Chapter 4: Probability and Combinatorics

DSCC 462

Computational Introduction to Statistics

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Probability

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    - Income of a selected individual
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- Probability is the mathematics of random occurrences

Events

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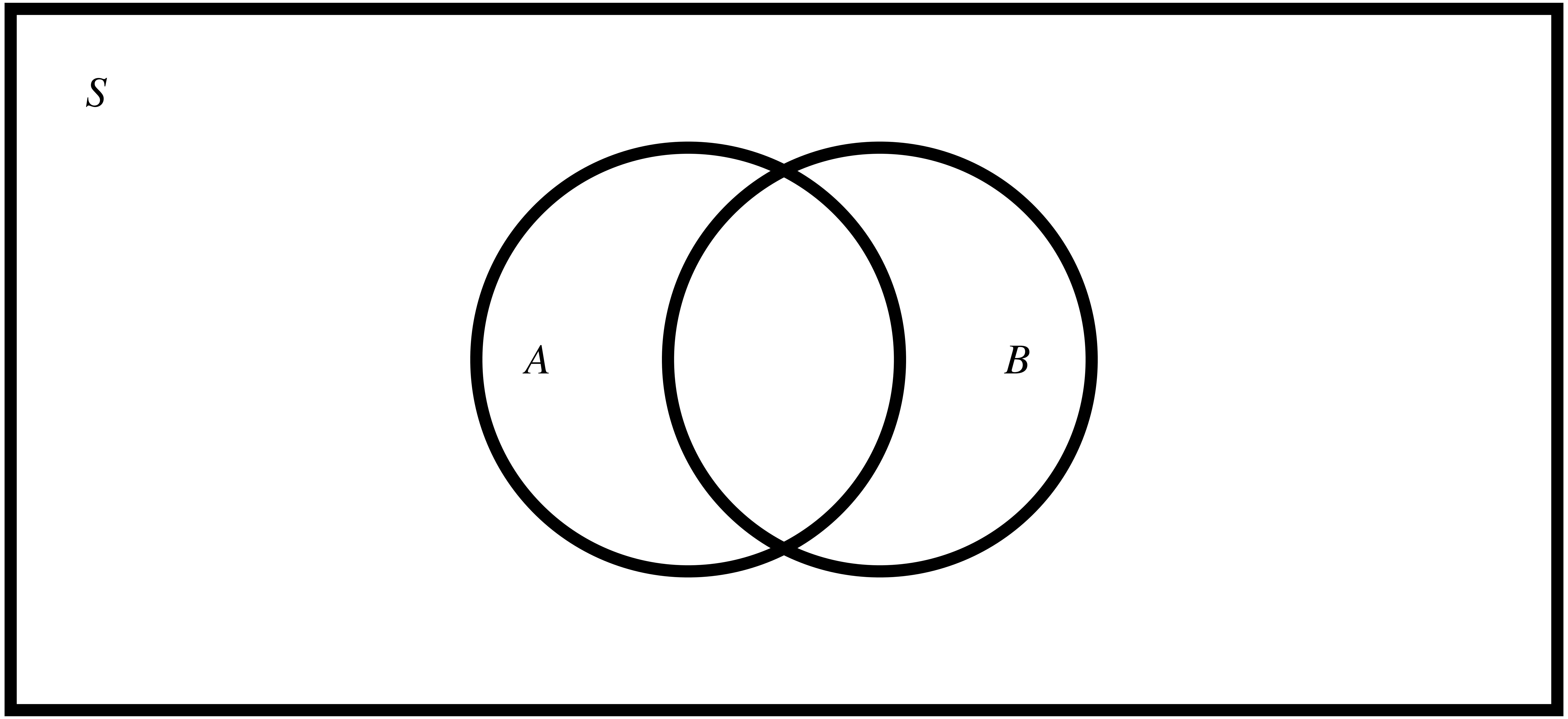
- Events are represented by uppercase letters: $A, B, C, \ldots$
- List the event in $\{\ \}$ brackets
- Example: $A = \{\text{roll an even number on a six-sided die}\} = \{2,4,6\}$

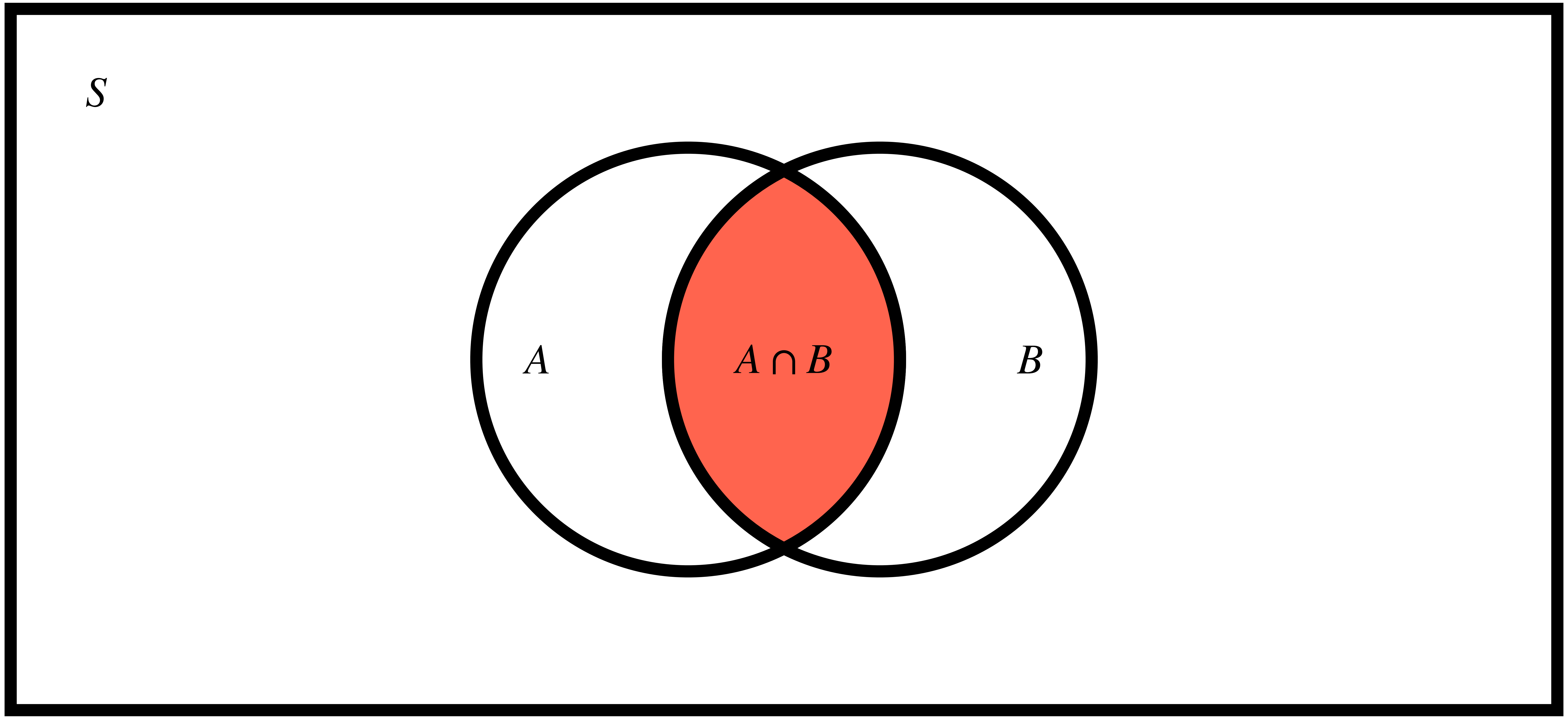
Operations on Events

- Let A and B be events, or subsets of S, where $A \subset S$ and $B \subset S$



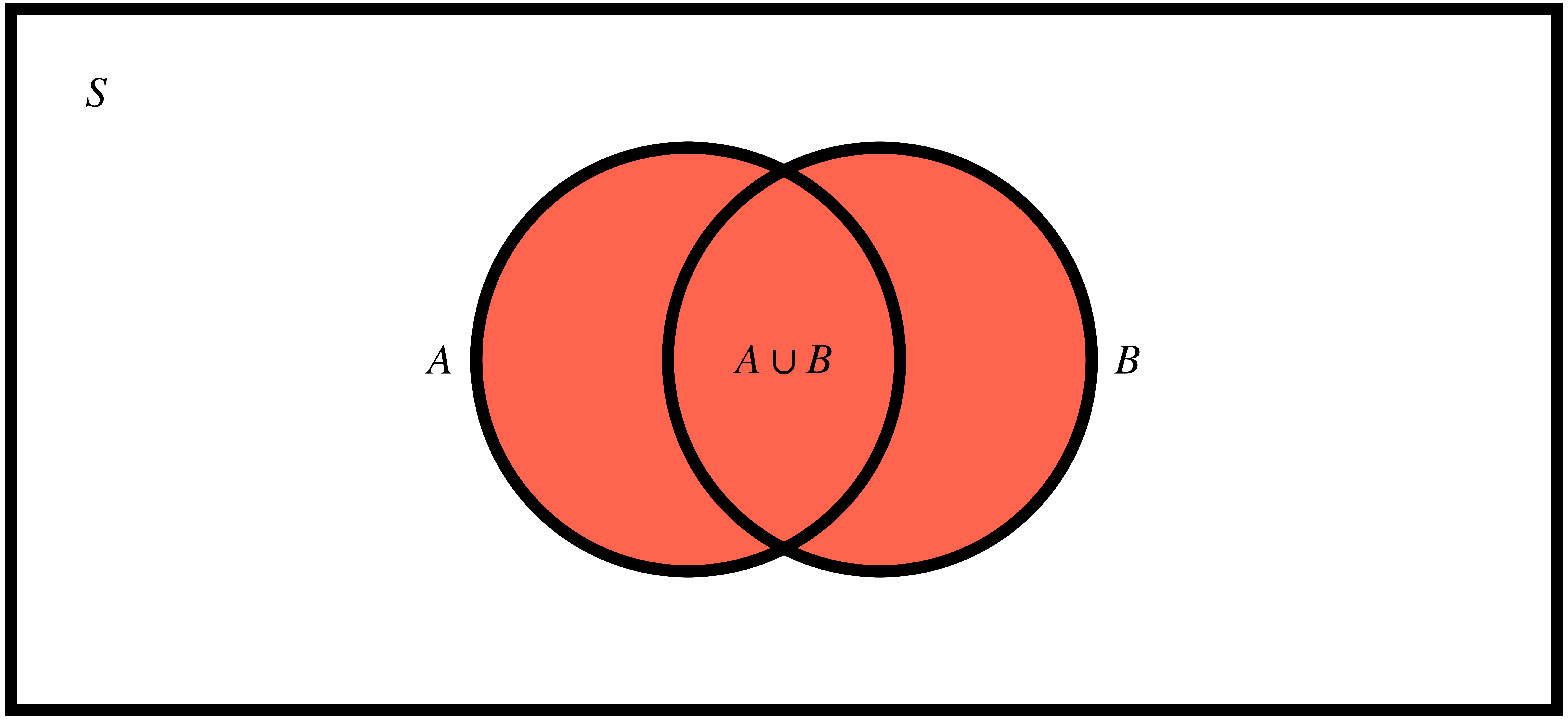
Intersection

- Intersection ($A \cap B$): The event "both A and B", or all elements in S in both A and B



