Sub:- Computational Logic and Discrete Structures

Unit 1
Chapter 1-SET THEORY

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Mathematics can be broadly classified into two categories:

- Continuous Mathematics
- Discrete Mathematics

Continuous Mathematics is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.

Discrete Mathematics, on the other hand, involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

Topics in Discrete Mathematics

Though there cannot be a definite number of branches of Discrete Mathematics, the following topics are almost always covered in any study regarding this matter:

- Sets, Relations and Functions
- Mathematical Logic
- Group theory
- Counting Theory
- Probability
- Mathematical Induction and Recurrence Relations
- Graph Theory
- Trees
- Boolean Algebra

We will discuss each of these concepts in the subsequent chapters of this tutorial.

German mathematician **G. Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.

Set theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.

Set – Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

Set is collection of well defined objects

Note : Well-defined means that it is possible to decide whether a given object belongs to given collection or not.

Kitchen

Kitchen is the most relevant example of sets. Our mother always keeps the kitchen well arranged. The plates are kept separate from bowls and cups. Sets of similar utensils are kept separately.

School Bags

School bags of children is also an example. There are usually divisions in the school bags, where the sets of notebooks and textbooks are kept separately.

Representation of a Set

Sets can be represented in two ways:

- Roster or Tabular Form
- Set Builder Notation

Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1: Set of vowels in English alphabet, A = {a,e,i,o,u}

Example 2: Set of odd numbers less than 10, B = {1,3,5,7,9}

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as $A = \{ x : p(x)\}$

Example 1: The set {a,e,i,o,u} is written as:

$A = \{ x : x \text{ is a vowel in English alphabet}\}$

Your task:-

Write in roster form

1. The set E is even positive integers less than 10

2. The set P is set of positive integers less than 100

Your task:-

Write in set builder form:-

1. The set A is Even positive integers less than 10

Some Important Sets

N: the set of all natural numbers = {1, 2, 3, 4, ...}

Z: the set of all integers = {....., -3, -2, -1, 0, 1, 2, 3, ...}

Z$^+$: the set of all positive integers

Q: the set of all rational numbers

R: the set of all real numbers

W: the set of all whole numbers

Cardinality of a Set

Cardinality of a set S, denoted by $|S|$, is the number of elements of the set. If a set has an infinite number of elements, its cardinality is ∞.

Example: $|\{1, 4, 3,5\}| = 4$, $|\{1, 2, 3,4,5,\ldots\}| = \infty$

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set

A set which contains a definite number of elements is called a finite set.

Example: $S = \{x \mid x \in N \text{ and } 70 > x > 50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example: $S = \{x \mid x \in N \text{ and } x > 10\}$

Subset

A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

Example 1: Let, X = { 1, 2, 3, 4, 5, 6 } and Y = { 1, 2 }. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Example 2: Let, X = {1, 2, 3} and Y = {1, 2, 3}. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Your task:-

Set A= {m, n, o, p, q}

Set B= {k, l, m, n, o, p, q, r}

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

Example: Let, X = {1, 2,3,4,5, 6} and Y = {1, 2}. Here set Y is a proper subset of set X as at least one element is more in set X. Hence, we can write $Y \subset X$.

Universal Set

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example: We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Empty Set or Null Set

An empty set contains no elements. It is denoted by ∅. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example: $\emptyset = \{x \mid x \in N \text{ and } 7 < x < 8\}$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example: $S = \{x \mid x \in N, 7 < x < 9\}$

Equal Set

If two sets contain the same elements they are said to be equal.

Example: If $A = \{1, 2, 6\}$ and $B = \{6, 1, 2\}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example: If A = {1, 2, 6} and B = {16, 17, 22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$

Overlapping Set

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$

Example: Let, $A = \{1, 2, 6\}$ and $B = \{6, 12, 42\}$. There is a common element '6', hence these sets are overlapping sets.

Disjoint Set

If two sets C and D are disjoint sets as they do not have even one element in common. Therefore, $n(A \cup B) = n(A) + n(B)$

Example: Let, $A = \{1, 2, 6\}$ and $B = \{7, 9, 14\}$, there is no common element, hence these sets are overlapping sets.

