Chapter 9: Inference for Variances

DSCC 462

Computational Introduction to Statistics

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- However, variances are essential to inference of means
- We may be interested in determining:
  - Whether a population variance is equal to a predetermined value
  - Whether two population variances are equal

Sampling Distribution of Variance

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  $\frac{x_i - \mu}{\sigma} \sim$
- Now, examine s^2/σ^2

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χ^2_n

$\frac{x_i - \mu}{\sigma} \sim N(0,1)$

z

χ^2_1

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- Rewrite the sample variance as (approximately) $s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \mu)^2$
- Now, examine s^2/σ^2
- We have $\frac{s^2}{\sigma^2} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2$
- Therefore, $\frac{s^2}{\sigma^2} = \frac{1}{n-1} \cdot \boxed{\sum_{i=1}^{n} Z_i^2}$, where Z_i is a standard normal random variable

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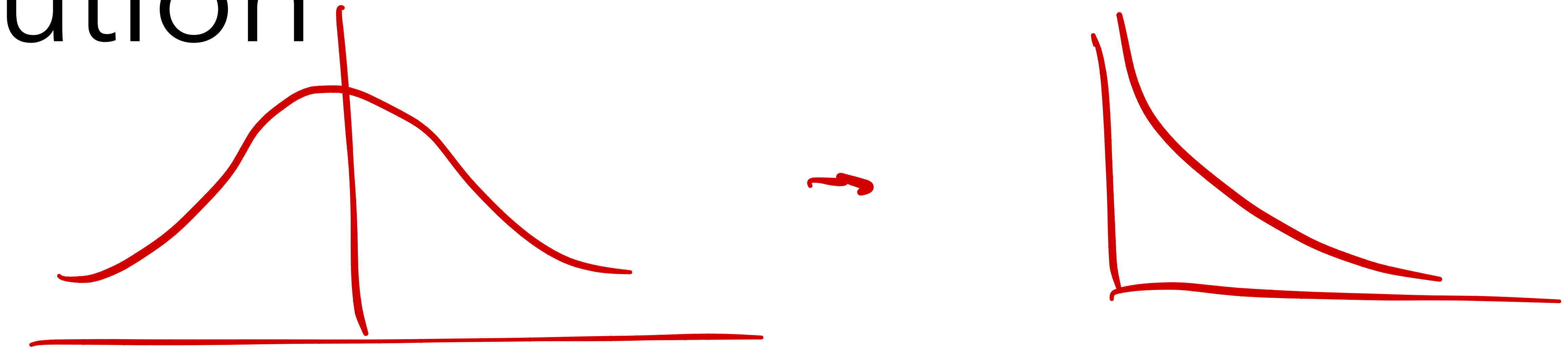