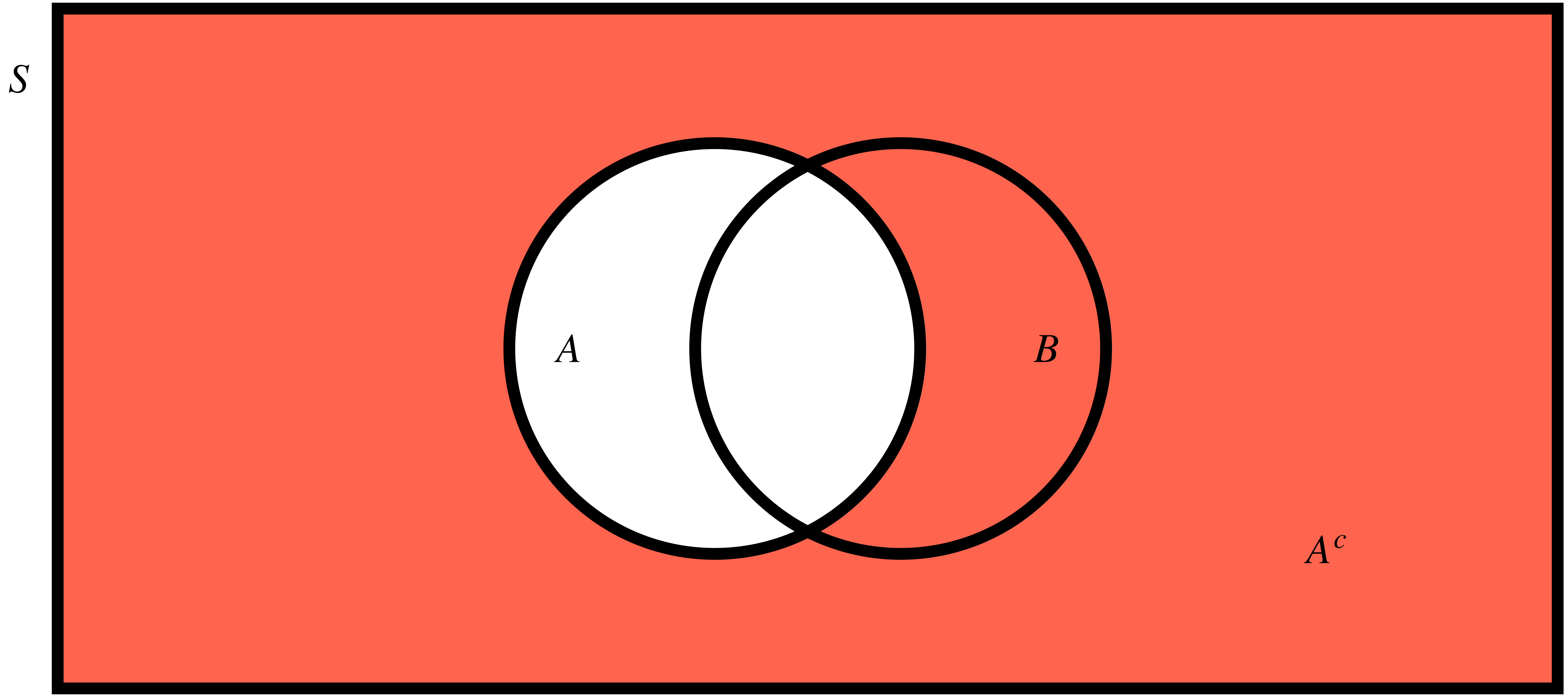
Union

- Union ($A \cup B$): The event "either A or B", or all elements in S in either A or B



Complement

- Complement (A^c, $\overline{A}$, or A'): The event "not A", or all elements in S not in A



Operations Example

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- Suppose we have the following, where $A \subset S, B \subset S,$ and $C \subset S$:

$$S = \{1,2,3,4,5,6,7,8\}$$

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- Evaluate the following expressions:

$$A \cap B =$$

$$(A \cup C) \cap B =$$

$$A^c \cap C =$$

$$(A \cap B^c) \cup C =$$

Operations on Events: De Morgan’s Laws

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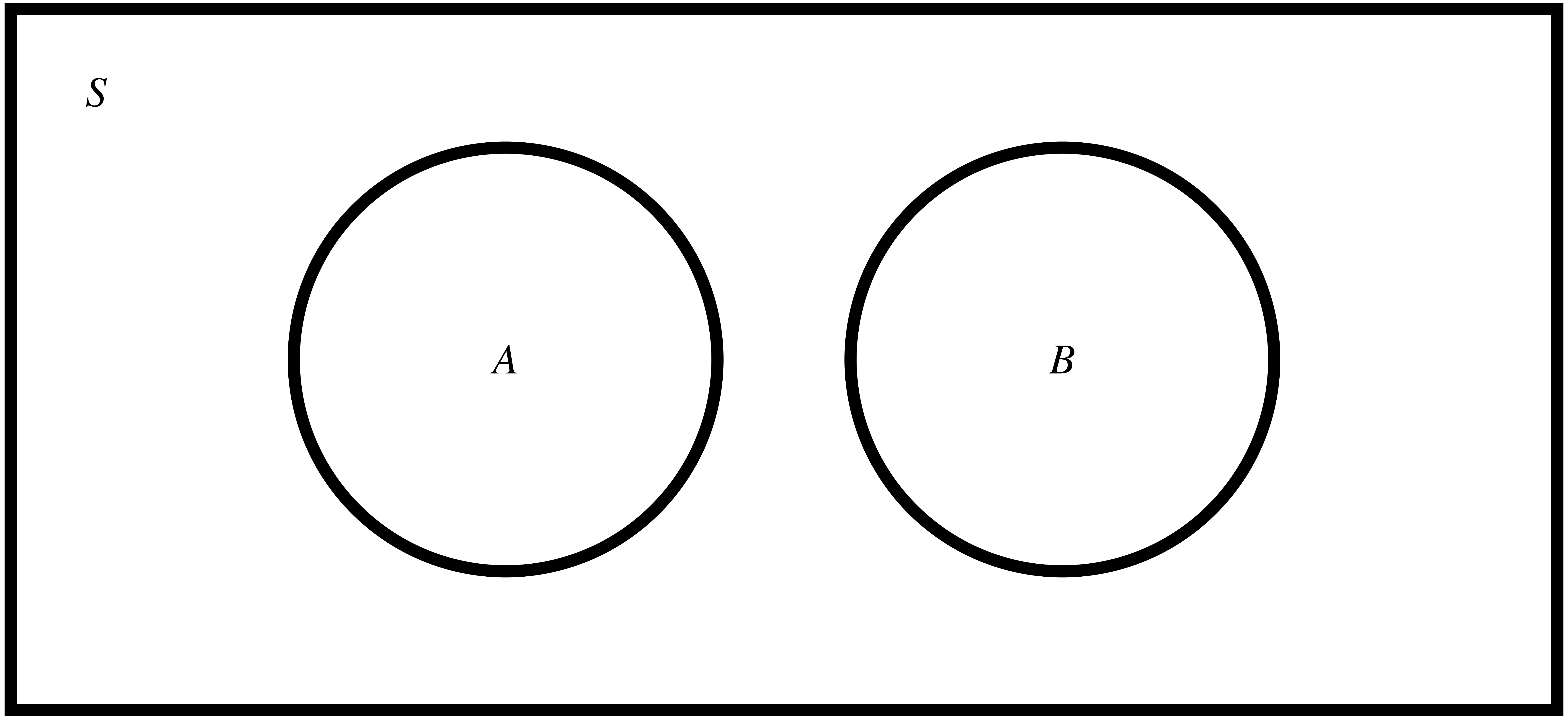
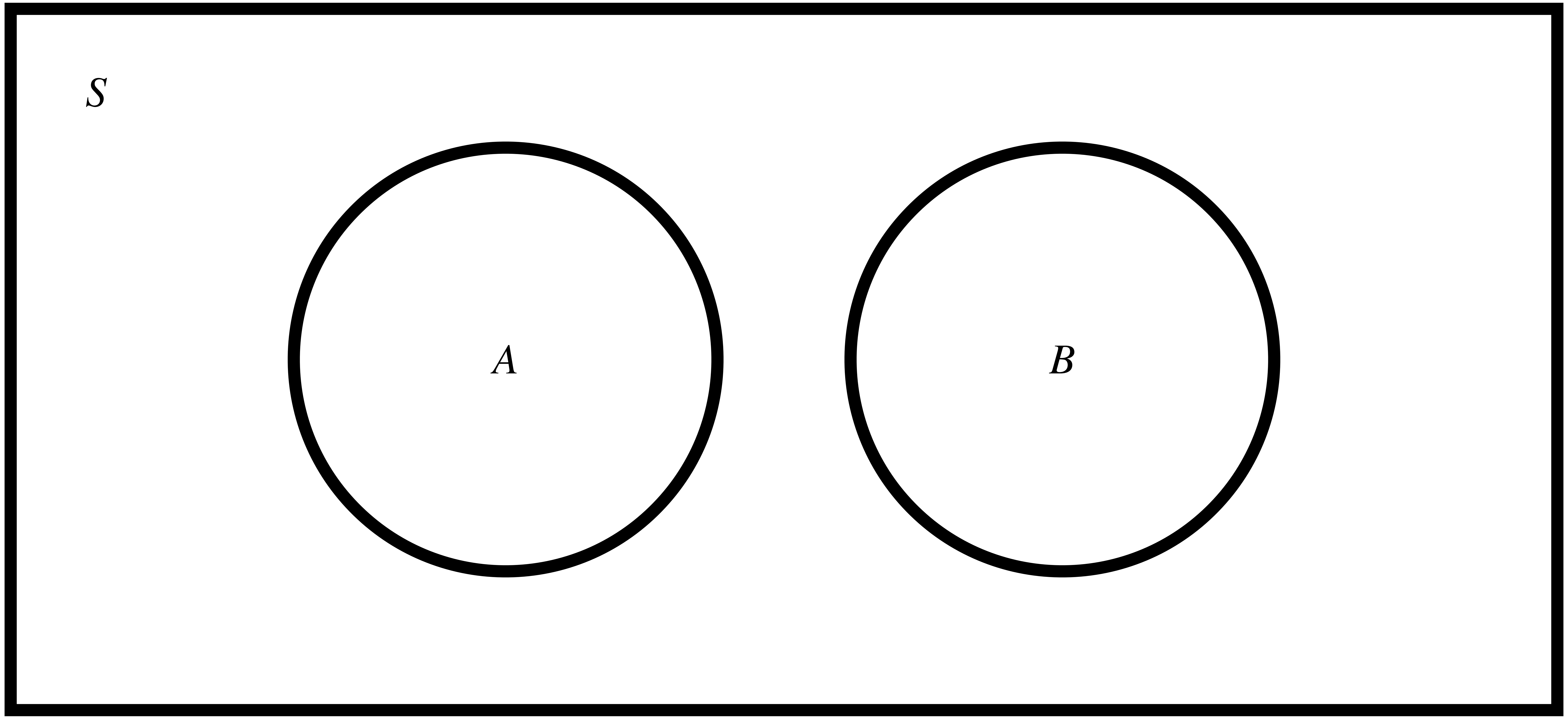


