Chapter 5: Distributions

DSCC 462

Computational Introduction to Statistics

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Random Variables

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  - Examples: Time required to run a mile

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- Let X be a random variable defined over sample space S_X
- For any $E \subseteq S_X$, we can define $p_X(E) = \Pr(X \in E)$

$p_X(E)$

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- The support S_X consists of all x for which $p_X(x) > 0$ (all achievable outcomes)

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$3 \times \frac{1}{8}$

HTT
THT
TTH

PMF

x	Pr(X = x)
0	$1/8$ 125
1	$3/8$ 375
2	$3/8$
3	$1/8$ 125

Discrete Probability Distribution: Example

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- Fill in the following table:
- Probability distribution tables resemble relative frequency distribution tables: probability of each outcome is the relative frequency distribution of each outcome in a large number of trials

x	Pr(X = x)
0	
1	
2	
3	

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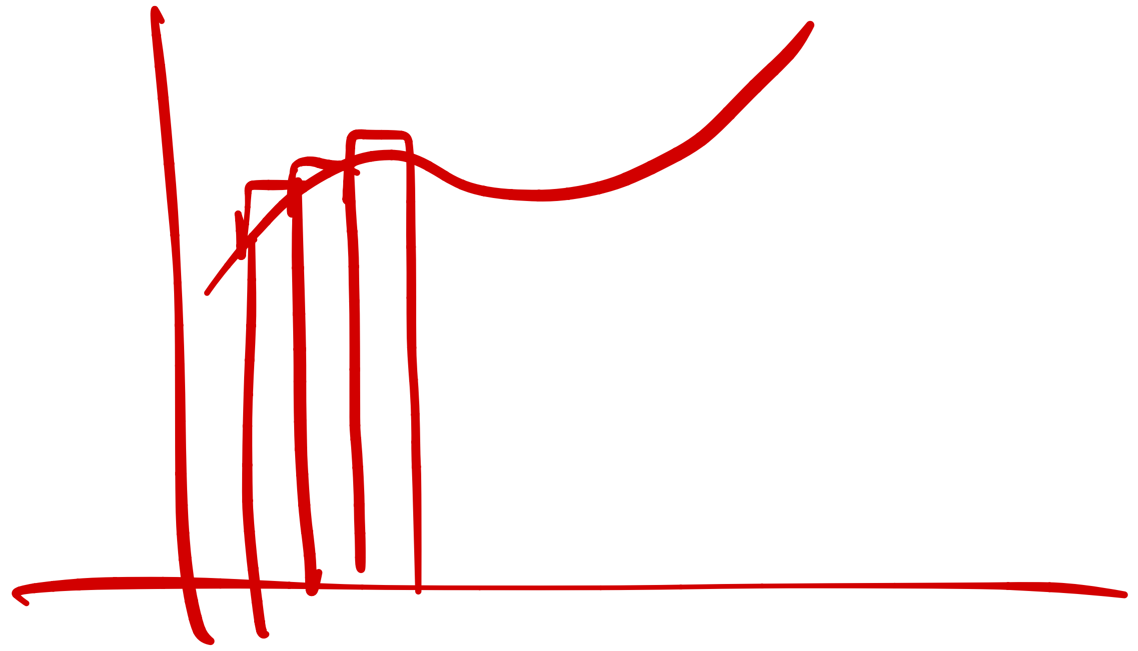

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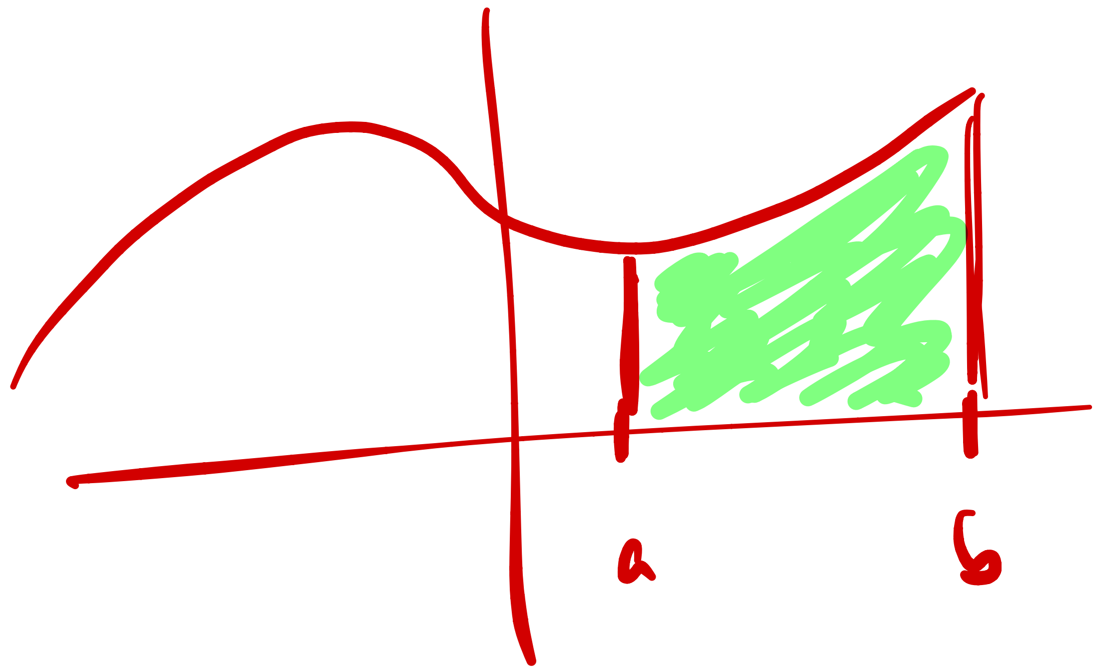
$$\Pr(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

$$= \text{Area under } f \text{ between } a \text{ and } b$$



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$$\int_{S_x} f(x)\, dx = \int_{S_x} \frac{g(x)}{NC}\, dx = \frac{1}{NC} \boxed{\int_{S_x} g(x)\, dx} = 1.$$

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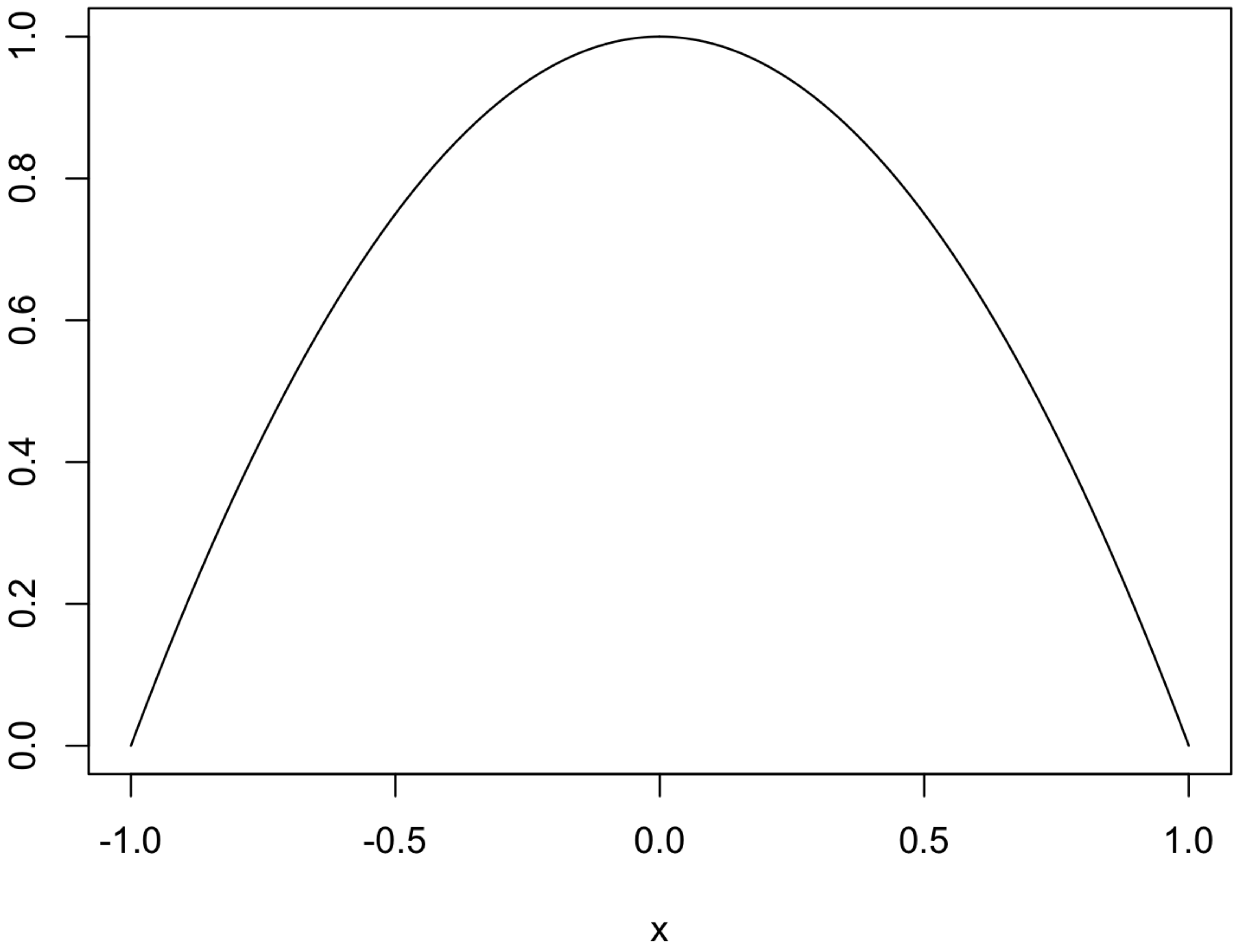
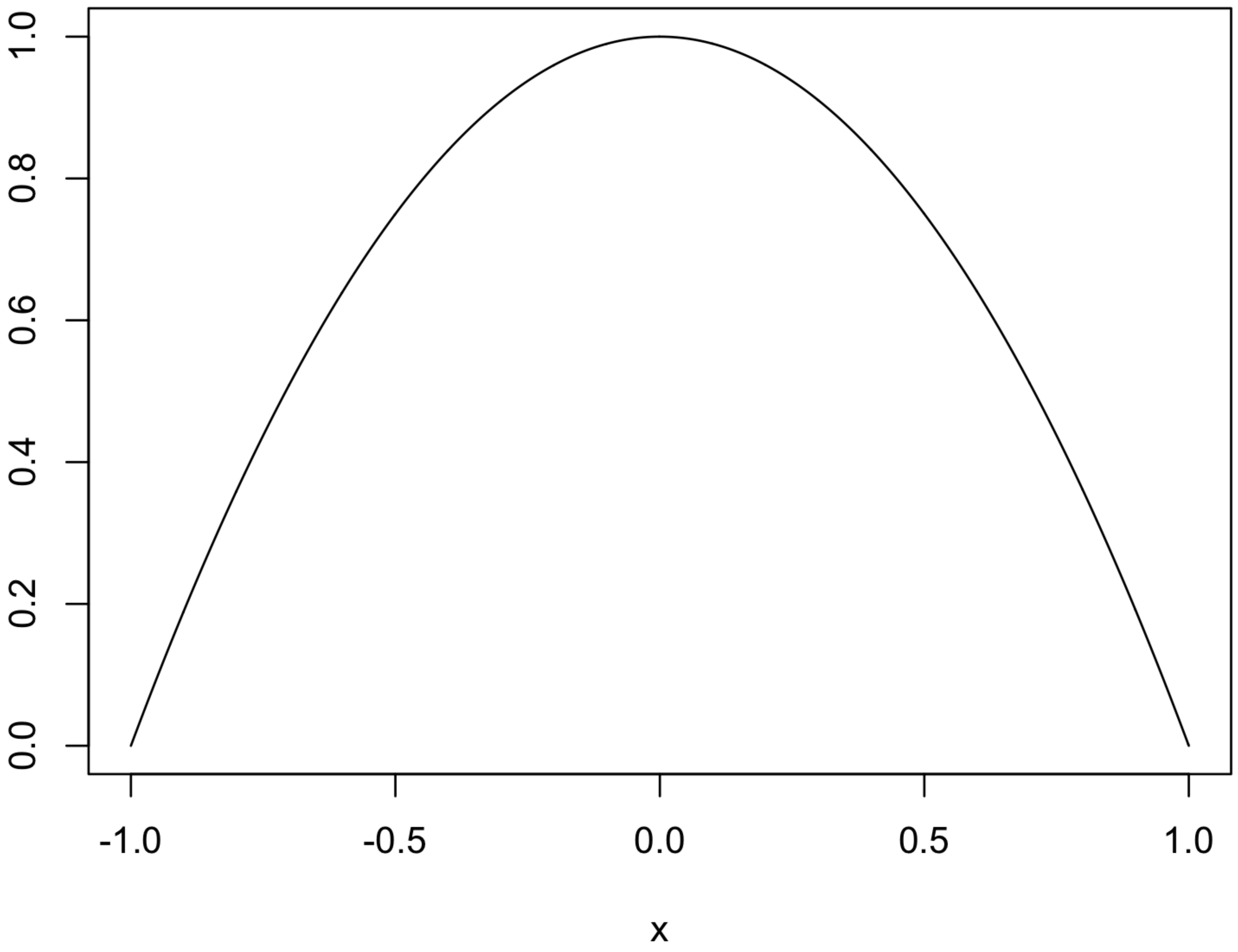
- *Normalization constant*: 1/denominator

Normalization: Example



Normalization: Example

- Suppose that we generally know that probability is distributed according to the following curve:



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- Is this a proper density?

$$\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} 1 - x^2 \, dx = \left[x - \frac{x^3}{3} \Big|_{-1}^{1} \right]$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$

$$\frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

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$$f(x) = \frac{3}{4}(1-x^2)$$

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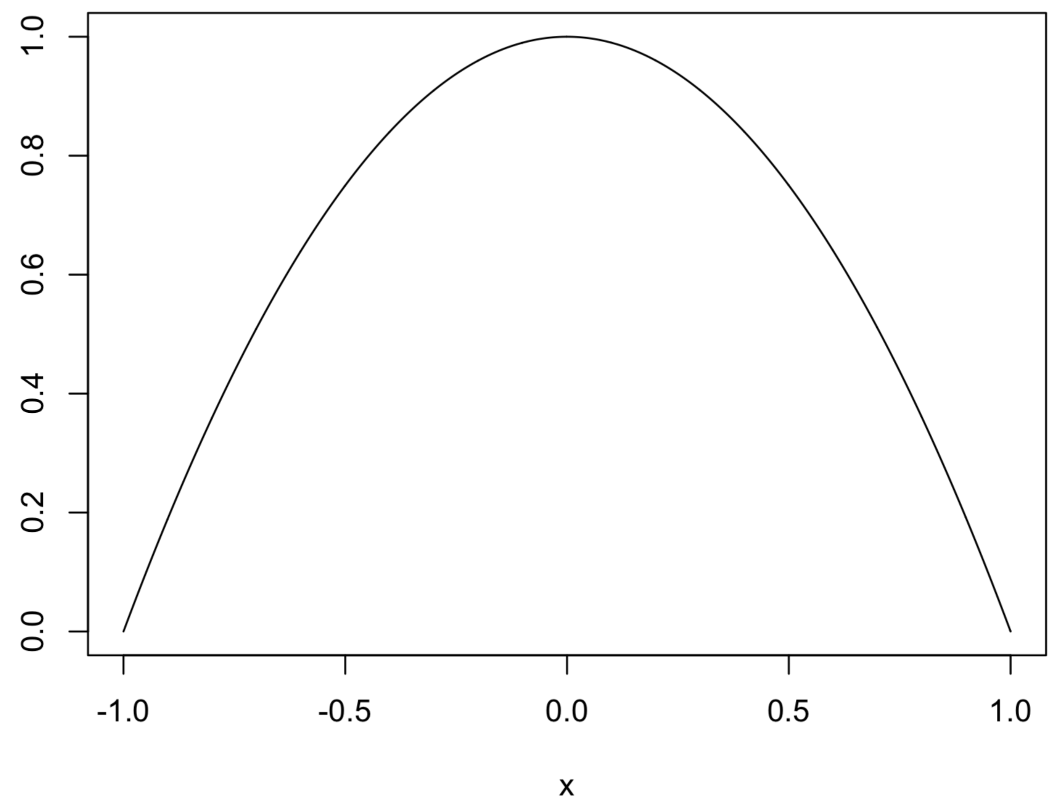
"  "

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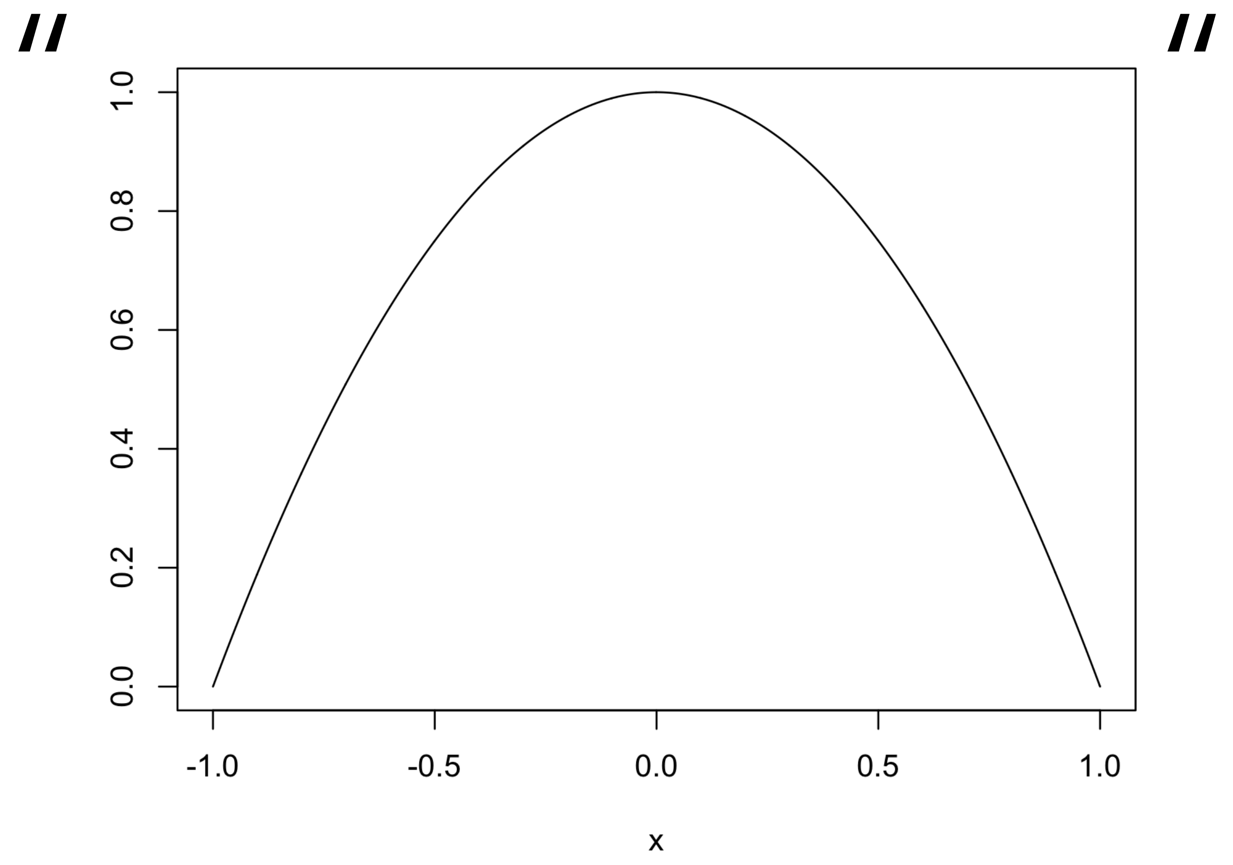

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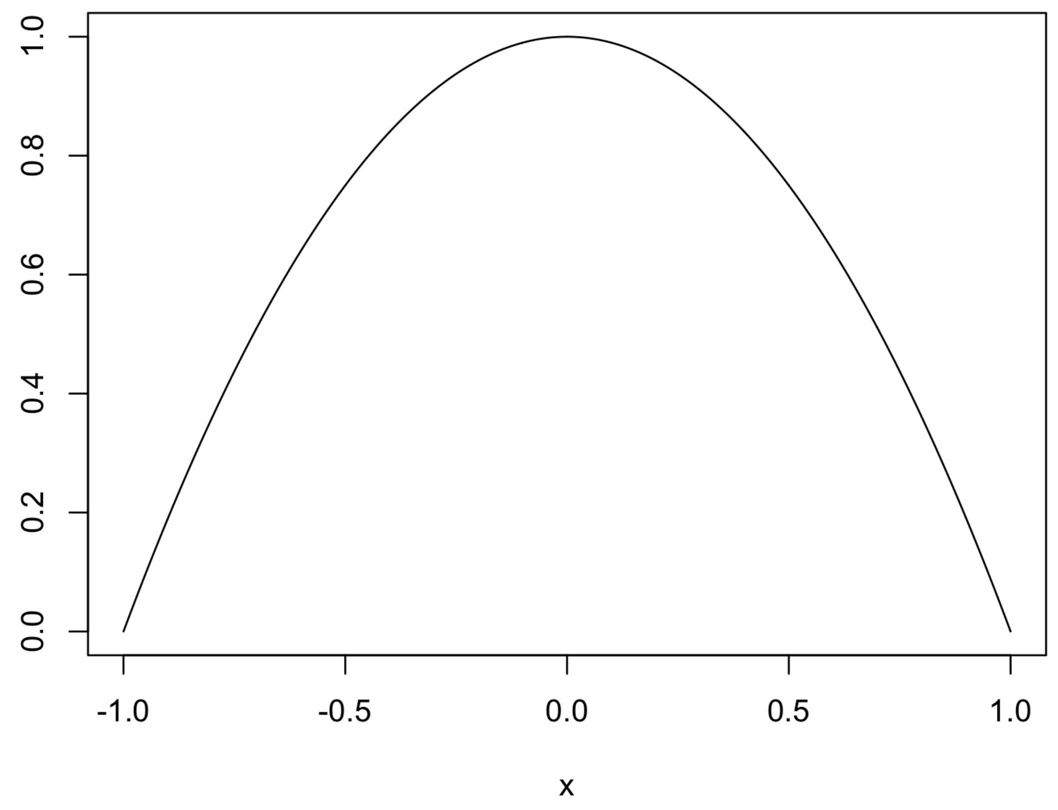

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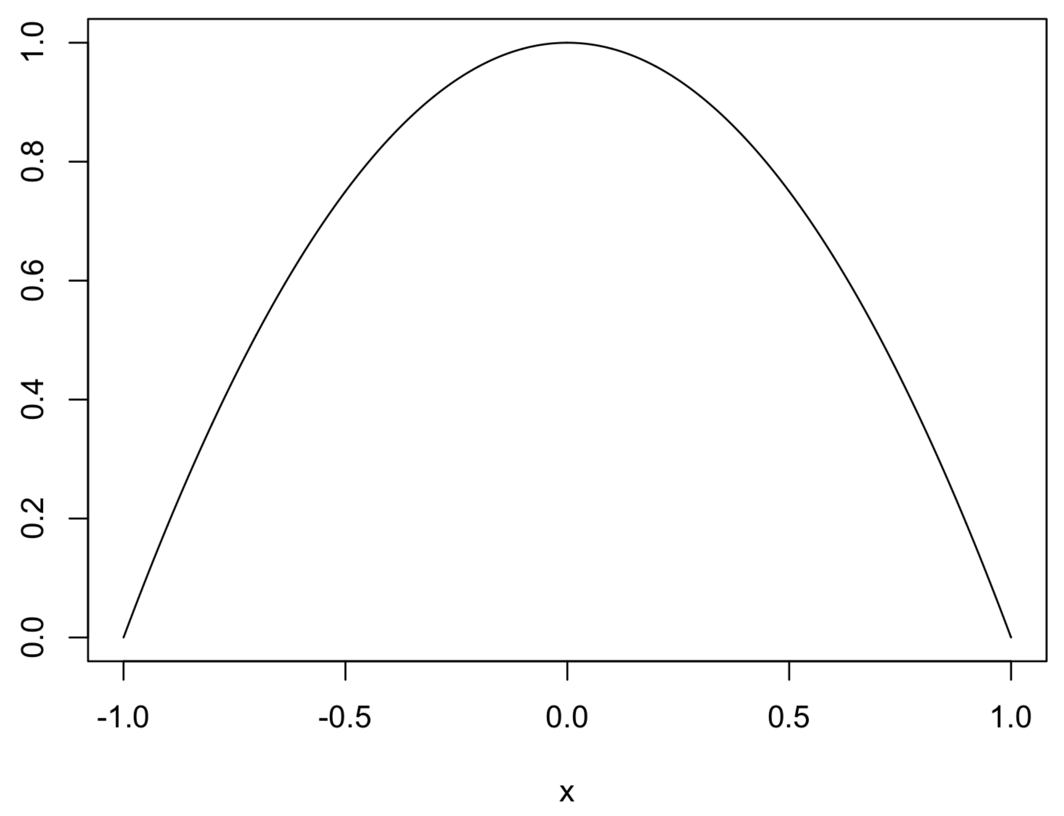

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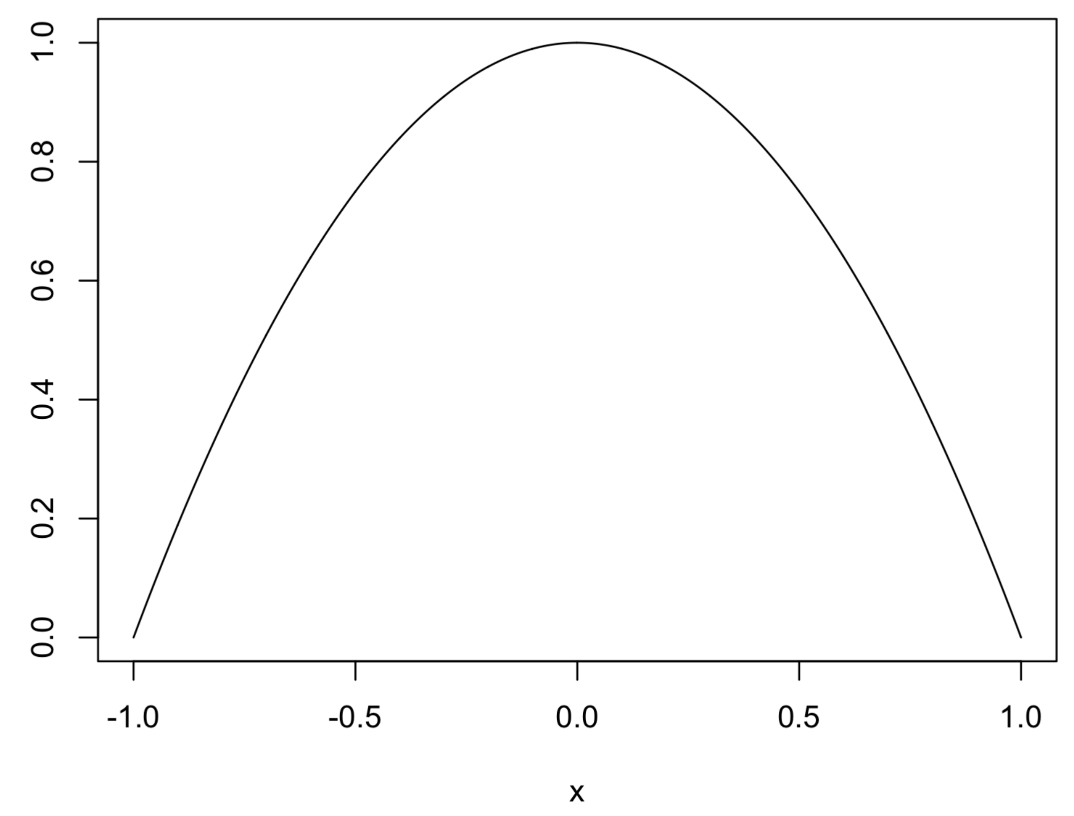

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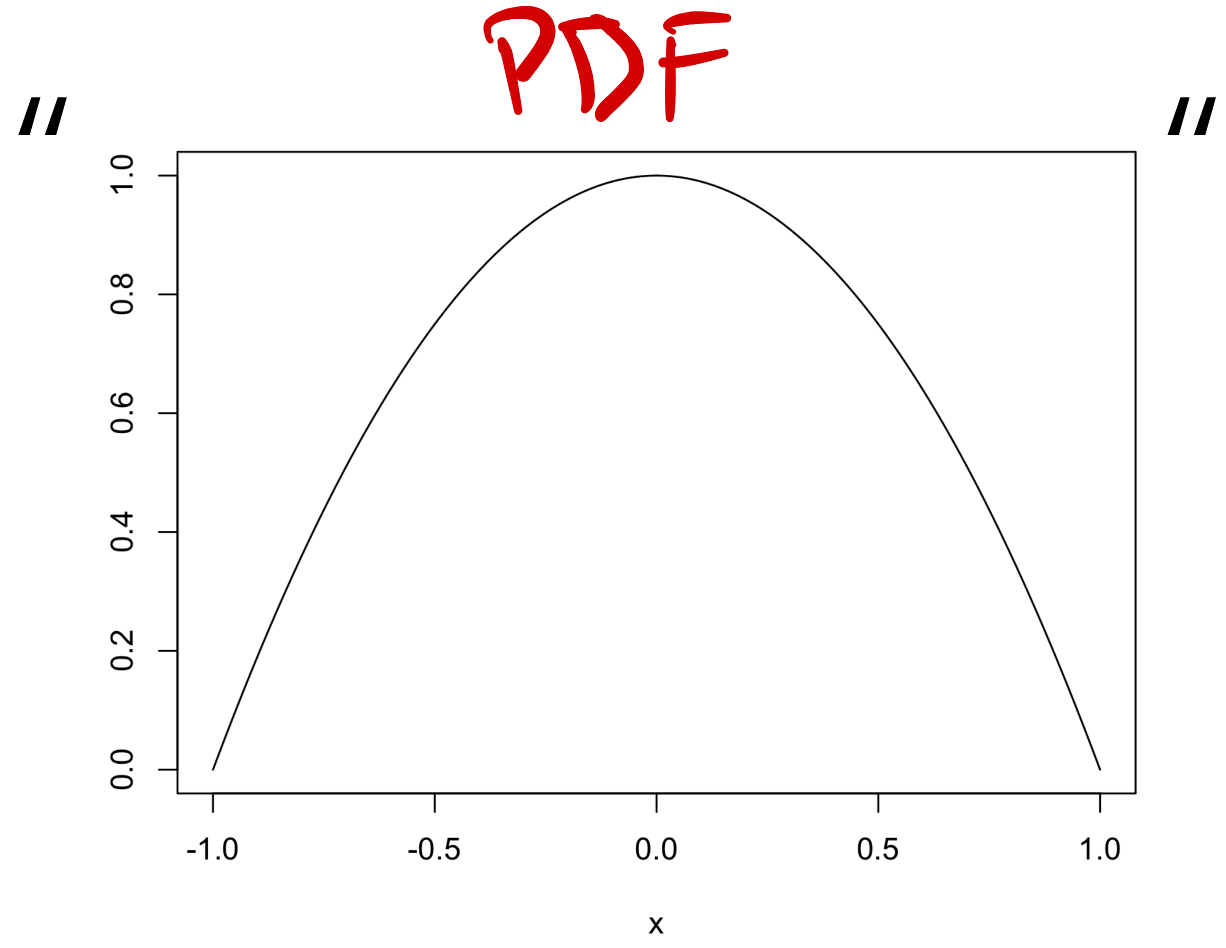

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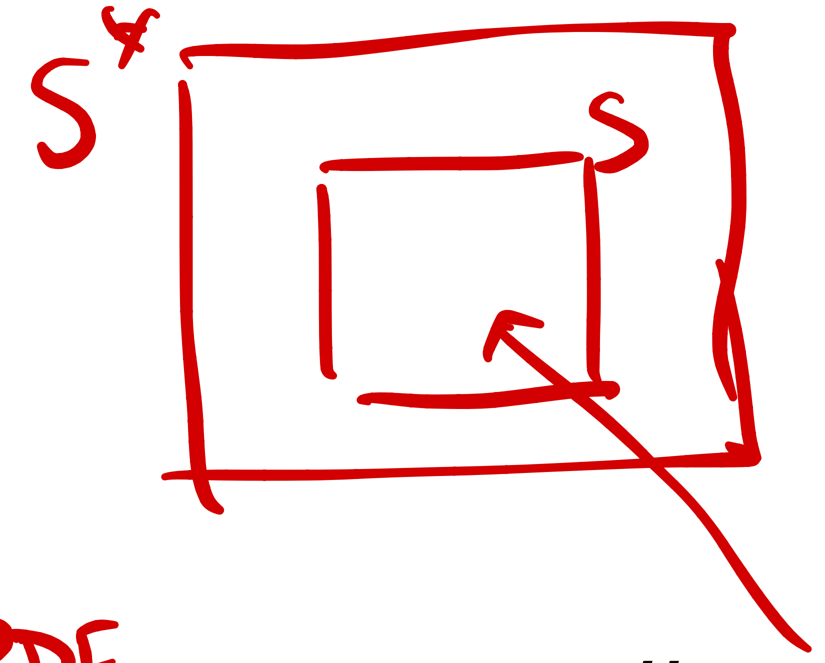

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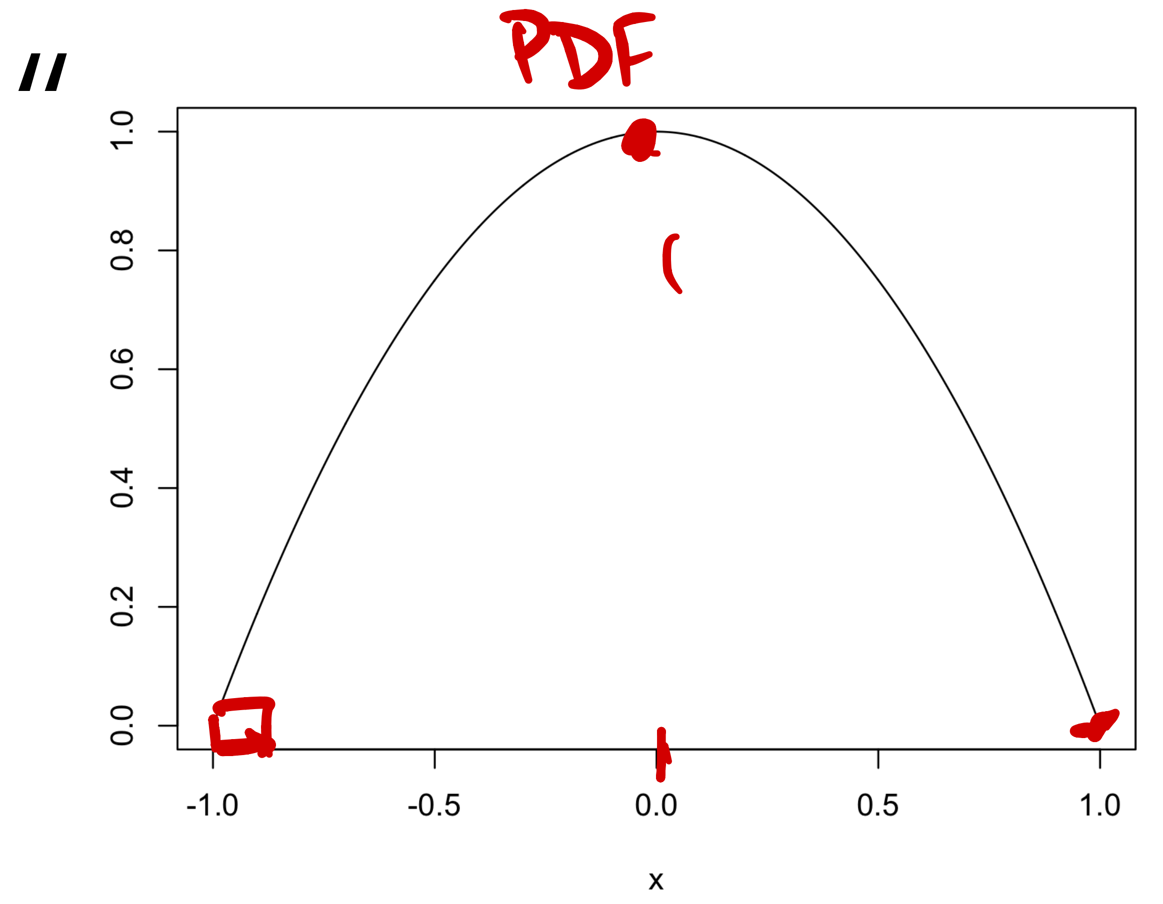

"PDF f

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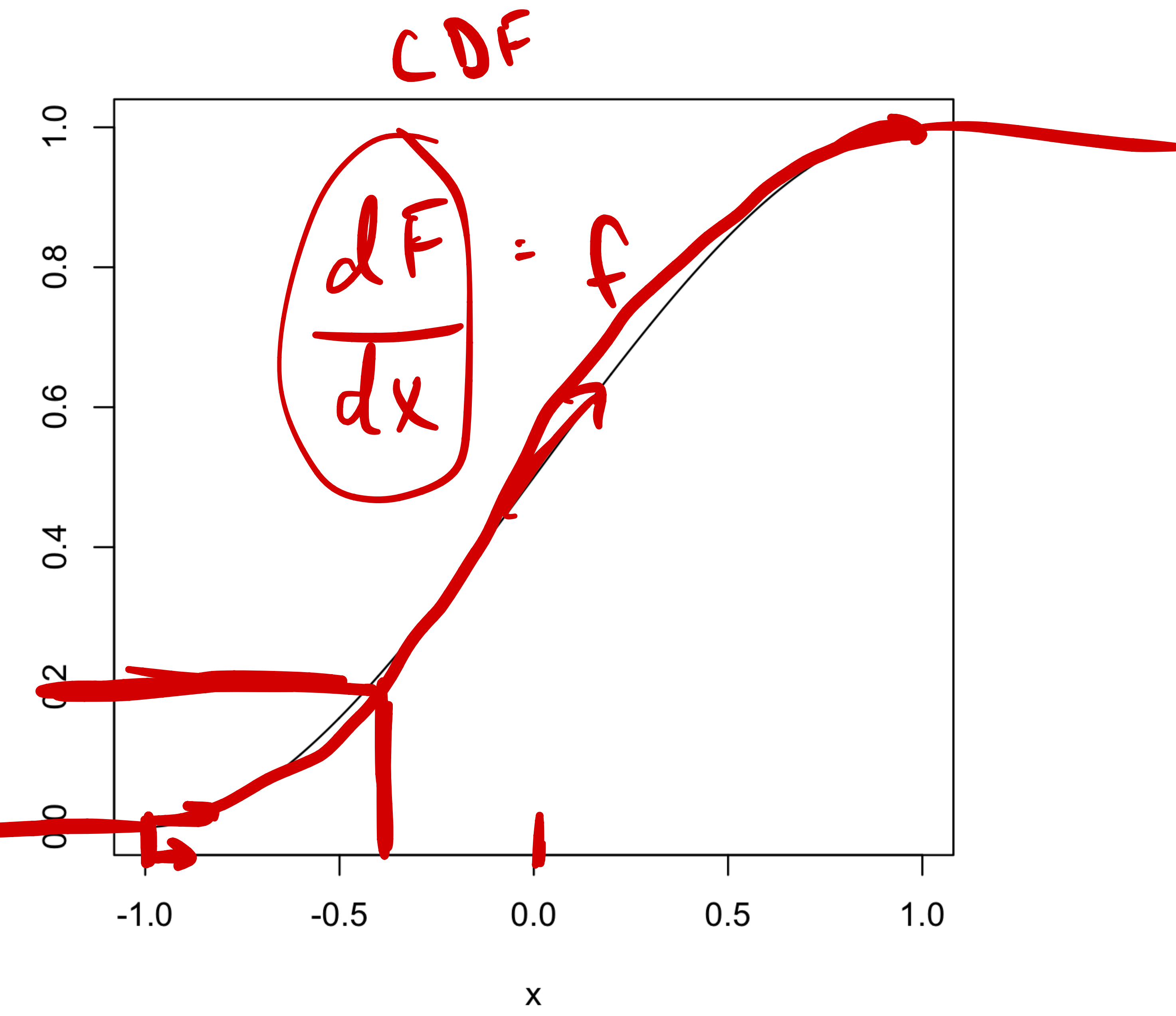

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$F(x) = Pr(X \leq x)$

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$F(85)$

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- Then, $\Pr(X \leq 85) = 0.72$
- We say that $q = 85$ is the $p = 0.72$ quantile of this distribution (also called the 72nd percentile)

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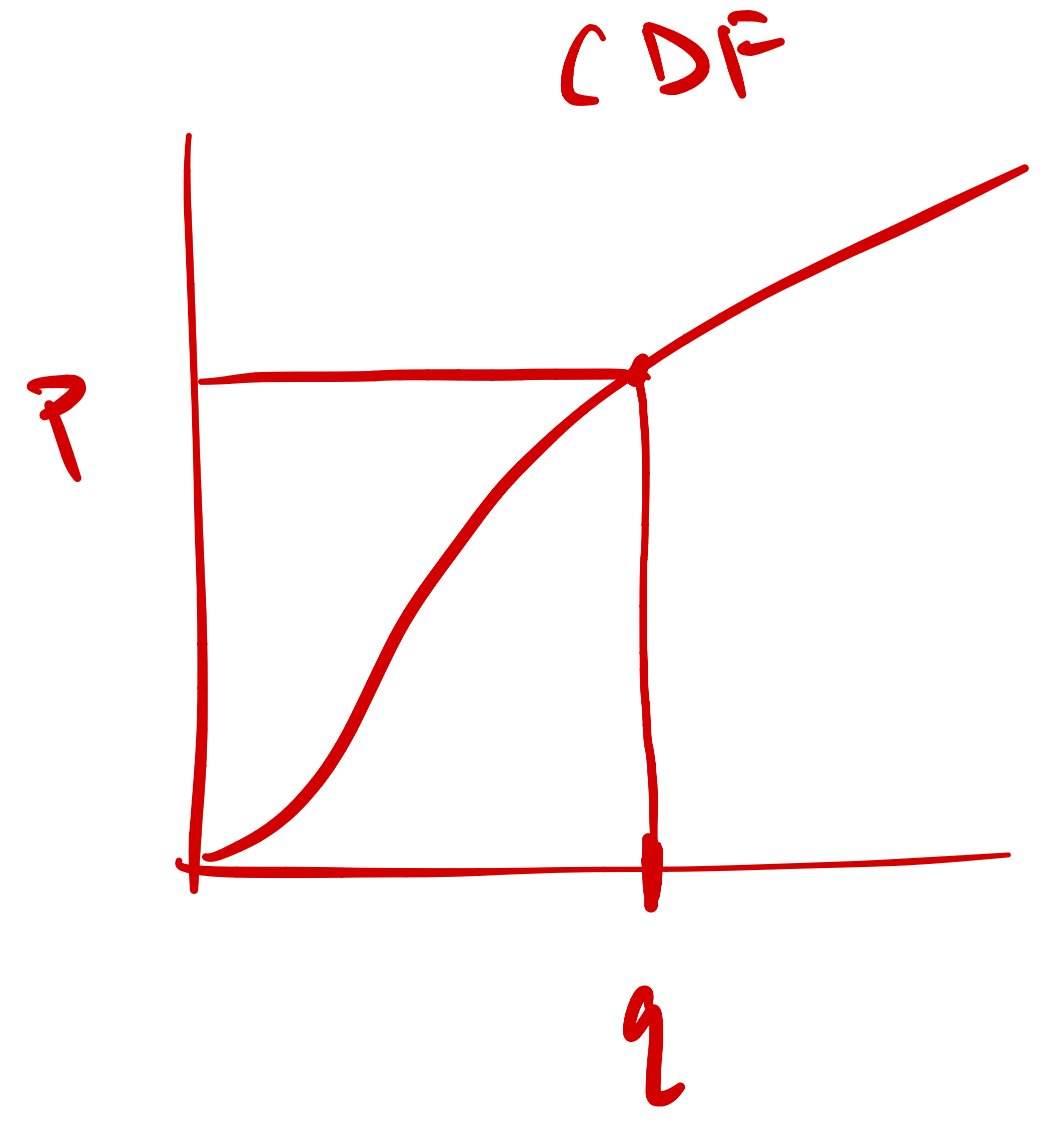

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qnorm

- Although $Q(p)$ is uniquely defined, the p-quantile may not be unique

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$S_x \in [-1, 1]$

- What is the 0.25-quantile?

$F(0.25)$ find x s.t. $F(x) = 0.25$

$$Q(p) = \{\min x \in S_x : \underbrace{Pr(X \le x)}_{F(x)} \ge p\}$$

$$F(x) = \boxed{\int_{-1}^{x} f(x)\,dx} = \frac{3}{4}\left(x - \frac{x^3}{3}\right) + \frac{1}{2}.$$

$$\frac{3}{4}\left(x - \frac{x^3}{3}\right) + \frac{1}{2} = \frac{1}{4} \rightarrow x = \{-1.53, \boxed{-0.35}, 1.879\}$$

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normalization $\int$

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$g(x) \rightarrow f(x) \rightarrow F(x)$

shape PDF CDF

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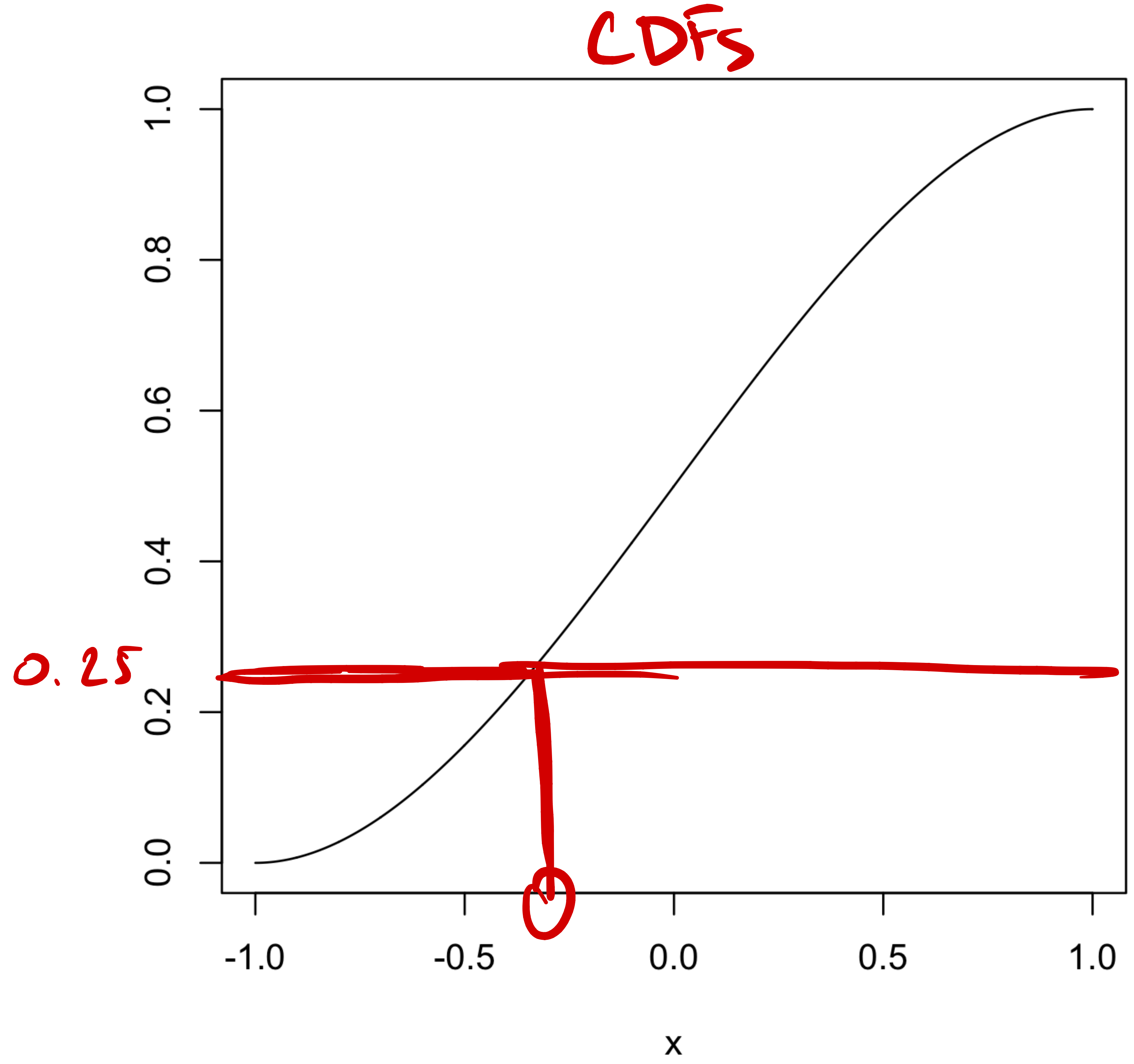

$g(x)$ over $[-1, 1]$

$\uparrow$

S_x

$g(x) = \frac{9}{} \; -x^2$ over $[-10, 10]$

$[-10, -3]$ $[3, 10]$

$f(x)$

Summarizing Probability Distributions

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Expected Value

3 coins $\frac{1}{2}$ H, $\frac{1}{2}$ T

X = #Hs

- If X is a discrete random variable:

$$\mu_X = E(X) = \sum_{x \in S_X} x \cdot \Pr(X = x)$$

$P(X=0) = 1/8$

$P(X=1) = 3/8$

$$E[X] = \sum_{x \in \{0,1,2,3\}} x \cdot \Pr(X=x)$$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \cdot \frac{1}{8}$$

$$\frac{3+6+3}{8} = \boxed{1.5}$$

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$$\sum_x \sum_y (x+y) \Pr(X=x, Y=y)$$

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- Q1: What is the expected number of heads when flipping a fair coin (1/2 H, 1/2 T)?

$$0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$$

- Q2: What is the expected number of heads when flipping an unfair coin (2/3 H, 1/3 T)?

$$0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \boxed{\frac{2}{3}}$$

- Q3: What is the expected number of heads when flipping three fair coins (1/2 H, 1/2 T) and two unfair coins (2/3 H, 1/3 T)?

$$3 \times \frac{1}{2} + 2 \times \boxed{\frac{2}{3}} = \frac{17}{6}$$

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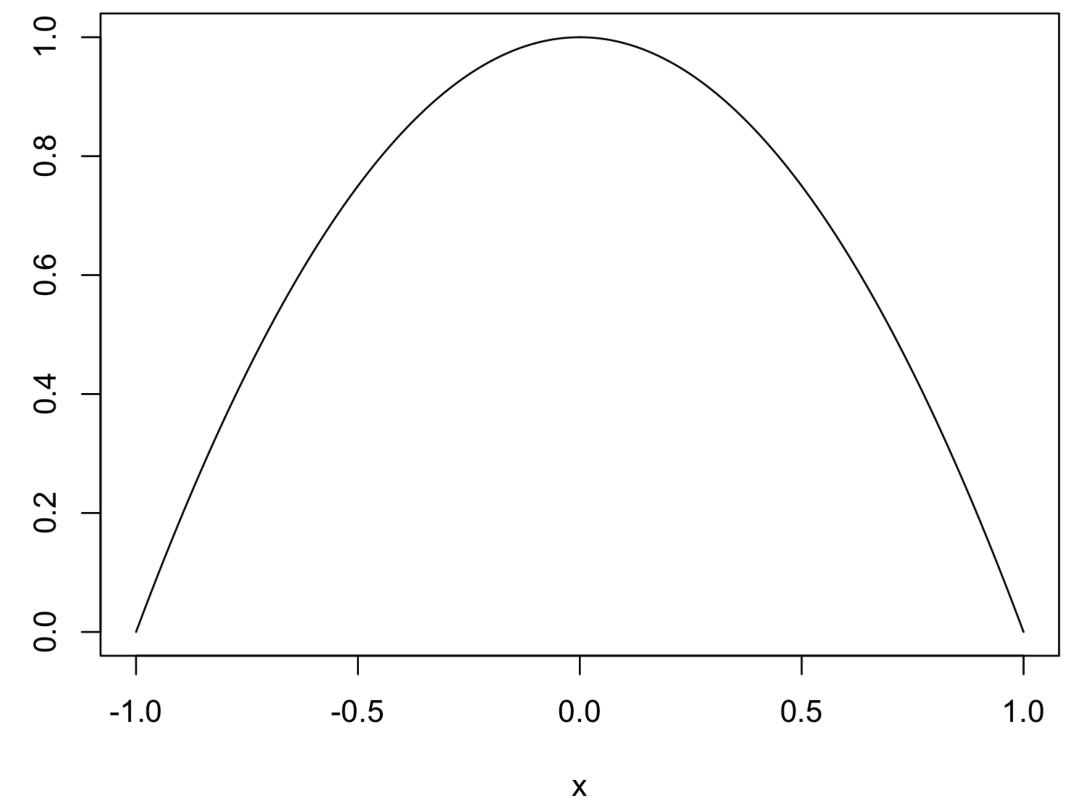

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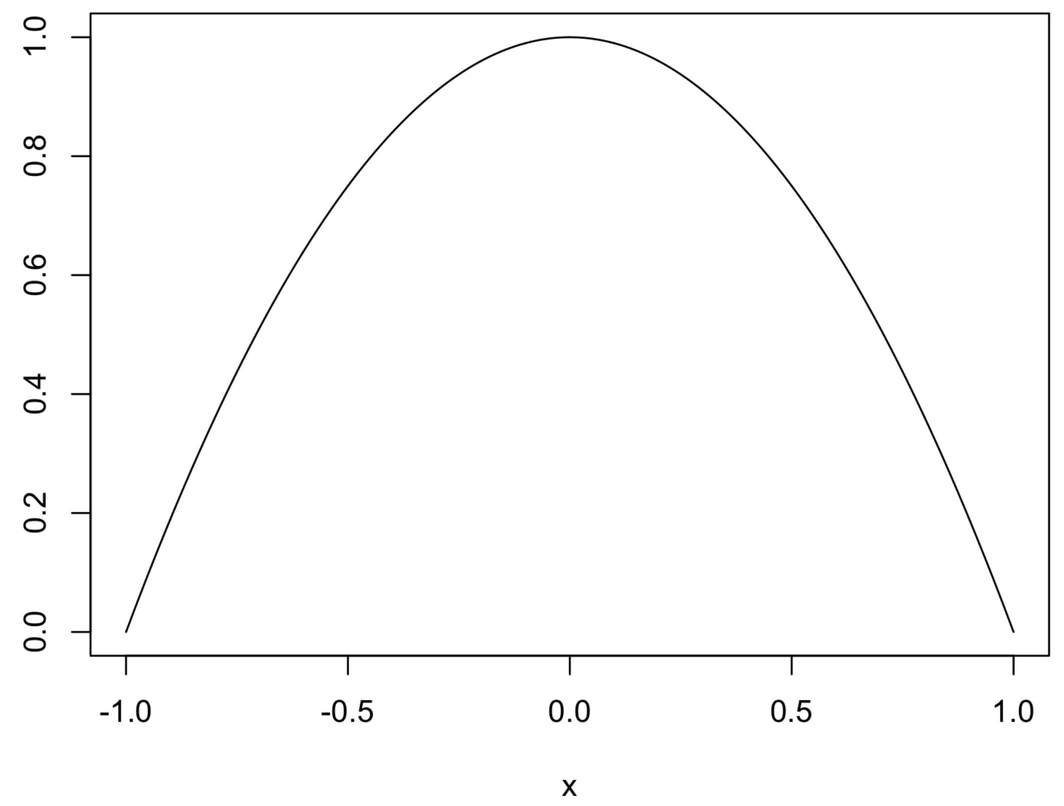

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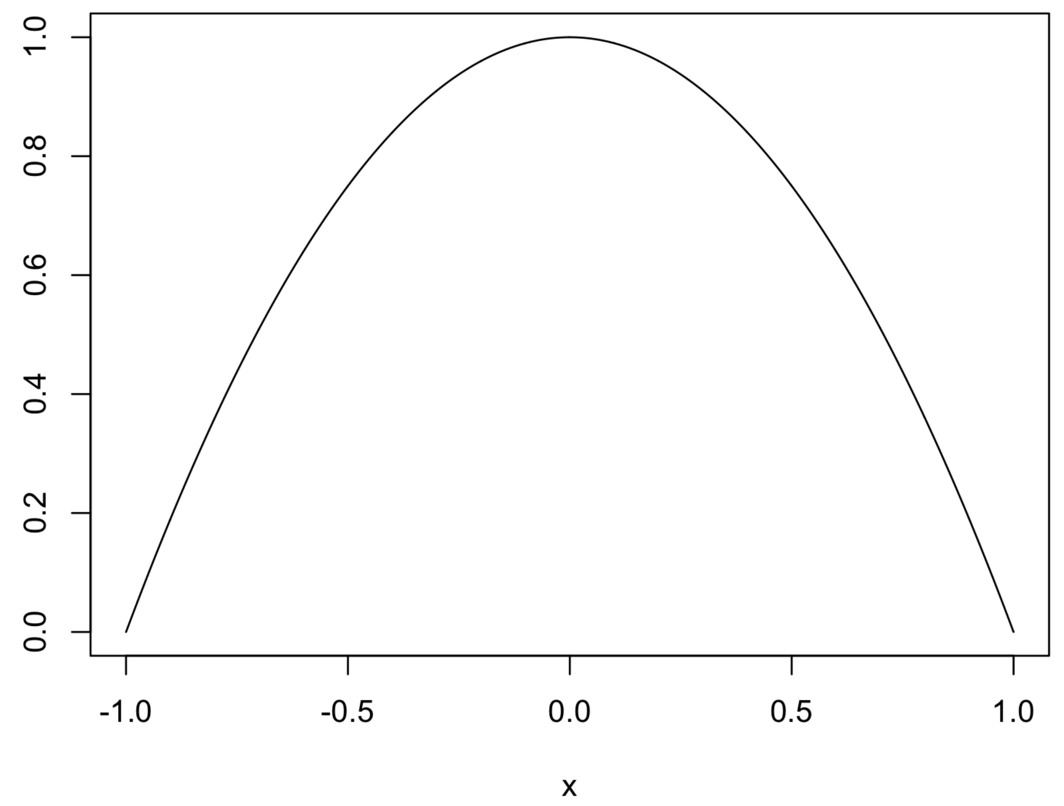

- What is $E(X)$?

- Intuition: By symmetry, $E(X) = 0$

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$$\boxed{\begin{aligned} var(X) &= E\left(\left(X - E(X)\right)^2\right) \\ &= E(X^2) - E(X)^2 \end{aligned}}$$

$$= E\left(X^2 - 2XE(X) + E(X)^2\right)$$

$$E(X^2) - E(2XE(X)) + E(E(X)^2)$$

$$E(X^2) - \underline{2E(X)E(X)} + \underline{E(X)^2}$$

$$= E(X^2) - E(X)^2$$

$$E(X^2) \geq E(X)^2$$

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- We use the notation $\sigma^2 = \sigma_X^2 = var(X)$
- The standard deviation is the square root of the variance: $\sigma = \sigma_X = \sqrt{var(X)}$

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$$0 : 1/4$$
$$1 : 1/2$$
$$2 : 1/4$$
$$E(X) = 1$$

$$= (0-1)^2 \cdot \frac{1}{4} + (1-1)^2 \cdot \frac{1}{2} + (2-1)^2 \cdot \frac{1}{4} = \boxed{\frac{1}{2}}$$

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- Q: What is the standard deviation of X?

$\sqrt{1/2} = \sqrt{2}/2$

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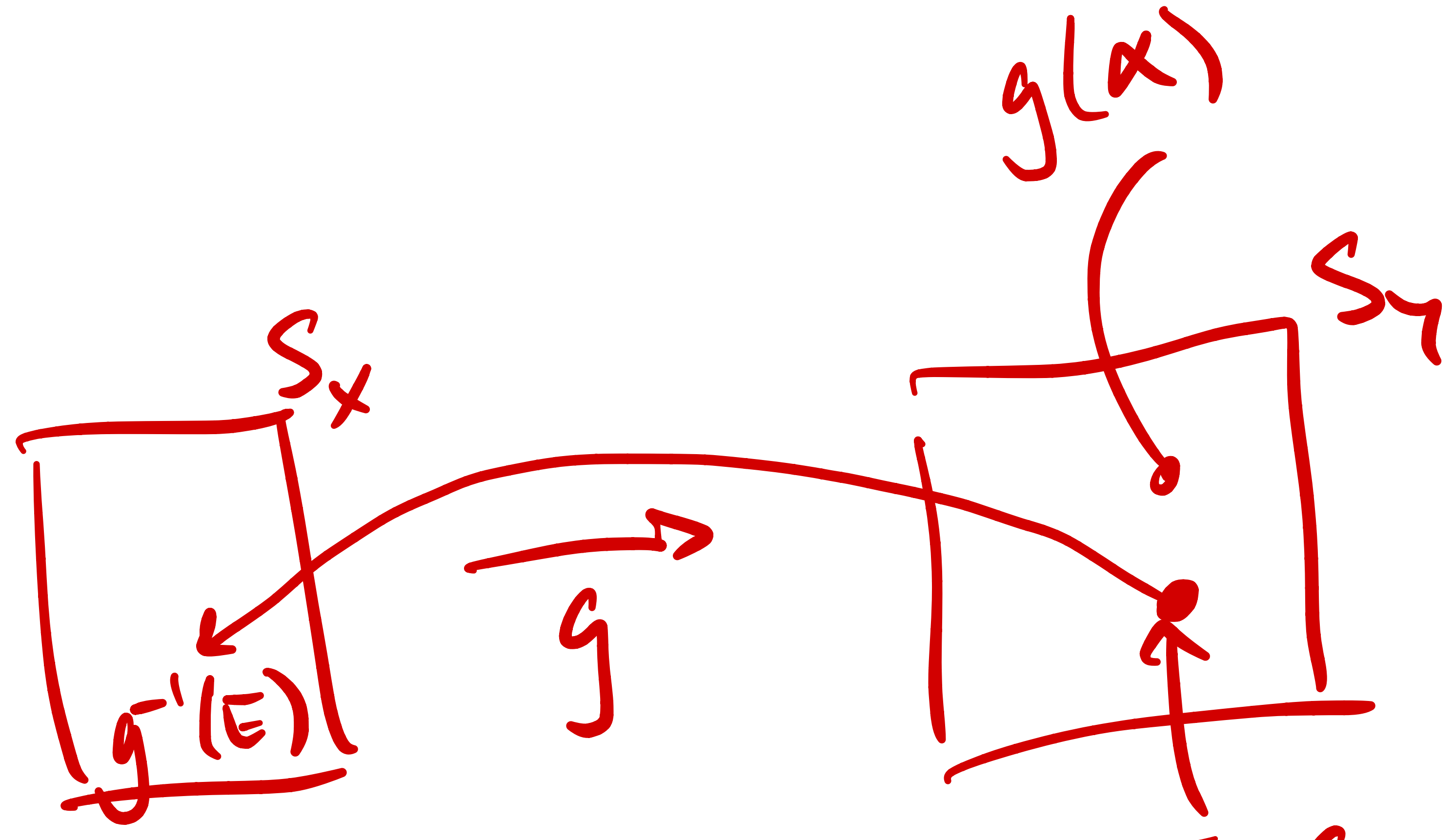

- Take random variable X and function $g(\,.\,)$
- We can obtain a new random variable: $Y = g(X)$
- This is what is called a *transformation of variables*
- In general, to get the distribution of Y, we have that for any event $E \subseteq S_Y$, we have $p_Y(E) = p_X(g^{-1}(E))$

Linear Transformations: Mean and Variance

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- Finding the mean of Y:

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$$\begin{aligned}
\sigma_Y^2 &= var(Y) = E((aX + b - E(aX + b))^2) \\
&= E((aX + \cancel{b} - aE(X) - \cancel{b})^2) = E((aX - aE(X))^2) \\
&= E(a^2(X - E(X))^2) = a^2E((X - E(X))^2) \\
&= a^2 \cdot var(X) = \boxed{a^2 \cdot \sigma_X^2}
\end{aligned}$$

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$\Sigma\, x \cdot Pr(X = x)$

General Transformations: Mean

- If we have $Y = g(X)$ for general $g(X)$, then we have:

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$Pr(X=1) = Pr(Y=1^2)$

$Pr(X=2) = Pr(Y=2^2)$

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let cost = g(x)

↑Pr = Pr

$$g(E(X)) = \left(\frac{1+2+3+4+5+6}{6}\right)^2 = (3.5)^2 = 12.25$$

\$ – \$

Independence

$$P(A, B) = P(A) \times P(B)$$

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  - Example: Roll two dice, or flip three fair coins

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- Let X and Y be two random variables with means μ_X and μ_Y, respectively

die : $\{1, \ldots, 6\}$

X

$E = \{$ either a 1 or 6 $\}$

X takes values in S_X

$E \subseteq S_X$

$S_X = \{\underline{1}, \ldots, \underline{6}\}$

1/6

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$$cov(X,Y) = \sigma_{XY} = E(\underbrace{(X-\mu_X)}_{}\underbrace{(Y-\mu_Y)}_{}) = \underbrace{E(XY) - E(X)E(Y)}_{}$$

$$Var(X) = E((X-\mu_X)^2) = E(X^2) - E(X)^2$$

$$(X-\mu_X)(X-\mu_X)$$

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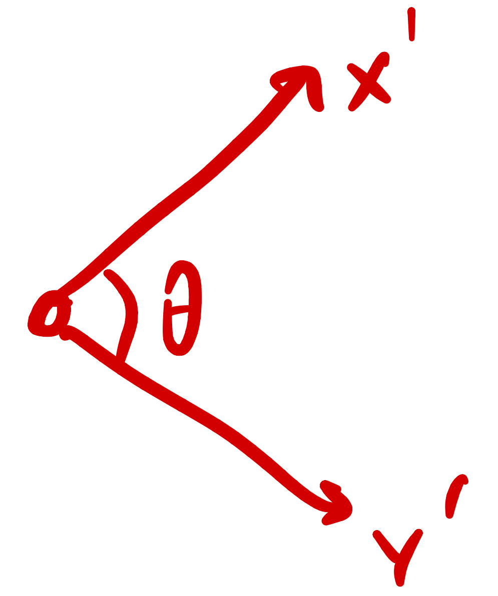
- Correlation (essentially normalized covariance):

Covariance

$$x' = (x_1 - \bar{x},\ x_2 - \bar{x},\ \ldots,\ x_n - \bar{x})$$

$$y' = (y_1 - \bar{y},\ \ldots,\ y_n - \bar{y})$$



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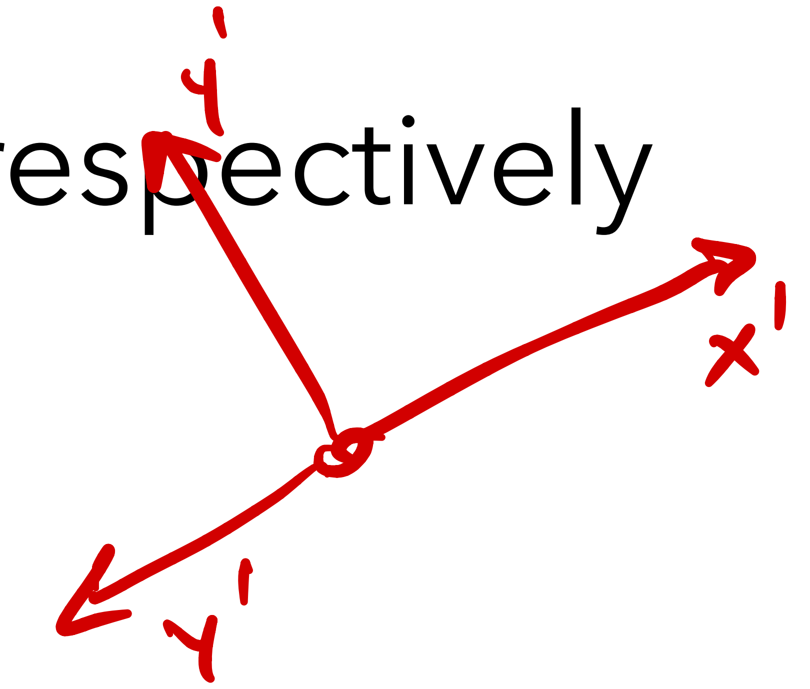

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$$corr(X, Y) = \rho = \rho_{XY} = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

SAMPLE

$$\frac{\overbrace{\sum_i (x_i - \bar{x})(y_i - \bar{y})}^{x' \cdot y'}}{\sqrt{\underbrace{\left(\sum_i (x_i - \bar{x})^2\right)}_{\|x'\|} \underbrace{\left(\sum_i (y_i - \bar{y})^2\right)}_{\|y'\|}}} \approx \cos\theta$$

Properties of Covariance

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  - $cov(X, Y) = cov(Y, X)$

Linear Combinations

$$Y = aX + b$$

$$Z = aX + bY$$

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$a\mu_X + b\mu_Y$

$E[Z] = E[aX + bY] = E[aX] + E[bY] = aE[X] + bE[Y]$

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- The mean of Z is

$Z = aX$

μ

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- The variance of Z is $\mathrm{var}(Z) = \mathrm{cov}(Z, Z) = \mathrm{cov}(aX + bY, aX + bY)$

$$\sigma_Z^2 = \underbrace{a^2\sigma_X^2} + \underbrace{b^2\sigma_Y^2} + 2ab\,\underbrace{\mathrm{cov}(X, Y)}$$

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- Notation: $Y \sim Bern(p)$

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$\sum x \cdot P(Y=x)$

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$E(Y) = \underline{1} \cdot \underline{p} + \underline{0}(\underline{1-p}) = p$

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- What is $var(Y)$ in terms of p? pq

$$var(Y) = E((Y - E(Y))^2) = \underbrace{E(Y^2)}_{E(Y) = p} - \underbrace{E(Y)^2}_{p}$$

$$Y^2 : \begin{cases} 0^2 & (1-p) \\ 1^2 & p \end{cases}$$

$$= p - p^2 = p(1-p)$$

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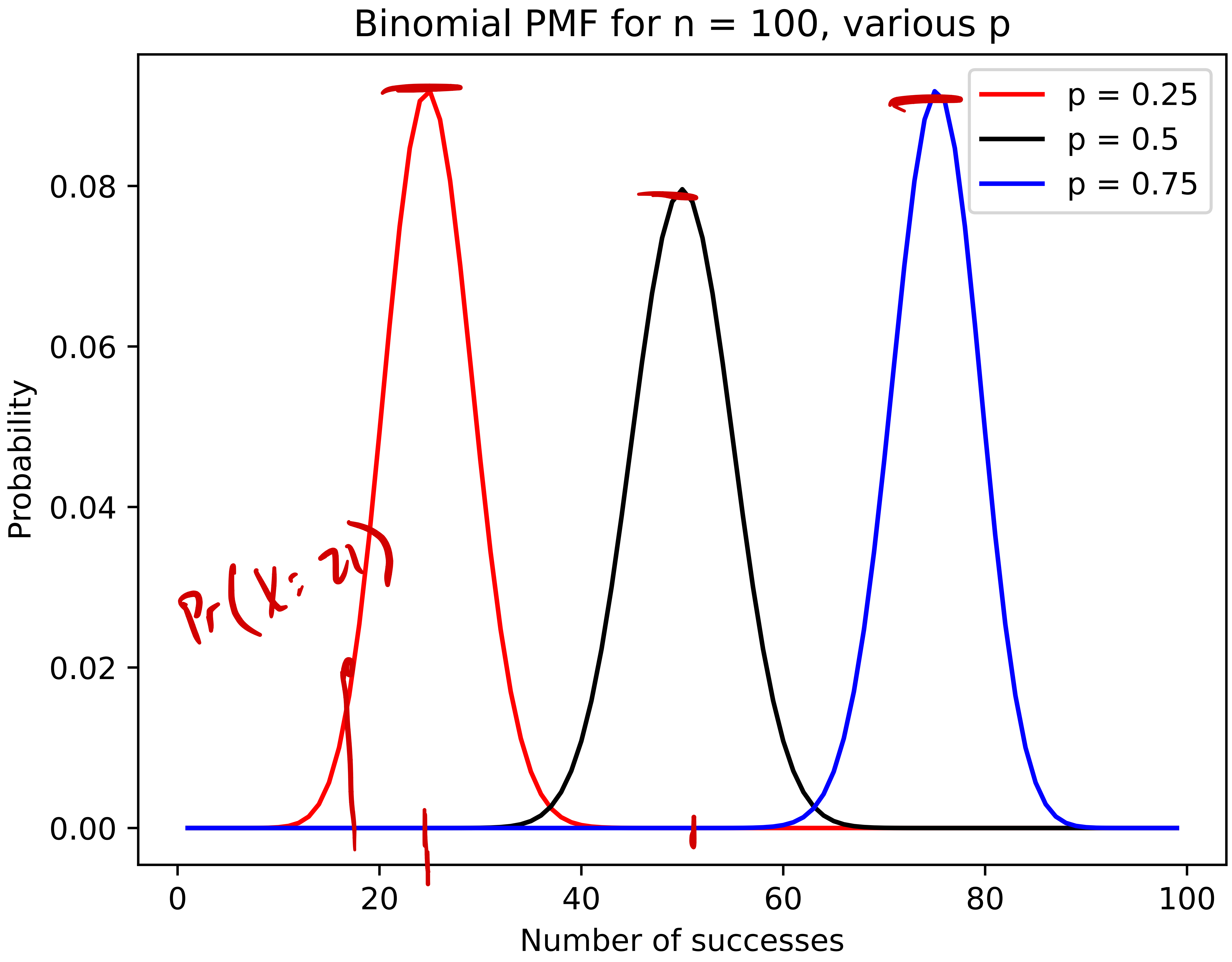
Binomial Distribution



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    - Probability that the other $n - x$ trials fail: $(1-p)^{n-x}$

Binomial Coefficients

$n = 3$

$p = 1/3$

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$\Pr(X = 2)$

- Q: What is the probability of getting exactly x out of n successes?
  - Choose which x trials succeed: $\binom{3}{2}$
  - Probability that these x trials succeed:
  - Probability that the other $n - x$ trials fail:

$$\begin{aligned} HHT &= 1/3 \cdot 1/3 \cdot 2/3 \\ HTH &= 1/3 \cdot 2/3 \cdot 1/3 \\ THH &= \underbrace{2/3 \cdot 1/3 \cdot 1/3} \end{aligned}$$

- In general, $\binom{n}{x} \boxed{p^x (1-p)^{n-x}}$

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dbinom ← PDF
p ← CDF
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$$= \sum E(X_i)$$

$$= \sum_{i=1}^{n} p$$

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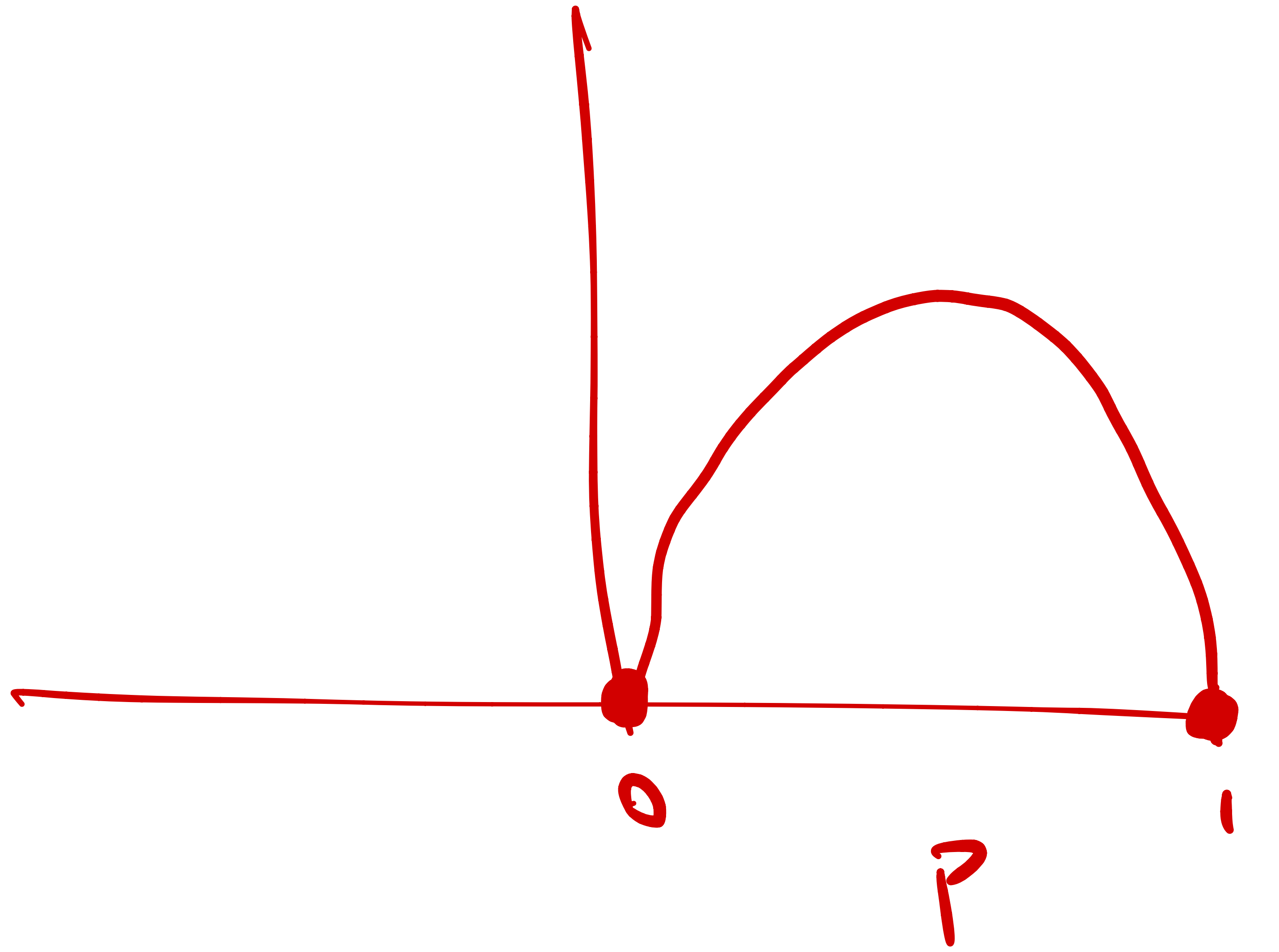

- Q: How does $var(X)$ change with $p \in [0,1]$?

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$$\frac{\lambda^{\#}}{\#!}$$

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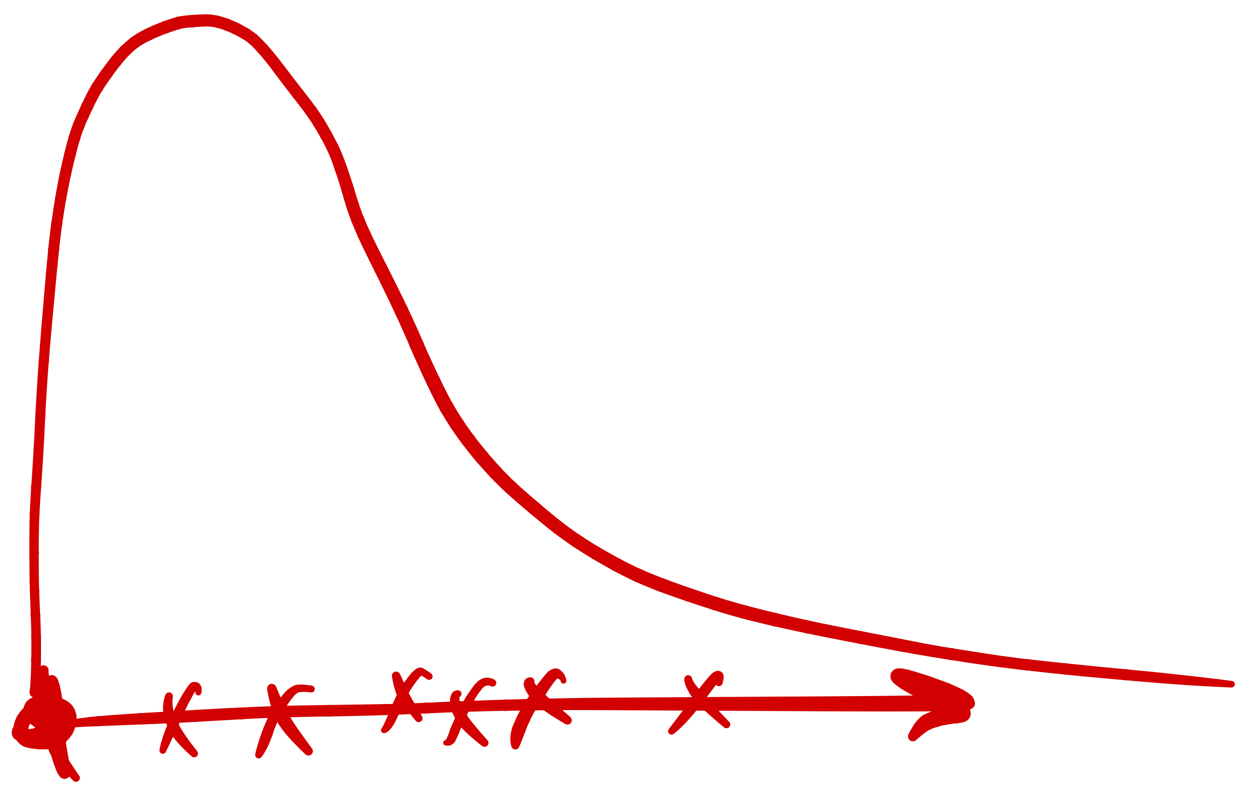

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  - For a Poisson distribution, both the mean and the variance are equal to λ

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$$P(X=0) = \frac{e^{-1.95}\, 1.95^{0}}{0!} = e^{-1.95} \approx 0.142$$

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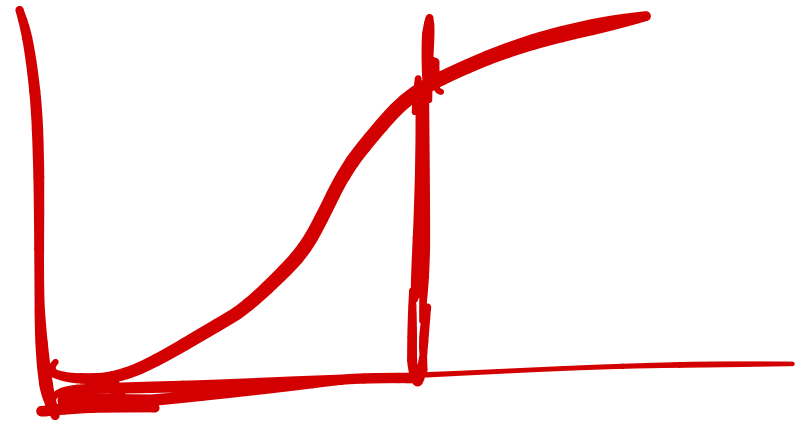
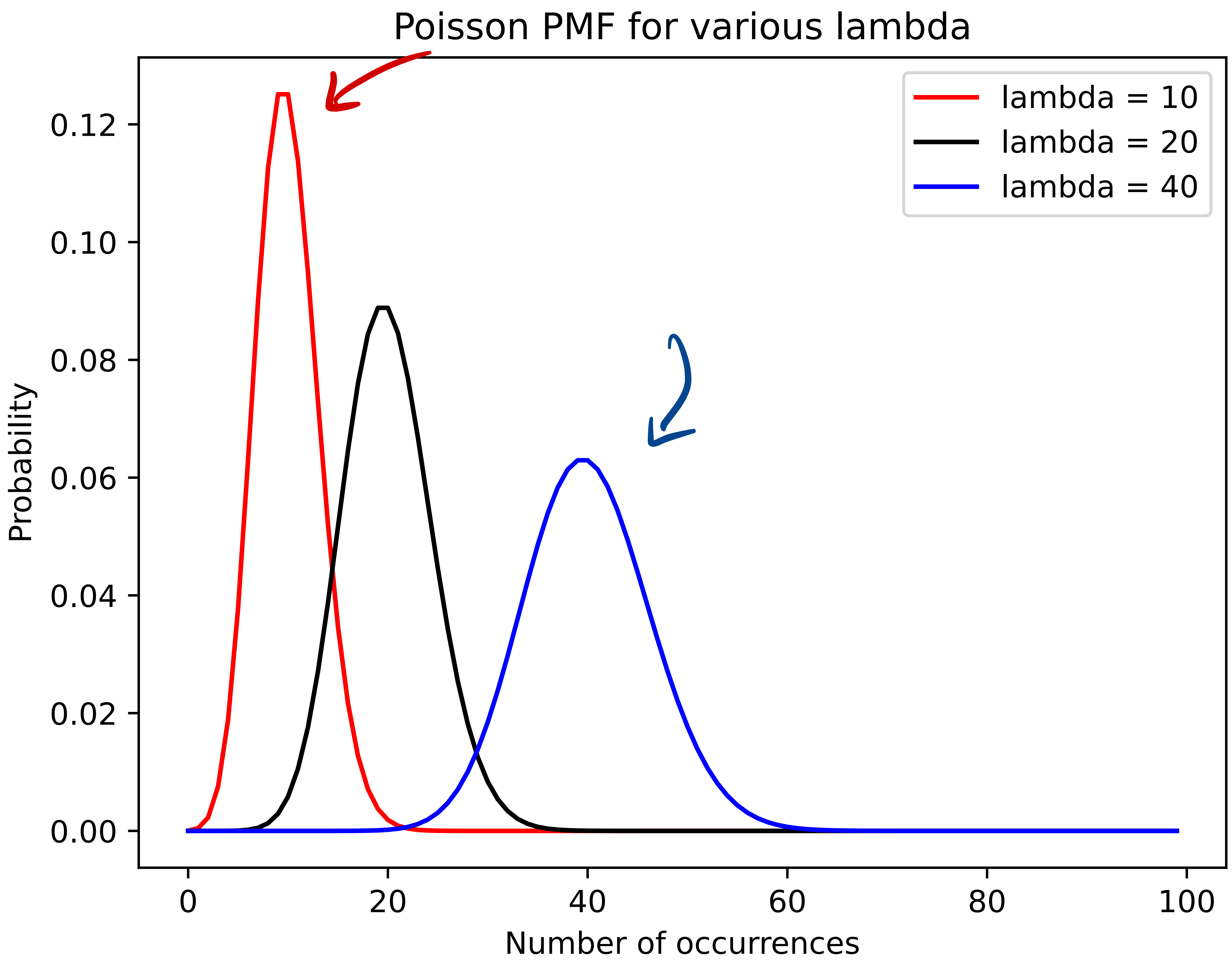


- Q: What is the probability of one person developing the disease in the next year?

$$Pr(X=1) = \frac{e^{-1.95}\,1.95^{1}}{1!} = 1.95 \times \quad \approx 0.277$$

Poisson Distribution: Visualized



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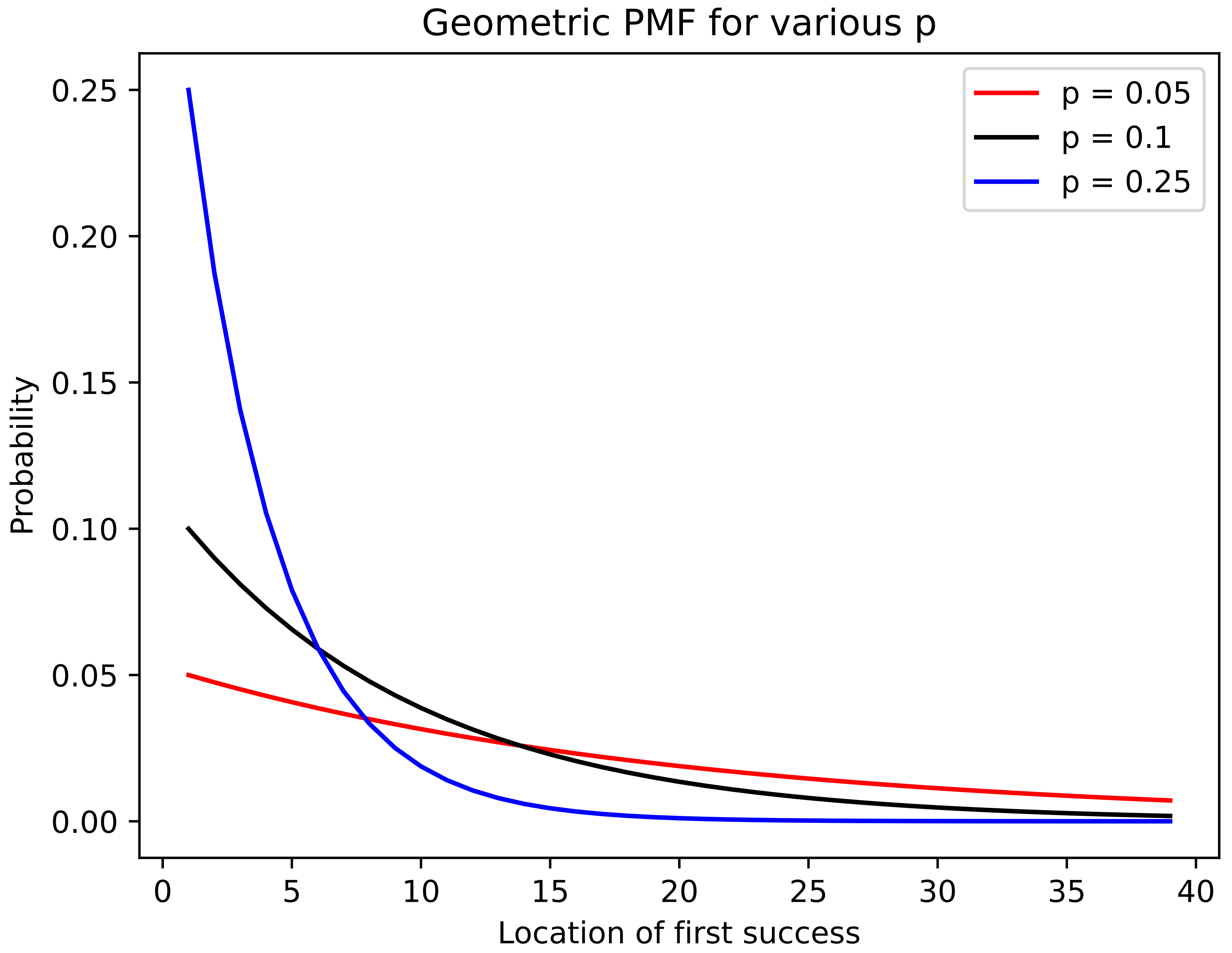

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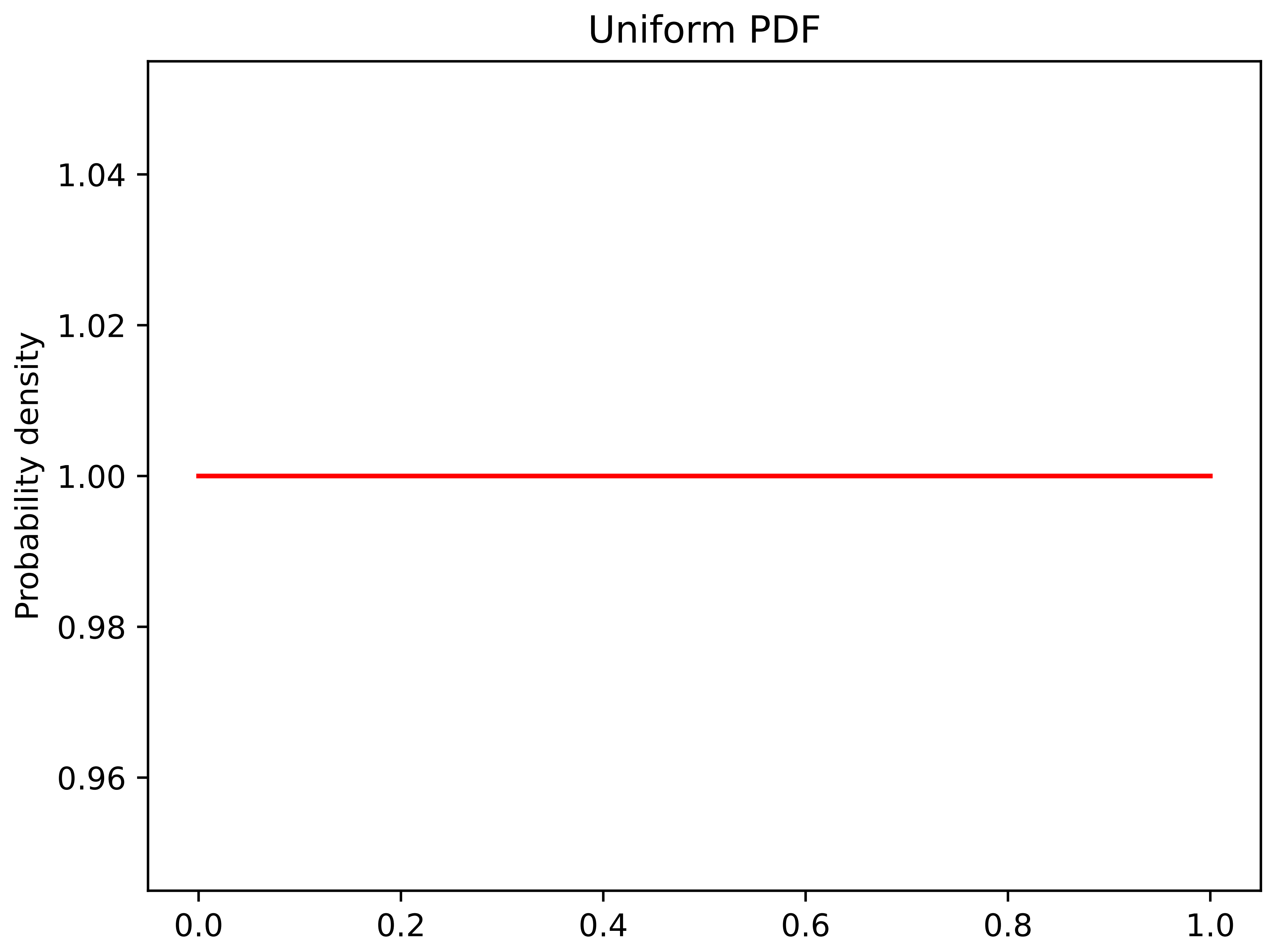

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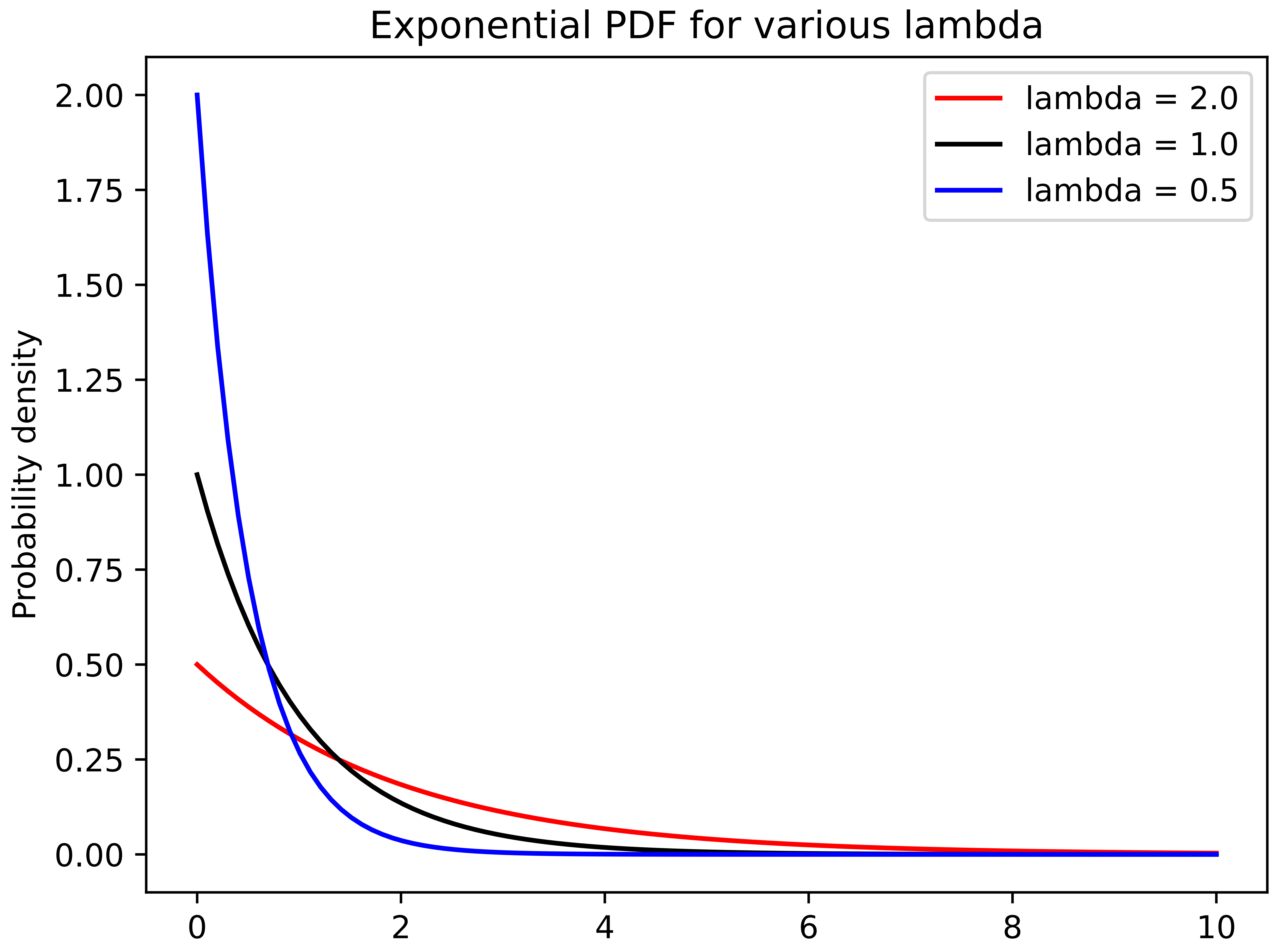

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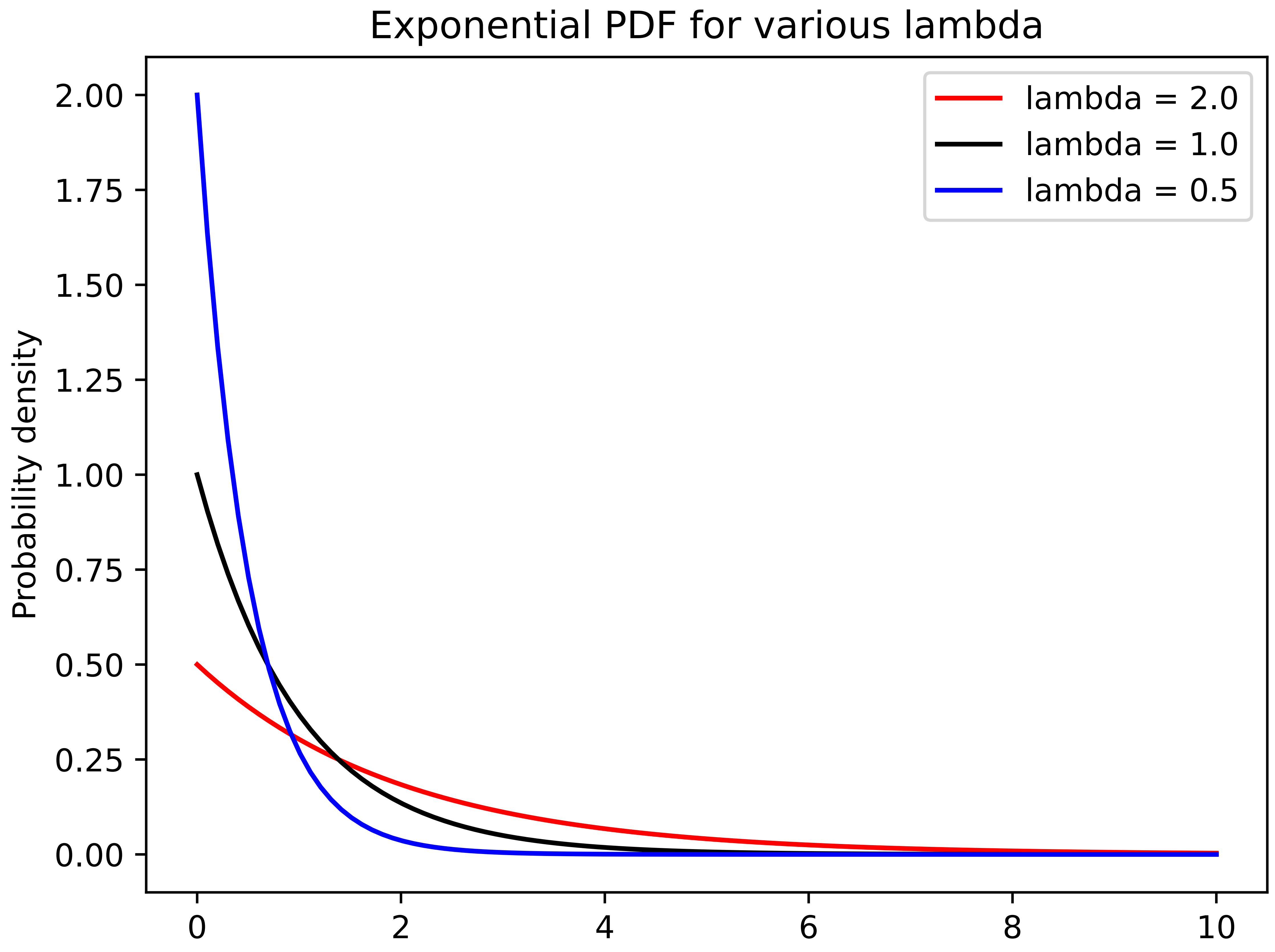

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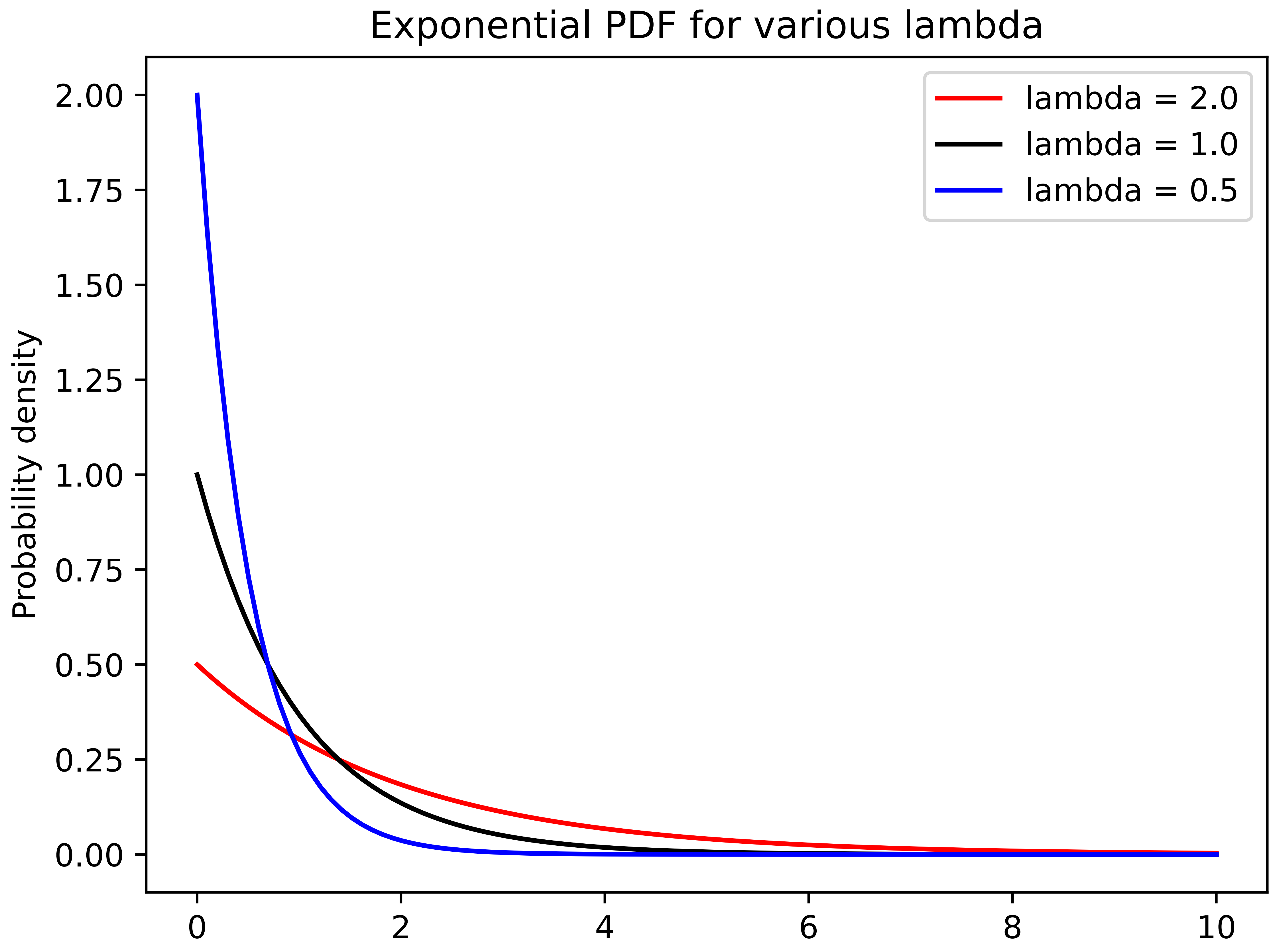

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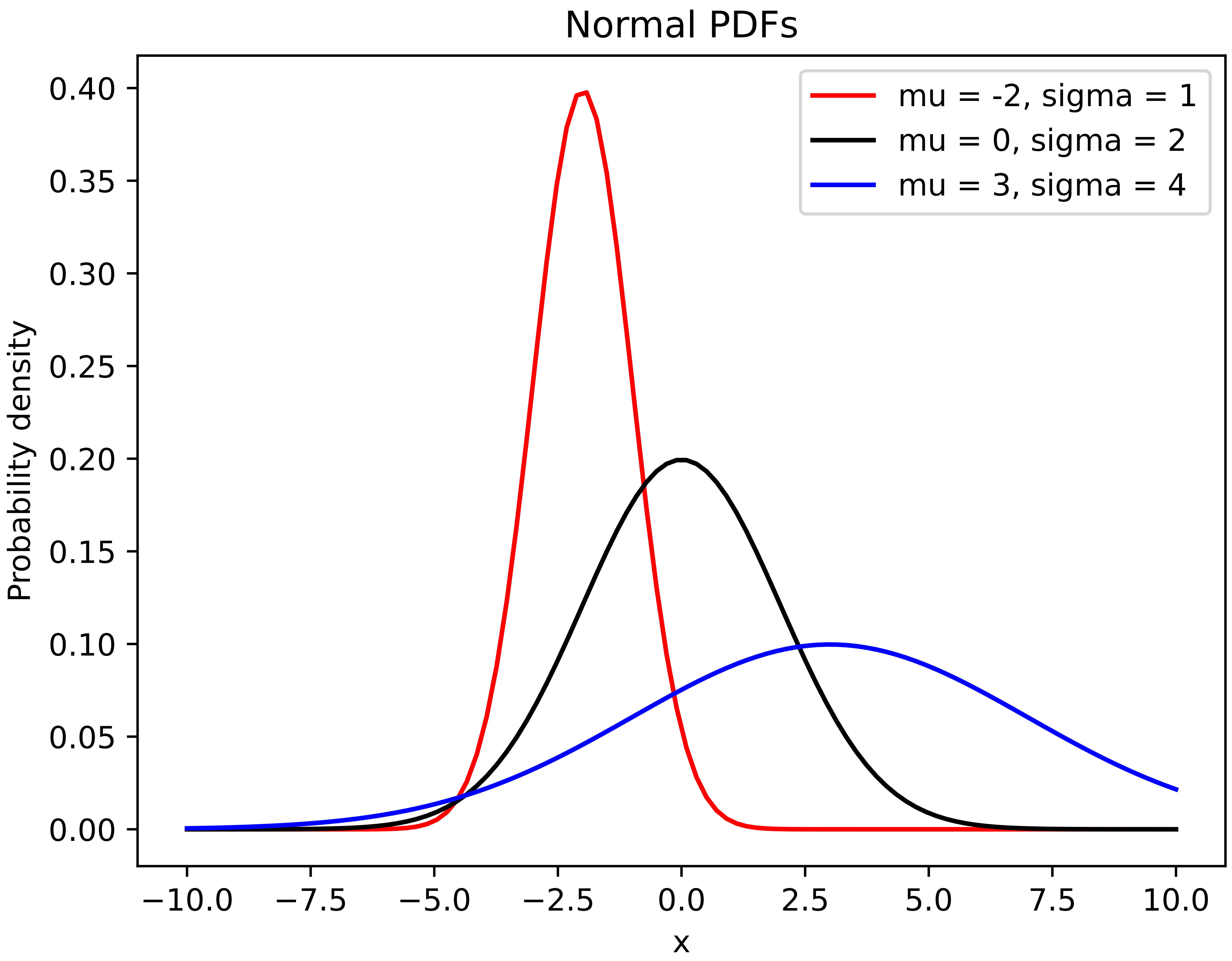
- Mean = median = mode = μ, variance = σ^2, standard deviation = σ

Normal Distribution: Visualization



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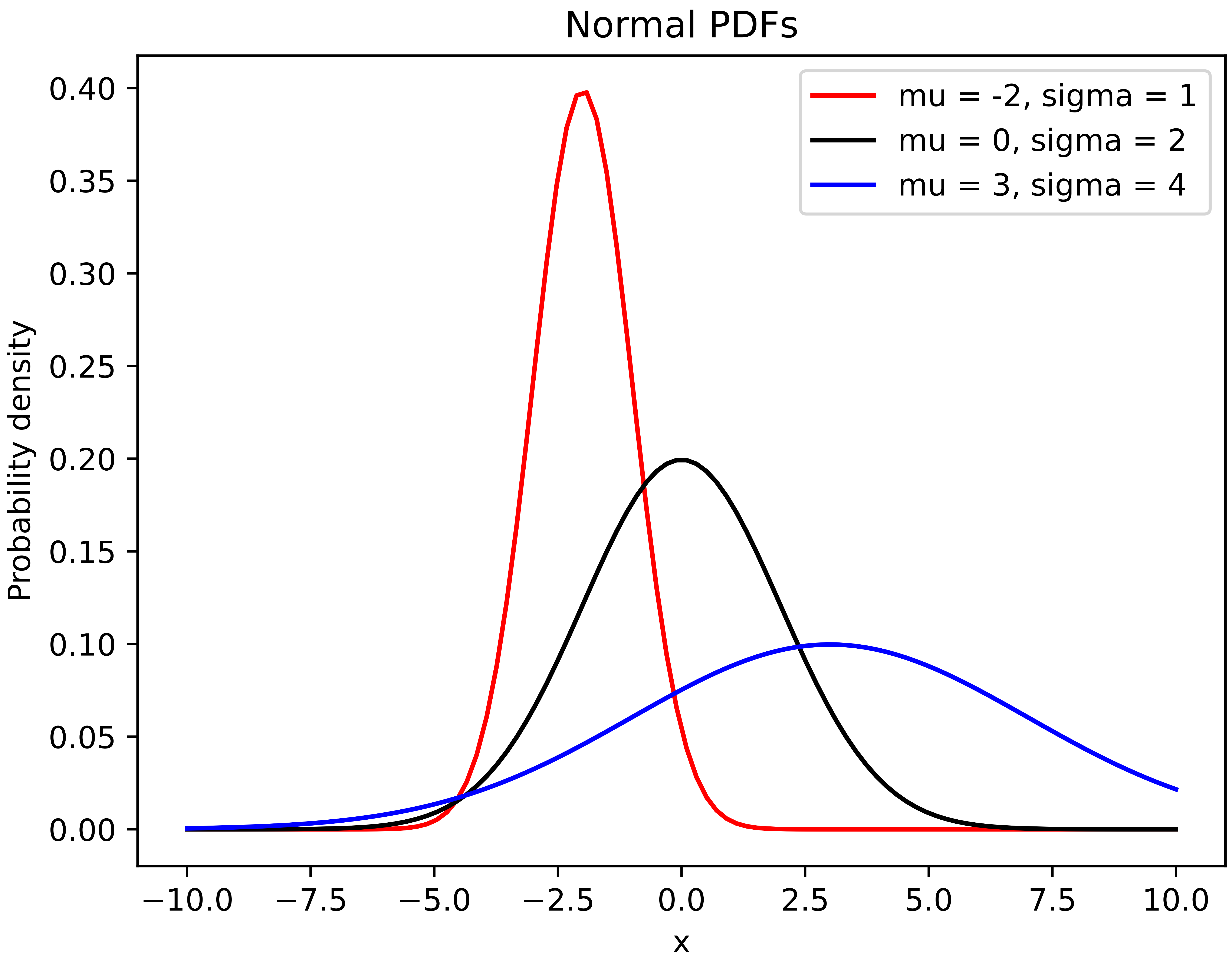

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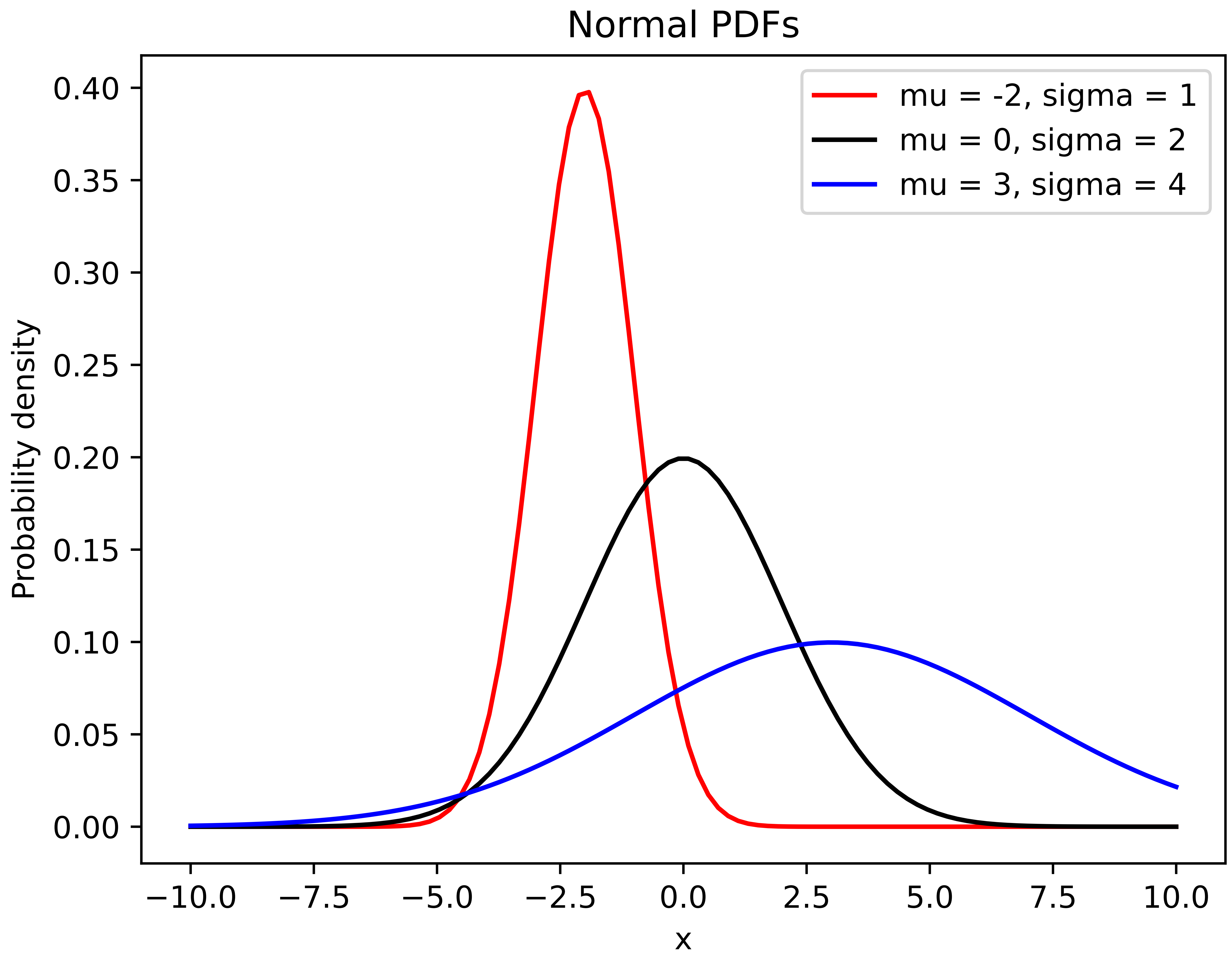

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- μ (center) and σ^2 (spread) fully define the normal distribution
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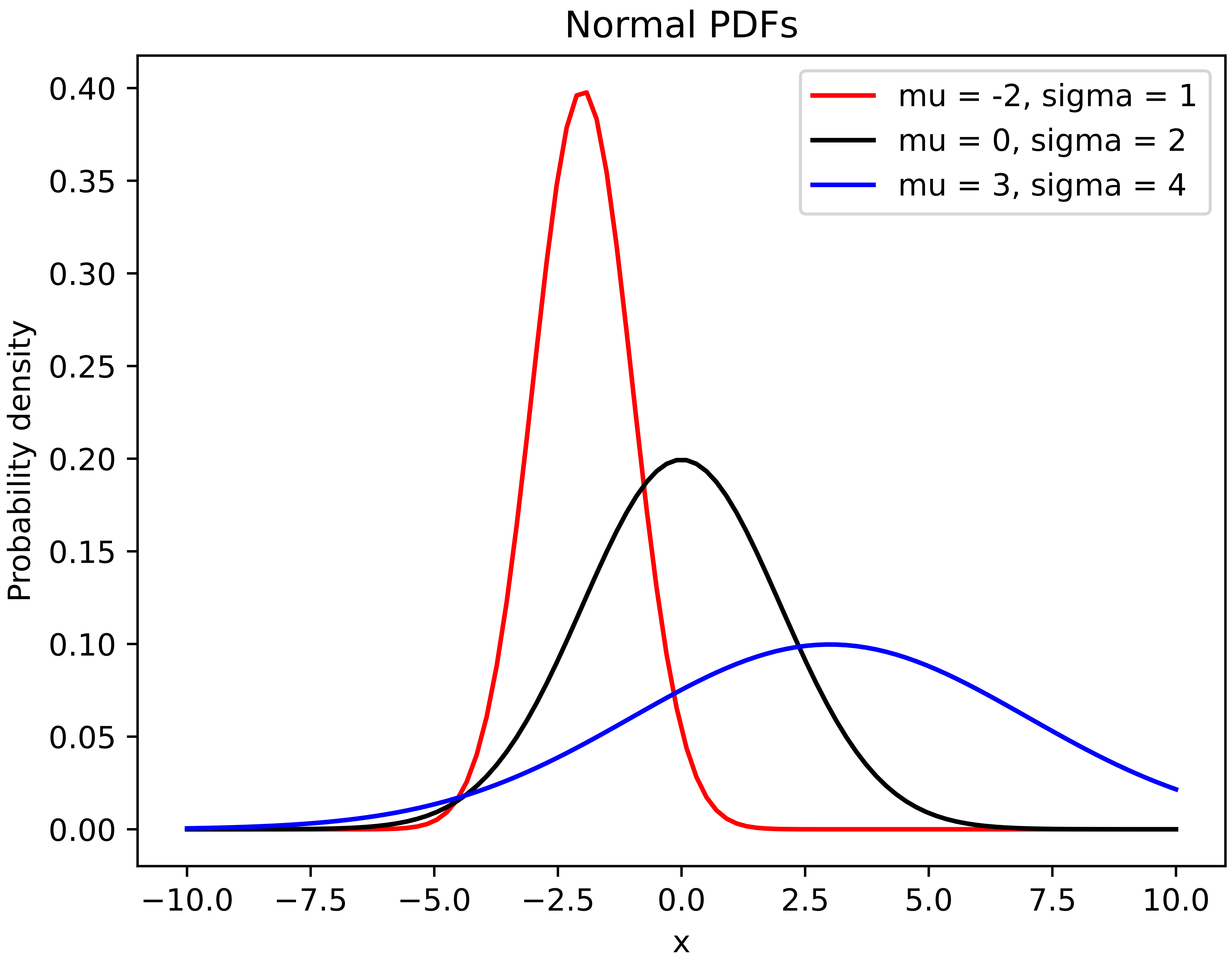
- When $\mu = 0$ and $\sigma^2 = 1$, we have the *standard normal distribution*



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- Empirical rule (68%, 95%, 99.7%) appears to be quite good

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- Directly in R: `qnorm(p, mean, sd)`; `qnorm(p)` for z value

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- Q2: Find the value that cuts off the upper 10% of the curve in American ED patient weights

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    - You take one random sample and get mean $\bar{x}_1$, take another random sample and get mean $\bar{x}_2$, and repeat until you have $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$
    - Take m random samples for a total of m sample means
- These m means form a distribution with mean μ and variance $\dfrac{\sigma^2}{n}$ where n is the sample size
- Key idea: $\overline{X}$ has its own distribution

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- Standard deviation of $\overline{X}$ is $\dfrac{\sigma}{\sqrt{n}}$; this is known as the **standard error**

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- If $X \sim N(\mu, \sigma)$, then $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

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- **Central Limit Theorem**: If the population we are sampling from is not normal, then the shape of the distribution of $\overline{X}$ will be normal as long as n is sufficiently large (typically $n \geq 30$ suffices)

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- In notation: $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

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$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

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- Q2: What is the standard error of the sample mean?

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- Q2: What is the standard error of the sample mean?
- Q3: What distribution does the sample mean follow?

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- Q1: What is the mean of the distribution of sample means?
- Q2: What is the standard error of the sample mean?
- Q3: What distribution does the sample mean follow?
- Q4: What is the probability that the sample mean of $n = 64$ house prices is greater than \$500,000?

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- These m proportions form a distribution with mean p
- Standard deviation of $\hat{p}$, known as standard error, is $\sqrt{\dfrac{p(1-p)}{n}}$

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  - $np \geq 5$
  - $n(1 - p) \geq 5$
- If both of these conditions are met, then $\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$