Homogeneous Coordinates

Group 80

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Linear Algebra

Karan Nijhawan
Daksh Shah
Sairam Babu

Abstract

This research report explores & dives deep into the topic of linear transformations in homogeneous coordinates within the realm of linear algebra. The report covers various types of linear transformations and their representation in homogeneous coordinates. Additionally, it discusses the integration of homogeneous coordinates in computer graphics, modeling, and perspective imaging. The report concludes with a comprehensive overview of the research conducted, including collaborative efforts & individual contributions.

Introduction to Homogeneous Coordinates

Homogeneous coordinates are a mathematical framework used in projective geometry and computer graphics. They extend Euclidean coordinates by adding an extra coordinate, typically denoted as w. This allows for the representation of points at infinity and simplifies geometric transformations. They are widely used in computer graphics, computer vision, and robotics. In this research project, we will explore the theory, applications, and most importantly, practical implementations of homogeneous coordinates

The key idea behind homogeneous coordinates is that a point (x, y) in Euclidean space can be represented as (wx, wy, w), where w is a non-zero scalar. The coordinate (wx, wy, w) is called the homogeneous representation of the point. By varying the value of w, different points in Euclidean space can be represented using the same homogeneous coordinates. This concept is often referred to as projective equivalence.

Homogeneous coordinates allow for the representation of points at infinity by assigning $w = 0$. This property is particularly useful in computer graphics, where the representation of parallel lines and vanishing points becomes straightforward. Additionally, homogeneous coordinates simplify geometric transformations, such as translation, rotation, scaling, and perspective projection, by representing them as simple matrix multiplications.

The advantages of using homogeneous coordinates include:

- **Simplicity and versatility:** provide a unified representation for points, lines, and transformations, enabling concise and elegant mathematical formulations
- **Handling points at infinity:** allow for the representation of points at infinity, facilitating operations involving parallel lines and vanishing points
- **Simplified transformations:** geometric transformations can be expressed as simple matrix multiplications, making them more intuitive and easier to implement

Homogenous coordinates form the foundation of many algorithms and techniques, such as 3D transformations, perspective projection, and image warping.

In this research project, we will also investigate specific use cases and algorithms where homogeneous coordinates play a crucial role, showcasing their significance in various domains.

Homogeneous Coordinates Example

In the context of homogeneous coordinates, let's consider a point $P(x,y)$ in 2D Euclidean space. To represent this point using homogeneous coordinates, we introduce an additional coordinate w. The homogeneous representation of the point P is given as (w_x, w_y, w).

Suppose we want to translate this point by a vector $T(t_x, t_y)$. The translation operation can be expressed using a 3×3 matrix called the translation matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_X & t_y & 1 \end{bmatrix}$$

To perform the translation, we multiply the homogeneous representation of the point P with the translation matrix:

$$\begin{pmatrix} w_x & w_y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{pmatrix} = \begin{pmatrix} w'_x & w'_y & 1 \end{pmatrix}$$

Geometric Transformations in Homogeneous Coordinates

In homogeneous coordinates, the following transformations are commonly used:

- **Translation:** involves shifting it by a specified amount in the x and y directions without bringing changes to its size or shape
- **Rotation:** involves changing its angle or orientation around a specified pivot point while preserving its size & shape
- **Scaling:** involves altering its size or dimensions uniformly or non-uniformly by a certain factor

These transformations are fundamentals in computer graphics, computer vision, designing, and modeling. By applying these transformations individually or in combination, a wide range of effects and manipulations can be achieved.

Why Homogeneous coordinates?

Consider a set of 2D points that need to be transformed using a rotation operation. The goal is to rotate the points around a given pivot point by a certain angle. How can homogeneous coordinates and matrix transformations be utilized to solve this problem effectively and apply it to real-world scenarios in computer graphics, robotics, and motion planning?

To solve this problem and extend it to real-world applications, we can further explore the power of homogeneous coordinates and matrix transformations.
In computer graphics, homogeneous coordinates and rotation transformations are commonly used to create stunning visual effects and animations.

In the field of robotics, homogeneous coordinates and rotations play a crucial role in defining robotic motions. In motion planning, they are used to plan the trajectory of moving objects. For example, in autonomous vehicles, the rotation of the vehicle around its center of mass is essential for making accurate turns and navigating through obstacles.

The solution approach remains the same as before, utilizing homogeneous coordinates and rotation matrices. Let P be the set of 2D points represented in homogeneous coordinates as:

$$\begin{bmatrix} x & y & w \end{bmatrix}$$

where x and y are the Euclidean coordinates, and w is the homogeneous coordinate

The rotation transformation matrix R is given by:

$$R = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To perform the rotation, we can compute the transformed homogeneous coordinates P' by multiplying the rotation matrix R with P:

$$P \cdot R = P'$$

After obtaining the transformed homogeneous coordinates P', we can convert them back to Euclidean coordinates by dividing each coordinate by w to obtain the final rotated points.
Similarly, we can use this simple geometric transformations in combinations to achieve solutions to variety of real-world problems in the fields like computer-graphics, robotics etc.