Events

- Null events are events that can never occur, represented as $\varnothing$



Events

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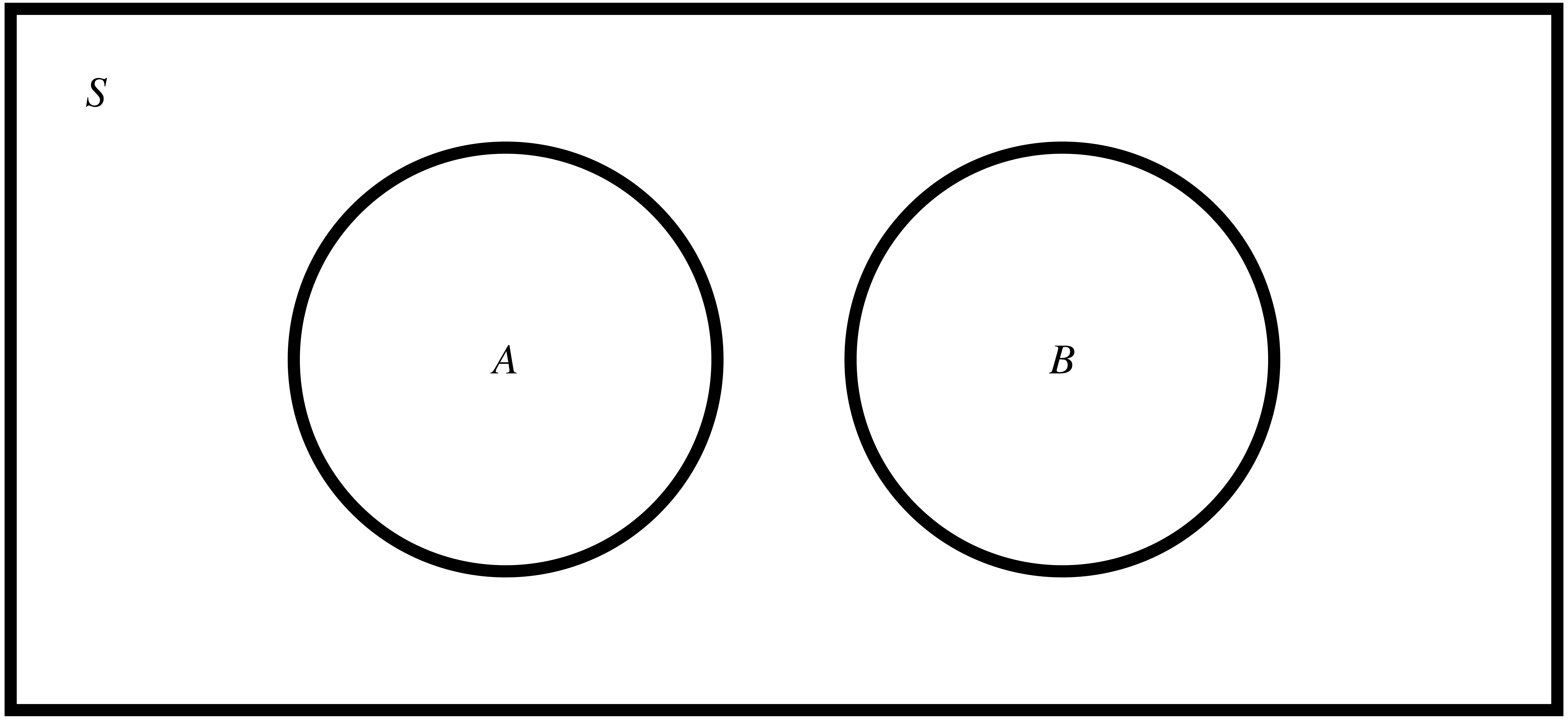
- Disjoint or mutually exclusive events are events that cannot occur simultaneously; A and B are disjoint if and only if $A \cap B = \varnothing$



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- Three types of cardinality:
  - Finite: $|A| < \infty$
  - Countable: $|A| = \infty$ but elements can be listed as $x_1, x_2, \ldots$
  - Uncountable: $|A| = \infty$ and elements cannot be listed as $x_1, x_2, \ldots$

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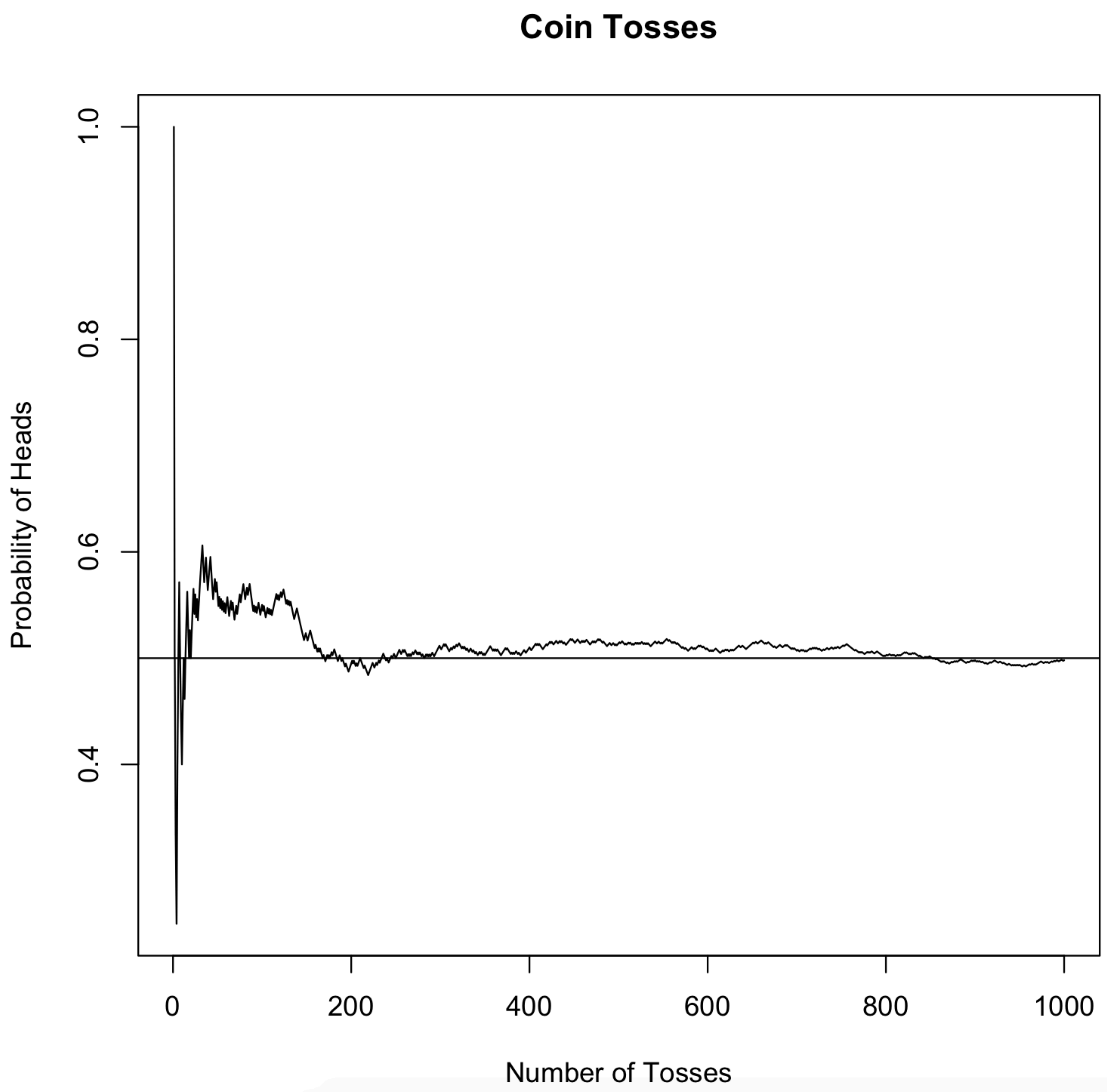
- Relative frequency of occurrence of an event when repeated many times
- $\Pr(A) = \dfrac{\text{\# of times } A \text{ occurs}}{\text{total \# of trials}}$

Probability



Probability Rules

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- $\Pr(\varnothing) = 0$
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- If $A \subset B$, then $\Pr(A) \leq \Pr(B)$

