Mathematical induction, is a technique for proving results or establishing statements for natural numbers. This part illustrates the method through a variety of examples.

Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below:

Step 1(Base step): It proves that a statement is true for the initial value.

Step 2(Inductive step): It proves that if the statement is true for the n^{th} iteration (or number n), then it is also true for $(n+1)^{th}$ iteration (or number $n+1$).

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

How to Do It

Step 1: Consider an initial value for which the statement is true. It is to be shown that the statement is true for n=initial value.

Step 2: Assume the statement is true for any value of $n=k$. Then prove the statement is true for $n=k+1$. We actually break $n=k+1$ into two parts, one part is n=k (which is already proved) and try to prove the other part.

Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Problem 2

$1 + 3 + 5 + \ldots + (2n-1) = n^2$ for $n=1, 2, \ldots$

Solution

Step 1: For $n=1$, $1 = 1^2$, Hence, step 1 is satisfied.

Step 2: Let us assume the statement is true for $n=k$.

Hence, $1 + 3 + 5 + \ldots + (2k-1) = k^2$ is true (It is an assumption)

We have to prove that $1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$ also holds

$$
\begin{aligned}
& 1 + 3 + 5 + \ldots + (2(k+1) - 1) \\
&= 1 + 3 + 5 + \ldots + (2k+2 - 1) \\
&= 1 + 3 + 5 + \ldots + (2k + 1) \\
&= 1 + 3 + 5 + \ldots + (2k - 1) \mathbf{+} (2k + 1) \\
&= k^2 + (2k + 1) \\
&= (k + 1)^2
\end{aligned}
$$

So, $1 + 3 + 5 + \ldots + (2(k+1) - 1) = (k+1)^2$ hold which satisfies the step 2.

Hence, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is proved.

Prove the proposition $P(n)$ that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$; that is,

$$P(n) = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

The proposition holds for $n = 1$ since:

$$P(1) : 1 = \tfrac{1}{2}(1)(1+1)$$

Assuming $P(k)$ is true, we add $k+1$ to both sides of $P(k)$, obtaining

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \tfrac{1}{2}k(k+1) + (k+1) \\ &= \tfrac{1}{2}[k(k+1) + 2(k+1)] \\ &= \tfrac{1}{2}[(k+1)(k+2)] \end{aligned}$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the Principle of Induction, P is true for all n.

Prove the following proposition (for $n \geq 0$):

$$P(n) \colon 1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

$P(0)$ is true since $1 = 2^1 - 1$. Assuming $P(k)$ is true, we add 2^{k+1} to both sides of $P(k)$, obtaining

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is $P(k+1)$. That is, $P(k+1)$ is true whenever $P(k)$ is true. By the principle of induction, $P(n)$ is true for all n.