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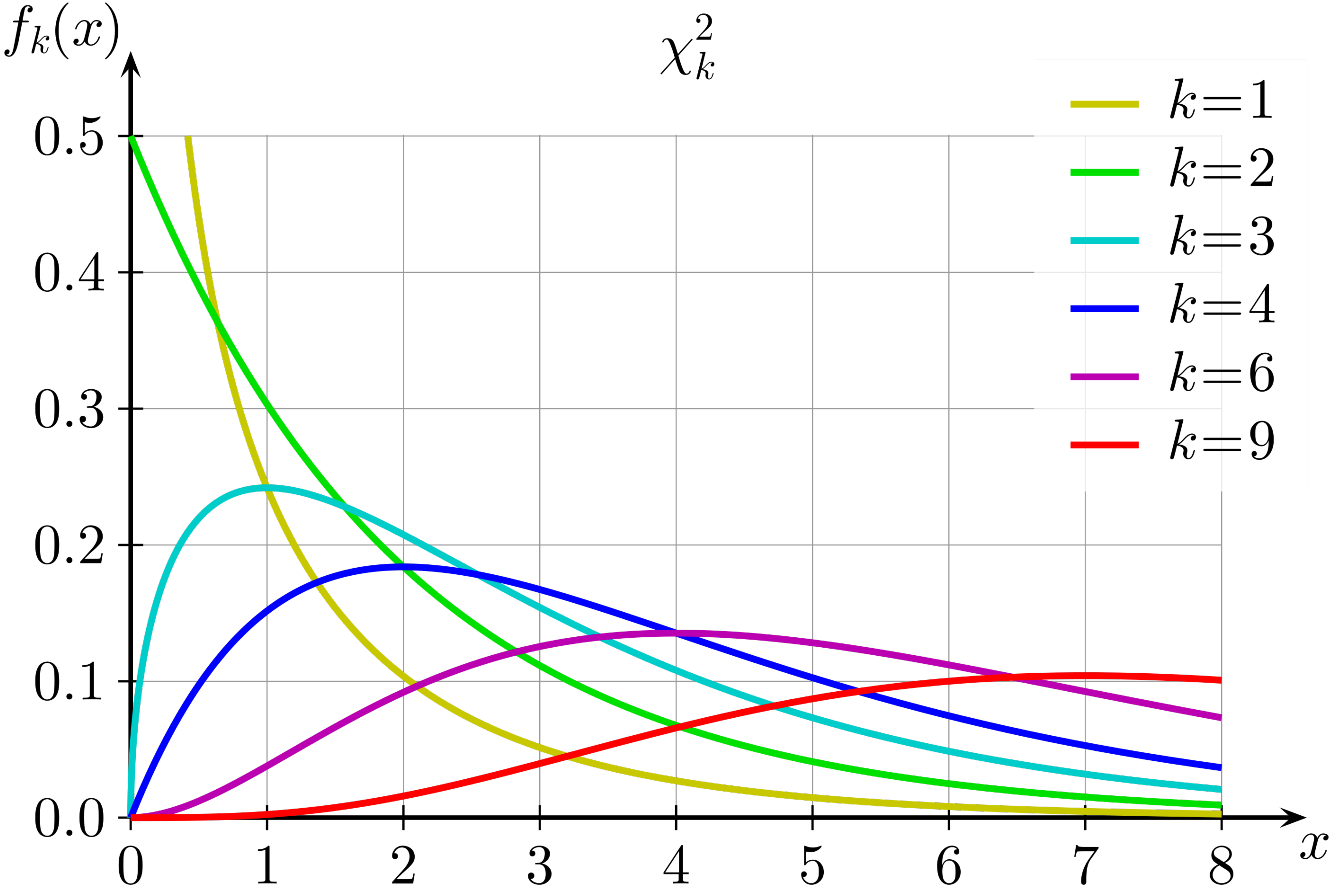
- Mean of χ_k^2 is k (degrees of freedom)
- Variance of χ_k^2 is $2k$ (twice the degrees of freedom)

Chi-squared (χ^2) distribution



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Sampling Distribution of Variance $\frac{1}{n-1} \sum (x_i - \bar{x})^2$

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- This means that $\boxed{(n-1) \cdot \frac{s^2}{\sigma^2}} \sim \chi^2_{n-1}$ (chi-squared distribution with $n-1$ dof)

Hypothesis Tests for One Population Variance

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- We have considered the case of comparing the *mean* of a population to a predetermined value
- We can also test whether the *variance* of a population is a specified value

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- What is our test statistic?

$$(n-1)\cdot\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$$

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- Recall that we know $(n-1)\cdot\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$, so it makes sense to define $T = \boxed{(n-1)\cdot\frac{s^2}{\sigma_0^2}}$ based on our null hypothesis

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- Once we have our test statistic, we can compare to the chi-square distribution to calculate a p-value