Mutual Exclusivity and Exhaustiveness

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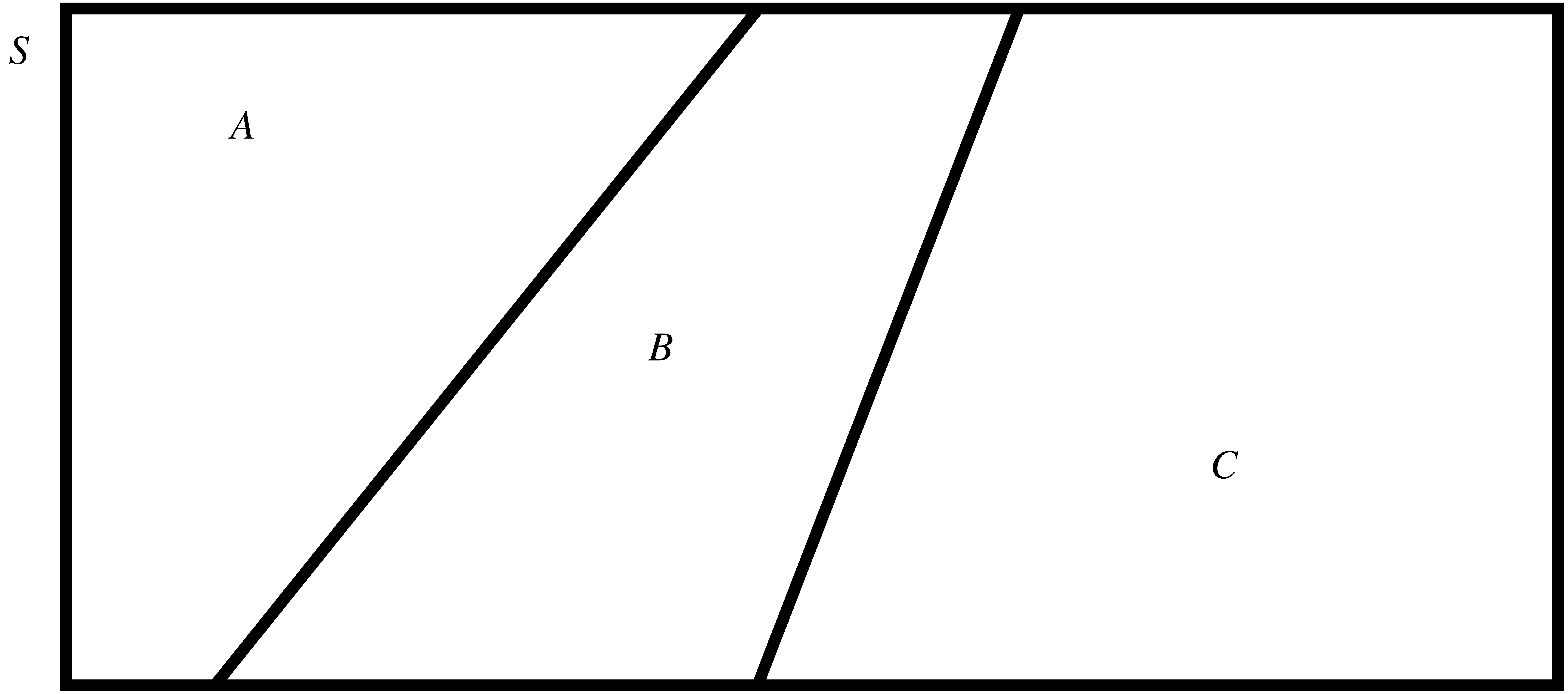

Addition Rule: Mutually Exclusive Events

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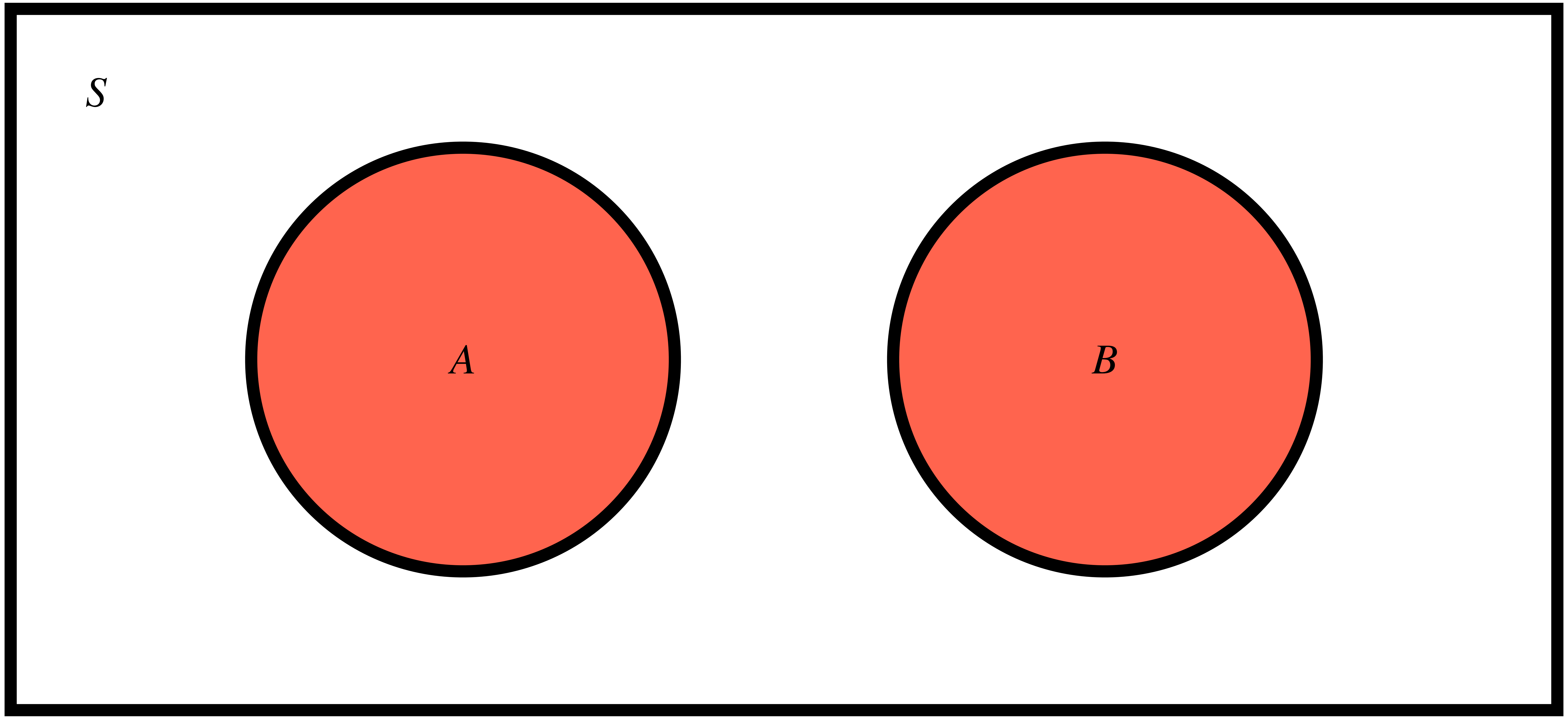

Addition Rule: General

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- In general, we have $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

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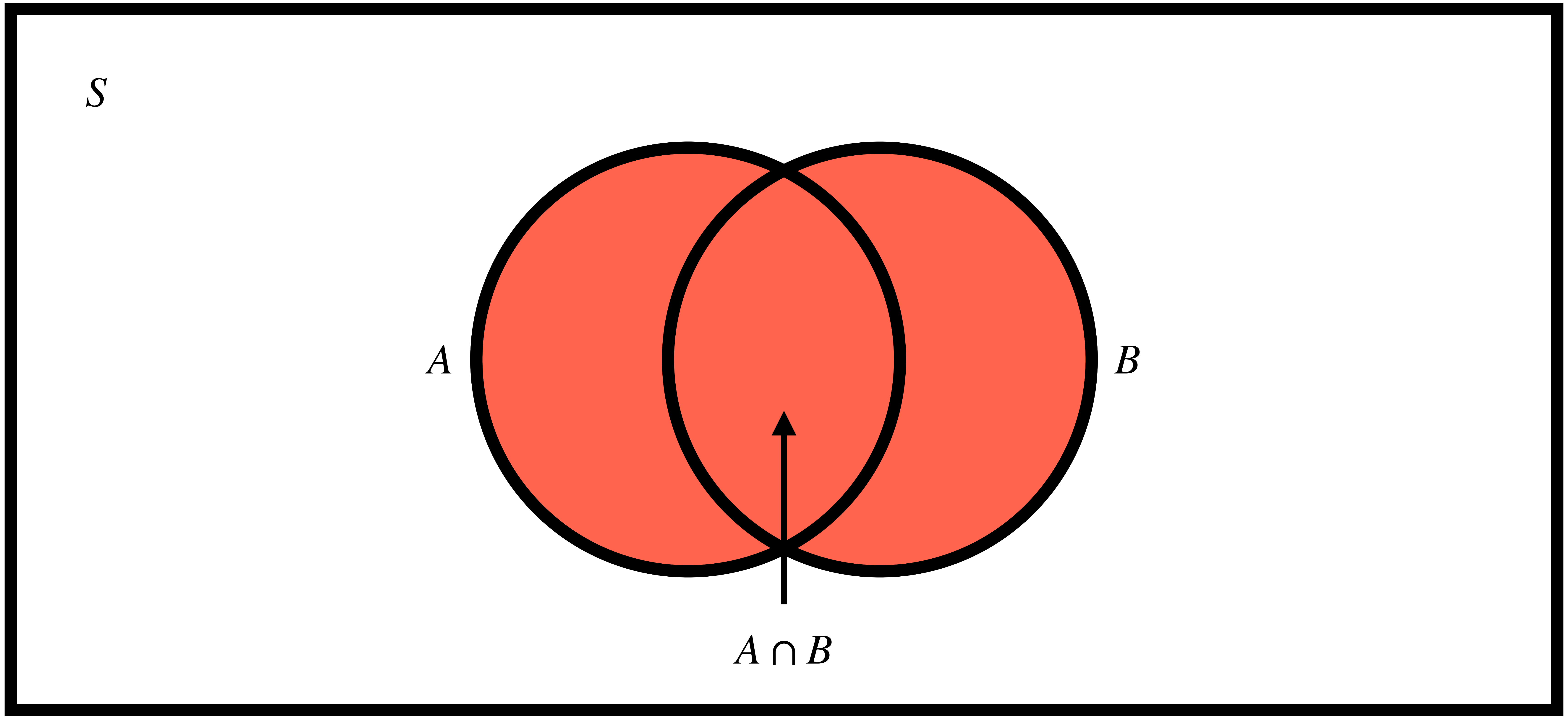

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- Suppose that 55% of cancer patients are female, 20% of cancer patients have previously undergone chemotherapy, and 15% of cancer patients are both female and have undergone chemotherapy

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- What is the probability that a patient is female or has undergone chemotherapy?

