# Chapter 14: Inference for Correlation

DSCC 462

Computational Introduction to Statistics

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# Final Project Announcements

- Teams are due by Friday: Use the Google Sheet (see Announcements) to sign up / find teammates
- Final project description is up on Blackboard
  - Setup: Ad dataset, company wants to understand their data…
  - Some open-ended questions, some less so (room for interpretation!)
- Datasets will be released once teams are formed (Friday)

# Plan for Today

- Introduce tools that allow us to go past our previous assumptions of independence when comparing random variables
- In particular, learn how to infer whether or not a linear relationship exists between two variables, $X$ and $Y$

# Correlation

- Recall that correlation tells us the degree to which two random variables are (linearly) associated or related

- We denote the true population correlation of $X$ and $Y$ as $\rho$ ("rho")

- We estimate $\rho$ with Pearson's coefficient of correlation, $r$ (Chapter 3)

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]\left[\sum_{i=1}^{n}(y_i - \bar{y})^2\right]}} = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

- Note that $-1 \leq r \leq 1$

# Correlation: Example

- Setup: Suppose we examine $n = 7$ subjects for which we have age and weight measurements
- We want to determine whether a significant linear relationship exists between age ($X$) and weight ($Y$)

# Correlation: Inference

- Question: Is the population correlation, $\rho$, between two variables, $X$ and $Y$, different from 0?

- In other words, we're investigating whether a linear relationship exists between $X$ and $Y$ (age and weight)

- Use the sample correlation $r$ for our statistic in hypothesis testing

# Correlation: Inference

- Hypotheses: $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$
- If pairs $(x_i, y_i)$ come from normally distributed $X$ and $Y$, then if we standardize $r$, we get a statistic that has a t distribution with $n - 2$ degrees of freedom
- Test statistic: $t = \dfrac{r - \rho}{SE(r)}$, where $SE(r) = \sqrt{\dfrac{1 - r^2}{n - 2}}$
- We thus get $t = \dfrac{r - \rho}{SE(r)} = r\sqrt{\dfrac{n - 2}{1 - r^2}}$

# Correlation: Inference Example

- Returning to setup: Suppose we examine $n = 7$ subjects for which we have age and weight measurements
- We want to determine whether a significant linear relationship exists between age ($X$) and weight ($Y$)
  - $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$
- We know that the correlation between weight and age for this sample is $r = 0.865$
- Test this null hypothesis at the $\alpha = 0.05$ significance level

# Correlation: Inference Example

- $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$, test at $\alpha = 0.05$
- The correlation between weight and age for this sample is $r = 0.865$
- Test statistic:
- p-value:
- Conclusion:

# Correlation: Inference Example

- $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$, test at $\alpha = 0.05$
- The correlation between weight and age for this sample is $r = 0.865$
- Test statistic:

$$t = r\sqrt{\frac{n-2}{1-r^2}} = 0.865 \cdot \sqrt{\frac{7-2}{1-0.865^2}} = 3.856$$

- p-value:

$p = 2 \cdot \Pr(T \geq 3.856) =$ `2*(1-pt(3.856,df=5))=0.0119`

- Conclusion: Since the p-value is less than $\alpha = 0.05$, we reject the null hypothesis and conclude that a linear relationship does exist between age and weight

