Chapter 16: Stochastic Processes

DSCC 462
Computational Introduction to Statistics

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Fall 2022

Final Project Announcement

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- Official announcement will be posted after class today

Plan for Today

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- Poisson processes (counting arrivals)
- Markov chains (any memoryless process)

Stochastic Processes

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  - Radioactive decay (when is random, how is not random)
  - Financial derivatives (when and how are both random)

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- Discrete time process: sequence $X_1, X_2, \ldots$
- Continuous time process: process on a subset $t \in [0,\infty)$

X_i $\qquad X_{7.5}$, $X_{6\pi}$

Counting Processes

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$s < \quad \leq t$

- Think of $N(t)$ as an arrival process
  - At $N(0) = 0$, no one has arrived yet; one person arrives at a time; no one leaves
- For $t > s$, we have $N(t) - N(s)$ is the number of arrivals in the interval $(s, t]$

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  - Number of arrivals in any interval of length $s > 0$ has a $Pois(\lambda s)$ distribution

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- The number of customers arriving at a department store can be modeled by a Poisson process with intensity $\boxed{\lambda = 10 \text{ customers / hour}}$
- Find the probability that there are $\underline{2}$ customers between $\underline{11{:}00 \text{ and } 11{:}20}$

$$\Pr(X = x) = \frac{e^{-\lambda s}(\lambda s)^x}{x!}$$

$$\lambda s = \frac{10}{3} \rightarrow \Pr(X = 2) = \frac{e^{-10/3}\left(\frac{10}{3}\right)^2}{2!} \approx \boxed{\underline{0.2}}$$

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- The number of customers arriving at a department store can be modeled by a Poisson process with intensity $\lambda = 10$ customers / hour

- Find the probability that there are 2 customers between $\underbrace{\text{11:00 and 11:20}}$

≈ 0.2

same!

- What about between $\underbrace{\text{12:15 and 12:35}}$?

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$$\frac{e^{-\lambda s}\cancel{(\lambda s)^x}}{\cancel{x!}}$$

$\geq$

- $\Pr(X_2 > s \mid X_1 > t) = \Pr(N(t+s) - N(t) = 0) = \underline{e^{-\lambda s}}$
  - If $X_2 > s$, then there were no arrivals between t and $t + s$

$$F(x) = 1 - e^{-\lambda s}$$

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  - If $X_2 > s$, then there were no arrivals between t and $t + s$
- Time between events is exponentially distributed with rate λ

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$$\Pr(X_n = j \mid \underbrace{X_1, X_2, \ldots, X_{n-1}}) = \Pr(X_n = j \mid \underbrace{X_{n-1}})$$

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- Intuitively: Probability of an event depends only on the previous state

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- In this lecture, we will consider Markov chains that are
  - Discrete time (time proceeds in jumps of one increment)
  - Time-homogeneous (probabilities do not change over time)
  - Finite state space
- There exist more complicated Markov chains, but they’re out of scope for today

Properties of Markov Chains

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- $\sum_j \Pr(X_n = j \mid X_{n-1} = i) = 1$ (outgoing edges from each state sum to 1)