Conditional Probability

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  - Example: What is the probability that it rains tomorrow given that it rained today?

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- **Conditional Probability**: The probability that event A will occur given that we already know the outcome of event B
- $\Pr(A \mid B) =$ probability of A given B

Multiplicative Rule

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- The *multiplicative rule of probability* tells us the following:

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- Rearranging yields *conditional probability expressions*:

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

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    - 450 students changed majors
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- Q1: What is the probability of changing majors given that you are a male?
- Q2: What is the probability of changing majors given that you are not a male?

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- Setup:
  - The probability that you will be sick tomorrow is 0.6
  - If you are sick tomorrow, the probability that you will be sick the next day is 0.7
  - If you are not sick tomorrow, the probability that you will be sick the next day is 0.2

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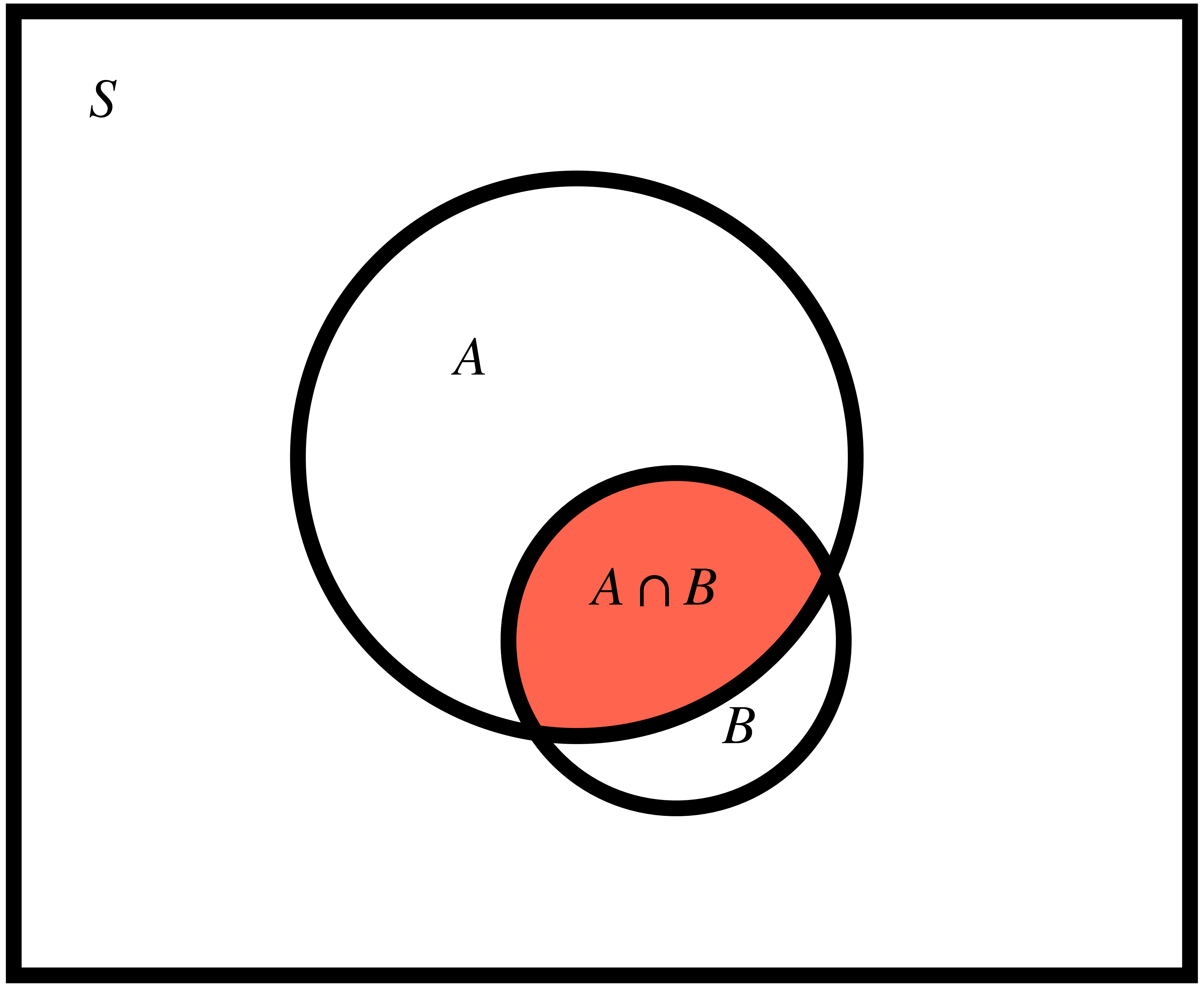
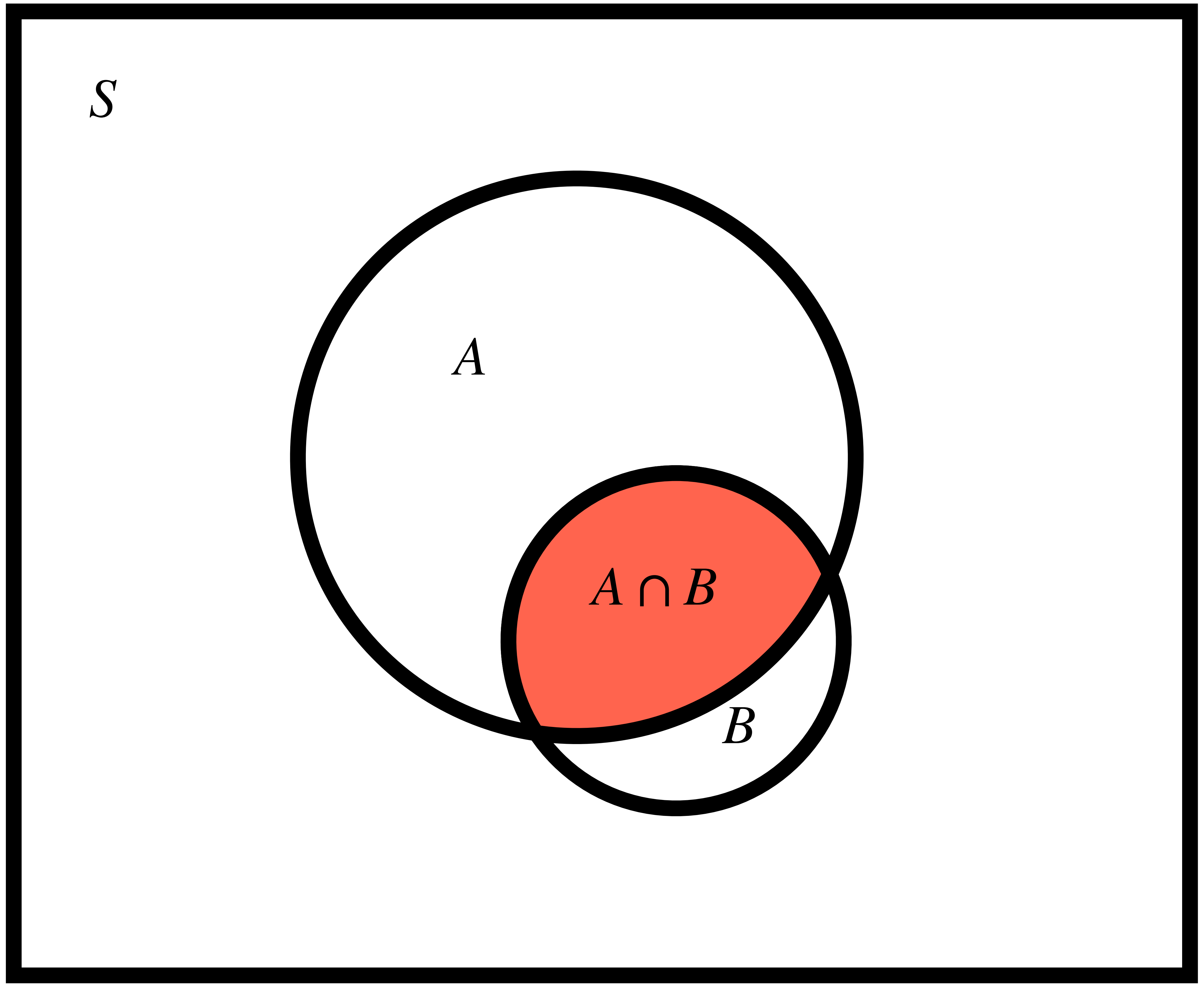
  - If you are not sick tomorrow, the probability that you will be sick the next day is 0.2
- Q1: What is the probability that you are sick tomorrow and the next day?
- Q2: What is the probability that you are not sick tomorrow but sick the following day?

