UNIT I
Chapter 2-RELATIONS

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Introduction to Relations

Definition: Let A and B be two sets. A binary relation R from A to B is a subset of AXB.

OR

$$R \subseteq AXB$$

Recall: $AXB = \{(a, b) \mid a \in A \text{ and } b \in B\}$
Usually we use the notation aRb to denote $(a, b) \in R$.

$a \not{R} b$ is used to denote $(a, b) \notin R$

Example: Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 4\}$
$AXB = \{(1, 0), (1, 1,), (1, 2), (1, 4), (2, 0), (2, 1), (2, 2), (2, 4), (3, 0), (3, 1), (3, 2), (3, 4)\}$

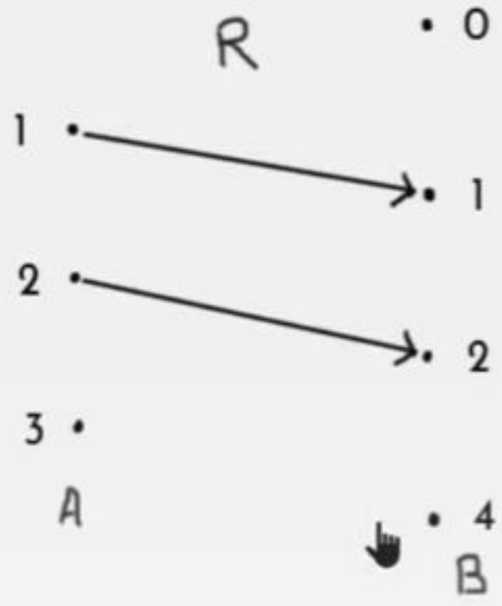
Let say R is the relation where $(a, b) \in R$ if and only if $a = b$ then
$R = \{(1, 1), (2, 2)\}$ and $R \subseteq AXB$

Example: Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 4\}$

$AXB = \{(1, 0), (1, 1,), (1, 2), (1, 4), (2, 0), (2, 1), (2, 2), (2, 4), (3, 0), (3, 1), (3, 2), (3, 4)\}$

Let say R is the relation where $(a, b) \in R$ if and only if $a = b$ then $R = \{(1, 1), (2, 2)\}$ and $R \subseteq AXB$

Graphical representation of ordered pairs.



Arrows are used to represent ordered pairs of relation R.

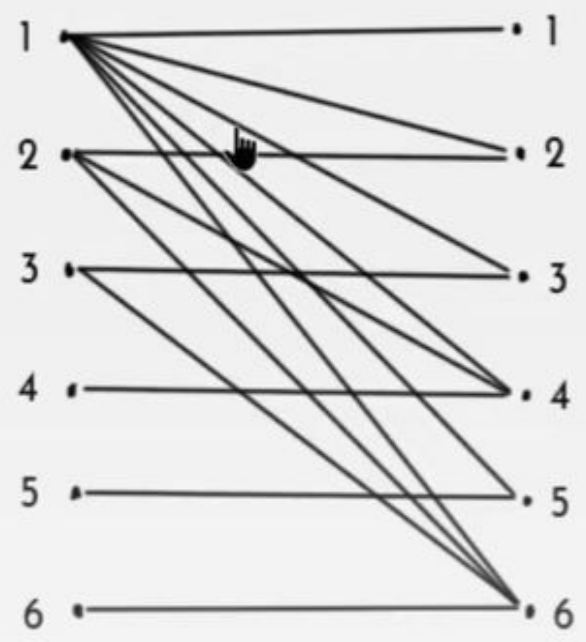
Relation from a set to itself

A relation on a set A is a relation from A to A.

Example: Let $R = \{(a, b) \mid a \text{ divides } b\}$ $\qquad A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

Graphical representation of relation R



A binary relation R on a single set A is a subset of $A \times A$.

For two distinct sets, A and B, having cardinalities *m* and *n* respectively, the maximum cardinality of a relation R from A to B is *mn*.

