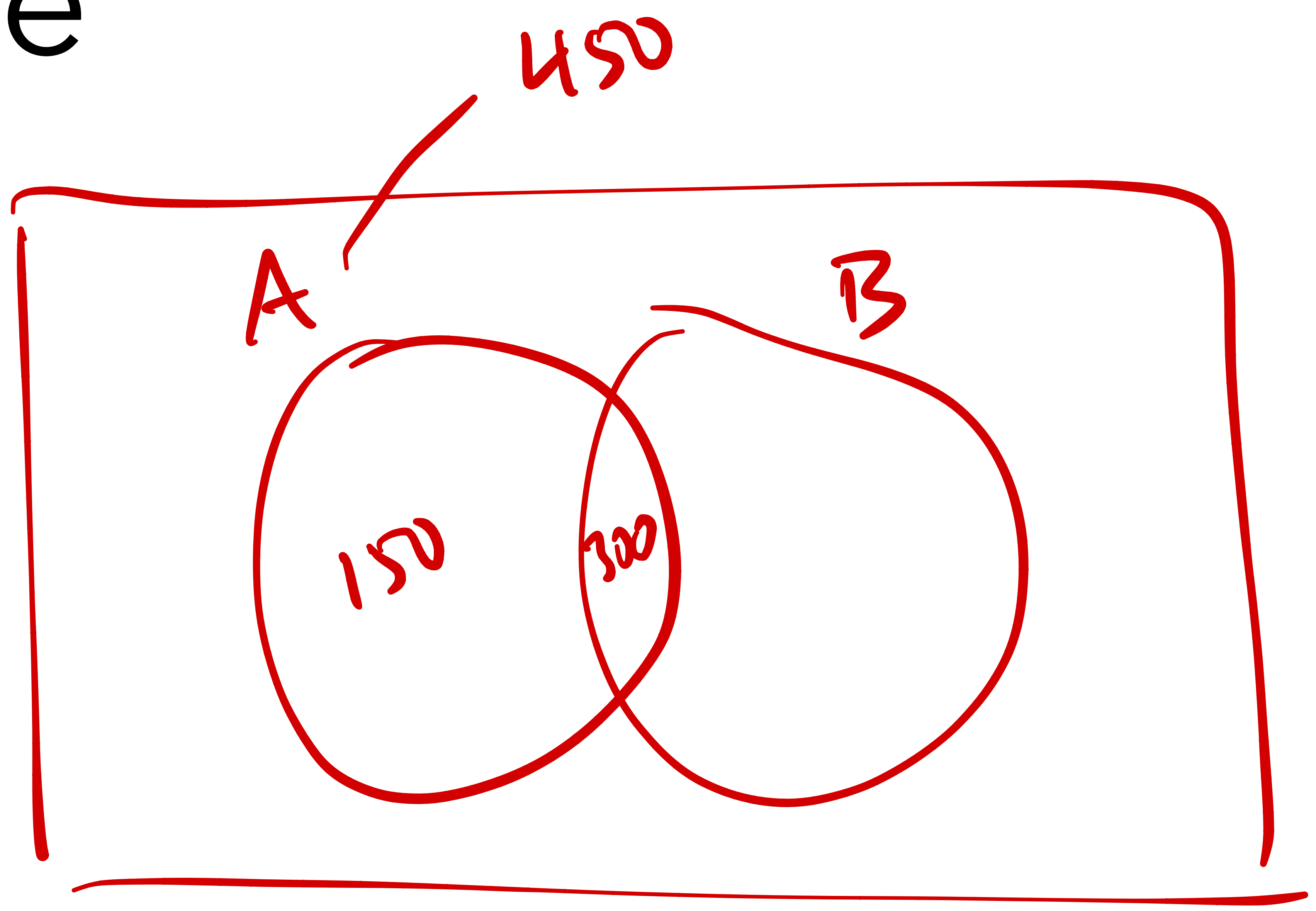

- Q1: What is the probability of changing majors given that you are a male?

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{300/10K}{3000/10K} = 1/10.$$

- Q2: What is the probability of changing majors given that you are not a male?

$$\Pr(A \mid B^c) = \frac{\Pr(A \cap B^c)}{\Pr(B^c)} = \frac{150}{7000} \sim 2\%$$

Multiplicative Rule Example

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- Setup:
  - The probability that you will be sick tomorrow is 0.6
  - If you are sick tomorrow, the probability that you will be sick the next day is 0.7
  - If you are not sick tomorrow, the probability that you will be sick the next day is 0.2