Conditional Probability



Conditional Probability

- Note, $\Pr(B \mid A) \neq 1 - \Pr(A \mid B)$



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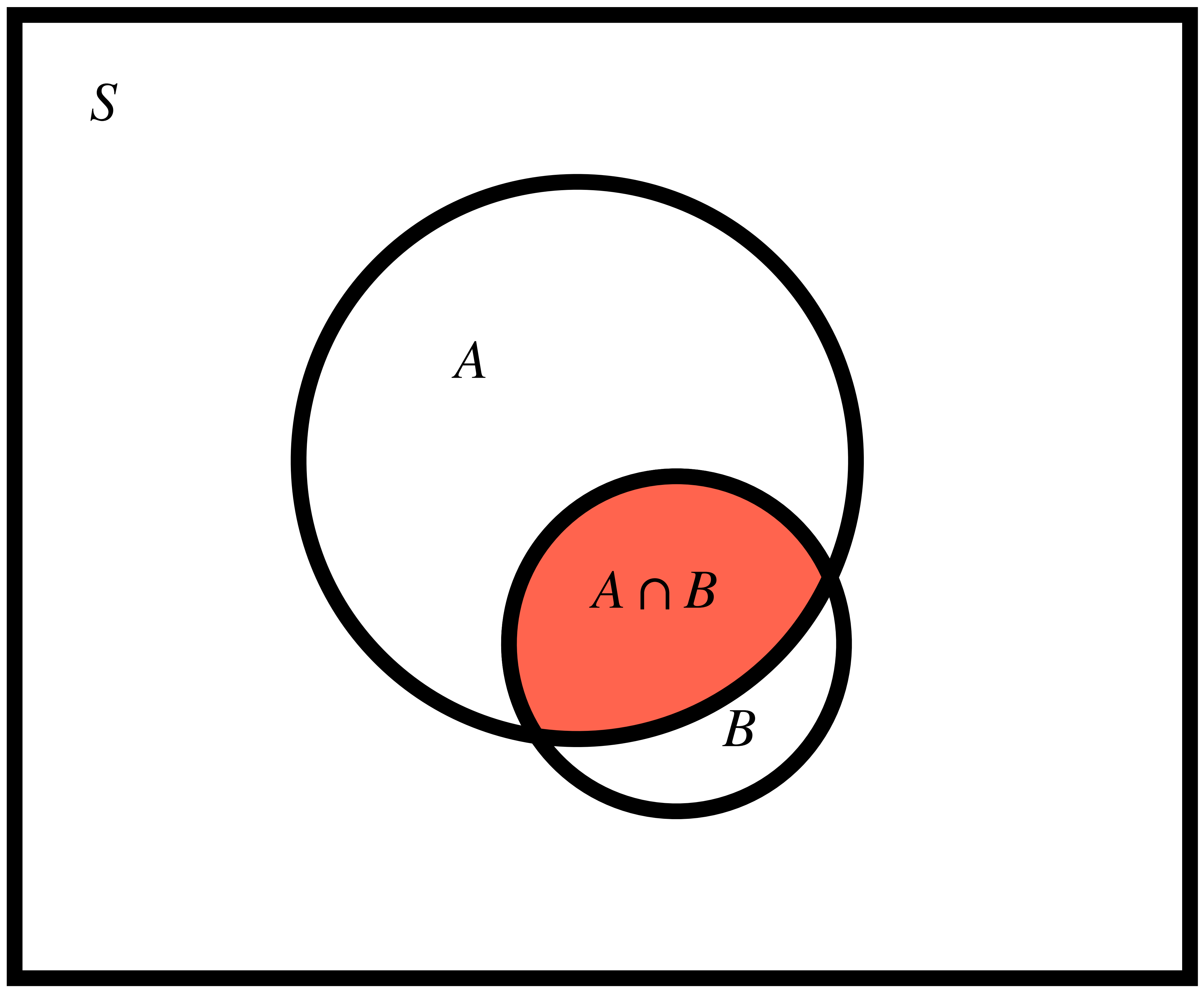
- Note, $\Pr(B\,|\,A) \neq 1 - \Pr(A\,|\,B)$
- Similarly, $\Pr(B\,|\,A) \neq 1 - \Pr(B\,|\,A^c)$



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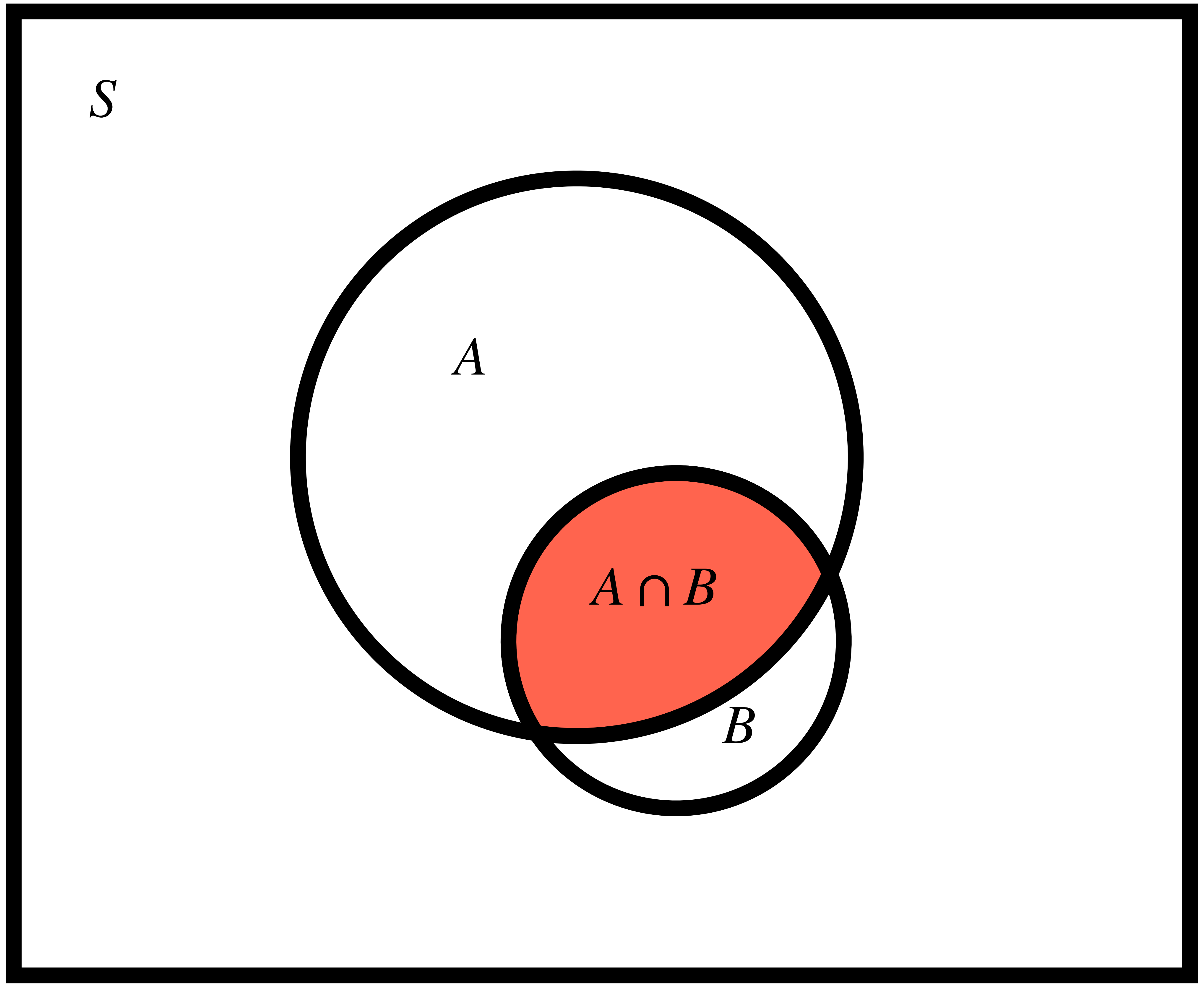
- Note, $\Pr(B \mid A) \neq 1 - \Pr(A \mid B)$
- Similarly, $\Pr(B \mid A) \neq 1 - \Pr(B \mid A^c)$
- But, $\Pr(B \mid A) = 1 - \Pr(B^c \mid A)$



Conditional Probability Example

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- Setup:
  - Consider a random experiment where 3 balls are randomly selected (without replacement) from 5 balls labeled 1, 2, 3, 4, 5. Sample space:

    123, 124, 125, 134, 135, 145
    234, 235, 245
    345

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- Let $A = \{1 \text{ is selected}\}$ and $B = \{5 \text{ is selected}\}$. What is $\Pr(A \mid B)$?

Independence

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  - If A and B are independent, then $\Pr(A \mid B) = \Pr(A)$ (and $\Pr(B \mid A) = \Pr(B)$)

Independence

- **Independence**: The outcome of one event has no effect on the outcome of another event
  - If A and B are independent, then $\Pr(A \mid B) = \Pr(A)$ (and $\Pr(B \mid A) = \Pr(B)$)
- This is because intersection is decomposable:
  - If A and B are independent, then $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$
  - From this, we see that $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{\Pr(A) \cdot \Pr(B)}{\Pr(B)} = P(A)$

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- Setup:
  - Suppose we flip a coin twice; tosses are independent
  - Let $A = \{\text{first flip is heads}\}$ and $B = \{\text{second flip is heads}\}$
  - $\Pr(A) = \Pr(B) = 1/2$

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  - Let $A = \{\text{first flip is heads}\}$ and $B = \{\text{second flip is heads}\}$
  - $\Pr(A) = \Pr(B) = 1/2$
- What is $\Pr(A \cap B)$ (probability that both flips are heads)?

Mutual Independence

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- Suppose we have n events, N. These n events are **mutually independent** iff, for every subset of events $M \subseteq N$, we have

$$\Pr\left(\bigcap_{i \in M} A_i\right) = \prod_{i \in M} \Pr(A_i)$$

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- Consider the case of $n = 3$. Events A_1, A_2, A_3 are independent iff the following hold:

$$\begin{aligned}
\Pr(A_1 \cap A_2) &= \Pr(A_1) \cdot \Pr(A_2) \\
\Pr(A_1 \cap A_3) &= \Pr(A_1) \cdot \Pr(A_3) \\
\Pr(A_2 \cap A_3) &= \Pr(A_2) \cdot \Pr(A_3) \\
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\Pr(A_1 \cap A_2 \cap A_3) &= \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3)
\end{aligned}$$

- If all but the last equality hold, A_1, A_2, A_3 are *pairwise independent*, but not mutually independent

Pairwise Independence: Example

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- Setup: Consider rolling a fair six-sided die. Consider the events $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{2,3\}$
  - $\Pr(A) = \Pr(B) = \Pr(C) =$
  - $\Pr(A \cap B) =$
  - $\Pr(A \cap C) =$
  - $\Pr(B \cap C) =$
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  - $\Pr(A \cap B) =$
  - $\Pr(A \cap C) =$
  - $\Pr(B \cap C) =$
  - $\Pr(A \cap B \cap C) =$
- These events are pairwise independent but not mutually independent

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Independence vs. Mutual Exclusivity

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- If A and B are mutually exclusive, then $\Pr(A \mid B) = 0$ and $\Pr(B \mid A) = 0$
- This is not the same thing as independence, where $\Pr(A \mid B) = \Pr(A)$ and $\Pr(B \mid A) = \Pr(B)$
- Independence: the other event still may occur; its probability is unaffected

