UNIT 2

Functions

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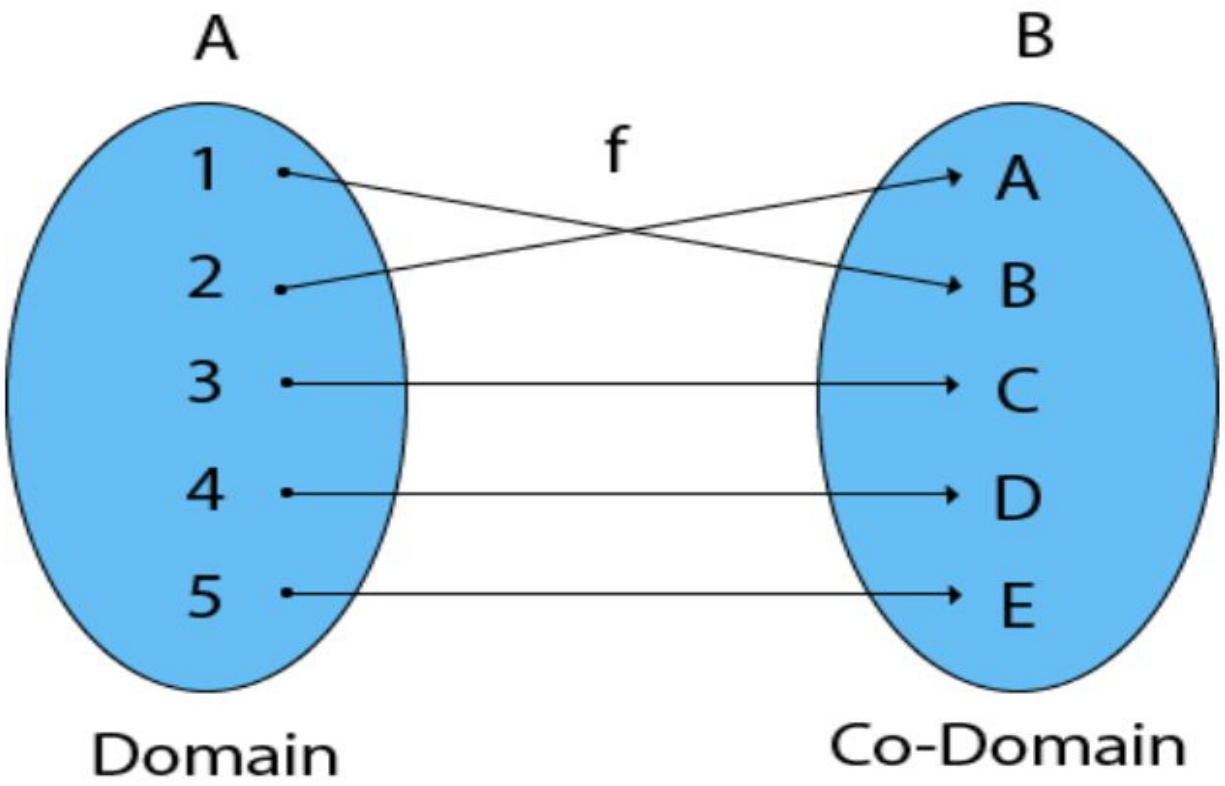
Functions

- Functions Defined on General Sets
- One-to-One and Onto
- Inverse Functions
- Composition of Functions
- Cardinality with Applications to Computability

Functions

It is a mapping in which every element of set A is uniquely associated at the element with set B. The set of A is called Domain of a function and set of B is called Co domain.



Domain, Co-Domain, and Range of a Function:

Domain of a Function: Let f be a function from P to Q. The set P is called the domain of the function f.

Co-Domain of a Function: Let f be a function from P to Q. The set Q is called Co-domain of the function f.

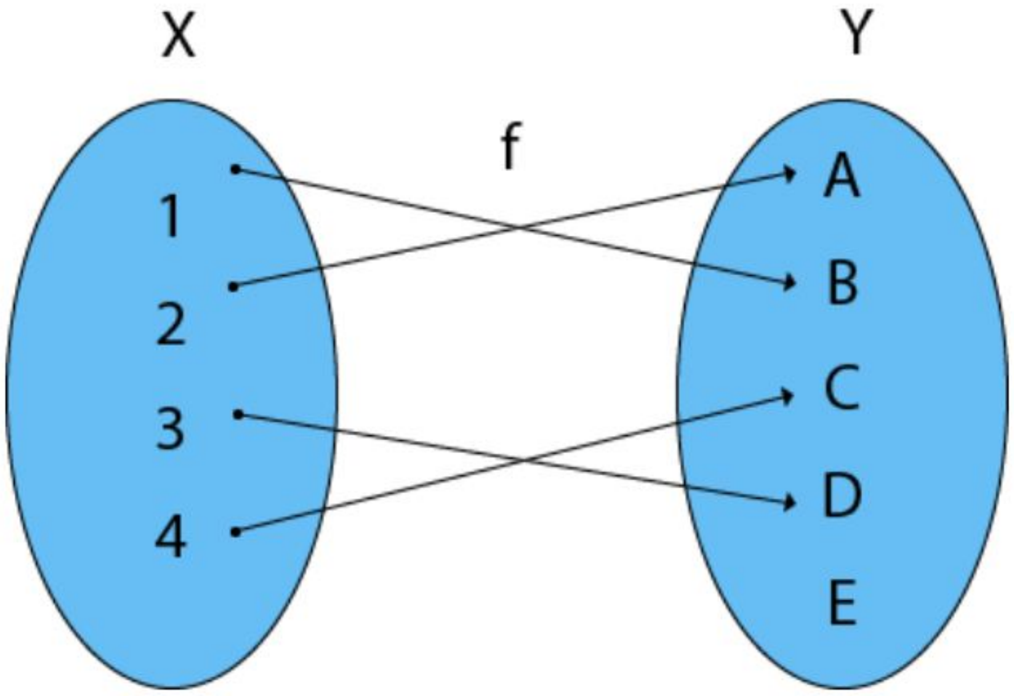
Range of a Function: The range of a function is the set of picture of its domain. In other words, we can say it is a subset of its co-domain. It is denoted as f (domain).