Domain and Range of Relation

Domain of Relation: The Domain of relation R is the set of elements in P which are related to some elements in Q, or it is the set of all first entries of the ordered pairs in R. It is denoted by DOM (R).

Range of Relation: The range of relation R is the set of elements in Q which are related to some element in P, or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).

Example:

> Let A = {1, 2, 3, 4}
>
> B = {a, b, c, d}
>
> R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.

Solution:

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DOM (R) = {1, 2}
RAN (R) = {a, b, c, d}
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Example:-

Let, A={1,2,9} B={1,3,7}

Case 1 − If relation R is 'equal to' then

R={(1,1)}

Dom(R) = {1}, Ran(R)={1}

Case 2 − If relation R is 'less than' then

R={(1,3),(1,7),(2,3),(2,7)}

Dom(R) = {1,2}, Ran(R)={3,7}

Case 3 − If relation R is 'greater than' then

R={(2,1),(9,1),(9,3),(9,7)}

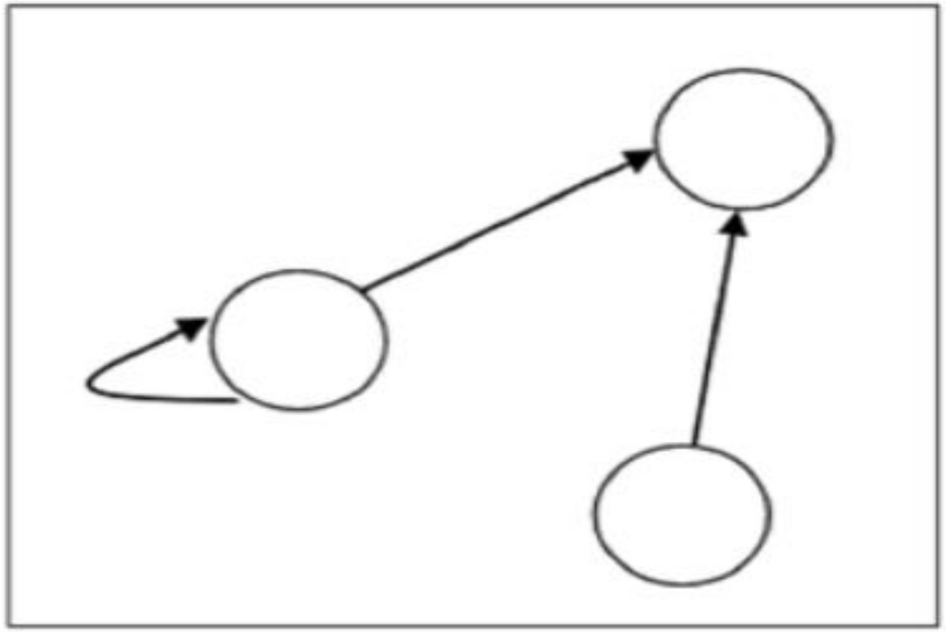
Dom(R) = {2,9}, Ran(R)={1,3,7}

Representation of Relations using Graph

A relation can be represented using a directed graph.

The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined. For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex ‘x’ to vertex ‘y’. If there is an ordered pair (x, x), there will be self- loop on vertex ‘x’.

Suppose, there is a relation $R = \{(1,1), (1,2), (3,2)\}$ on set $S = \{1,2,3\}$, it can be represented by the following graph −



Types of Relations (Part 1)

1. Reflexive Relation:

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
In other words, $\forall a((a, a) \in R)$.

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (4, 4)\}$

Relation R_1 is reflexive because it contains all ordered pairs of the form (a, a) for every element $a \in A$ i.e., R_1 has $(1, 1), (2, 2), (3, 3), (4, 4)$

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 4)\}$

Relation R_2 is not reflexive because the ordered pair $(3, 3)$ is not in R_2.

2. Irreflexive Relation:

A relation R on a set A is called irreflexive if $\forall a \in A, (a, a) \notin R$.

