Chapter 14: Inference for Correlation

DSCC 462
Computational Introduction to Statistics

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Plan for Today

- Introduce tools that allow us to go past our previous assumptions of independence when comparing random variables
- In particular, learn how to infer whether or not a linear relationship exists between two variables, X and Y

Correlation

- Recall that correlation tells us the degree to which two random variables are (linearly) associated or related
- We denote the true population correlation of X and Y as ρ ("rho")
- We estimate ρ with Pearson's coefficient of correlation, r (Chapter 3)

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]\left[\sum_{i=1}^{n}(y_i - \bar{y})^2\right]}} = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

- Note that $-1 \leq r \leq 1$

Correlation: Example

- Setup: Suppose we examine $n = 7$ subjects for which we have age and weight measurements
- We want to determine whether a significant linear relationship exists between age (X) and weight (Y)

Correlation: Inference

- Question: Is the population correlation, ρ, between two variables, X and Y, different from 0?
- In other words, we're investigating whether a linear relationship exists between X and Y (age and weight)
- Use the sample correlation r for our statistic in hypothesis testing

Correlation: Inference

- Hypotheses: $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$
- If pairs (x_i, y_i) come from normally distributed X and Y, then if we standardize r, we get a statistic that has a t distribution with $n - 2$ degrees of freedom
- Test statistic: $t = \dfrac{r - \rho}{SE(r)}$, where $SE(r) = \sqrt{\dfrac{1 - r^2}{n - 2}}$
- We thus get $t = \dfrac{r - \rho}{SE(r)} = r\sqrt{\dfrac{n - 2}{1 - r^2}}$

Correlation: Inference Example

- Returning to setup: Suppose we examine $n = 7$ subjects for which we have age and weight measurements
- We want to determine whether a significant linear relationship exists between age (X) and weight (Y)
  - $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$
- We know that the correlation between weight and age for this sample is $r = 0.865$
- Test this null hypothesis at the $\alpha = 0.05$ significance level

Correlation: Inference Example

- $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$, test at $\alpha = 0.05$
- The correlation between weight and age for this sample is $r = 0.865$
- Test statistic:
- p-value:
- Conclusion:

Correlation: Inference in R

- We can also calculate directly in R using the `cor.test()` function

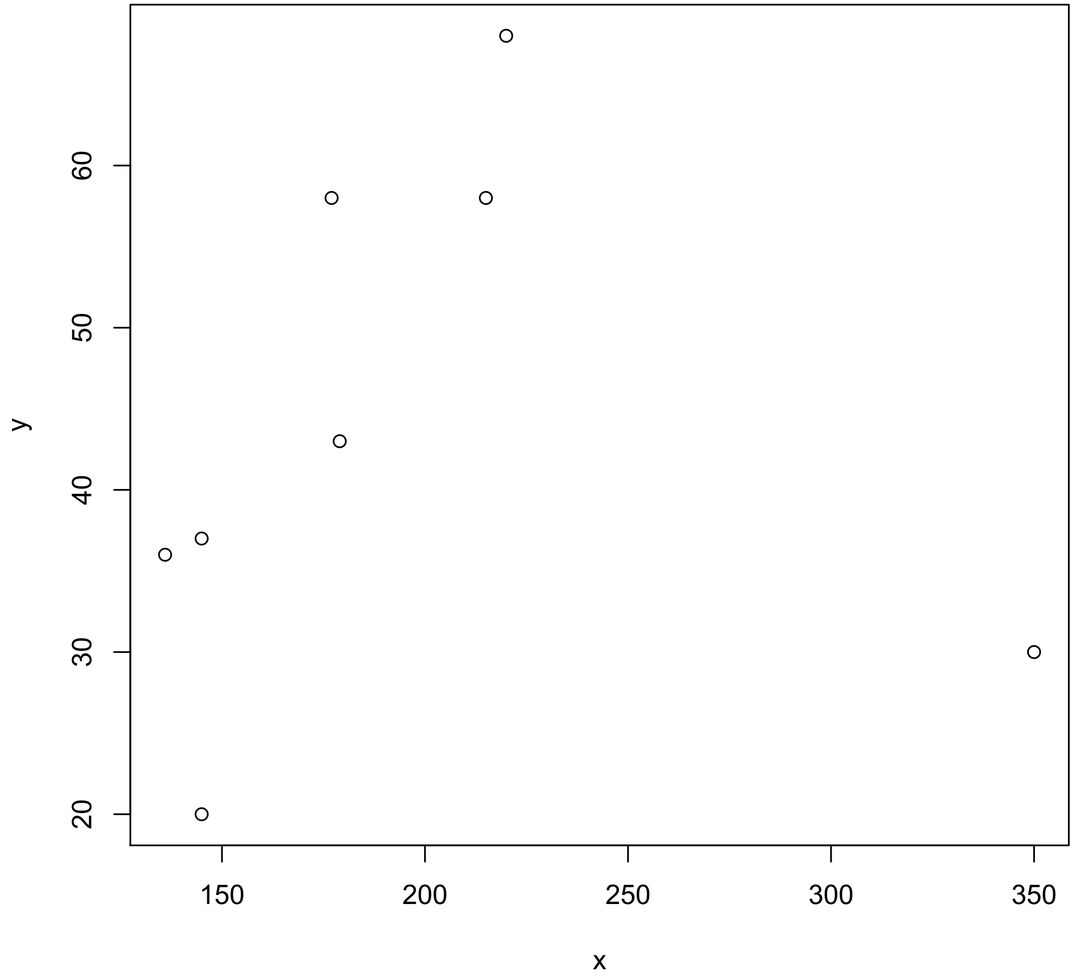
```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y)

    Pearson's product-moment correlation

data:  x and y
t = 3.856, df = 5, p-value = 0.01193
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.3213733 0.9798243
sample estimates:
      cor
0.8650727
```