Example: Find the Domain, Co-Domain, and Range of function.

Let x = {1, 2, 3, 4}

y = {a, b, c, d, e}

f = {(1, b), (2, a), (3, d), (4, c)}



Solution:

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Domain of function: {1, 2, 3, 4}
Range of function: {a, b, c, d}
Co-Domain of function: {a, b, c, d, e}
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Functions as a Set

If P and Q are two non-empty sets, then a function f from P to Q is a subset of P x Q, with two important restrictions

1. $\forall$ a $\in$ P, (a, b) $\in$ f for some b $\in$ Q
2. If (a, b) $\in$ f and (a, c) $\in$ f then b = c.

Note1: There may be some elements of the Q which are not related to any element of set P.

2. Every element of P must be related with at least one element of Q.

Example1: If a set A has n elements, how many functions are there from A to A?

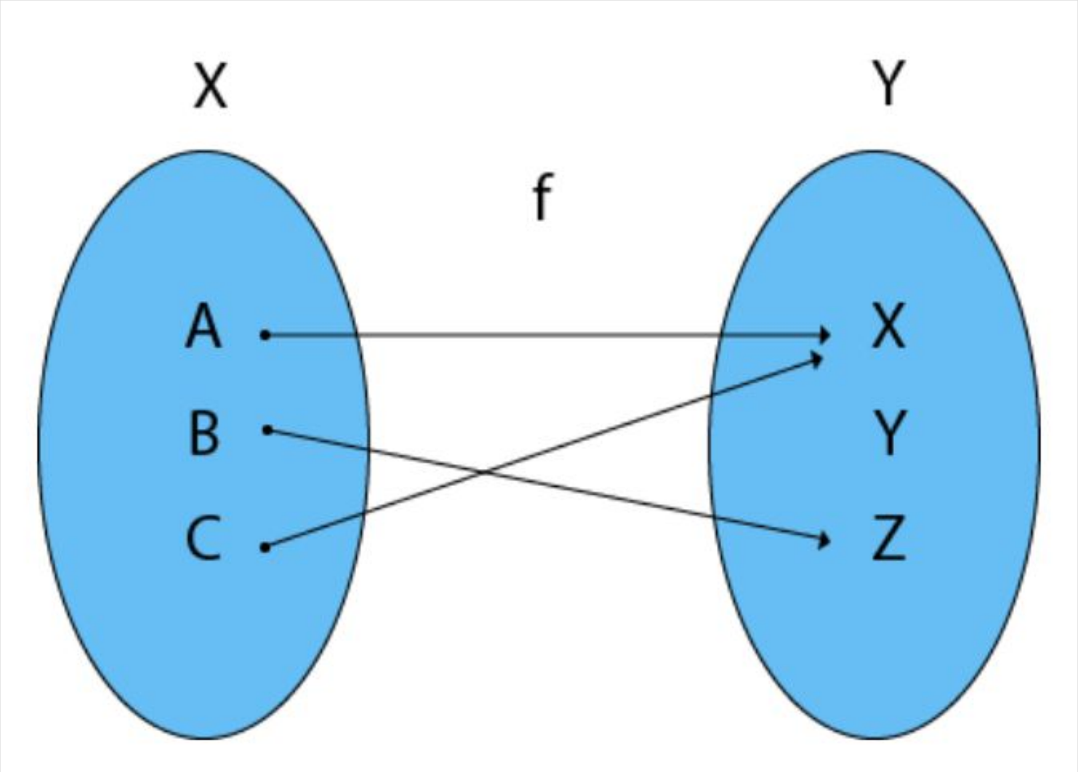
Solution: If a set A has n elements, then there are n^n functions from A to A.

Representation of a Function

The two sets P and Q are represented by two circles. The function f: P → Q is represented by a collection of arrows joining the points which represent the elements of P and corresponds elements of Q

Example1:

> Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and $f: X \to Y$ such that
>
> $f = \{(a, x), (b, z), (c, x)\}$



Example2: Let X = {x, y, z, k} and Y = {1, 2, 3, 4}. Determine which of the following functions. Give reasons if it is not. Find range if it is a function.

a. f = {(x, 1), (y, 2), (z, 3), (k, 4)

b. g = {(x, 1), (y, 1), (k, 4)

c. h = {(x, 1), (x, 2), (x, 3), (x, 4)

d. l = {(x, 1), (y, 1), (z, 1), (k, 1)}

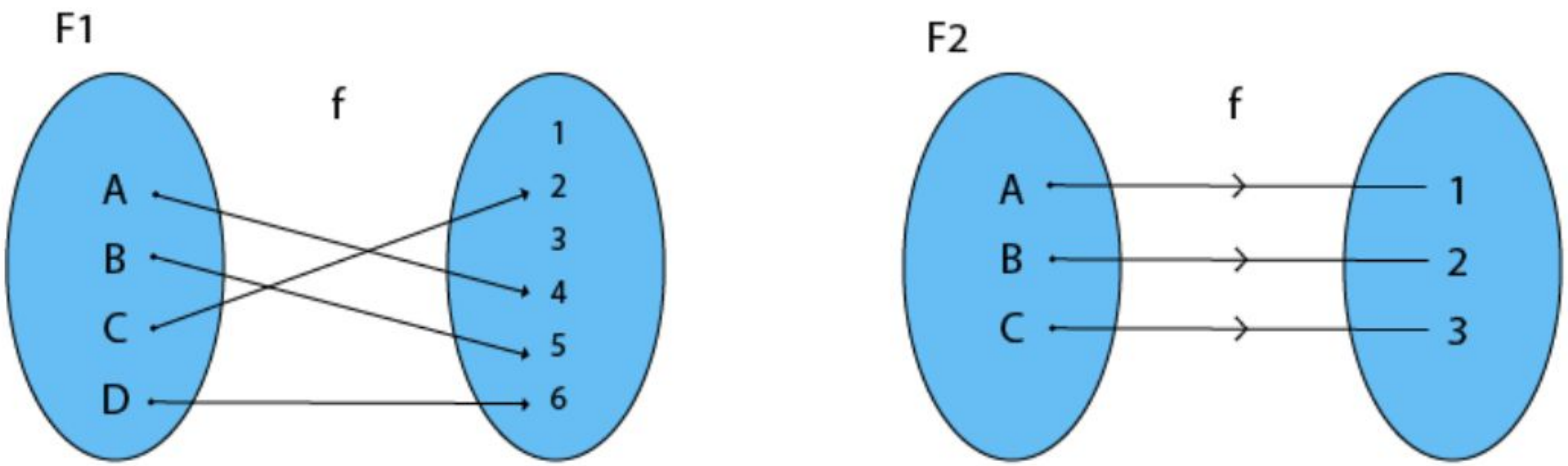
e. d = {(x, 1), (y, 2), (y, 3), (z, 4), (z, 4)}.