Example: $A = \{1, 2, 3, 4\}$

$R_3 = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$ is not irreflexive because (3, 3) and (4, 4) is there in R_3.

$R_4 = \{(1, 2), (2, 1)\}$ is irreflexive because $\forall a \in A, (a, a) \notin R_4$

3. Symmetric Relation:

A relation R on a set A is called symmetric if $(b, a) \in R$ holds when $(a, b) \in R$ for all $a, b \in A$
In other words, relation R on a set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$

Example: Relation $R_5 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is symmetric because for every $(a, b) \in R_5$ $(b, a) \in R_5$
like (1, 2) (2, 1) is in R_5.
There is no need to check for (1, 1), (2, 2).

Relation $R_6 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ is not symmetric because for (1, 2) there is no (2, 1) in R_6. Same is true for (1, 3) and (1, 4).

4. Antisymmetric Relation:

A relation R on a set A is called antisymmetric if $\forall a \forall b((a, b) \in R \wedge (b, a) \in R \rightarrow (a = b))$
Whenever we have (a, b) in R, we will never have (b, a) in R until or unless (a = b)

Example: Relation $R_7 = \{(1, 1), (2, 1)\}$ on set A is antisymmetric because (2, 1) is in R_7 but (1, 2) is not in R_7.

5. Transitive Relation:

A relation R on a set A is called transitive if $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a,c) \in R)$

Example: $A = \{1, 2, 3, 4\}$

$R_8 = \{(2, 1), (3, 1), (3, 2), (4, 4)\}$ is transitive because (3, 2), (2, 1), and (3, 1) are there in R_8.

$R_9 = \{(2, 1), (1, 3)\}$ is not transitive as (2, 1) and (1, 3) are there in R_9 but there is no (2, 3) in relation R_9.

6. Asymmetric Relation:

A relation R on a set A is called asymmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$

Example: $A = \{1, 2, 3, 4\}$

$R_{10} = \{(1, 1), (1, 2), (1, 3)\}$ is not an asymmetric relation because of (1, 1).

$R_{11} = \{(1, 2), (1, 3), (2, 3)\}$ is an asymmetric relation.

- A relation is an **Equivalence Relation** if it is reflexive, symmetric, and transitive.

Example – The relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$ on set

$A = \{1,2,3\}$ is an equivalence relation since it is reflexive, symmetric, and transitive.

Let $A = \{1, 2, 3, 4\}$ and
$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$.

Show that R is an Equivalence Relation.

Consider the following Relations on a set $\{1,2,3,4\}$:

$R1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R2 = \{(1,1), (1,2), (2,1)\}$

$R3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R4 = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,3), (3,4), (4,1), (4,4)\}$

Which of these relations are Reflexive, Symmetric or Transitive ?

For each of these relations on the set {1,2,3,4}, decide whether it is reflexive, whether it is symmetric, and whether it is transitive.

$R1 = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$

$R2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R3 \equiv \{(1,2),(2,1),(2,2),(1,1)\}$

$R4 = \{(1,1), (2,2), (3,3), (4,4)\}$

$R5 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$

$R6 \equiv \{(2,4), (4,2)\}$

Partial Order Relations

A relation R on a set A is called a partial order relation if it satisfies the following three properties:

1. Relation R is Reflexive, i.e. $aRa\ \forall\ a \in A$.
2. Relation R is Antisymmetric, i.e., aRb and $bRa \Rightarrow a = b$.
3. Relation R is transitive, i.e., aRb and $bRc \Rightarrow aRc$.

Example1: Show whether the relation (x, y) ∈ R, if, x ≥ y defined on the set of +ve integers is a partial order relation.

Solution: Consider the set A = {1, 2, 3, 4} containing four +ve integers. Find the relation for this set such as R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)}.

Reflexive: The relation is reflexive as for every $a \in A$. $(a, a) \in R$, i.e. $(1, 1), (2, 2), (3, 3), (4, 4) \in R$.

Antisymmetric: The relation is antisymmetric as whenever (a, b) and $(b, a) \in R$, we have $a = b$.

