Chapter 15: Regression III

DSCC 462
Computational Introduction to Statistics

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Fall 2022

Plan for Today

Plan for Today

- Transformations of variables

Plan for Today

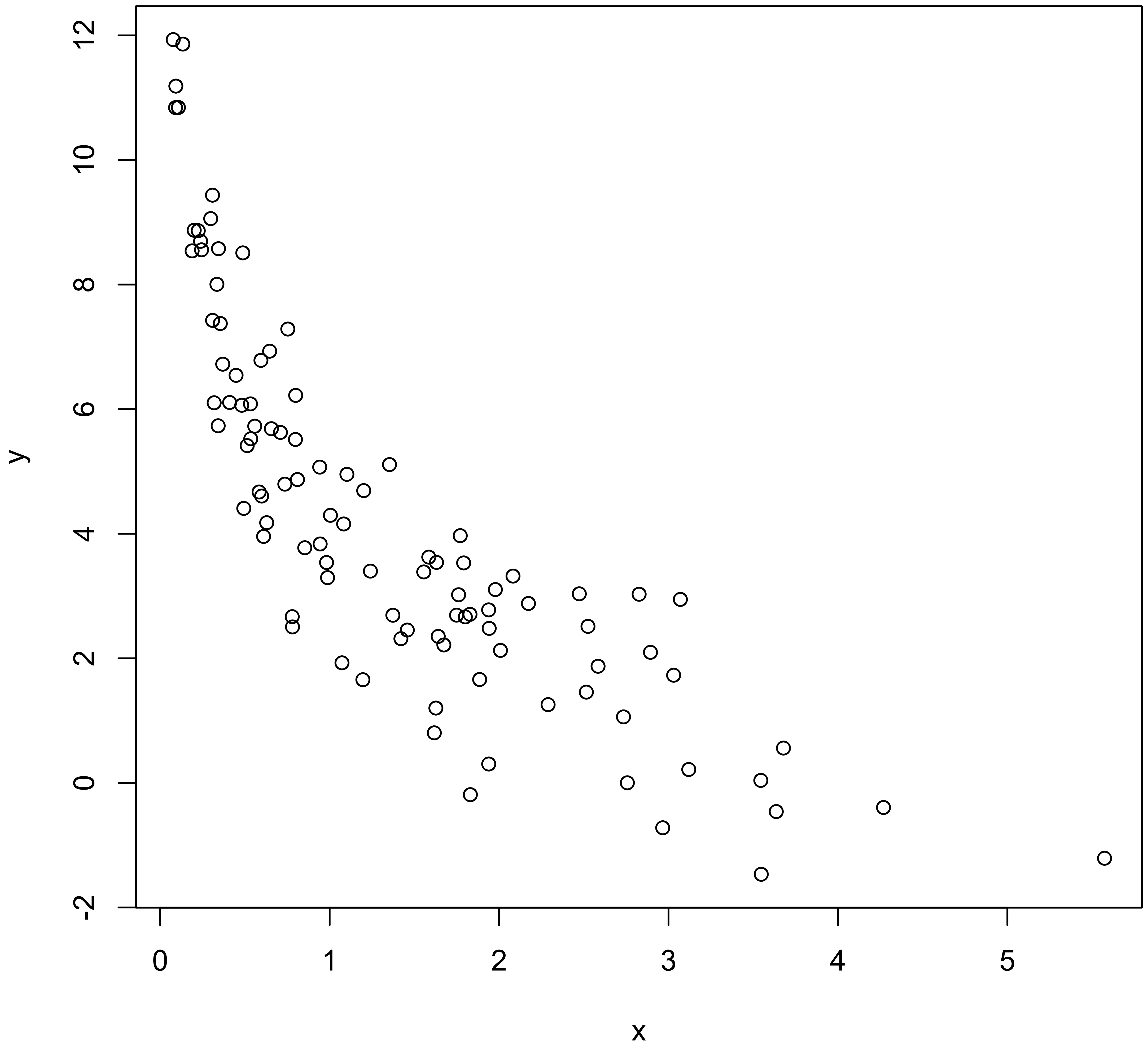
- Transformations of variables
- Categorical variables with multiple categories

Plan for Today

- Transformations of variables
- Categorical variables with multiple categories
- Predicting binary random variables (logistic regression)

Transformations

Transformations



Transformations

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- Or, we will use the natural log: $\ln(x)$ or $\ln(y)$ – corresponds to a choice of $p = 0$ in the above power transformations
- To determine which transformation is a good place to start, we use the (*Tukey) ladder of powers*

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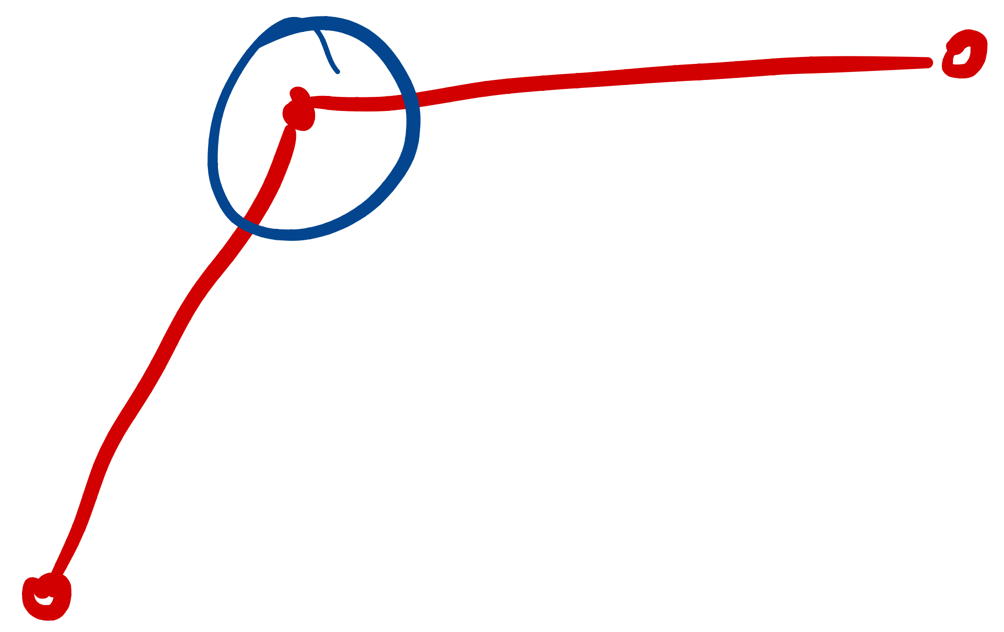

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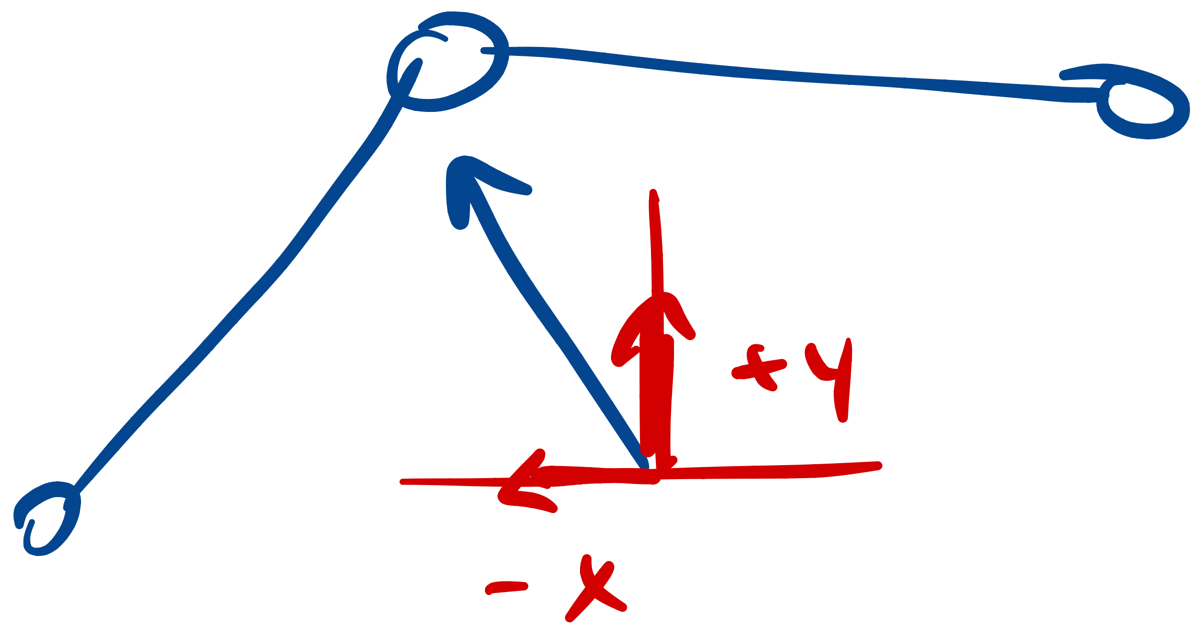

- As originally described by Tukey:
  - Visually divide your data into even thirds, take median x and y value in each third to get three reference points
  - Draw lines connecting consecutive reference points
  - Draw an arrow toward the "elbow" of the lines, then use this direction to decide how to transform data