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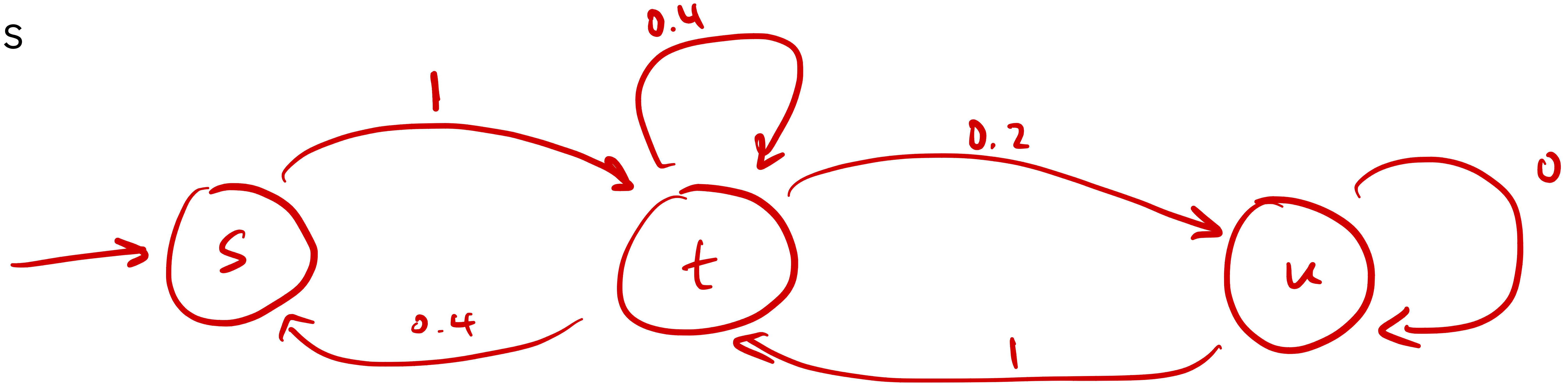
- Can express Markov chains as a graph
  - Nodes represent states and directed edges represent transitions between states



Transition Matrices

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- $$P = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} j_1 & \quad\quad & j_m \end{array} \\ i & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

(i, j)

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$\Sigma \qquad = 1$

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- We assume that transition probabilities do not change with time (time homogeneous)

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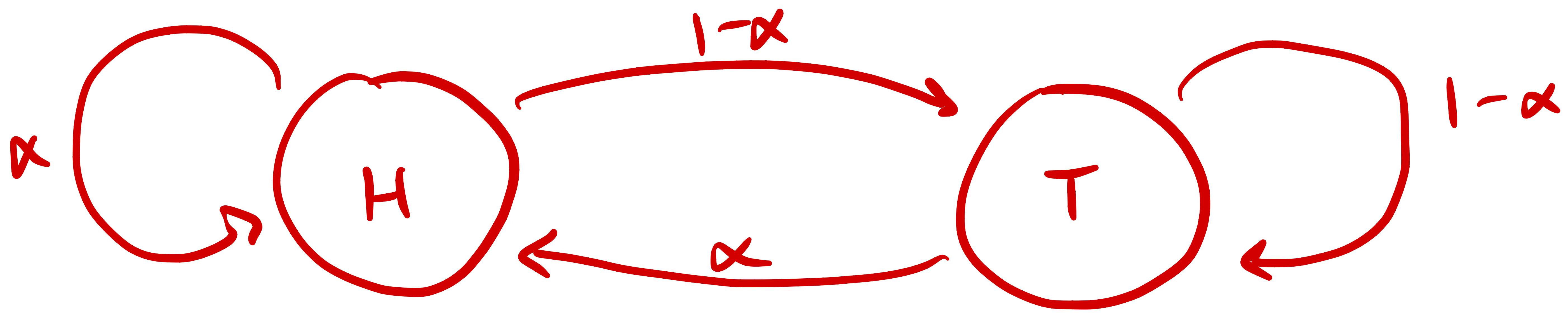

- Question 2: How can we express this as a transition matrix?

$$
\begin{array}{c|cc}
 & H & T \\
\hline
H & \alpha & 1-\alpha \\
T & \alpha & 1-\alpha
\end{array}
\qquad\qquad
\begin{array}{c|cccc}
 & 0 & 1 & \cdots & n \\
\hline
0 & & & & \\
1 & & & & \\
\vdots & & & & \\
n & & & &
\end{array}
$$