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Multiplicative Rule Example

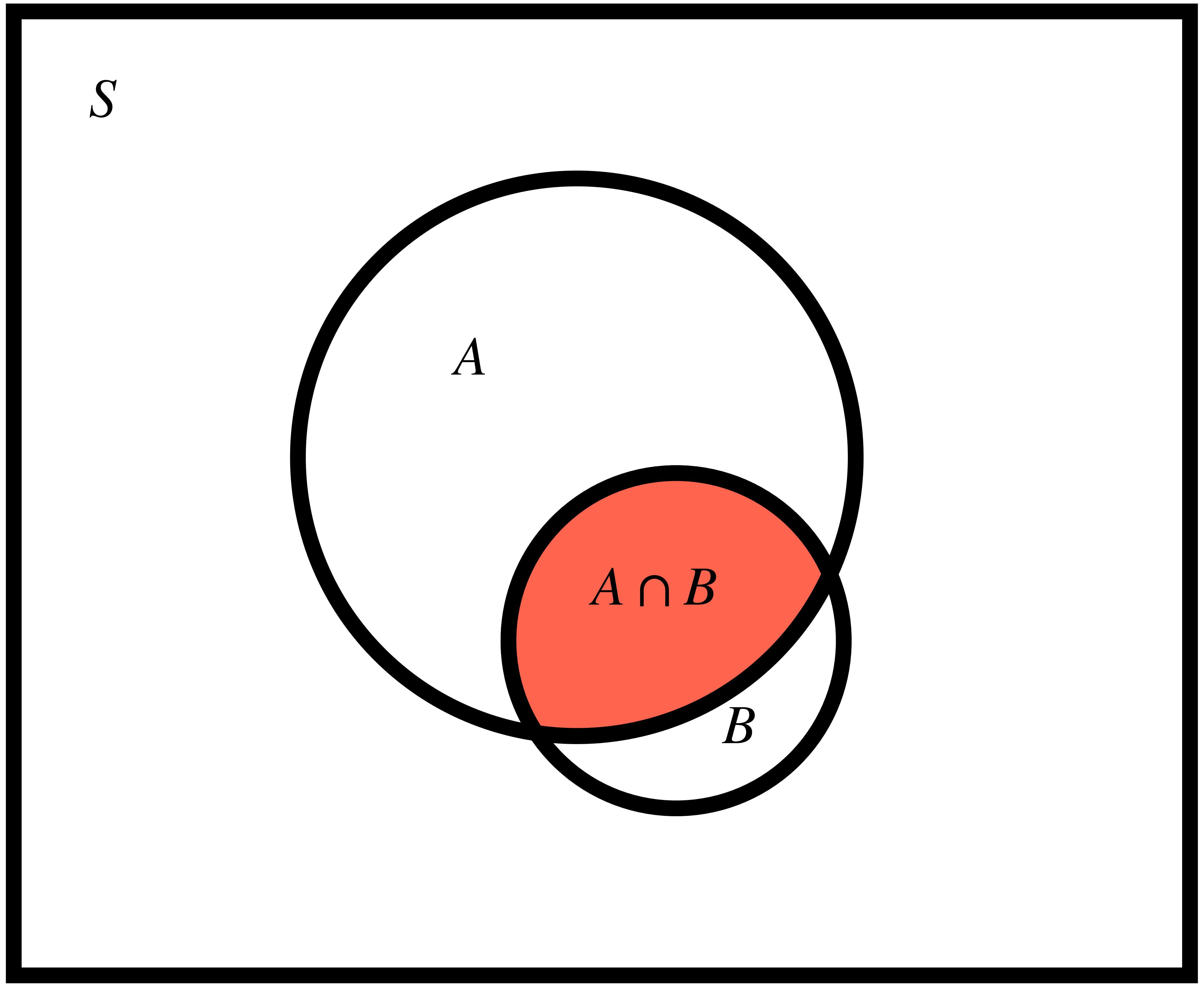
- Setup:
  - The probability that you will be $\overbrace{\text{sick tomorrow}}^{A}$ is 0.6
  - If you are sick tomorrow, the probability that you will be $\overbrace{\text{sick the next day}}^{B}$ is 0.7
  - If you are not sick tomorrow, the probability that you will be sick the next day is 0.2
- Q1: What is the probability that you are sick tomorrow and the next day?

$$P(A \cap B) = P(B|A) \cdot P(A) = 0.7 \times 0.6 = \underline{0.42}$$

- Q2: What is the probability that you are not sick tomorrow but sick the following day?

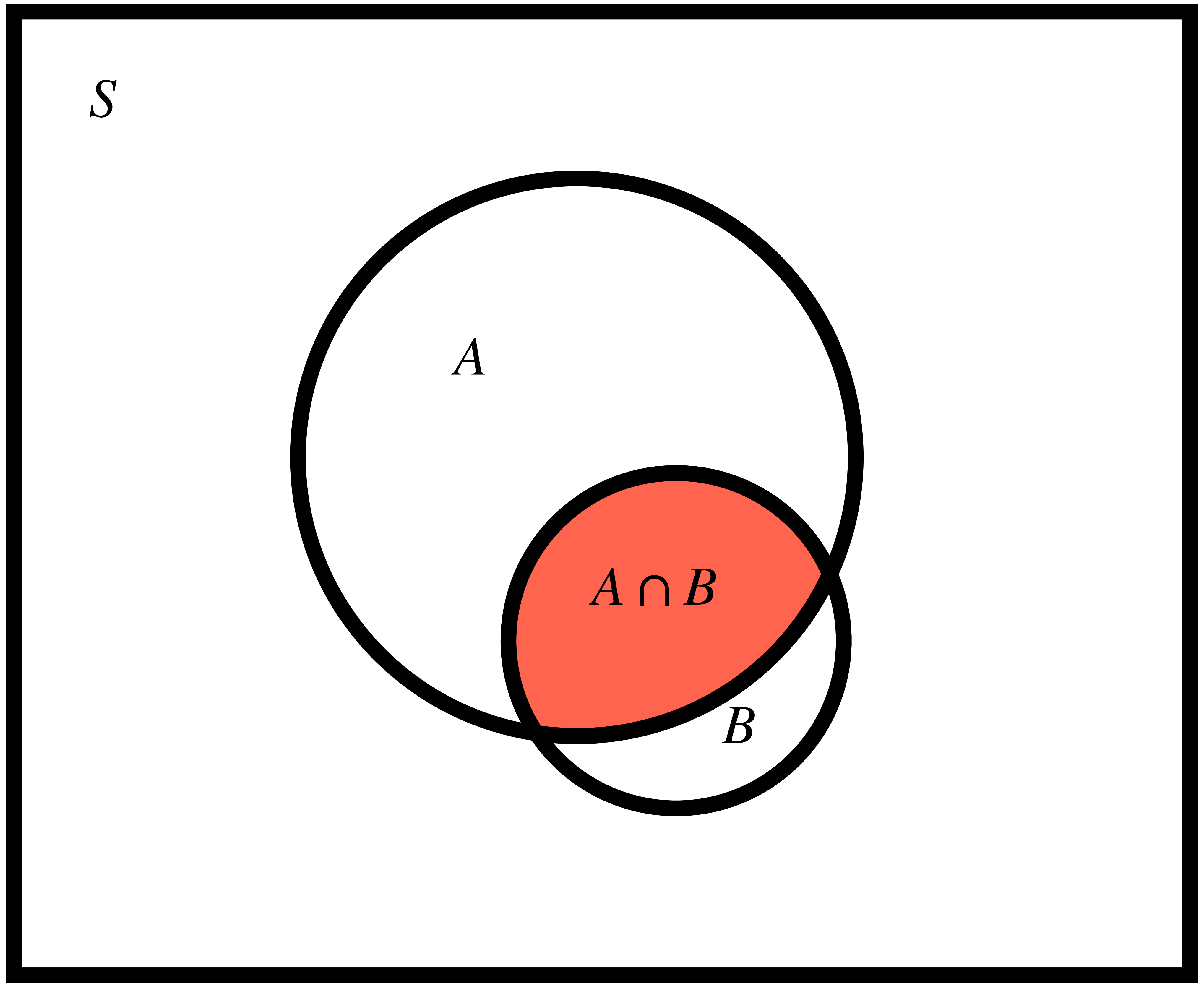
$$Pr(A^c \cap B) = Pr(B|A^c) \cdot P(A^c) = (1-0.6)(0.2) = \underline{0.08}$$