Transitive: The relation is transitive as whenever (a, b) and $(b, c) \in R$, we have $(a, c) \in R$.

1. $A = \{1, 2, 3, 4\}$, $B = (x, y, z)$. Let R be a relation from set A to B given by,

$R = \{ (1, x) , (2, z) , (3, x) , (3, y) , (3, z) \}$

Find (a) R^{-1} and (b) Domain of R & Range of R

Solution :

$A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

$R = \{(1, x) , (2, z) , (3, x) , (3, y) , (3, z) \}$

a) $\therefore$ R^{-1} and $\{ (x, 1) , (z, 2) , (x, 3) , (y, 3) , (z, 3) \}$

b) Domain of $R = \{ 1, 2, 3 \}$

c) Range of $R = \{ x, y, z \}$

1. $A = \{ 1, 2, 3, 4, 6 \}$

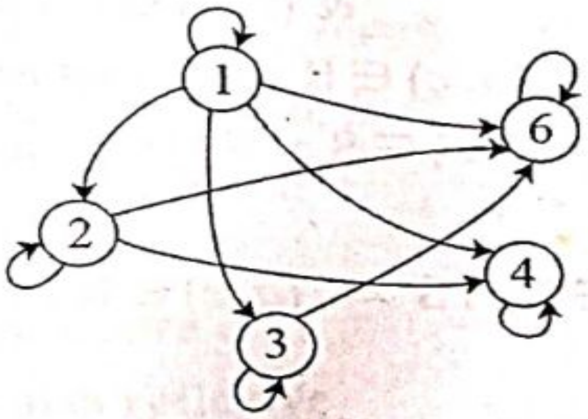
Let R be a relation defined on set A such that $x\text{R}y$ iff "x completely divides y". Write R, draw the digraph and write the matrix for R.

Solution :

$A = \{ 1, 2, 3, 4, 6 \}$

$\therefore \quad R = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6) \}$

Digraph :



Matrix for R

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Composition of Relations:-

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C indicated by $R \circ S$ and defined by:

1. $a\,(R \circ S)c$ **if for** some $b \in B$ we have aRb and bSc.
2. $R \circ S = \{(a, c) \mid \text{there exists } b \in B$ **for** $\text{which } (a, b) \in R \text{ and } (b, c) \in S\}$

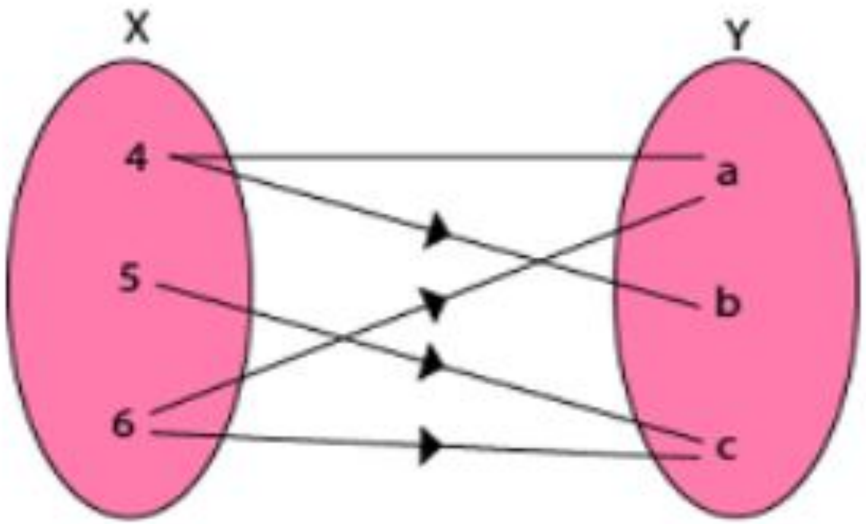
The relation $R \circ S$ is known the composition of R and S.

