PMATH 450/650
Assignment 1

1. Let $X = \mathbb{N}$ and $\mathcal{A} = P(\mathbb{N})$. Find all measures on $\mathcal{A}$.

2. (a) Let $f : X \to Y$ be a function and let $(X, \mathcal{A})$ be a measurable space. Let $\mathcal{F} \subseteq P(Y)$ be such that $f^{-1}(F) \in \mathcal{A}$ for all $F \in \mathcal{F}$. Prove that $f^{-1}(B) \in \mathcal{A}$ for every $B \in \sigma(\mathcal{F})$.

   (b) Let (X, d) and (Y, d') be metric spaces. Suppose $f : X \to Y$ is a function such that $f^{-1}(A) \in \mathcal{B}(X)$ for every open set $A \subseteq Y$. Prove that $f^{-1}(A) \in \mathcal{B}(X)$ for every $A \in \mathcal{B}(Y)$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A_k \in \mathcal{A}$ for every $k \in \mathbb{N}$.

   (a) Prove that
   $$\limsup A_k := \{x \in X : x \in A_k \text{ for infinitely many } k\} \in \mathcal{A}$$
   and
   $$\liminf A_k := \{x \in X : x \in A_k \text{ for all but finitely many } k\} \in \mathcal{A}.$$

   (b) Prove that
   $$\mu(\liminf A_k) \leq \liminf \mu(A_k).$$

   (c) If $\mu(\cup A_k) < \infty$, prove that
   $$\mu(\limsup A_k) \geq \limsup \mu(A_k).$$

   (d) Suppose $\sum \mu(A_k) < \infty$. Prove that $\mu(\limsup A_k) = 0$.

4. Let $(X, \mathcal{A})$ be a measurable space. A measure μ on $\mathcal{A}$ is called σ-finite if there exists $A_i \in \mathcal{A}$, $i \in \mathbb{N}$, such that $\mu(A_i) < \infty$ and $X = \cup A_i$. A measure μ on $\mathcal{A}$ is called semifinite if whenever $A \in \mathcal{A}$ such that $\mu(A) = \infty$, there exists $F \in \mathcal{A}$ with $F \subseteq A$ and $0 < \mu(F) < \infty$.

   (a) Prove that every σ-finite measure is semifinite.

   (b) Give an example of a measure which is semifinite but not σ-finite.

   (c) Let $(X, \mathcal{A}, \mu)$ be a measure space, where μ is semifinite. Prove that for $A \in \mathcal{A}$,

$$\mu(A) = \sup\{\mu(B) : B \in \mathcal{A}, B \subseteq A, \mu(B) < \infty\}.$$