Law of Total Probability

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- Then, for any event E, the law of total probability states the following:

$$\begin{aligned}
\Pr(E) &= \Pr(E \cap A_1) + \Pr(E \cap A_2) + \ldots + \Pr(E \cap A_n) \\
&= \Pr(E \mid A_1) \cdot \Pr(A_1) + \Pr(E \mid A_2) \cdot \Pr(A_2) + \ldots + \Pr(E \mid A_n) \cdot \Pr(A_n)
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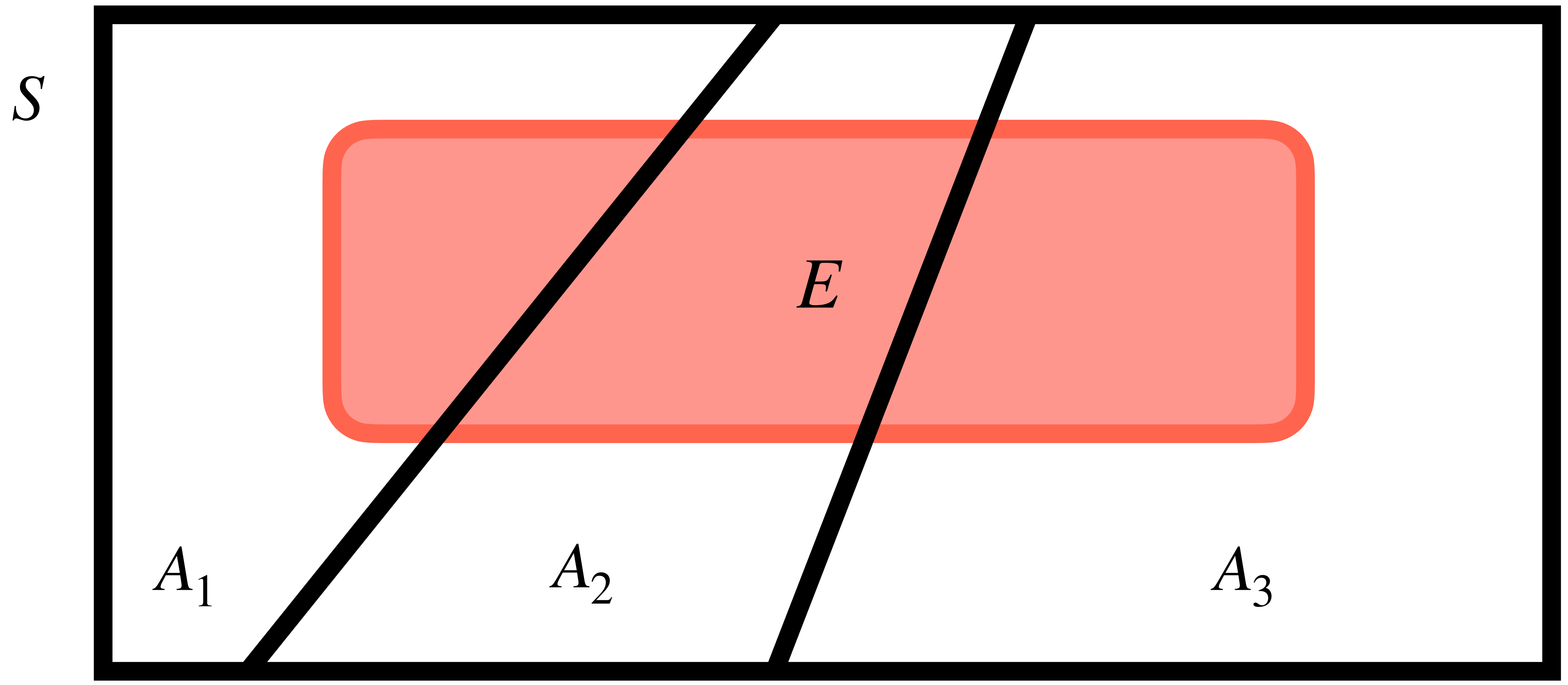

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$$\underbrace{\Pr(A \mid B)}_{\text{Posterior}} = \frac{\overbrace{\Pr(B \mid A)}^{\text{Likelihood}} \cdot \Pr(A)}{\Pr(B)} = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B \mid A) \cdot \Pr(A) + \Pr(B \mid A^c) \cdot \Pr(A^c)}$$

Posterior Likelihood Prior

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  - Given that you do not have diabetes, there is a 35% chance you are overweight
  - 10% of people have diabetes

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  - Given that you have diabetes, there is a 70% chance you are also overweight
  - Given that you do not have diabetes, there is a 35% chance you are overweight
  - 10% of people have diabetes
- Q: Given that a randomly selected person is overweight, what is the probability that he has diabetes?

Diagnostic Tests

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- Apply Bayes' theorem to diagnostic testing and screening

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- Assume that we run a screening test on a patient to determine if they have the disease, with two mutually exclusive and exhaustive outcomes:
  - T^+: the test is positive
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- Typically, we are interested in $\Pr(D_1 \mid T^+)$

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  - $\Pr(T^- \mid D_1) = 1 - \text{Sensitivity}$
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    - $\Pr(T^- \mid D_1) = 1 -$ Sensitivity
- **Specificity**: Probability of a negative test result given that the individual tested does not have the disease (true negative):
    - $\Pr(T^- \mid D_2)$
- **False positive probability**: Probability of a positive test result given that the individual tested does not have the disease (false positive):
    - $\Pr(T^+ \mid D_2) = 1 -$ Specificity

Positive Predictive Value (PPV)

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$$\Pr(D_1 \mid T^+) = \frac{\Pr(D_1 \cap T^+)}{\Pr(T^+)}$$

$$= \frac{\Pr(T^+ \mid D_1) \cdot \Pr(D_1)}{\Pr(T^+ \mid D_1) \cdot \Pr(D_1) + \Pr(T^+ \mid D_2) \cdot \Pr(D_2)}$$

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- What are $\Pr(D_1)$ and $\Pr(D_2)$?
    - $\Pr(D_1)$: probability of having the disease, or prevalence of the disease
    - $\Pr(D_2) = 1 - \Pr(D_1)$

Negative Predictive Value (NPV)

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  - $\Pr(D_2 \mid T^-)$

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- **Negative predictive value (NPV)**: The probability that a person with a negative test result actually does not have the disease
    - $\Pr(D_2 \mid T^-)$
- Using Bayes' Rule, sensitivity, and specificity:

$$\Pr(D_2 \mid T^-) = \frac{\Pr(D_2 \cap T^-)}{\Pr(T^-)}$$

$$= \frac{\Pr(T^- \mid D_2) \cdot \Pr(D_2)}{\Pr(T^- \mid D_2) \cdot \Pr(D_2) + \Pr(T^- \mid D_1) \cdot \Pr(D_1)}$$

Diagnostic Tests: Example

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- Cancer test has the following properties:
  - The test gives a positive result 95% of the time when the patient has cancer
  - The test gives a negative result 90% of the time when the patient does not have cancer
  - About 12% of patients have cancer

Diagnostic Tests: Example

$c \quad c^c$

pos $\quad$ neg

- Cancer test has the following properties:
  - The test gives a positive result 95% of the time when the patient has cancer
  - The test gives a negative result 90% of the time when the patient does not have cancer
  - About 12% of patients have cancer
- Q: A patient tested positive for cancer. What is the probability that they have cancer?

$$Pr(c \mid pos) = \frac{Pr(pos \mid c)\, Pr(c)}{Pr(pos)} = \frac{Pr(pos \mid c)\, Pr(c)}{Pr(pos \mid c)\, Pr(c) + Pr(pos \mid c^c)\, Pr(c^c)}$$

$$= \frac{.95 \times .12}{.95 \times .12 + (1-.9)(1-.12)} \approx \underline{56\%}$$

Combinatorics

Counting Outcomes

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    - Where N is the total number of outcomes in E and D is the total number of outcomes in S
- We're going to learn how to count the number of outcomes

Ordered vs. Unordered Selection

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- **Unordered selection** of size n from sample space S: select n distinct objects from S where order of selection does not matter
    - Care about the names of choices (think of it as a set)

Rule of Product

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Rule of Product: Example

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- How many valid three-digit numbers (i.e., between 100 and 999, inclusive) have three different digits and only a single odd number in the middle?

① Choose middle digit $= 5 \leftarrow 1, 3, 5, 7, 9$

② " first " $= 4 \leftarrow 2, 4, 6, 8$

③ " last " $= 4 \leftarrow \{0, 2, 4, 6, 8\} \setminus \{(2)\}$

80

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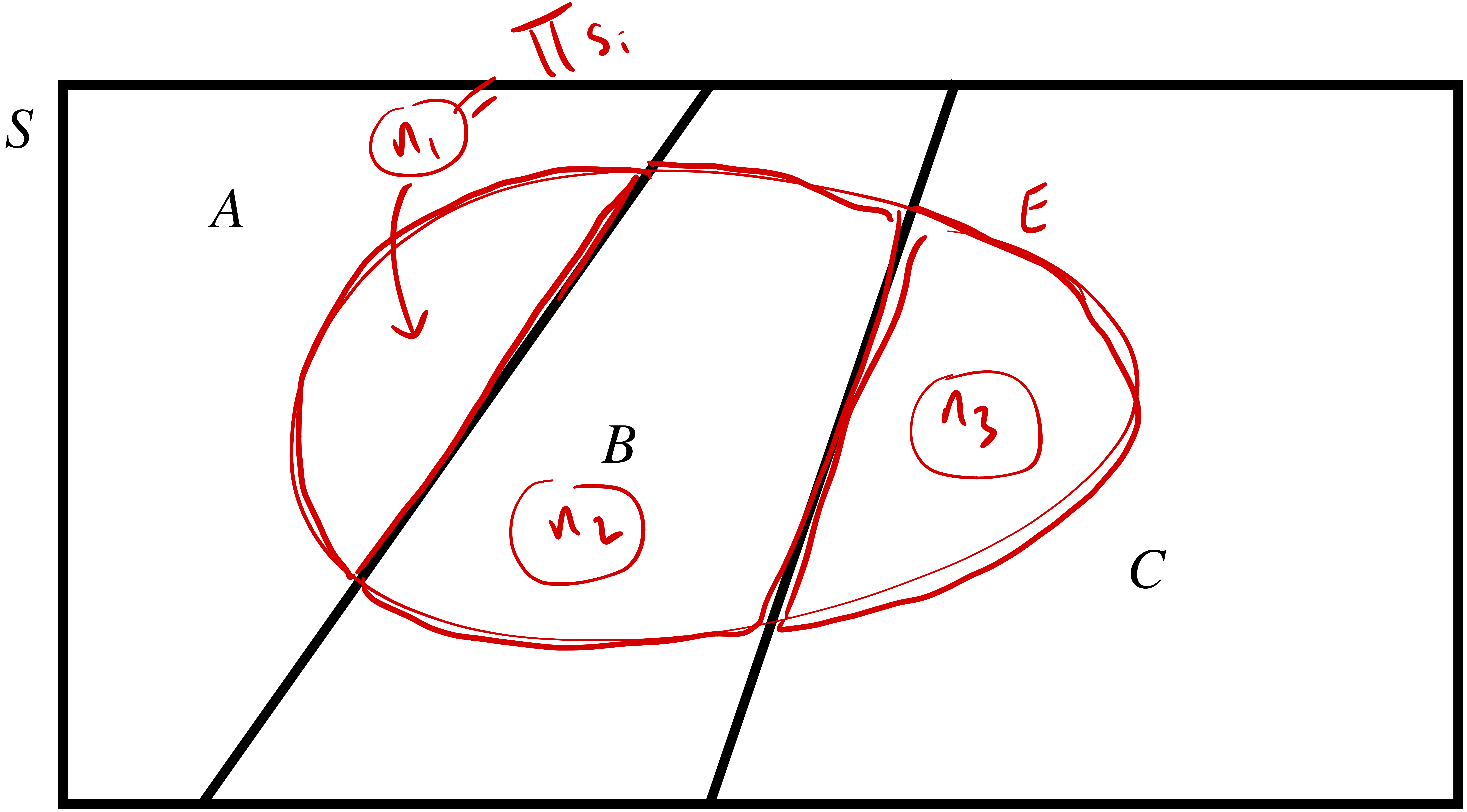
- Then, there are $n_1 + n_2 + \ldots + n_m$ distinct ways to perform the entire procedure
- Often, use the rule of sum (tree method) and the rule of product together