# Correlation: Inference in R

- We can also calculate directly in R using the `cor.test()` function

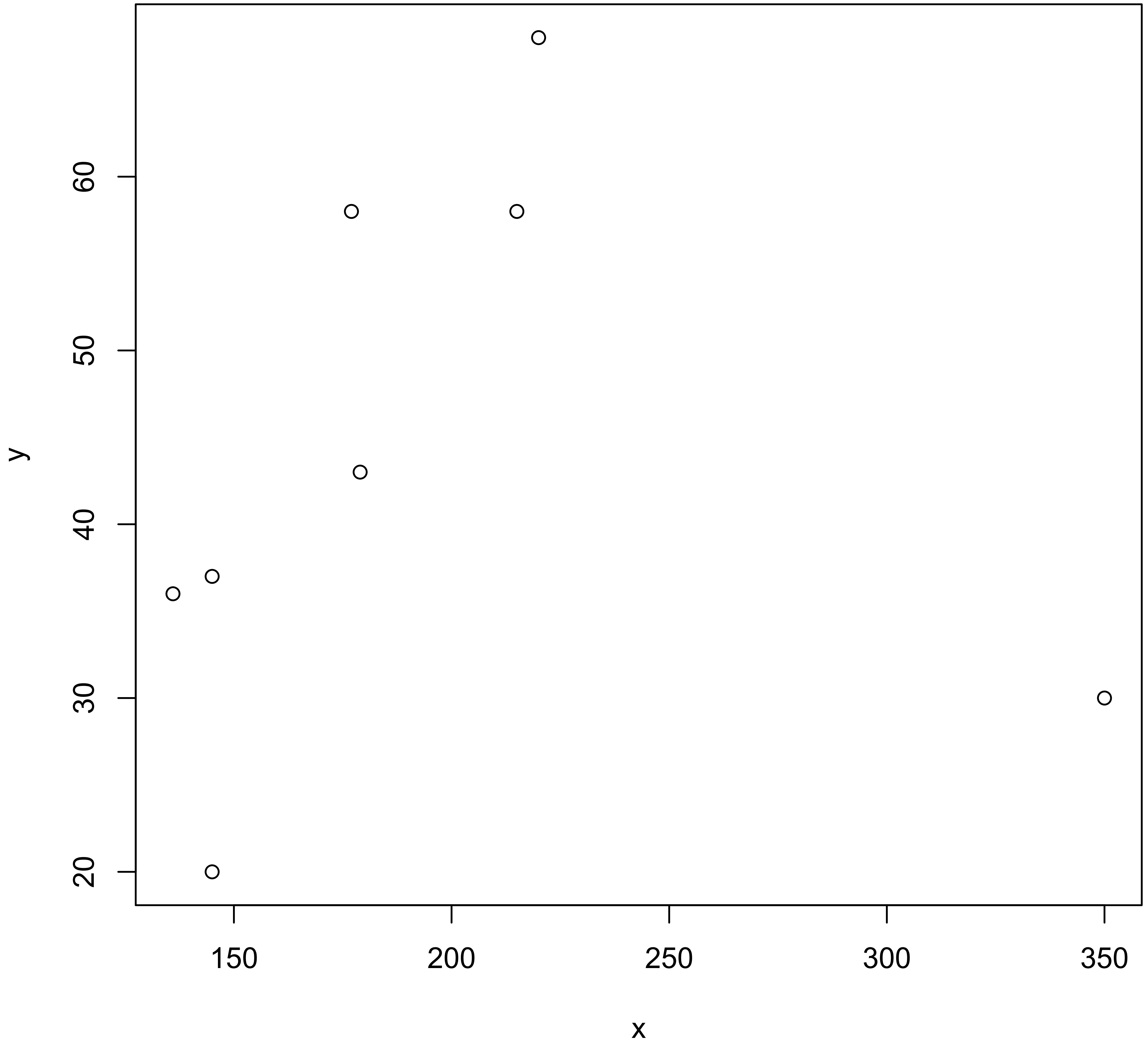
```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y)

	Pearson's product-moment correlation

data:  x and y
t = 3.856, df = 5, p-value = 0.01193
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.3213733 0.9798243
sample estimates:
      cor
0.8650727
```

# Correlation Limitations

- Only describes linear relationships
  - We could be missing a nonlinear relationship if we don't examine the scatterplot
- Hypothesis testing only works for the null hypothesis $H_0 : \rho = 0$
  - For any $\rho \neq 0$, normality assumptions are not met and our hypothesis testing procedures are invalid
- Correlation can be very sensitive to outliers and can thus give misleading results when outliers are present

# Correlation Limitations

- Suppose we have another subject who is 30 years old with a weight of 350 pounds



# Correlation Limitations

```
> x<-c(220,215,179,145,145,177,136, 350)
> y<-c(68,58,43,37,20,58,36, 30)
> plot(x,y)
> cor(x,y)
[1] 0.06260467
```

# Spearman's Rank Correlation Coefficient

- Need a more robust measure that isn't as sensitive to outliers
- Instead of using the actual observations, we rank the data and then use the *ranks* as our data
  - If multiple values are the same, assign the average rank
- Rank all the $x$ values, and call these ranks $x_r$
- Rank all the $y$ values, and call these ranks $y_r$
- Compute the Pearson's correlation coefficient for this ranked data $(x_r, y_r)$ instead of the actual data
- This is *Spearman's correlation coefficient*

# Spearman's Rank Correlation Coefficient

- Calculate Spearman's correlation coefficient as follows:

$$r_s = \frac{\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)(y_{ri} - \bar{y}_r)}{\sqrt{\left[\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)^2\right]\left[\sum_{i=1}^{n} (y_{ri} - \bar{y}_r)^2\right]}}$$

- Note that $-1 \leq r_s \leq 1$
- Same interpretations of association between $X$ and $Y$

# Spearman’s Rank Correlation Coefficient

- Consider the following age and weight measurements for the 7 subjects

| Patient | Weight | Rank | Age | Rank |
|---|---|---|---|---|
| 1 | 220 | | 68 | |
| 2 | 215 | | 58 | |
| 3 | 179 | | 43 | |
| 4 | 145 | | 37 | |
| 5 | 145 | | 20 | |
| 6 | 177 | | 58 | |
| 7 | 136 | | 36 | |