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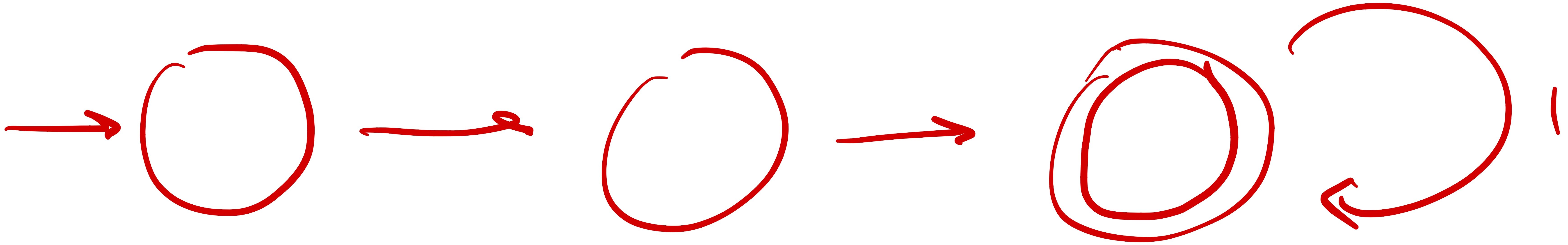
- Pictorially:



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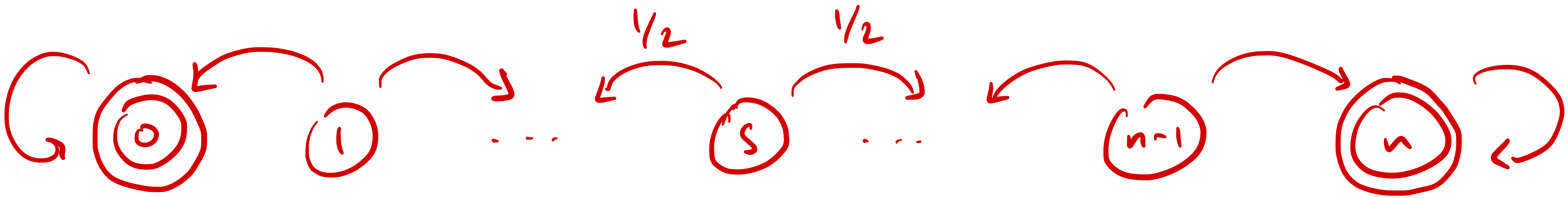
- Express this as a graph:



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Given s, ± 1 transitions

$\Pr(\text{hit } n \text{ before } 0)$

- Question: What is the probability of winning? (Key idea: recurrences!)

$R_i = \Pr(\text{winning starting from } \$i)$

$R_i - R_{i-1} = R_{i+1} - R_i$

$R_1 - 0 = R_2 - R_1$

$2R_1 = R_2$

$\rightarrow \quad R_k = k \cdot R_1$

$R_0 = 0$

$R_n = 1$

$R_n = n \cdot R_1 = 1$

$R_1 = 1/n$

$R_k = k/n$

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Here, P_{ij}^k is the $(i, j)^{th}$ entry in P^k

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$$\Pr(X_{n+k} = j \mid X_n = i) = P_{ij}^k \qquad P_{ij}^2$$

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- Let's consider $k = 2$ transitions

$$\Pr(X_{n+2} = j \mid X_n = i) = \sum_h \Pr(X_{n+2} = j \mid \boxed{X_{n+1} = h, \cancel{X_n = i}}) \leftarrow \text{step 2}$$

$$\times \Pr(X_{n+1} = h \mid X_n = i) \leftarrow \text{step 1}$$

step 1 h

$$\sum_h P_{ih} \cdot P_{hj} = \sum_h \underbrace{\Pr(X_{n+2} = j \mid X_{n+1} = h)}_{P_{hj}} \underbrace{\Pr(X_{n+1} = h \mid X_n = i)}_{P_{ih}}$$

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- Get as a result of matrix multiplication!

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- Path: $i_0 \to i_1 \to \ldots \to i_{m-1} \to i_m$
- Because every transition is given, we can just use the product rule
- $\Pr(X_m = i_m, X_{m-1} = i_{m-1}, \ldots, X_1 = i_1 \mid \boxed{X_0 = i_0}) = p_{i_0,i_1} \cdot p_{i_1,i_2} \cdot \ldots \cdot p_{i_{m-1},i_m}$

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  - A Markov chain is ergodic if all its states are ergodic ("best behaved chains")

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$$\pi = \pi P$$

Stationary Distributions

- Long-run behavior of a Markov chain is given by the **stationary distribution** $\pi = (\pi_1, \ldots, \pi_n)$ where π_i is the steady-state probability that the chain is in state i