Rule of Sum (OR) and Rule of Product (AND)



Factorials

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$5! = 5 \times 4 \times 3 \times 2 \times 1$

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  - $n! = n \cdot (n-1) \cdot \ldots \cdot 1$

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5 4 3 2 1

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- In R: use `factorial(x)`

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  - Ordered selection
- There are n ways to select the first object, $n-1$ ways to select the second object, and so on until we have $n-k+1$ ways to select the final object

$$P(n,k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

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$P(26,4) = 26 \times 25 \times 24 \times 23 \sim 360K$

- Q2: How many ways are there of assigning three students among seven orientation groups, where each student must go to a different group?

$P(7,3) = 7 \times 6 \times 5 = 210$

Combination

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3

$\{1, 2, 3, 4, 5\}$

1, 2, 3

3!

2, 1, 3

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- *Binomial coefficient*

Combination: Example (Poker Hands)

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(3c, 3h, 3s, 3d)

$\binom{52}{5} \approx 2.6$ million

- Q2: What is the probability of getting $\boxed{\text{four of the same } \cancel{\text{kind}}}$?

value #

$13 \cdot \binom{4}{4}$

n_1: # ways of drawing 4 of same #

n_2: choose 5^{th} card. ← 48.

$$\frac{13 \cdot \binom{4}{4} \cdot 48}{\binom{52}{5}} \approx \boxed{0.00024}$$

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- Q1: What is the probability that there are two pairs of balls which have the same number?

3p 3y   13p 13y

denom: $\binom{70}{4}$

#1 #2

num: $\binom{35}{2}\binom{2}{2}\binom{2}{2}$

#1 #2

$$\to \frac{\binom{35}{2}}{\binom{70}{4}} \approx 0.00065$$

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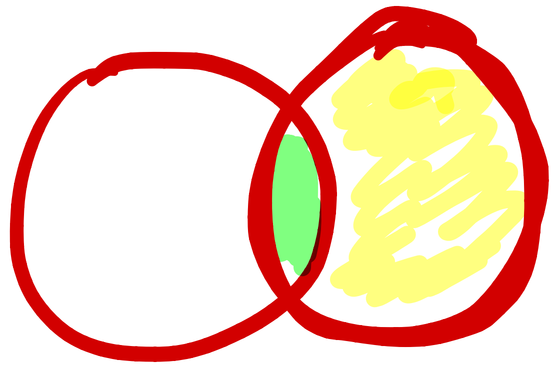
- Q1: What is the probability that there are two pairs of balls which have the same number?

denom: $\binom{70}{4}$

pair: $35\binom{2}{2} = 35$

diff: $\binom{34}{2} \times 2 \times 2$

- Q2: What is the probability that there is exactly one pair of balls with matching numbers?



$(35)\binom{2}{2}\cdot\binom{68}{2} - \binom{35}{2}$

Combination: Example (Urn)

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Combination: Example (Urn)

$2_y \quad 3_y \quad 4_y \quad 5_y$

- Setting: An urn contains 35 yellow balls (numbered 1-35) and 35 pink balls (numbered 1-35). Four balls are chosen at random
- Q3: What is the probability that the balls are all the same color and consecutively numbered?

denom: $\binom{70}{4}$

1, 2, 3, 4
2, 3, 4, 5
3, 4, 5, 6
...
30, 31, 32, 33
⋮
32, 33, 34, 35

num:
y: $\underline{32}$
p:

$$\frac{2 \times 32}{\binom{70}{4}} = 7 \times 10^{-5}$$

Stars and Bars: Intuition

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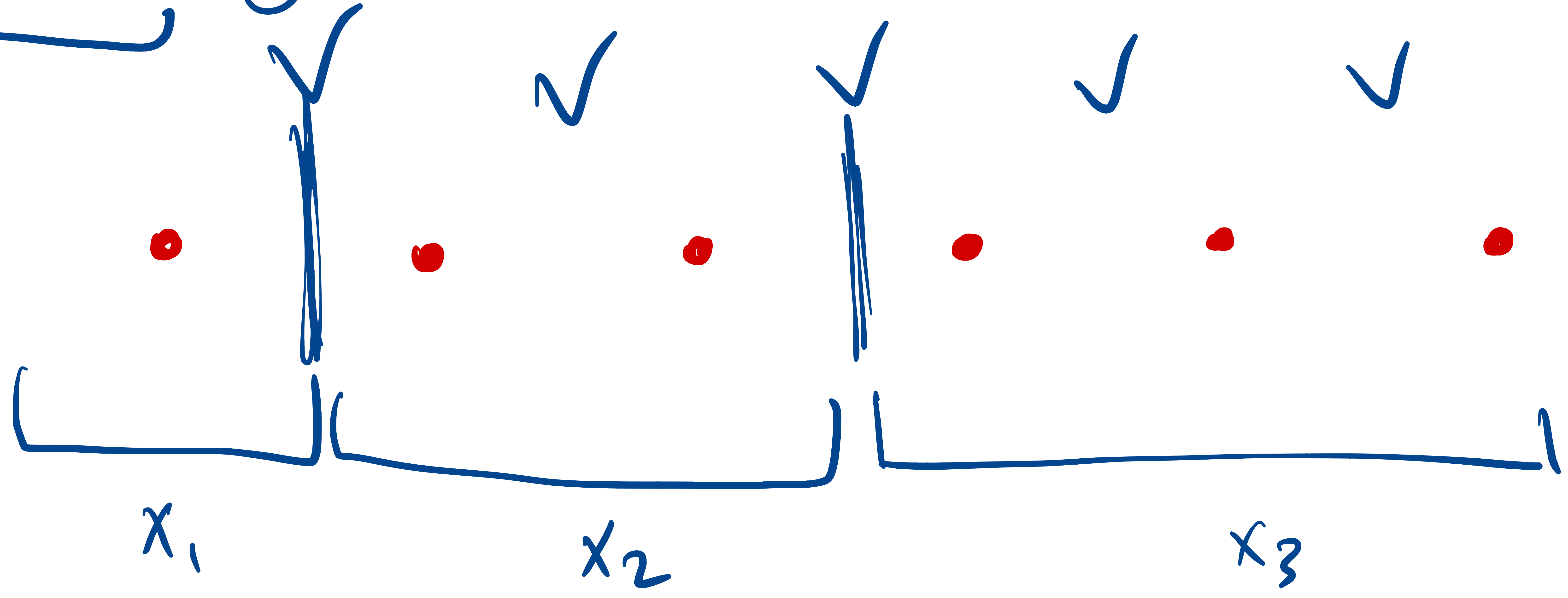
$x_1 = 1 \quad x_2 = 2 \quad x_3 = 3$

- How many ways are there of choosing three *positive* numbers, x_1, x_2, x_3, such that $\underbrace{x_1 + x_2 + x_3}_{k} = \overset{n}{6}$?



$$= \binom{n-1}{k-1} = \binom{5}{2} = 10$$

Stars and Bars: Intuition

- How many ways are there of choosing three *positive* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?

$$\star \quad \star \quad \star \quad \star \quad \star \quad \star$$

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- How many ways are there of choosing three *nonnegative* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?

7

4

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$$\binom{n+(k-1)}{k-1}$$

| ★ ★ ★ ★ ★ ★ |

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$$\Big|\, \star \quad \star \;\Big|\; \star \quad \star \quad \star \quad \star$$

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- How many ways are there of choosing three *nonnegative* numbers, x_1, x_2, x_3, such that $x_1 + x_2 + x_3 = 6$?
  - $\binom{6+3-1}{3-1} = \binom{8}{2}$: | ★ ★ | ★ ★ ★ ★

Stars and Bars: More Formally

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x_1 , x_2 , x_3 #s

- Suppose there are n objects and k bins. Bins are distinguishable, but objects are not. The only thing we care about is the number of objects in each bin.

- If each bin has to have at least one object in it:
  - Total number of ways = $\binom{n-1}{k-1}$ (think of filling in gaps between objects)

- For nonnegative (not positive) constraints:
  - Total number of ways = $\binom{n+k-1}{k-1}$ (think of arranging n objects and $k-1$ dividers)

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$\binom{5}{3} = 10$

- Q1: How many different requests are possible if at least one child must choose each flavor?

n = stars = children = 6

k = bars = 3

x | x | x x x | x

v s ch ca

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- Q1: How many different requests are possible if at least one child must choose each flavor?

- Q2: How many different requests are possible without this restriction?

$n + (k-1)$

$k-1$

$$\binom{n+k-1}{k-1} = \binom{9}{3} = 84$$

x | x || x x x x