Solution:

1. It is a function. Range (f) = {1, 2, 3, 4}
2. It is not a function because every element of X does not relate with some element of Y i.e., Z is not related with any element of Y.
3. h is not a function because h (x) = {1, 2, 3, 4} i.e., element x has more than one image in set Y.
4. d is not a function because d (y) = {2, 3} i.e., element y has more than image in set Y.

Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

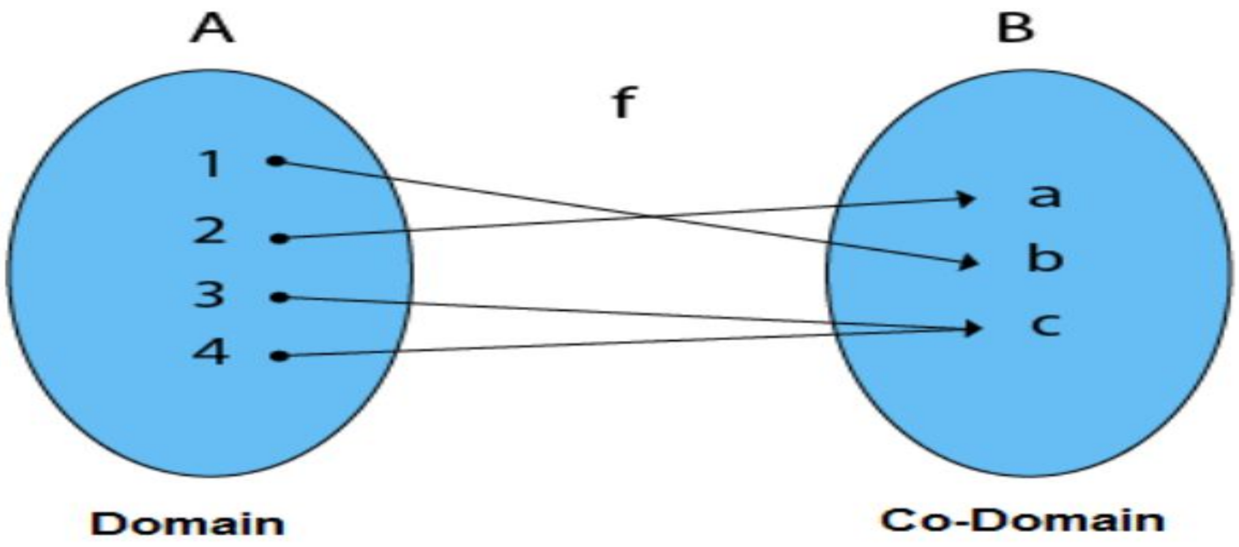


F1 and F2 show one to one Function

2. Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

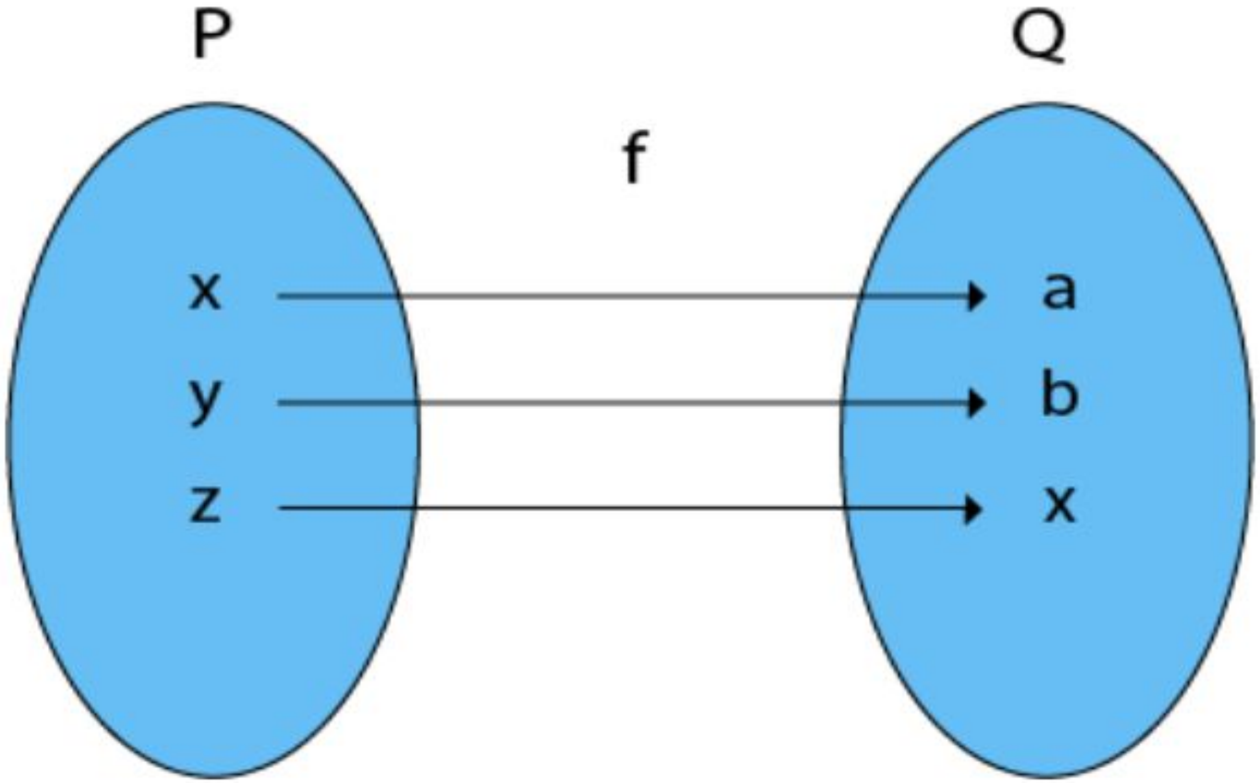
Example: Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