Conditional Probability



Conditional Probability

- Note, $\Pr(B \mid A) \neq 1 - \Pr(A \mid B)$



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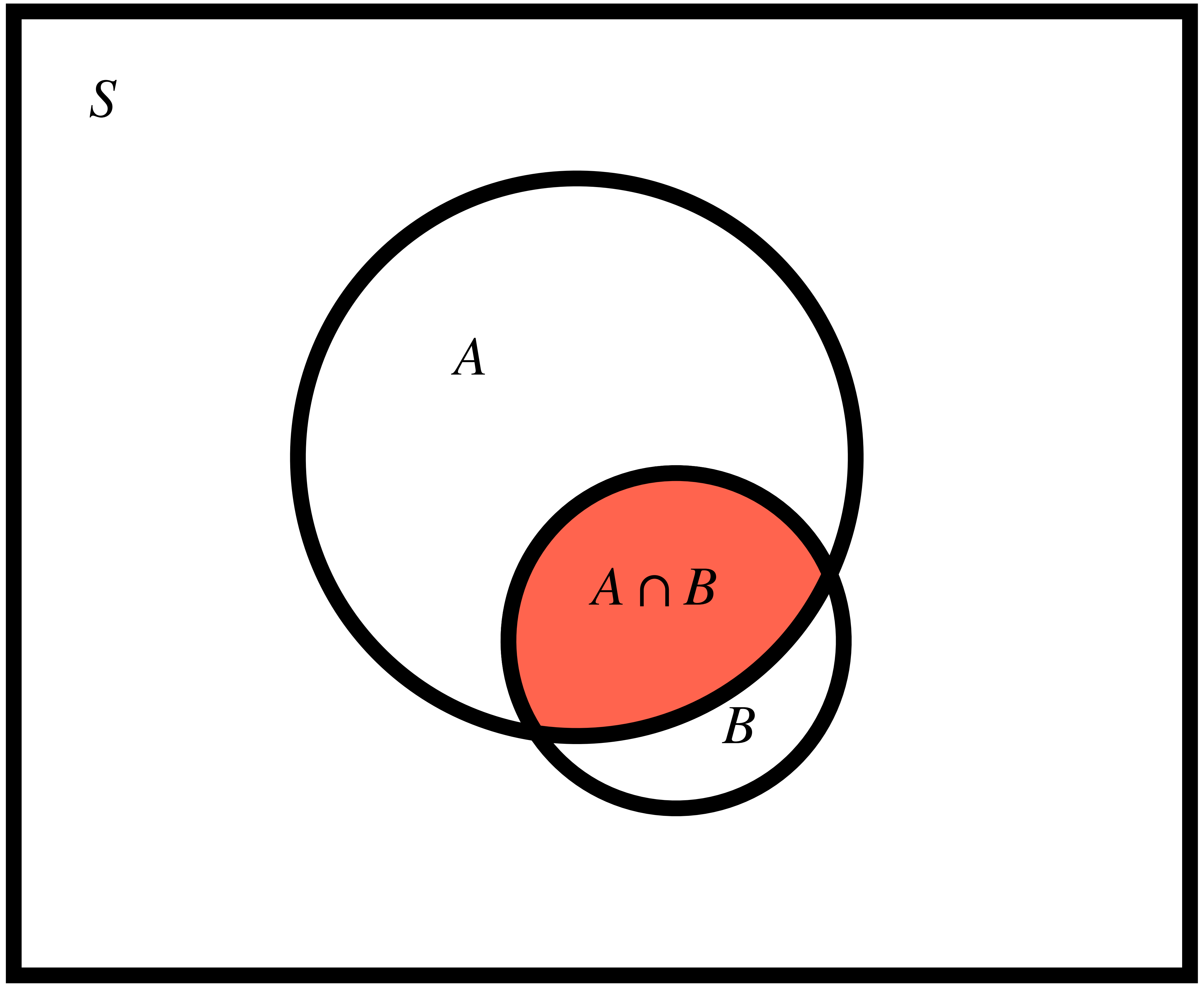
- Note, $\Pr(B\,|\,A) \neq 1 - \Pr(A\,|\,B)$
- Similarly, $\Pr(B\,|\,A) \neq 1 - \Pr(B\,|\,A^c)$



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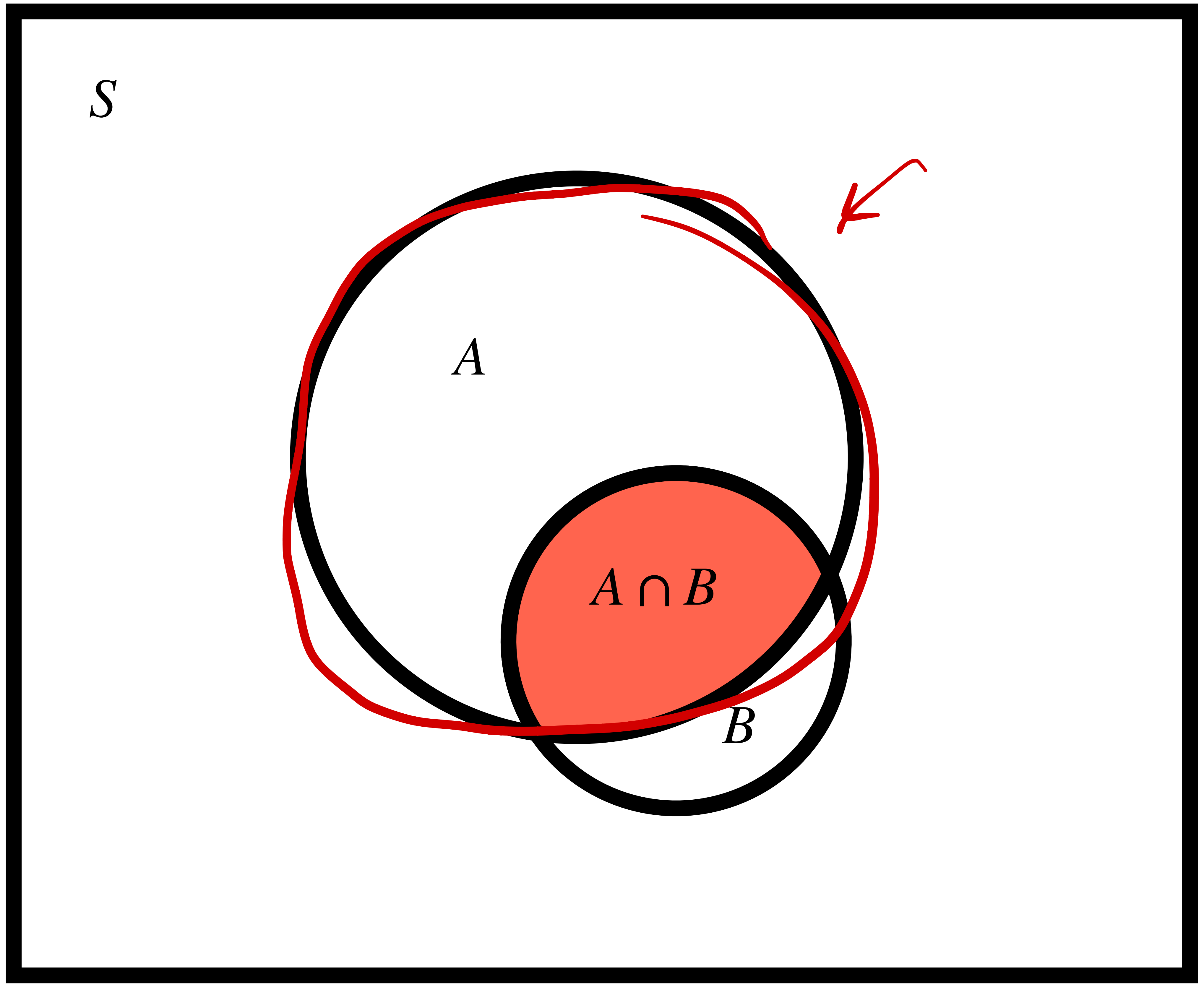
- Note, $\Pr(B\,|\,A) \neq 1 - \Pr(A\,|\,B)$
- Similarly, $\Pr(B\,|\,A) \neq 1 - \Pr(B\,|\,A^c)$
- But, $\underbrace{\Pr(B\,|\,A) = 1 - \Pr(B^c\,|\,A)}$

