Chapter 8: Hypothesis Testing with Two Samples

DSCC 462

Computational Introduction to Statistics

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Comparison of Two Means

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- Must determine whether the two samples are paired or independent
  - Paired: Weight before and after surgery for a group of men
  - Independent: Height of Americans compared to height of Canadians

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- *Matched pairing*: Match two individuals with similar demographics / characteristics and compare their differences in response
    - Make the pair as similar as possible with respect to important characteristics (e.g., age, gender, socioeconomic status, etc.) depending on the setting

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- In general, pairing makes comparisons more *precise*

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- In the situation of paired samples, our data often is the difference between elements in each pair
  - Sample 1: $x_{11}, x_{21}, \ldots, x_{n1}$
  - Sample 2: $x_{12}, x_{22}, \ldots, x_{n2}$
  - Difference: $d_1 = x_{11} - x_{12}, d_2 = x_{21} - x_{22}, \ldots, d_n = x_{n1} - x_{n2}$

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- These 40 people then run on a treadmill for 30 minutes, sit for 5 minutes, and then have their heart rate measured
- Sample 1: “before” heart rate
- Sample 2: “after” heart rate
- Goal: We are interested in how heart rate changed after running; we care about *before – after*

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- Mean: $\bar{x}_d$, sample standard deviation s_d (unknown true σ_d)
- Standard error: $\dfrac{s_d}{\sqrt{n}}$
- Assumption: $\bar{x}_d \sim N\left(\mu_d, \sigma_d/\sqrt{n}\right)$

$n = 40$

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- Reject H_0 if $p \leq \alpha$

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d

Before	**After**	**Difference**
55	60	-5
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- Does heart rate significantly change after running on a treadmill?
- Test at the $\alpha = 0.01$ significance level

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- $\bar{x}_d = \dfrac{-5 - 13 - 4 - 17 - 13}{5} = -10.4$
- $s_d^2 = \dfrac{(-5 + 10.4)^2 + (-13 + 10.4)^2 + (-4 + 10.4)^2 + (-17 + 10.4)^2 + (-13 + 10.4)^2}{5 - 1} = 31.8$

Paired Samples: Example

$d = b - a$

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- $s_d = \sqrt{31.8} = 5.64$

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- Calculate t statistic: $t = \dfrac{\bar{x}_d - \mu_d}{s_d/\sqrt{n}} = \dfrac{-10.4 - 0}{5.64/\sqrt{5}} = -4.12$

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- In R: `2*pt(-4.12,4) = 0.0146`

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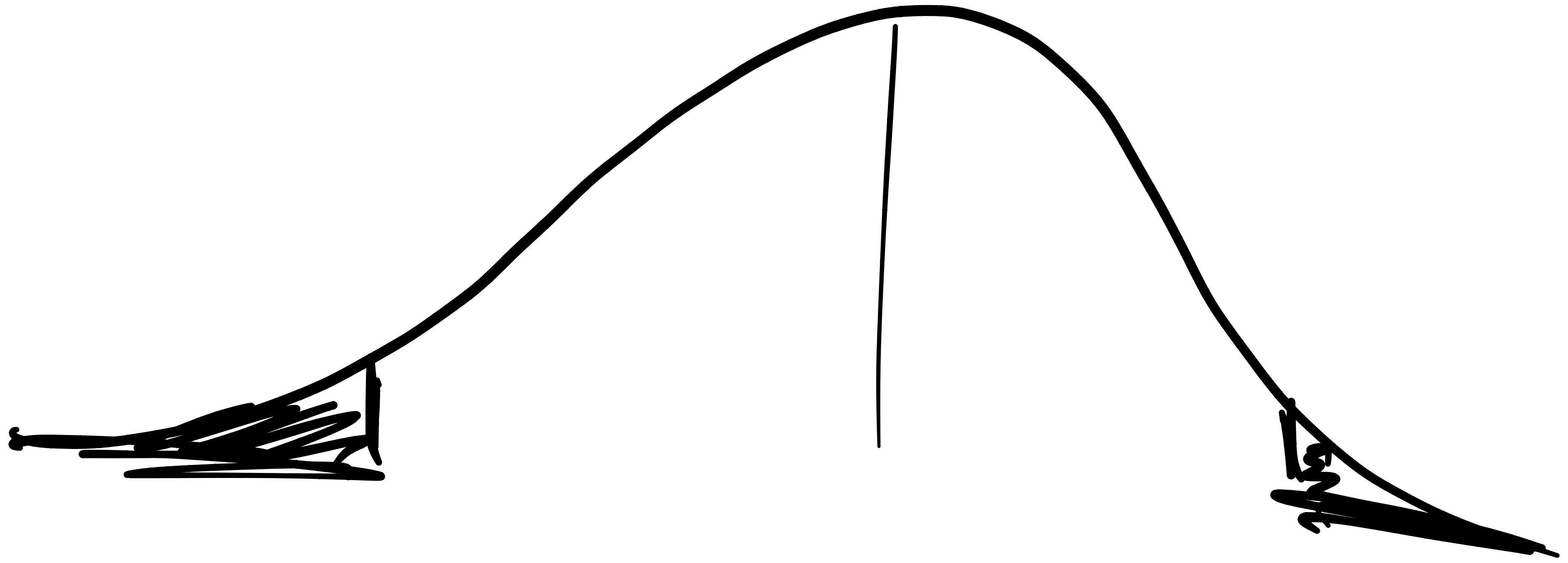
Paired Samples: Example