Example:

> Consider P = {x, y, z}
>
> Q = {a, b, c}
>
> and f: $P \rightarrow Q$ such that
>
> f = {(x, a), (y, b), (z, c)}

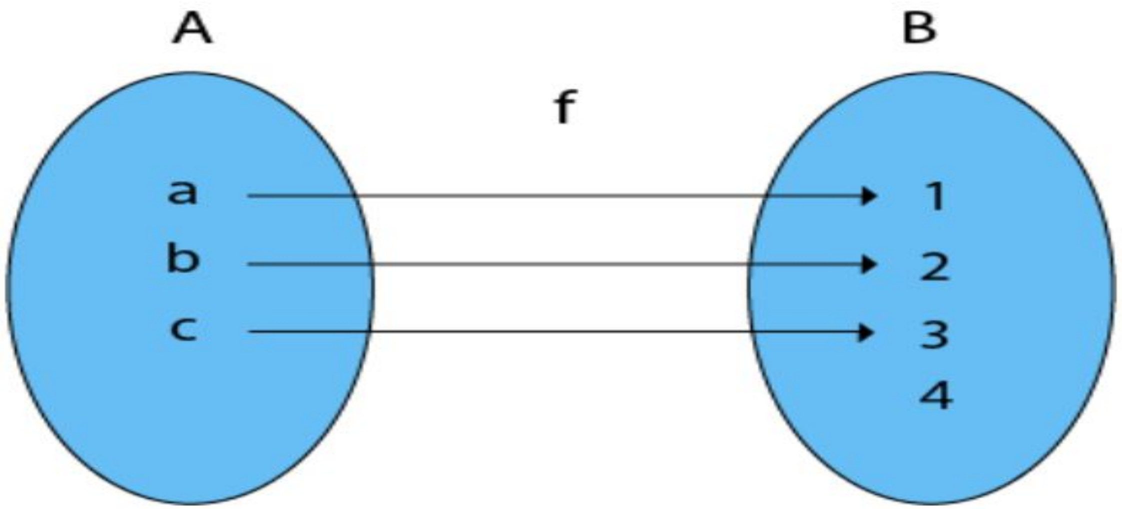
The f is a one-to-one function and also it is onto. So it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

Example:

> Consider, A = {a, b, c}
>
> B = {1, 2, 3, 4} and f: A → B such that
>
> f = {(a, 1), (b, 2), (c, 3)}
>
> In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1, 2, 3, 4}

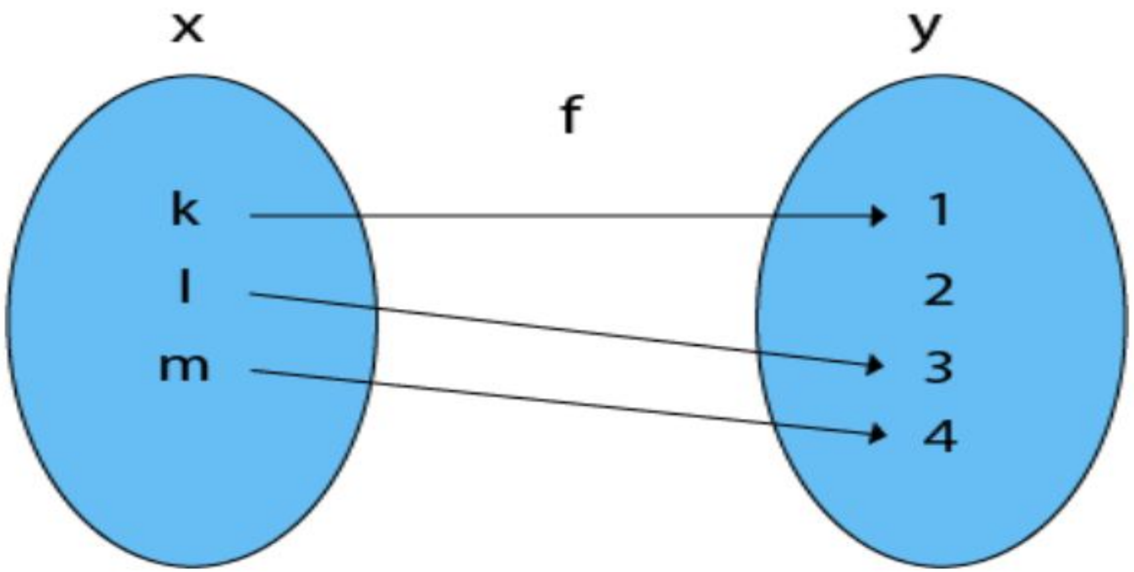
Therefore, it is an into function



5. One-One Into Functions: Let f: X → Y. The function f is called one-one into function if different elements of X have different unique images of Y.