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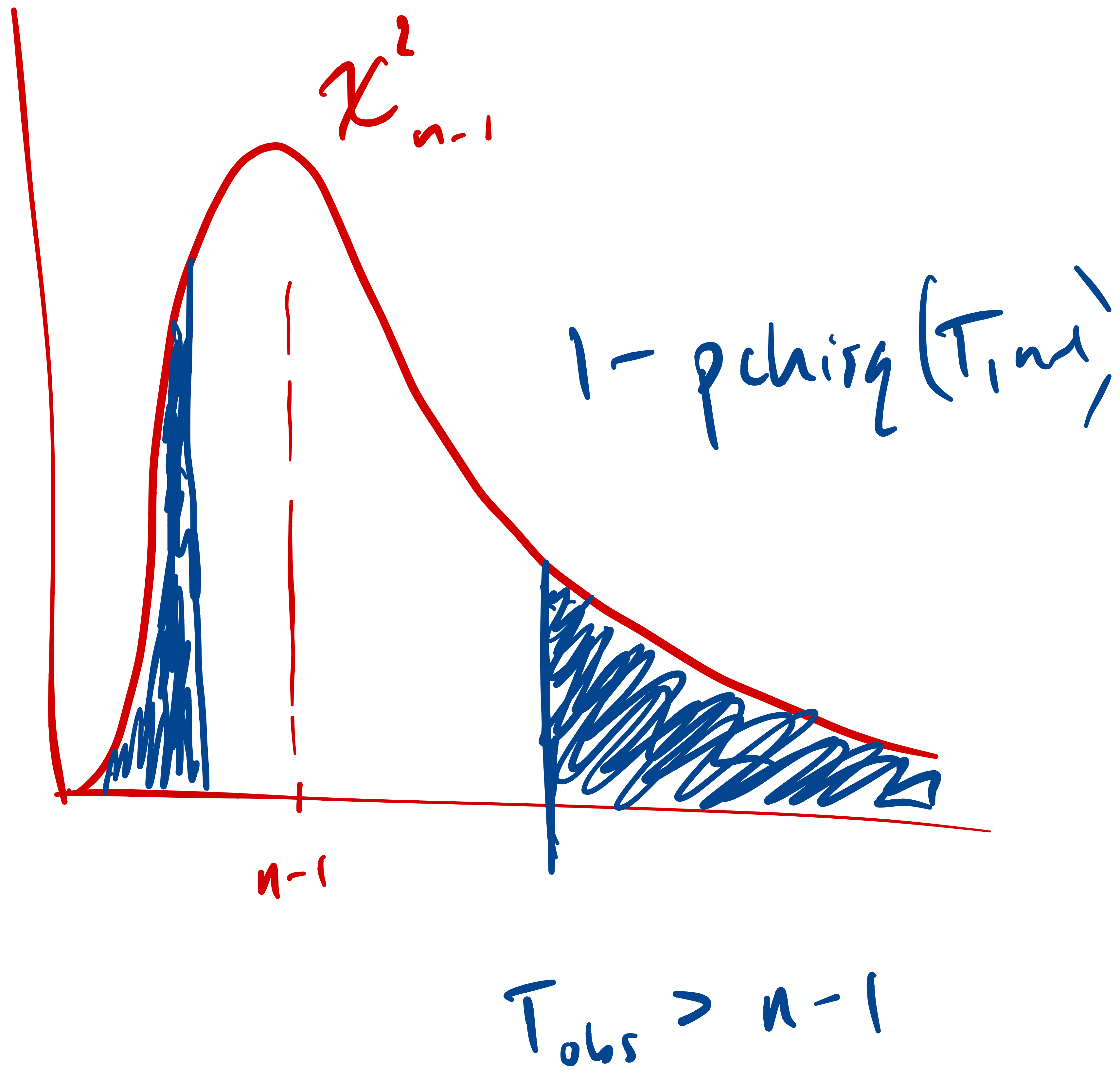
    - If $T_{obs} > n-1$, we have `2*(1-pchisq(Tobs,n-1))`



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- If the p-value is less than α, reject the null hypothesis and conclude that the population variances are unequal to each other
- If the p-value is greater than α, fail to reject the null hypothesis and conclude that the population variances are equal to each other

Hypothesis Tests for One Population Variance: Example

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- Setup: In 2010, the distribution of the amount of time spent per table at The Cheesecake Factory is normally distributed with $\mu = 65$ minutes and $\sigma^2 = 121$ minutes². This year, we took a sample of $n = 45$ guests and found that the sample variance $s = 196$ minutes². Has the variance changed? Evaluate at $\alpha = 0.01$

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- Hypotheses: $H_0 : \sigma^2 = 121$ vs. $H_1 : \sigma^2 \neq 121$
- Calculate the T statistic: $T = (n-1) \cdot \frac{s^2}{\sigma^2} = 44 \cdot \frac{196}{121} = 71.3 > 44$
- Calculate the p-value: $2 \times (1 - \text{pchisq}(71.3, 44)) = 0.0115$
- Conclusion: $p > \alpha$, fail to reject.