$\Pr(B) = 1 - \Pr(B^c)$



Conditional Probability Example

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- Setup:
  - Consider a random experiment where 3 balls are randomly selected (without replacement) from 5 balls labeled 1, 2, 3, 4, 5. Sample space:

123, 124, 125, 134, 135, 145
234, 235, 245
345

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  - Consider a random experiment where 3 balls are randomly selected (without replacement) from 5 balls labeled 1, 2, 3, 4, 5. Sample space:

    123, 124, 125, 134, 135, 145
    234, 235, 245
    345

- Let $A = \{1 \text{ is selected}\}$ and $B = \{5 \text{ is selected}\}$. What is $\Pr(A \mid B)$?

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{3}{6} = \frac{1}{2}.$$

Independence

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- **Independence**: The outcome of one event has no effect on the outcome of another event
    - If A and B are independent, then $\Pr(A \mid B) = \Pr(A)$ (and $\Pr(B \mid A) = \Pr(B)$)

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- **Independence**: The outcome of one event has no effect on the outcome of another event
  - If A and B are independent, then $\Pr(A \mid B) = \Pr(A)$ (and $\Pr(B \mid A) = \Pr(B)$)
- This is because intersection is decomposable:

$Pr(B|A) \cdot Pr(A) = Pr(A) \cdot Pr(B)$

  - If A and B are independent, then $\boxed{\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)}$
  - From this, we see that $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{\Pr(A) \cdot \Pr(B)}{\Pr(B)} = P(A)$

Independence Example

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- Setup:
  - Suppose we flip a coin twice; tosses are independent
  - Let $A = \{\text{first flip is heads}\}$ and $B = \{\text{second flip is heads}\}$
  - $\Pr(A) = \Pr(B) = 1/2$

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  - $\Pr(A) = \Pr(B) = 1/2$
- What is $\Pr(A \cap B)$ (probability that both flips are heads)?

$1/4$

$\underline{1/2^4}$

Mutual Independence

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- Suppose we have n events, N. These n events are **mutually independent** iff, for every subset of events $M \subseteq N$, we have

$$\Pr\left(\bigcap_{i \in M} A_i\right) = \prod_{i \in M} \Pr(A_i)$$

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- Consider the case of $n = 3$. Events A_1, A_2, A_3 are independent iff the following hold:

$$\begin{aligned}
\Pr(A_1 \cap A_2) &= \Pr(A_1) \cdot \Pr(A_2) \\
\Pr(A_1 \cap A_3) &= \Pr(A_1) \cdot \Pr(A_3) \\
\Pr(A_2 \cap A_3) &= \Pr(A_2) \cdot \Pr(A_3) \\
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\end{aligned}\right]$$

$$\left.\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3)\right]$$

- If all but the last equality hold, A_1, A_2, A_3 are *pairwise independent*, but not mutually independent

Pairwise Independence: Example

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- Setup: Consider rolling a fair ~~six~~ 4-sided die. Consider the events $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{2,3\}$
  - $\Pr(A) = \Pr(B) = \Pr(C) =$ 1/2
  - $\Pr(A \cap B) =$ 1/4 = Pr(A) · Pr(B)
  - $\Pr(A \cap C) =$ 1/4 "
  - $\Pr(B \cap C) =$ 1/4 "
  - $\Pr(A \cap B \cap C) =$ $0 \neq (1/2)^3$

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- Setup: Consider rolling a fair six-sided die. Consider the events $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{2,3\}$
    - $\Pr(A) = \Pr(B) = \Pr(C) =$
    - $\Pr(A \cap B) =$
    - $\Pr(A \cap C) =$
    - $\Pr(B \cap C) =$
    - $\Pr(A \cap B \cap C) =$
- These events are pairwise independent but not mutually independent

Independence vs. Mutual Exclusivity

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- Independence and mutual exclusivity are not the same thing

Independence vs. Mutual Exclusivity

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$A \qquad A^c$

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- Independence: the other event still may occur; its probability is unaffected

Law of Total Probability

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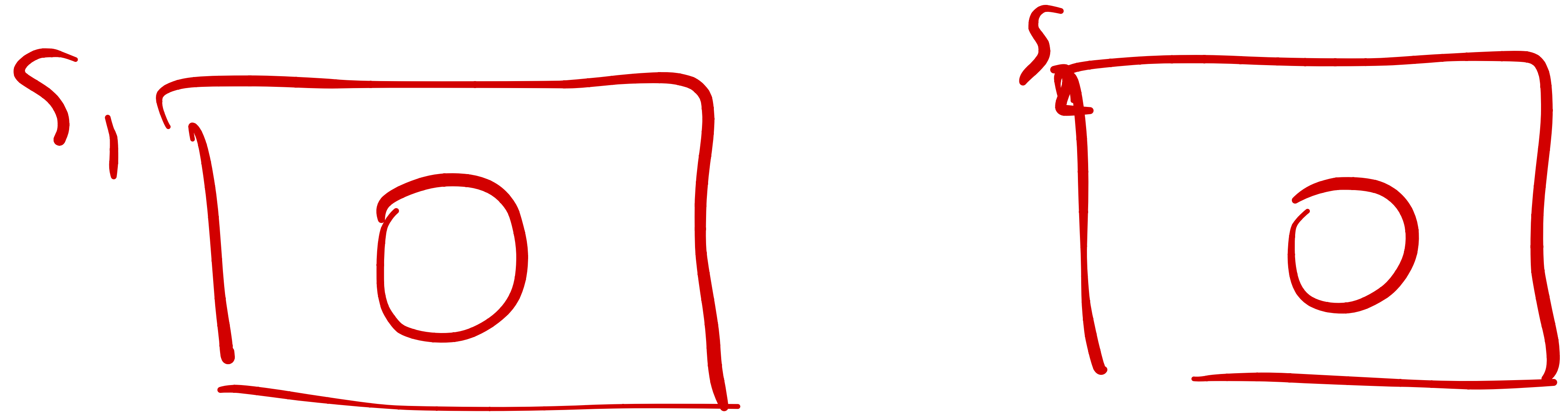
- Consider a collection of mutually exclusive and exhaustive events $A_1, A_2, \ldots, A_n$ that *partitions* the sample space S