- Let $N_i(k)$ be the number of transitions into state i after the k^{th} transition

$$\pi_i = \lim_{k \to \infty} \frac{N_i(k)}{k}$$

- The stationary distribution is *invariant* by the transition matrix:

$$\pi = \pi P$$

- Any ergodic Markov chain has a unique stationary distribution

Stationary Distributions: Example

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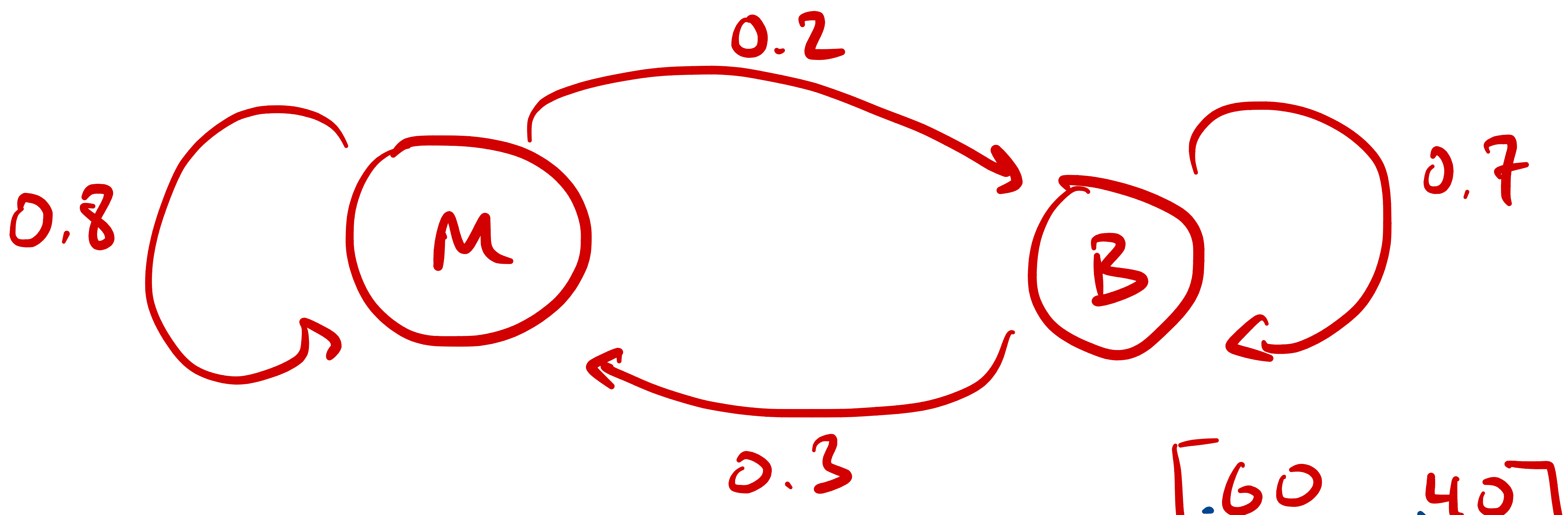
- A population of 100 people either go to McDonald's or Burger King for fast food every day. Every day, 20 ~~10~~% of people who went to McDonald's the previous day switch to Burger King, and 30% ~~20~~% of people who went to Burger King the previous day switch to McDonald's. After a while, the $\boxed{\text{proportion}}$ of the population that goes to each restaurant stabilizes. What is the stationary distribution?



m = # ppl at McD's

b = # ... BK

$$[.60 \quad .40] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \stackrel{?}{=} [60 \quad 40]$$

$$\pi = \pi P$$

$$\Big[\overset{48}{60 \times 0.8} + \overset{12}{40 \times 0.3}, \ \overset{12}{60 \times 0.2} + \overset{28}{40 \times 0.7}\Big] = [60 \quad 40]$$