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$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \cancel{\bar{x}}\,\mu)^2 \sim \chi^2_n$$

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$$\frac{1}{n-1} \sum (x_i - \bar{x})^2 \sim \chi^2_{n-1}$$

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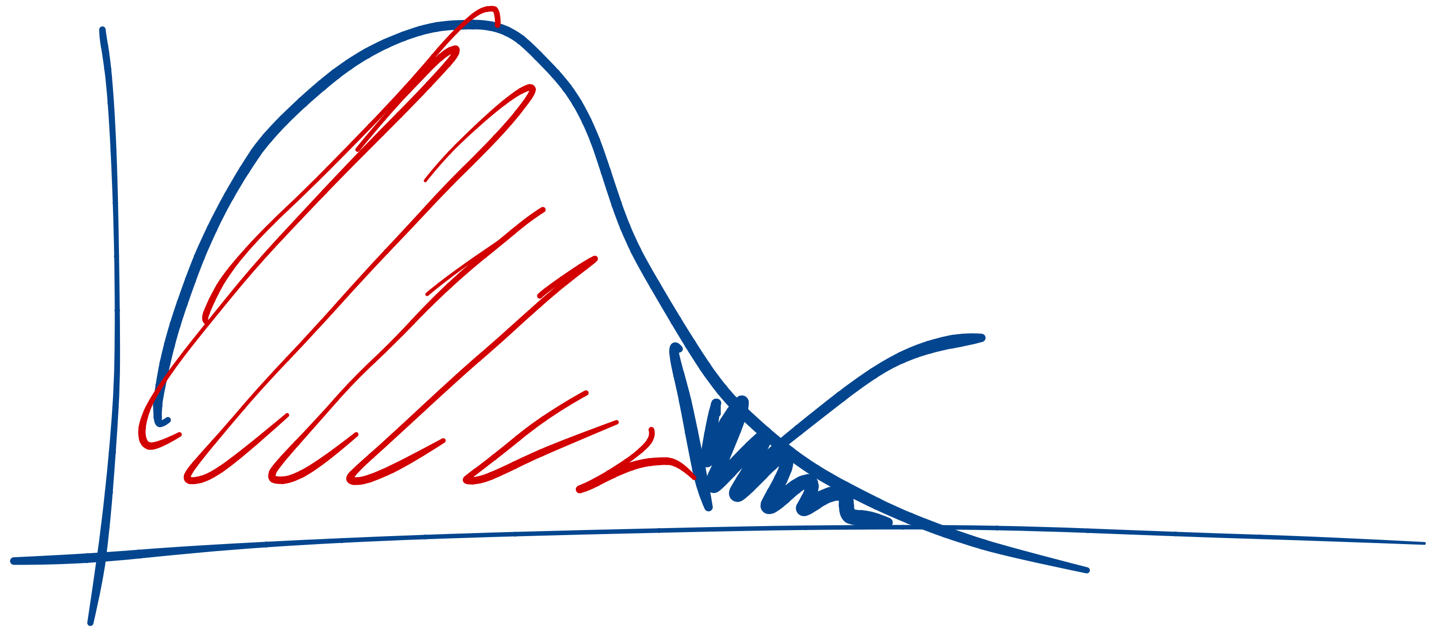
- Hypotheses: $H_0: \sigma^2 \le 121$ $\quad H_1: \sigma^2 > 121$
- Calculate the T statistic: $T = (n-1)\frac{s^2}{\sigma^2} = 71.3$



- Calculate the p-value: $1 - \text{pchisq}(71.3, 44) \approx 0.0055$
- Conclusion: $p < \alpha \rightarrow$ Reject. σ^2 Increased since 2010

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qchisq(α/2, n-1)

$$\Pr\left(\chi^2_{\alpha/2,n-1} \le (n-1)\cdot\frac{s^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right) = \alpha$$

$$\implies \Pr\left(\frac{1}{\chi^2_{\alpha/2,n-1}} \ge \frac{\sigma^2}{(n-1)\cdot s^2} \ge \frac{1}{\chi^2_{1-\alpha/2,n-1}}\right) = \alpha$$

$$\implies \Pr\left(\frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2,n-1}}\right) = \alpha$$

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$$\Pr\left(\chi^2_{\alpha/2,n-1}\le(n-1)\cdot\frac{s^2}{\sigma^2}\le\chi^2_{1-\alpha/2,n-1}\right)=\alpha$$

$$\implies\ \Pr\left(\frac{1}{\chi^2_{\alpha/2,n-1}}\ge\frac{\sigma^2}{(n-1)\cdot s^2}\ge\frac{1}{\chi^2_{1-\alpha/2,n-1}}\right)=\alpha$$

$$\implies\ \Pr\left(\frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2,n-1}}\le\sigma^2\le\frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2,n-1}}\right)=\alpha$$

qchisq(1 - α/2, n-1) qchisq(α/2, n-1)