Transformations

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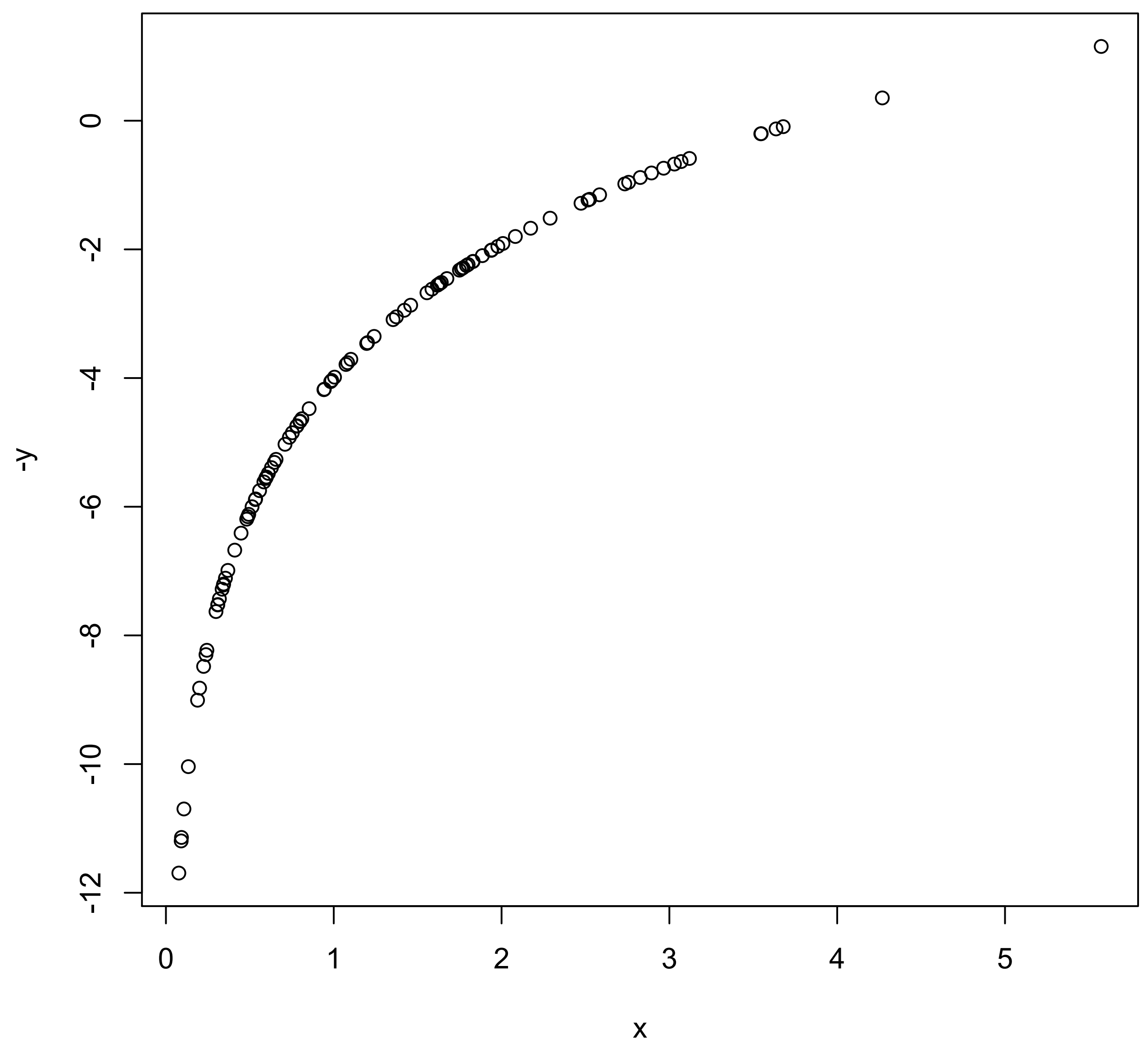

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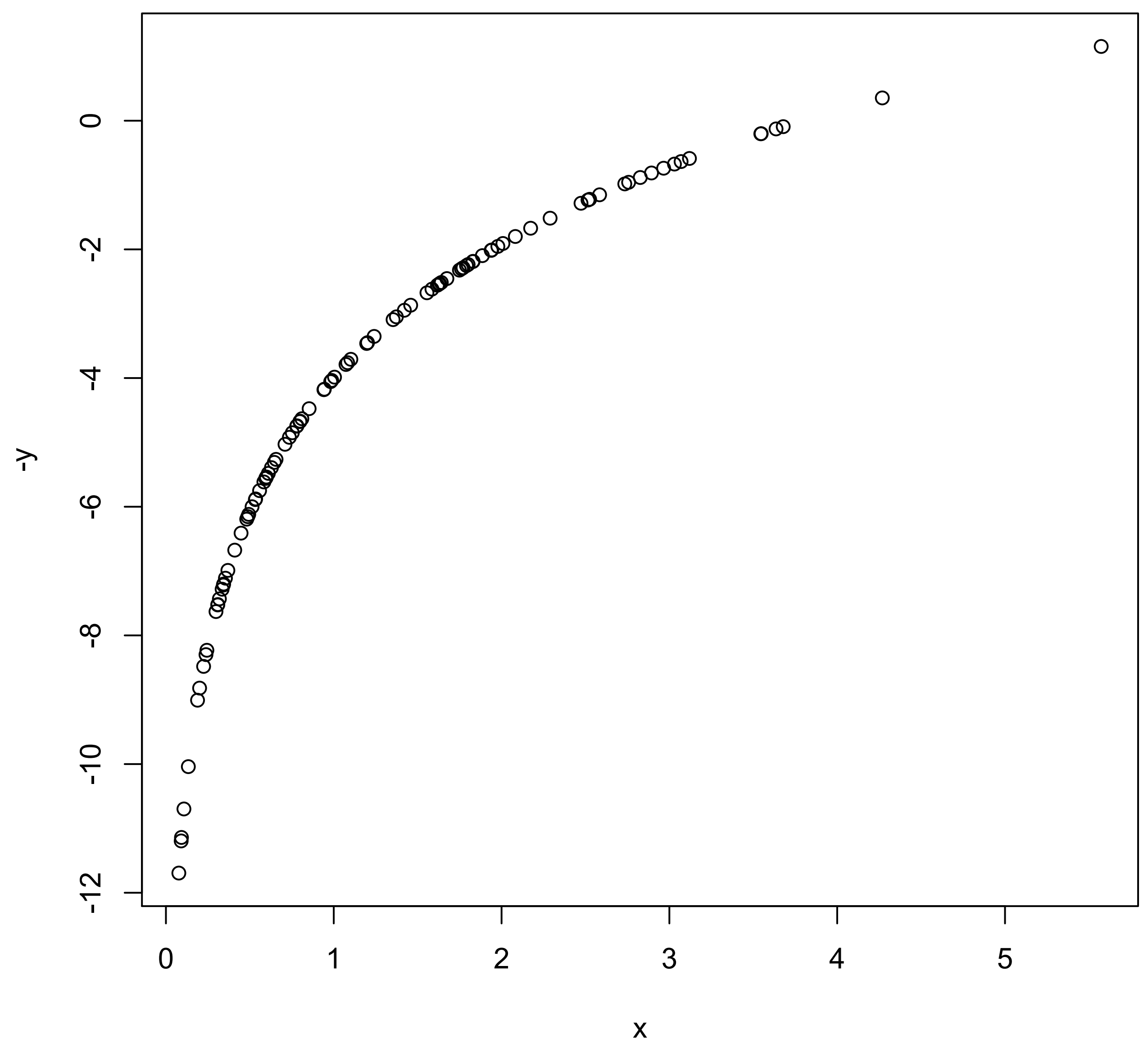

