Optimization In Finance
(Portfolio Management)

IE-402:Optimization

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<u>GROUP NUMBER: 34</u>

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Index

Abstract:

Investors at any time and of any investment area are faced with the problem of conflicting objective of minimizing risks and simultaneously maximizing returns.

The Portfolio Optimization service helps investment managers understand the optimal trade-offs between risk and reward based on changes in the portfolio. The framework is built upon a flexible mathematical model which allows for solving a wide range of investment problems based on the objectives and constraints.

When investing in the stock market ,the first problem investors have is the selection of stocks. Selecting the stocks that simultaneously offer high return and low risk is a difficult problem that is worth investigating. The Traditional Modern Portfolio Theory (MPT) fails to live up to our expectations. The traditional calculation is unable to calculate the coefficient of variation, and merely considers the relationship between each pair of stocks, so it cannot accurately assess portfolio risk. Therefore, this report compares two great Mathematical Models which are used in daily life by the stock brokers for maximum profit for their customers . Compared with the traditional method, the proposed method significantly reduces the computation complexity because the complexity does not increase when the portfolios stock number increases. In addition, we utilize the sliding window to avoid the over-fitting problem, which is common in this field, and test the effect of all kinds of training and testing periods. The experimental results with the example and case study shows the detailed analysis of these explained mathematical models.

Introduction And Background:

Portfolio optimization is a problem of selecting the number of assets to invest where to maximize the expected return and minimize the risk(variance) at the same time by using a portfolio model that was proposed in 1952, the mean-variance model proposed by Markowitz as modern portfolio theory (MPT). The assets are selected as the portions of these investments and are called portfolio weights, which are decision variables of the optimization algorithm. Portfolio objectives are as follows: i) income or the expected return and ii) risk variance. It is desirable to maximize the portfolio return and minimize the portfolio risk.

Kelly criterion is a formula for bet sizing that leads almost surely to higher wealth compared to any other strategy in the long run (i.e. the limit as the number of bets goes to infinity). The Kelly bet size is found by maximizing the expected value of the logarithm of wealth, which is equivalent to maximizing the expected geometric growth rate.

Portfolio Rate of Return

Let *Asset Return*$_i$ denote the rate of return for asset and let *Weight*$_i$ denote the asset's weight in the portfolio.

The portfolio rate of return is the weighted sum of asset rates of return:

$$Portfolio\ Return = \sum_i Weight_i \cdot Asset\ Return_i$$

Where,

$$\sum_i Weight_i = 1$$

For long-only portfolios, weights should be strictly greater than zero and for short-only it could be either positive or negative.

Portfolio's mean rate of return is given by

$$E[Portfolio\ Return] = \sum_i Weight_i \cdot E[Asset\ Return_i]$$

Where as variance is given by

$$Var[Portfolio\ Return] = \sum_{i,j} Weight_i \cdot Weight_j \cdot Cov[Asset\ Return_i, Asset\ Return_j]$$

Markowitz Mathematical Model

A popular form of portfolio optimization is due to Markowitz which maximizes the portfolio's mean return and minimizes the variance. Such portfolios are called mean-variance optimal. The return variance is commonly believed to measure investment risk so the mean-variance optimal portfolios are thought to maximize the mean return while minimizing risk.
That is it

$$\text{Maximize: } E[Portfolio\ Rate\ of\ Return]$$

$$\text{Minimize: } Var[Portfolio\ Rate\ of\ Return]$$

Mathematical formulation of Markowitz's mean-variance analysis:

Our aim is to $\text{minimize } \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$

With respective to constraints $\sum_{i=1}^{N} w_i = 1.$ and $\sum_{i=1}^{N} w_i \overline{R}_i = \mu_P$.
Given the target expected rate of return of portfolio μP, find the portfolio strategy that minimizes σ_P^2.

Solution by Lagrangian-

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1\left(\sum_{i=1}^{N} w_i - 1\right) - \lambda_2\left(\sum_{i=1}^{N} w_i \overline{R}_i - \mu_P\right)$$

Where λ_1 and λ_2 are Lagrangian multipliers.

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \overline{R}_i = 0, i = 1, 2, \cdots, N. \tag{1}$$

$$\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{N} w_i - 1 = 0; \tag{2}$$

$$\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{N} w_i \overline{R}_i - \mu_P = 0. \tag{3}$$

From Eq. (1), the portfolio weight admits solution of the form

$$\boldsymbol{w}^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu}) \tag{4}$$

where $\mathbf{1} = (1 \quad 1 \cdots 1)^T$ and $\boldsymbol{\mu} = (\overline{R}_1 \quad \overline{R}_2 \cdots \overline{R}_N)^T$.

To determine λ_1 and λ_2

$$1 = \mathbf{1}^T \Omega^{-1} \Omega \boldsymbol{w}^* = \lambda_1 \mathbf{1}^T \Omega^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}. \tag{5}$$

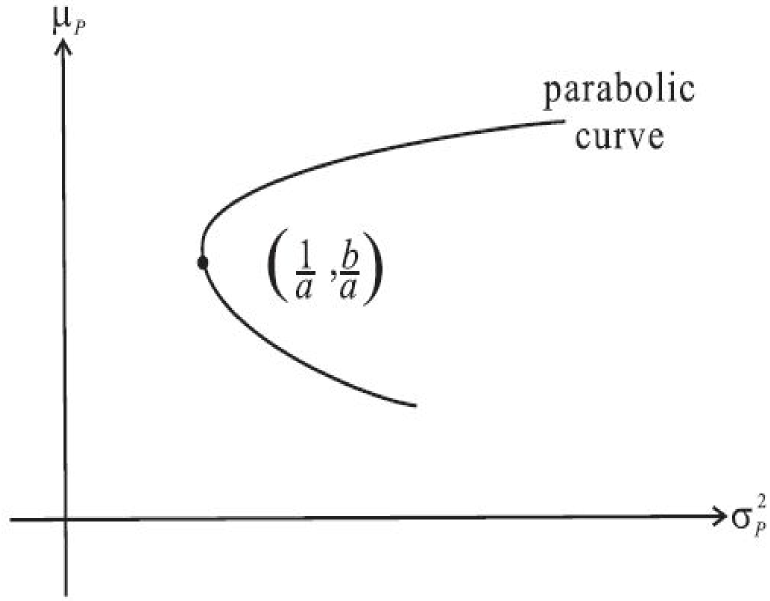
$$\mu_P = \boldsymbol{\mu}^T \Omega^{-1} \Omega \boldsymbol{w}^* = \lambda_1 \boldsymbol{\mu}^T \Omega^{-1} \mathbf{1} + \lambda_2 \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}. \tag{6}$$

Write $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}$ and $c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}$, we have

$$1 = \lambda_1 a + \lambda_2 b \text{ and } \mu_P = \lambda_1 b + \lambda_2 c.$$

Solving for λ_1 and λ_2 : $\lambda_1 = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \dfrac{a\mu_P - b}{\Delta}$, where $\Delta = ac - b^2$.

The parabolic curve is generated by varying the value of the parameter μP .



For example:

Asset	Possible Returns			Mean	Stdev
A	(4%)	(5%)	(6%)	(5%)	1%
B	5%	10%	15%	10%	5%

Minimum variance portfolio:

Asset A Weight	5/6
Asset B Weight	1/6
Portfolio Mean	(2.5%)
Portfolio Stdev	0%

In the shown example, there are two assets A and B whose possible returns are -4%,-5%,-6% and 5%,10%,15% respectively.When minimising variance ,weights of A and B results in ⅚ and ⅙ with mean equal to -2.5.But we can see that when weight of B=1,A=0,mean comes out to 10. So clearing an investor could instead have chosen a portfolio consisting entirely of asset B which would always give a positive return of either 5%, 10% or 15%. So clearly minimising variance according to Markovitz theory fails here by a huge margin.

Kelly Mathematical Model

Kelly proposed a formula for bet sizing that leads almost surely to higher wealth compared to any other strategy in the long run (i.e. the limit as the number of bets goes to infinity). In this , the idea is to maximise mean logarithmic growth-rate which results into sure profits in the long run .

So,

$$Kelly\ Criterion = E[\log(1 + Portfolio\ Rate\ of\ Return)]$$

$$= E\left[\log\left(1 + \sum_i Asset\ Weight_i \cdot Asset\ Rate\ of\ Return_i\right)\right]$$

Now,our objective is to find weights that maximize the Kelly Criterion.This can be solved by a numerical optimization method.

Mathematical Formulation for Kelly Criterion:

Well our average rate of return is determined by E(log(X)). If we're betting a fraction x of our total bankroll then: E(log(X))=plog(1+bx)+(1−p)log(1−x). To maximize this we need to find where the derivative is 0.
First let us find the derivative:

$$\frac{d}{dx}(E(\log(X))) = \frac{d}{dx}(p\log(1+bx)+(1-p)\log(1-x))$$

$$= pb\frac{1}{1+bx} + (1-p)(-1)\frac{1}{1-x}$$

$$= \frac{pb}{1+bx} - \frac{1-p}{1-x}$$

Now let us find where it is 0:

$$0 = \frac{d}{dx}(E(\log(X)))$$

$$0 = \frac{pb}{1+bx} - \frac{1-p}{1-x}$$

$$0 = pb(1-x) - (1-p)(1+bx)$$

$$0 = pb - pbx - 1 - bx + p + pbx$$

$$bx = pb + p - 1$$

$$x = \frac{pb-(1-p)}{b}$$

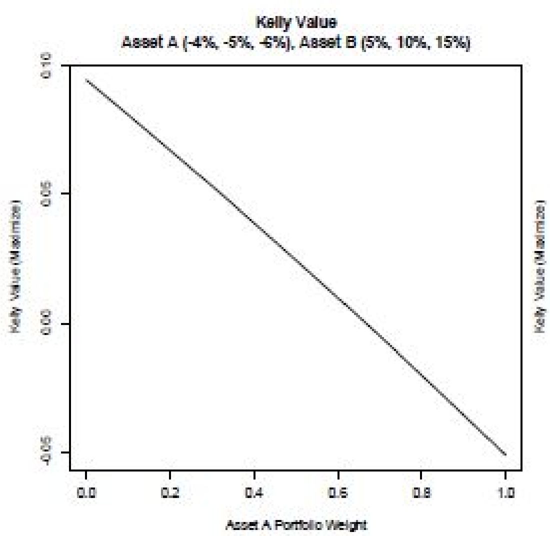
Example:

Asset A = -4%,-5% ,-6%;
Asset B = 5%,10%,15%

According to Kelly,

$$Kelly\ Criterion = \\ = E[\log(1 + Asset\ A\ Weight \cdot (-4\%, -5\%, -6\%) \\ + Asset\ B\ Weight \cdot (5\%, 10\%, 15\%))]$$

When we maximise the kelly criterion , it gives weight of Asset A =0, and weight of AssetB =1 which was likely to come from our observations.Kelly value comes out to be 0.1
Here is a graph how kelly value changes according to weight of Asset A.



So for the above example, by the kelly criterion ,right answer is obtained.

Markowitz vs Kelly Portfolio Optimization:

Markowitz portfolios do NOT minimize risk even if given the true probability distribution of asset returns.
Kelly portfolios maximize average long-term returns if given the true return distribution. But Kelly portfolios have other problems
.
The **efficient frontier** proposed by **Markowitz** in which we try to maximize the return with a certain risk, i.e. it focuses on containing the assigned risk.

On the other hand, there is **Kelly's method** proposed by **John Kelly and Ed Thorpe** that tries to maximise the expectation of the log utility of wealth, i.e. maximizing return is the focus.

Case Study - KOSPI200

As we are comparing the two Mathematical model , a great case study which was published in the Proceedings of the 2014 International Conference on Industrial Engineering and Operations Management Bali, Indonesia, January 7– 9, 2014 named as KOSPI200. The expected return with Kelly Mathematical Model can be seen as 0.8790% and with the Markowitz Mathematical Model can be seen as 0.8100% over the return of 200 stocks included in KOSPI 200.For further studies , reference links are provided at the end of this report.

Conclusion :

So finally, using our knowledge in the course and by the references , with the intensive effort of the group members, we conclude this report by making certain observations for the two mathematical Models.

Markowitz Model:

- Markowitz portfolios do not maximize return and minimize risk as we all believe, even when given the true probability distribution of returns are there.
- Markowitz portfolios are diversified which may give an illusion of safety.

Kelly Model:

- Kelly portfolios may underperform in the “short run” but will have the best performance on average in the “long run”.
- Kelly portfolio optimization does what it is supposed to: Favours assets with better return distributions.
- Kelly portfolios are often concentrated in few assets. So if the return distributions are incorrect then Kelly overweighs the wrong assets.
- By limiting the weight assets in Kelly , Diversification can take place int this model.

We have shown the optimization portfolios with the basic example and also for stocks listed in the *KOSPI* 200 using the Kelly’s and a Markowitz’s model to empirically see the characteristics of two portfolio models. And it was found that investing in the stocks using the Kelly portfolio model generated much higher rate of return and risk on investment, and less diversified portfolio than a traditional investing the strategies like the Markowitz portfolio model.

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