$H_0: \mu_d \geq 0$

$H_1: \mu_d < 0$

- $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d \neq 0$ ← 2-tailed
- Calculate t statistic: $t = \dfrac{\bar{x}_d - \mu_d}{s_d/\sqrt{n}} = \dfrac{-10.4 - 0}{5.64/\sqrt{5}} = -4.12$
- Degrees of freedom: df = 5 - 1 = 4
- In R: `2*pt(-4.12,4) = 0.0146`
- Since $0.0146 > \alpha = 0.01$, we fail to reject H_0
- There is insufficient evidence to conclude that the average heart rate after running on a treadmill for 30 minutes is significantly different than the average sitting heart rate at the $\alpha = 0.01$ significance level



n=5

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- Continuing with the heart rates example, we can get the following 99% confidence interval:

$$\bar{x}_d \pm t_{\alpha/2}\frac{s_d}{\sqrt{n}} = -10.4 \pm \overbrace{4.60}^{t_{0.01/2,\,4}} \cdot \frac{5.65}{\sqrt{5}}$$

$$= (-21.62, 1.62)$$

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$$= (-21.62, 1.62)$$

- I am 99% confident that the interval (-21.62, 1.62) captures the true mean before-after difference in heart rate for subjects running on a treadmill

Paired Samples: R Code

```
> before <- c(55, 62, 61, 72, 57)
>  after <- c(60, 75, 65, 89, 70)
>  d <- before-after
> t.test(d)

	One Sample t-test

data:  d
t = -4.1239, df = 4, p-value = 0.01457
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -17.401928  -3.398072
sample estimates:
mean of x
    -10.4

> t.test(before, after, paired=T)

	Paired t-test

data:  before and after
t = -4.1239, df = 4, p-value = 0.01457
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
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sample estimates:
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- Ten pairs of data were taken measuring zinc concentration in bottom water and surface water (two layers of water from same sources)
- Based on this sample, the mean difference in zinc concentrations between the bottom water and surface water was $\bar{x}_d = 0.0804$ and the standard deviation was $s_d = 0.9023$

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- Ten pairs of data were taken measuring zinc concentration in bottom water and surface water (two layers of water from same sources)
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b - s

- Q: Does the true average zinc concentration in the bottom water exceed that of surface water?

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b has more than top.

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- Calculate t: $t = \dfrac{\bar{x}_d - \mu_d}{s_d/\sqrt{n}} = \dfrac{0.0804 - 0}{0.0923/\sqrt{10}} = \underline{2.88}$

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- R: `1-pt(2.88,9) = 0.0091`

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- R: `1-pt(2.88,9) = 0.0091`
- Since $0.0091 < \alpha = 0.05$, we reject H_0
- There is sufficient evidence to conclude that, on average, the bottom zinc concentration is higher than the surface zinc concentration

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  - $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$

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$\bar{x} \sim \bar{X}$