> Consider, X = {k, l, m}
>
> Y = {1, 2, 3, 4} and f: X → Y such that
>
> f = {(k, 1), (l, 3), (m, 4)}

The function f is a one-one into function

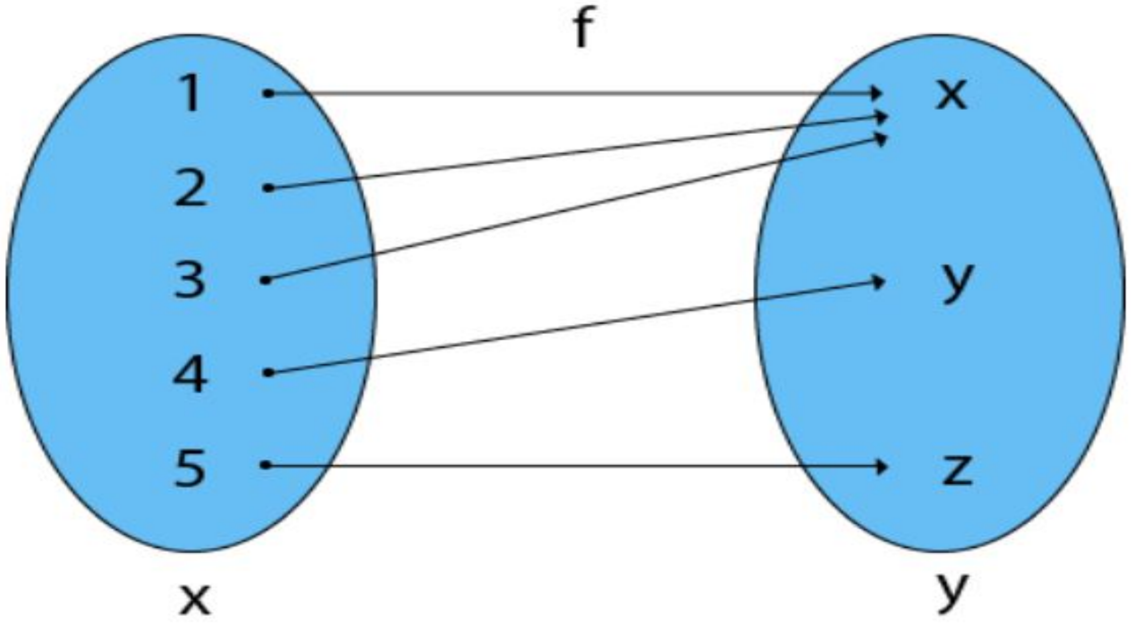


6. Many-One Functions: Let f: X → Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

Example:

> Consider X = {1, 2, 3, 4, 5}
> Y = {x, y, z} and f: X → Y such that
> f = {(1, x), (2, x), (3, x), (4, y), (5, z)}

The function f is a many-one function

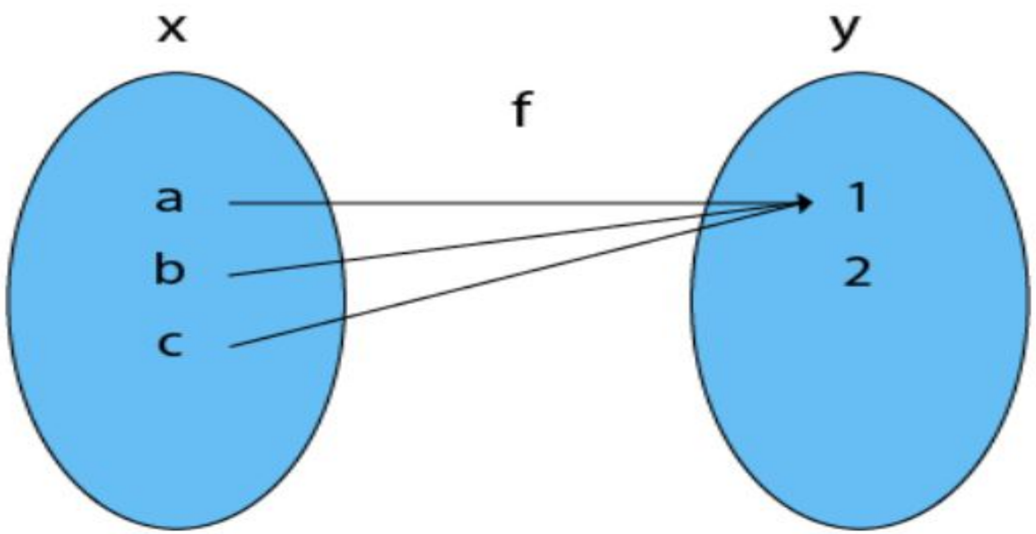


7. Many-One Into Functions: Let f: X → Y. The function f is called the many-one function if and only if is both many one and into function.

Example:

> Consider X = {a, b, c}
> Y = {1, 2} and f: X → Y such that
> f = {(a, 1), (b, 1), (c, 1)}

As the function f is a many-one and into, so it is a many-one into function.

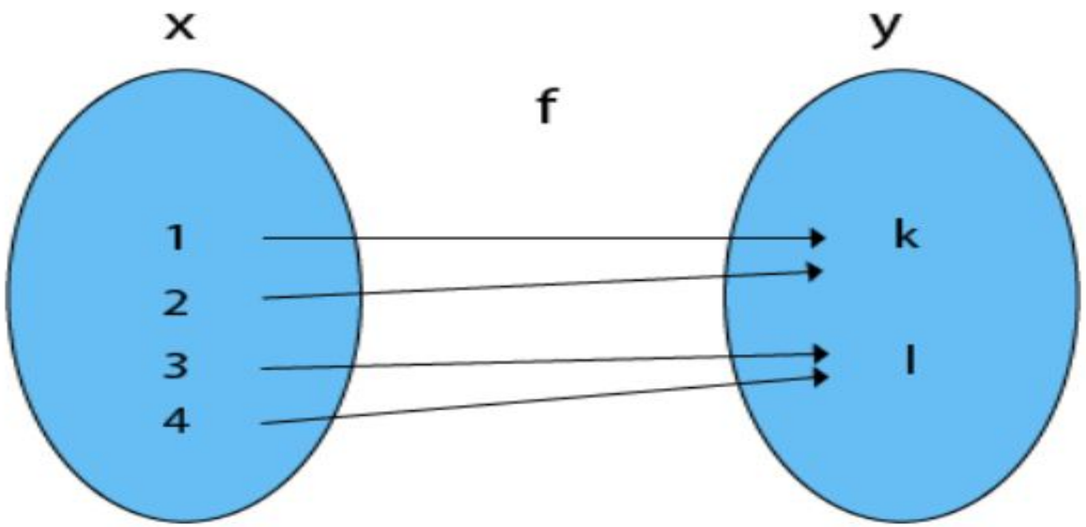


8. Many-One Onto Functions: Let f: X → Y. The function f is called many-one onto function if and only if is both many one and onto.