- Therefore, the interval $\left(\dfrac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2,n-1}},\dfrac{(n-1)\cdot s^2}{\chi^2_{\alpha/2,n-1}}\right)$ contains σ^2 with probability $1-\alpha$

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  - $\left(0, \dfrac{(n-1) \cdot s^2}{\chi^2_{\alpha,n-1}}\right)$

$\chi^2_{\alpha,n-1}$

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- One-sided, upper:
  - $\left(0,\ \dfrac{(n-1)\cdot s^2}{\chi^2_{\alpha,n-1}}\right)$
- One-sided, lower:
  - $\left(\dfrac{(n-1)\cdot s^2}{\chi^2_{1-\alpha,n-1}},\ \infty\right)$

Confidence Intervals for One Population Variance: Example

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- Let's say we are interested in the amount of water the average person drinks per day. We sample a group of $n = 40$ people and find that the sample variance is $s^2 = 60$ oz^2. What is a two-tailed 95% confidence interval for the variance of this distribution?

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- What is α?

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$(n-1) \cdot \frac{s^2}{\sigma^2}$

$\frac{60}{40.26}$ $\frac{98.92}{60}$

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$(40.26, 98.92)$

- What is α? $\alpha = 0.05$
- CI: $\left(\frac{(n-1) \cdot s^2}{\chi^2_{1-\alpha/2,\, n-1}}, \frac{(n-1) \cdot s^2}{\chi^2_{\alpha/2,\, n-1}} \right) = \left(\frac{39 \cdot 60}{\chi^2_{0.975,\, 39}}, \frac{39 \cdot 60}{\chi^2_{0.025,\, 39}} \right)$