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- The two underlying populations are independent and normally distributed
- The first population has mean μ_1 and the second population has mean μ_2
- Test whether the two populations are identical
  - $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$ $\rightarrow$ $\bar{x}_1 \sim \bar{X}_1 = \bar{x}_2 \sim \bar{X}_2$
  - Or, equivalently, $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 \neq 0$

$N(\mu, \sigma)$

$\bar{x}_1 - \bar{x}_2 \sim \bar{X}_1 - \bar{X}_2$

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  - Unequal population variances: $\sigma_1^2 \neq \sigma_2^2$

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- Suppose we are interested in the difference between the two population means
- By extension of the CLT, $\overline{X}_1 - \overline{X}_2$ is approximately normal with mean $\mu_1 - \mu_2$ and standard error $\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$

$$\text{Var}(\bar{X}_1 - \bar{X}_2) = \text{Var}(\bar{X}_1) + \text{Var}(\bar{X}_2)$$

$$\sigma^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$\sigma = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Equal, Known Variances

- Consider situations where it is either known or reasonable to assume that the two population variances are the same, and we know $\sigma^2 = \sigma_1^2 = \sigma_2^2$

- Suppose we are interested in the difference between the two population means

- By extension of the CLT, $\overline{X}_1 - \overline{X}_2$ is approximately normal with mean $\mu_1 - \mu_2$ and standard error $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

$$\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}$$

- Because we assume $\sigma^2 = \sigma_1^2 = \sigma_2^2$, the standard error is $\sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$ $= \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

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- Because we assume $\sigma^2 = \sigma_1^2 = \sigma_2^2$, the standard error is $\sqrt{\sigma^2 \left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}$
- Then, we can use a z-test with $z = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ as our test statistic

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  - $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{(n_1 - 1) + (n_2 - 1)}$
- In other words, s_p^2 is the weighted average of the two sample variances, s_1^2 and s_2^2, where the weights are the degrees of freedom for each sample

Equal, Unknown Variances

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- We can use a t-test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ $= \sqrt{\dfrac{s_p^2}{n_1} + \dfrac{s_p^2}{n_2}}$

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- Reject H_0 if $p \leq \alpha$

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- Setup: A study examines whether targeted political ads makes potential voters more likely to donate money to candidates
  - A sample of 54 potential voters who saw targeted political ads had mean donation amount $\bar{x}_1 = 240$ USD with standard deviation $s_1 = 30$ USD