Law of Total Probability



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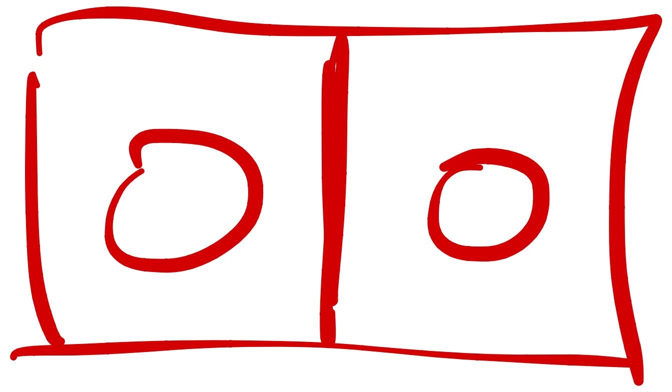
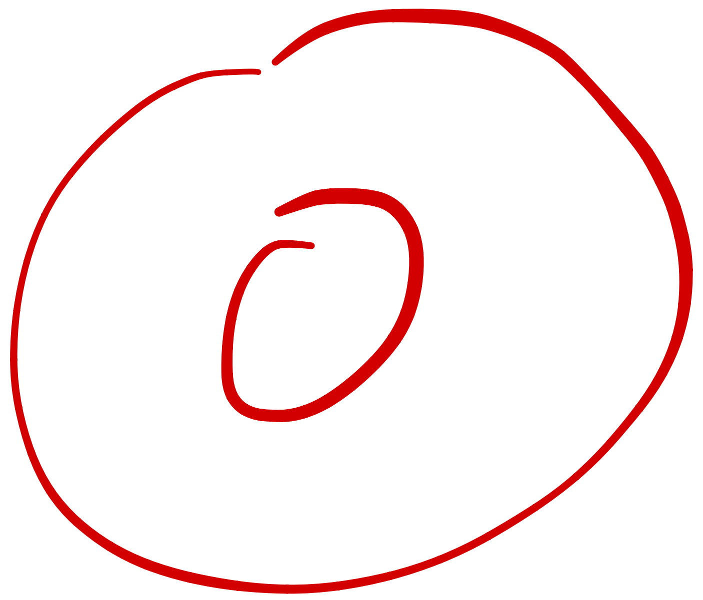
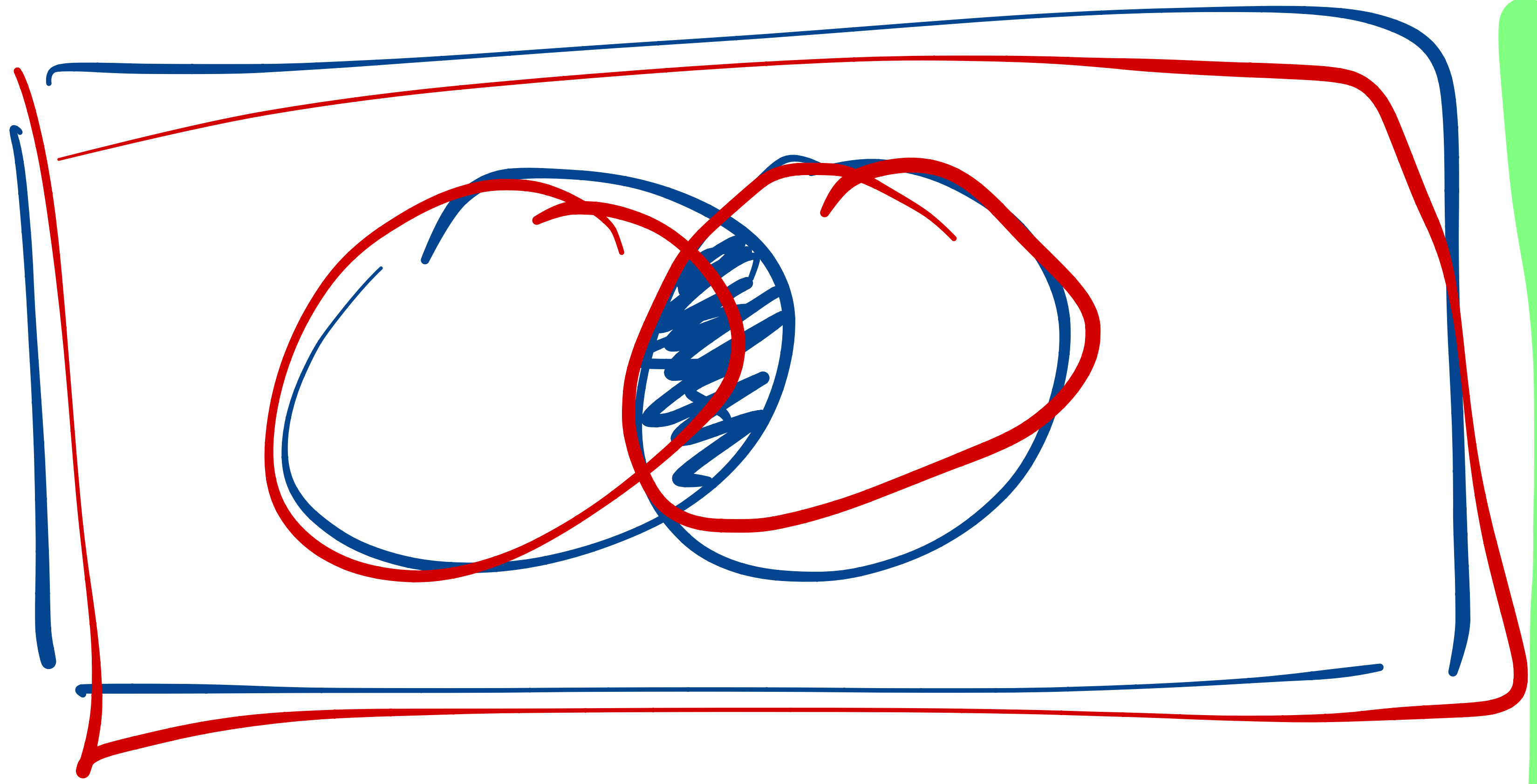
- Then, for any event E, the law of total probability states the following:



$$\begin{aligned}\Pr(E) &= \Pr(E \cap A_1) + \Pr(E \cap A_2) + \ldots + \Pr(E \cap A_n) \\ &= \Pr(E\,|\,A_1) \cdot \Pr(A_1) + \Pr(E\,|\,A_2) \cdot \Pr(A_2) + \ldots + \Pr(E\,|\,A_n) \cdot \Pr(A_n)\end{aligned}$$





$$\boxed{\begin{aligned}\Pr(A|B) &= \Pr(A) \\ \Pr(B|A) &= \Pr(B)\end{aligned}}$$

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$P(A \cap B) = Pr(A|B)\,Pr(B)$

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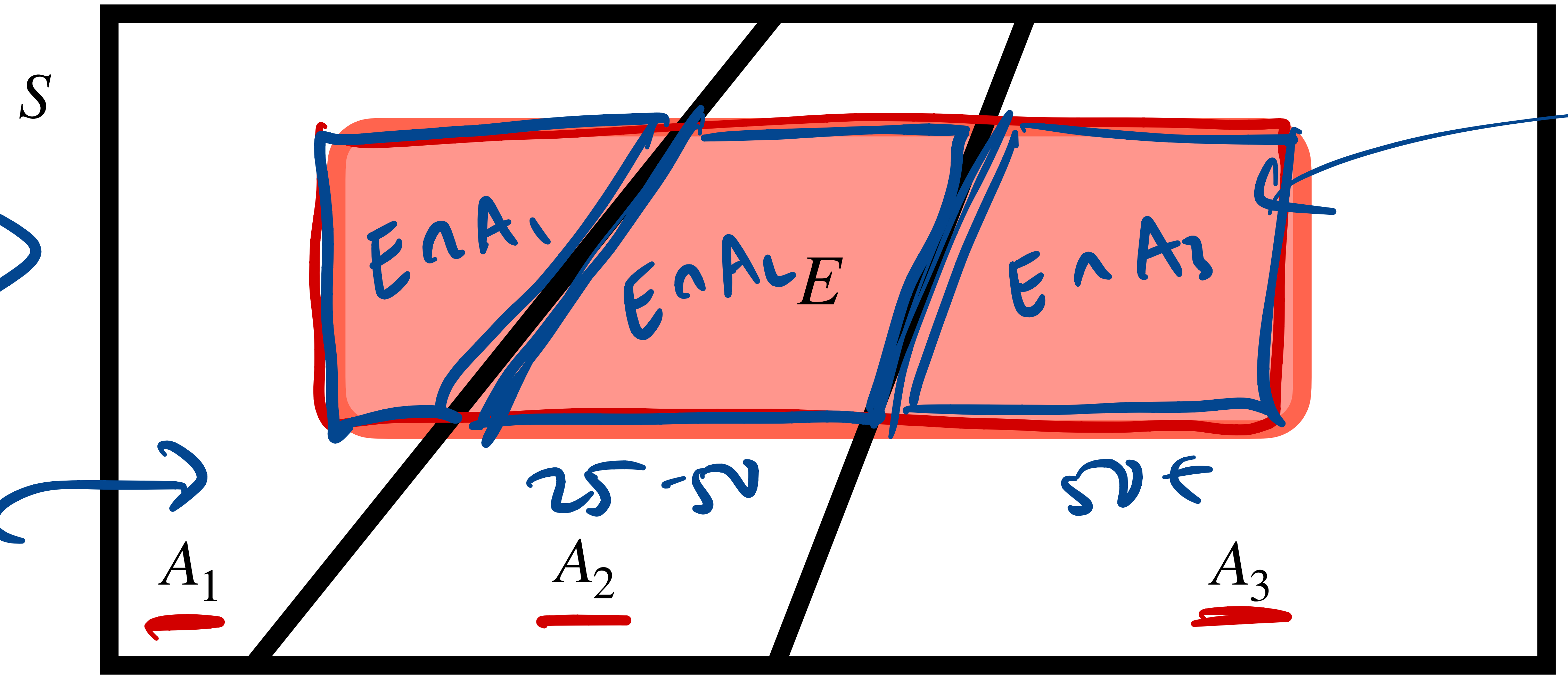

0-25

Pr(cancer)

Bayes' Theorem

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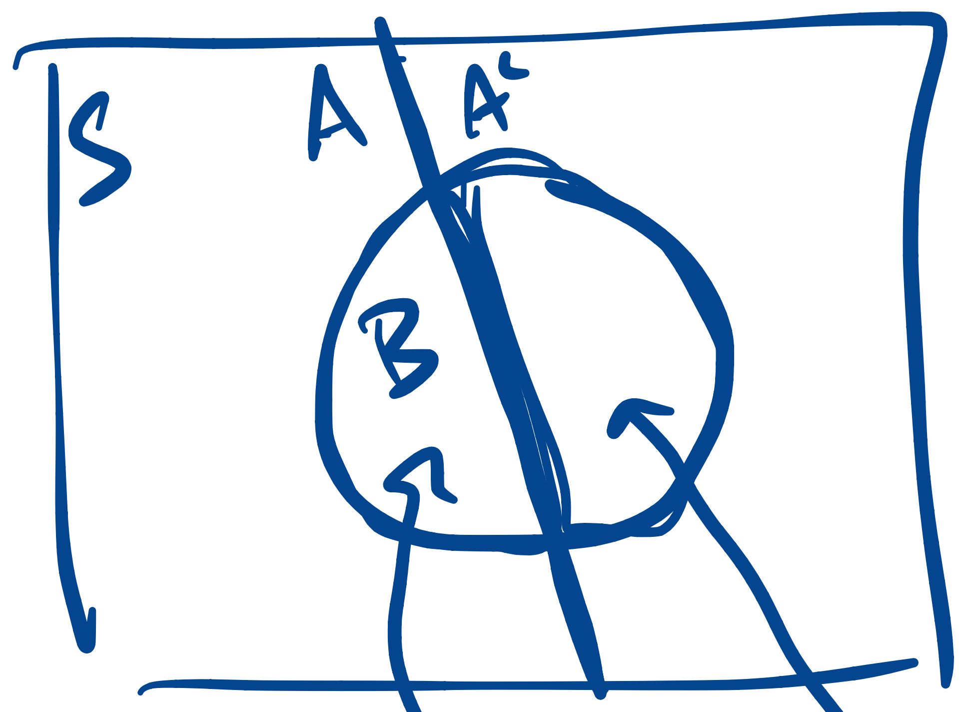
Bayes' Theorem