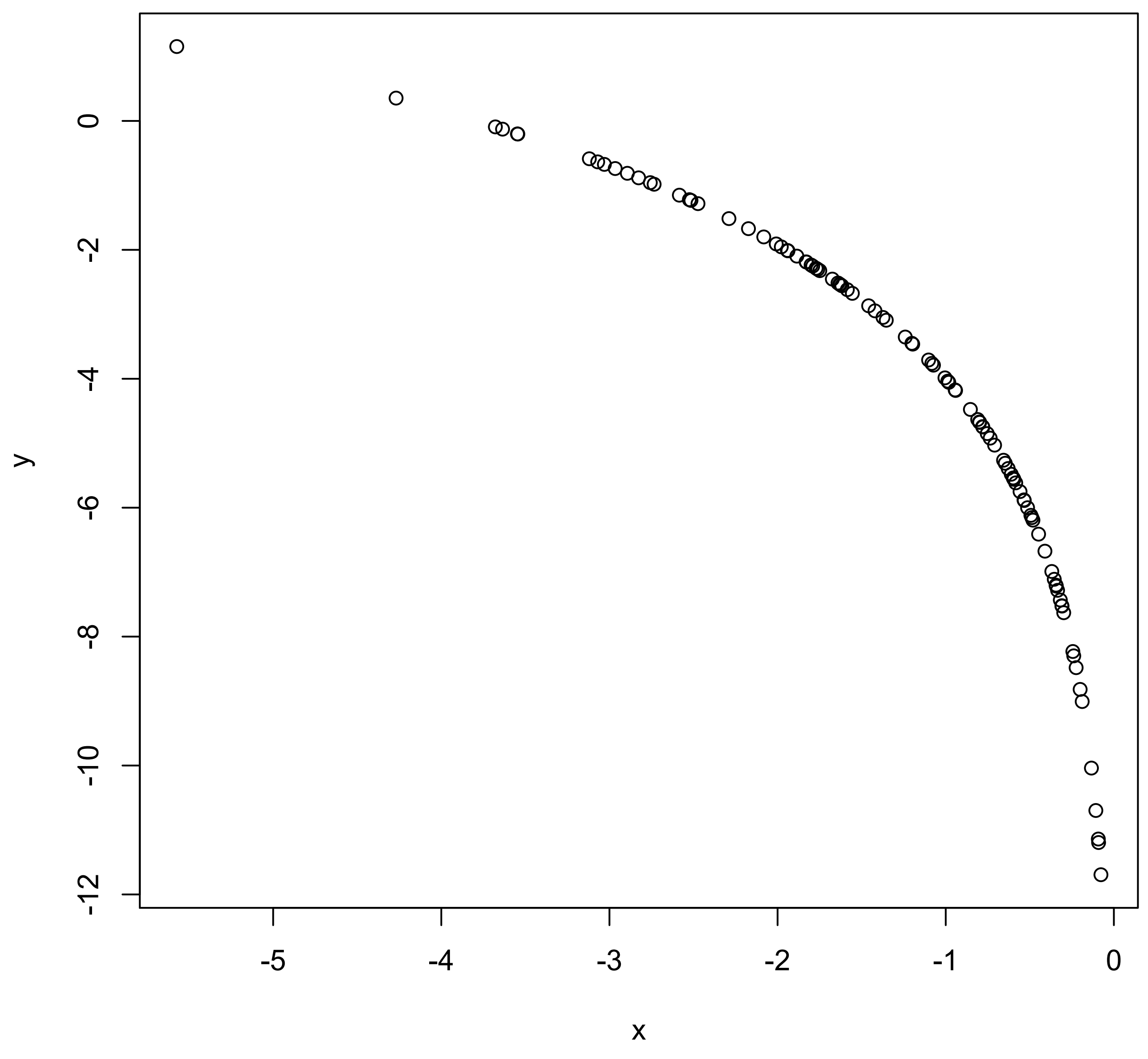

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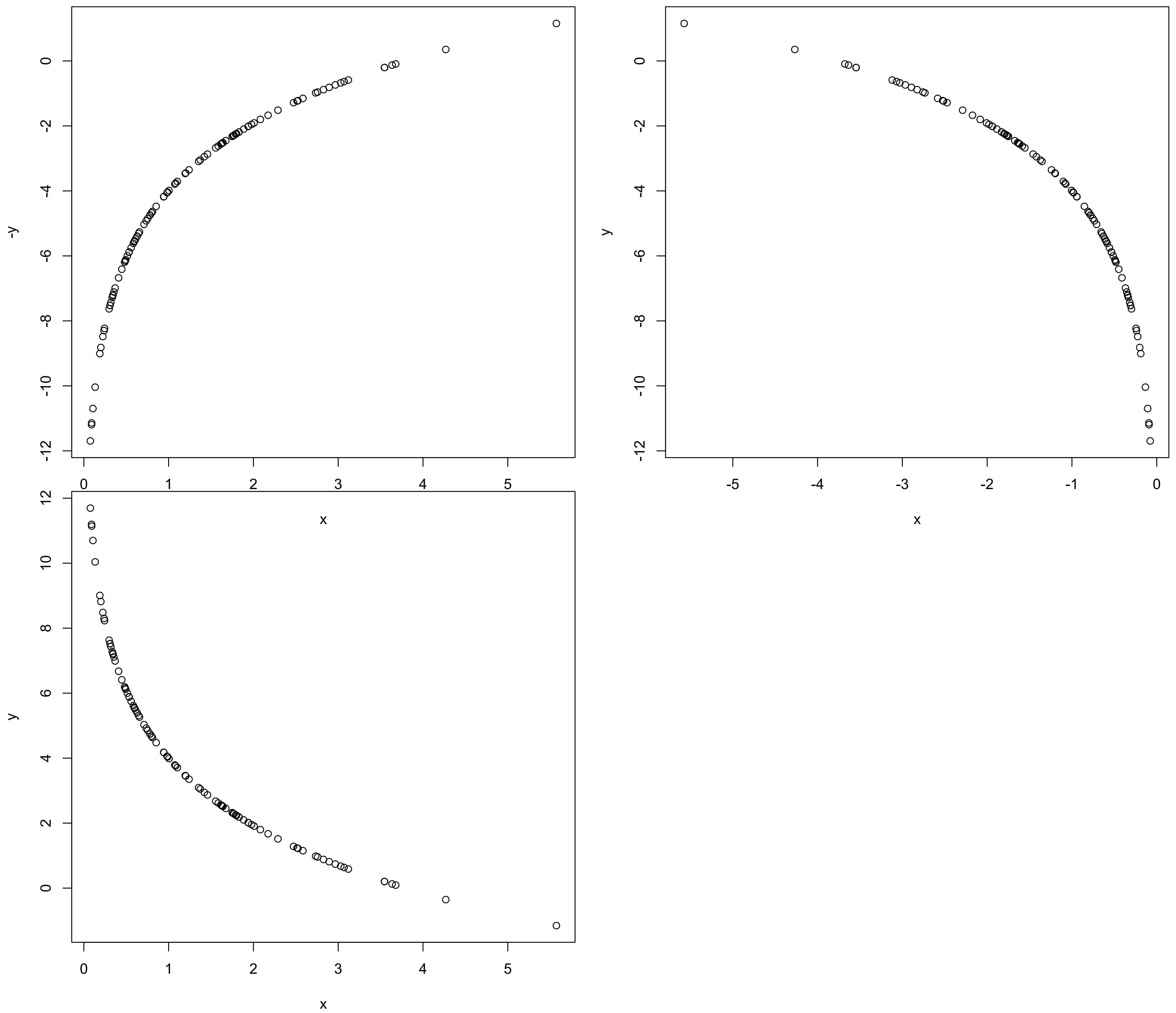
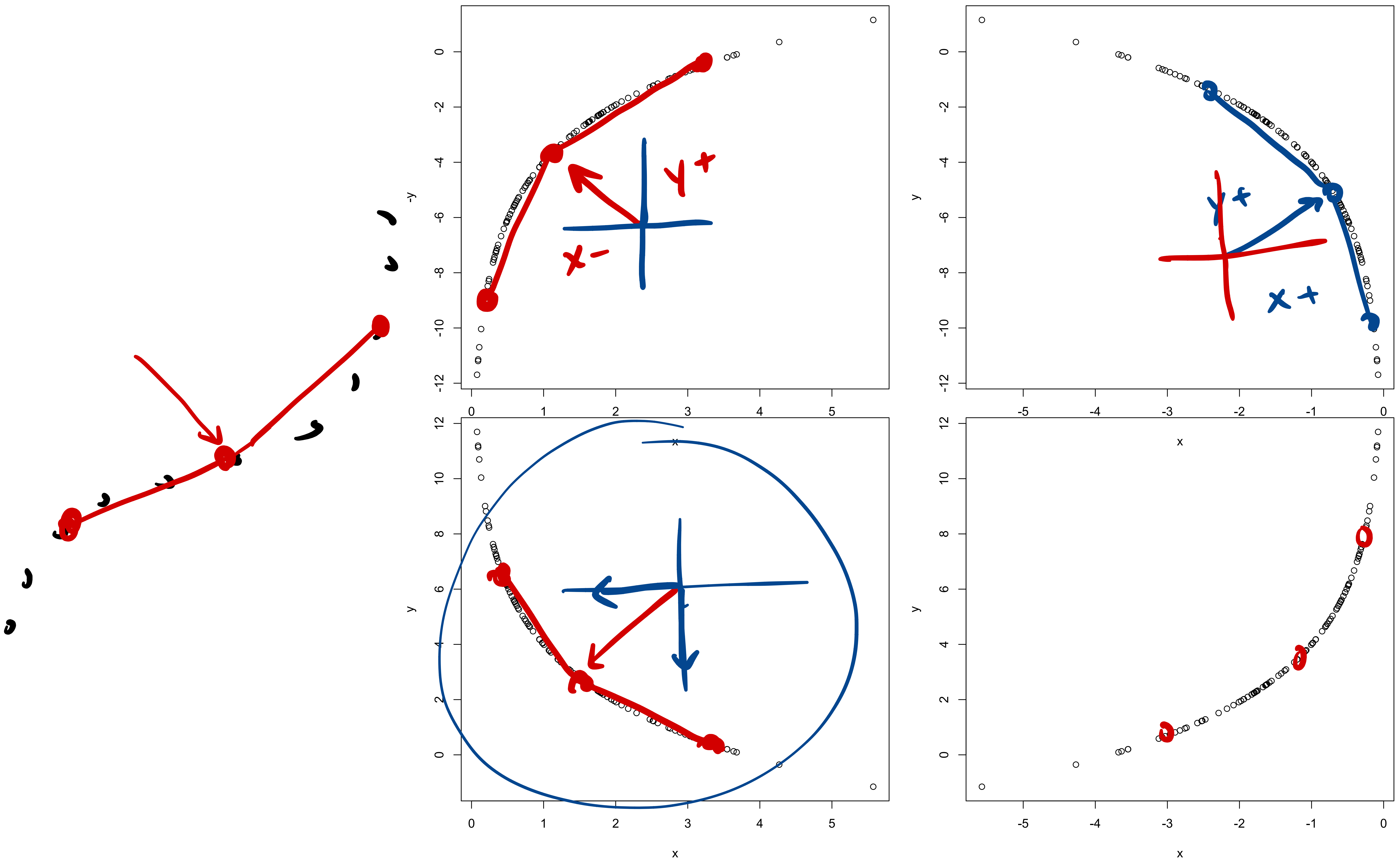