Correlation Limitations

- Only describes linear relationships
  - We could be missing a nonlinear relationship if we don't examine the scatterplot
- Hypothesis testing only works for the null hypothesis $H_0 : \rho = 0$
  - For any $\rho \neq 0$, normality assumptions are not met and our hypothesis testing procedures are invalid
- Correlation can be very sensitive to outliers and can thus give misleading results when outliers are present

Correlation Limitations

- Suppose we have another subject who is 30 years old with a weight of 350 pounds



Correlation Limitations

```
> x<-c(220,215,179,145,145,177,136, 350)
> y<-c(68,58,43,37,20,58,36, 30)
> plot(x,y)
> cor(x,y)
[1] 0.06260467
```

Spearman's Rank Correlation Coefficient

- Need a more robust measure that isn't as sensitive to outliers
- Instead of using the actual observations, we rank the data and then use the *ranks* as our data
  - If multiple values are the same, assign the average rank
- Rank all the x values, and call these ranks x_r
- Rank all the y values, and call these ranks y_r
- Compute the Pearson's correlation coefficient for this ranked data (x_r, y_r) instead of the actual data
- This is *Spearman's correlation coefficient*

Spearman's Rank Correlation Coefficient

- Calculate Spearman's correlation coefficient as follows:

$$r_s = \frac{\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)(y_{ri} - \bar{y}_r)}{\sqrt{\left[\sum_{i=1}^{n} (x_{ri} - \bar{x}_r)^2\right]\left[\sum_{i=1}^{n} (y_{ri} - \bar{y}_r)^2\right]}}$$

- Note that $-1 \leq r_s \leq 1$
- Same interpretations of association between X and Y

Spearman's Rank Correlation Coefficient

- Consider the following age and weight measurements for the 7 subjects

Patient	Weight	Rank	Age	Rank
1	220		68	
2	215		58	
3	179		43	
4	145		37	
5	145		20	
6	177		58	
7	136		36	

Spearman's Rank Correlation Coefficient

- Using this information, we can calculate r_s as follows:
  - $\bar{x}_r =$
  - $\bar{y}_r =$
  - $r_s =$
- Recall that Pearson's correlation coefficient was given as 0.865. How does Spearman's rank correlation coefficient compare?

Spearman's Rank Correlation Coefficient: R

- In R:

```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor(x,y, method="spearman")
[1] 0.8727273
>
> x1 <- rank(x)
> y1 <- rank(y)
> x1; y1
[1] 7.0 6.0 5.0 2.5 2.5 4.0 1.0
[1] 7.0 5.5 4.0 3.0 1.0 5.5 2.0
> cor(x1,y1)
[1] 0.8727273
```

Spearman's Rank Correlation Coefficient: Interpretation

- Spearman's rank correlation coefficient is a measure of concordance of ranks for the outcomes x and y
- If measurements of X and Y are ranked in the same order for each variable, then $r_s = 1$
- If measurements of X and Y are ranked in the reverse order from each other, then $r_s = -1$
- If there is no linear correspondence between the ranks, then $r_s = 0$

Spearman's Rank Correlation Coefficient: Outliers

- What is the Spearman rank correlation coefficient when we have an outlier in our data?

```
> x<-c(220,215,179,145,145,177,136,350)
> y<-c(68,58,43,37,20,58,36,30)
> cor(x,y)
[1] 0.06260467
> cor(x,y, method="spearman")
[1] 0.373494
```

Spearman's Rank Correlation Coefficient: Inference

- We can similarly perform hypothesis tests for ρ based on r_s
- If the sample size is large enough (generally $n \geq 10$) and if we can assume that pairs of ranks (x_{ri}, y_{ri}) are chosen randomly, then we can test the null hypothesis $H_0 : \rho = 0$ vs. the alternative hypothesis $H_1 : \rho \neq 0$
- Use a similar test statistic: $t_s = r_s \cdot \sqrt{\frac{n-2}{1-r_s^2}}$
- Compare t_s to a t distribution with $n - 2$ degrees of freedom

Spearman's Rank Correlation Coefficient: Inference

- Although we only have $n = 7$, let's test $H_0 : \rho = 0$ vs. $H_1 : \rho \neq 0$ at $\alpha = 0.05$
- Calculating our t-statistic:

- Calculating the p-value:

- Conclusion:

Spearman's Rank Correlation Coefficient: R

```
> x<-c(220,215,179,145,145,177,136)
> y<-c(68,58,43,37,20,58,36)
> cor.test(x,y,method="spearman",exact=FALSE)

    Spearman's rank correlation rho

data:  x and y
S = 7.1273, p-value = 0.01035
alternative hypothesis: true rho is not equal to 0
sample estimates:
      rho
0.8727273
```