3 Blue 1 Brown



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Posterior Likelihood Prior

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- Setup:
  - Given that you have diabetes, there is a 70% chance you are also overweight
  - Given that you do not have diabetes, there is a 35% chance you are overweight
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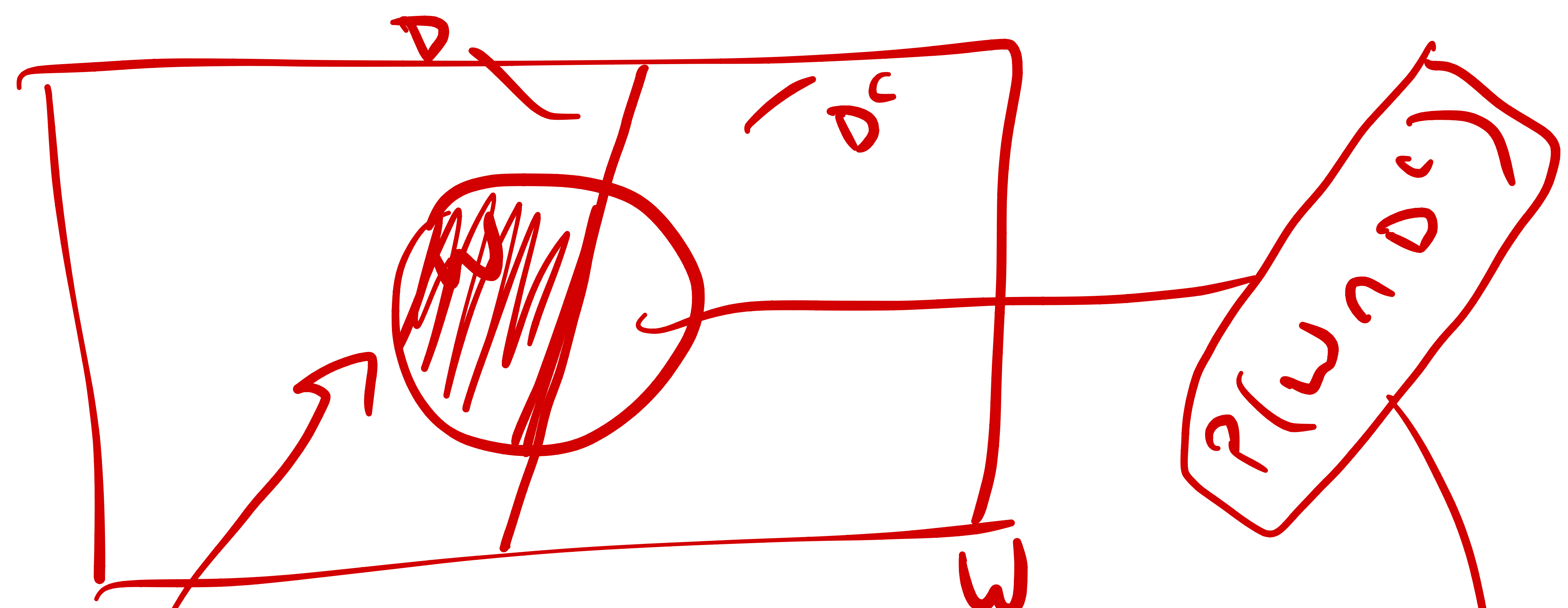

- Setup:
  - Given that you have diabetes, there is a 70% chance you are also overweight
  - Given that you do not have diabetes, there is a 35% chance you are overweight
  - 10% of people have diabetes
- Q: Given that a randomly selected person is overweight, what is the probability that he has diabetes?

$$P(D|W) = \frac{P(W|D)\,P(D)}{P(W)} = \frac{.7 \times .1}{\underbrace{P(W|D)}_{.70} \times \underbrace{P(D)}_{.10} + \underbrace{P(W|D^c)}_{.35}\,\underbrace{P(D^c)}_{.90}}$$

Diagnostic Tests

(Bayes' IRL)

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  - T^+: the test is positive
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- Typically, we are interested in $\Pr(D_1 \mid T^+)$ (true positive rate of a test)

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- What are $\Pr(D_1)$ and $\Pr(D_2)$?
  - $\Pr(D_1)$: probability of having the disease, or prevalence of the disease
  - $\Pr(D_2) = 1 - \Pr(D_1)$

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$$= \frac{\Pr(T^- \mid D_2) \cdot \Pr(D_2)}{\Pr(T^- \mid D_2) \cdot \Pr(D_2) + \Pr(T^- \mid D_1) \cdot \Pr(D_1)}$$

Diagnostic Tests: Example

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- Cancer test has the following properties:
  - The test gives a positive result 95% of the time when the patient has cancer
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\Pr(C \mid pos) &= \frac{\Pr(C \cap pos)}{\Pr(pos)} \\
&= \frac{\Pr(pos \mid C) \cdot \Pr(C)}{\Pr(pos \mid C) \cdot \Pr(C) + \Pr(pos \mid C^c) \cdot \Pr(C^c)} \\
&= \frac{0.95 \cdot 0.12}{0.95 \cdot 0.12 + (1 - 0.90) \cdot (1 - 0.12)} \\
&= 0.5644
\end{aligned}
$$

Combinatorics

Counting Outcomes

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  - Where N is the total number of outcomes in E and D is the total number of outcomes in S
- We're going to learn how to count the number of outcomes

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    - Care about the names of choices (think of it as a set)

Rule of Product

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Rule of Product: Example

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- How many valid three-digit numbers (i.e., between 100 and 999, inclusive) have three different digits and only a single odd number in the middle?

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- How many valid three-digit numbers (i.e., between 100 and 999, inclusive) have three different digits and only a single odd digit in the center?