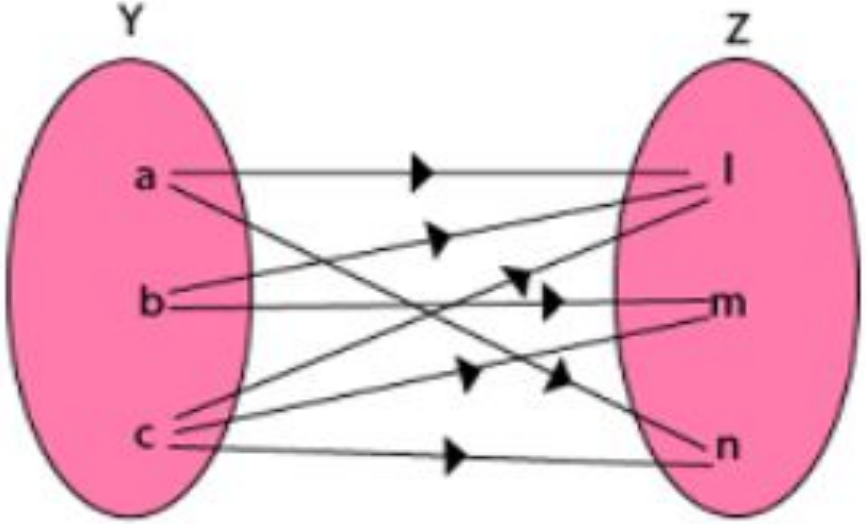
Let R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always represented. Also, $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on.

Let $X = \{4, 5, 6\}$, $Y = \{a, b, c\}$ and $Z = \{l, m, n\}$. Consider the relation R_1 from X to Y and R_2 from Y to Z.

$R_1 = \{(4, a), (4, b), (5, c), (6, a), (6, c)\}$

$R_2 = \{(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)\}$





Find the composition of relation **(i)** $R_1 \circ R_2$ **(ii)** $R_1 \circ R_1^{-1}$

Solution:

(i) The composition relation R_1 o R_2 as shown in fig:

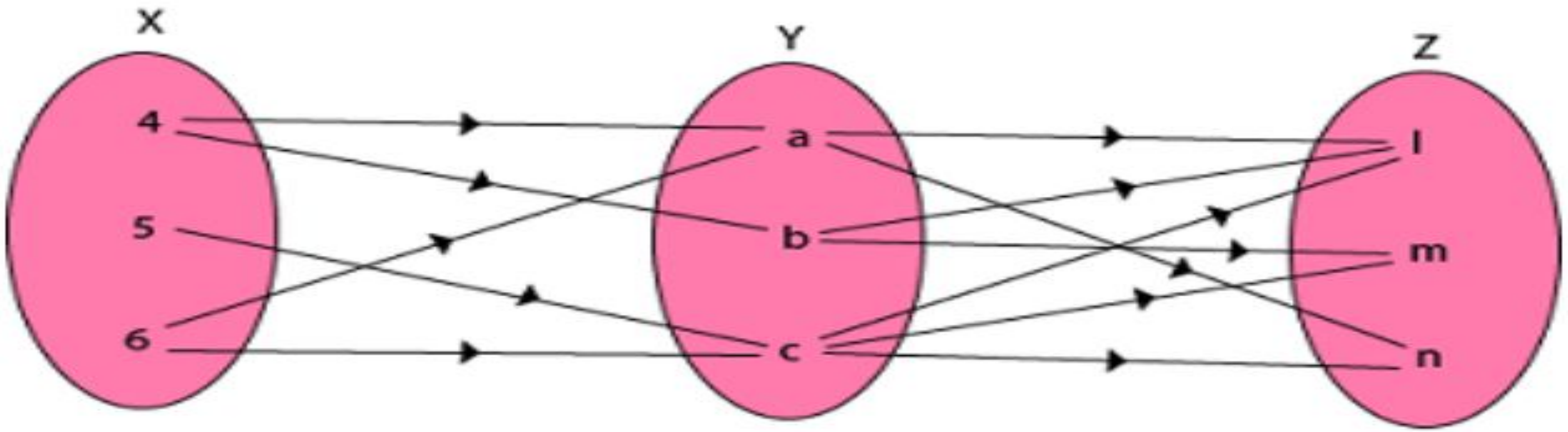


Fig : R_1 o R_2

R_1 o R_2 = {(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)}

(ii) The composition relation $R_1 o R_1^{-1}$ as shown in fig:

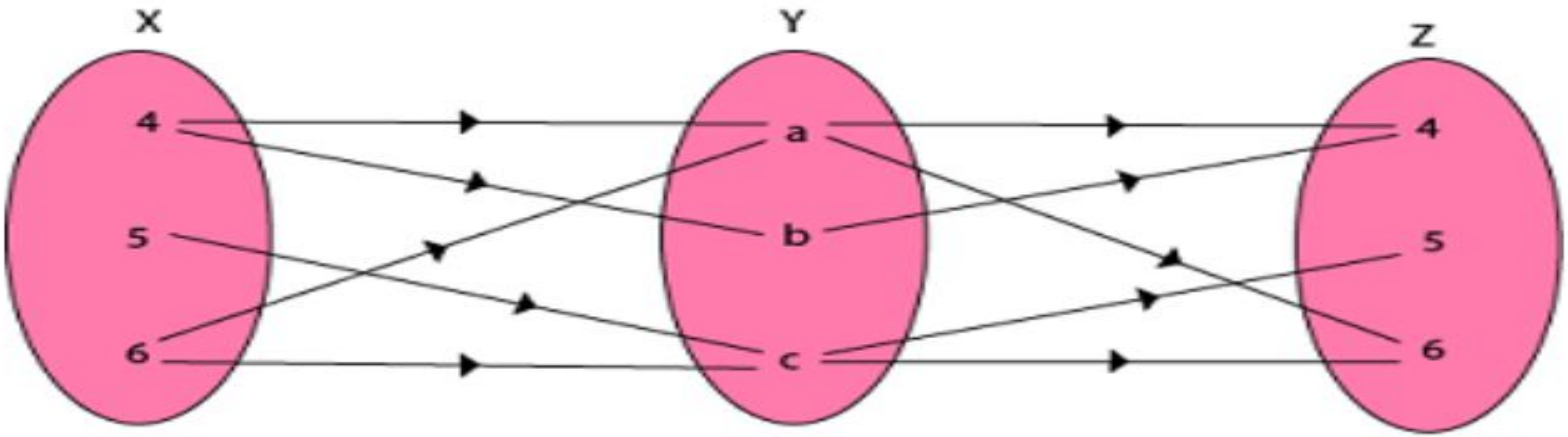


Fig : R_1 o R_1^{-1}

R_1o R_1^{-1} = {(4, 4), (5, 5), (5, 6), (6, 4), (6, 5), (4, 6), (6, 6)}

Let $A=\{1,2,3\}$, $B=\{p,q,r\}$, $C=\{x,y,z\}$

$R=\{(1,p) , (1,r) , (2,p) , (2,q) \}$

$S=\{(p,y) , (q,x) , (q,y) (r,z) \}$

Compute:- RoS

Let $A=\{1,2,3\}$ $B=\{a,b,c\}$ and $C=\{x,y,z\}$

$R=\{(1,b) , (2,a) , (2,c)\}$

$S=\{(a,y) , (b,x) , (c,y) , (c,z)\}$

Compute:- RoS

Given:-

R={(1,2) , (3,4) , (2,2)}

S={(4,2) , (2,5) , (3,1) , (1,3)}

Find:- RoS , SoR , Ro(SoR) , RoR, SoS

Let $A=\{1,2,3,4\}$

$R=\{(1,2)\ ,\ (1,1)\ ,\ (1,3)\ ,\ (2,4)\ ,\ (3,2)\}$ and

$S=\{(1,4)\ ,\ (1,3)\ ,\ (2,3)\ ,\ (3,1)\ ,\ (4,1)\}$

Find:- SoR, RoS, SoS

$A=\{1,2,3,4\}$, $B=\{a,b,c,d\}$, $C=\{x,y,z\}$

$R=\{(1,a) , (2,d) , (3,a) , (3,d)\}$

$S=\{(b,x) , (b,z) , (c,y) , (d,z)\}$

Find:- SoR, RoR, RoS

Closure of Relations

1. Reflexive Closure

The reflexive closure $R^{(r)}$ of a relation R is the smallest reflexive relation that contains R as a subset.