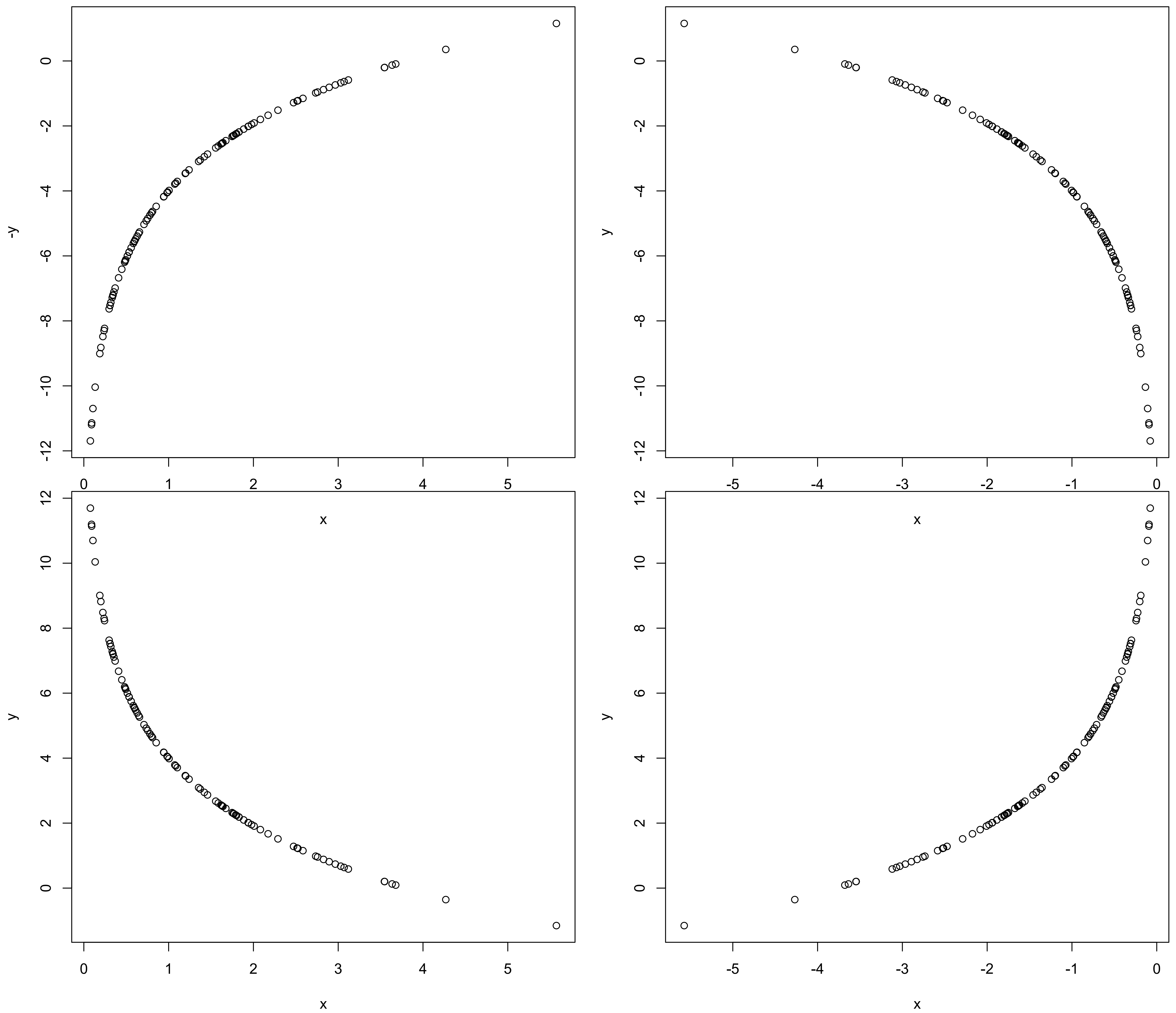
Transformations

x^p y^p



Transformations

Decrease exponent of x
Increase exponent of y



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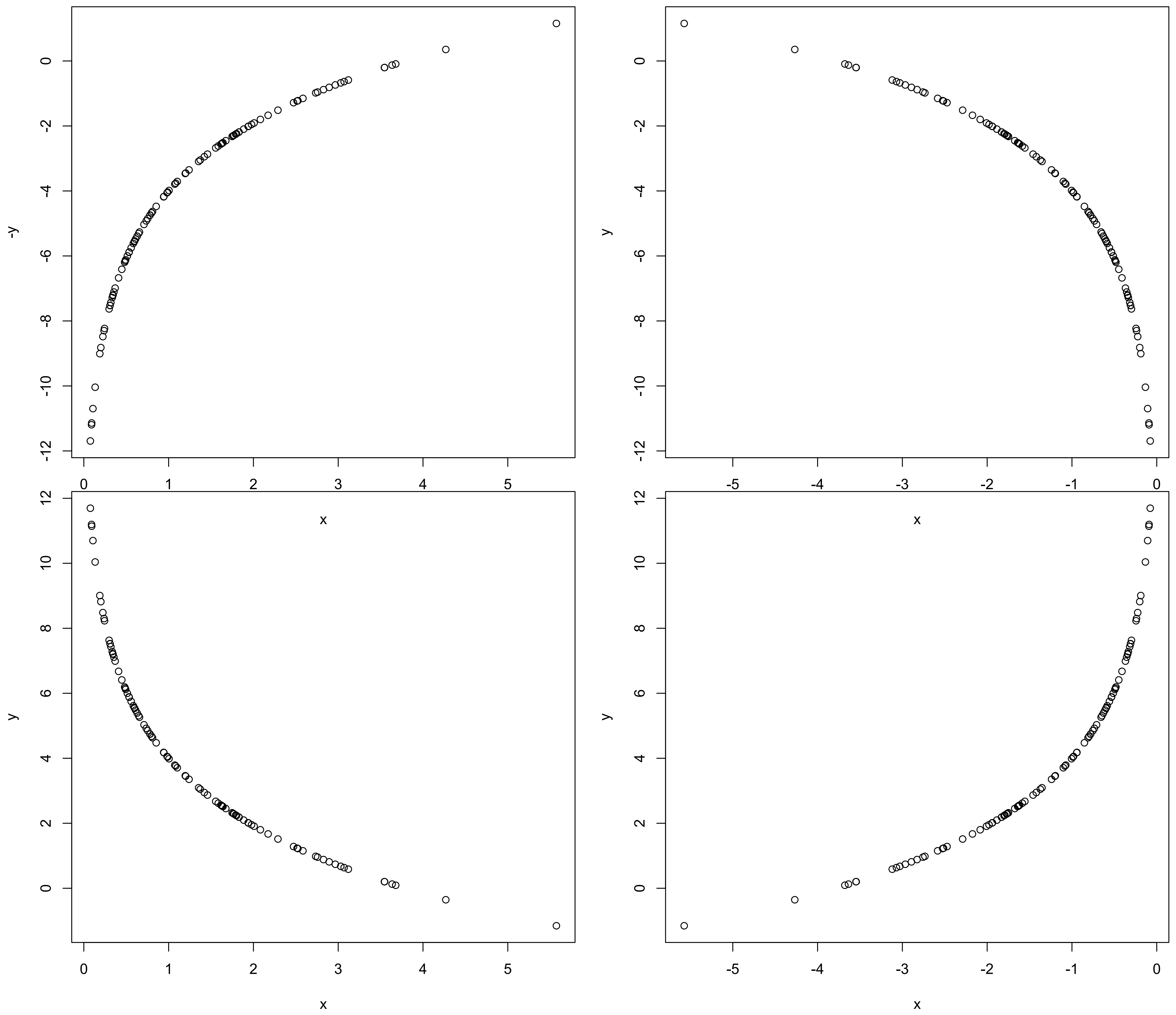

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Transformations

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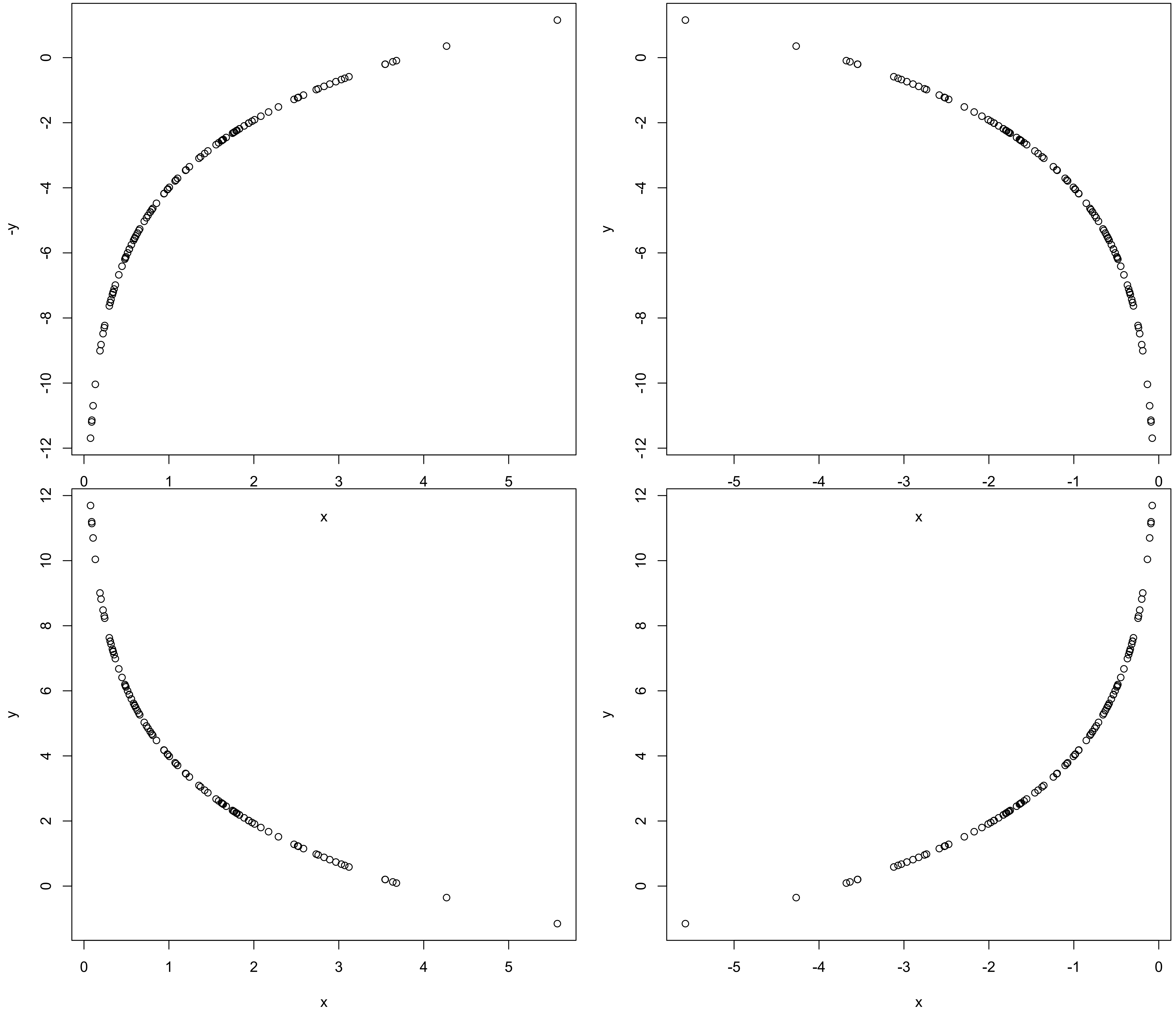

