Independence vs. Mutual Exclusivity

- Independence and mutual exclusivity are not the same thing
- If A and B are mutually exclusive, then $\Pr(A \mid B) = 0$ and $\Pr(B \mid A) = 0$
- This is not the same thing as independence, where $\Pr(A \mid B) = \Pr(A)$ and $\Pr(B \mid A) = \Pr(B)$
- Independence: the other event still may occur; its probability is unaffected

Multiplicative Rule

- The *multiplicative rule of probability* tells us the following:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B \,|\, A)$$

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- Rearranging yields *conditional probability expressions*:

$$\Pr(B \,|\, A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

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Mutual Independence

- Suppose we have n events, N. These n events are **mutually independent** iff, for every subset of events $M \subseteq N$, we have

$$\Pr\left(\bigcap_{i \in M} A_i\right) = \prod_{i \in M} \Pr(A_i)$$

- Consider the case of $n = 3$. Events A_1, A_2, A_3 are independent iff the following hold:

$$\begin{aligned}
\Pr(A_1 \cap A_2) &= \Pr(A_1) \cdot \Pr(A_2) \\
\Pr(A_1 \cap A_3) &= \Pr(A_1) \cdot \Pr(A_3) \\
\Pr(A_2 \cap A_3) &= \Pr(A_2) \cdot \Pr(A_3) \\
\Pr(A_1 \cap A_2 \cap A_3) &= \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3)
\end{aligned}$$

- If all but the last equality hold, A_1, A_2, A_3 are *pairwise independent*, but not mutually independent

Bayes' Theorem

- Let's say you have an idea of $\Pr(B \mid A)$ but want to know about $\Pr(A \mid B)$
- Recall that $\Pr(A \mid B) \cdot \Pr(B) = \Pr(B \mid A) \cdot \Pr(A) = \Pr(A \cap B)$
- Rearranging yields Bayes' Theorem:

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B \mid A) \cdot \Pr(A) + \Pr(B \mid A^c) \cdot \Pr(A^c)}$$

Posterior   Likelihood   Prior

Stars and Bars: Intuition

- How many ways are there of choosing three *positive* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?
  - $\binom{6-1}{3-1} = \binom{5}{2}$: ★ | ★ ★ | ★ ★ ★
- How many ways are there of choosing three *nonnegative* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?
  - $\binom{6+3-1}{3-1} = \binom{8}{2}$: | ★ ★ | ★ ★ ★ ★

Thus, we only need to choose k – 1 of the n + k – 1 positions to be bars (or, equivalently, choose n of the positions to be stars). (https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics))

Independence

- **Independence**: The outcome of one event has no effect on the outcome of another event
    - If A and B are independent, then $\Pr(A \mid B) = \Pr(A)$ (and $\Pr(B \mid A) = \Pr(B)$)
- This is because intersection is decomposable:
    - If A and B are independent, then $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$
    - From this, we see that $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{\Pr(A) \cdot \Pr(B)}{\Pr(B)} = P(A)$

Stars and Bars: More Formally

- Suppose there are n objects and k bins. Bins are distinguishable, but objects are not. The only thing we care about is the number of objects in each bin.

- If each bin has to have at least one object in it:

  - Total number of ways = $\binom{n-1}{k-1}$ (think of filling in gaps between objects)

- For nonnegative (not positive) constraints:

  Thus, we only need to choose k – 1 of the n + k – 1 positions to be bars (or, equivalently, choose n of the positions to be stars). (https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics))

  - Total number of ways = $\binom{n+k-1}{k-1}$ (think of arranging n objects and $k-1$ dividers)

Stars and Bars: Example

- Setup: Six children are choosing ice cream flavors from {vanilla, strawberry, chocolate, caramel}. Each child picks exactly one flavor. Requests are placed in the form: {# vanilla, # strawberry, # chocolate, # caramel}.

- Q1: How many different requests are possible if at least one child must choose each flavor?

- Q2: How many different requests are possible without this restriction?

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  Stars: Children
  Bars: Flavor dividers

$$\binom{6-1}{4-1} = \binom{5}{3} = 10$$

- Q2: How many different requests are possible without this restriction?

$$\binom{6+4-1}{4-1} = \binom{9}{3} = 84$$