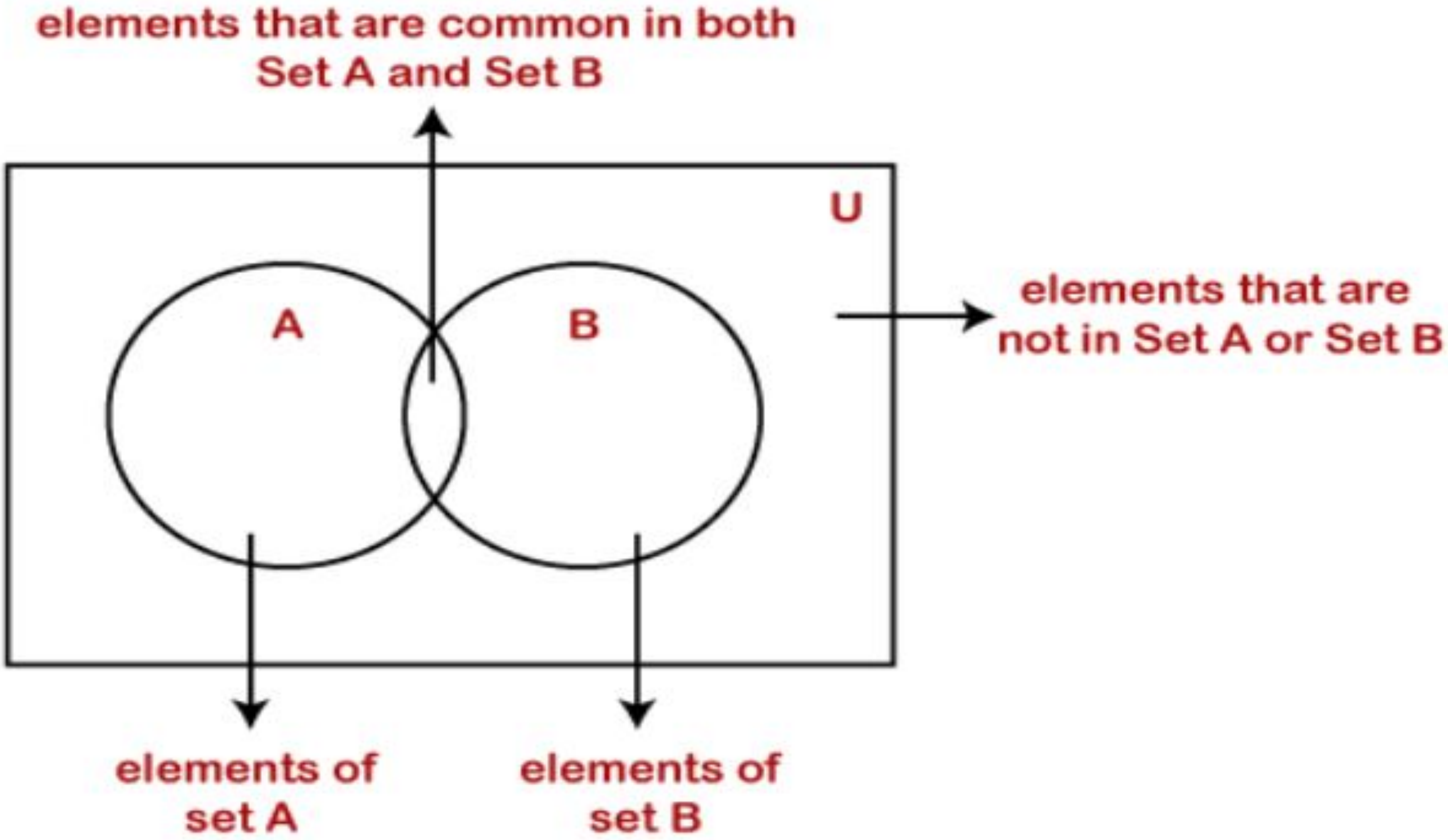
Your task:-

Identify the type of set:-

1. $S = \{x \mid x \in N, 7 < x < 9\}$
2. $A = \{13, 12, 10\}$ and $B = \{12, 14, 16\}$
3. $A = \{11, 12, 16\}$ and $B = \{7, 9, 14\}$

Set Operations

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.


elements that are common in both
Set A and Set B
U
A
B
elements that are
not in Set A or Set B
elements of
set A
elements of
set B

Set Union

The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$.

Example: If $A = \{10, 11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $A \cup B = \{10, 11, 12, 13, 14, 15\}$. (The common element occurs only once)