# Spearman’s Rank Correlation Coefficient

- Consider the following age and weight measurements for the 7 subjects

| Patient | Weight | Rank | Age | Rank |
|---|---|---|---|---|
| 1 | 220 | 7 | 68 | 7 |
| 2 | 215 | 6 | 58 | 5.5 |
| 3 | 179 | 5 | 43 | 4 |
| 4 | 145 | 2.5 | 37 | 3 |
| 5 | 145 | 2.5 | 20 | 1 |
| 6 | 177 | 4 | 58 | 5.5 |
| 7 | 136 | 1 | 36 | 2 |

# Spearman's Rank Correlation Coefficient

- Using this information, we can calculate $r_s$ as follows:
  - $\bar{x}_r =$
  - $\bar{y}_r =$
  - $r_s =$
- Recall that Pearson's correlation coefficient was given as 0.865. How does Spearman's rank correlation coefficient compare?

# Spearman's Rank Correlation Coefficient

- Using this information, we can calculate $r_s$ as follows:
  - $\bar{x}_r = 4$
  - $\bar{y}_r = 4$
  - $$r_s = \frac{\sum_{i=1}^{n}(x_{ri} - 4)(y_{ri} - 4)}{\sqrt{\left[\sum_{i=1}^{n}(x_{ri} - 4)^2\right]\left[\sum_{i=1}^{n}(y_{ri} - 4)^2\right]}} = 0.8727$$
- Recall that Pearson's correlation coefficient was given as 0.865. How does Spearman's rank correlation coefficient compare? Matches closely because we had normally distributed data and no obvious outliers

# Spearman's Rank Correlation Coefficient: R

- In R:

```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor(x,y, method="spearman")
[1] 0.8727273
>
> x1 <- rank(x)
> y1 <- rank(y)
> x1; y1
[1] 7.0 6.0 5.0 2.5 2.5 4.0 1.0
[1] 7.0 5.5 4.0 3.0 1.0 5.5 2.0
> cor(x1,y1)
[1] 0.8727273
```

# Spearman’s Rank Correlation Coefficient: Interpretation

- Spearman’s rank correlation coefficient is a measure of concordance of ranks for the outcomes $x$ and $y$
- If measurements of $X$ and $Y$ are ranked in the same order for each variable, then $r_s = 1$
- If measurements of $X$ and $Y$ are ranked in the reverse order from each other, then $r_s = -1$
- If there is no linear correspondence between the ranks, then $r_s = 0$

# Spearman’s Rank Correlation Coefficient: Outliers

- What is the Spearman rank correlation coefficient when we have an outlier in our data?

```
> x<-c(220,215,179,145,145,177,136,350)
> y<-c(68,58,43,37,20,58,36,30)
> cor(x,y)
[1] 0.06260467
> cor(x,y, method="spearman")
[1] 0.373494
```

# Spearman's Rank Correlation Coefficient: Inference

- We can similarly perform hypothesis tests for $\rho$ based on $r_s$
- If the sample size is large enough (generally $n \geq 10$) and if we can assume that pairs of ranks $(x_{ri}, y_{ri})$ are chosen randomly, then we can test the null hypothesis $H_0 : \rho = 0$ vs. the alternative hypothesis $H_1 : \rho \neq 0$
- Use a similar test statistic: $t_s = r_s \cdot \sqrt{\dfrac{n-2}{1-r_s^2}}$
- Compare $t_s$ to a t distribution with $n - 2$ degrees of freedom

# Spearman's Rank Correlation Coefficient: Inference

- Although we only have $n = 7$, let's test $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$ at $\alpha = 0.05$
- Calculating our t-statistic:
- Calculating the p-value:
- Conclusion:

# Spearman's Rank Correlation Coefficient: Inference

- Although we only have $n = 7$, let's test $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$ at $\alpha = 0.05$
- Calculating our t-statistic:

$$t_s = r_s \cdot \sqrt{\frac{n-2}{1-r_s^2}} = 0.8727 \cdot \sqrt{\frac{7-2}{1-0.8727^2}} = 3.997$$

- Calculating the p-value:

$p = 2\Pr(T > 3.997) =$ `2*(1-pt(3.997,df=5))` $= 0.01035$

- Conclusion:

$p < \alpha = 0.05$, so we reject $H_0$ and conclude nonzero correlation

# Spearman’s Rank Correlation Coefficient: R

```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y,method="spearman",exact=FALSE)

	Spearman's rank correlation rho

data:  x and y
S = 7.1273, p-value = 0.01035
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.8727273
```