Transformations

1, 1   1, 1

2, 6   4, 36

Decrease exponent of x

Increase exponent of y

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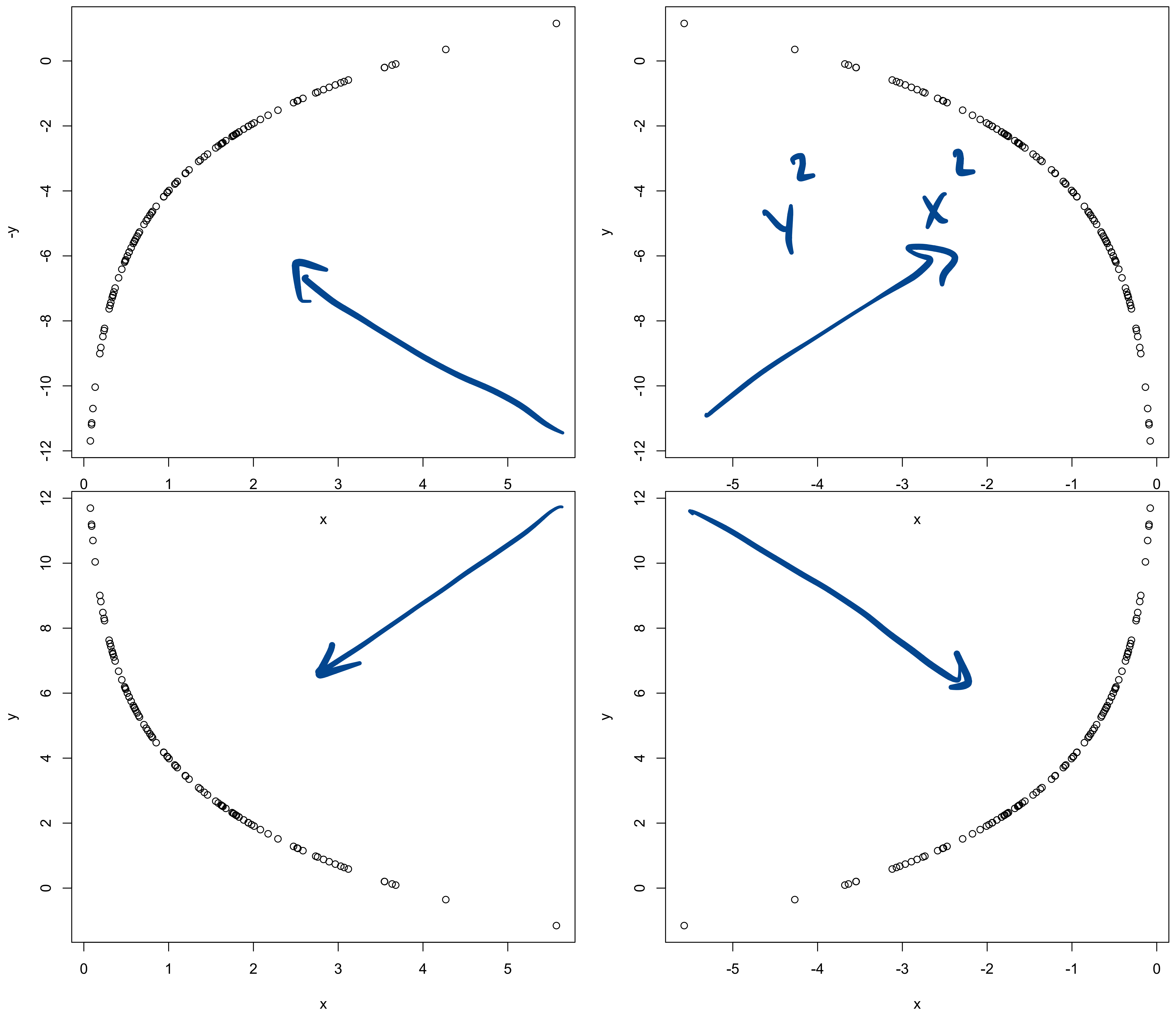
Increase exponent of y



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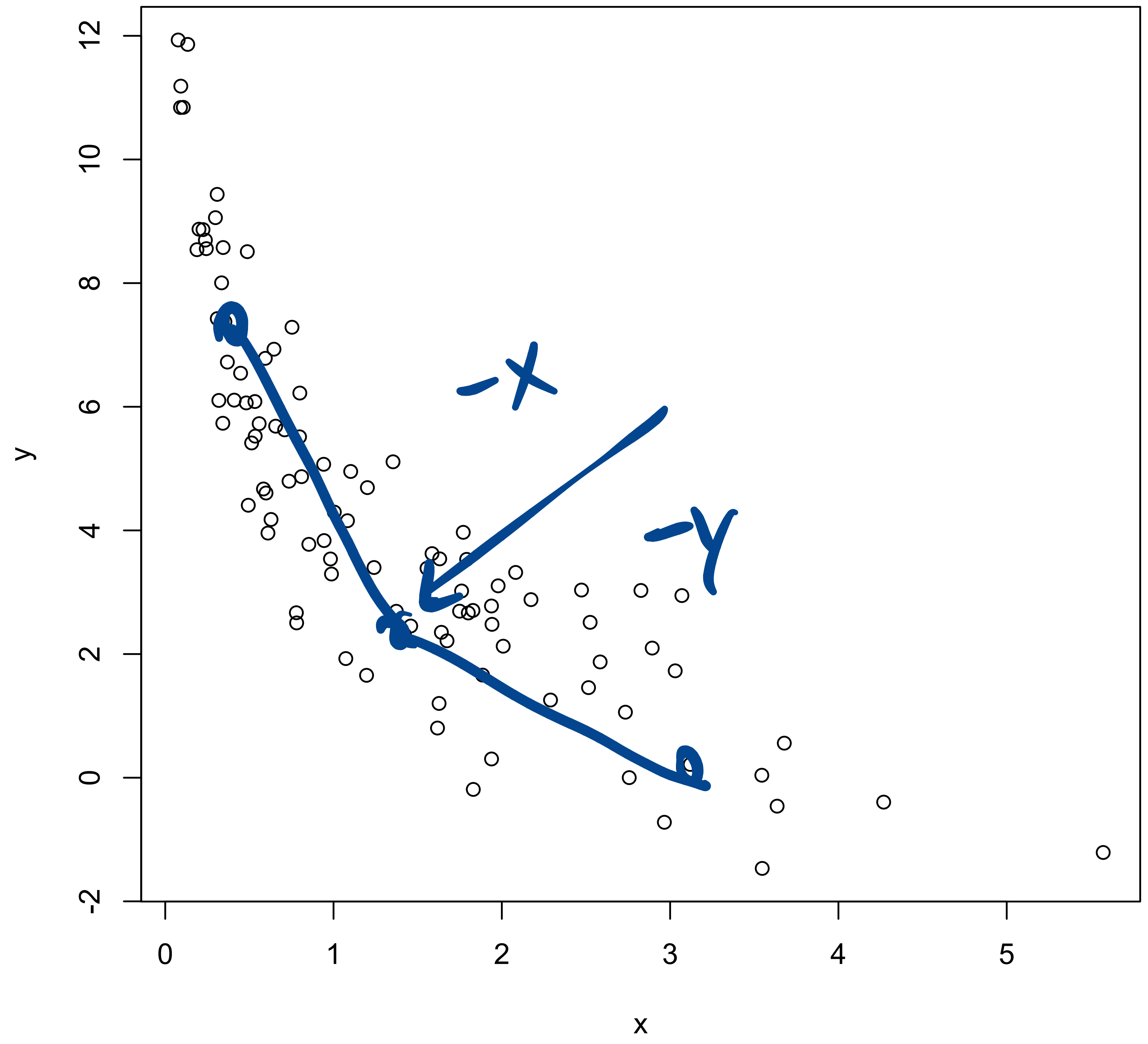
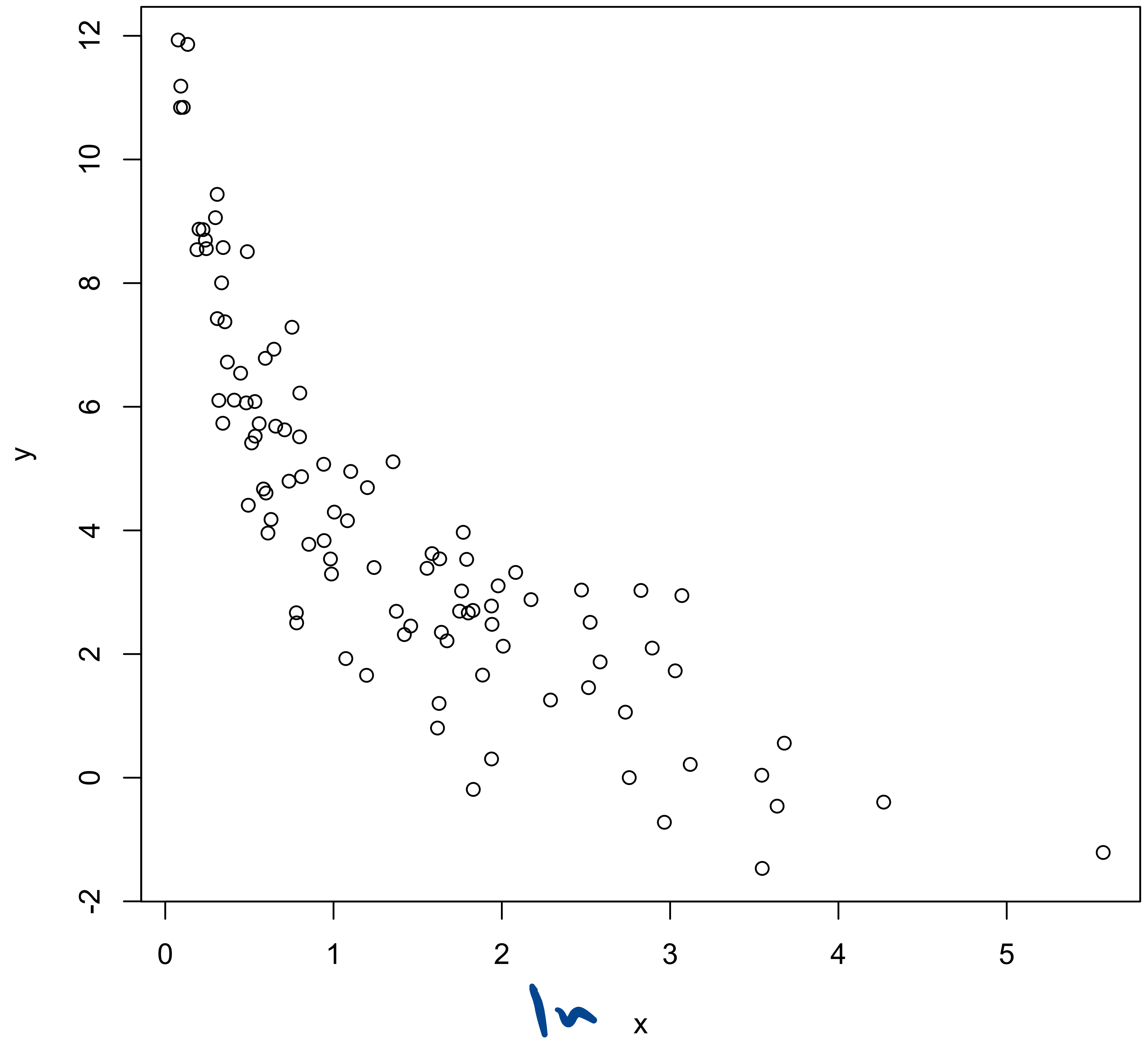