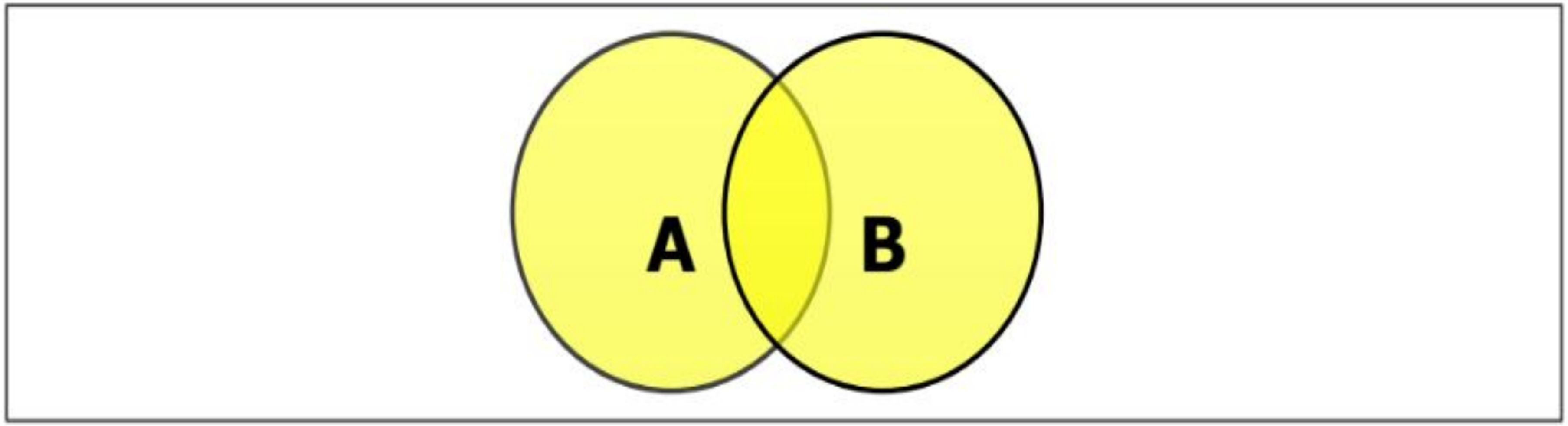


Figure: Venn Diagram of $A \cup B$

$A = \{ x, y , z \}$

$B = \{ 2, 5 \}$

$A \cup B = ?$

Set Intersection

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$.

Example: If $A = \{11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $A \cap B = \{13\}$.

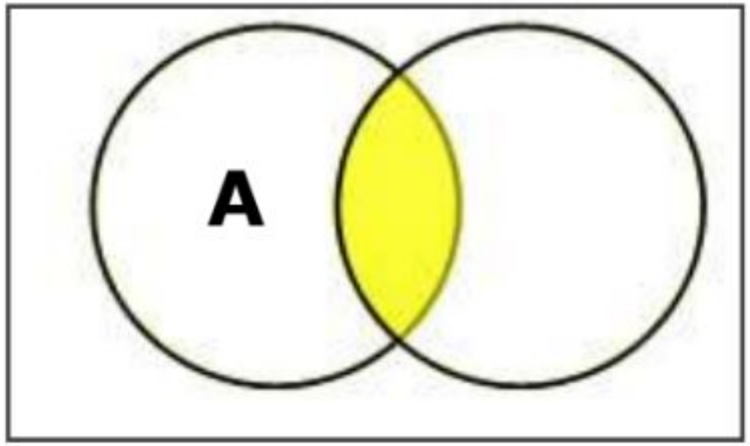


Figure: Venn Diagram of $A \cap B$

A = {1, 2, 3},

B = {2, 4, 5}

then

Find A n B.

Set Difference/ Relative Complement

The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence, $A-B = \{x \mid x \in A \text{ AND } x \notin B\}$.

Example: If $A = \{10, 11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $(A-B) = \{10, 11, 12\}$ and $(B-A) = \{14,15\}$. Here, we can see $(A-B) \neq (B-A)$

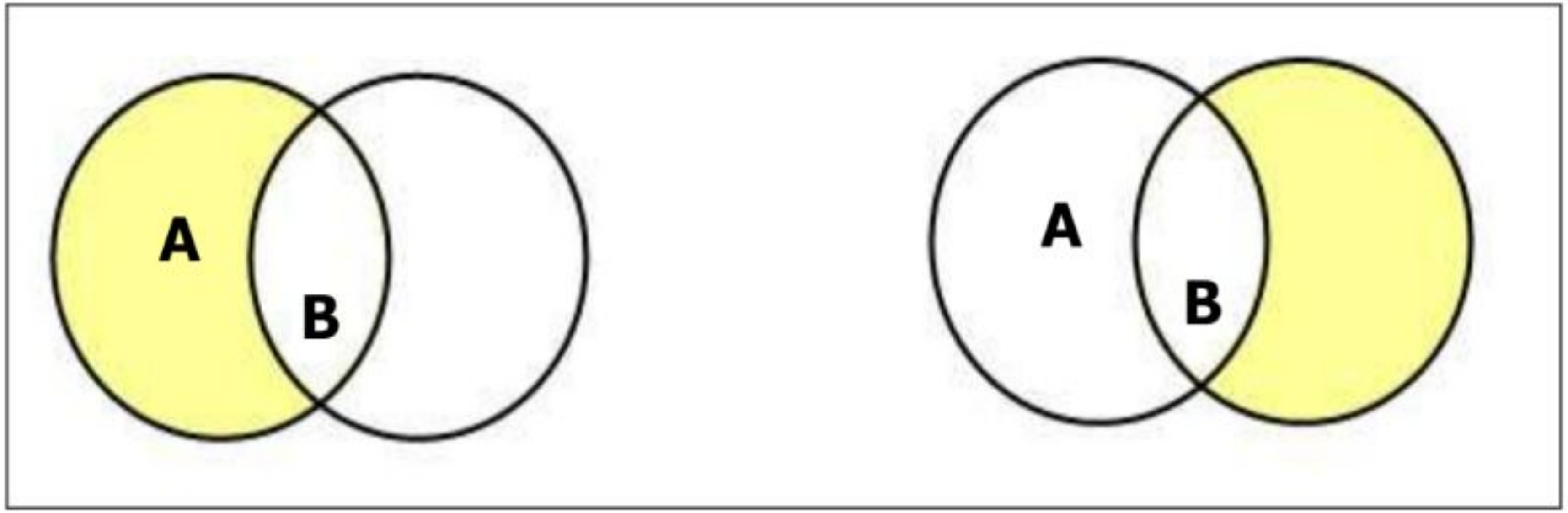


Figure: Venn Diagram of A – B and B – A

$A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

Find,

$A - B$

$B - A$

Complement of a Set

The complement of a set A (denoted by A’) is the set of elements which are not in set A. Hence, $A' = \{x \mid x \notin A\}$.