Example:

> Consider X = {1, 2, 3, 4}
> Y = {k, l} and f: X → Y such that
> f = {(1, k), (2, k), (3, l), (4, l)}

The function f is a many-one (as the two elements have the same image in Y) and it is onto (as every element of Y is the image of some element X). So, it is many-one onto function



Identity Functions

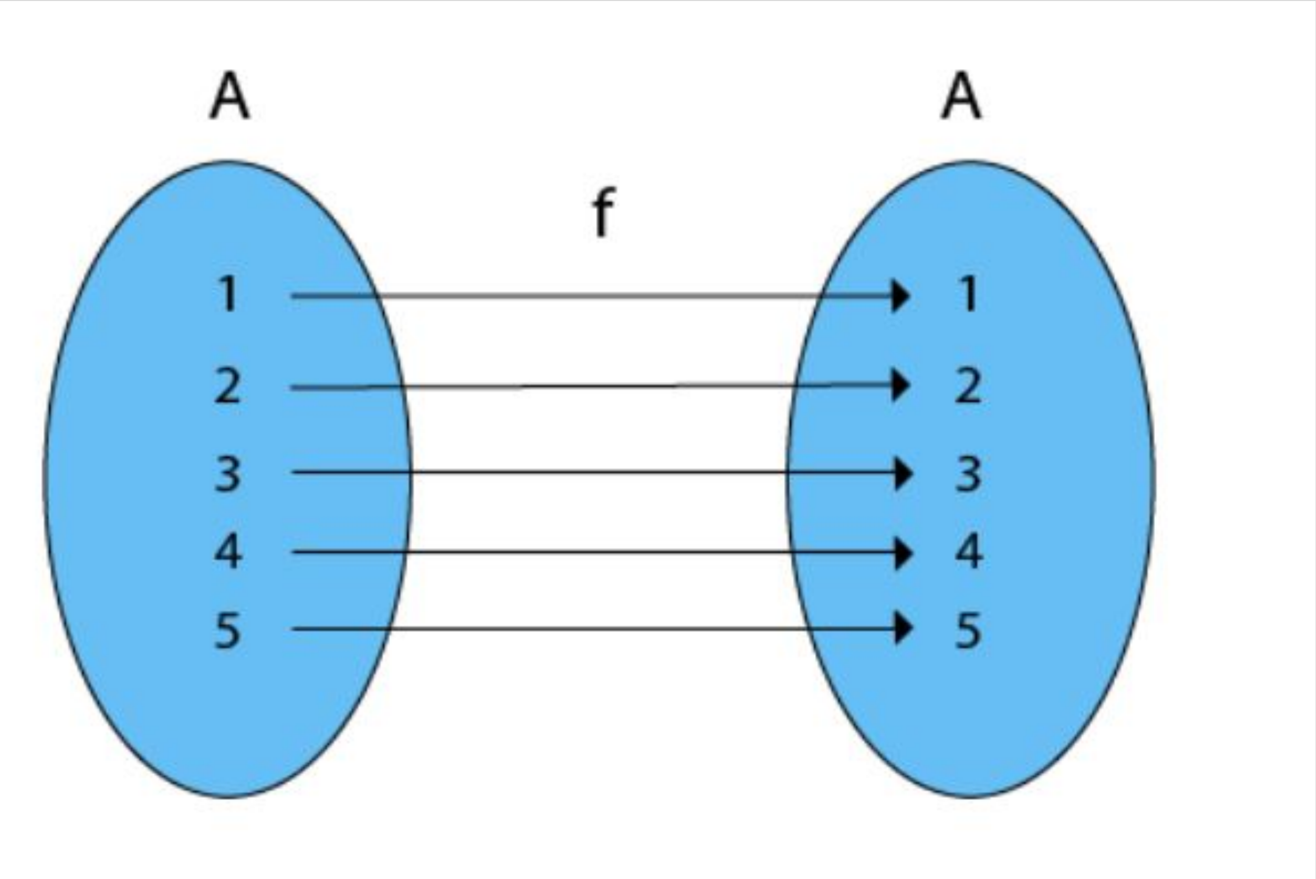
The function f is called the identity function if each element of set A has an image on itself i.e. $f(a) = a \ \forall \ a \in A$.

It is denoted by I.

Example:

> Consider, $A = \{1, 2, 3, 4, 5\}$ and $f: A \rightarrow A$ such that
>
> $f = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.

The function f is an identity function as each element of A is mapped onto itself. The function f is a one-one and onto


A
A
f
1
2
3
4
5
1
2
3
4
5

Invertible (Inverse) Functions

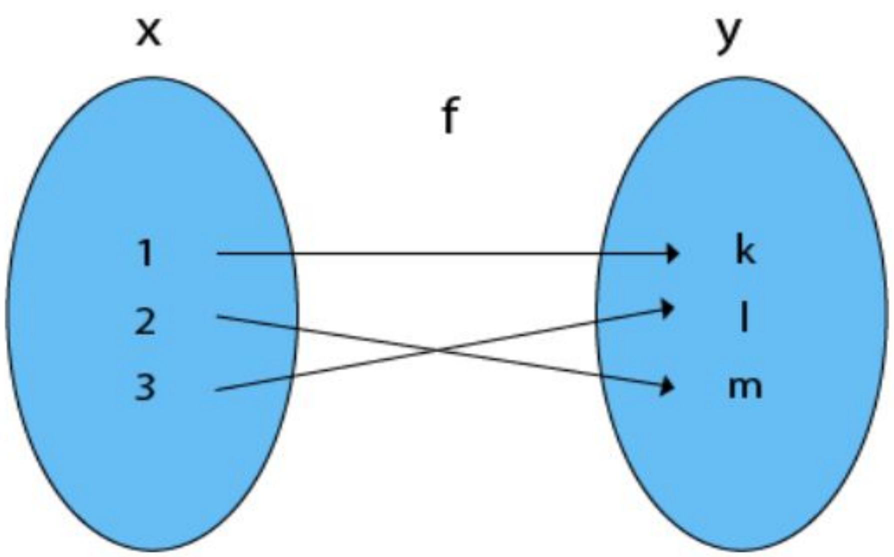
A function $f: X \rightarrow Y$ is invertible if and only if it is a bijective function.