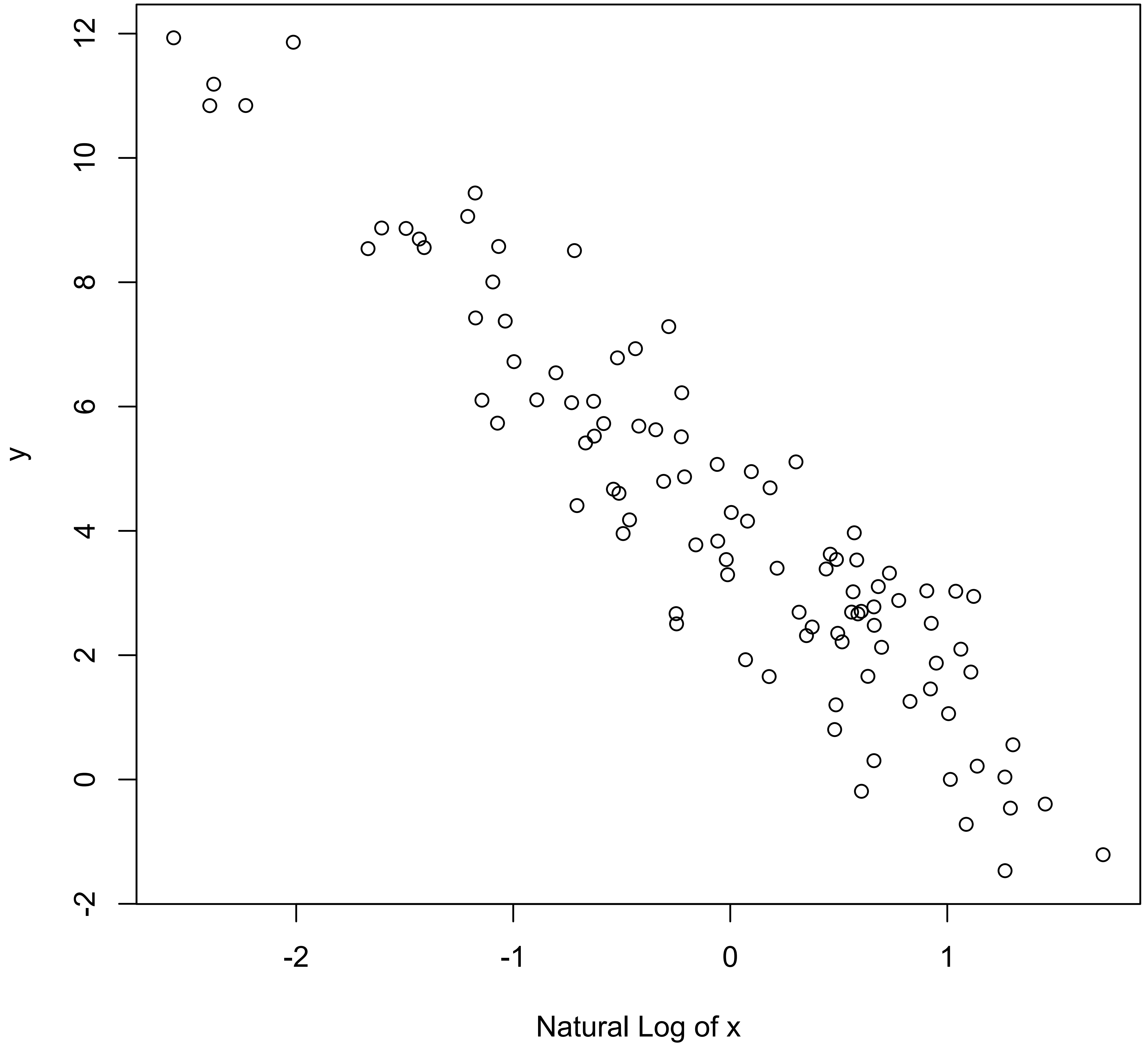
Transformations

- Consider the following plot
- A natural log transformation of x may be appropriate, since we need to decrease the exponent



Transformed Data

$$\hat{y} = \hat{\beta}_1 \cdot \ln x + \hat{\beta}_0$$



Interpretation of Logarithmic Variables

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$\ln x + 1$

$= \ln x + \ln e$

$= \ln(x \cdot e)$

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  - $\hat{\beta}_1$ is the increase in $\hat{y}$ associated with multiplying x by $e \approx 2.718$

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$x \to x(1.01)$

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$$\underbrace{\hat{\beta}_1 \left[\ln(x)\right.}\left. + \ln(1+p) \right] + \hat{\beta}_0$$

$$\ln(1+p) \approx p$$

$$\ln(1.01) \approx 0.01$$

$$\hat{y}^* = \underbrace{\hat{\beta}_1 \ln(x) + \hat{\beta}_0} \boxed{+ \hat{\beta}_1 \ln(1+p)}$$

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0.01

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$\hat{y}$

- **Interpretation**: $p \cdot 100\,\%$ increase in $x \to \hat{y}$ increases by $\hat{\beta}_1 \cdot p$
  - If x increases by 1%, $\hat{y}$ increases by $\hat{\beta}_1/100$

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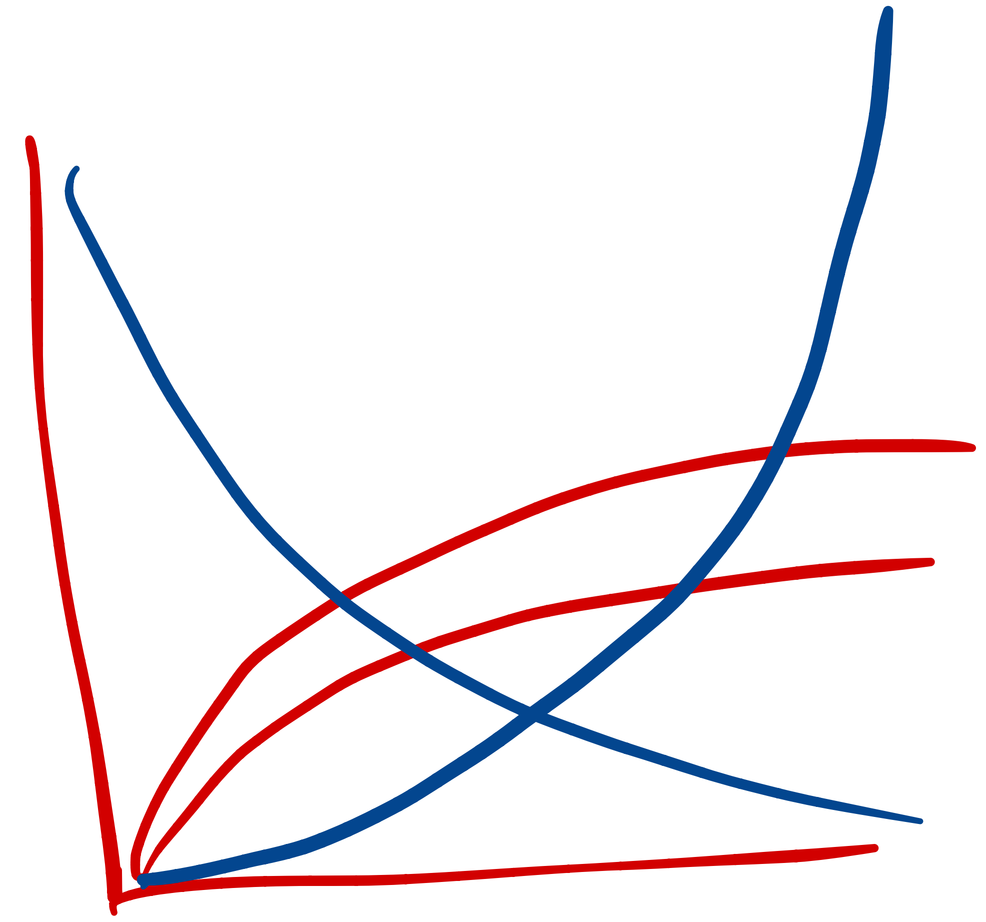
$$\hat{y}\, e^{\hat{\beta}_1} \approx \hat{y}\,(1 + \hat{\beta}_1) = \hat{y} + \boxed{\hat{\beta}_1 \cdot \hat{y}}$$

Interpretation of Logarithmic Variables



$$y = \hat{\beta}_1 \cdot \sqrt{x} + \hat{\beta}_0$$

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% %

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- How can we use these variables in linear regression?

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  - This implies that there is an ordering over categories, and stipulates that changing eye color from brown to amber is three times as meaningful as changing eye color from brown to blue
  - We need another approach

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  - For each data point, exactly one of these variables will be 1, and the rest will be 0
- Extension of "dummy coding" from previous lecture (for a single binary variable)
  - R does this by default!