More specifically, $A' = (U-A)$ where U is a universal set which contains all objects.

Example: If $A = \{x \mid x \text{ belongs to set of odd integers}\}$ then $A' = \{y \mid y \text{ does not belong to set of odd integers}\}$

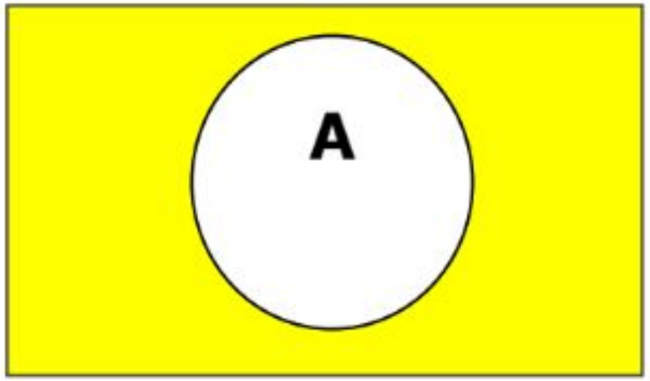


Figure: Venn Diagram of A'

A = {a, e, i, o, u} (where the universal set is the set of letters of the English alphabet).

Then

A’ = ?

Let A be the set of positive integers greater than 10 (with universal set the set of all positive
integers).
Then A’ = ?