Consider the bijective (one to one onto) function $f: X \rightarrow Y$. As f is a one to one, therefore, each element of X corresponds to a distinct element of Y. As f is onto, there is no element of Y which is not the image of any element of X, i.e., range = co-domain Y.

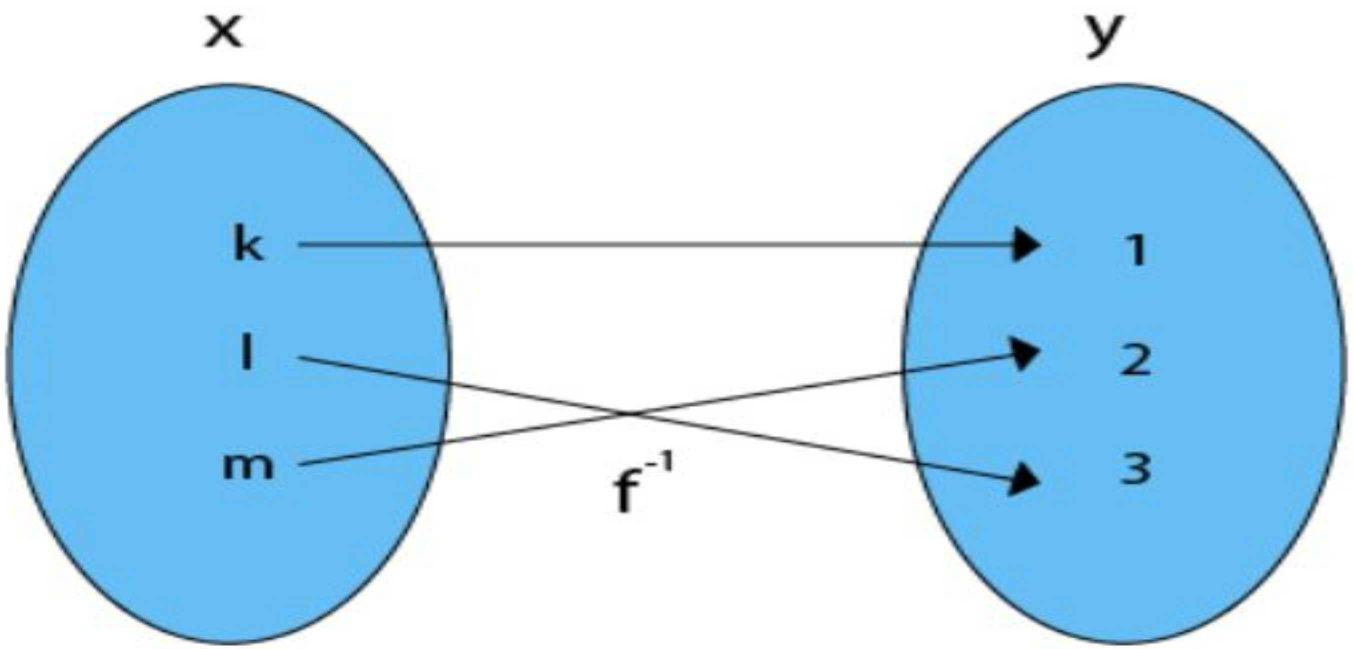
The inverse function for f exists if f^{-1} is a function from Y to X.

Example:

> Consider, X = {1, 2, 3}
>
> Y = {k, l, m} and f: X→Y such that
>
> f = {(1, k), (2, m), (3, l)}



The inverse function of f is shown in fig:



$A = \{1,2,3,4\}$
$B = \{1,2,3,4\}$

$f = \{ (1,1) , (2,3) , (3,4) , (4,2) \}$

f inverse

Compositions of Functions

Consider functions, $f: A \rightarrow B$ and $g: B \rightarrow C$. The composition of f with g is a function from A into C defined by $(gof)(x) = g[f(x)]$ and is defined by gof.

To find the composition of f and g, first find the image of x under f and then find the image of f (x) under g.

Example

Let $f(x) = x + 2$ and $g(x) = 2x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution

$(f \circ g)(x) = f(g(x)) = f(2x) = 2x+2$

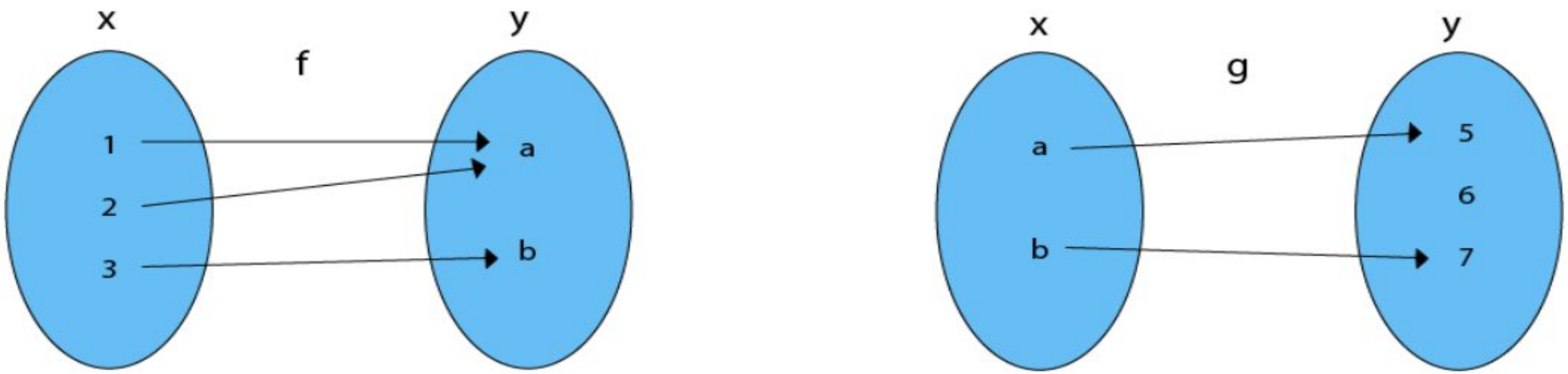
$(g \circ f)(x) = g(f(x)) = g(x+2) = 2(x+2) = 2x+4$