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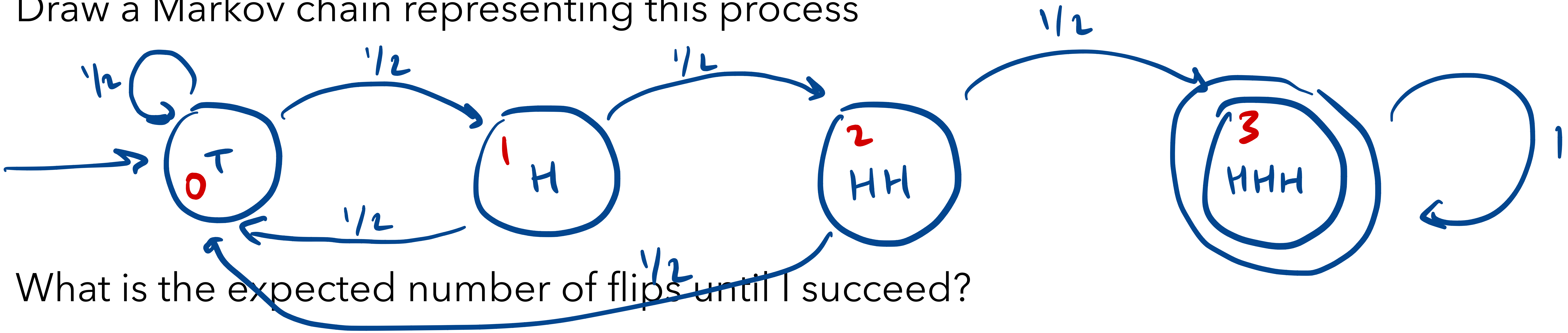

- What is the expected number of flips until I succeed?

X_i = exp # flips until HHH starting at i.

$$\boxed{X_0 = 1 + \tfrac{1}{2} X_0 + \tfrac{1}{2} X_1}$$

$$X_1 = 1 + \tfrac{1}{2} X_0 + \tfrac{1}{2} X_2$$

$$X_2 = 1 + \tfrac{1}{2} X_0 + \tfrac{1}{2} \underbrace{X_3}_{0}$$

TBD ...

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- What is the expected number of flips until I succeed?

  $X_0 = 1 + (1/2)(X_0 + X_1)$
  $X_1 = 1 + (1/2)(X_0 + X_2)$
  $X_2 = 1 + (1/2)(X_0 + X_3)$
  $X_3 = 0$
  Solving yields $X_0 = X_1 + 2$, $X_1 = X_2 + 4$, $X_1 = 12$, so $X_0 = 14$

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- Often applied to birth and death processes, with the requirement that state transitions can only occur between adjacent integers (or radioactive decay)

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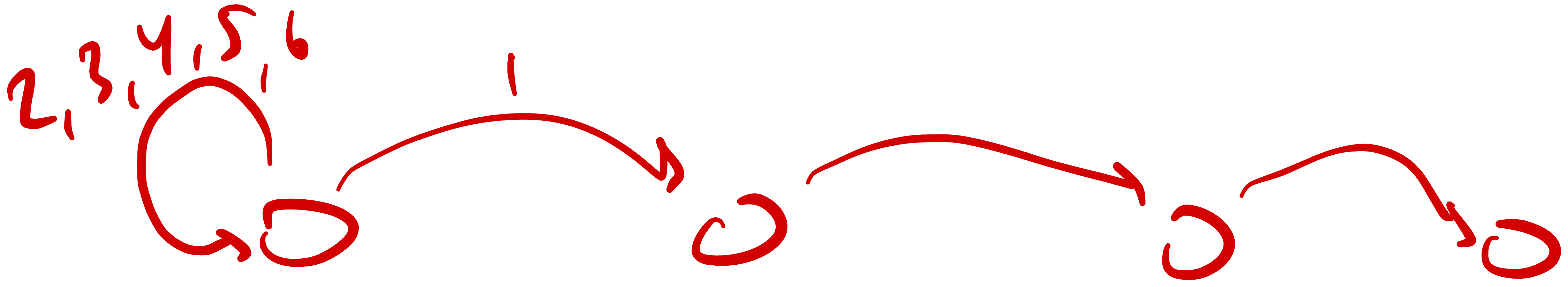