  - A sample of 49 potential voters who did not see targeted political ads had mean donation amount $\bar{x}_2 = 220$ USD with standard deviation $s_2 = 48$ USD

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- Setup: A study examines whether targeted political ads makes potential voters more likely to donate money to candidates
  - A sample of 54 potential voters who saw targeted political ads had mean donation amount $\bar{x}_1 = 240$ USD with standard deviation $s_1 = 30$ USD
  - A sample of 49 potential voters who did not see targeted political ads had mean donation amount $\bar{x}_2 = 220$ USD with standard deviation $s_2 = 48$ USD
  - Is there a significant difference (at the $\alpha = 0.05$ significance level) in average donations between potential voters who saw targeted political ads and those who did not, assuming equal variances?

Q

Equal, Unknown Variances: Example

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- Hypotheses: $H_0 : \mu_1 - \mu_2 = 0$, $H_1 : \mu_1 - \mu_2 \neq 0$, significance $\alpha = 0.05$

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- Calculate the pooled (sample) variance:

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{(n_1-1)+(n_2-1)} = \frac{(54-1)30^2 + (49-1)48^2}{54-1+49-1}$$

$$= 1567$$

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1567

- Next, calculate the t-statistic:

$$t = \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(220 - 240) - 0}{\sqrt{1567\left(\frac{1}{59} + \frac{1}{48}\right)}} = \boxed{2.56}$$

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n_1 n_2

- We compare this t-value with a t distribution with (54 - 1) + (49 - 1) = 101 df

$t = 2.561$

Equal, Unknown Variances: Example

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- In R: `2*(1-pt(2.561,101)) = 0.012`

t df

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- In R: `2*(1-pt(2.561,101)) = 0.012`
- Since the p-value is 0.012 < 0.05, we reject the null hypothesis
- There is significant evidence to conclude that the average donation amount from potential voters who saw targeted political ads is significantly different from the average donation amount from potential voters who did not see targeted political ads

Equal, Unknown Variances: Example

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- What if we want a $(1-\alpha)\,\%$ confidence interval for $\mu_1 - \mu_2$?

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$$= (4.50, 35.50)$$

σ_1 ✗ σ_2 ✗ s_1 ✓ s_2 ✓

Equal, Unknown Variances: Example

ad

no ad

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$$= (4.50, 35.50)$$

- We are 95% confident that the interval (4.50, 35.50) captures the true difference in average donation amounts

Unequal Variance (Welch t-test)

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  t, df

  - Use an approximation

Unequal Variance (Welch t-test)

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" nu

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pt (t , v)

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Unequal Variance (Welch t-test): Example

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- Setup: Are average blood calcium levels different for people over 60 years old compared to 10-30 year olds?