Hence, $(f \circ g)(x) \neq (g \circ f)(x)$

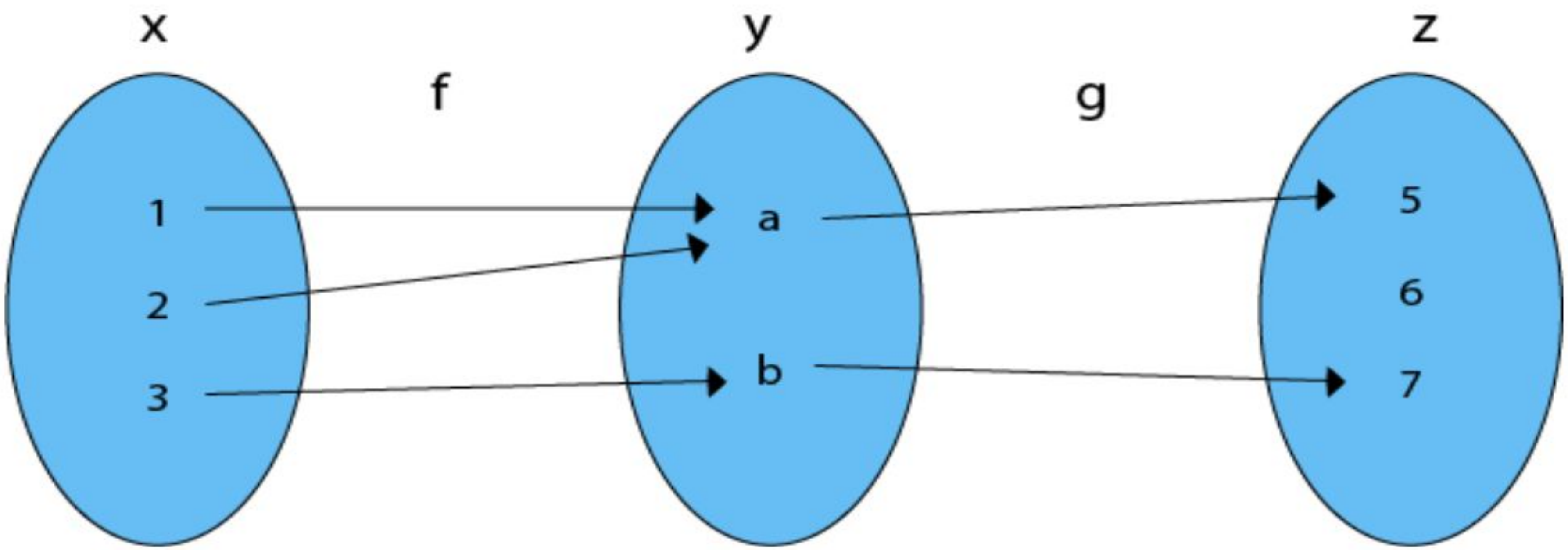
Example1:

> Let X = {1, 2, 3}
>
> Y = {a, b}
>
> Z = {5, 6, 7}.

Consider the function f = {(1, a), (2, a), (3, b)} and g = {(a, 5), (b, 7)} as in figure. Find the composition of gof.



Solution: The composition function gof is shown in fig:

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(gof) (1) = g [f (1)] = g (a) = 5, (gof) (2) = g [f (2)] = g (a) = 5
(gof) (3) = g [f (3)] = g (b) = 7.
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Example2: Consider f, g and h, all functions on the integers, by $f(n) = n^2$, $g(n) = n + 1$ and $h(n) = n - 1$.

Determine **(i)** hofog **(ii)** gofoh **(iii)** fogoh.

Solution:

(i) hofog (n) = n + 1,

hofog (n + 1) = $(n+1)^2$

h $[(n+1)^2]$ = $(n+1)^2 - 1 = n^2 + 1 + 2n - 1 = n^2 + 2n$.

(ii) gofoh (n) = n - 1, gof (n - 1) = $(n-1)^2$

g $[(n-1)^2]$ = $(n-1)^2 + 1 = n^2 + 1 - 2n + 1 = n^2 - 2n + 2$.

(iii) fogoh (n) = n - 1

fog (n - 1) = (n - 1) + 1

f (n) = n^2.

Let $A = B = C = R$ and let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be defined by $f(a) = a + 1$, $g(b) = b^2 + 2$ and $h(c) = 7c - 2$. Find (a) $(g \circ f)(-2)$ (b) $(f \circ g)(-2)$ (c) $g \circ f(a)$ (d) $f \circ f(a)$ (e) $(g \circ h) \circ f(3)$ (f) $g \circ (h \circ f)(3)$

Solution:

(a) $(g \circ f)(-2) = g(f(-2)) = g((-2) + 1) = g(-1) = (-1)^2 + 2 = 3$

(b) $(f \circ g)(-2) = f(g(-2)) = f((-2)^2 + 2) = f(6) = 6 + 1 = 7$

(c) $g \circ f(a) = g(f(a)) = g(a + 1) = (a + 1)^2 + 2 = a^2 + 2a + 3$

(d) $f \circ f(a) = f(f(a)) = f(a + 1) = (a + 1) + 1 = a + 2$

(e) $(g \circ h) \circ f(3) = (g \circ h)(f(3)) = (g \circ h)(3 + 1) = (g \circ h)(4) = g(h(4))$
$= g(7(4) - 2) = g(28 - 2) = g(26) = (26)^2 + 2 = 678$

(f) $g \circ (h \circ f)(3) = g \circ (h(f(3))) = g((h(3 + 1))) = g(h(4)) = g(26) = (26)^2 + 2 = 678$

We can observe from e and f that $(g \circ h) \circ f(3) = g \circ (h \circ f)(3)$

i.e. **Composition is associative.**

Note:

- If f and g are one-to-one, then the function (gof) (gof) is also one-to-one.
- If f and g are onto then the function (gof) (gof) is also onto.
- Composition consistently holds associative property but does not hold commutative property.

THANK YOU