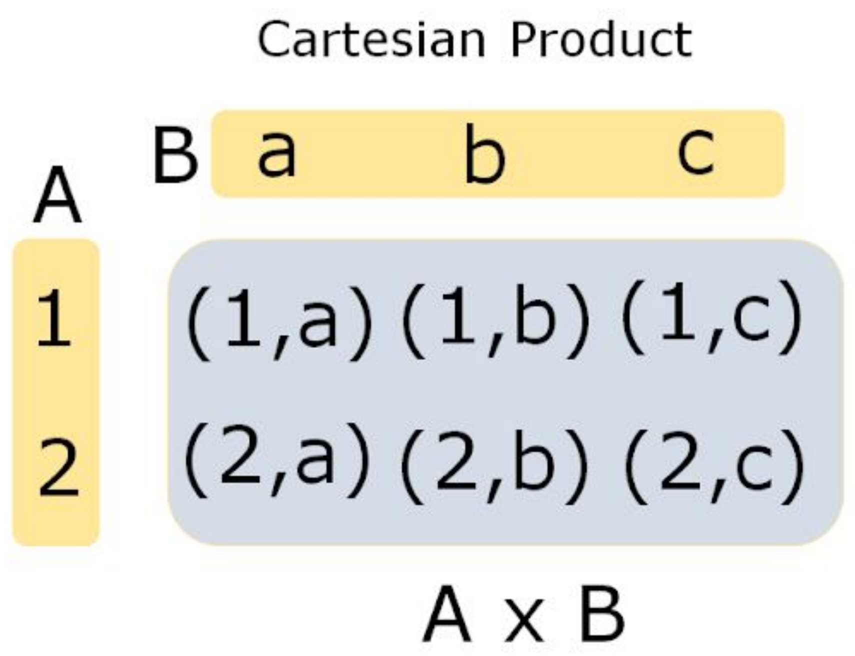
Cartesian Product / Cross Product

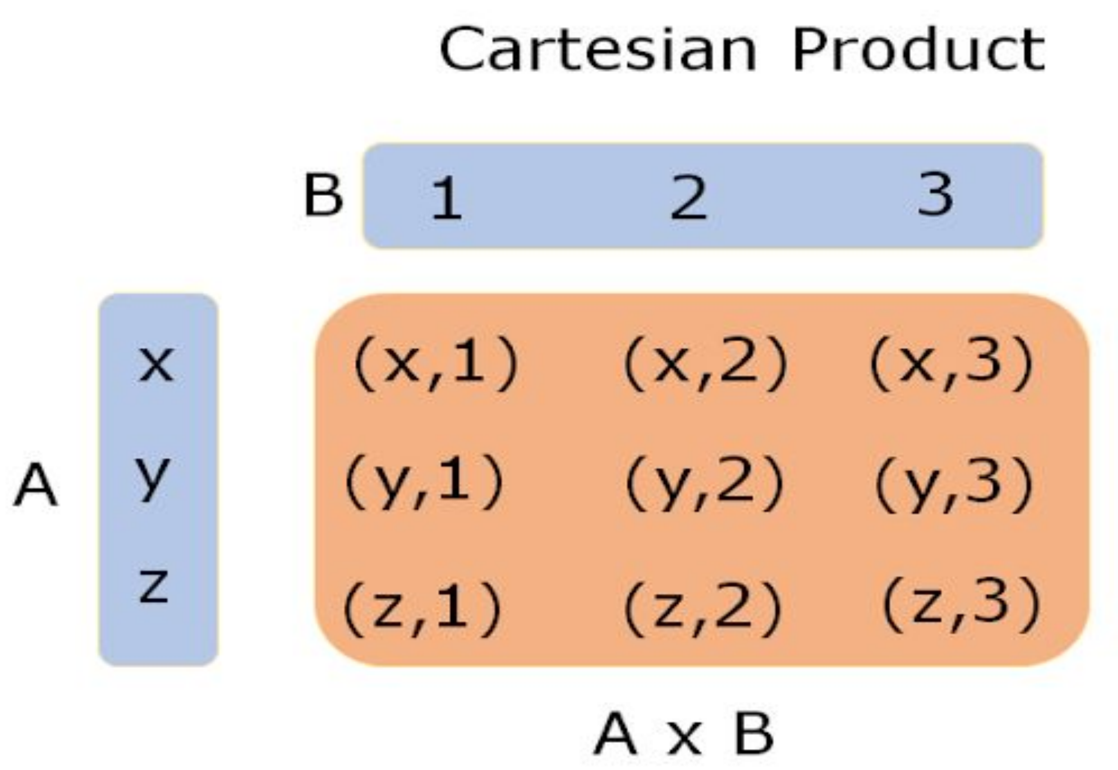
The Cartesian product of n number of sets $A_1, A_2.....A_n$, defined as $A_1 \times A_2 \times..... \times A_n$, are the ordered pair $(x_1,x_2,....x_n)$ where $x_1 \in A_1$, $x_2 \in A_2$, ... $x_n \in A_n$

Example: If we take two sets A= {a, b} and B= {1, 2},

The Cartesian product of A and B is written as: A×B= {(a, 1), (a, 2), (b, 1), (b, 2)}

The Cartesian product of B and A is written as: B×A= {(1, a), (1, b), (2, a), (2, b)}


Cartesian Product
B a b c
A
1
2
(1,a) (1,b) (1,c)
(2,a) (2,b) (2,c)
A x B


Cartesian Product
B 1 2 3
A
x
y
z
(x,1) (x,2) (x,3)
(y,1) (y,2) (y,3)
(z,1) (z,2) (z,3)
A x B

Cartesian Product:

A={1,2,3}

B={a,b,c}
Find:-

A x B=

B x A=

A x A=

A x A x A=

B x B= ?

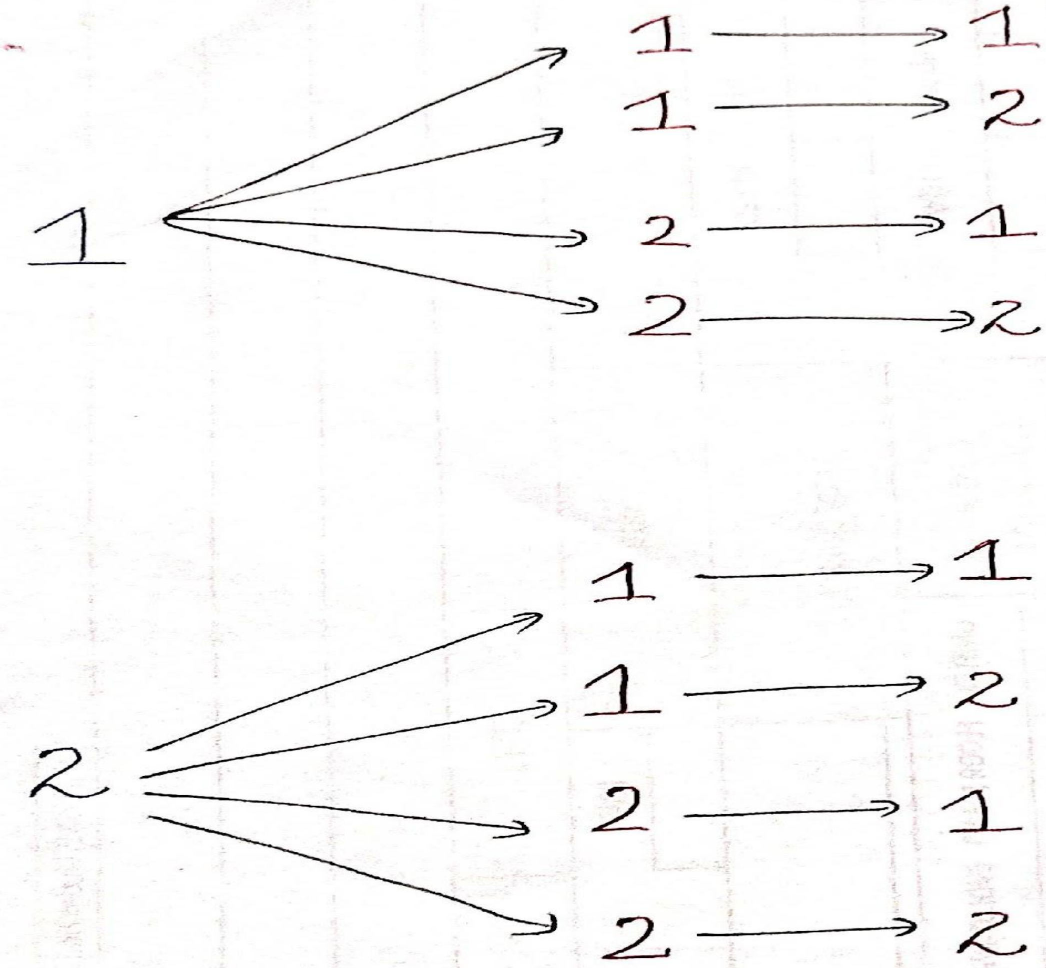
B x B x B= ?

