ADVANCED COUNTING TECHNIQUES, RECURSION

PIGEON–HOLE PRINCIPLE

Statement

If '$n = m + 1$' pigeons are assigned 'm' pigeon–holes and $m < n$, then atleast one of pigeon–holes will contain two or more pigeons.

1. **Show that if 7 numbers from 1 to 12 are selected then atleast two of them will upto 13.**

Solution :

The given set of integers = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

The disjoint subsets of the given set of 2 elements, such that their sum is 13 are,

{1, 12}, {2, 11}, {3, 10}, {4, 9}, {5, 8}, {6, 7}

No. of pigeons = no. of integers selected from given set

$\therefore \quad n = 7$

No. of pigeon–holes = no. of subsets fromed satisfying the given condition

$\therefore \quad m = 6$

$\therefore \quad m < n$, by pigeon–hole principle, atleast two of the selected numbers lie in the same subset and hence add up to 13.

2. **If 13 people are assembled in a room, then atleast two of them will have their birthday in the same month.**

Solution

These are 12 months in a year.

No. of pigeons = no. of people in room

$\therefore \quad n = 13$

No. of pigeon–holes = no. of months in a year.

$\therefore \quad m = 12$

By pigeon–hole principle, atleast two people would have their birthday in the same month.

3. **Show that in a room of people, shaking hands, there are atleast two people who have shaken hands the same no. of time.**

Solution :

Let these be 'x' people in a room.

Each person will shake hands 0, 1, 2, ..., $(x - 1)$ times.

i.e. total handshakes / person = x

Now '0' and $(n - 1)$ handshakes are simultaneously not possible as one cannot shakehand no one and everyone.

$\therefore$ No. of handshakes / person = $(x - 1)$

$m < n$, $\therefore$ by pigeon–hole principle, atleast two people are there who have shaken hands same number of times.

4. Show that in any set of 11 integers, there are atleast two integers whose difference is divisible by 10.

Solution :

Any integer will have one of the following digits in the Unit's place {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

No. of pigeons = No. of integers selected

$\therefore \quad n = 11$

No. of pigeon–holes = No. of possible digits in the Unit's place

$\therefore \quad m = 10$

$m < n$, by pigeon–hole principle, atleast two integers will end with same digit and hence their difference is divisible by 10.

5. Show that if 8 positive integers are chosen, then atleast two of them will have same remainder when divided by 7.

Solution :

When any positive integer is divided by 7, the possible remainder are,

{0, 1, 2, 3, 4, 5, 6}

No. of pigeons = No. of positive integers selected

$\therefore \quad n = 8$

No. of pigeon–holes = no. of remainders possible when +ve integer is divided by 7.

$\therefore \quad m = 7$

$m < n$, by pigeon–hole principle, these are atleast two positive integers which have same remainder when divided by 7.

PROBLEMS FOR PRACTICE

1. Show that 5 numbers from 1 to 8 are chosen, the atleast 2 of them will add upto 9.
2. A student has offered six subjects and the college does not work on weekends. Show that the student must study atleast two subjects on one of the days.
3. Consider a regular hexagon of length one unit. Show that if 7 points are selected inside the hexagon, then there are atleast two points which are not more than one unit apart.
4. Consider an equilateral Δ of side 1 unit. If 5 points are selected inside the triangle, then show that there are atleast two points which are not more than 0.5 unit a part.
5. Show that if 11 integers are selected from the set {1, 2, 3, ..., 20}, then atleast one of them will be the multiple of other.
6. Show that there does not exists 7 lectures each of 30 minutes from 10 am to 1 pm.
7. Show that if 10 points are selected inside an equilateral Δ of length 3 units, these are atleast 2 points which are not more than 1 unit apart.
8. Show that if five points are selected inside a square of side length 1 unit, then there are atleast two points which are at the most $\left(\frac{1}{\sqrt{2}}\right)$ unit apart.

9. Consider six people, where any two of them are either friends or strangers. Show that these are three of them which are mutual friends or mutual strangers.

10. Show that in a gathering of 20 people, these are two people who know the same number of people.

11. Assume these are 3 girls and 5 boys at a party. Show that if these people are lined in a row, then atleast 2 boys will be next to each other.