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  - Relatively simple to set up
  - Easy to map features to binary 0/1s
- Drawbacks:
  - Relatively primitive (there are other more expressive ways of encoding categorical variables)
  - If the variable has many categories, have to create many dummy variables

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  - E.g., how do age, weight, and/or eye color predict height?
- Now, what if the variable we want to predict is binary?
  - E.g., how do age, weight, and/or eye color predict diabetes? 0/1

Odds

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- For an event E, odds$(E) = \frac{\Pr(E)}{\Pr(E^c)}$ (the probability of the event occurring divided by the probability of the event not occurring)

$$\Pr(E) = p$$

$$\text{odds}(E) = \frac{p}{1-p}$$

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- If we are told that the odds of event E happening are x to y, then we have $\text{odds}(E) = \dfrac{x}{y}$

Odds $\in [0, \infty]$ instead of $p \in [0,1]$

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- Extracting probabilities: $\Pr(E) = \dfrac{x^{\,100}}{\underset{101}{x+y}}$, and $\Pr(E^c) = \dfrac{y^{\,1}}{\underset{101}{x+y}}$

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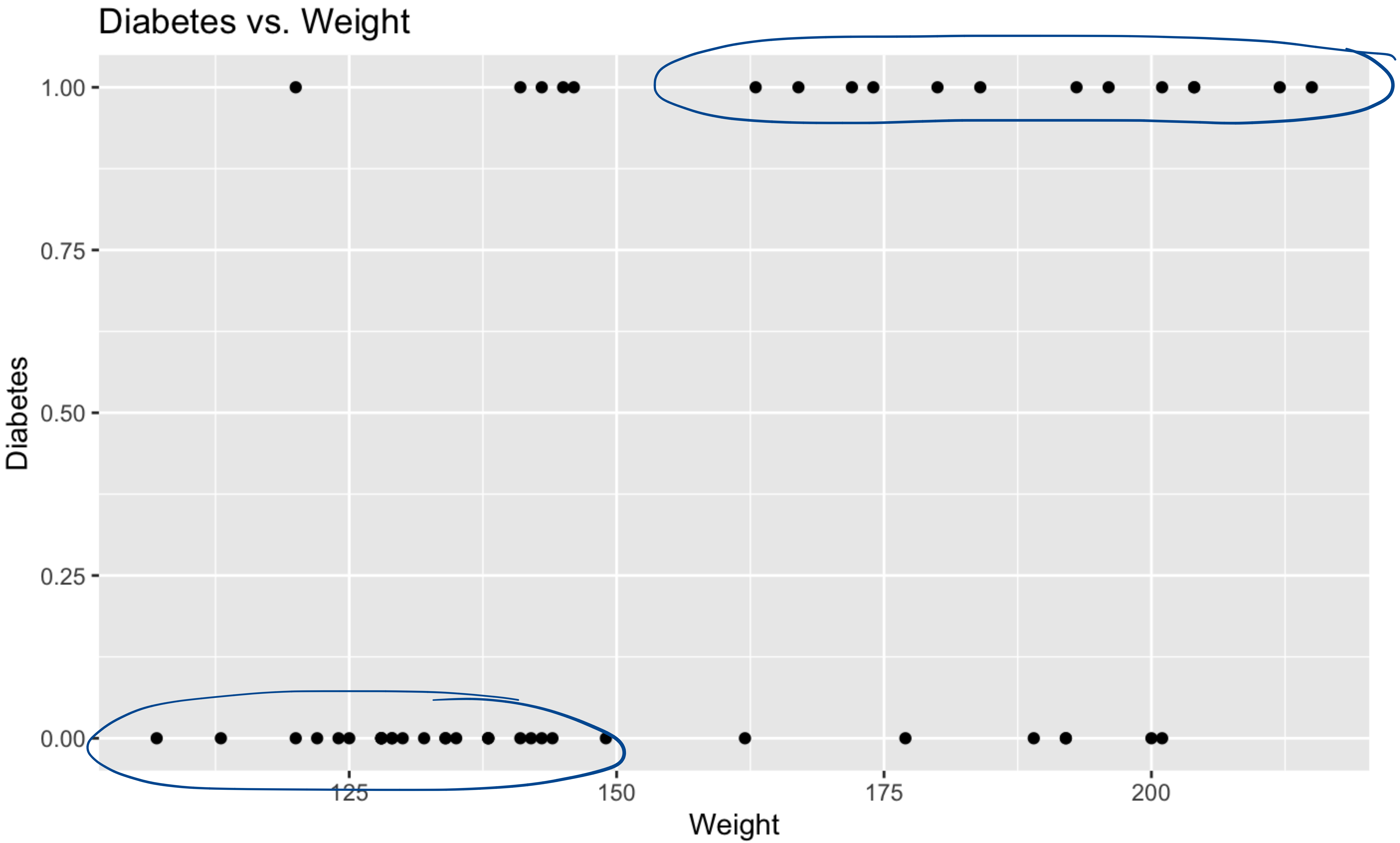
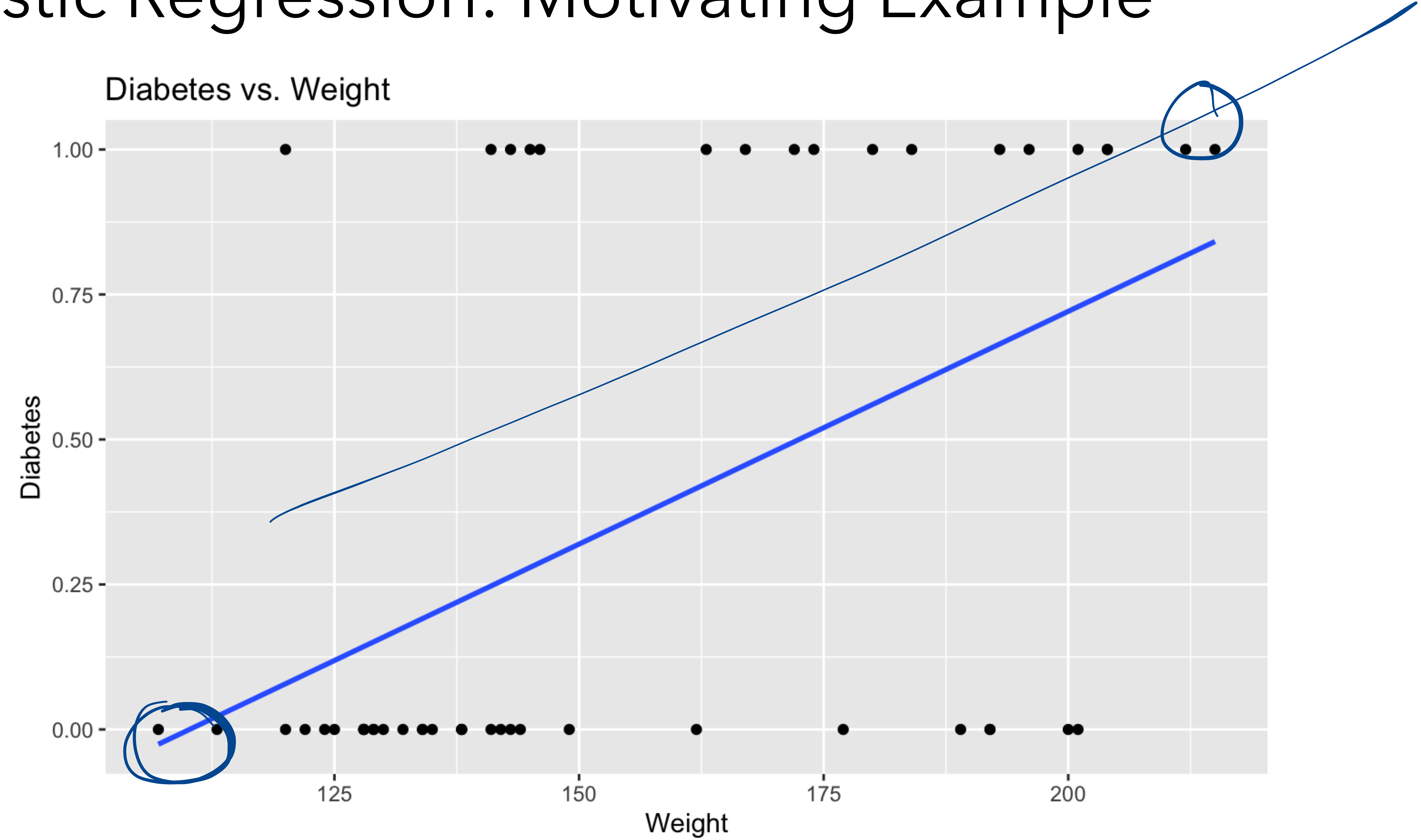
- Suppose that we would like to predict whether or not someone has diabetes based on various measurements about them
- Response variable: diabetes (binary)
- Explanatory variables: weight, height, age, sex
  - For now, just consider weight for simple logistic regression