Break this down into $m = 3$ tasks

Task 1: Select an odd (center) digit, $n_1 = 5$

Task 2: Select a first (even) digit that is not 0, $n_2 = 4$

Task 3: Select a last (even) digit, $n_3 = 4$

Total: $n_1 \cdot n_2 \cdot n_3 = 5 \cdot 4 \cdot 4 = 80$

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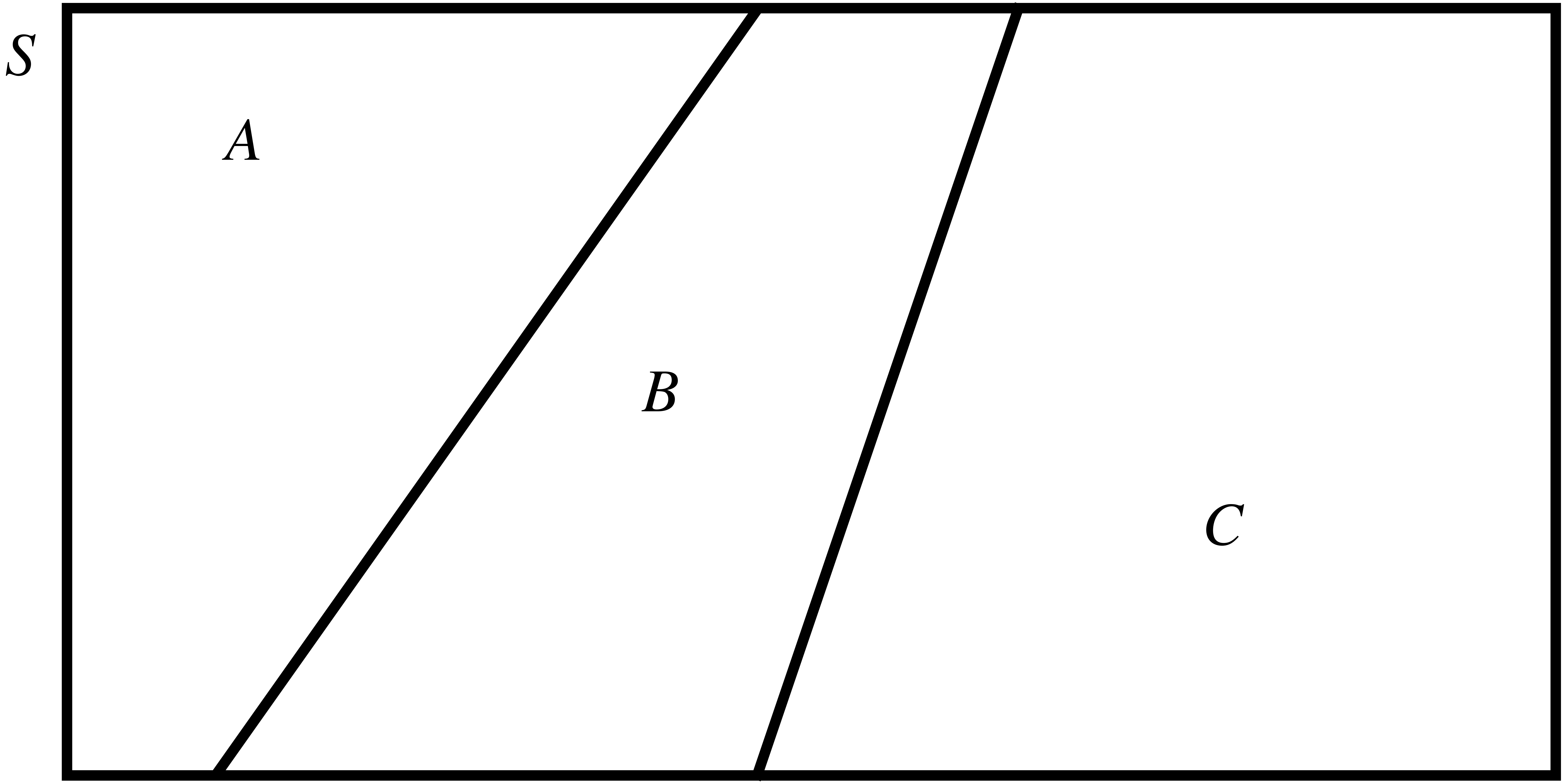
- Often, use the rule of sum (tree method) and the rule of product together

Rule of Sum (OR) and Rule of Product (AND)



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- In R: use `factorial(x)`

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- Suppose we want to select and order k objects from a total of n objects
  - Ordered selection
- There are n ways to select the first object, $n-1$ ways to select the second object, and so on until we have $n-k+1$ ways to select the final object

$$
\begin{aligned}
P(n,k) &= n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \\
&= \frac{n!}{(n-k)!}
\end{aligned}
$$

Permutation: Example

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- Q1: How many four-letter “words” are there where each letter is distinct?

Permutation: Example

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- Q2: How many ways are there of assigning three students to seven orientation groups, where each student must go to a different group?

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- *Binomial coefficient*

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- Q2: What is the probability of getting four of the same kind?

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- Setting: An urn contains 35 yellow balls (numbered 1-35) and 35 pink balls (numbered 1-35). Four balls are chosen at random

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- Setting: An urn contains 35 yellow balls (numbered 1-35) and 35 pink balls (numbered 1-35). Four balls are chosen at random
- Q1: What is the probability that there are two pairs of balls which have the same number?

Combination: Example (Urn)

- Setting: An urn contains 35 yellow balls (numbered 1-35) and 35 pink balls (numbered 1-35). Four balls are chosen at random
- Q1: What is the probability that there are two pairs of balls which have the same number?
- Q2: What is the probability that there is exactly one pair of balls with matching numbers?

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Combination: Example (Urn)

- Setting: An urn contains 35 yellow balls (numbered 1-35) and 35 pink balls (numbered 1-35). Four balls are chosen at random
- Q3: What is the probability that the balls are all the same color and consecutively numbered?

Stars and Bars: Intuition

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$$\Big|\; \star \quad \star \;\Big|\; \star \quad \star \quad \star \quad \star$$

Stars and Bars: Intuition

- How many ways are there of choosing three *positive* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?
  - $\binom{6-1}{3-1} = \binom{5}{2}$: $\quad \star \;\Big|\; \star \quad \star \;\Big|\; \star \quad \star \quad \star$
- How many ways are there of choosing three *nonnegative* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?
  - $\binom{6+3-1}{3-1} = \binom{8}{2}$: $\quad \Big|\, \star \quad \star \;\Big|\; \star \quad \star \quad \star \quad \star$

Stars and Bars: More Formally

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  - Total number of ways = $\binom{n-1}{k-1}$ (think of filling in gaps between objects)
- For nonnegative (not positive) constraints:
  - Total number of ways = $\binom{n+k-1}{k-1}$ (think of arranging n objects and $k-1$ dividers)

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Stars and Bars: Example

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- Q1: How many different requests are possible if at least one child must choose each flavor?
- Q2: How many different requests are possible without this restriction?