Let A be a non-empty set. Let R be a relation defined on A, then the reflexive closure of R is given by

$$R^{(r)} = R \cup \Delta_A \qquad \text{where } \Delta_A = \{(a, a) ; \forall\, a \in A\}$$

2. Symmetric Closure

The symmetric closure $R^{(s)}$ is the smallest symmetric relation that contains R as a subset.

Let A be a non-empty set. Let R be a relation defined on set A, then the symmetric closure of R is given by

$$R^{(s)} = R \cup R^{-1}$$

3. Transitive Closure

The transitive closure $R^{(t)}$ is the smallest transitive relation that contains R as a subset.

$$R^{\infty} = R \cup R^2 \cup R^3 \cup \ldots\ldots \cup R^n.$$

Let R be a relation on A={1,2,3,4}
R={(1,1) , (1,3) , (2,4) , (3,1) , (3,3) , (4,3)}.

Find reflexive , symmetric and transitive closures.

Solution:

Consider equality relation Δ on A.

$\Delta = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

We can see that R is not reflexive since ordered pairs (2, 2) and (4, 4) are not in R.

$$\text{Reflexive closure} = R \cup \Delta$$
$$= \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (2, 2), (4, 4)\}$$

Consider inverse relation R^{-1}.

$R^{-1} = \{(1, 1), (3, 1), (4, 2), (1, 3), (3, 3), (3, 4)\}$

We can see that R is not symmetric since (4, 2) and (3, 4) are not in R.

$\therefore$ Symmetric closure $= R \cup R^{-1}$

$= \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (4, 2), (3, 4)\}$

and consider $R^2 = R \circ R = \{(1, 1), (1, 3), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$

$R^3 = R^2 \circ R = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$

$R^4 = R^3 \circ R =$ Same as R^3

$\therefore$ Transitive closure $= R \cup R^2 \cup R^3 \cup R^4$

$= \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (2, 3), (4, 1), (2, 1)\}$

Consider the given relation R on the set

$A=\{1,2,3,4\}$

$R=\{(1,1)\ ,\ (1,3)\ ,\ (2,4)\ ,\ (3,1)\ ,\ (3,3)\ ,\ (4,3)\ \}$

Find reflexive and symmetric closure.

Symmetric Closure of $R = R' = R \cup R^{-1}$

$R^{-1} = \{(1, 1), (3, 1), (4, 2), (1, 3), (3, 3), (3, 4)\}$

$R' = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 3), (4, 4)\} \cup \{(1, 1), (3, 1), (4, 2), (1, 3), (3, 3), (3, 4)\}$

$R' = \{(1, 1), (1, 3), (3, 1), (2, 2), (2, 4), (4, 2), (4, 3), (3, 4), (4, 4)\}$

$\therefore$ Symmetric closure $(R) = \{(1, 1), (1, 3), (3, 1), (2, 2), (2, 4), (4, 2), (4, 3), (3, 4), (4, 4)\}$

Transitive Closure: Let R be a relation on a set A with n elements then

Example 19: Consider the relation R on $A = \{1, 2, 3\}$

$R = \{(1, 2), (2, 3), (3, 3)$. Find transitive closure

Solution: Here $|A| = 3$

$\therefore R^2 = \{(1, 3), (2, 3), (3, 3)\}$ and $R^3 = ((1, 3), (2, 3), (3, 3)\}$

Transitive $(R) = R \cup R^2 \cup R^3$

$$= \{(1, 2), (1, 3), (2, 3), (3, 1)\}$$

THANK YOU