Logistic Regression: Motivating Example



Logistic Regression: Motivating Example



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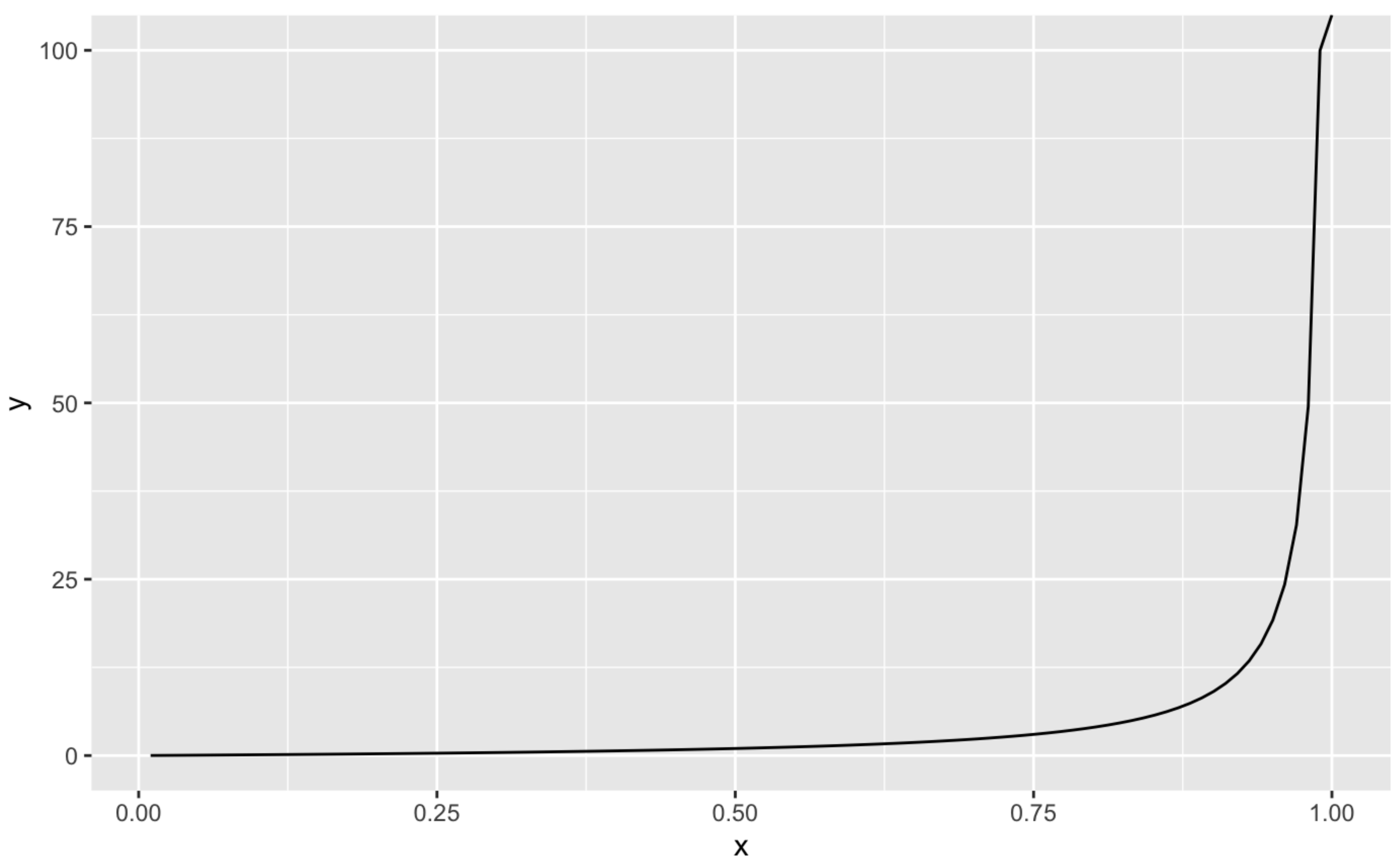

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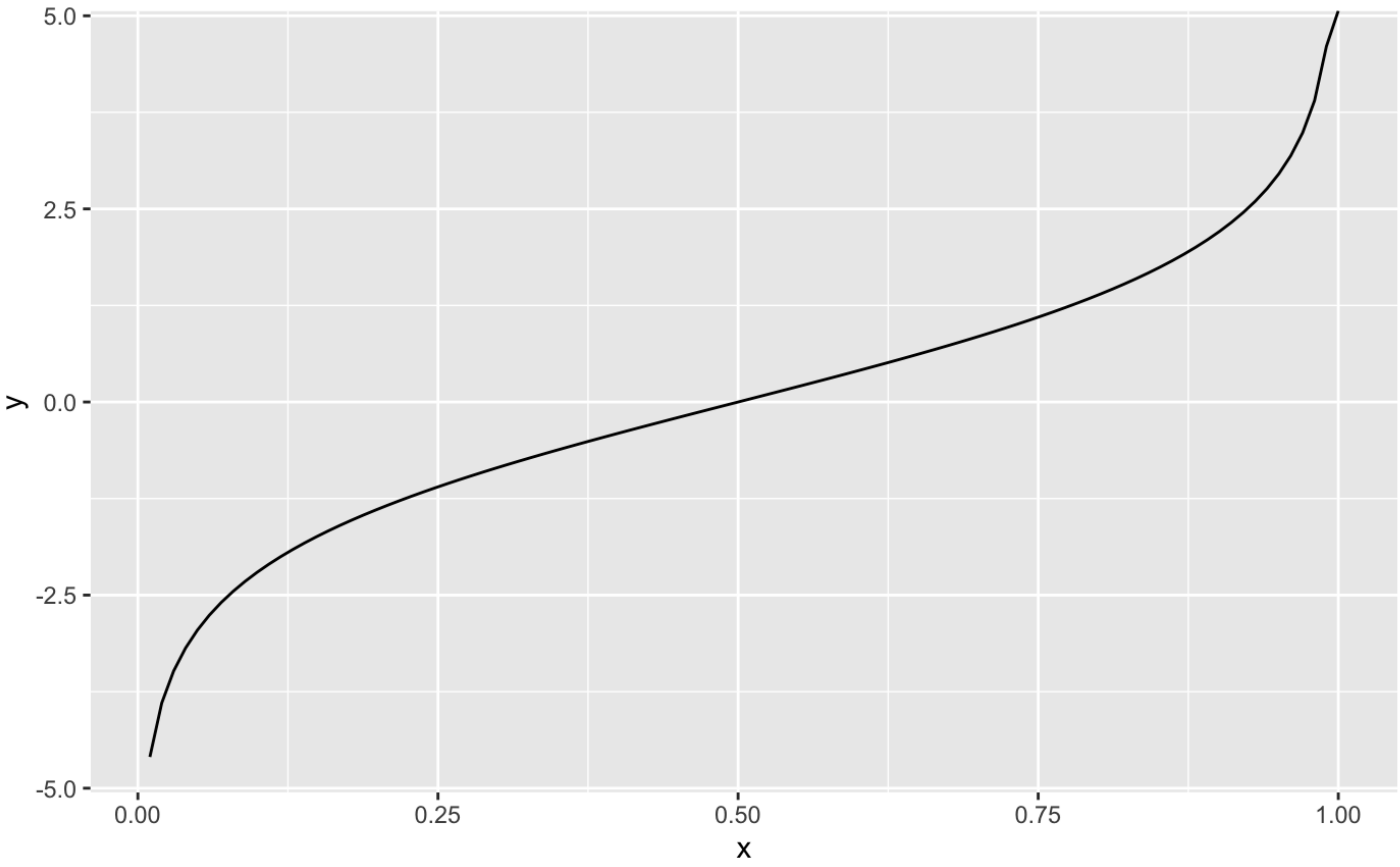

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- With the log-odds function, we can run “normal” regression now

Logistic Regression: R

$\log\left(\frac{p}{1-p}\right) = \hat{\beta}_1 x + \hat{\beta}_0$

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logit1 <- glm(diabetes~weight, data=diabetes_data, family="binomial")
summary(logit1)
```

```
Call:
glm(formula = diabetes ~ weight, family = "binomial", data = diabetes_data)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.6557  -0.6705  -0.5614   0.8154   2.0965

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -6.75790    1.94453  -3.475  0.00051 ***
weight       0.03898    0.01190   3.275  0.00106 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 65.342  on 49  degrees of freedom
Residual deviance: 51.925  on 48  degrees of freedom
AIC: 55.925

Number of Fisher Scoring iterations: 4
```

new

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$$\log\left(\frac{p}{1-p}\right) = \overset{\hat{\beta}_0}{-6.758} + \overset{\hat{\beta}_1}{0.039} \cdot \text{weight}$$

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Logit

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- Interpretation of slope: Change in log-odds ratio per unit change in predictor (in this case, weight) – often relatively unintuitive

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  - Conclusion: If $\underline{p} < \alpha$, we reject the null hypothesis and conclude that the model is useful

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- Must compare AIC or BIC values for different models on the *same* data (absolute values are not comparable between different settings)
  - Lower is better
  - Still must check residuals! Lowest AIC/BIC score doesn't mean the model is necessarily good (all models may be bad)

Logistic Regression: R

54

+height + weight * height

```
```{r}
logit1 <- glm(diabetes~weight, data=diabetes_data, family="binomial")
summary(logit1)
```
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```
Call:
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Deviance Residuals:
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(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 65.342  on 49  degrees of freedom
Residual deviance: 51.925  on 48  degrees of freedom
AIC: 55.925

Number of Fisher Scoring iterations: 4
```

```
```{r}
logit2 <- glm(diabetes~height, data=diabetes_data, family="binomial")
summary(logit2)
```
```

```
Call:
glm(formula = diabetes ~ height, family = "binomial", data = diabetes_data)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.3582  -0.8968  -0.6762   1.1324   1.9529

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -10.85616    4.78450  -2.269   0.0233 *
height        0.15441    0.07136   2.164   0.0305 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 65.342  on 49  degrees of freedom
Residual deviance: 60.039  on 48  degrees of freedom
AIC: 64.039

Number of Fisher Scoring iterations: 4
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- $p = \Pr(|Z| > |z|) =$ `2*pnorm(-abs(z))`
- If $p < \alpha$, reject H_0

Logistic Regression: Interactions

```{r}
logit2 <- glm(diabetes~weight+height+sex+weight*sex, data=diabetes_data, family="binomial")
summary(logit2)
```

```
Call:
glm(formula = diabetes ~ weight + height + sex + weight * sex,
    family = "binomial", data = diabetes_data)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.5068  -0.6900  -0.4197   0.8994   2.5298

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -14.95394   10.19871  -1.466    0.143
weight        0.10436    0.06891   1.514    0.130
height       -0.01234    0.09401  -0.131    0.896
sexM         14.77105   11.11219   1.329    0.184
weight:sexM  -0.09556    0.07487  -1.276    0.202

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 65.342  on 49  degrees of freedom
Residual deviance: 49.841  on 45  degrees of freedom
AIC: 59.841

Number of Fisher Scoring iterations: 5
```

$\hat{\beta}_i$

Generalized Linear Models

lm( ) → glm( )

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    - For logistic regression: $g(p) = logit(p) = \underline{\log\left(\frac{p}{1-p}\right)}$

$\log\left(\frac{p}{1-p}\right) = \eta$

$=$

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  - A function g that relates the linear model to the parameter of the outcome distribution
    - $g(p) = \eta$ or $p = g^{-1}(\eta)$
    - For logistic regression: $g(p) = logit(p) = \log\left(\frac{p}{1-p}\right)$
- In R: `glm()` command

?glm( )

q

p = # explanatory