$A = \{1, 2\}$

$A = \{1, 2\}$

$A = \{1, 2\}$

$A \times A \times A$



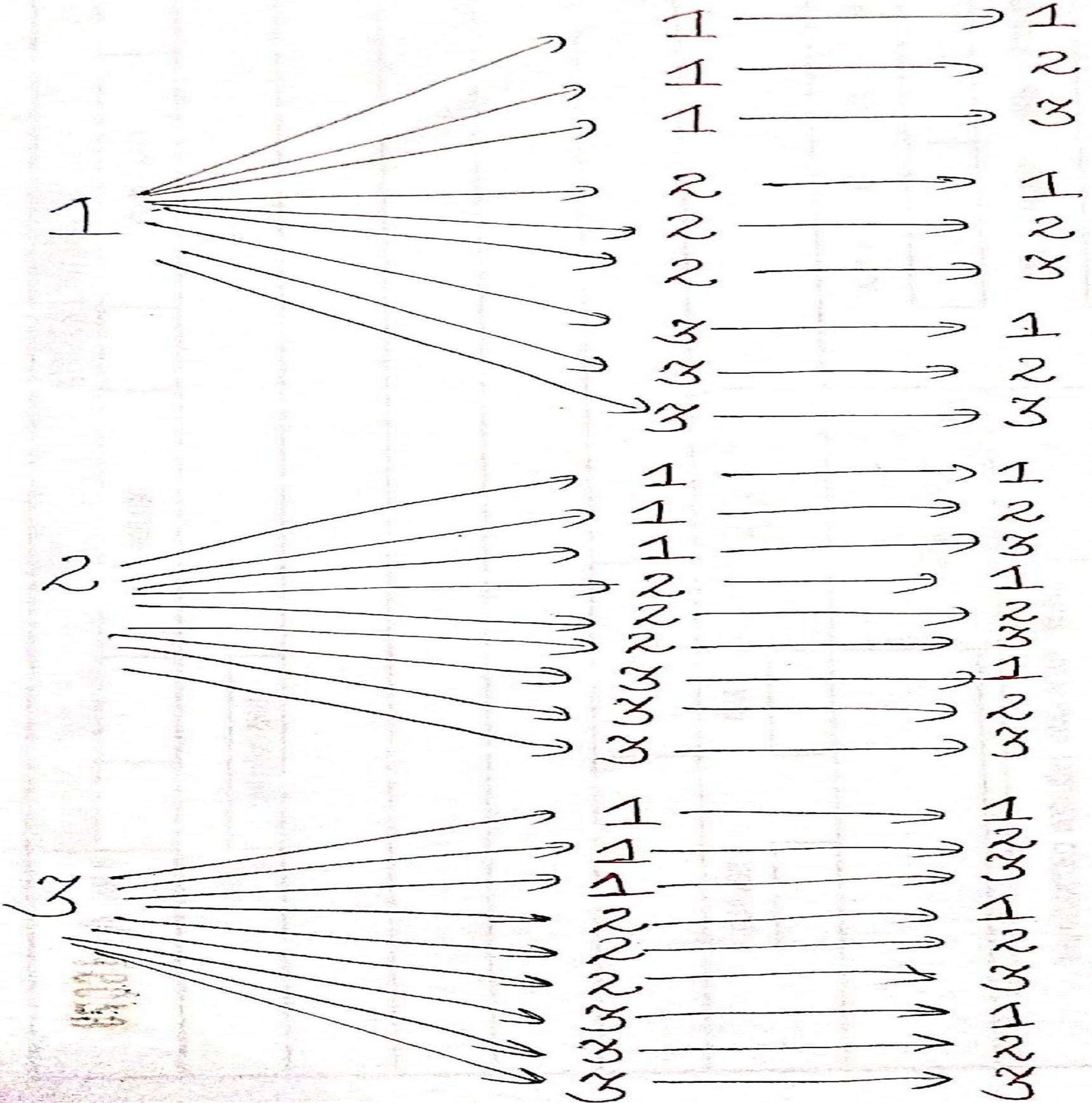
$\{111,$
$112,$
$121,$
$122,$
$211,$
$212,$
$221,$
$222\}$

$A = \{1, 2, 3\}$

$A = \{1, 2, 3\}$

$A = \{1, 2, 3\}$

$A \times A \times A$



$\{111,$
$112,$
$113,$
$121,$
$122,$
$123,$
$131,$
$132,$
$133,$
$211,$
$212,$
$213,$
$221,$
$222,$
$223,$
$231,$
$232,$
$233,$
$311,$
$312,$
$313,$
$321,$
$322,$
$323,$
$331,$
$332,$
$333\}$

Let $A = \{1, 2, 3\}$,

$B = \{3, 4\}$

$C = \{4, 5, 6\}$

Find $A \times (B \cap C)$

Let A = {a,b,c} and B= {p,q}.