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- We have talked about testing whether the means of the two independent populations are equal to each other
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Hypothesis Tests for Two Population Variances

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F distribution

- However, from the above, we have that $\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim \boxed{\frac{\chi^2_{n_1-1}/(n_1-1)}{\chi^2_{n_2-1}/(n_2-1)}}$

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- It turns out that this type of distribution has a name: F distribution

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- Once we have our test statistic, we can compare to the F distribution to calculate a p-value

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    - If $F_{obs} \leq 1$, we have `2*pf(`$\texttt{F}_{\texttt{obs}}$`,`$\texttt{n}_1$`-1,`$\texttt{n}_2$`-1)`

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    - R: `pf(Fobs,n1-1,n2-1)`
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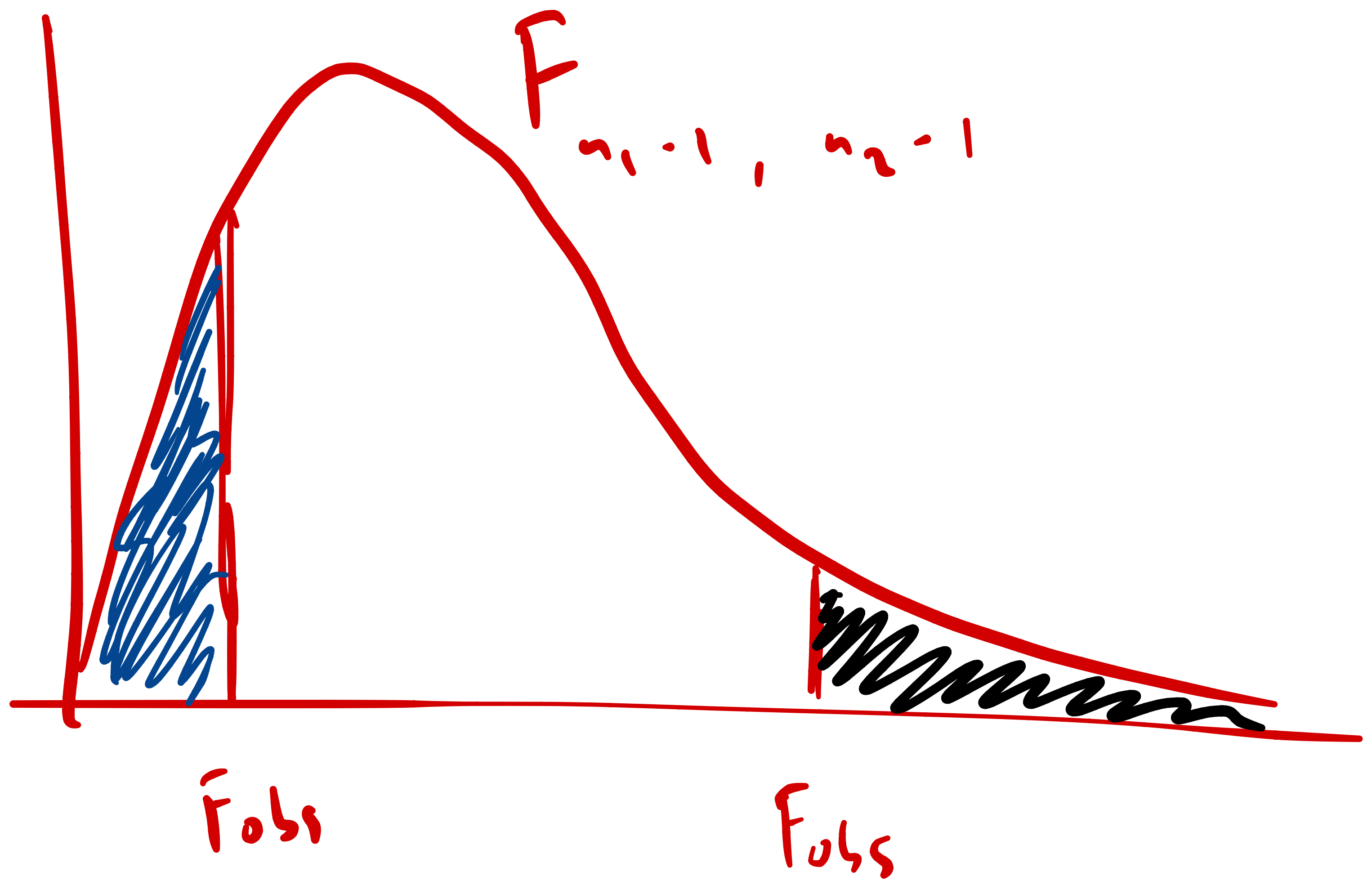
    - R: `1-pf(Fobs,n1-1,n2-1)`
  - Two-tailed hypothesis: $p = \Pr(|T| \geq |T_{obs}|)$
    - If $F_{obs} \leq 1$, we have `2*pf(Fobs,n1-1,n2-1)`
    - If $F_{obs} > 1$, we have `2*(1-pf(Fobs,n1-1,n2-1))`