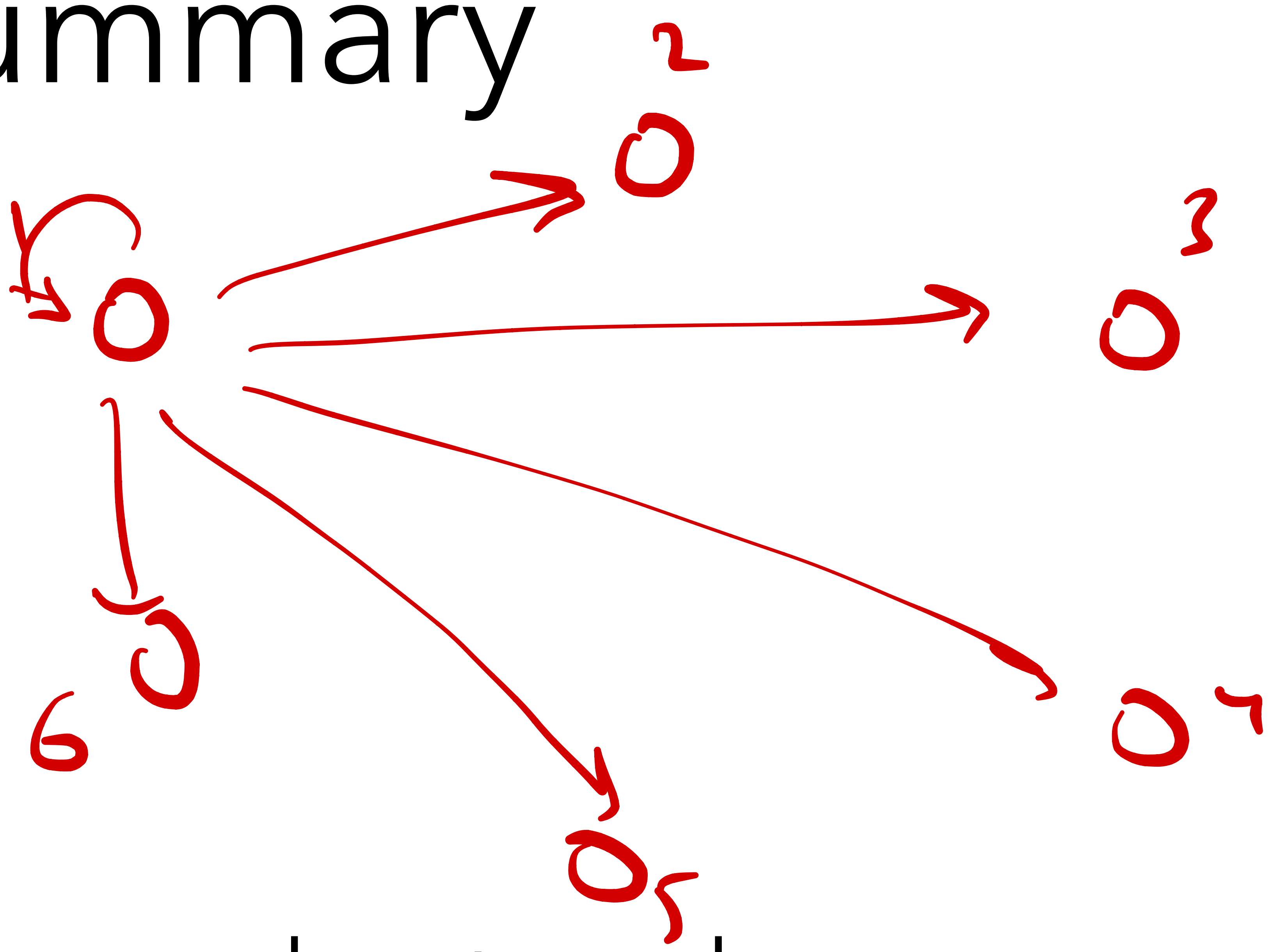

- Randomness is common
- These processes give us tools to reason about randomness
- Many extensions of these frameworks are foundational for advanced topics in machine learning, data science, physics, etc.

$m = 0.8m + 0.3b$