Find the cartesian product of sets A and B.

(i) A X B

(ii) B X A

(iii) A x A

(iv) B X B

1) Let $A=\{p,q,r,s\}$ $B=\{r,s,t,u\}$

Find $(A \times B)\times(B-A)$

2) $A=\{p,q\}$ $B=\{q,r\}$ $C=\{r,s,t\}$

Find

i) $A\times(B \cup C)$

ii) $(A\times B)\cup(A\times C)$

iii) $A\times(B \cap C)$

iv) Verify $(A-B)\times C=(A\times C)-(B\times C)$

Classes of sets:-

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a *set of sets*. Whenever such a situation occurs, to avoid confusion, we will speak of a *class* of sets or *collection* of sets rather than a *set* of sets. If we wish to consider some of the sets in a given class of sets, then we speak of *subclass* or *subcollection*.

EXAMPLE 1.9 Suppose $S = \{1, 2, 3, 4\}$.

(a) Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

That is, the elements of A are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.

(b) Let B be the class of subsets of S, each which contains 2 and two other elements of S. Then

$$B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of B are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

Power Set

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2^n. Power set is denoted as P(S).

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets.

Example:

For a set S = {a, b, c, d} let us calculate the subsets:

- Subsets with 0 elements: {ø} (the empty set)
- Subsets with 1 element: {a}, {b}, {c}, {d}
- Subsets with 2 elements: {a,b}, {a,c}, {a,d}, {b,c}, {b,d},{c,d}
- Subsets with 3 elements: {a,b,c},{a,b,d},{a,c,d},{b,c,d}
- Subsets with 4 elements: {a,b,c,d}

Hence, P(S) =

{ {ø},{a}, {b}, {c}, {d},{a,b}, {a,c}, {a,d}, {b,c},
{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},{a,b,c,d} **}**

$| P(S) | = 2^4 = 16$

Note: The power set of an empty set is also an empty set.
$| P(\{\emptyset\}) | = 2^0 = 1$

$A = \{ \text{mango, lemon, apple} \}$

$P(A)=?$

If $n(C)=8$, what is $P(C)$?

2. If a set A has 3 elements then find the number of elements in power set of set A.

a) 1

b) 2

c) 8

d) 27

The cardinality of the power set of {0, 1, 2 . . ., 10} is __________.

(A) 1024

(B) 1023

(C) 2048

(D) 2043

Partitions

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say P_1, P_1, ... P_n that satisfies the following three conditions −

- P_i does not contain the empty set.

  $$[\ P_i \neq \{\ \emptyset\ \} \text{ for all } 0 < i \leq n\ ]$$

- The union of the subsets must equal the entire original set.

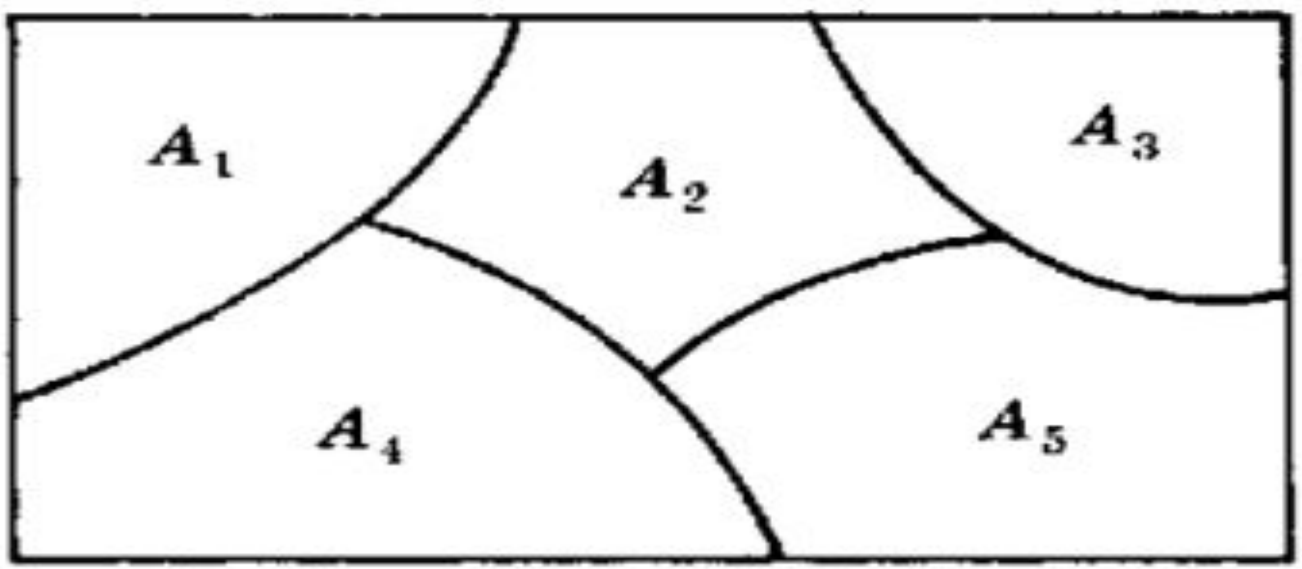
  $$[\ P_1 \cup P_2 \cup \ldots \cup P_n = S\ ]$$