$$F = \frac{\chi^2_{n_1-1} / (n_1-1)}{\chi^2_{n_2-1} / (n_2-1)}$$

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- If the p-value is less than α, reject the null hypothesis and conclude that the population variances are unequal to each other
- If the p-value is greater than α, fail to reject the null hypothesis and conclude that the population variances are equal to each other

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$F_{n_1-1,\,n_2-1,\,\alpha}$

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- Two-tailed hypothesis:
  - Reject if $F_{obs} \leq F_{n_1-1,n_2-1,\alpha/2}$ or $F_{obs} \geq F_{n_1-1,n_2-1,\cancel{\alpha/2}}$ $1-\alpha/2$

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- Take a sample of $n_1 = 10$ Civics and $n_2 = 20$ Accords
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- Evaluate at $\alpha = 0.05$

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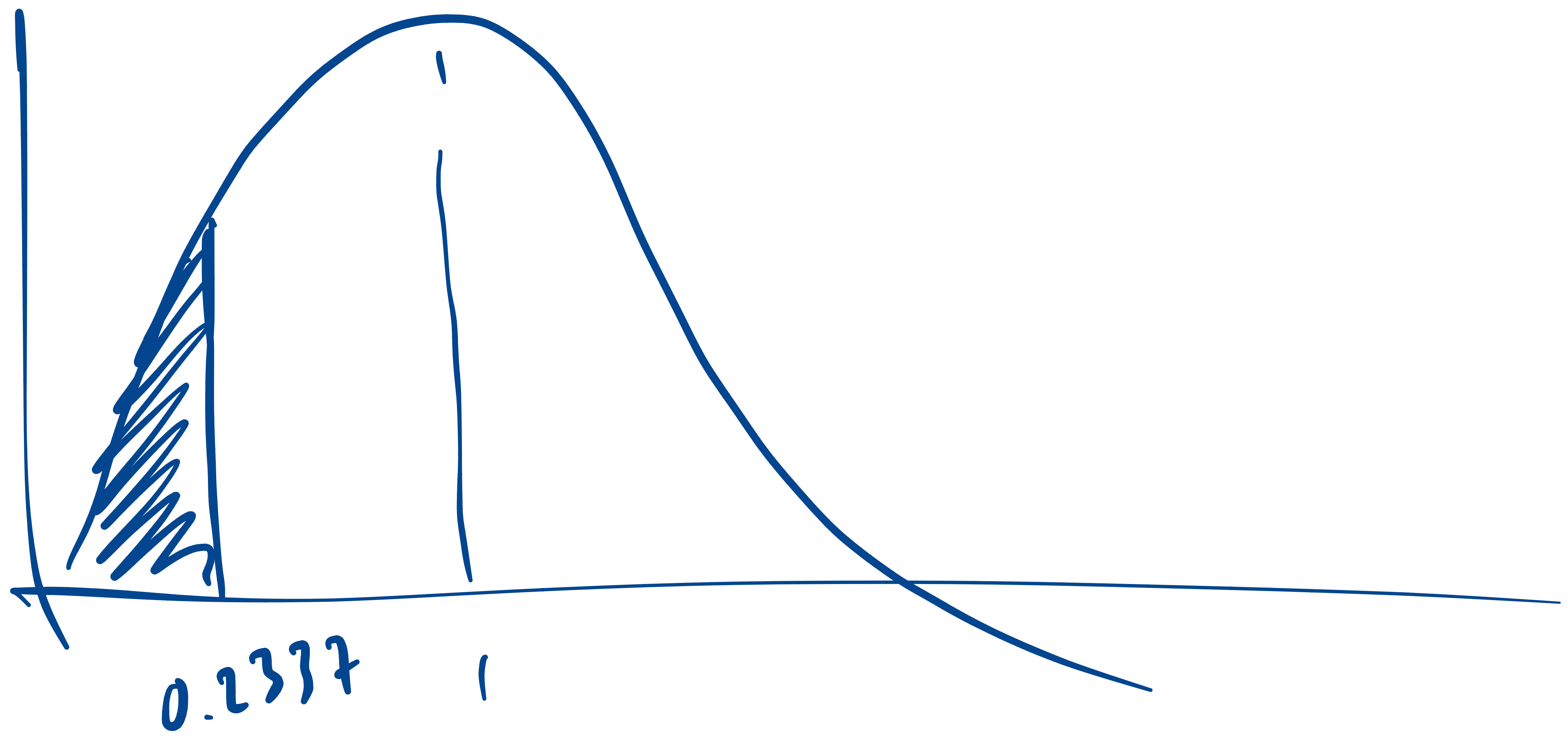
$$F = \frac{s_1^2}{s_2^2} = \frac{2.33}{9.47} \approx 0.2337$$

$$p = 2 \times pf(0.2337, 9, 19)$$

$$= \underline{0.03}$$

- Conclusion:



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- Similarly, we could have calculated the critical values for the F-test statistic and used them to complete the test:

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$$F = 0.2337$$

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$$F_{n_1-1,n_2-1,\alpha/2} = \texttt{qf(0.025,9,19)} = 0.2715$$
$$F_{n_1-1,n_2-1,1-\alpha/2} = \texttt{qf(0.975,9,19)} = 2.880$$

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- Since $F = 0.2337 < 0.2715$, we reject the null hypothesis and conclude $\sigma_1^2 \neq \sigma_2^2$

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$$\left(\frac{1}{F_{1-\alpha/2}}\cdot\frac{s_1^2}{s_2^2}, \frac{1}{F_{\alpha/2}}\cdot\frac{s_1^2}{s_2^2}\right)$$

$$\frac{1}{\chi^2_{n-1,\,1-\alpha/2}} \times \cdots \quad , \quad \frac{1}{\chi^2_{n-1,\,\alpha/2}} \times \cdots$$

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- A one-sided upper $(1-\alpha)\cdot 100\,\%$ confidence interval for σ_1^2/σ_2^2 is given by
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- A one-sided lower $(1-\alpha)\cdot 100\,\%$ confidence interval for σ_1^2/σ_2^2 is given by
$$\left(\frac{1}{F_{1-\alpha}}\cdot\frac{s_1^2}{s_2^2}, \infty\right)$$

Confidence Intervals for Two Population Variances: Example

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- What is α? $\alpha = 0.05$

  $(0, 0.689)$

- CI: $\left(0, \frac{1}{F_\alpha} \cdot \frac{s_1^2}{s_2^2}\right) = \left(0, \frac{1}{0.339} \cdot \frac{2.33}{9.97}\right)$

  $qf(0.05, 9, 19) = 0.339$