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  - Sample standard deviation: 0.92
- Test at $\alpha = 0.05$ significance level

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Unequal Variance (Welch t-test): Example

- Test at $\alpha = 0.05$ significance level
- Let μ_1 be the average blood calcium level for people over 60 years old

Unequal Variance (Welch t-test): Example

- Test at $\alpha = 0.05$ significance level
- Let μ_1 be the average blood calcium level for people over 60 years old
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Unequal Variance (Welch t-test): Example

- Test at $\alpha = 0.05$ significance level
- Let μ_1 be the average blood calcium level for people over 60 years old
- Let μ_2 be the average blood calcium level for people between 10-30 years old
- Hypotheses: $H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 \neq 0$

Unequal Variance (Welch t-test): Example

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$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \cancel{(\mu_1 - \mu_2)}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{9.3 - 10.6}{\sqrt{\frac{1.86^2}{15} + \frac{0.92^2}{7}}} = -2.19$$

- Calculate degrees of freedom (ν):

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}\right]} = \frac{\left(\frac{1.86^2}{15} + \frac{0.92^2}{7}\right)^2}{\left[\frac{\left(\frac{1.86^2}{15}\right)^2}{14} + \frac{\left(\frac{0.92^2}{7}\right)^2}{6}\right]} = 19.82$$

- R:

15 7

$$p = 2 \times pt(-2.19, 19.82) = 0.0407$$

Unequal Variance (Welch t-test): Example

- Calculate t-statistic:

- Calculate degrees of freedom (ν):

- R:

- Since the p-value is less than $\alpha = 0.05$, we reject the null hypothesis and conclude a difference in average blood calcium level between the two age groups

Unequal Variance (Welch t-test): Example

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- Similarly, we can calculate a $(1-\alpha)\,\%$ confidence interval for $\mu_1 - \mu_2$ as

$$\underline{(\bar{x}_1 - \bar{x}_2)} \pm \underline{t_{\alpha/2}}\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)} \qquad SE$$

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- We are $(1-\alpha)\,\%$ confident that this interval contains the true mean difference $\mu_1 - \mu_2$

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- We are $(1-\alpha)\,\%$ confident that this interval contains the true mean difference $\mu_1 - \mu_2$

- For our blood calcium example, a 95% confidence interval for the difference in average blood calcium levels for 60+ year olds and 10-30 year olds is

Unequal Variance (Welch t-test): Example

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- We are $(1-\alpha)\,\%$ confident that this interval contains the true mean difference $\mu_1 - \mu_2$

- For our blood calcium example, a 95% confidence interval for the difference in average blood calcium levels for 60+ year olds and 10-30 year olds is

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)} = (9.3 - 10.6) \pm 2.09\sqrt{\left(\frac{1.86^2}{15} + \frac{0.92^2}{7}\right)} = (-2.54, -0.06)$$

Unequal Variance (Welch t-test): Example