Real World Applications

Homogeneous coordinates showcase the use of linear algebra in various domains. In this project we will showcase how the following domains use linear algebra.

The domains that will be discussed in detail in the project include:

- **Computer Graphics:** Homogeneous coordinates play a critical role in computer graphics by providing a unified framework for representing points, lines, and transformations. They enable efficient rendering of 3D objects, handling of perspective projection, and efficient computation of transformations such as translation, rotation, and scaling.

- **Modeling:** Homogenous coordinates provide a powerful mathematical framework for modeling, allowing for accurate representation, efficient geometric transformations, perspective projection, and intersection calculations. Using this we can transform 3-D space to 2-D linearly and then work accordingly. Modeling here refers to usage in several domains:

  - Medical Industry: *Medical Industry uses the transformations to get a scanned 3D body organ on a 2D screen, still looking as 3D, making it easy to be analysed by the doctor*

  - 3D Printing: *In 3D printing, transformations and homogeneous coordinates are used to accurately position, resize, and manipulate 3D models before they are printed to keep the model within the printing volume. Additionally, they are used in mesh manipulation to deform and align the model as needed.*
    Finally, they play a crucial role in slicing the model into layers and generating the corresponding G-code instructions for the printer, ensuring precise printing of each layer

- **Perspective Imaging:** Homogeneous coordinates are crucial in projective imaging for creating images with a 3D look.
  They enable perspective projection, which simulates how objects appear in 3D space on a 2D image plane. Homogeneous coordinates also allow for the representation of vanishing points and horizon lines, adding depth to the images.
  Overall, homogeneous coordinates are essential for achieving a convincing sense of depth and a three-dimensional appearance in projective imaging.
  Also, in modern-world, they are used a lot by AI applications as well as artists in order to create images with creative ideas that might be difficult to imagine being made by bare hands

FUTURE PLANS AND TIMELINE

Python Code Scripts for Transformation Visualization

- **Objective:** to make the process of learning more interactive by adding visual demonstrations
- **Approach:** developing Python code scripts of each transformation on objects or images, taking examples of translation, rotation, scaling & projection

Mathematical Explanations of Transformations

- **Objective:** to provide clear mathematical explanations alongside code scripts to deepen the understanding of transformations
- **Approach:** explaining underlying principles, equations, and matrices associated with each transformation step-by-step

Exploring Computer Graphics and Modeling

- **Objective:** to delve deeper into computer graphics and modeling concepts, showcasing their mathematical foundations
- **Approach:** by implementing mathematical principles in computer graphics through rendering techniques, shading models, and surface representations using working examples of Python code scripts

3D to 2D Conversions in Medical Sciences and Perspective Imaging

- **Objective:** to explore 3D to 2D conversions in medical sciences and perspective imaging
- **Approach:** by employing mathematical algorithms and techniques to create realistic 3D renderings from medical imaging data and achieve depth perception in perspective images, supported by Python code scripts

Integration of Electrical Component Designing and 3D Printing in Modeling

- **Objective:** to incorporate electrical component designing and 3D printing into modeling
- **Approach:** by utilizing the intersection of mathematics, computer-aided design (CAD), and 3D printing technologies to create functional electrical model designs, enabling prototyping and rapid product development

By incorporating these elements into our future plans, we will demonstrate a comprehensive understanding of the subject matter and showcase our intention to explore the practical applications of homogeneous coordinates and related concepts.

WORK DISTRIBUTION

For this research project on homogeneous coordinates, we divided the work among our team members based on our individual expertise and interests. The distribution of work was as follows:

- **Daksh Shah:** Led the mathematical aspect of the project, focusing on the understanding and application of linear algebra in homogeneous coordinates.
  He explored the theoretical foundations, derived equations and matrices for transformations as well as worked out on custom-made examples that could be incorporated in the project for better understanding.
- **Karan Nijhawan:** Conducted extensive research in the field of computer graphics, investigating techniques and algorithms used in rendering, shading, and lighting calculations.
  He explored how homogeneous coordinates integrate with computer graphics, creating visually appealing 3D scenes and animations, some of which are also included in the project as examples.
- **Sairam Babu:** Explored the intersection of homogeneous coordinates with medical sciences and electrical component design. He researched over the applications of homogeneous coordinates in medical imaging, focusing on 3D modeling and visualization.
  Additionally, he explored the integration of electrical component designing with 3D printing technology.

Throughout the project, all team members collaborated & worked very closely, sharing their findings, insights, and code implementations.
This collaborative approach ensured the accuracy and coherence of the overall research project. In this way, we are aiming to produce a comprehensive research project that encompasses the mathematical foundations as well as highly useful & trending real-world applications of HOMOGENOUS COORDINATES!