- The intersection of any two distinct sets is empty.

  $$[\ P_a \cap P_b = \{\ \emptyset\ \}, \text{ for } a \neq b \text{ where } n \geq a,\ b \geq 0\ ]$$

The subsets in a partition are called **cells**.

Venn diagram of a partition of the rectangular set S of points into five cells, A1, A2, A3, A4, A5.



Let $A = \{a, b, c, d\}$. Examples of partitions of A are:

- $\{\{a\}, \{b\}, \{c, d\}\}$
- $\{\{a, b\}, \{c, d\}\}$
- $\{\{a\}, \{b\}, \{c\}, \{d\}\}$

Let S = { a, b, c, d, e, f, g, h }

One probable partitioning is { a }, { b, c, d }, { e, f, g, h }

Another probable partitioning is { a, b }, { c, d }, { e, f, g, h }

EXAMPLE 1.11 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

1.5 ALGEBRA OF SETS, DUALITY

Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in Table 1-1. In fact, we formally state this as:

Theorem 1.5: Sets satisfy the laws in Table 1-1.

Table 1-1 Laws of the algebra of sets

Idempotent laws:	(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws:	(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	(5a) $A \cup \emptyset = A$	(5b) $A \cap \mathbf{U} = A$
	(6a) $A \cup \mathbf{U} = \mathbf{U}$	(6b) $A \cap \emptyset = \emptyset$
Involution laws:	(7) $(A^{\mathrm{C}})^{\mathrm{C}} = A$	
Complement laws:	(8a) $A \cup A^{\mathrm{C}} = \mathbf{U}$	(8b) $A \cap A^{\mathrm{C}} = \emptyset$
	(9a) $\mathbf{U}^{\mathrm{C}} = \emptyset$	(9b) $\emptyset^{\mathrm{C}} = \mathbf{U}$
DeMorgan's laws:	(10a) $(A \cup B)^{\mathrm{C}} = A^{\mathrm{C}} \cap B^{\mathrm{C}}$	(10b) $(A \cap B)^{\mathrm{C}} = A^{\mathrm{C}} \cup B^{\mathrm{C}}$

Example 1: Prove that i) $(A\cup B)\cap(A\cup B^c)=A$ and

ii) $(A\cap B)\cup(A\cap B^c)=A$.

Solution: L.H.S.= $(A\cup B)\cap(A\cup B^c)$

$= A\cup(B\cap B^c)$ (Distributive law)

$= A\cup\phi$ ($B\cap B^c=\phi$ complement law)

$= A$

$=$ R.H.S

Hence $(A\cup B)\cap(A\cup B^c)=A$.

Similarly, we can prove $(A\cap B)\cup(A\cap B^c)=A$.

Example 2: If U = {x\x is a natural number less than 20} is the universal
set, A = {1, 3, 4, 5, 9}, B = {3, 5, 7, 9, 12}. Verify that De Morgan's laws.

Solution: De Morgan's laws can be state as i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$,

ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

By listing method,

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19},

and A = {1, 3, 4, 5, 9},

$\overline{A}$ = {2, 6, 7, 8, 10, 11, 12, 13,14,15,16,17,18,19},

and B = {3,5,7,9,12},

$\overline{B}$ = {1, 2, 4, 6, 8, 10, 11, 13, 14, 15, 16, 17, 18,19}

$A \cup B$ = {1, 3, 4, 5, 7, 9, 12}

$(\overline{A \cup B})$ = {2, 6, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19}

Also $(\overline{A} \cap \overline{B})$ = {2, 6, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19}

Hence $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Now $(A \cap B) = \{3, 5, 9\}$,

$$(\overline{A \cap B}) = \{1, 2, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$$

Also $(\bar{A} \cup \bar{B}) = \{ 1, 2, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$

Hence $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Example 3: If u={1,2,3,4,5,6,7,8,9,10,11,12} is the universal set.
A={2,3,5,8,10}, and
B={4,5,7,8,9,11}, find.
i). A-B, ii). B-A, iii). $(\overline{A-B})$.

Solution:- i). A-B={2,3,10}
ii). B-A={4,7,9,11}
iii). $(\overline{A-B})=\{1,4,5,6,7,8,9,11,12\}$.

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Thank you!