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- I am 95% confident that the interval (-2.54, -0.06) captures the true difference in average blood calcium levels between 60+ year olds and 10-30 year olds

Unequal Variance (Welch t-test): Example

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- Setup: Do babies born in Rochester have larger birthweights than babies born in Buffalo?

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- Let μ_1 be the mean birthweight of babies born in Rochester
- Let μ_2 be the mean birthweight of babies born in Buffalo

Unequal Variance (Welch t-test): Example

- Setup: Do babies born in Rochester have larger birthweights than babies born in Buffalo?

- Let μ_1 be the mean birthweight of babies born in Rochester

- Let μ_2 be the mean birthweight of babies born in Buffalo

- Assume unequal variances

Unequal Variance (Welch t-test): Example

- Setup: Do babies born in Rochester have larger birthweights than babies born in Buffalo?
- Let μ_1 be the mean birthweight of babies born in Rochester
- Let μ_2 be the mean birthweight of babies born in Buffalo
- Assume unequal variances
- Let $\alpha = 0.05$ be our significance level

Unequal Variance (Welch t-test): Example

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- Hypotheses: $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$, samples of sizes $n_1 = 140$ and $n_2 = 172$

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- $\bar{x}_1 = 8.2$ lbs, $\bar{x}_2 = 7.9$ lbs, $s_1^2 = 1.4$ lbs^2, and $s_2^2 = 1.1$ lbs^2

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- Calculating $t =$

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- Calculating $t =$
- Calculating $\nu =$

Unequal Variance (Welch t-test): Example

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- $\bar{x}_1 = 8.2$ lbs, $\bar{x}_2 = 7.9$ lbs, $s_1^2 = 1.4$ lbs², and $s_2^2 = 1.1$ lbs²
- Calculating $t =$ $\dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \dfrac{0.3}{\sqrt{\frac{1.4^2}{140} + \frac{1.1^2}{172}}} = 2.1$
- Calculating $\nu =$

$n_1 = 140$

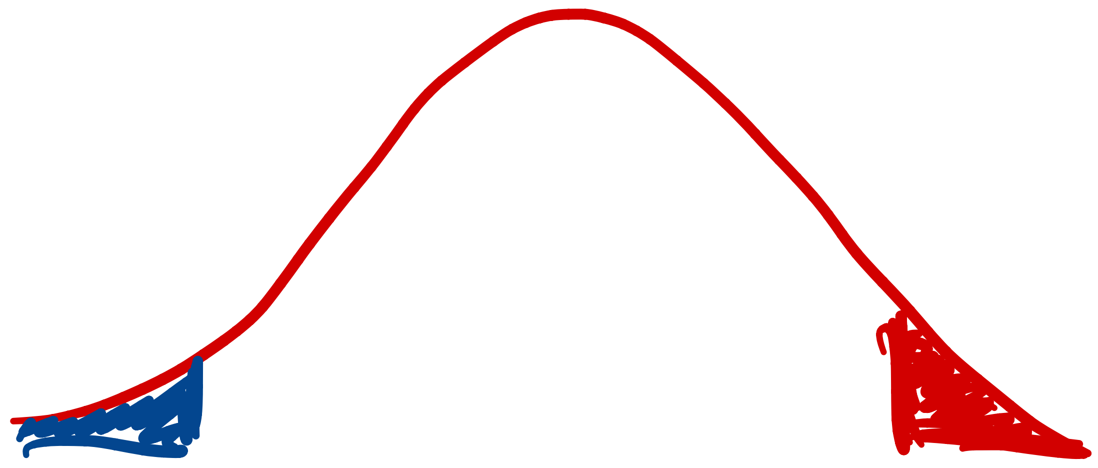
$n_2 = 172$

$\nu = 260.353$

- In R: $p = 1 - pt(2.1, 260.363) = \underline{0.018} < \alpha = 0.05$

$t > 0$

$t < 0$



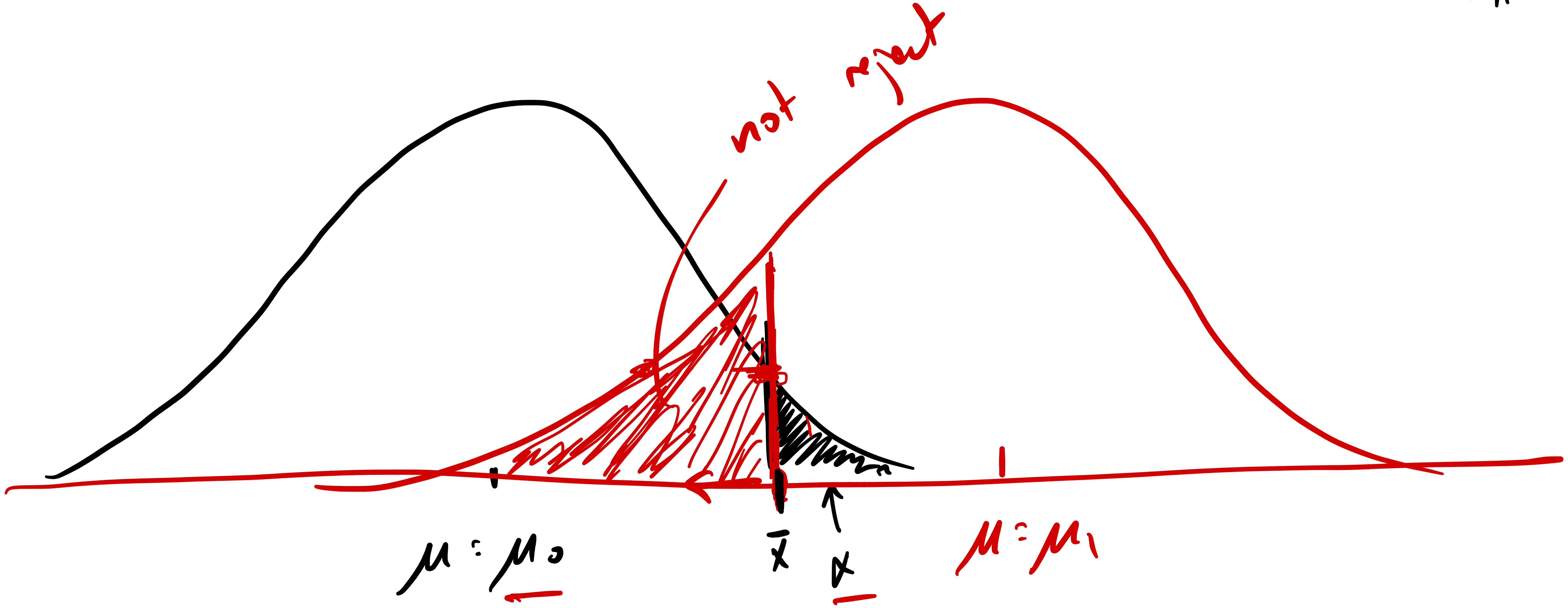
$1 - pt(t)$

$pt(t)$

Unequal Variance (Welch t-test): Example

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

- Hypotheses: $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$, samples of sizes $n_1 = 140$ and $n_2 = 172$
- $\bar{x}_1 = 8.2$ lbs, $\bar{x}_2 = 7.9$ lbs, $s_1^2 = 1.4$ lbs², and $s_2^2 = 1.1$ lbs²
- Calculating $t =$



- Calculating $\nu =$
- In R:
- Conclusion:

Unequal Variance (Welch t-test): Example

- Hypotheses: $H_0 : \mu_1 \leq \mu_2$ vs. $H_1 : \mu_1 > \mu_2$, samples of sizes $n_1 = 140$ and $n_2 = 172$
- $\bar{x}_1 = 8.2$ lbs, $\bar{x}_2 = 7.9$ lbs, $s_1^2 = 1.4$ lbs², and $s_2^2 = 1.1$ lbs²
- Calculating $t =$

$$Pr(\text{Dell} \mid \text{batt} > 8.5) = \frac{Pr(\text{batt} > 8.5 \mid \text{Dell})\, Pr(\text{Dell})}{Pr(\text{batt} > 8.5)}$$

- Calculating $\nu =$

$$Pr(\text{batt} > 8.5 \mid \text{Dell})\, Pr(\text{Dell}) + Pr(\text{batt} > 8.5 \mid \text{Mac})\, Pr(\text{Mac})$$

- In R:
- Conclusion: