Homotopy Theory of Simplicial Sets

Vincent Siebler

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1 Motivation

Let $\mathcal{C}$ be a category and W a class of morphisms.

Definition 1.1. A localisation of $\mathcal{C}$ at W is a category $\mathcal{C}[W^{-1}]$ together with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that $\forall f \in W$, we get that $\gamma(f)$ is an isomorphism in $\mathcal{C}[W^{-1}]$.

Example 1.2.
- A ring considered as category and the localisation at an ideal. Can this be extended to rings with more than one object?
- The derived category of an abelian category is the localisation with respect to the quasi isomorphisms.

Proposition 1.3. *Let $\mathcal{C}[W^{-1}]$ as in **??** 1.1. For any category $\mathcal{D}$ the functor*

$$\begin{aligned} j^*\colon \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) &\to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\ (\mathcal{C}[W^{-1}] \to \mathcal{D}) &\longmapsto (\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}] \xrightarrow{F} \mathcal{D}) \end{aligned}$$

is an equivalence.

Theorem 1.4. *Set-theoretic issues aside, loacalisations always exist.*

Example 1.5.
- Let Top be the category with objects given by topological spaces and morphisms given by continuous maps. Let W be the weak homotopy equivalences, that is morphisms $f : X \to Y$ such that the induced maps on path components

$$\pi_0(f)\colon \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$$

  and for all points $x \in X$ and for all $n \geqslant 1$ with $n \in \mathbb{N}$ the morphism f induces an an isomorphism on homotopy groups

$$\pi_n(f, x)\colon \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x))$$

  in Grp. The result of the localisation is called the homotopy category $\mathcal{H} : \mathrm{Top}[W^{-1}]$.
- The localisation at all morphism of the category $\mathcal{C}[\mathcal{C}^{-1}]$ is a groupoid.

Remark 1.6. The takeaway is the general paradigm, that the localisation is the <u>truncation</u> of a richer mathematical structure.

1.1 Exercises

Exercise 1. For a group G let Set_G be the category of sets with a right G action i.e.

- the objects are tuples (X, ρ_X) where X is a set and $\rho\colon X \times G \to X$ is a map satisfying for all $g, h \in G$ and $x \in X$ that $\rho_X(x, gh) = \rho_X(\rho_X(x, g), h)$ and furthermore $\mathrm{id}_X = \rho_X(-, e)$ for e the neutral element of G, and

- a morphism $\phi\colon (X, \rho_X) \to (Y, \rho_Y)$ is given by a map $\phi\colon X \to Y$ satisfying $\phi \circ \rho_X = \rho_Y \circ (\phi \times \mathrm{id}_G)$.

Recall that we view G as a category BG with one object $\star$ and $\mathrm{Hom}_{BG}(\star, \star) = G$. Show that there is and isomorphism of categories between Set_G and $\widehat{BG}$, the category of presheaves over BG.

Exercise 2. For a set Y, show that there is an isomorphism of functors $\mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}$

$$\mathrm{Hom}_{\mathrm{Set}}(- \times Y, ?) \cong \mathrm{Hom}_{\mathrm{Set}}(-, \mathrm{Hom}_{\mathrm{Set}}(Y, ?)).$$

Exercise 3. Let $\eta\colon F \to G$ be a natural transformation of two functors $F, G\colon \mathcal{A} \to \mathcal{B}$. Show that η is a natural isomorphism, i.e. there exists a natural transformation $\eta'\colon G \to F$ such that $\eta' \circ \eta = \mathrm{id}_F$ and $\eta \circ \eta' = \mathrm{id}_G$, if and only if for every $a \in \mathcal{A}$ the morphism $\eta_a\colon F(a) \to G(a)$ is an isomorphism.

Exercise 4. Fix an object $x \in \mathcal{C}$ of a category $\mathcal{C}$. The slice category $\mathcal{C}/x$ of $\mathcal{C}$ over x is defined as follows.

- The objects are tuples (a, π) with $a \in \mathcal{C}$ an object and a morphism $\pi\colon a \to x$.
- A morphism $f\colon (a, \pi) \to (b, \rho)$ is given by a morphism $f\colon a \to b$ such that $\pi = \rho \circ f$.

After convincing yourself that this defines a category, do the following.

(a) Show that there exists a final object in $\mathcal{C}/x$, i.e. an object (e, ρ) such that for any object (a, π) there is a unique morphism $f\colon (a, \pi) \to (e, \rho)$.

(b) Define the coslice category $x/\mathcal{C}$ of elements under x.

(c) Describe $(\mathcal{C}/x)^{\mathrm{op}}$ as a slice or coslice category.

2 Presheaves and the Yoneda lemma

Throughout we will fix a small category $\mathcal{A}$.

Definition 2.1. Let $\widehat{\mathcal{A}} \coloneqq \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$ be the category of contravariant functors from $\mathcal{A}$ to the category of sets. This category will be called the category of presheaves on $\mathcal{A}$. By definition $X \in \widehat{\mathcal{A}}$ consists of the following data:

- $\forall a \in \mathcal{A}$ a set $X_a \coloneqq X(a) \in \mathrm{Set}$ This set will be called the fibre of X at a.
- $\forall u \colon b \to a \in \widehat{\mathcal{A}}(b, a)$ a map of sets $u^* = X(u) \colon X_a \to X_b$ such that functoriality constraints are satisfied:
- (Unitality) For all objects $a \in \mathcal{A}$ a morphism $(\mathrm{id}_a)^* = X(\mathrm{id}_a) \colon X_a \to X_a$ such that $(\mathrm{id}_a)^* = \mathrm{id}_{X_a}$.
  (Composition) For all composition of morphisms in $\mathcal{A}$ a composition of the induced morphisms on the fibres:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[rr, bend right, "v \circ u"'] & b \arrow[r, "v"] & c \end{tikzcd} \qquad \begin{tikzcd} & X_b \arrow[dl, "u^*"'] & \\ X_a & & X_c \arrow[ul, "v^*"'] \arrow[ll, "(v \circ u)^*"] \end{tikzcd}$$

  which is equivalent to $u^* \circ v^* = (v \circ u)^*$.

Remark 2.2. Throughout we are going to talk alot alot about presheaves, especially representable ones, what happens when we introduce sheaves and maybe use sheaves in the later theory.

Example 2.3. Let M be a monoid, then BM is the category with a single object $\mathrm{Ob}(BM) \coloneqq \{*\}$ and morphisms $BM(*, *) \times BM(*, *) \to BM(*, *) = M$. A presheaf $X \in \widehat{BM}$ consists of a set $X = X_* \in \mathrm{Set}$ and for each $m \in M$ a morphism $m^* : X \to X$ that we denote on elements by left multiplication $m^*(x) = x \cdot m$. Moreover a morphism $e_m^* \colon X \to X$ such that $e_m^* = \mathrm{id}_{X*}$, that is $x \cdot e_M = x$ for all $x \in X$. At last for any diagram of morphisms in BM, a diagram in Set

$$\begin{tikzcd} & * \arrow[dr, "u"] & \\ * \arrow[ur, "m"] \arrow[rr, "um"'] & & * \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\cdot m"'] & \\ X & & X \arrow[ul, "\cdot u"'] \arrow[ll, "\cdot (um)"] \end{tikzcd}$$

which means that $x \cdot (nm) = (xn) \cdot m$.

Definition 2.4. For every $a \in \mathcal{A}$, let $\mathcal{A}$ be the functor,

$$\mathcal{A}(-,a)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd}
b \arrow[r, mapsto] & \mathcal{A}(b,a) \arrow[d, "u^*"] & b \xrightarrow{v} a \\
c \arrow[u, "u"] \arrow[r, mapsto] & \mathcal{A}(c,a) & c \xrightarrow{u} b \xrightarrow{v} a
\end{tikzcd}$$

is the presheaf represented by a $\in \mathcal{A}$.

Let $X, Y \in \widehat{\mathcal{A}}$ be presheaves. By definition a morphism $f\colon X \to Y$ is a natural transformation of functors $\eta\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$, that is for every $a \in \mathcal{A}$ a morphism of sets $\eta_a : X_a \to Y_a$ such that the usual naturality constraint holds, that is

$$\begin{tikzcd}
a \arrow[d, "u"] & X_a \arrow[r, "\eta_a"] & Y_a \\
b & X_b \arrow[u, "u^*"] \arrow[r, "\eta_b"] & Y_b \arrow[u, "u^*"']
\end{tikzcd}$$

commutes for every morphism u in $\mathcal{A}$, so we have $u^* \circ \eta_b = \eta_a \circ u^*$.

Example 2.5. Let M be a monoid and $X, Y \in \widehat{BM}$. A morphism $f\colon X \to Y$ consists of a function $f = f^*\colon X = X_* \to Y_* = Y$ such that, the following diagram commutes

$$\begin{tikzcd}
* \arrow[d, "m"] & X \arrow[r, "f"] & Y \\
* & X \arrow[u, "m"] \arrow[r, "f"] & Y \arrow[u, "m"']
\end{tikzcd}$$

for $m \in BM(*,*)$.

Theorem 2.6. *Yoneda lemma version 1 Let a be an object in $\mathcal{A}$ and $X \in \widehat{\mathcal{A}}$ a presheaf. Then*

$$\phi = \phi_{a,x}\colon \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-,a), X) \longrightarrow X_a$$

$$f \longmapsto f_a(\mathrm{id}_a)$$

is bijective.

Proof. We first observe that the following square commutes

$$\begin{tikzcd}
b \arrow[d, "u"] & \mathcal{A}(b,a) \arrow[r, "f_b"] & X_b \\
a & \mathcal{A}(a,a) \arrow[u, "u^*"] \arrow[r, "f_a"] & X_a \arrow[u, "u^*"']
\end{tikzcd}$$

which means $f_b(u^*(\mathrm{id}_a)) = f_b(u) = u^*(f_a(\mathrm{id}_a)) = u^*(\phi(f))$. So without evaluating at an object we get $f_b(-) = (-)^*(\phi(f)) = X(-)(\phi(f))$. Let us first show that ϕ is injective, suppose $f, g\colon \mathcal{A}(-,a) \to X$ such that $\phi(f) = \phi(g)$. For $b \in \mathcal{A}$ we get morphisms $f_b, g_b\colon \mathcal{A}(b,a) \to X_b$ and for any morphism $u : b \to a$ in $\mathcal{A}$ we get that $f_b(u) = u^*(\phi(f)) = u^*(\phi(g)) = g_b(u)$ and thus $f = g$. Let us now show that ϕ is surjective. Let $x \in X_a$ and $f^\times := (f_b^\times : \mathcal{A}(b,a) \to X_b \mid b \in \mathcal{A})$ given on any morphism $u\colon b \to a$ by $u^*(x)$. We need to prove that these are indeed the components of a natural transformation $f\colon \mathcal{A}(-,a) \to X$.

$$\begin{tikzcd}
b \arrow[d, "u"] & \mathcal{A}(b,a) \arrow[r, "f_b^\times"] & X_b \\
a & \mathcal{A}(c,a) \arrow[r, "f_a^\times"] \arrow[u, "u^*"] & X_c \arrow[u, "u^*"']
\end{tikzcd}$$

The square commutes, which gives for any $c \xrightarrow{\nu} a$ in $\mathcal{A}$, that

$$f_b^\times(u^*(\nu)) = f_b^\times(\nu \circ u) = (\nu \circ u)^*(x) = u^*(f_c^\times(\nu)) = \\ u^*(\nu^*(x)) = (u^* \circ \nu^*)(x) = (\nu \circ u)^*(x).$$

□

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Theorem 2.7. *The functor $\mu : \mathcal{A} \to \widehat{\mathcal{A}}$*

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d] & \widehat{a} = \mathcal{A}(-,a) \arrow[d, "\widehat{u}"] & \mathcal{A}(c,a) \arrow[d, "\widehat{u}_c"] \\
b \arrow[r, mapsto] & \widehat{b} = \mathcal{A}(-,b) & \mathcal{A}(c,b)
\end{tikzcd}$$

that sends an object of a to the the presheaf of morphisms into a is fully faithfull called the Yoneda embedding.

Proof. We first have to show that μ is a functor. So let $u\colon a \to b$ be a morphism. The claim is that $\widehat{u}\colon \widehat{a} \to \widehat{b}$ is a natural transformation.

$$\begin{tikzcd}
c \arrow[d, "v"] & \mathcal{A}(c,a) \arrow[r, "u_*"] & \mathcal{A}(c,b) \\
d & \mathcal{A}(d,a) \arrow[r, "u_*"] \arrow[u, "\nu^*"] & \mathcal{A}(d,b) \arrow[u, "\nu^*"']
\end{tikzcd}$$

The square commutes, which means that morphisms are mapped to natural transformations under μ, thus μ is actually a functor. Next let us show that μ is fully faithfull, which means that the μ is a bijection on hom-sets

$$\begin{tikzcd}
\mu\colon \mathcal{A}(a,b) \arrow[r] & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, \widehat{b}). \arrow[l, dashed, bend right, "\phi"']
\end{tikzcd}$$

We claim that

$$\phi \circ \mu = \mathrm{id}\,.$$

Let $u\colon a \to b$, then

$$\phi(\widehat{u}) = \widehat{u}_a(\mathrm{id}_a) = u \circ \mathrm{id}_a = u$$

which proves the above claim. ☐

Remark 2.8. There is the contravariant Yoneda embedding as well, given by

$$\mu_{\mathcal{A}^{\mathrm{op}}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \widehat{\mathcal{A}}^{\mathrm{op}} = \mathrm{Fun}(\mathcal{A}, \mathrm{Set})$$

$$a \longmapsto \mathcal{A}^{\mathrm{op}}(-, a) = \mathcal{A}(a, -).$$

Proposition 2.9. *Let $X \in \widehat{\mathcal{A}}$ consider the presheaf*

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, ?), X)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$\mathcal{A} \overset{\mu}{\hookrightarrow} \widehat{\mathcal{A}} \xrightarrow{\mathrm{Hom}_{\widehat{\mathcal{A}}}(-, X)} \mathrm{Set}$$

$$a \mapsto \mathcal{A}(-, a) \longmapsto \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, a), X)$$

Then

$$\phi_{?,X}: \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, ?), X) \to X$$
$$\phi_{?,X} = (\phi_{a,X}\colon \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, a), X) \xrightarrow{\sim} X_a \mid a \in \mathcal{A})$$

is a natural isomorphism of presheaves.

Proof. We only need to prove naturality since $\phi_{a,X}$ is an isomorphism for every a by **??** 2.6. So let us look at the following square

$$\begin{tikzcd} a \arrow[d, "u"] & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, a), X) \arrow[r, "\phi"] & X_a \\ b & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-, b), X) \arrow[u, "? \circ \widehat{u}"] \arrow[r, "\phi"] & X_b \arrow[u, "u^*"'] \end{tikzcd}$$

For $f\colon \widehat{b} \to X$ the commutative square yields

$$\begin{tikzcd} & u^*(f_b(\mathrm{id}_b)) = f_a(u) & (f \circ \widehat{u}) \arrow[r] & (f \circ \widehat{u})_a(\mathrm{id}_a) \\ f \arrow[r] & f_b(\mathrm{id}_b) \arrow[u, mapsto] & f \arrow[u, mapsto] & \end{tikzcd}$$

where $f_a\colon \mathcal{A}(a, b) \xrightarrow{f_a} X_a$. Notice that $(f \circ \widehat{u})_a(\mathrm{id}_a) = f_a \circ \widehat{u}_a(\mathrm{id}_a) = f_a(u \circ \mathrm{id}_a) = f_a(u)$ which means both compositions are the same, so the square commutes. ☐

Definition/Proposition 2.10. Let $X \in \widehat{\mathcal{A}}$ then the following are equivalent:

1. There $\exists a \in \mathcal{A}$ such that $\exists f : \widehat{a} \to X$ that is an isomorphism in $\widehat{\mathcal{A}}$.
2. $\exists a \in \mathcal{A}$ and $\exists x \in X_a$ such that $\forall b \in \mathcal{A}$, we have that
$$\mathcal{A}(b, a) \to X_b \qquad u \mapsto u^*(x)$$
is an isomorphism.
3. There $\exists a \in \mathcal{A}$ and $\exists x \in X_a$ such that $\forall b \in \mathcal{A}$ and $\forall u \in \mathcal{A}(b, c)$ we have that $\exists! y \in X_b$ such that $u^*(x) = y$.

We call the pair $(a \in \mathcal{A}, x \in X_a)$ a representation of X and $a \in \mathcal{A}$ a representing object and $x \in X_a$ a universal element.

Proof. This can be deduced from the previous proposition. $\square$

Proposition 2.11. *For an element $a \in \mathcal{A}$ the isomorphism $\phi_{a,X} \colon \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, X) \xrightarrow{\sim} X$ is natural in X.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} X & \qquad & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, X) & \xrightarrow{\phi} & X_a \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \circ ?} & & \downarrow{\scriptstyle f_a} \\ Y & & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, Y) & \xrightarrow{\phi} & Y_b \end{array}$$

which evaluates on an element $g \colon \widehat{a} \to X$ to

$$\begin{array}{ccccccc} g & \longmapsto & \phi(g) = g_a(\mathrm{id}_a) & \qquad & g & & \\ & & \big\downarrow & & \big\downarrow & & \\ & & f_a(g_a(\mathrm{id}_a)) & & (f \circ g) & \longmapsto & \phi(f \circ g) = (f \circ g)_a(\mathrm{id}_a) \end{array}$$

comparing the two outcomes, we get the following equalities

$$f_a(g_a(\mathrm{id}_a)) = (f_a \circ g_a)(\mathrm{id}_a) = (f \circ g)_a(\mathrm{id}_a)$$

which yields the result. $\square$

Theorem 2.12. *The Yoneda lemma Let $\mathcal{A}$ be a small category. The functions*

$$\phi_{a,X} \colon \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, X) \xrightarrow{\phi} X_a$$

$$f \longmapsto f_a(\mathrm{id}_a)$$

are natural in $a \in \mathcal{A}^{\mathrm{op}}$ and $X \in \widehat{\mathcal{A}}$ seperately. Hence they yield an isomorphism of functors.

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \times \widehat{\mathcal{A}} \arrow[rr, bend left, "\mathrm{Hom}_{\widehat{\mathcal{A}}}(\mu(?),-)"] \arrow[rr, bend right, "\mathrm{ev}"'] \arrow[rr, phantom, "\Downarrow"] & & \mathrm{Set}
\end{tikzcd}$$

Given on and object (a, X) as follows.

$$\begin{tikzcd}
& \mathrm{Hom}_{\widehat{\mathcal{A}}}(\widehat{a}, X) \arrow[d, "\phi_{(a,X)}"] \\
(a, X) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
& X_a
\end{tikzcd}$$

2.1 Exercises

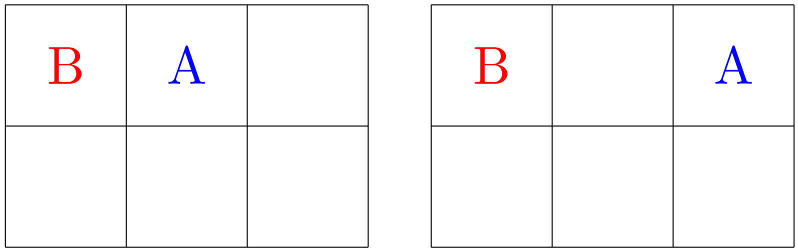
Exercise 1. Alice and Bob are randomly placed in a 3×2-grid on two different squares.

		B
A		

Every turn they are moving to an orthogonally adjacent square which is not currently occupied and they never move to the same square.

(a) Describe the groupoid of configurations by connecting two configurations if they are connected by a single turn.

(b) Which of the following configurations can be reached from the above? How many connected components does the groupoid have?



(c) Describe what additional information the groupoid holds over the equivalence classes of configurations up to moves.

Exercise 2. For an (associative and unital) ring R let BR be the category associated to the multiplicative monoid of R. Show that there is an isomorphism of categories $\psi\colon \operatorname{mod}_R \to \operatorname{Fun}_{\mathbb{Z}}((BR)^{\mathrm{op}}, \mathrm{Ab})$ relating the category of right R-modules into the category of '$\mathbb{Z}$-linear presheaves over BM', i.e. contravariant $\mathbb{Z}$-linear functors from BR to the category of abelian groups Ab.

Exercise 3. Let R be a ring and let ${}_RR_R$ be R viewed as an R-R-bimodule.

(a) Show that for any R-R-bimodule N and any $M \in \operatorname{mod} R$ that $\operatorname{Hom}_{\operatorname{mod} R}(N, M)$ carries the structure of a right R-module via $(fr)(x) := f(rx)$. Deduce that N induces a functor

$$\operatorname{Hom}_{\operatorname{mod} R}(N, -)\colon \operatorname{mod} R \to \operatorname{mod} R.$$

(b) Show that there is an isomorphism of functors $\operatorname{Hom}_{\operatorname{mod} R}({}_RR_R, -) \cong \operatorname{id}_{\operatorname{mod} R}$ given by evaluation at $1 \in R$.

Exercise 4. Consider four sets A, B, C and E and assume that $A, B \subseteq C$ with the inclusions ι_A and ι_B.

(a) Show that for any two maps $\phi_A\colon E \to A$ and $\phi_B\colon E \to B$ such that $\iota_A \circ \phi_A = \iota_B \circ \phi_B$ there is a unique map $\phi\colon E \to A \cap B$ such that $\phi_A = i_A \circ \phi$ and $\phi_B = i_B \circ \phi$ for i_A and i_B the respective inclusions of $A \cap B$.

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!\phi"] \arrow[drr, bend left, "\phi_A"] \arrow[ddr, bend right, "\phi_B"'] & & \\
& A \cap B \arrow[r, "i_A"] \arrow[d, "i_B"] & A \arrow[d, "\iota_A"] \\
& B \arrow[r, "\iota_B"] & C
\end{tikzcd}$$

(b) Give an example of two maps $f_A\colon A \to D$ and $f_B\colon B \to D$ such that $A \cap B$ does not have the above property for f_A and f_B instead of ι_A and ι_B.

(c) Show that in this more genral setting that

$$A \times_D B := \{(a,b) \in A \times B \mid f_A(a) = f_B(b)\}$$

with the canonical projections p_A, p_B satisfying the universal property from before.

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!\phi"] \arrow[drr, bend left, "\phi_A"] \arrow[ddr, bend right, "\phi_B"'] & & \\
& A \times_D B \arrow[r, "p_A"] \arrow[d, "p_B"] & A \arrow[d, "f_A"] \\
& B \arrow[r, "f_B"] & D
\end{tikzcd}$$

(d) What is the relation between $A \cap B$ and $A \times_C B$ for part (a)?

3 Limits and Colimits

Let D be a small category, $\mathcal{C}$ be a category and $F\colon D \to \mathcal{C}$ a functor (a D-shaped diagram in $\mathcal{C}$). For example let D be given by

$$10 \xrightarrow{g} 11 \xleftarrow{f} 01$$

and let $F\colon D \to \mathcal{C}$ be a functor. We get a diagram

$$F(10) \xrightarrow{F(g)} F(11) \xleftarrow{F(f)} F(01).$$

Definition 3.1. A <u>cone over F</u> is a pair $(X, (\phi_a)_{a\in A})$ consisting of

1. $X \in \mathcal{C}$,
2. $(\phi_a\colon X \to F(a) \mid a \in A$,

such that $\forall u\colon a \to b$ in $\mathcal{A}$

$$\begin{tikzcd} & X \arrow[dl, "\phi_a"'] \arrow[dr, "\phi_b"] & \\ F(a) \arrow[rr, "F(u)"'] & & F(b) \end{tikzcd}$$

And $\phi_b = F(u) \circ \phi_a$.

Cones form a category $\mathcal{C}/F$ with morphisms given by

$$f\colon (X, (\phi_a)_{a\in A}) \to (Y, (\sigma_a)_{a\in A})$$

given by $f\colon X \to Y$ such that for all $a \in A$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_a"'] & & Y \arrow[dl, "\sigma_a"] \\ & F(a) & \end{tikzcd}$$

A <u>limit (cone) of F</u> is a final object in $\mathcal{C}/F$. Explicitely $(\lim F, (\sigma_a)_{a\in A})$ is a limit of F if for all cones $(X, (\phi_a)_{a\in A})$, there exists a unique $f\colon X \to \lim F$ in $\mathcal{C}$ such that for all $a \in A$ the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_a"'] & & \lim F \arrow[dl, "\phi_a"] \\ & F(a) & \end{tikzcd}$$

Example 3.2. Let $A = \phi$ and $F\colon \phi \to \mathcal{C}$. A limit is an object $\mathbb{1} \in \mathcal{C}$ such that for all $X \in \mathcal{C}$ there exists a unique $f\colon X \to \mathbb{1}$ that is $\mathbb{1}$ is a final object in $\mathcal{C}$.

Example 3.3. Let $A = \{1, 2\} \xrightarrow{F} \mathcal{C}$ be a functor the limit cone in $\mathcal{C}$ is given by the product, that is a cone (Y, π_i) in $\mathcal{C}$.

$$\begin{tikzcd}
 & & F(2) \\
X \arrow[r, "\exists!"] \arrow[urr, bend left, "\phi_2"] \arrow[drr, bend right, "\phi_1"'] & Y \arrow[ur, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
 & & F(1)
\end{tikzcd}$$

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If $(X, \overline{\rho}) = \overline{X}$ and $(Y, \overline{\sigma}) = \overline{Y}$ are limits of $F\colon A \to \mathcal{C}$. Then there is a unique isomorphism of cones $\overline{X} \to \overline{Y}$. It is enough to prove the statement for final objects, by definition of the limit cone. Let $X, Y \in \mathcal{D}$ be final objects. Then there exists a unique morphism $f\colon X \to Y$ in $\mathcal{D}$ since Y is final. Then there exists a unique morphism $f\colon Y \to X$ in $\mathcal{D}$ since X is final. But then $g \circ f(X) = X$ must be $g \circ f = \mathrm{id}_X$ since X is final.

Proposition 3.4. *Suppose that* $(\lim F, \overline{\sigma})$ *is a limit of* $F\colon \mathcal{A} \to \mathcal{C}$ *and* $f\colon X \to \lim F$ *is an isomorphism. Then* $(X, (\sigma_a f\colon X \to F(a))a \in \mathcal{A})$ *is a limit cone.*

Proof. For all $u\colon a \to b$ in A, we have get the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "\sigma_a \circ f"'] \arrow[dr, "\sigma_b \circ f"] & \\
F(a) \arrow[rr, "F(u)"'] & & F(b)
\end{tikzcd}$$

which means that

$$F(u) \circ (\sigma_a \circ f) = \sigma_b \circ f.$$

Thus $(X, (\sigma_a \circ f)_{a \in A})$ is indeed a cone and $f\colon (X, (\sigma \circ f)_{a \in A}) \to (\lim_A F, \overline{\sigma})$ is an isomorphism of cones since f is an isomorphism and since

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\sigma_a \circ f"'] & & \lim_A F \arrow[dl, "\sigma_a"] \\
 & F(a) &
\end{tikzcd}$$

commutes for all a in A. ∎

Definition/Proposition 3.5. A category $\mathcal{C}$ is complete if for all small categories A and functors $F : A \to \mathcal{C}$ a limit of F exists. The category Set is complete.

Proof. Let A be a small category and $F\colon A \to \mathrm{Set}$ a diagram. Let

$$\lim_A F := \{\bar{X} = (X_a)_{a \in A} \in \prod_{a \in A} F(a) \mid \forall u\colon a \to b \text{ in } A, F(u)(X_a) = X_b\}$$

This this is a subset of a product, it comes with projections. We get that $(\lim_A F, (\pi_a : \bar{X} \to X_a)_{a\in A})$ is a cone over F since for all morphisms $u\colon a \to b$ in A we get that the following diagram

$$
\begin{tikzcd}
& \lim_A F \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
F(a) \arrow[rr, "F(a)"'] & & F(b)
\end{tikzcd}
$$

equates to

$$
\begin{tikzcd}
& \bar{X} \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
X_a \arrow[rr, "F(u)(X_a)"'] & & X_b
\end{tikzcd}
$$

Now let $(X, (\rho_a\colon X \to F(a) \mid a \in A))$ be another cone over F. Define $\bar{\rho} : X \to \prod_{a\in A} F(a)$ by $x \mapsto (\rho_a(x))_{a\in A}$. Notice that $\bar{\rho}$ factors through $\lim_A F \subseteq \prod_{a\in A} F(a)$ since for all $x \in X$ and for all morphisms $u\colon a \to b$, we have that $F(u)(\rho_a(x)) = \rho_b(x)$ since the following diagram commutes

$$
\begin{tikzcd}
& X \arrow[dl, "\rho_a"'] \arrow[dr, "\rho_b"] & \\
F(a) \arrow[rr, "F(u)"'] & & F(b)
\end{tikzcd}
$$

Thus $\bar{\rho} \to \lim_A F$ is well defined. Observe that $\bar{\rho}$ is actually a morphism of cones, since

$$
\begin{tikzcd}
X \arrow[rr, "\bar{\rho}"] \arrow[dr, "\rho_b"'] & & \lim F \arrow[dl, "\pi_b"] \\
& F(b) &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "\bar{\rho}"] \arrow[dr, mapsto] & & (\rho_a(x))_{a\in A} \arrow[dl, mapsto, "\pi_b"] \\
& \rho_b(x) &
\end{tikzcd}
$$

Finally if $f\colon (X, (\rho_a)_{a\in A}) \to (\lim_A F, (\pi_a)_{a\in A})$ is a morphism of cones, then (by definition) we get for all $a \in A$

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\rho_a"'] & & \lim_A F \arrow[dl, "\pi_a"] \\
& F(a) &
\end{tikzcd}
$$

that is for all $x \in X$ we get $\pi_a(f(x)) = \rho_a(x)$, so $f = \bar{\rho}$. $\square$

Definition 3.6. A functor G preserves limits of shape A if for all functors $F\colon A \to \mathcal{C}$, G sends limit cones of F to limit cones of $G \circ F$. A functor G preserves limits if for all small categories A we have that G preserves limits of shape A.

Remark 3.7. Consider the example of the covariant Hom-functor. Let $F\colon A \to \mathcal{C}$ and $X \in \mathcal{C}$. Consider the covariant functor $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Set}$ and a limit $\sigma_a\colon \lim_A F \to F(a)$. We can put these together to obtain a cone $\mathrm{Hom}_{\mathcal{C}}(X, \sigma_a)\colon \mathrm{Hom}_{\mathcal{C}}(X, \lim F) \xrightarrow{\sigma_a \circ ? = (\sigma_a)_*} \mathrm{Hom}_{\mathcal{C}}(X, F(a))$.

Theorem 3.8. *The functor* $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Set}$ *preserves limits.*

Proof. Consider the map

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_A F) \xrightarrow{\quad\phi\quad} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, F(a))$$

$$(X \xrightarrow{f} \lim_A F) \longmapsto (\sigma_a \circ f\colon X \xrightarrow{f} \lim_A F \xrightarrow{\sigma_a} F(a))_{a\in A}$$

This is a morphism of cones, now we need to show it is bijective. For injectivity, assume there are two morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} \lim_A F$ such that $\phi(f) = \phi(g)$. Then for all $a \in A$ we get that $\sigma_a \circ f = \sigma_a \circ g$ and for all morphisms $a \to b$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] \arrow[rrr, shift left, "f"] \arrow[rrr, shift right, "g"'] & & & \lim_A F \arrow[dl, "\sigma_a"'] \arrow[dr, "\sigma_b"] & \\
F(a) \arrow[rr, "F(u)"] & & F(b) & F(a) \arrow[rr, "F(u)"] & & F(b)
\end{tikzcd}$$

which means f and g are morphisms of cones, but by the uniqueness of a morphism into a limit, such that the above commutes, we get that $f = g$. For the surjectivity, let $(f_a\colon X \to F(a))_{a\in A}$ be morphism indexed by A and take $\lim_{a\in A} \mathrm{Hom}(X, F(a))$. This means that for all morphism $u\colon a \to b$ in A, we have that $\mathrm{Hom}_{\mathcal{C}}(X, F(u))(f_a) = f_b$, so that $(X, (f_a\colon \to F(a))_{a\in A})$ is a cone over F. Thus there exists a unique morphism into the limit cone $\psi : (X, (f_a)_{a\in A}) \to (\lim_A F, (\sigma_a)_{a\in A})$, where $\sigma_a \circ \psi = f_a$ that is $\phi(f) = (f_a)_{a\in A}$. ☐

Theorem 3.9. *The Yoneda embedding* $\mu\colon \mathcal{A} \to \widehat{A} = \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$ *preserves limits.*

Proof. The proof is just an application of **??** 3.8 to the Yoneda embedding from **??** 2.7. ☐

3.1 Exercise

Exercise 1. Show that two objects $a, b \in A$ in a category A are isomorphic if and only if their respresentable presheaves $\mathrm{Hom}_A(-, a)$ and $\mathrm{Hom}_A(-, b)$ are isomorphic in $\widehat{A}$.

Exercise 2. Consider a functor $F\colon A \to \mathcal{C}$ from a small category A.

1. Show that if A is an initial object $\varnothing$, then the limit of F exists.

2. Show that if A has a final object e, then the colimit of F exists.

Exercise 3. Let $F\colon A \to \mathrm{Set}$ be a functor from a small category to the category of sets. Recall that we have shown in the lecture that the limit of F exists and is given by

$$\lim_A F := \left\{ x \in \prod_{a\in A} F(a) \mid \forall u\colon a \to b \quad F(u)(x_a) = x_b \right\}$$

together with the canonical projections.

(a) Show that the inclusion $\lim_A F \subseteq \prod_{a\in A} F(a)$ exhibits $\lim_A F$ as the equalizer (= limit of the following diagram)

$$\prod_{a\in A} F(a) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{\substack{u\colon s\to t \\ \text{in } A}} F(t)$$

where $\phi(x)_u = F(u)(x_{s(u)})$ and $\psi(x)_u = x_{t(u)}$.

(b) Let $\coprod$ denote the disjoint union of sets. Assume that the coequalizer (= colimit of the following diagram)

$$\coprod_{\substack{u\colon s\to t \\ \text{in } A}} F(t) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{a\in A} F(a)$$

exists where for $y \in F(s(v)) \subseteq \coprod_{\substack{u\colon s\to t \\ \text{in } A}} F(t)$ we have $\phi(y) = F(v)(y) \in F(t(v)) \subseteq \coprod_{a\in A} F(a)$ and $\psi(y) = y \in F(s(v)) \subseteq \coprod_{a\in A} F(a)$. Show that it is a colimit of F with the canonical maps fro $F(a)$.

Exercise 4.

(a) Show that the disjoint union of sets defines a coproduct in the category of sets, i.e. show that for every family of sets $(U_i)_{i\in I}$ their disjoint union $\coprod_{i\in I} U_i$ together with the canonical inclusion $U_j \subseteq \coprod_{i\in I} U_i$ is the colimit of the functor $U\colon I \to \mathrm{Set}$ assigning to each $i \in I$ the set U_i. Here I is some indexing set.

(b) Show that any coequalizer (= colimit of the following diagram)

$$U \overset{\phi}{\underset{\psi}{\rightrightarrows}} V$$

exists in Set by considering the smallest equivalence relation on V such that $v \sim v'$ whenever there is some $u \in U$ such that $\phi(u)$ and $\psi(u) = v'$.

(c) Conclude using Exercise 2.3 that Set is cocomplete, i.e. every small colimit exists.

4 Adjunctions

Lecture 22.10

Everytime you encounter some free object, in the sense that it is freely generated from some other object of some other category, like a free group on some set, you are most likely going to use the properties of the object stemming from an adjunction.

Definition 4.1. $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ An adjunction $L \dashv R$ is a natural isomorphism $\phi_{c,d}\colon \operatorname{Hom}_{\mathcal{D}}(Lc, d) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(c, Rd)$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$.

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\operatorname{Hom}_{\mathcal{D}}(L(-),-)"] \arrow[r, bend right, "\operatorname{Hom}_{\mathcal{C}}(-,R(-))"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathrm{Set} \end{tikzcd}$$

That is explicitely, we have commutative squares for all pairs of morphisms:

$$\begin{tikzcd} (c,d) \arrow[d, shift left, "\mathrm{id}_d"] & \operatorname{Hom}_{\mathcal{D}}(Lc,d) \arrow[r, "\phi_{c,d}"] \arrow[d, "g\circ ?"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd) \arrow[d, "R(g)\circ ?"] \\ (c',d) \arrow[u, shift left, "f"] & \operatorname{Hom}(Lc,d') \arrow[r, "\phi_{c,d'}"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd') \end{tikzcd}$$

which means that for all $f\colon Lc \to d$ and all $g\colon d \to d'$ we have that $R(g) \circ \bar{f} = \overline{g \circ f}$, where the closure operator denotes the image of an element under ϕ

$$\begin{tikzcd} (c,d) \arrow[d, shift left, "\mathrm{id}_g"] & \operatorname{Hom}_{\mathcal{D}}(Lc,d) \arrow[r, shift left, "\phi_{c,d}"] \arrow[d, "?\circ Lf"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd) \arrow[l, shift left, "\bar{(-)}"] \arrow[d, "?\circ f?"] \\ (c',d) \arrow[u, shift left, "f"] & \operatorname{Hom}(Lc',d) & \operatorname{Hom}_{\mathcal{C}}(c',Rd) \arrow[l, "\bar{(-)}"] \end{tikzcd}$$

which means that for all $k\colon c \to Rd$ and all $f\colon c' \to c$ we have that $\bar{k} \circ Lf = \overline{k \circ f}$.

Remark 4.2. For $c \in \mathcal{C}$ consider η_c given as follows

$$\operatorname{Hom}_{\mathcal{D}}(Lc, Lc) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(c, RLc)$$

$$\mathrm{id}_{Lc} \longmapsto \eta_c := \overline{\mathrm{id}_{Lc}}$$

as well as for any $d \in \mathcal{D}$ consider ϵ_d given as follows

$$\operatorname{Hom}_{\mathcal{C}}(Rc, Rc) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(LRd, d)$$

$$\mathrm{id}_{Rc} \longmapsto \epsilon_d := \overline{\mathrm{id}_{Rd}}$$

Proposition 4.3. *Notice that $\eta = (\eta_c\colon c \to RLc \mid c \in \mathcal{C})$ is a natural transformation $\eta\colon \mathbb{1}_{\mathcal{C}}\ RL$ we call this the unit of the adjunction and similarly $\epsilon = (\epsilon_c\colon c \to LRd \mid d \in \mathcal{C})$ is a natural transformation $\epsilon\colon LR \to \mathbb{1}_{\mathcal{D}}$ and is a called the counit of the adjunction.*

Proof. Consider the following square:

$$\begin{tikzcd} c \arrow[d, "f"] \\ c' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] & RLc \arrow[d, "RLf"] \\ c' \arrow[r, "\eta_{c'}"'] & c' \end{tikzcd}$$

and the resulting equation

$$\begin{aligned} & RLf \circ \eta_c = \eta'_c \circ f && (1)\\ \iff & \overline{RLf \circ \eta_c} = \overline{\eta_{c'} \circ f} && (2)\\ \iff & \overline{\overline{Lf}} = \overline{\mathrm{id}_{c'} \circ Lf} && (3)\\ \iff & Lf = \mathrm{id}_f \circ Lf && (4) \end{aligned}$$

□

Proposition 4.4. *The unit $\eta\colon \mathbb{1}_{\mathcal{C}} \Rightarrow RL$ aand the counit $\epsilon\colon LR \Rightarrow \mathbb{1}_{\mathcal{D}}$ satisfy the triangle identities:*

$$\begin{tikzcd} & LRLc \arrow[dr, "\epsilon_{Lc}"] & \\ Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc \end{tikzcd} \qquad \begin{tikzcd} & RLRd \arrow[dr, "R(\epsilon_d)"] & \\ Rd \arrow[ur, "\eta_{Rd}"] \arrow[rr, "\mathrm{id}_{Rd}"'] & & Rd \end{tikzcd}$$

For all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Proof. If we take the triangle for the unit above and apply the bar-operator we obtain the following

$$\begin{tikzcd} & LRLc \arrow[dr, "\epsilon_{Lc}"] & \\ Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc \end{tikzcd} \quad \overset{\overline{(-)}}{\longleftrightarrow} \quad \begin{tikzcd} & RLc \arrow[dr, "\mathrm{id}_{RLc}"] & \\ c \arrow[ur, "\eta_c"] \arrow[rr, "\overline{\mathrm{id}_c} = \eta_c"'] & & RLc \end{tikzcd}$$

The second triangle clearly commutes, the argument for the counit is analogous.

□

Proposition 4.5. *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be two functors between categories and suppose there exist natural transformations $\eta\colon \mathbb{1}_{\mathcal{C}} \Rightarrow RL$ and $\eta\colon LR \Rightarrow \mathbb{1}_{\mathcal{D}}$ that satisfy the triangle identities. Then the following defines an adjunction $L \dashv R$*

$$\phi\colon \mathrm{Hom}_{\mathcal{D}}(Lc, d) \rightleftarrows \mathrm{Hom}_{\mathcal{C}}(c, Rd)\colon \psi$$

$$Lc \xrightarrow{g} d \ \substack{\longmapsto \\ \longleftarrow\!\shortmid} \ R(g) \circ \eta_c$$

Proof. For $g\colon Lc \to d$ we have that

$$g = (\psi \circ \phi)(g) = \psi(R(g) \circ \eta_c) = \epsilon_d \circ LR(g) \circ L(\eta_c)$$

Now we have the following the diagram

$$\begin{tikzcd}
 & LRLc \arrow[rr, "LRg"] \arrow[dr, "\epsilon_{Lc}"] & & LRd \arrow[dr, "\epsilon_d"] & \\
Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc \arrow[rr, "g"'] & & d
\end{tikzcd}$$

The triangle commutes due to the triangle identities and the square commutes by the naturality of the counit. thus the whole diagram commutes, which means that ψ is an inverse to ϕ which yields the statement. □

Definition 4.6. For $c \in \mathcal{C}$ and $R\colon \mathcal{D} \to \mathcal{C}$ a functor, define the category c/R with objects given by tuples (c, f) where $f\colon c \to R(d)$ is a morphism for some $d \in \mathcal{D}$ and morphisms are given by

$$\begin{tikzcd}
d \in \mathcal{D} \arrow[d, "g"] & c \arrow[r, "f"] \arrow[d, equal] & Rd \arrow[d, "R(g)"] \\
d' \in \mathcal{D} & c \arrow[r, "f'"'] & Rd'
\end{tikzcd}$$

Dually for $L\colon \mathcal{C} \to \mathcal{D}$ and $d \in \mathcal{D}$ we define L/d as tuples (d, g) where $g\colon Lc \to g$ is a morphism and morphisms are given by

$$\begin{tikzcd}
c \in \mathcal{C} \arrow[d, "f"] & Lc \arrow[r, "g"] \arrow[d, "Lf"'] & d \arrow[d, equal] \\
c' \in \mathcal{C} & Lc' \arrow[r, "g'"'] & d
\end{tikzcd}$$

Notice that given an adjunction $L \dashv R$ we have that $\forall c \in \mathcal{C}(c, c \xrightarrow{\eta_c} RLc)$ is in c/R and $\forall \in \mathcal{D}(d, LRd \xrightarrow{\epsilon_d} d)$ is in L/d.

Proposition 4.7. *The object $(c, c \xrightarrow{\eta_c} RLc)$ is initial in c/R and $(d, LRd \xrightarrow{\epsilon_d} d)$ is final in L/d.*

Proof. Consider the following commutative triangle

$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[dr, "\overline{\overline{f}}=f"'] & RLc \arrow[d, "R(\overline{f})"] \\
 & Rd
\end{tikzcd}$$

where $\overline{\overline{f}} = \overline{R(\overline{f}) \circ \eta_c}$. Thus the morphism f of the object (c, f) uniquely determines the morphism $R(\overline{f})$. The argument for final object is dual to this one. □

Lecture 24.10

Proposition 4.8. *Let*

$$\mathcal{C} \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathcal{D} \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathcal{E}$$

be adjunctions then their composition

$$\mathcal{C} \underset{R_1 \circ R_2}{\overset{L_1 \circ L_2}{\rightleftarrows}} \mathcal{D}$$

is an adjunction as well.

Proof. Consider the following transformations, given by the adjunction isomorphisms

$$\mathrm{Hom}_{\mathcal{E}}(L_2 L_1 c, e) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(L_1 c, R_2 e) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, R_1 R_2 e)$$

□

Remark 4.9. Given an adjunction

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

then the following is an adjunction as well

$$\mathcal{C}^{\mathrm{op}} \underset{R^{\mathrm{op}}}{\overset{L^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{op}}$$

and the unit $\eta_c \colon c \to RLc$ in $\mathcal{C}$ corresponds to the counit $c \leftarrow RLc$ in $\mathcal{C}^{\mathrm{op}}$

Proposition 4.10. *Let $L_1; L_2 \colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : R$ be functors. Suppose that $L_1 \dashv R$ and $L_2 \dashv R$ are adjunctions, then it follows that $L_1 \cong L_2$.*

Proof. We need to construct a natural isomorphism $\phi \colon L_1 \xrightarrow{\sim} L_2$. Let $\eta^{(1)} \colon \mathbb{1} \to RL_1$ and $\eta^{(2)} \colon \mathbb{1}_{\mathcal{C}} \to RL_2$. By the uniqueness of initial objects in c/R, for $c \in \mathcal{C}$ **??** 4.7 we obtain

$$\begin{tikzcd}
 & c \arrow[dl, "\eta_c^{(1)}"'] \arrow[d, "\eta_c^{(2)}"] \arrow[dr, "\eta_c^{(1)}"] & \\
RL_1 c \arrow[r, dashed, "Rg"'] & RL_2 c \arrow[r, dashed, "Rh"'] & RL_1 c
\end{tikzcd}$$

By the uniqueness of a morphism out of an initial object we obtain for the composition that $L_1 c \xrightarrow{g} L_2 c \xrightarrow{h} L_1 c$ is given by $h \circ g = \mathrm{id}_{L_1 c}$. Similarly one obtains $g \circ h = \mathrm{id}_{L_2 c}$. Thus $g =: U_c \colon L_1 c \xrightarrow{\sim} L_2 c$. We now have to check that U_c is actually a natural transformation of functors. Consider the following diagram

$$\begin{tikzcd}
c \arrow[d, "f"] & L_1 c \arrow[r, "U_c"] \arrow[d, "L_1 f"'] & L_2 c \arrow[d, "L_2 f"] \\
c' & L_1 c' \arrow[r, "U_c'"'] & L_2 c'
\end{tikzcd}$$

apply R to it

$$\begin{tikzcd}
 & c \arrow[dl, "\eta_c^{(1)}"'] \arrow[dr, "\eta_c^{(2)}"] \arrow[dd, "f"] & \\
RL_1c \arrow[rr, "R(U_c)"] \arrow[dd, "RL_1(f)"'] & & RL_2c \arrow[dd, "RL_2(f)"] \\
 & c' \arrow[dl, "\eta_{c'}^{(1)}"'] \arrow[dr, "\eta_{c'}^{(2)}"] & \\
RL_1c' \arrow[rr, "R(U_{c'})"] & & RL_2(c')
\end{tikzcd}$$

This yields the following equations

$$\begin{aligned}
R(U_{c'} \circ L_1(f)) \circ \eta_c^{(1)} &= R(U_{c'}) \circ \eta_{c'}^{(1)} \circ f && (5)\\
&= \eta_{c'}^{(2)} \circ f && (6)\\
&= RL_2(f) \circ \eta_c^{(2)} && (7)\\
&= RL_2(f) \circ R(U_c) \circ \eta_c^{(1)} && (8)\\
&= R(L_2(f) \circ U_c) \circ \eta_c^{(1)} && (9)
\end{aligned}$$

And thus results in the following commutative triangle

$$\begin{tikzcd}
c \arrow[r, "\eta_c^{(1)}"] \arrow[dr, bend right, "\eta_{c'}^{(2)} \circ f"'] & RL_1c \arrow[d, shift right, "R(U_{c'} \circ L_1(f))"'] \arrow[d, shift left, "R(L_2(f) \circ U_c)"] \\
 & RL_2(c)
\end{tikzcd}$$

Both compositions give an initial object in c/R by **??** 4.7, thus by the uniqueness of an initial object they are equal in c/R and thus $U_{c'} \circ L_1(f) = L_2(f) \circ U_c$. $\square$

Proposition 4.11. *Let* $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ *be an adjunction then* L *preserves colimits that exist in* $\mathcal{C}$ *and* R *preserves limits that exist in* $\mathcal{D}$.

Proof. Let $X\colon A \to \mathcal{C}$ be a diagram that admits a colimit in $\mathcal{C}$, $\operatorname{colim} X_a \in \mathcal{C}$ and $a \in A$.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(L(\operatorname{colim}_{a\in A} X_a), d) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_{a\in A} X_a, Rd) && (10)\\
&\xrightarrow{\sim} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X_a, Rd) && (11)\\
&\xrightarrow{\sim} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(LX_a, d) && (12)\\
& && (13)
\end{aligned}$$

This exhibits $L(\operatorname{colim}_{a\in A} X)$ as a colimit of $X\colon A \xrightarrow{X} \mathcal{C} \xrightarrow{L} \mathcal{D}$. $\square$

Let A be a small category and $\mathcal{C}$ a category. Consider the functor $\mathrm{const}_A : \mathcal{C} \to \mathrm{Fun}(A, \mathcal{C})$ that maps each object of $\mathcal{C}$ to the functor $F_c(a) = c$ for all $a \in A$ and each morphism to id_c.

Proposition 4.12. *Suppose that there exists $L\colon \mathrm{Fun}(A, \mathcal{C}) \to \mathcal{C}$ a left adjoint to const_A. Then for all $X\colon A \to \mathcal{C}$ the unit $\eta_X : X \to \mathrm{const}_A(LX)$ exhibits LX as a colimit of X.*

Proof. We know by **??** 4.7 that $\eta_X : X \to \mathrm{const}_A(LX)$ is initial in X/const_A. Notice that the objects of X/const_A are pairs $(c \in \mathcal{C}, \rho\colon X \Rightarrow \mathrm{const}_A(c))$. Thus $\bar{\rho} = (\rho_a\colon X_a \to c \mid a \in A)$ is such that

$$\begin{tikzcd} a \arrow[d, "u"] & X_a \arrow[r, "\rho_a"] \arrow[d, "X_u"'] & c \arrow[d, "\mathrm{id}_c"] \\ b & X_b \arrow[r, "\rho_b"] & c \end{tikzcd}$$

Let furthermore $f\colon c \to c'$ be a morphism inducing a morphism in X/const_A

$$\begin{tikzcd} X \arrow[r, Rightarrow, "\bar{\rho}"] \arrow[d, equal] & \mathrm{const}_A(c) \arrow[d, "\mathrm{const}_A(f)"] \\ X \arrow[r, Rightarrow, "\bar{\sigma}"] & \mathrm{const}_A(c') \end{tikzcd}$$

given evaluated on objects $a \in A$ by

$$\begin{tikzcd} X_a \arrow[r, "\phi_a"] \arrow[d, equal] \arrow[dr, "\sigma_a"] & c \arrow[d, "f"] \\ X_a \arrow[r, "\sigma_a"'] & c' \end{tikzcd}$$

The two diagrams above tell us, that c together with the ϕ_a is a cocone and combining this with the universality of the initial object, we obtain that LX is a colimit. ∎

Proposition 4.13. *Let $\mathcal{C} \leftarrow \mathcal{D} : R$ be such that for all $c \in \mathcal{C}$ the functor $\mathrm{Hom}_{\mathcal{C}}(c, R(-))\colon \mathcal{D} \to \mathrm{Set}$ is corepresentable by an object $L(c) \in \mathcal{D}$ via $\phi_c\colon \mathrm{Hom}_{\mathcal{D}}(L(c), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(c, R(-))$. Then the association $c \mapsto L(c)$ can be promoted to a functor $L\colon \mathcal{C} \to \mathcal{D}$ that is adjoint to R via ϕ.*

Proof. We need to define L on morphisms. For $c \xrightarrow{f} c'$ in $\mathcal{C}$ consider the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(L(c), -) \arrow[r, "\phi_c"] & \mathrm{Hom}_{\mathcal{C}}(c, R(-)) \\ \mathrm{Hom}_{\mathcal{D}}(L(c'), -) \arrow[r, "\phi_{c'}"] \arrow[u, dashed, "L(f)^*"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "f^*"'] \end{tikzcd}$$

By Yoneda we obtain an object $d \in \mathcal{D}$ such that

$$\begin{tikzcd}
L(c) \arrow[d, "L(f)"'] \arrow[dr] & \\
L(c') \arrow[r] & d
\end{tikzcd}$$

Now we need to prove that this actually defines a functor $L\colon \mathcal{C} \to \mathcal{D}$. For $c \in \mathcal{C}$ we have that $L(\mathrm{id}_c) = \mathrm{id}_{L(c)}$ by construction. Let $c \xrightarrow{f} c' \xrightarrow{g} c''$ be in $\mathcal{C}$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(c), -) \arrow[r, "\phi_c"] & \mathrm{Hom}_{\mathcal{C}}(c, R(-)) \\
\mathrm{Hom}_{\mathcal{D}}(L(c'), -) \arrow[r, "\phi'_c"] \arrow[u, "L(f)^*"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "f^*"'] \\
\mathrm{Hom}_{\mathcal{D}}(L(c''), -) \arrow[r, "\phi''_c"] \arrow[u, "L(g)^*"] \arrow[uu, bend left=60, "L(g \circ f)"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "g^*"'] \arrow[uu, bend right=60, "g \circ f"']
\end{tikzcd}$$

The uniqueness given by Yoneda, implies

$$L(g \circ f) = L(g) \circ L(f)$$

Let $\phi_{c,d} = (\phi_c)_d$ be the isomorphism $\mathrm{Hom}_{\mathcal{D}}(L(c), d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, R(d))$. We need to show that $\varphi\colon \mathrm{Hom}_{\mathcal{D}}(L(-), ?) \to \mathrm{Hom}_{\mathcal{C}}(-, R(?))$ is a natural transformation of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$. Thus let $(c, d), (c', d') \in \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ be objects and $f\colon c' \to c$ as well as $g\colon d \to d'$ morphisms. Then the commutativity of the following diagram yields the result.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(c), d) \arrow[r, "\varphi_{c,d}"] \arrow[d, "g_* \circ \mathrm{id}^*_{L(c)}"] \arrow[dd, bend right=60, "g_* \circ L(f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(c, R(d)) \arrow[d, "R(g)_* \circ \mathrm{id}^*_c"] \arrow[dd, bend left=60, "R(g)_* \circ f^*"] \\
\mathrm{Hom}_{\mathcal{D}}(L(c), d') \arrow[r, "\varphi_{c,d'}"] \arrow[d, "L(f)^*"] & \mathrm{Hom}_{\mathcal{C}}(c, R(d')) \arrow[d, "f^*"] \\
\mathrm{Hom}_{\mathcal{D}}(L(c'), d') \arrow[r, "\varphi_{c',d'}"] & \mathrm{Hom}_{\mathcal{C}}(c', R(d'))
\end{tikzcd}$$

Since the two inner squares as well as the triangles on the sides commute, we obtain that the whole diagram commutes. $\square$

Proposition 4.14. *Suppose that $\mathcal{C}$ has all colimits of shape A. Then $\mathrm{const}_A\colon \mathcal{C} \to \mathrm{Fun}(A, \mathcal{C})$ admits a left adjoint.*

Proof. For $X \in \mathrm{Fun}(A, \mathcal{C})$ we have $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{a \in A} X, c) \xrightarrow{\sim} \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(Xa, c) = \mathrm{Hom}_{X/\mathrm{const}_A}(X, \mathrm{const}_A(c))$ since $\overline{\phi}\colon (\phi\colon X_a \to c \mid a \in A)$ belongs to the right hand side if and only if for all $a \xrightarrow{u} b$ in A, we have that

$$\begin{tikzcd}
& c & \\
X_a \arrow[ur, "\phi_a"] \arrow[rr, "X_u"'] & & X_b \arrow[ul, "\phi_b"']
\end{tikzcd}$$

That is $(c, \overline{\phi})$ is a cone under X, that is $\overline{\phi} : X \to \mathrm{const}_A(c)$. □

4.1 Exercises

Exercise 1. Consider two small categories A and B and a functor $F\colon B \to \mathrm{Fun}(A, \mathcal{C})$. Assume further that $\lim_B(\mathrm{ev}_a \circ F)$ exists in $\mathcal{C}$ for every $a \in A$.

(a) Show that a cone C of F is a limit cone if and only if for every $a \in A$ the evaluation $C(a)$ is a limit cone of $\mathrm{ev}_a \circ F$.

(b) Deduce that for any small category A the category of presheaves $\widehat{A}$ is complete and cocomplete.

Bonus We aim to show that the converse of the above is not true in general, i.e if not all $\lim_B(\mathrm{ev}_a \circ F)$ exists, then $\lim_B F$ might still exist (and consequentially $(\lim_B F)(a)$ is not a limit cone of $\mathrm{ev}_a \circ F$ for some $a \in A$.

For this we will reformulate the property of a morphism being a monomorphism in terms of a pullback and show that there might exist monomorphisms in $\mathrm{Fun}(A, \mathcal{C})$ which are not pointwise monomorphisms. Fix $A := \{0 < 1\}$ to be the category with two objects and one morphism between them. Then $\mathrm{Fun}(A, \mathcal{C})$ is the morphism category in $\mathcal{C}$. We now take $\mathcal{C}$ to be the category which does contain a non-monomorphism, explicitely let $\mathcal{C}$ be given by

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y \arrow[r, "h"] & z \end{tikzcd}$$

with the relation $h \circ f = h \circ g$.

- Show that a morphism $u\colon s \to t$ in a category is a monomorphism if and only if

$$\begin{tikzcd} s \arrow[r, "\mathrm{id}_s"] \arrow[d, "\mathrm{id}_s"] & s \arrow[d, "u"] \\ s \arrow[r, "u"] & u \end{tikzcd}$$

is a pullback diagram.

- Show that there is a monomorphism $u\colon f \to h$ in $\mathrm{Fun}(A, \mathcal{C})$ such that u_1 is not a monomorphism.
- Describe a B and a functor $F\colon B \to \mathrm{Fun}(A, \mathcal{C})$ giving the desired counterexample.

Exercise 2. Let $F\colon A \to \mathcal{C}$ be a functor from a small category. Let $\tilde{F}\colon A \to \mathcal{C} \to \widehat{\mathcal{C}}$ be the composition of F with the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$. Recall from Exercise 3.1 that the limit $\lim_A \tilde{F}$ exists.

(a) Show that there is a bijection between representations of $\lim_a \tilde{F}$ and limit cones of F.

(b) Deduce that the limit of F exists if and only if $\lim_A \tilde{F}$ is representable, i.e. there exists some $c_F \in \mathcal{C}$ such that $\lim_A \tilde{F} \cong \mathrm{Hom}_{\mathcal{C}}(-, c_F)$.

(c) Conclude that the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ preserves limits.

Exercise 3. Show that the inclusion $\iota\colon \mathrm{Gpd} \to \mathrm{Cat}$ of small groupoids into the category of small categories has a right adjoint given by "forgetting" all non-isomorphisms in a given category.

Exercise 4. Let $L \dashv R$ be an adjoint pair of functors where $L\colon \mathcal{C} \to \mathcal{D}$. Recall that we define unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to R \circ L$ the image of $\mathrm{id}_{L(c)}$ under the adjunction isomorphism

$$\phi_{(c,L(c))}\colon \mathrm{Hom}_{\mathcal{D}}(L(c), L(c)) \cong \mathrm{Hom}_{\mathcal{C}}(c, R(L(c)))$$

for every $c \in \mathcal{C}$ and the counit $\epsilon\colon L \circ R \to \mathrm{id}_{\mathcal{D}}$ as the image of $\mathrm{id}_{R(d)}$ under the adjunction isomorphism

$$\phi^{-1}_{(R(d),d)}\colon \mathrm{Hom}_{\mathcal{C}}(R(d), R(d)) \cong \mathrm{Hom}_{\mathcal{D}}(L(R(d)), d)$$

for every $d \in \mathcal{D}$. Show the following

1. The assignment ϵ is indeed a natural transformation.

2. Without using $L \dashv R$, show that $\tilde{\phi}_{(c,d)} := \eta_c^* \circ R_{L(c),d}$ defines a natural transformation

$$\tilde{\phi}\colon \mathrm{Hom}_{\mathcal{D}}(L(-), ?) \to \mathrm{Hom}_{\mathcal{C}}(-, R(?)).$$

3. There is an adjunction $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$ between the opposite categories with uni $\epsilon^{\mathrm{op}}\colon \mathrm{id}_{\mathcal{D}^{\mathrm{op}}} \to L^{\mathrm{op}} \circ R^{\mathrm{op}}$ and counit $\eta^{\mathrm{op}}\colon R^{\mathrm{op}} \circ L^{\mathrm{op}} \to \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

4. For any $d \in \mathcal{D}$ the category L/d has the final object $(R(d), \epsilon_d\colon LR(d) \to d)$.

5. Given a second adjunction $L' \dashv R'$ with $L'\colon \mathcal{D} \to \mathcal{A}$ and a unit η' and counit ϵ', the counit of the composed adjunction $L' \circ L \dashv R \circ R'$ is given by $\epsilon'_{(-)} \circ L'(\epsilon_{R'(-)})$.

6. Any right adjoint R' of L, $L \dashv R'$, is isomorphic to R.

5 Extending functors by colimits

Consider $X \in \mathrm{Set} = \widehat{\mathbb{1}} = \mathrm{Fun}(\mathbb{1}^{op}, \mathrm{Set})$ with $\mathbb{1}$ the category $\{* \circlearrowleft \mathrm{id}\}$. The following diagram

$$\begin{tikzcd} \{(*,y)\} \arrow[dr, hook] & \cdots & \{(*,z)\} \arrow[dl, hook'] \\ & X \cong \coprod_{x\in X}\{x\} & \end{tikzcd}$$

exhibits X as colimit. Now consider a small category A, a presheaf $X \in \widehat{A}$, a morphism $u\colon a \to b$ in A and elements $s \in X_a, t \in X_b$ with

$$\begin{tikzcd} X_a & X_b \arrow[l, "u^*"'] \\ s & t \arrow[l, mapsto] \end{tikzcd}$$

Owing to Yoneda, we have a commutative diagram in $\widehat{A}$:

$$\begin{tikzcd} \widehat{a} \arrow[rr, "\widehat{u}"] \arrow[dr, "s"'] & & \widehat{b} \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Replacing $\mathbb{1}$ with a small category A we can generalize the construction from the beginning.

Definition 5.1. The <u>category of elements of X</u>, denoted $\int^A X$ has as objects the pairs $(a \in A, s \in X_a) = (a \in A, \widehat{a} \xrightarrow{s} X)$ with morphisms

$$\begin{tikzcd} a \arrow[d, "\forall u \in A"'] & \widehat{a} \arrow[r, "s"] \arrow[d, "u^*"'] & X \arrow[d, equal] \\ b & \widehat{b} \arrow[r, "t"'] & X \end{tikzcd}$$

given that $u^*(t) = s$. Note that there is a canonical projection $can\colon \int^A X \to A$.

We will see that the presheaf $X \in \widehat{A}$ acts as a colimit with $\int^A X$ as the indexing category.

Example 5.2.
- $A = \mathbb{1}, X \in \mathrm{Set}$. Then $\int^{\mathbb{1}} X = \{(*, s \in X) | s \in X\}$. A morphism $(*, s \in X) \xrightarrow{\mathrm{id}^*} (*, t \in X)$ requires $s = t$.
- M: monoid $\rightsquigarrow \widehat{BM} = \mathrm{Fun}(* \circlearrowleft M^{op}, \mathrm{Set})$. We have the following morphisms in $\int^{BM} X$:

  $$(*, x \in X) \xrightarrow{m \in M} (*, y \in X)$$

  with $m^*(y) = y \cdot m = x$, i.e. morphisms exist precisely within orbits.

- $b \in A \rightsquigarrow \int^A A(-, b)$. The morphisms are given by

$$\begin{tikzcd} a \arrow[d, "\forall u \in A"'] & (a \in A, f \in A(a,b)) \arrow[r, equal] \arrow[d, "u"'] & (a \in A, f\colon a \to b) \arrow[d, "u"] \\ a' & (a' \in A, g \in A(a',b)) \arrow[r, equal] & (a' \in A, g\colon a' \to b) \end{tikzcd}$$

  i.e. $u^*(g) = g \circ u = f$.

Consider the composite

$$\begin{tikzcd} \int^A X \arrow[r] & A \arrow[r, hook, "\mu"] & \widehat{A} \ni X \\ (a,s) \arrow[r, mapsto] & a \arrow[r, mapsto] & \widehat{a} \end{tikzcd}$$

The presheaf X has a canonical cone structure under this diagram, which is what we alluded to before:

$$\begin{tikzcd} (a,s) \arrow[d, "\substack{\forall u \in \int^A X \\ (u^*(t) = s)}"'] \\ (b,t) \end{tikzcd} \qquad \begin{tikzcd} \widehat{a} \arrow[rr, "\widehat{u}"] \arrow[dr, "s"'] & & \widehat{b} \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Proposition 5.3. *The cocone $(X, (s\colon \widehat{a} \to X)_{(a,s) \in \int^A})$ is a colimit of the composition $\int^A X \xrightarrow{can} A \xrightarrow{\mu} \widehat{A}$.*

Proof. Consider a cocone $(Y \in \widehat{A}, (\mu_{a,s} X\colon \widehat{a} \to Y)_{(a,s) \in \int^A})$, we need to prove that there exists a unique morphism of cocones $f\colon (X, (s)_{(a,s) \in \int^A}) \mapsto (Y, (\mu_{a,s})_{(a,s) \in \int^A X})$. Consider the tuple: $f\colon (f_a\colon X_a \to Y_a \mid a \in A)$ where $f_a\colon X_a \to Y_a$ and $f_a(s) = \mu_{(a,s)}$ to prove $f : X \to Y$ is a natural transformation.

$$\begin{tikzcd} a \arrow[d, "u"] \\ b \end{tikzcd} \qquad \begin{tikzcd} X_a \arrow[r, "f_a"] & Y_a \\ X_b \arrow[u, "u^*"] \arrow[r, "f_b"'] & Y_b \arrow[u, "u^*"'] \end{tikzcd} \qquad \begin{tikzcd} u^*(t) \arrow[r, mapsto] & \mu_{a,u^*(t)} \\ t \arrow[u, mapsto] \arrow[r, mapsto] & \mu_{b,u^*(t)} \arrow[u, mapsto, "u^*(\mu_{b,t})"] \end{tikzcd}$$

We need to prove that $f\colon (X, (s)_{(a,s) \in \int^A X}) \to (Y, (\mu_{a,s})_{(a,s) \in \int^A X})$ is a morphism of cocones. That is to show, that for $(a,s) \in \int^A X$

$$\begin{tikzcd} & \widehat{a} \arrow[dl, "s"'] \arrow[dr, "\mu_{a,s}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, so $f \circ s = \mu_{a,s}$, which means that $f_a(s) = \mu_{a,s}$, but this is true by the definition of f. $\square$

Remember that we have a natural isomorphism of functors $\mathrm{Hom}_{\widehat{A}}(A(-,?),X) \xrightarrow{\sim} X$. Suppose now we are given a functor $u\colon A \to \mathcal{C}$ and $\mathcal{C}$ has all small colimits. Consider the functor $u^*\colon \mathcal{C} \to \widehat{A}$, where $u^*(c) = \mathrm{Hom}_{\mathcal{C}}(u(-),c)$ which can be considered to be the composition $A^{\mathrm{op}} \xrightarrow{u^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,c)} \mathrm{Set}$.

Theorem 5.4. *Kan The functor $u^*\colon \mathcal{C} \to \widehat{A}$ admits a left adjoint $u_!\colon \widehat{A} \to \mathcal{C}$. Moreover $\exists! \phi\colon u_! \circ \mu \xrightarrow{\sim} u$ natural morphism such that for all $a \in A$ and $c \in \mathcal{C}$.*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(u_!(\widehat{a}),c) \arrow[d, "adj."] & \mathrm{Hom}_{\mathcal{C}}(u(a),c) \arrow[l, "\phi_a^*"'] \arrow[d, equal] \\
\mathrm{Hom}_{\widehat{A}}(\widehat{a},u^*(c)) \arrow[r, "\sim Yoneda"] & u^*(c)_a
\end{tikzcd}$$

Proof. It is enough to prove that for $X \in \widehat{A}$ the functor

$$\mathrm{Hom}_{\widehat{A}}(X,u^*(-))\colon \mathcal{C} \to \mathrm{Set}$$

is corepresentable. Consider the functor given by the composition $\int^A X \xrightarrow{p} A \xrightarrow{u} \mathcal{C}$. Since $\mathcal{C}$ is cocomplete, we may choose a colimit $(u_!(X),(f_{a,s}\colon u(a) \to u_!(X))_{(a,s)\in\int^A X})$. Then $\mathrm{Hom}_{\mathcal{C}}(u_!(X),c) \xrightarrow{\sim} \lim_{(a,s)\in\int^A X} \mathrm{Hom}_{\mathcal{C}}(u(a),c)$, since the Hom functor takes colimits in the first entry to limits of the Hom functor. Now by Yoneda **??** 2.6 $\lim_{(a,s)\in\int^A X} \mathrm{Hom}_{\mathcal{C}}(u(a),c) \xrightarrow{\sim} \lim_{(a,s)\in\int^A X} \mathrm{Hom}_{\widehat{A}}(\widehat{a},u^*(c))$ which is isomorphic to $\mathrm{Hom}_{\widehat{A}}(X,u^*(c))$. ☐

Remark 5.5. Note $u_!$ preserves colimits and u^* preserves limits, since they are the components of an adjunction.

Proposition 5.6. *Let $F\colon \widehat{A} \to \mathcal{C}$ be colimit preserving. Then $F \cong (F\circ\mu)_!$ and in particular it admits a right adjoint.*

Proof. For $X \in \widehat{A}$. We have $F(X) = F(\mathrm{colim}_{(a,s)\in\int^A X}\widehat{a}) \xrightarrow{\sim} \mathrm{colim}_{(a,s)\in\int^A X} F(\widehat{a}) = \mathrm{colim}_{(a,s)\in\int^A X}(F\circ\mu)(a) = (F\circ\mu)_!(X)$. The last equality can be found in the proof right above. ☐

Example 5.7. Let $1_{\widehat{A}}\colon \widehat{A} \to \widehat{A}$, then $\mathbb{1}_{\widehat{A}} \cong \mu_!$ where μ is the Yoneda embedding.

Lecture 31.10

Let us take a look at another application of the adjunction constructed above, that is to internal Hom functors. For $Y \in \widehat{A}$ let

$${}_{\times}Y\colon \widehat{A} \to \widehat{A}, X \mapsto X \times Y$$

preserves colimits and thus by **??** 5.6 admits a right adjoint $\underline{\mathrm{Hom}}_{\widehat{A}}(Y,Z)_a\colon \widehat{A} \to \widehat{A}$. Let $\underline{\mathrm{Hom}}(Y,Z)_a = \mathrm{Hom}_{\widehat{A}}(\widehat{a}\times Y,Z)$ we obtain

$$\mathrm{Hom}_{\widehat{A}}(X\times Y,Z) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{A}}(X,\underline{\mathrm{Hom}}_{\widehat{A}}(X\times Y,Z)).$$

Now for any $W \in \widehat{A}$, $\mathrm{Hom}_{\widehat{A}}(W, \underline{\mathrm{Hom}}_{\widehat{A}}(X \times Y, Z)) \cong \mathrm{Hom}_{\widehat{A}}(W \times (X \times Y), Z) \cong \mathrm{Hom}_{\widehat{A}}((W \times X) \times Y, Z) \cong \mathrm{Hom}_{\widehat{A}}(W \times X, \underline{\mathrm{Hom}}_{\widehat{A}}(Y, Z))$. Let $u \colon \int^A X \to \widehat{A}/X$, where $\widehat{A}/X$ is the category of presheaves over X, be the functor induced by Yoneda. Then $u_! \colon \widehat{\int_X^A} \to \widehat{A}/X$ is colimit preserving.

Theorem 5.8. *The functor $u_!$ given above is an equivalence of categories.*

Proof. At first observe that $u \colon \int^A X \to \widehat{A}/X$ is fully faithfull, since it is given by the composition of the Yoneda embedding with an isomorphism. Secondly $u_!(\widehat{(a, s)}) = u(a, s) = (\widehat{a} \xrightarrow{s} X)$ satisfies that $\mathrm{Hom}_{\widehat{A}/X}(u_!(\widehat{(a, s)}), -) \colon \widehat{A}/X \to \mathrm{Set}$ preserves small colimits. The third observation is that the collection $\{u_!(\widehat{(a, s)}) \mid (a, s) \in \int^A X\} \subseteq \widehat{A}/X$ generates under small colimits the whole category. Put together we get that $u_! \colon \widehat{\int^A X} \to \widehat{A}/X$ is an equivalence of categories. □

Consider the following $F \colon \widehat{A} \to \mathcal{D}$ where $\mathcal{D}$ is cocomplete and F a functor that preserves colimits.

Aim We want to prove that F is an equivalence and are going to do so in 3 steps:

1. $F \circ \mu \colon A \to \mathcal{D}$ is fully faithfull,
2. for all $a \in A$ the functor $\mathrm{Hom}_{\mathcal{D}}(F(\widehat{a}), -)$ preserves colimits,
3. and $\{F(\widehat{a}) \in \mathcal{D} \mid a \in A\} \subseteq \mathcal{D}$ generates under small colimits.

Suppose we are given functors $\widehat{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{C}$ and a natural transformation $\eta : F \to G$.

Proposition 5.9. *The natural transformation $\eta : F \to G$ is an isomorphism of functors, if and only if for all $a \in A$ the induced morphism $F(\widehat{a}) \xrightarrow{\eta_{\widehat{a}}} G(\widehat{a})$ is an isomorphism.*

Proof. Consider $\mu(A) \subseteq * = \{X \in \widehat{A} \mid \eta_X \colon FX \to GX$ is an iso $\subseteq \widehat{A}$. By the density theorem it is enough to prove that this category is closed under colimits. Let $\underline{X} \colon I \to X \subseteq \widehat{A}$ be a diagram. Consider $\mathrm{colim}_I \underline{X} \in \widehat{A}$. We need to prove $\eta_{\mathrm{colim}_I \underline{X}} \colon F(\mathrm{colim}_I \underline{X}) \to G(\mathrm{colim}_I \underline{X})$ is an isomorphism. To do so consider the diagram:

$$\begin{tikzcd}
\mathrm{colim}_I F\underline{X} \arrow[r, dashed, "\mathrm{colim}_I \eta_{\underline{X}}"] \arrow[d, "\sim"] & \mathrm{colim}_I G\underline{X} \arrow[d, "\sim"] \\
F(\mathrm{colim}_I X) \arrow[r, "\eta_{\mathrm{colim}_I \underline{X}}"] & G(\mathrm{colim}_I \underline{X})
\end{tikzcd}$$

□

5.1 Exercises

Exercise 1. Let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories.

(a) For $d \in \mathcal{D}$ describe the category of elements of the presheaf $\mathrm{Hom}_{\mathcal{D}}(L(-), d) \in \widehat{\mathcal{C}}$.

(b) Show that L admits a right adjoint if and only if for every $d \in \mathcal{D}$ the category of elements $\int^{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(L(-), d)$ admits a final object.

Exercise 2. Let A be a small category and $a \in A$ some object.

(a) Show that $\mathrm{Hom}_{\widehat{A}}(\widehat{a}, -)\colon \widehat{A} \to \mathrm{Set}$ preserves colimits, i.e. for any functor $F\colon I \to \widehat{A}$ the canonical map

$$\mathrm{colim}_I(\mathrm{Hom}_{\widehat{A}}(\widehat{a}, -) \circ F) \to \mathrm{Hom}_{\widehat{A}}(\widehat{a}, \mathrm{colim}_I F)$$

is an isomorphism.

(b) Deduce that $\mathrm{Hom}_{\widehat{A}}(\widehat{a}, -)$ admits a right adjoint and describe it.

Exercise 3. Let $u\colon A \to B$ be a functor between small categories. Let $u*\colon \widehat{B} \to \widehat{A}$ denote the functor obtained by precomposition with u.

(a) Show that u^* preserves colimits.

(b) Deduce that there exists a right adjoint $u^* \dashv u_*$.

(c) Give an explicit description of u_*.

(d) Confirm directly that $u^* \dashv u_*$ by giving the adjunction isomorphism explicitly.

Exercise 4. Let X be a presheaf over a small category A. Recall from the lecture the canonical functor.

$$u\colon \int^A X \to \widehat{A}/X$$

sending $(a, s \in X_a)$ to $s\colon \widehat{a} \to X$, where $\int^A X$ is the category of elements of X. Hence, by extending by colimits we obtain a functor

$$u_!\colon \widehat{\int^A X} \to \widehat{A}/X$$

which we aim to show is equivalence, most of which was done in the lecture. Show the remaining claims.

(a) The slice category $\widehat{A}/X$ is cocomplete.

(b) For any $(a, s\colon a \to X) \in \int^A X$ the functor $\mathrm{Hom}_{\widehat{A}/X}(u_!(\widehat{(a,s)}), -)$ preserves colimits.

(c) Any $(Y, f\colon Y \to X)$ can be obtained as a colimit of a diagram in the essential image of u.

Exercise 5. Let $\mathcal{C}$ and $\mathcal{D}$ be two cocomplete categories where $\mathcal{C}$ is small and let $D\colon I \to \mathcal{C}$ be a diagram in $\mathcal{C}$.

(a) Show that there is a natural transformation of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D} \to \mathcal{D}$

$$can\colon \operatorname{colim}_I D^*(-) \to \mathrm{ev}_{\operatorname{colim}_I D}\,.$$

(b) Deduce that if F and G are two isomorphic funtors in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, then F is colimit preserving if and only if G is.

Exercise 6. Consider an adjunction $L \dashv R$ where $L\colon \mathcal{C} \to \mathcal{D}$.

(a) Show that for a $d \in \mathcal{D}$ the counit ϵ_d at d is an isomorphism if and only if R induces a natural isomorphism

$$R_{d,-}\colon \mathrm{Hom}_{\mathcal{D}}(d, -) \to \mathrm{Hom}_{\mathcal{C}}(R(d), R(-)).$$

(b) Deduce that R is fully faithful if and only if the counit $\epsilon\colon L \circ R \to \mathrm{id}_{\mathcal{D}}$ is an isomorphism.

(c) Give the dual statement to (b) and give a proof reducing the statement to (b).

(d) Show that if R is fully faithful, then $c \in \mathcal{C}$ is in the essential image of R if and only if the unit morphism η_c is an isomorphism at c.

Exercise 7. Let $u\colon A \to B$ be a functor between small categories. Recall from the lecture and Exercise 4.3 that we have a triple of adjunctions $u_! \dashv u^* \dashv u_*$ where $u^*\colon \widehat{B} \to \widehat{A}$. Assume further that u is fully faithful.

(a) Show that u_* is fully faithful.

(b) Show that $u_!$ is fully faithful.
(Hint: Show that the class $\{X \in \widehat{A} \mid \eta_X \text{ is invertible }\}$ is closed under colimits and contains all representable presheaves.)

6 Simplicial sets

Definition 6.1. The simplicial category Δ has objects $[n] \coloneqq \{0 < 1 < 2 < \ldots < n\}$ for $n \geqslant 0$ and morphisms $\Delta(m,n) \coloneqq \mathrm{Hom}_\Delta([m],[n]) \coloneqq \{f\colon [m] \to [n], \text{order preserving}\}$.

Definition 6.2. The category of simplicial sets is given by $\mathrm{Set}_\Delta = \mathrm{sSet} = \widehat{\Delta} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Remark 6.3. An alternative definiton of a simplicial set X can be given as follows:

- For all $n \geqslant m$ a set X_n called the n-simplices of X.
- For all $0 \leqslant i \leqslant n$ morphisms $d_i\colon X_n \to X_{n+1}$ called the face maps.
- For all $0 \leqslant i \leqslant n$ morphisms $s_i\colon X_n \to X_{n+1}$ called the degeneracy maps.
- The face and degeneracy maps satisfy the following identities:

$$\begin{array}{ccc} \underset{i<j}{d_i d_j = d_{j-1} d_i} & \underset{i<j}{d_i s_j = s_{j-1} d_i} & d_j s_j = \mathrm{id} = d_{j+1} s_j \\ \underset{i>j+1}{d_i s_j = s_j d_{i-1}} & \underset{i \leqslant j}{s_i s_j = s_{j+1} s_i} & \end{array}$$

Example 6.4. 1. For an arbitrary simplicial set we often write

$$\begin{tikzcd} X\colon \ldots & X_3 \arrow[r, shift left=3] \arrow[r, shift left=2, leftarrow] \arrow[r, shift left=1] \arrow[r, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r, shift left=1, leftarrow] \arrow[r] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=2] & X_1 \arrow[r, shift left=1] \arrow[r, leftarrow] \arrow[r, shift right=1] & X_0 \end{tikzcd}$$

where the arrows correspond to the face and boundary maps.

- $[0] = \{0\}$
- $[1] = \{ \begin{tikzcd} 0 \arrow[loop left, "id"] \arrow[r, "10"] & 1 \arrow[loop right, "id"] \end{tikzcd} \}$
- $[2] = \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\}$
- $[3] = \left\{ \begin{tikzcd} & 1 \arrow[dr] \arrow[dd] & \\ 0 \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \right\}$

Let $d^i\colon [n-1] \to [n]$ be the unique order preserving injective map not having $i \in [n]$ in its image for all $0 \leqslant i \leqslant n$.

$$[0] = \{0\} \xrightarrow{d^0} \{0 \to 1\} = [1]$$

$$[0] = \{0\} \xrightarrow{d^0} \{0 \to 1\} = [1]$$

$$\left\{ [1] = \begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd} \right\} \xrightarrow{d^0} \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\} = [2]$$

We obtain for any simplicial set X a diagram

$$\begin{tikzcd} X_n \arrow[r, "d_i"] & X_{n-1} \\ \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \arrow[u, "\wr"] \arrow[r, "? \circ d^i_*"] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n-1}, X) \arrow[u, "\wr"] \end{tikzcd}$$

and thus for any $x \in X_n$ a diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "x"] & X \\ \Delta^{n-1} \arrow[u, "d^i_*"] \arrow[ur] & \end{tikzcd} .$$

For $n = 2$ this looks explicitely as follows

$$\begin{tikzcd} & 1 \arrow[dr] & & \\ 0 \arrow[ur] \arrow[rr] & & 2 \arrow[r, "x"] & X \\ & & & \\ 0 \arrow[rr] & & 1 \arrow[uur] & \end{tikzcd}$$

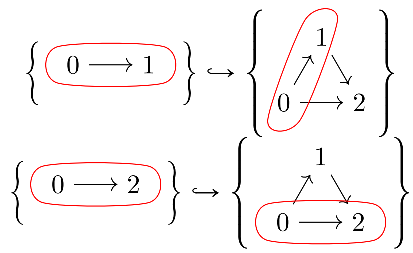
Definition 6.5. The category Δ_{big} has as objects the finite non-empty total orders with order preserving maps between them

$$\Delta \leftrightarrows \Delta_{\mathrm{big}} \ni I = \{i_0 < i_1 < \ldots < i_n\}$$

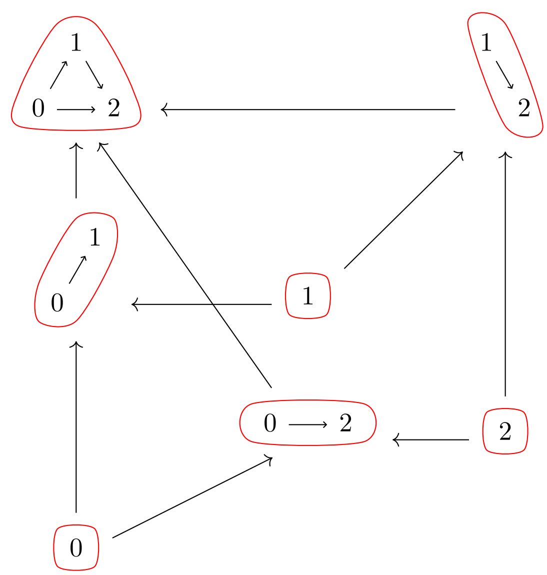
This time we take a closer look at the diagrams that arise from the inclusions of partial orders.

$$\{1\} \longhookrightarrow \{0 \to 1\}$$

$$\{0\} \longhookrightarrow \{0 \to 1\}$$



These inclusions yield the following poorly organized faces of a square



which looks applied to a simplicial set as follows

$$\begin{tikzcd}
X_2 \arrow[rr] \arrow[dr] \arrow[dd] & & X_{\{1\to 2\}} \arrow[dr] \arrow[dd] & \\
& X_{\{0\to 2\}} \arrow[rr] \arrow[dd] & & X_{\{2\}} \\
X_{\{0\to 1\}} \arrow[rr] \arrow[dr] & & X_{\{1\}} & \\
& X_{\{0\}} & &
\end{tikzcd}$$

We furthermore have the co-degeneracy maps $s^i\colon [n+1] \to [n]$ which are the unique order preserving surjective map, that take the value i twice. This can be visualised as follows:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \dots \arrow[r] & i-1 \arrow[r] \arrow[d] & i \arrow[r] \arrow[d] & i+1 \arrow[r] \arrow[dl] & \dots \arrow[r] & n+1 \arrow[dl] \\
0 \arrow[r] & 1 \arrow[r] & \dots \arrow[r] & i-1 \arrow[r] & i \arrow[r] & \dots \arrow[r] & n &
\end{tikzcd}$$

and we obtain the following simplex diagram

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r] \arrow[dr, dashed] & \Delta^n \arrow[d, "x"] \\
& X
\end{tikzcd}$$

Example 6.6. Let $I \in \mathrm{Set}$, we can define the constant simplicial set associated to I, where for all $n \geqslant 0$ we have $I_n = I$, the faces and degeneracies are the identity functor.

Example 6.7. Let $\mathcal{C}$ be a small category. We define $N(\mathcal{C}) \in \mathrm{Set}_\Delta$ as follows. Let $N(\mathcal{C})_0 \coloneqq \mathrm{Ob}(\mathcal{C})$, $N(\mathcal{C})_1 = \mathrm{Mor}(\mathcal{C})\colon$ set of morphisms in $\mathcal{C}$ with the face and boundary maps given as follows

$$\begin{tikzcd}
N(\mathcal{C})_1 \arrow[r, shift left=2, "d_0 = target"] \arrow[r, shift right=2, "d_1 = source"'] & N(\mathcal{C})_0 \arrow[l, "s_1" description]
\end{tikzcd}$$

where $s_1(X) = X \xrightarrow{\mathrm{id}_X} X$ for an $X \in \mathrm{Ob}(\mathcal{C})$. Now

$$N(\mathcal{C})_2 = \left\{ \begin{tikzcd}
& f_1 \arrow[dr, "f_{21}"] & \\
f_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"] & & f_2
\end{tikzcd} \mid f_{20} = f_{21} \circ f_{10} \right\}$$

and we have degeneracies and codegeneracies

$$\begin{tikzcd}
N(\mathcal{C})_2 \arrow[r, shift left=4, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=4, "d_2"'] & N(\mathcal{C})_1 \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "s_1"]
\end{tikzcd} .$$

Now lastly

$$N(\mathcal{C})_3 = \left\{ \begin{tikzcd}
& f_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{13}"] & \\
f_0 \arrow[ur, "f_{10}"] \arrow[rr, dashed, "f_{20}"] \arrow[dr, "f_{30}"'] & & f_2 \arrow[dl, "f_{23}"] \\
& f_3 &
\end{tikzcd} \mid \forall i \leqslant j \leqslant k f_{kj} \circ f_{ji} = f_{ki} \right\}.$$

This gives a functor $N(\mathcal{C})\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ where $[n] \mapsto \mathrm{Fun}([n], \mathcal{C})$ and thus a simplicial set. Note that the nerve has a left adjoint given by the truncation functor.

Example 6.8. Let X be a topological space we define $\mathrm{Sing}(X)$ to be its associated singular simplicial set.

- Let $\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$, where $|\Delta^n| := \{v \in \mathbb{R}^{n+1}_{\leqslant 0}, |\sum_{i=0}^n v_i| = 1\}$.
- For a morphism $\sigma\colon [m] \to [n]$ we obtain a morphism $\sigma_*\colon |\Delta^m| \to |\Delta^n|$, where $\sigma_*(v)_i = \sum_{j \in \sigma^{-1}(i)} v_j$.

Note that Sing has a left adjoint given by the geometric realisation functor.

Theorem 6.9. *For all $X \in \mathrm{Set}_\Delta$ $|X| = \mathrm{colim}_{\substack{([n],x) \in \int_X^\Delta \\ X\colon \Delta^n \rightarrowtail X}} |\Delta^n|$ is a CW complex.*

Definition 6.10. An element x of simplicial set X is <u>degenerate</u> if $x \in \operatorname{im} s_i$ for some i.

Remark 6.11. Let $N([n]) = \mathrm{Hom}_{\mathrm{Cat}}([?], [n]) \cong \mathrm{Hom}_\Delta([?], [n]) = \Delta^n$. Furthermore for $\mathcal{P}$ a poset, let $N(\mathcal{P})_n = \mathrm{Hom}_{Poset}([n], \mathcal{P})$. Take now a morphism $[n] := \{0 < 1 < 2 < \ldots < n\} \xrightarrow{\sigma} \mathcal{P}$ and $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n) \in \mathcal{P}^{n+1}$ such that $\sigma_0 \leqslant \sigma_1 \leqslant \ldots \leqslant \sigma_n$, we call this a chain in the poset. Let us now compute $\Delta^2 = N([2])$, we denote the chain $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ by $\sigma_0\sigma_1 \ldots \sigma_n$ in the following table.

Δ^2_0		Δ^2_1			Δ^2_2	
$\boxed{0}$	00	$\boxed{01}$	$\boxed{02}$	000	001	002
$\boxed{1}$		11	$\boxed{12}$		011	$\boxed{012}$
$\boxed{2}$			22	111	112	022
					122	
				222		

The encircled chains correspond to non-degenerate simplices.

Definition 6.12. Let $X \in \mathrm{Set}_\Delta$ we write $\dim X \leqslant k$ if $\forall n > k$ we have that $X_n = X_n^{degeneracies}$.

We have a functor that goes from simplicial sets to simplicial groups and is similar to the free functor from set to abelian groups.

$$\begin{tikzcd} X \in \mathrm{Set}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) \arrow[d, "\mathbb{Z}"] \\ \mathrm{Ab}_\Delta = (\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab})) \end{tikzcd}$$

We write $(\mathbb{Z}X)_n = \mathbb{Z}\langle X_n \rangle$. Let now $Y \in \mathrm{Ab}_\Delta$ be a simplicial abelian group, we associate to it a chain complex $C(Y) \in \mathrm{Ch}_{\geqslant 0}(\mathrm{Ab})$, where $C(Y)_n := Y_n$ and the differential is given as

$$\begin{aligned} C(Y)_n &\to C(Y)_{n-1} \\ x &\mapsto \partial(x) = \sum_{i=0}^{n} (-1)^i d_i(x). \end{aligned}$$

Remark 6.13. 1. If we now take $X \in \mathrm{Top}$ we can take $\mathrm{Sing}(X) \in \mathrm{Set}_\Delta$ then $\mathbb{Z}\,\mathrm{Sing}(X) \in \mathrm{Ab}_\Delta$ and then $C(\mathbb{Z}\,\mathrm{Sing}(X)) \in \mathrm{Ch}_{\geqslant 0}$ with its n-th component being given by $\mathbb{Z}\langle \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)\rangle \in \mathrm{Ab}$. This complex is called the Moore complex and its associated homology groups give the integral singular homology $H_*(X;\mathbb{Z})$ of the space X. Furthermore we can define the homology groups $H_n(Y, A)$ of a simplical set Y with coefficients in an abelian group A to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

2. Let $\mathcal{A}$ be an exact category. Then $\mathcal{A}$ has an associated category $Q\mathcal{A}$ with objects those of $\mathcal{A}$ and arrows given by equivalence classes of diagrams

$$\bullet \twoheadleftarrow \bullet \to \bullet$$

where both arrows are parts of exact sequences of $\mathcal{A}$, and composition is represented by pullback. Then $K_{i-1}(\mathcal{A}) := \pi_i|BQ\mathcal{A}|$ defines the K-groups of $\mathcal{A}$ for $i \geqslant 1$; in particular $\pi_i|BQ\,\mathrm{proj}(R)| = K_{i-1}(R)$, the i^{th} algebraic K-group of the ring R, here $\mathrm{proj}(R)$ denotes the category of finitely generated projectives over R.

Definition 6.14. The normalized Moore chain of Y is the chain complex with components $\overline{C}(Y)_n := \bigcap_{i=1}^{n} \ker d_i$ and differentials $\partial_n = d_0^n$.

Theorem 6.15. *(Dold-Kan Correspondences) Let $\Delta \to \mathrm{Ch}_{\geqslant 0}$ be given by $[n] \mapsto \overline{C}(\mathbb{Z})\Delta^n$, then there exists an adjunction.*

$$\bar{C}\colon \mathrm{Ab}_\Delta \underset{\sim}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Ch}_{\geqslant 0}(\mathrm{Ab}) : DK$$

6.1 Exercises

Exercise 1. Show that the functor $\widehat{\Delta} \to \mathrm{Set}_\Delta$ which remembers only the face and degeneracy maps and is the identity on morphisms is an isomorphism of categories. In particular,

- show that the functor is well defined by showing that the co-face and co-degeneracy maps satisfy the cosimplicial identities.

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \quad \text{if } i < j \\ s^j s^i &= s^i s^{j+1} \quad \text{if } i \leqslant j \end{aligned}$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i \in \{j, j+1\} \\ d^{i-1} s^j & \text{if } i > j = 1 \end{cases}$$ Recall that the co-face maps $d^i :=$ d^i_n and co-degeneracy maps $s^i := s^i_n$ are defined as follows for each $n \in \mathbb{N}_+$ respective $n \in \mathbb{N}_0$ and $0 \leqslant i \leqslant n$.

$$\begin{aligned} d^i = d^i_n \colon [n-1] &\to [n] \\ k &\mapsto \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leqslant k \end{cases} \end{aligned}$$

$$\begin{aligned} s^i = s^i_n \colon [nn1] &\to [n] \\ k &\mapsto \begin{cases} k & \text{if } k \leqslant i \\ k-1 & \text{if } i < k \end{cases} \end{aligned}$$

- For the inverse, show that a morphism $\sigma \colon [m] \to [n]$ in Δ can be uniquely written as

$$\sigma = d^{i_s}_n \circ d^{i_{s-1}}_{n-1} \circ \cdots \circ d^{i_1}_{n-s+1} \circ s^{j_1}_{m-t} \circ \cdots \circ s^{j_{t-1}}_{m-2} \circ s^{j_t}_{m-1}$$

  with $0 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n$ and $0 \leqslant j_1 < j_2 < \cdots < j_t < m$ and $n - s = m - t$.

Exercise 2.

Fix $n \in \mathbb{N}_+$. Define the simplicial subset $\partial\Delta^n \subseteq \Delta^n$ by

$$\partial\Delta^n([m]) := \{\sigma \colon [m] \to [n] \text{ non-surjective }\}$$

and similarly, define for $0 \leqslant k \leqslant n$ the simplicial subset $\Lambda^n_k \subseteq \partial\Delta^n$ by

$$\Lambda^n_k([m]) := \{\sigma \in \partial\Delta^n([m]) \mid \sigma([m]) \neq [n] \backslash \{k\}\}.$$

(a) Confirm that the above are indeed simplicial subsets.

(b) Show one of the following

- $\partial\Delta$ is the smallest simplicial subset of Δ^n such that $\{d^i_n \mid 0 \leqslant i \leqslant n\} \subseteq \partial\Delta^n([n-1])$.
- Λ^n_k is the smallest simplicial subset of Δ^n such that $\{d^i_n \mid 0 \leqslant i \leqslant n \wedge i \neq k\} \subseteq \Lambda^n_k([n-1])$.

Recall that we may view Δ as a category of posets, so that we may compute the nerve of (subsets of) $[n]$ as partially ordered set. Moreover, we may associate to $[n]$ the category $\mathrm{Sub}_*([n])$ of non-empty proper full subcategories, i. e. $[n] \notin \mathrm{Sub}_*([n])$, and morphisms given by inclusions. Similarly, we define for $k \in [n]$ the full subcategory $\mathrm{Sub}^k_*([n]) := \{k \in E \in \mathrm{Sub}_*([n]) \subseteq \mathrm{Sub}_*([n])$.

(c) Show one of the following.

(a)
$$\partial\Delta^n \cong \bigcup_{E \in \mathrm{Sub}_*([n])} N(E) := \operatorname{colim}_{E \in \mathrm{Sub}_*([n])} N(E)$$

(b)
$$\partial\Delta^n \cong \bigcup_{E \in \mathrm{Sub}_*([n])} N(E) := \operatorname{colim}_{E \in \mathrm{Sub}_*([n])} N(E)$$

(d) Justify the notation $\bigcup$.

Exercise 3. Let $\Delta_{\leqslant n}$ be the full subcategory of Δ of the elements $[0], [1], \ldots, [n]$. The inclusion $\iota_n \colon \Delta_{\leqslant n} \to \Delta$ induces a truncation functor $\mathrm{Tr}_n := (\iota_n)^* \colon \mathrm{Set}_\Delta \to \widehat{\Delta_{\leqslant n}}$.

(a) Show that Tr_n admits both a left and a right adjoint, $\mathrm{sk}_n \dashv \mathrm{Tr}_n \dashv \mathrm{cosk}_n$, which are both fully faithful.

(b) Deduce that we have an adjunction $\mathbf{sk_n} := \mathrm{sk}_n \circ \mathrm{tr}_n \dashv \mathrm{cosk}_n \circ \mathrm{tr}_n =: \mathbf{cosk_n}$ of endofunctors of Set_Δ.

The essential image of sk_n are called the n-skeletal simplicies while the essential image of $\mathbf{cosk_n}$ are the n-coskeletal simplices.

(c) Show that a simplicial set X is n-skeletal if and only if

$\mathrm{tr}_n \colon \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, -) \to \mathrm{Hom}_{\widehat{\Delta_{\leqslant n}}}(\mathrm{tr}_n X, \mathrm{tr}_n(-))$ is an isomorphism of functors.

(d) Show that a simplicial set Y is n-coskeletal if and only if

$\mathrm{tr}_n \colon \mathrm{Hom}_{\mathrm{Set}_\Delta}(-, Y) \to \mathrm{Hom}_{\widehat{\Delta_{\leqslant n}}}(\mathrm{tr}_n(-), \mathrm{tr}_n Y)$ is an isomorphism of functors.

Exercise 4.

(a) Show that a map $F \colon X \to N(\mathcal{C})$ to the nerve of a category $\mathcal{C}$, is completely determined by a map $u \colon X_1 \to \mathrm{Mor}(\mathcal{C})$ such that

(a) for all $x \in X_0$ we have that $u(s_0(x))$ is an identity and

(b) for any 2-simplex $\sigma \in X_2$ we have that $u(d_1(\sigma)) = u(d_0(\sigma)) \circ u(d_2(\sigma))$.

(b) Deduce that the nerve of a category is 2-coskeletal.

(c) Conclude that the nerve $N \colon \mathrm{Cat} \to \mathrm{Set}_\Delta$ is fully faithful.

Exercise 5. Consider the functor $\mathrm{Op}\colon \Delta \to \Delta$ which is the identity on objects and for $\sigma : [n] \to [m]$

$$\mathrm{Op}(\sigma)(i) := m - \sigma(n-i)$$

Let $(-)^{\mathrm{op}} := \mathrm{Op}^*\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ be the corresponding involution of the category of simplicial sets.

(a) Show that Op is a well defined involution.

(b) For a simplicial set X, describe the face and degeneracy maps of X^{op}.

(c) Show that for a category $\mathcal{C}$ we have an isomorphism $N(\mathcal{C}^{\mathrm{op}}) \cong N(\mathcal{C})^{\mathrm{op}}$.

(d) Show that for a topological space there is an isomorphism $\mathrm{Sing}(X) \cong \mathrm{Sing}(X)^{\mathrm{op}}$ for the singular complex of X.

Lecture 12.11

7 Connected components & the fundamental groupoid I

Consider the following functors

$$
\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathrm{Gpd} \arrow[l, bend left, "\pi_0"] \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "L"] \arrow[r, bend left, "N"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend left, "\tau"]
\end{tikzcd}
$$

These functors admit left adjoints: Given $G \in \mathrm{Gpd}$ we let $\pi_0(G) \in \mathrm{Set}$ be its set of isoclasses.

Proposition 7.1. *The canonical functor $G \to \iota(\pi_0(G))$, $G \in \mathrm{Gpd}$ are the components of the unit of an adjunction*

$$
\pi_0 \colon \mathrm{Gpd} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Set} : \iota
$$

Proof. Let $J \in \mathrm{Set}$ and let F be a functor $F \colon G \to \iota(J)$. We observe that $\forall f \colon X \to Y$ in G we have $F(f) = \mathrm{id}_{F(x)} = \mathrm{id}_{F(y)}$ in other words, F is a constant on isomorphism-classes in G, hence F factors uniquely through $G \xrightarrow{\eta_g} \iota(\pi_0(G))$.

$$
\begin{tikzcd}
G \arrow[r, "\eta_g"] \arrow[dr, "\forall F"'] & \iota(\pi_0(G)) \arrow[d, "\iota(\bar{F})"] \\
& \iota(J)
\end{tikzcd}
$$

This gives an isomorphism $\mathrm{Hom}_{\mathrm{Set}}(\pi_0(G), -) \cong \mathrm{Hom}_{\mathrm{Gpd}}(G, \iota(-))$, which means we have an adjunction $\pi_0 \dashv \iota$ with unit η. $\square$

For $\mathcal{C} \in \mathrm{Cat}$ we define $L\mathcal{C} \in \mathrm{Gpd}$ as follows: Consider the quiver with vertices $\mathrm{Ob}(\mathcal{C})$ and arrows $\mathrm{Mor}(\mathcal{C}) \amalg \{f^- \mid f \in \mathrm{Mor}(\mathcal{C})\}$. For $f \in \mathrm{Mor}(\mathcal{C})$

$$
\begin{aligned}
s(f) &= \text{domain } f & \quad X \xrightarrow{f} Y \\
t(f) &= \text{codomain of } f \\
s(f^-) &= \text{codomain of } f & \quad X \xleftarrow{f^-} Y \\
t(f^-) &= \text{domain of } f
\end{aligned}
$$

Consider the quotient of the path category of the above quiver by the relation generated by

- $\forall X \in \mathrm{Mor}(\mathcal{C}), \mathrm{id}_X \sim e_X :$ lazy path at X
- $\forall f, g \in \mathrm{Mor}(\mathcal{C})$ composable $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \sim \cdot \xrightarrow{g \circ f} \cdot$

- $\forall f \in \mathrm{Mor}(\mathcal{C})$

$$\cdot \xrightarrow{f} \cdot \xrightarrow{f^-} \cdot \sim \mathrm{id}$$
$$\cdot \xrightarrow{f^-} \cdot \xrightarrow{f} \cdot \sim \mathrm{id}$$

  that is $[f^-] = [f]^{-1}$

Theorem 7.2. *The canonical functors $\gamma = \gamma_{\mathcal{C}} \colon \mathcal{C} \to L\mathcal{C}$, $\mathcal{C} \in \mathrm{Cat}$, form the components of the unit of an adjunction*

$$L\colon \mathrm{Cat} \longrightarrow \mathrm{Gpd} : j$$

Proof. A functor $F\colon \mathcal{C} \to j(G)$, $G \in \mathrm{Gpd}$ nessecarily inverts all maps in $\mathcal{C}$ hence the functor $\bar{F}(x) := F(x)\colon L\mathcal{C} \to j(G)$ $\bar{F}(f) := F([f])$ $([f^-]) \bar{:=} F(f)^{-1}$ is well defined and is the unique functor such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma_{\mathcal{C}}"] \arrow[dr, "F"'] & j(L\mathcal{C}) \arrow[d, "j(\bar{F})"] \\ & j(G) \end{tikzcd}$$

commutes, giving $L \dashv j$ with unit η. □

Consider now for $X \in \mathrm{Set}_\Delta$, the quiver with vertices X_0 and arrows X_1 with $d_1 = s\colon, X_1 \to X_0$ and $d_0 = t\colon X_1 \to X_0$. We have that $\tau(X)$ is the quotient of the path category of this quiver modulo the relation

$$\begin{tikzcd} & \cdot \arrow[dr, "f_{21}"] & \\ \cdot \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & \cdot \end{tikzcd} \in X_2 \implies [f_{20}] = [f_{21}] \circ [f_{10}].$$

Proposition 7.3. *The canonical maps $\eta_X \colon X \to N(\tau X)$, $X \in \mathrm{Set}_\Delta$ form the components of the unit of an adjunction*

$$\tau\colon \mathrm{Set}_\Delta \underset{\perp}{\rightleftarrows} \mathrm{Cat}\colon N$$

Definition 7.4. We also have the composite adjunction

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] \arrow[rrr, bend right=60] & \mathrm{Gpd} \arrow[l, bend left, "\pi_0"] \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "L"] \arrow[r, bend left, "N"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend left, "\tau"] \arrow[lll, bend right=60, "\pi_0"'] \end{tikzcd}$$

$$\pi_0 := \pi_0 \circ L \circ \tau\colon \mathrm{Set}_\Delta \underset{\perp}{\rightleftarrows} \mathrm{Cat}\colon N \circ j \circ \iota$$

Definition 7.5. For $X \in \mathrm{Set}_\Delta$ we call $\pi_0(X) \in \mathrm{Set}$ is a connected component. For $I \in \mathrm{Set}$ we have that

$$N(j(\iota(I)))_n = N(\iota(I))_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \iota(I)) \cong I$$

which means that

$$N \circ j \circ \iota = \mathrm{const}_\Delta$$

By uniqueness of adjoints we obtain that $\pi_0 \cong \mathrm{colim}_\Delta$ is adjoint to $\mathrm{const}_\Delta \cong N \circ j \circ \iota$ where $\pi_0(X) \cong X_0/\sim$ and $X \sim Y$ if and only if there exists a zigzag of edges in X. Consider now the composite adjunction

$$\begin{tikzcd}
\mathrm{Gpd} \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=60, "\pi_1"] & \mathrm{Cat} \arrow[l, bend left, "L"] \arrow[r, bend left, "N"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend left, "\tau"] \arrow[ll, bend left=60, "N"]
\end{tikzcd}$$

Where $\pi_1 := L \circ \tau \colon \mathrm{Set}_\Delta \underset{\perp}{\overset{\mathrm{Gpd}}{\rightleftarrows}} \mathrm{Gpd} \colon N \circ j = N_{|\mathrm{Gpd}} = N$

Definition 7.6. For $X \in \mathrm{Set}_\Delta$ we call $\pi_1(X) \in \mathrm{Gpd}$ its fundamental groupoid. For $x \in X_0$ we define $\pi_1(X, x) \in \mathrm{Grp}$ as

$$s_0(x) = [1_x] \in \mathrm{Hom}_{\pi_1(X)}(x, x)$$

Let $f \colon X \to Y$ in Set_Δ be a morphism then we get a functor $\pi_1(X) \to \pi_1(Y)$ given as

$$\pi_1(f) \colon \pi_1(X, x) = \mathrm{Hom}_{\pi_1(X)}(x, x) \longrightarrow \mathrm{Hom}_{\pi_1(Y)}(f_0(x), f_0(x)) = \pi_1(Y, f_0(x))$$

Lecture:14.11

7.1 connected component

We have the adjunction $\pi_0 \colon \mathrm{Set}_\Delta$

$$\pi_0(X) \cong \mathrm{coeq}(X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0)$$

For $X \in \mathrm{Set}_\Delta$ we have the unit morphism $\eta_X \colon X \to \pi_0(X)$. When η_X is an isomorphism we say that X is discrete.

We want to show that $X \in \mathrm{Set}_\Delta$ decomposes as the disjoint union of the connected simplicial subsets, called its connected components.

Construction 7.7. For $C \in \pi_0(X) = X_0/\sim$ (in particular $C \subseteq X_0$) let $X(C)_n := \{\sigma \in X_n \mid \forall \Delta \xrightarrow{f} \Delta^n \xrightarrow{\sigma} X, \sigma \circ f \in C\}$. Notice that $X(C) \subseteq X$ is a simplicial subset since for $\sigma \in X(C)_n$ and $\tau \colon \Delta^m \to \Delta^n$ we have that

$$\begin{tikzcd} \Delta^m \arrow[r, "\tau"] & \Delta^n \arrow[r, "\sigma"] & X \\ \Delta^0 \arrow[u, "v"] \arrow[ur] & & \end{tikzcd}$$

Moreover, $X(C) \subseteq X$ is a summand in the sense that $X \cong X(C) \amalg (X \backslash X(C))$ for $X_n \backslash X(C)_n$ also defines a simplicial subset.

$$X_n \backslash X(C)_n = \{\sigma \in X_n \mid \forall \Delta \xrightarrow{f} \Delta^n \xrightarrow{\sigma} X, \sigma \circ f \notin C\}$$

Proposition 7.8. *The canonical map*

$$\begin{tikzcd} \amalg_{C \in \pi_0(X)} X(C) \arrow[r, hook] & X \\ X(C) \arrow[u, hook] \arrow[ur, "\iota_C"] & \end{tikzcd}$$

is an isomorphism of simplicial sets.

Remark 7.9. Notice that $\pi_0(X(C)) = \{C\}$ as well as $\amalg_{C \in \pi_0(X)} \pi_0(X(C)) = \pi_0(\amalg_{C \in \pi_0(X)} X(C)) \xrightarrow{\sim} \pi_0(X)$

Proposition 7.10. *For any simplicial set $X \in \mathrm{Set}_\Delta$, that is not equal to the empty set, the following are equivalent*

1. $\pi_0(X) = \{*\}$
2. *if $X \cong Y \amalg Z$ then $Y = \varnothing$ or $Z = \varnothing$*

When this is the case we call X connected.

Proof.

1. "1) $\implies$ 2)" We argue by contrapositive. Let

$$X \cong Y \amalg Z$$

such that neither Y nor Z are the empty simplicial set. Then we have that

$$\pi_0(X) \cong \pi_0(Y) \amalg \pi_0(Z) \neq \{*\}$$

2. "1) $\implies$ 2)" We argue again by contrapositive.

$$\amalg_{C \in \pi_0(X)} X(C) \xrightarrow{\sim} X$$

where at least two of the summands on the left are nonempty.

□

Proposition 7.11. *Let $\varnothing \neq X \in \mathrm{Set}_\Delta$ and $S \subseteq X$ a connected component, that is $\pi_0(S) = \{*\}$ and $X = S \amalg (X \backslash S)$. Then $\exists ! C \in \pi_0(X)$ such that $S = X(C)$.*

Proposition 7.12. *Let X, Y be connected simplicial sets then $X \times Y$ is connected.*

Proof. Let $(x, y), (x', y') \in (X \times Y)_0 = X_0 \times Y_0$ and $X \xleftarrow{f} X'' \xrightarrow{g} X'$ with $f, g \in X_1$ as well as $y \xrightarrow{h} y'$ with $h \in Y_1$.

$$\begin{tikzcd}
(x,y) \arrow[d, "{(1_x,h)}"'] & (x',y') \\
(x,y') & (x'',y) \arrow[l, "{(f,1_{y'})}"] \arrow[u, "{(g,1_{y'})}"']
\end{tikzcd}$$

Warning! 7.13. The collection of simplicial sets is not closed under infinite products. Take the nerve of the natural numbers $N(\mathbb{N})$ as well ordered set and let X be the associated simplicial set, then $S = \prod_{n \in \mathbb{Z}_{\geqslant 0}} X$ is not connected, since the element $i = (0, 0, 0, \dots)$ and $j = (0, 1, 2, 3, \dots)$ have no edge between one another, since edges in the product are finite compositions of tuples of edges in the components.

□

7.2 Exercises

Exercise 1. Recall that we defined the connected component functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ as

$$\pi_0(X) := \mathrm{colim}_\Delta X$$

which yields a left adjoint to the constant diagram functor $\mathrm{const} : \mathrm{Set} \to \mathrm{Set}_\Delta$.

(a) Show that $\pi_0(\Delta^n)$ is a one point set for any $n \in \mathbb{N}$.

(b) Recall from Exercise 6.2 the boundaries $\partial\Delta^n$ and horns Λ^n_k of Δ^n. Compute $\pi_0(\partial\Delta^n)$ and $\pi_0(\Lambda^n_k)$.

By definition the connected components of a small category $\mathcal{C}$ are defined as the coequalizer

$$\mathrm{Mor}(\mathcal{C}) \underset{\text{source}}{\overset{\text{target}}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$$

(c) Show that for any simplicial set X, the connected components $\pi_0(X)$ of X agree with the connected components of the category of elements $\int^\Delta X$.

(d) Show that the connected components of a category $\mathcal{C}$ agree with the connected components of its nerve $\pi_0(N(\mathcal{C}))$.

Exercise 2. For a simplicial set X, let $\mathbf{Sk_n}(X)$ be the smallest simplicial subset of X such that $(\mathbf{Sk_n}(X))_m = X_m$ for $m \leqslant n$.

1. Show that there is a natural isomorphism $\mathbf{Sk_n}(X) \cong \mathbf{sk_n}(X)$, i.e. $\mathbf{Sk_n}$ describes the n-skeleton functor from Exercise 6.2.

Consider for any simplicial set X the canonical functor $\mathbf{sk}_X \colon \mathbb{N}_0 \to \mathrm{Set}_\Delta$ with $\mathbf{sk}_X(n) := \mathbf{sk}_n(X)$ and morphisms induced by the inclusion $\mathbf{sk}_n(X) \subseteq X$. We call the image of $\mathbf{sk}_X$ the skeletal filtration of X. For convenience, we set $\mathbf{sk}_{-1}(X)$ to the empty presheaf.

(b) Argue that the morphisms in the skeletal filtration of X are monomorphisms and show that $X \cong \mathrm{colim}_{\mathbb{N}}\, \mathbf{sk}_X$.

(c) Recall that $\sigma \in X_n$ can be viewed as a morphism $\sigma : \Delta^n \to X$. Observe that σ factors through $\mathbf{sk}_n(X)$ and that the precomposition of σ with the inclusion $\partial\Delta^n \subseteq \Delta$ factors through $\mathbf{sk}_{n-1}(X)$. Show that these maps assemble into a pushout diagram.

$$\begin{tikzcd}
\coprod_{\sigma \in X_n^{nd}} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in X_n^{nd}} \Delta^n \arrow[d] \\
\mathbf{sk_{n-1}}(X) \arrow[r] & \mathbf{sk_n}(X)
\end{tikzcd}$$

Here $X_n^{nd} := X_n \backslash (\mathbf{sk_{n-1}(X)})_n$ denotes the set of non-degenerate n-simplicies in X

(d) Show that the geometric realisation of a simplicial set is a CW complex.

8 Kan complexes

lecture 19.11

The aim of this chapter is to introduce a calss of simplicaial sets that behave simultaneously as nerves of groupoids and singular sets of spaces.

Definition 8.1. Let $n \geqslant 1$ and $0 \leqslant k \leqslant n$. The k-th horn $\Lambda^n_k \subseteq \Delta^n$ is the simplicial subset generated by $\{d^i\colon [n-1] \to [n] \mid 0 \leqslant i \leqslant n; i \neq n\} = \Delta(n-1, n) = (\Delta^n)_{n-1}$, or equivalently $\Lambda^n_k = \operatorname{colim}_{\varnothing \neq k \in I \subsetneq [n]} \Delta^I$ where $\Delta^I \cong \Delta^{|I|-1}$.

Remark 8.2. We will sometimes refer to Λ^n_k as the n-horn at position k.

Example 8.3. Let us give a list for the horns up to dimension 3

- For $n = 1$ we have the two horns $\Lambda^1_0 = 0$ and $\Lambda^1_1 = 1$.
- For $n = 2$ we have the following 3 horns, given here with their embedding into the standard 2-simplex

$$\Lambda^2_0\colon \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \hookrightarrow \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$\Lambda^2_1\colon \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \hookrightarrow \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$\Lambda^2_2\colon \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd} \hookrightarrow \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Remark 8.4. The horn $\Lambda^n_k \subseteq \Delta^n$ enjoys the following universal property: For $X \in \mathrm{Set}_\Delta$ the map

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_k, X) \arrow[r, hook] & \prod_{\substack{0\leqslant i \leqslant \\ i \neq k}} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n-1}, X) \\ \sigma \arrow[r, mapsto] & (\sigma \circ d^i)_{\substack{0 \leqslant i \leqslant n \\ i \neq k}} \end{tikzcd}$$

is injective, with image the subset of tuples $(\sigma_0, \sigma_1, \ldots, \sigma_{k-1}, \cdot, \sigma_{k+1}, \ldots, \sigma_n) \in (X_{n-1})^n$ such that for all $0 \leqslant i < j \leqslant n$, $i \neq k$ $d_(\sigma_j) = d_{j-1}(\sigma_i)$ with $I = [n]\backslash\{i\}$

and $J = [n]\backslash\{j\}$ the following diagrams commute

$$\begin{tikzcd} & \Delta^{I\cap J} \arrow[dl, hook] \arrow[dr, hook] & \\ \Delta^I \arrow[dr] & & \Delta^J \arrow[dl] \\ & \Lambda^n_k & \end{tikzcd} \qquad \begin{tikzcd} & \Delta^{n-2} \arrow[dl, "d^{j-1}"'] \arrow[dr, "d^i"] & \\ \Delta^{n-1} \arrow[dr, "d^i"'] & & \Delta^{n-1} \arrow[dl, "d^j"] \\ & \Delta^n & \end{tikzcd}$$

Definition 8.5. Let $X \in \mathrm{Set}_\Delta$, X is a Kan complex ($= \infty$-groupoid) if for all $n \geqslant 1$ and all morphisms $\sigma\colon \Lambda^n_k \to X$ we have the following diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{can}"'] & X \\ \Delta^n \arrow[ur, dashed, "\exists\hat{\sigma}"'] & \end{tikzcd}$$

Remark 8.6. Notice that the morphism $\widehat{\sigma}$ need not be unique.

Recall the following adjunctions

$$\begin{tikzcd} \mathrm{Gpd} \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "L"] \arrow[r, bend left, "N"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend left, "\tau"] \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, bend left, "\mathrm{Sing}"] \end{tikzcd}$$

Proposition 8.7. *Let $X \in \mathrm{Top}$. Then $\mathrm{Sing}(X)$ is a Kan complex.*

Proof. For all $n \geqslant 1$ and $0 \leqslant k \leqslant n$ we want the following diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\iota"'] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed, "\exists\tilde{\sigma}"'] & \end{tikzcd}$$

Now after applying the geometric realization functor $|\cdot|$ to the above diagram, we obtain a diagram

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "\overline{\sigma}"] \arrow[d, hook, shift right, "|\iota|"'] & X \\ {|\Delta^n|} \arrow[u, shift right, "\exists r"'] \arrow[ur, "\alpha=\bar{\sigma}\circ r"'] & \end{tikzcd}$$

where r is a continuous retraction of Δ^n onto $|\Lambda^n_k|$. Then we apply the adjunction to the following composition

$$|\Lambda^n_k| \xrightarrow{|\iota|} |\Delta^n| \xrightarrow{\alpha} X = |\Lambda^n_k| \xrightarrow{|\sigma|} X$$

to obtain

$$\Lambda^n_k \xrightarrow{\iota} \Delta^n \xrightarrow{\overline{\alpha}} \operatorname{Sing}(X) = \Lambda^n_k \xrightarrow{\overline{\overline{\sigma}}=\sigma} \operatorname{Sing}(X)$$

which then gives the desired horn extension:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & \operatorname{Sing}(X) \\ \Delta^n \arrow[ur, "\overline{\alpha}=:\tilde{\sigma}"'] & \end{tikzcd}$$

□

Definition 8.8. Let $X \in \mathrm{Set}_\Delta$, it is called an inner Kan complex (= ∞-category) if for all $n \geqslant 2$ and $0 < k < n$ we have the following diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{can}"'] & X \\ \Delta^n \arrow[ur, dashed, "\exists\hat{\sigma}"'] & \end{tikzcd}$$

Remark 8.9. Let $n \geqslant 0$ and $\Delta_{\leqslant n} := \{[m] \in \Delta \mid 0 \leqslant m \leqslant n\} \xhookrightarrow{\iota} \Delta$ as well as $\mathrm{tr}_n = \iota^*$. We have the following adjunction

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, "\mathrm{tr}_n"] & \mathrm{Set}_{\Delta_{\leqslant n}} \arrow[l, bend right, "\mathrm{sk}_n"'] \arrow[l, bend left, "\mathrm{cosk}_n"] = \mathrm{Fun}(\Delta^{\mathrm{op}}_{\leqslant n}, \mathrm{Set}) \end{tikzcd}$$

and call $X \in \mathrm{Set}_\Delta$ n-coskeletal if the following equivalent conditions hold:

1. $X \in \mathrm{Im}(\mathrm{cosk}_n)$
2. $X \xrightarrow{\sim} \mathrm{cosk}_n(\mathrm{tr}_n)$
3. $\forall Y \in \mathrm{Set}_\Delta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta_{\leqslant n}}}(\mathrm{tr}_n Y, \mathrm{tr}_n X)$

Also note that for $\mathcal{C} \in \mathrm{Cat}$ we have that $N(\mathcal{C}) \in \mathrm{Set}_\Delta$ is 2-coskeletal.

Proposition 8.10. *Let $\mathcal{C} \in \mathrm{Cat}$. The nerve of $\mathcal{C}$ is an inner Kan complex.*

Proof. Let $n \geqslant 2$ and $0 < k < n$, consider the horn extension diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed, "\exists\hat{\sigma}?"'] & \end{tikzcd}$$

By coskeletality of $N(\mathcal{C})$ it is enough to solve the 2-truncated extension

$$\begin{tikzcd}
\mathrm{tr}_n(\Lambda^n_k) \arrow[r, "\mathrm{tr}_2\,\sigma"] \arrow[d, "\mathrm{tr}_2(\iota)"] & \mathrm{tr}_2(N(\mathcal{C})) \\
\mathrm{tr}_n(\Delta^n) \arrow[ur, "\mathrm{tr}_2 \circ \exists \tilde{\sigma}?"'] &
\end{tikzcd}$$

For $n \geqslant 4$ we have that $\mathrm{tr}_2(\Lambda^n_k) = \mathrm{tr}_2(\Delta^n)$ and hence the problem is trivial in that case. For $n = 2$ we have that $0 < k < 2$, thus $k = 1$, so we consider the horn at 1 and the corresponding horn extension problem

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
0 \arrow[ur] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2_1 \arrow[r, "\sigma"] \arrow[d] & N(\mathcal{C}) \\
\Delta^n \arrow[ur, dashed, "\tilde{\sigma}"'] &
\end{tikzcd}$$

So σ is explicitely given as

$$\begin{tikzcd}
 & X_1 \arrow[dr, "\sigma_{21}"] & \\
X_0 \arrow[ur, "\sigma_{10}"] & & X_2
\end{tikzcd}$$

so we can choose $\sigma_{21} \circ \sigma_{10}$ to complete the horn to a full simplex giving the desired horn extension. For $n = 3$ we have the $k = 1, 2$, we are going to consider the case $k = 1$ explicitly. We get the following simplex diagramm

$$\begin{tikzcd}
 & X_1 \arrow[dr, "\sigma_{21}"] \arrow[dd, "\sigma_{31}" near end] & \\
X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}" near start] \arrow[dr, "\sigma_{30}"'] & & X_2 \arrow[dl, "\sigma_{32}"] \\
 & X_3 &
\end{tikzcd}$$

where the red triangle is not commuting a priori. The simplex gives the following identities

$$\sigma_{30} = \sigma_{32} \circ \sigma_{20} \iff \sigma_{30} = \sigma_{32} \circ (\sigma_{21} \circ \sigma_{10}) = (\sigma_{32} \circ \sigma_{21}) \circ \sigma_{10} = \sigma_{31} \circ \sigma_{10} = \sigma_{30}$$

which yields the commutativity of the bottom simplex. $\square$

Proposition 8.11. *Let $\mathcal{C} \in \mathrm{Cat}$. Then the following are equivalent:*

1. *$\mathcal{C}$ is a groupoid,*

2. $N(\mathcal{C})$ is a Kan complex.

Proof. "2) $\implies$ 1)" Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. The horns

$$\begin{tikzcd} & Y \arrow[dr, dashed, "\exists g"] & \\ X \arrow[ur, "f"] \arrow[rr, "1_X"'] & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, dashed, "\exists h"] \arrow[rr, "1_Y"'] & & Y \end{tikzcd}$$

extend to 2-simplices of $N(\mathcal{C})$ since $N(\mathcal{C})$ is a Kan complex. So we get that $gf = \mathrm{id}_X$ and $fh = \mathrm{id}_Y$. Thus $\mathcal{C}$ is a groupoid. "1) $\implies$ 2)" We already know that $N(\mathcal{C})$ is an inner Kan complex. "1) $\implies$ 2)" We already know that $N(\mathcal{C})$ is an inner Kan complex. It is enough to consider the cases $n = 2$, $k = 0, 2$ and $n = 3$ $k = 0, 3$ by 2-coskeletality of $N(\mathcal{C})$. For $n = 2$ and $k = 0$ we have that the diagram

$$\begin{tikzcd} & X_1 & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"'] & & X_2 \end{tikzcd}$$

in $\mathcal{C}$, extends to

$$\begin{tikzcd} & X_1 \arrow[dr, dashed, "\sigma_{20}\circ\sigma_{10}^{-1}"] & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"'] & & X_2 \end{tikzcd}$$

where $\sigma_{20} \circ \sigma_{10}^{-1}$ exists since $\mathcal{C}$ is a groupoid. The case for $k = 2$ is done analogously. For $n = 3$ and $k = 2$, consider the following 2-simplex

$$\begin{tikzcd} & X_1 \arrow[dr, "\sigma_{21}"] \arrow[dd, "\sigma_{31}"] & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"] \arrow[dr, "\sigma_{30}"'] & & X_2 \arrow[dl, "\sigma_{32}"] \\ & X_3 & \end{tikzcd}$$

which gives the following chain of identities:

$$\sigma_{31} \circ \sigma_{10} = \sigma_{32} \circ \sigma_{21} \circ \sigma_{10} = \sigma_{32} \circ \sigma_{20} = \sigma_{30}$$

$\square$

Lecture 21.11

Example 8.12. 1. For all $X \in \mathrm{Top}$, $\mathrm{Sing}(X) \in \mathrm{Set}_\Delta$ is a Kan complex.

2. For all $\mathcal{C} \in \mathrm{Cat}$, $N(\mathcal{C})$ is an inner Kan complex.

3. $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

4. For all $n \geqslant 0$, $\Delta^n = N([n])$ is an inner Kan complex and furthermore Δ^n is a Kan complex if and only if $n = 0$.

5. If M is a monoid then $N(BM)$ is an inner Kan complex and furthermore $N(BM)$ is a Kan complex if and only if M

Definition 8.13. The category of simplicial groups is $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grp})$. Thus $X \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grp})$ consists of the following data:

- For all $n \geqslant 0$ a group X_n with a neutral element $e_n \in X_n$.
- For all $0 \leqslant i \leqslant n$ $\begin{array}{c} d_i \colon X_n \to X_{n-1} \text{ face map} \\ s_i \colon X_n \to X_{n+1} \text{ degeneracy map,} \end{array}$ satisfy the simplicial identities and are group homomorphisms.

Proposition 8.14. *Let X be a simplicial group. Then the underlying simplicial set of X is a Kan complex.*

Proof. The case $n = 1$ is trivial. We illustrate the argument for $n = 2$ and $k = 1$. Suppose given a horn

$$\begin{tikzcd} & x_1 \arrow[dr, "x_{21}"] & \\ x_0 \arrow[ur, "x_{10}"] \arrow[rr, dashed, "\exists w"'] & & x_2 \end{tikzcd}$$

Form the degenerate 2-simplex $y =: s_0(x_{21})$

$$\begin{tikzcd} x_1 \arrow[dr, "x_{21}"] & \\ x_1 \arrow[u, dashed, "1_{x_1}"] \arrow[r, "x_{21}"'] & x_2 \end{tikzcd}$$

Let $z := d_2(y) = 1_{X_1} \in X_1$. Consider now $s_1(x_{10}z^{-1}) = s_1(x_{10}d_2(y)^{-1}) = d_2(y^{-1})$.

$$s_1(x_{10}z^{-1}) : \quad \begin{tikzcd} & x_1x_1^{-1} = e_0 \arrow[dr, "1_{e_0}=s_0(e_0)=e_1"] & \\ x_0x_1 \arrow[ur, "x_{10}1_{x_1}^{-1}"] \arrow[rr, "x_{10}1_{x_1}^{-1}"'] & & x_1x_1^{-1} = e_0 \end{tikzcd}$$

Let $w := s_1(x_{10}z^{-1})y \in x_2$ and we get the chain of equalities

$$d_0(w) = d_0((s_1(x_{10}z^{-1}))y) = d_0(s_1(x_{10}z^{-1}))d_0(y) = e_1x_{21} = x_{21}$$

as well as

$$d_2(w) = d_2(s_1(x_0z^{-1}))d_2(y) = (x_{10}1_{x_1}^{-1})1_{x_1} = x_{10}e_1 = x_{10}$$

For the general case, consider a horn

$$(x_0, x_1, \dots, x_{k-1}, \bullet, x_{k+1}, \dots, x_n)$$

in X. Where $x_i \in X_{n-1}$ and $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ and $i, j \neq k$. Suppose that there exists a $y \in X_n$ such that for all $0 \leqslant i < k$ and for all $l \leqslant i \leqslant n$. Then $w := s_{l-2}(x_{l-1} \cdots d_{l-1}(y)^{-1})y$ satisfies $d_i(w) = x_i$ for all $0 \leqslant i < k$ and all $l - 1 \leqslant i \leqslant n$. ☐

Recall the Dold-Kan correspondence.

$$\begin{tikzcd}
\tau\colon \mathrm{Ab}_\Delta \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\text{forgetfull}"'] & \mathrm{Ch}_{\leqslant}(\mathrm{Ab})\colon Dk \arrow[d, hook] & \\
\mathrm{Set}_\Delta & \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\gamma"] & D(\mathrm{Ab})
\end{tikzcd}$$

Proposition 8.15. *Let X be a Kan complex ($X \in \mathrm{Set}_\Delta$). Then $x, y \in X_0$ satisfy $[x] = [y]$ in $\pi_0(X)$ if and only if there exists a $\sigma \in X_1$ such that $d_0(\sigma) = y$ and $d_1(\sigma) = x$.*

Proof. At first, when there is a σ that relates X to Y we can complete this to 1-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "d_0(\tau)"] & \\
X \arrow[ur, "\sigma"] \arrow[rr, "1_X"'] & & X
\end{tikzcd}$$

and obtain $d_0(\tau)$ that relates Y to X. Also if X relates to Y and Y relates to Z, we obtain a 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "\tau"] & \\
X \arrow[ur, "\sigma"] \arrow[rr, "d_1(\alpha)"'] & & Z
\end{tikzcd}$$

where $d_1(\alpha)$ relates X to Z. ☐

8.1 Exercises

Exercise 1.

(a) Show that for $n \in \mathbb{N}_0$ we have $\mathbf{sk}_n(\Delta^{n+1}) \cong \partial\Delta^{n+1}$.

(b) Show that for every horn Λ_k^m we have that

$$\mathbf{sk_n}(\Lambda_k^m) \cong \begin{cases} \Lambda_k^m & \text{if } m \leqslant n+1 \\ \mathbf{sk}_n(\Delta^m) & \text{if } m > n+1 \end{cases}$$

for $0 \leqslant k \leqslant m \in \mathbb{N}_+$.

(c) Show that for a Kan comlex K and any $n \in \mathbb{N}_0$, the n-coskeleton $\mathbf{cosk}_n(K)$ is a Kan complex.

Exercise 2.

(a) Show that the class of Kan complexes and the class of inner Kan complexes are closed under set indeced products.

(b) Show that a set indexed product of connected Kan complexes is connected.

(c) Give an example that an infinite product of connected inner Kan complexes is not necessarily connected. (Recall that the nerve of a category is an inner Kan complex.)

9 Fundamental groupoid revisited

Recall the adjunction $\mathrm{Set}_\Delta \underset{N}{\overset{\tau}{\rightleftarrows}} \mathrm{Gpd}$ and suppose that $X \in \mathrm{Set}_\Delta$ is a Kan complex.

Construction 9.1. Boardman-Vogt Construction
The homotopy category of X is called $\mathrm{Ho}(X)$ is defined as follows:

- $\mathrm{Ob}(\mathrm{Ho}(X)) = X_0$
- $\hom_{\mathrm{Ho}(X)}(X, Y) = \{f \in X_1 \mid d_1(f) = X, d_0(f) = Y\}/\sim$

Recall that a 2-simplex of X

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

should be a "witness" of the fact the h is a composition of f and g.

Problem 9.2. Why are there different choices of relation $\sim$? Given as follows

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "g"'] & & Y \end{tikzcd} = f \sim_1 g \qquad \begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Y \end{tikzcd} = f \sim_2 g$$

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd} = f \sim_3 g \qquad \begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \end{tikzcd} = f \sim_4 g$$

where $1_X := s_0(X)$. What remains to be shown is that all these are equivalence relations and are in fact the same.

Lecture 26.11

Proposition 9.3. *The four relations above are the same and are equivalence relations.*

Proof. We show $(f \sim_3 g \implies f \sim_1 g)$, thus let

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd} = \sigma \in X_2$$

We can glue the following 2-simplices

$$\begin{tikzcd} & X \arrow[dr, "1_X"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "1_X"'] & & X \end{tikzcd} = c^2(X) \qquad \begin{tikzcd} X \arrow[rr, "1_X"] \arrow[dr, "g"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd} = s_0(g)$$

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "1_X"] & \\ & X \arrow[dl, "g"] \\ Y & \end{tikzcd} = \sigma$$

to obtain a 3-horn

$$\begin{tikzcd} & X \arrow[dr, "1_X"] \arrow[dd, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, dashed, "1_X"] \arrow[dr, "g"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd} = \Lambda^3_2$$

The red 2-simplex is exactly the one of the equivalence relation $f \sim_1 g$. Since X is an inner Kan complex by assumption, we have the desired horn extension. The other direction were part of an exercise and will be included here eventually. What remains to be shown, is that it is an equivalence relation.

- (Reflexivity) Let $X \xrightarrow{f} Y$ be in X_1, then we have the 2-simplex

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

  which means that $\sim$ is associative.

- (Symmetry) We have the following chain of equivalences

$$f \sim g \iff f \sim_1 g \iff f \sim_3 g \iff g \sim_1 f \iff g \sim f$$

- (Transitivity) Let $f \sim g$ and $g \sim h$. Consider the following diagrams

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "1_Y"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "f"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "1_Y"] \\
Y &
\end{tikzcd}$$

which glued at 1_Y and f result in the following horn

$$\begin{tikzcd}
& X \arrow[dr, "f"] \arrow[dd, "f"] & \\
X \arrow[ur, "1_X"] \arrow[rr, dashed, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "1_Y"] \\
& Y &
\end{tikzcd}$$

By the horn filling property of the inner Kan complex for a 3-horn at position 2 we get the desired $f \sim g$.

□

Proposition 9.4. *The composition law in* $\mathrm{Ho}(X)$ *is well defined, unital and associative.*

Proof. Suppose that $f \sim f'$. From now on we write $gf \sim h$ to mean that there exists a 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd} \in X_2$$

We prove that if $gf \sim h$, $gf' \sim h'$ and $gf' \sim h$ it follows that $h \sim h'$. So consider the 2-simplices

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[dd, "h"'] \arrow[dd, phantom, shift left=4ex, "\rho"] & \\
& Y \arrow[dl, "g"] \\
Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dr, "f"] \arrow[d, phantom, "\sigma"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "f'"'] & {} & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "f'"] \arrow[dr, "h'"'] & {} \arrow[d, phantom, "\tau"] & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

which can be glued to obtain a 3-horn

$$\begin{tikzcd}
& X \arrow[dr, "f"] \arrow[dd, "h"] & \\
X \arrow[ur, "1_X"] \arrow[rr, dashed, "f'"] \arrow[dr, "h'"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Now since this is a horn at position 2 and since X is an inner Kan complex, we obtain a horn extension

$$\begin{tikzcd}
\Lambda^3_2 \arrow[r, "\gamma"] \arrow[d] & X \\
\Delta^3 \arrow[ur, dashed, "\exists\tilde{\gamma}"'] &
\end{tikzcd}$$

which gives the missing 2-simplex

$$\begin{tikzcd}
& X \arrow[dr, "h"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "h'"] & & Z
\end{tikzcd}$$

which yields $h \sim h'$ by the definition of the relation. Next we show Unitality, we have 2-simplices

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dr, "1_Y"] & \\
X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

on equivalence classes this gives $[f] \circ [1_X] = [f] = [1_Y] \circ [f] = [f]$, which exactly means that we have found a unit with respect to the composition. Lastly we show Associativity, for that we choose a composition $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$.

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "u"] \arrow[dr, "v"'] & & Z \arrow[dl, "h"] \\
& W &
\end{tikzcd}$$

which gives the equations

$$[v] = [h] \circ [u] = [h] \circ ([g] \circ [f]). \tag{14}$$

Now we also have a simplex corresponding to the composition of g and h,

$$\begin{tikzcd}
Y \arrow[dr, "g"] \arrow[dd, "\alpha"'] & \\
& Z \arrow[dl, "h"] \\
W &
\end{tikzcd}$$

putting these three 2-simplices together we obtain a 3-horn.

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\
X \arrow[ur, "f"] \arrow[rr, dashed, "u"] \arrow[dr, "v"'] & & Z \arrow[dl, "h"] \\
& W &
\end{tikzcd}$$

Now the missing simplex corresponds to

$$[v] = [\alpha] \circ [f] = ([h] \circ [g]) \circ [f], \tag{15}$$

since X is an inner Kan complex, the horn can be filled and we obtain associativity by combining equations (1) and (2). □

Remark 9.5. The result, that $\mathrm{Ho}(X)$ is a well defined category is known as Joyal's coherence lemma.

Proposition 9.6. *Let $X \in \mathrm{Set}_\Delta$ be a Kan complex, then $\mathrm{Ho}(X)$ is a groupoid.*

Proof. Let $f \in \mathrm{Ho}(X)$ be a morphism. Then we have a simplex

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
Y \arrow[ur, dashed, "\exists g"] \arrow[rr, "1_Y"'] & & Y
\end{tikzcd}$$

which corresponds to $[g] \circ [f] = [1_Y]$. □

Corollary 9.7. *Let $X \in \mathrm{Set}_\Delta$ be a Kan complex, then $L\,\mathrm{Ho}(X) \cong \mathrm{Ho}(X)$.*

Proof. Consider the adjunction

$$\mathrm{Cat} \underset{j}{\overset{L}{\rightleftarrows}} \mathrm{Gpd}$$

where j is just the inclusion. Then the counit ϵ of the adjunction is the desired isomorphism, since $L\,\mathrm{Ho}(X)$ actually means $jL\,\mathrm{Ho}(X)$, that is considering $L\,\mathrm{Ho}(X)$ as a category not a groupoid. □

Proposition 9.8. *Let $X \in \mathrm{Set}_\Delta$ be an inner Kan complex, then* $\mathrm{Ho}(X) \cong \tau X$*, where τ is the right adjoint of the nerve functor.*

Proof. Let $\mathcal{D} \in \mathrm{Cat}$, since $N(\mathcal{D}) \in \mathrm{Set}_\Delta$ is 2-coskeletal, we have a natural bijection:
$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(X, N(\mathcal{D})) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{tr}_2\, X, \mathrm{tr}_2(N(\mathcal{D}))$$

Consider a morphism $\mathrm{tr}_2(X) \xrightarrow{f} \mathrm{tr}_2(N(\mathcal{D}))$ given by

$$\begin{tikzcd}
\mathrm{tr}_2 \arrow[d, "f"] & X\colon \dots & X_2 \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d, "f_2"] & X_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_1"] & X_0 = \mathrm{Ob}(\mathrm{Ho}(X)) \arrow[d, "f_0"] \\
\mathrm{tr}_2(N(\mathcal{D})) & N(\mathcal{D})\colon \dots & N(\mathcal{D})_2 \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & N(\mathcal{D})_1 \arrow[r, shift left] \arrow[r, shift right] & N(\mathcal{D})_0 = \mathrm{Ob}(\mathcal{D})
\end{tikzcd}$$

Note that $N(\mathcal{D})_1 = \mathrm{Mor}(\mathcal{D})$. We have for any $\alpha \in X_1$ that $f_0(d_0(\alpha)) = \mathrm{target}(f_1(\alpha))$ and that for any $d_1(\alpha) \xrightarrow{\alpha} d_0(\alpha)$ that $f_0(d_1(\alpha)) = \mathrm{source}(\alpha)$. Now for 2-simplices we have

$$\alpha \sim \beta \quad \begin{tikzcd}
 & x \arrow[dr, "\alpha"] & \\
x \arrow[ur, "1_x"] \arrow[rr, "\beta"'] & & y
\end{tikzcd} \quad \overset{f_2}{\longmapsto} \quad \begin{tikzcd}
 & f_0(x) \arrow[dr, "f_1(\alpha)"] & \\
f_0(x) \arrow[ur, "\mathrm{id}_{f_0(x)}"] \arrow[rr, "f_1(\beta)"'] & & f_0(y)
\end{tikzcd}$$

Thus $f_1(\alpha) = f_1(\beta)$ which results in

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{tr}_2(X), \mathrm{tr}_2(N(\mathcal{D})) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Ho}(X), \mathcal{D})$$

□

Corollary 9.9. *Let $X \in \mathrm{Set}_\Delta$ be a Kan complex. Then $\pi_1(X) = \mathrm{Ho}(X)$ and for all $x \in X_0$ it holds that $\pi_1(X, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(X)}(X, X)$.*

Proof. We know that

$$\pi_1(X, x) = \mathrm{Hom}_{\pi_1(X)}(x, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(X)}(x, x)$$

as well as
$$\pi_1(X) = L(\tau X) \xrightarrow{\sim} L(\mathrm{Ho}(X)) \xrightarrow{\sim} \mathrm{Ho}(X).$$
by putting the previous two corollaries together. □

9.1 Exercises

Exercise 1. Fix an inner Kan complex X. For edges $f, g, h \in X_1$ we say $g \cdot f \sim h$ if there exists a 2 simplex such that $d_0(\sigma) = g, d_1(\sigma) = h$ and $d_2(\sigma) = f$.

1. Show Joyal's Coherence Lemma:

   Consider $\alpha : \mathbf{sk}_1(\Delta^3) \to X$ with $\alpha_1(f_{21}) \cdot \alpha_1(f_{10}) \sim \alpha_1(f_{20})$ and $\alpha_1(f_{32}) \cdot \alpha_1(f_{21}) \sim \alpha_1(f_{31})$. Then $\alpha_1(f_{31}) \cdot \alpha_1(f_{10}) \sim \alpha_1(f_{30})$ if and only if $\alpha_1(f_{32}) \cdot \alpha_1(f_{20}) \sim \alpha_1(f_{30})$. Here we denote by f_{ji} the unique morphism $f_{i,j} : [1] \to [3]$ with image $\{i < j\} \subseteq [3]$.

We define the following four relations on X_1.

$$\begin{aligned} f \sim_1 g &\iff f \cdot s_0(d_1(f)) \sim g \\ f \sim_3 g &\iff g \cdot s_0(d_1(g)) \sim f \end{aligned}$$

$$\begin{aligned} f \sim_2 g &\iff f \cdot s_0(d_0(f)) \sim g \\ f \sim_4 g &\iff g \cdot s_0(d_0(g)) \sim f \end{aligned}$$

(b) Show that $f \sim_1 g \iff f \sim_2 g$ and that $f \sim_1 g \implies f \sim_3 g$.

(c) Deduce that all four relations agree.

(d) Conclude that the relation $\simeq := \sim_1$ is an equivalence relation on X_1.

Exercise 2. Show that the homotopy category $\mathrm{Ho}(N(\mathcal{C}))$ of the nerve of a small category $\mathcal{C}$ is isomorphic to $\mathcal{C}$.

Exercise 3. Let G be a simplicial group. Let $n \in \mathbb{N}_+$ and $0 < l < n+1$. For $y \in G_{n+1}$ we say $x \in G_n$ is l-compatible if $d_i(x) = d_l(d_{i+1}(y))$ if $l \leqslant i$. Moreover, we say x is (k,l)-compatible for $0 \leqslant k < l$ if additionally $d_i(x) = d_{l-1}(d_i(y))$ for $0 \leqslant i < k$.

(a) Show that if x is (k,l)-compatible with $y \in G_{n+1}$, then we have for $y' := s_{l-1}(x \cdot (d_l(y))^{-1}) \cdot y \in G_{n+1}$ that

$$d_i(y') = \begin{cases} d_i(y) & \text{if } k < l \\ x & \text{if } i = l \\ d_i(y) & \text{if } i > l \end{cases}$$

(b) Give the notion of cocompatibility which is dual to compatibility.

(c) Show that the underlying simplicial set of G is a Kan complex.

Exercise 4. Recall from Exercise 6.2 the definition of the n-coskeleton of a simplicial set X and $n \in \mathbb{N}_0$.

(a) Show that if X is n-coskeletal, then for any $m > n$ we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^m, X)$$

induced by the inclusion $\partial\Delta^m \subseteq \Delta^m$.

(b) Show that if for some $m \in \mathbb{N}_+$ the inclusion $\partial\Delta^m \subseteq \Delta^m$ induces an isomorphism for X as above, then for any morphism $Y \to X$ of simplicial sets, the component $f_m \colon Y_m \to X_m$ is completely determined by $\mathrm{tr}_{m-1}(f)$.

(c) Conclude that a simplicial set X is n-coskeletal if and only if for every $m > n$ the inclusion $\partial\Delta^m \subseteq \Delta^m$ induces an isomorphism $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^m, X)$.

10 Kan Fibrations

Aim: Formalize the idea of a "family of Kan complexes parametrized by a simplicial set", i.e. suitable morphisms $X \xrightarrow{p} Y$ in Set_Δ whose fibres are Kan complexes. That is we have a pullback diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

<u>Terminology</u>: Let $L \xrightarrow{i} K$ and $X \xrightarrow{p} Y$ be morphisms. We say for the notation $(i \boxslash p)$ that i has the left lifting property with respect to p (equivalently p has the right lifting property with respect to i). If there exists for every $L \xrightarrow{f} X$ and $K \xrightarrow{g} Y$ a morphism $h : K \to X$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ K \arrow[r, "g"] \arrow[ur, "h"] & Y \end{tikzcd}$$

Let $X \in \mathrm{Set}_\Delta$ be a Kan complex. Then

$$\{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\} \boxslash (X \to \Delta^0)$$

is given by

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & X \arrow[d, "!"] \\ \Delta^n \arrow[r, "!"'] \arrow[ur, "\tilde{\sigma}"] & \Delta^0 \end{tikzcd}$$

Definition 10.1. A morphism $X \xrightarrow{p} Y$ is a Kan fibration if

$$\{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\} \boxslash (X \xrightarrow{p} Y)$$

Remark 10.2. We may as well write $X \xrightarrow{p} Y \in \{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}^{\boxslash}$.

Lemma 10.3. *Let $i \boxslash p$ and*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

be a pullback-square, then $i \boxslash p'$.

Proof. Consider the diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"'] & X' \arrow[r, "u"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ K \arrow[r, "g"'] \arrow[ur, dashed, "h"] \arrow[urr, dashed, "r"'] & Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

where r exists since i has the left lifting property with respect to p and h exists by the universal property of the pullback. Now $u \circ h \circ i = r \circ i = u \circ f$ and $p' \circ h \circ i = q \circ i = p' \circ f$, since the morphism given by the pullback is unique we get that $h \circ i = f$. ☐

Lecture 3.12

Formalise the notion of a locally constant family of Kan complexes relative to a base simplicial set.

Definition 10.4. Let $X \xrightarrow{p} Y$ be a Kan fibration. By the preceding Lemma, we get that for every $y \in Y_0$ the fibre X_y is a Kan complex. Now consider Δ^0 as the simplicial set where $\Delta^0_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}([n],[0]) = \{[n] \to [0]\}$ is the unique map into the final object. Furthermore we have the constant map $c = c^n : Y_0 \to Y_n$, $y \mapsto c^n(y)$. Let $X \xrightarrow{p} Y$ be a Kan fibration. Where

$$\begin{tikzcd} \Lambda^1_1 = \Delta_0^{\{1\}} \arrow[r, "x"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{\{0,1\}} \arrow[r, "f"] \arrow[ur, dashed, "\exists \tilde{f}"] & Y \end{tikzcd}$$

Proposition 10.5. *Let $L \xrightarrow{i} K$ be a morphism and suppose that $X \xrightarrow{p} Y$, $Y \xrightarrow{g} Z$ have the right lifting property with respect to i, then the composition $g \circ p$ has the right lifting property with respect to i.*

Proof. This is just a simple matter of writing down the square for g and using the obtained morphism to write down the square p obtaining a morphism that fullfills the desired property. ☐

Corollary 10.6. *Suppose that $X \xrightarrow{p} Y$ is a Kan fibration and Y is a Kan complex, then X is a Kan complex.*

Proof. Since $X \xrightarrow{p} Y$ is a Kan fibration and $Y \to \Delta^0$ is a Kan fibration, we get by **??** 10.5 that $X \to \Delta^0$ is a Kan fibration, thus we get that X is a Kan complex. ☐

Proposition 10.7. *Let $X^{(i)} \xrightarrow{p^i} Y^{(i)}$ with $i \in I$ be a set indexed family of Kan fibrations. Then $\prod X^{(i)} \xrightarrow{\prod p^i} \prod Y^{(i)}$ is a Kan fibration.*

Proof. Consider the diagram arising from the assumptions

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"] & \prod X^{(i)} \arrow[r, "\prod \pi_j"] \arrow[d, "\prod p^i"] & X^{(j)} \arrow[d, "p^j"] \\
\Delta^n \arrow[r, "\tau"'] \arrow[ur, dashed, "\exists?"] \arrow[urr, dashed, "h_j"'] & \prod Y^{(i)} \arrow[r, "\pi_j"'] & Y^{(j)}
\end{tikzcd}$$

Since p_j is a Kan fibration, there exists $h_j \colon \Delta^n \to X^{(j)}$ such that $\begin{cases} p_j h_j = \pi_j \tau \\ h_j \iota = \pi_j \sigma_j. \end{cases}$

The universal property of the product gives $h\colon \Delta^n \to \prod X^{(i)}$ such that $\pi_j \circ h = h_j$. Now we have that $\pi_j(h \circ \iota) = h_j \circ \iota = \pi_i \circ \sigma$ and thus $h \circ \iota = \sigma$ as well as $\pi_j(\prod p_i \circ h) = p_j \circ \pi_j \circ h = p_j h_j = \pi_j \tau$ and thus $\prod p_i \circ h = \tau$, where the equalities follow from the uniqueness of the morphism given by the pullback. $\square$

Proposition 10.8. *Let* $\ldots \to X^{(n)} \xrightarrow{p_n} \ldots \to X^{(1)} \xrightarrow{p_1} X^{(0)}$ *be a "tower" of Kan fibrations, that is for all* $i \in \mathbb{N}, p_i$ *is a Kan fibration. Then* $X^\infty := \lim X^{(n)} \xrightarrow{\pi_0} X^{(0)}$ *is a Kan fibration.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & X^{(\infty)} \arrow[d, "\pi_0"] \\
\Delta^n \arrow[r, "\tau"'] \arrow[ur, dashed, "\exists?"] & X^{(0)}
\end{tikzcd}$$

The idea is to construct a cone of the tower with apex Δ^n and then use the universal property X^∞.

$$\begin{tikzcd}
 & & & \Delta^n \arrow[lll, dashed, bend right] \arrow[dll, "h_2"'] \arrow[dl, "h_1"] \arrow[d, "h_0 := \tau_0"] \\
\ldots \arrow[r, "p_2"'] & X^{(2)} \arrow[r, "p_1"'] & X^{(1)} \arrow[r, "p_0"'] & X^{(0)}
\end{tikzcd}$$

Suppose that $0 \leqslant s \leqslant t$ and that we have constructed $h_s \colon \Delta^n \to X^{(s)}$ such that $p_s h_s = h_{s-1}$ and consider the case for $t+1$, where the diagram following on the left side is used to construct the top morphism in the right diagram.

$$\begin{tikzcd}
 & \Lambda^n_k \arrow[d, "\sigma"] \\
 & X^{(\infty)} \arrow[dl, "\pi_{t+1}"'] \arrow[d, "\pi_t"] \\
X^{(t+1)} \arrow[r, "p_{t+1}"'] & X^{(t)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^n_k \arrow[r, "\pi_{t+1} \circ \sigma"] \arrow[d, "\iota"'] & X^{t+1} \arrow[d, "p_{t+1}"] \\
\Delta^n \arrow[r, "h_t"'] \arrow[ur, "\exists h_{t+1}"] & X^{(t)}
\end{tikzcd}$$

Since p_{t+1} is a Kan fibration, h_{t+1} exists and $h_{t+1} \circ \iota = \pi_{t+1} \circ \sigma$ as well as $p_{t+1} \circ h_{t+1} = h_t$. By the universal property of $X^{(\infty)}$ we get

$$\begin{tikzcd} \exists! h\colon \Delta^n \arrow[r] \arrow[dr, "h_t"'] & X^{(\infty)} \arrow[d, "\pi_t"] \\ & X^{(t)} \end{tikzcd}$$

such that for all $t, \pi_t \circ h = h_t$. Also for all t, $\pi_t \circ h \circ \iota = h_t \circ \iota = \pi_t \sigma$, since this holds in particular for the case $t = 0$ the statement is proven. $\square$

10.1 Exercises

Exercise 1. Recall that for a category $\mathcal{C}$ and a class of morphisms $\mathcal{F} \subseteq \mathrm{Mor}(\mathcal{C})$ the class of morphisms with the left lifting property with respect to $\mathcal{F}$ is

$$l(\mathcal{F}) := \left\{ i\colon x \to y \mid \forall \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"] & x \arrow[d, "p \in \mathcal{F}"] \\ b \arrow[r, "g"] & y \end{tikzcd} \exists h \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"] & x \arrow[d, "p"] \\ b \arrow[r, "g"] \arrow[ur, dashed, "h"] & y \end{tikzcd} \right\}$$

where diagrams commute, i.e. $p \circ f = g \circ i, f = h \circ i$ and $g = p \circ h$.

(a) Give the explicit description of the class of morphisms with the right lifting property of $\mathcal{F}, l(\mathcal{F})$, which is defined dually, i.e.

$$r(\mathcal{F}) := (l(\mathcal{F}^{\mathrm{op}}))^{\mathrm{op}}$$

where $\mathcal{F}^{\mathrm{op}}$ denotes the same class of morphisms but viewed in the opposite category $\mathcal{C}^{\mathrm{op}}$.

(b) Show that $l(r(l(\mathcal{F}))) = l(\mathcal{F})$.

From now on let $\mathcal{C} = \mathrm{Set}$ and consider the inclusion $\iota : \varnothing \to \{\star\}$ of the empty set into a set with one element.

(c) Compute the set of morphisms with the right lifting property for $\iota, r(\{\iota\})$, in Set.

(d) Show that $l(r(\{\iota\}))$ is the class of injective maps.

(e) Show that (d) is equivalent to the axiom of choice.

$$\forall X (\varnothing \notin X \implies \exists \varphi\colon X \to \bigcup_{A \in X} A \forall A \in C \varphi(A) \in A)$$

Exercise 2. Let $\mathcal{F}$ be a class of morphisms in a cocomplette category $\mathcal{C}$. We say that an object $x \in \mathcal{C}$ is a retract of an object $y \in \mathcal{C}$ if there exist two morphisms $j : x \to y$ and $q : y \to x$ such that $q \circ j = \mathrm{id}_x$.

(a) Describe retracts in the category of morphisms $\mathrm{Fun}([1], \mathcal{C})$ where we assume $\mathcal{C}$ to be small.

(b) Show that $l(\mathcal{F})$ is closed under retracts in the sense (a).

(c) Show that $l(\mathcal{F})$ is closed under pushouts, i.e. for any pushout square

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"] & x' \arrow[d, "f'"] \\ y \arrow[r, "g'"] & y' \end{tikzcd}$$

we have that if $f \in l(\mathcal{F})$, then $f' \in l(\mathcal{F})$.

(d) Show that $l(\mathcal{F})$ is closed under composition.

(e) Show that $l(\mathcal{F})$ is closed under (countable) transfinite composition, i.e. for any functor $F : \mathbb{N}_0 \to \mathcal{C}$ with $f_n := F(n < n+1) \in l(\mathcal{F})$ we have that the canonical map $F(0) \to \mathrm{colim}_{\mathbb{N}} F$ is in $l(\mathcal{F})$.

Note that (e) can be generalised to arbitrary well-ordered sets (ordinals) in place of $\mathbb{N}$ using transfinite induction.

Exercise 3. Let $\mathcal{F}$ be a class of morphisms on a cocomplete category $\mathcal{C}$. Assume that $\mathcal{F}$ includes all identity morphisms, is closed under pushouts and (countable) transfinite composition in the sense of Exercise 9.2.

(a) Show that $\mathcal{F}$ contains all isomorphisms.

(b) Show that $\mathcal{F}$ is closed under composition.

(c) Deduce that $\mathcal{F}$ is closed under finite coproducts, i.e. for two morphisms $f : x \to y$ and $f' : x' \to y'$ with both $f, f' \in \mathcal{F}$, we have that the induced map $f \amalg f' : x \amalg x' \to y \amalg y'$ is also in $l(\mathcal{F})$.

(d) Show that $\mathcal{F}$ is closed under countable coproducts by expressing a morphism $\coprod_{n \in \mathbb{N}} f_n$ as suitable transfinite composition.

Again, assuming that $\mathcal{F}$ is closed under arbitrary transfinite compositions, we can generalise the above argument to show that $\mathcal{F}$ is closed under set indexed coproducts.

Exercise 4. Let A be a small category. Recall that a morphism $f : y \to z$ in A is a monomorphism, if for any two $g, g' : x \to y$, we have that $f \circ g = f \circ g'$ if and only if $g = g'$.

(a) Show that a morphism in Set is a monomorphism if and only if it is injective.

(b) Deduce from Exercise 3.1 that a morphism of presheaves $f : X \to Y$ in $\widehat{A}$ is a monomorphism if and only if f_a is a monomorphism for all $a \in A$.

(c) Conclude that the class of monomorphism in $\widehat{A}$ is closed under retracts, pushouts, (countable) transfinite composition and coproducts. In particular, the class of monomoprhism in $\widehat{A}$ is saturated.

11 Anodyne extensions

Definition 11.1. We say that a class $\mathcal{M}$ of morphisms in $\mathcal{C}$ is a saturated class if

1. $\mathcal{M}$ contains all isomorphisms,
2. $\mathcal{M}$ is closed under compositions,
3. if $i : L \to K$ is in $\mathcal{M}$ then so is the pushout i' given by the pushout diagram

$$\begin{tikzcd} L \arrow[r, "\sigma"] \arrow[d, "i"] & L' \arrow[d, "i'"] \\ K \arrow[r] & K' \end{tikzcd}$$

4. $\mathcal{M}$ is closed under coproducts and
5. for any sequence of objects and morphism

$$L^{(0)} \to L^{(1)} \to L^{(2)} \to \dots$$

in $\mathcal{M}$, the morphism $L^{(0)} \to L^{(\infty)} = \operatorname{colim} L^{(n)}$ is in $\mathcal{M}$.

We say that $\mathcal{M}$ is a saturated class of monomorphisms if all $i \in \mathcal{M}$ are monomorphisms.

Definition 11.2. The saturation of a class of monomorphisms is the intersection of all such containing it. We write $\overline{\mathcal{M}}$ for its saturation.

Definition 11.3. The class of anodyne extensions is $\overline{\{\Lambda^n_k \xrightarrow{i^n_k} \Delta^n \mid 1 \leqslant n, 0 \leqslant k \leqslant n\}} =$ An.

Remark 11.4. The term anodyne extensions translates to something like harmless extensions.

Proposition 11.5. *The class* $\mathrm{An}^{\boxslash}$ *is equal to the Kan fibrations.*

Proof. ($\subseteq$) Let $p \in \mathrm{An}^{\boxslash}$, then for all $n \geqslant 0$ there exists $0 \leqslant k \leqslant n$ such that $i_{n,k} \boxslash p : p \in \mathrm{KanFib}$

($\supseteq$) Consider ${}^{\boxslash}\mathrm{KanFib}$, this is a saturated class. By unitality of $\mathrm{An} \subseteq {}^{\boxslash}\mathrm{KanFib}$ it holds that for all $i \in \mathrm{An}$ and all $p \in \mathrm{KanFib}$ $i \boxslash p$ which means that for all $p \in \mathrm{KanFib}$ we have that $p \in \mathrm{An}^{\boxslash}$ $\square$

11.1 Exercises

Exercise 1.

We define the class of trivial Kan fibrations to be $r(\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}_0\})$ in Set_Δ. Furthermore, let $\mathcal{F}$ be the smallest saturated class containing all boundary inclusions $\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}_0\}$.

(a) Show that $r(\mathcal{F})$ is the class of trivial Kan fibrations.

(b) Show that $\mathcal{F}$ contains all monomorphisms. Proceed as follows.

- For a fixed monomorphism $\iota : X \to Y$ construct a filtration of Y by $X \cup \mathbf{sk_{n-1}}(Y)$ starting with $X = X \cup \mathbf{sk_{-1}}(Y)$, similar to the skeletal filtration from Exercise 7.2. Here $X \cup \mathbf{sk_n}(Y)$ is defined as pushouts of $\mathbf{sk_n}(Y)$ and the image of ι along their intersection.
- Observe that ι is the transfinite composition of this filtration.
- Show that the morphisms $X \cup \mathbf{sk_{n-1]}}(Y) \to X \cup \mathbf{sk_n}(Y)$ is pushout of morphisms in $\mathcal{F}$ for all $n \in \mathbb{N}_0$ by construction a similar pushout diagram as for the skeletal filtration.

(c) Deduce that $\mathcal{F}$ is the class of monomorphisms.

(d) Conclude that a trivial Kan fibration is a Kan fibration.

Exercise 2.
We have shown in the lecture that the saturated closure of the horn inclusions

$$\mathcal{B}_1 := \{\Lambda^n_k \subseteq \Delta^n \mid 0 \leqslant k \leqslant n \in \mathbb{N}_+\}$$

agrees with the saturated closure of

$$\mathcal{B}_2 := \{(\Delta^1 \times \partial\Delta^n) \cup (\{\epsilon\} \times \Delta^n) \subseteq (\Delta^1 \times \Delta^n) \mid n \in \mathbb{N}_0, epsilon \in \Delta^1_0 = \{0,1\}\}.$$

We call this class the anodyne extensions **An**. Show that **An** agrees with the saturated closure of

$$\mathcal{B}_3 := \{(\Delta^1 \times X) \cup (\{\epsilon\} \times Y) \subseteq (\Delta^1 \times Y) \mid \epsilon \in \Delta^1_0 = \{0,1\}, X \to Y \text{ monomorphism } \}.$$

12 Simplicial Homotopy

Aim: Introduce the notion of homotopy between maps of simplicial sets.

Definition 12.1. Let $f, g\colon K \to X$ be morphisms of simplicial sets

A homotopy $h\colon f \to g$ is a morphism $\Delta^1 \times K \xrightarrow{h} X$ such that

$$\begin{tikzcd}
\Delta^{\{0\}} \times K \cong K \arrow[d, "i_0 \times \mathrm{id}"'] \arrow[dr, "f"] & \\
\Delta^1 \times K \arrow[r, "h"] & X \\
\Delta^{\{1\}} \times K \cong K \arrow[u, "i_1 \times \mathrm{id}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Example 12.2. Let $x, y\colon \Delta^0 \to X$ be vertices of X. A homotopy $h\colon x \to y$ is a map h such that

$$\begin{tikzcd}
\Delta^{\{0\}} \times K \cong K \arrow[d, "i_0 \times \mathrm{id}"'] \arrow[dr, "x"] & \\
\Delta^1 \times K \arrow[r, "h"] & X \\
\Delta^{\{1\}} \times K \cong K \arrow[u, "i_1 \times \mathrm{id}"] \arrow[ur, "y"'] &
\end{tikzcd}$$

that is $h \in X_1$ such that $d_1(h) = x$ and $d_0(h) = y$.

12.1 Adjoint description of homotopy

Let $f, g \in \mathrm{Hom}(K, X) = \underline{\mathrm{Hom}}(K, X)_0$ and $h\colon f \mapsto g$, that is $h \in \mathrm{Hom}(\Delta^1 \times K, X) = \underline{\mathrm{Hom}}(K, X)_1$ such that $d_1(h) = f$ and $d_0(h) = g$.

Upshot Homotopy of maps $K \to X$ is an equivalence relation if $\underline{\mathrm{Hom}}(K, X)$ is a Kan complex. So take the following lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"] & \underline{\mathrm{Hom}(K, X)} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \Delta^0
\end{tikzcd}$$

which equates to the following lifting problem due to the adjunction of the inner hom and the product.

$$\begin{tikzcd}
\Lambda^n_k \times K \arrow[r, "\overline{\sigma}"] \arrow[d, "i \times \mathrm{id}_K"'] & X \\
\Delta^n \times K \arrow[ur, dashed, "\overline{\rho}"'] &
\end{tikzcd}$$

If now X were a Kan complex, then we would obtain a lift if $i \times \mathrm{id}_K$ is an anodyne extension.

Definition 12.3. Let $i\colon L \hookrightarrow K$ is a monomorphism in Set_Δ. Let $f, g\colon K \to X$ be such that $f \circ i = g \circ i$ we also write ($f_{|L} = g_{|L}$). A homotopy $h\colon f \to g$ (rel L) is a homtopy $h\colon f \to g$ such that

$$\begin{tikzcd}
\Delta^1 \times L \arrow[r, "\pi_L"] \arrow[d, "\mathrm{id} \times i"'] \arrow[dr, "1_\alpha"] & L \arrow[d, "f_{|L}=\alpha=g_{|L}"] \\
\Delta^1 \times K \arrow[r, "h"] & X
\end{tikzcd}$$

where 1_α is given by $\Delta^1 \times L \xrightarrow{(!, \mathrm{id}_L)} \Delta^0 \times L \xrightarrow{\alpha} X$ and ! is the unique map into the terminal object.

12.2 Adjoint description of relative homotopy

Consider the pullbackdiagram

$$\begin{tikzcd}
\Delta^1 \arrow[drrr, bend left, "h"] \arrow[dr, dashed, "\overline{h}"] \arrow[ddr, bend right] & & & \\
& \underline{\mathrm{Hom}}(K, X)_\alpha \arrow[rr] \arrow[d] & & \underline{\mathrm{Hom}}(K, X) \arrow[d, "i^*"] \\
& \Delta^0 \arrow[rr, "\alpha"] & & \underline{\mathrm{Hom}}(L, X)
\end{tikzcd}$$

A morphism $\overline{h}$ into the fiber induces a homotopy h that is constant on L, thus this pullback gives us the construction of homotopy relative the simplical subset L.

We want that given a Kan complex X, to get that $\underline{\mathrm{Hom}}(K, X) \xrightarrow{i^*} \underline{\mathrm{Hom}}(L, X)$ is a Kan fibration and finally get that $\underline{\mathrm{Hom}}(K, X)_\alpha$ is a Kan complex and that homotopy (rel L) is the equivalence relation on vertices of $\underline{\mathrm{Hom}}(K, X)_\alpha$.

Let us now prove that $\underline{\mathrm{Hom}}(K, X) \xrightarrow{i^*} \underline{\mathrm{Hom}}(L, X)$ is a Kan fibration.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d] & \underline{\mathrm{Hom}}(K, X) \arrow[d, "i^*"] \\
\Delta^n \arrow[r, "\tau"] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(L, X)
\end{tikzcd}$$

Using two seperate adjunctions for the upper and lower triangle in the square above, we obtain the following two triangles

$$\begin{tikzcd}
\Lambda^n_k \times K \arrow[r, "\overline{\sigma}^K"] \arrow[d, "i \times \mathrm{id}"'] & K \\
\Lambda^n \times K \arrow[ur, dashed, "\overline{\rho}^k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \overline{\mathrm{Hom}}(K, X) \times L \arrow[d, "\overline{i^*}^L"] \\
\Delta^n \times L \arrow[ur, dashed, "\rho \times L"] \arrow[r, "\underline{\tau}^L"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^n_k \times L \arrow[r, "i\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times i"] & \Delta^n \times L \arrow[r, dashed] \arrow[d, "\mathrm{id}\times i"] \arrow[dr, "\overline{\tau}^L"] & \underline{\mathrm{Hom}}(K,X)\times L \arrow[d, "\overline{i^*}^L"] \\
\Lambda^n_k \times K \arrow[r, "i\times\mathrm{id}"'] \arrow[rr, bend right, "\overline{\sigma}^K"'] & \Delta^n\times K \arrow[r, dashed, "\overline{\rho}^k"'] & X
\end{tikzcd}$$

Now we take the pushout of three objects in the left square and assemble with the appropriate morhisms into the following diagram.

$$\begin{tikzcd}
(\Lambda^n_k \times K) \amalg_{\Lambda^n_k \times L} (\Delta^n \times L) \arrow[r, "(\overline{\sigma}^K, \overline{\tau}^L)"] \arrow[d, "(i\times \mathrm{id}, \mathrm{id}\times i)"'] & X \\
\Delta^n \times K \arrow[ur, dashed, "\overline{\rho}^K"'] &
\end{tikzcd}$$

We want to show that $(i \times \mathrm{id}, \mathrm{id} \times i)$ is an anodyne extension, because then $\overline{\rho}^K$ exists, since X is a Kan complex, and this backtracks to i^* being a Kan fibration. Lecture 10.12

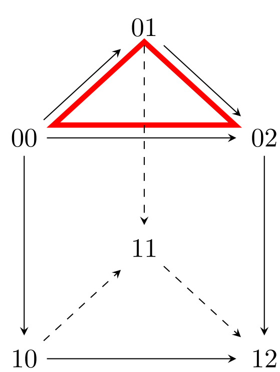
Proposition 12.4. *The following classes have the same saturated closure*

$$B_1 := \{\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leqslant n, 0 \leqslant k \leqslant n\}$$
$$B_2 := \{(\Delta^1 \times \partial\Delta^n) \cup (\Delta^{\{l\}} \times \Delta^n) \overset{i}{\hookrightarrow} \Delta^1 \times \Delta^n \mid 0 \leqslant n, l \in \{0,1\}\}$$
$$B_3 := \{(\Delta^1 \times L) \cup (\Delta^{\{0\}} \times K) \hookrightarrow \Delta^1 \times K \mid L \subseteq K\}$$

where the product $\Delta^1 \times \Delta^n = N([1]) \times N([n]) \cong N([1] \times [n])$ *and* $[1] \times [n]$ *is given by the following diagram:*

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 01 \arrow[r] \arrow[d] & \dots \arrow[r] & 0n \arrow[d] \\
10 \arrow[r] & 11 \arrow[r] & \dots \arrow[r] & 1n
\end{tikzcd}$$

As an example for $l = 1$ and $n = 2$ the domain of a morphism in B_2 is given by the following diagram, where the red triangle indicates 2-simplex with its interior removed.



Proof. Note that $\overline{\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \leqslant 0\}}$ is given by the monomorphisms. ($\overline{B_2} \subseteq \overline{B_1}$) We exhibit i as a finite composition of anodyne extensions

Let $(\Delta^1 \times \Delta^n)^{(-1)} := (\Delta^1 \times \partial\Delta^n) \cup (\Delta^l \times \Delta^n)$ and $(\Delta^1 \times \Delta)^{(n)} := \Delta^1 \times \Delta^n$ so that we obtain the following coposition of morphisms.

$$(\Delta^1 \times \Delta^n)^{(-1)} \hookrightarrow (\Delta^1 \times \Delta^n)^{(0)} \hookrightarrow (\Delta^1 \times \Delta^n)^{(1)} \hookrightarrow \ldots \hookrightarrow (\Delta^1 \times \Delta^n)^{(n)}$$

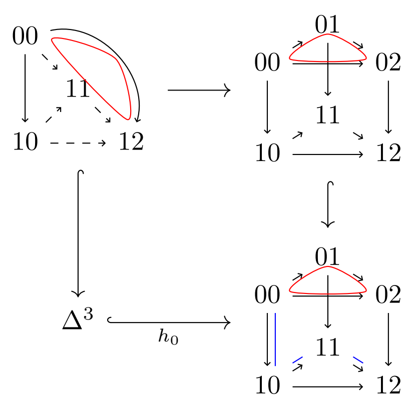
The non-degenerate (n+1)-simplices in $\Delta^1 \times \Delta^n$ are chains of the form

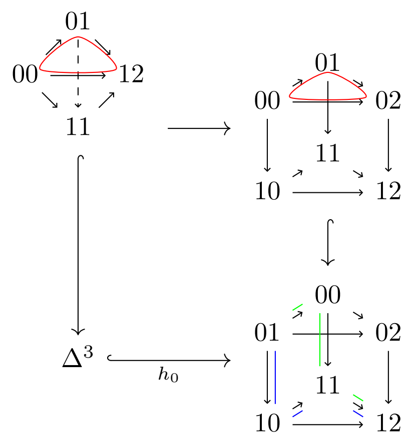
$$\begin{tikzcd}
00 \arrow[r] & 01 \arrow[r] & 02 \arrow[r] & \ldots \arrow[r] & 0k \arrow[d] & & & \\
& & & & 1k \arrow[r] & 1(k+1) \arrow[r] & \cdots \arrow[r] & 1n
\end{tikzcd}$$

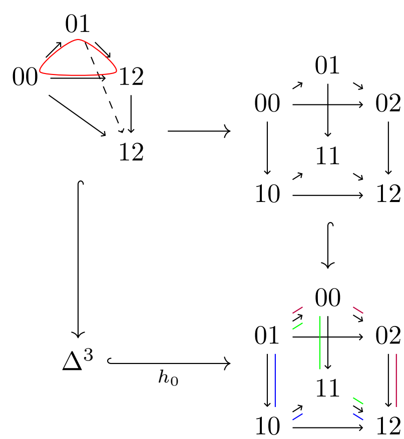
for $0 \leqslant k \leqslant n$. We have that $(\Delta^1 \times \Delta^n)^{(i)} \subseteq \Delta^1 \times \Delta^n$ is the smallest subset containing $(\Delta^1 \times \partial\Delta^1)^{(-1)}$ and the non degenerate $(n+1)$ simplices h_j for $0 \leqslant j \leqslant i$. Which means it is given by the following pushout square

$$\begin{tikzcd}
\Lambda^{n+1}_{i+1} \arrow[r] \arrow[d, hook, "B_1 \ni"'] & (\Delta^1 \times \Delta^n)^{(i-1)} \arrow[d, hook, "\in \overline{B}_1"] \\
\Delta^{n+1} \arrow[r, "h_i"] & (\Delta^1 \times \Delta^n)^{(i)}
\end{tikzcd}$$

Let us take a look at the example for the case $n = 2$ in detail. We iteratively add 3-simplices into $\partial\Delta^2 \times \Delta^1 \cup \Delta^2 \times \Delta^1$.







The colors in the bottom right simplex are the 3-simplices (chains) that we glued into the the product in each of the steps and the red encircled triangles symbolize holes, that is missing 2-simplices.

For the case $(\overline{B}_1 \subseteq \overline{B}_2 = \overline{B}_3)$ we exhibit each horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ as a retract of a map in $\overline{B}_3$ for $0 \leqslant k \leqslant n$.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "\mathrm{id}"] & (\Delta^1 \times \Lambda^n_k) \cup (\Delta^1 \times \Delta^1) \arrow[r] \arrow[d, "\in B_3"] & \Lambda^n_k \arrow[d] \\
\Lambda^n \arrow[r] & \Delta^1 \times \Delta^n \arrow[r] & \Delta^n
\end{tikzcd}$$

The vertical maps in the bottom row are given by the following composition of maps of partially ordered sets

$$[n] \xrightarrow{s} [1] \times [n] \xrightarrow{r} [n]$$

where the image of s is given as follows

$$
\begin{tikzcd}[row sep=small]
& 01 \arrow[r] & 02 \arrow[r] & \dots \arrow[r] & 0k \arrow[dr] & & & & \\
s\colon & & & & & 1(k+1) \arrow[r] & 1(k+2) \arrow[r] & \dots \arrow[r] & 1n
\end{tikzcd}
$$

and the image of the map r is given by

$$
r\colon \quad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & \dots \arrow[r] & k-1 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \dots \arrow[r] & k \arrow[d] \\
k \arrow[r] & k \arrow[r] & k \arrow[r] & \dots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \dots \arrow[r] & n
\end{tikzcd}
$$

□

Corollary 12.5. *Let $S \overset{\iota}{\hookrightarrow} T$ be an anodyne extension and $L \overset{i}{\hookrightarrow} K$ be an arbitrary inclusion. Then*

$$(S \times K) \cup (T \times L) \xrightarrow{incl.} T \times K$$

is an anodyne extension. In particular for $S = \Lambda^n_k$ and $T = \Delta^n$ we obtain that

$$(\Lambda^n_k \times K) \cup (\Delta^n \times L) \xrightarrow{incl.} \Delta^n \times K$$

is an anodyne extension.

Proof. For $L \overset{i}{\to} K$ define $\mathcal{M} := \{S \overset{i}{\hookrightarrow} T \mid (S \times K) \cup (T \times L) \to T \times K \text{ is anodyne}\}$ Notice that $\mathcal{M}$ is a saturated class hence if $\mathcal{M} \supseteq B_3 \implies \mathcal{M} = \overline{B}_3$. Consider $L' \hookrightarrow K'$ a monomorphism, that gives rise to $S = (\Delta^1 \times L') \cup (\Delta^{\{l\}} \times K') \to \Delta^1 \times K' = T$

$$
\begin{tikzcd}
(((\Delta^1 \times L') \cup (\Delta^{\{l\}} \times K')) \times K) \arrow[r, "\in \mathrm{An}"] \arrow[d, "\cong"] & (\Delta^1 \times K') \times K \arrow[d, "\cong"] \\
\Delta^1 \times (L' \times K \cup K' \times L) \cup (\Delta^{\{l\}} \times (K' \times K)) \arrow[r] & \Delta^1 \times (K' \times K)
\end{tikzcd}
$$

Where the vertical map in the top row is in B_3 and thus an anodyne extension. It follows the bottom row is an anodyne extension as well, which we wanted to show. □

Let now X be a Kan complex.

Definition 12.6. Let $x \in X_0$ and $n \geqslant 1$. We define

$$
\pi_n(X, x) = \left\{ \Delta^n \xrightarrow{\alpha} X \;\middle|\; \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \Delta^0 \arrow[d] \\ \Delta^n \arrow[r] & X \end{tikzcd} \right\} \Big/ \text{htpy relative } \partial\Delta^n
$$

Definition 12.7. The (pointed) simplicial n-sphere is

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
\Delta^0 \arrow[r, "x"] \arrow[dr, bend right, "x"'] & \Delta^n/\partial\Delta^n = S^n \arrow[dr, "\overline{\alpha}"] & \\
& & X
\end{tikzcd}$$

Proposition 12.8. *There are pullback squares* ($\partial\Delta^n \overset{i}{\hookrightarrow} \Delta^n$)

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S^n, X)_x \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(S^n, X) \arrow[r] \arrow[d, "\mathrm{ev}_*"] & \underline{\mathrm{Hom}}(\Delta^n, X) \arrow[d, "i^*"] \\
\Delta^0 \arrow[r, "x"] \arrow[rr, bend right, "c(x)"'] & X \cong \underline{\mathrm{Hom}}(\Delta^0, X) \arrow[r] & \underline{\mathrm{Hom}}(\partial\Delta^n, X)
\end{tikzcd}$$

where the vertical morphisms are Kan Fibrations and $\underline{\mathrm{Hom}}(S^n, X)_x$ *is a Kan complex. Moreover the induced map*

$$\underline{\mathrm{Hom}}(S^n, X)_x \to \underline{\mathrm{Hom}}(S^n, X)_{c(x)}$$

is an isomorphism (between Kan complexes, since X is a Kan complex).

Proof. Consider the following commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^m \times S^n, X) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^m \times \Delta^n, X) \arrow[d] \\
\mathrm{Hom}(\Delta^m \times \Delta^0, X) \arrow[r] & \mathrm{Hom}(\Delta^m \times \partial\Delta^n, X)
\end{tikzcd}$$

this is a pullback since the Hom-functor sends colimits to limits and thus the diagram from **??** 12.7 to a pullback. The left square is a pullback by construction and thus by the Pasting lemma the outer square is a pullback as well. □

Lecture 12.11

Proposition 12.9. *Let X be a Kan complex and $x \in X_0$. Then the "old" and "new" definitions of $\pi_1(X, x)$ agree.*

Proof. Consider the square

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\Delta^1, X)_{c(x)} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "i_*"] \\
\Delta^0 \arrow[r, "c(x)=(x,x)"] & \underline{\mathrm{Hom}}(\partial\Delta^1, X) \cong X \times X
\end{tikzcd}$$

and let $f, g\colon \Delta^1 \to X$ such that $f_{|\partial\Delta^1} = (x, x) = g_{|\partial\Delta^1}$. A homotopy $h\colon f \to g$ (rel $\partial\Delta^1$) is by definition

$$\begin{tikzcd}
\Delta^0 \times \Delta^1 \arrow[d] \arrow[dr, bend left, "f"] & \\
\Delta^1 \times \Delta^1 \arrow[r, "h"] & X \\
\Delta^1 \times \Delta^1 \arrow[u] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

the homotopy yields the following commutative square

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, "u"] & x \arrow[d, equal] \\
x \arrow[r, "g"'] & x
\end{tikzcd}$$

∎

<u>Aim</u>: To prove that $\pi_n(X, x)$ is a group for $n \geqslant 1$ (that is abelian for $n \geqslant 2$) as well as functoriality. Fix (X, x) where X is a Kan complex $x \in X_0$, take n-simplices $\alpha, \beta\colon \Delta^n \to X$ representatives of classes in $\pi_n(X, x)$ that is $\alpha_{|\partial\Delta^n} = c(x) = \beta_{|\partial\Delta^n}$. For $n = 1$ this yields a horn, which can be extended since X is a Kan complex.

$$\begin{tikzcd}
& x \arrow[dr, "\beta"] & \\
x \arrow[ur, "\alpha"] \arrow[rr, dashed, "\gamma"'] & & x
\end{tikzcd}$$

For the general case we obtain a map from the horn (given on its n-simplices) as follows

$$\begin{tikzcd}
\Lambda_n^{(n+1)} \arrow[r, "{(x,x,\ldots,x,\alpha,\bullet,\beta)}"] \arrow[d, hook] & X \\
\Delta^{(n+1)} \arrow[ur, dashed, "\exists\sigma"'] &
\end{tikzcd}$$

where $d_n\sigma =: \gamma$ is the composition of α and β. Now we define the multiplicative law on $\pi_n(X, x)$ by

$$\pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$$
$$([\alpha], [\beta]) \mapsto [\gamma]$$

Proposition 12.10. *The above binary operation is well defined.*

Proof. We have $\alpha, \alpha', \beta, \beta'\colon \Delta^n \to X$ are representatives in $\pi_n(X, x)$ and

$$h_{n-1}\colon \Delta^1 \times \Delta^n \to X \text{ htpy } \alpha \to \alpha' (\mathrm{rel}\, \partial\Delta^n)$$
$$h_{n-1}\colon \Delta^1 \times \Delta^n \to X \text{ htpy } \beta \to \beta' (\mathrm{rel}\, \partial\Delta^n)$$

Choose $w, w' \colon \Delta^{n+1} \to X$ such that

$$\begin{aligned} \partial w &= (x, \dots, x, \alpha, \gamma, \beta) \\ \partial w' &= (x, \dots, x, \alpha', \gamma', \beta') \end{aligned}$$

Putting all this together we obtain

$$\begin{tikzcd} (\Delta^{(n+1)} \times \partial\Delta^1) \cup (\Delta^1 \times \Lambda^{n+1}_n) \arrow[rr, "{(w,w',(x,\dots,x,h_{n-1},\bullet,h_n))}"] \arrow[d, hook] & & X \\ \Delta^1 \times \Delta^{(n+1)} \arrow[urr, "w''"'] & & \end{tikzcd}$$

where w'' exists, since on the one summand we just have an inclusion and on the other we use that X is a Kan complex. This way the composite

$$h_n \colon \Delta^1 \times \Delta^n \xrightarrow{1 \times d^n} \Delta^1 \times \Delta^{n+1} \xrightarrow{w''} X$$

is a homotopy from γ to γ' (rel $\partial\Delta^n$). $\square$

Let us take a look at the case $n = 1$ explicitely.

$$\Delta^1 \times \Delta^2 = \begin{tikzcd}[column sep=small, row sep=small] & 01 \arrow[dr] \arrow[dd] & \\ 01 \arrow[ur] \arrow[rr] \arrow[dd] & & 02 \arrow[dd] \\ & 11 \arrow[dr] & \\ 10 \arrow[ur] \arrow[rr] & & 12 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=small, row sep=small] & x \arrow[dr, "\alpha"] \arrow[dd, equal] & \\ x \arrow[ur, "\beta"] \arrow[rr, "\gamma"] \arrow[dd, equal] \arrow[r, phantom, "w"] & & x \arrow[dd, equal] \\ & x \arrow[dr, "\alpha'"] & \\ x \arrow[ur, "\beta'"] \arrow[rr, "\gamma'"'] \arrow[r, phantom, "w'"] & & x \end{tikzcd}$$

Proposition 12.11. *For all $n \geqslant 1$ it holds that $\pi_n(X, x)$ is a group.*

Proof. (Unitality) The neutral element in $\pi_n(X, x)$ is given by $[c(x)]$, where

$$c^n(x) \colon \quad \begin{tikzcd} & \Delta^0 \arrow[dr, "x"] & \\ \Delta^n \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

Let $w \coloneqq s_n(\alpha) \colon \Delta^{n+1} \to X$ for $\alpha \colon \Delta^n \to X$ such that $d_{n+1}(w) = d_{n+1}(s_n(\alpha)) = \alpha = d_n(s_n(\alpha)) = d_n(w)$.

$$\begin{tikzcd} \Lambda^{n+1}_k \arrow[r, "{(x,x,\dots,x,c^n(x),\bullet,\alpha)}"] \arrow[d, hook] & X \\ \Delta^{n+1} \arrow[ur, "w"'] & \end{tikzcd}$$

where $[c^n(x)][\alpha] = [\alpha]$.

(Associativity) Let $\alpha, \beta, \gamma \colon \Delta^n \to X$ be representatives of classes in $\pi_n(X, x)$. Choose $w_{n-1}, w_{n+1}, w_{n+2} \colon \Delta^{n+1} \to X$ such that

$$\begin{aligned} \partial w_{n-1} &= (x, \dots, x, \alpha, u, \beta); [\alpha][\beta] = [u] \\ \partial w_{n+1} &= (x, \dots, x, u, v, \gamma); [u][\gamma] = [v] \\ \partial w_{n+2} &= (x, \dots, x, \beta, \omega, \gamma); [\beta][\gamma] = [\omega] \end{aligned}$$

Let $q = (c^{n+1}(x), \dots, c^{n+1}(x), w_{n-1}, \bullet, w_{n+1}, w_{n+2})$, we obtain a diagram

$$\begin{tikzcd} \Lambda^{n+2}_n \arrow[r, "q"] \arrow[d, hook] & X \\ \Delta^{n+2} \arrow[ur, "\exists \tilde{w}"'] & \end{tikzcd}$$

Now let us define $w_n := d_n(\tilde{w})$ and thus $\partial w_n = (x, \dots, x, \alpha, v, \omega); [\alpha][\omega] = [v]$. This results in

$$([\alpha][\beta])[\gamma] = [u][\gamma] = [v] = [\alpha][w] = [\alpha]([\beta][\gamma])$$

(Inverses) Let $\Delta^n \xrightarrow{\alpha} X$ be a representative of a calass in $\pi_n(X, x)$. Consider the following horn-extension diagram

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(x,\dots,x,\bullet,c^n(x),\alpha)}"] \arrow[d, hook] & X \\ \Delta^{n+2} \arrow[ur, "\exists w"] & \end{tikzcd}$$

thus we get the equation $[d_{n-1}(w)][\alpha] = [c^n(x)]$

<u>Aim</u>: Establish the functoriality of $\pi_n(X, x)$ and let (X, x) and (Y, y) be pairs such that Y, X are Kan complexes $x \in X_0$ and $y \in Y_0$ and $f \colon X \to Y$ such that $f_0(x) = y$.

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\Delta^n, X)_{c^n(x)} \arrow[rr] \arrow[dr, "f_*"] \arrow[dd] & & \underline{\mathrm{Hom}}(\Delta^n, X) \arrow[dr, "f_*"] & \\ & \underline{\mathrm{Hom}}(\Delta^n, Y)_{c^n(y)} \arrow[rr] \arrow[dd] & & \mathrm{Hom}(\Delta^n, Y) \arrow[dd] \\ \Delta^0 \arrow[rr, "c^n(x)"] \arrow[dr, equal] & & \underline{\mathrm{Hom}(\partial\Delta^n, X)} \arrow[dr, "f^*"] & \\ & \Delta^0 \arrow[rr, "c^n(y)"] & & \underline{\mathrm{Hom}(\partial\Delta^n, X)} \end{tikzcd}$$

For $i \colon \partial\Delta^n \to \Delta^n$. Then $\pi_0(f_*) =: \pi_n(f) \colon \pi_n(X, x) \to \pi_n(Y, y)$ is a morphism of pointed sets and gives the functoriality. $\square$

Definition 12.12. Let $\Delta^0/\mathrm{Set}_\Delta$ be the category of pointed sets.

Proposition 12.13. *Let $f\colon (X,x) \to (Y,y)$ be a morphism of pointed Kan complexes, then*
$$\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,y)$$
is a group homomorphism.

Proof. Let $\alpha, \beta\colon \Delta^n \to X$ be representatives of classes in $\pi_n(X,x)$ and $w\colon \Delta^{n+1} \to X$ where $\partial w = (x,\dots,x,\alpha,\gamma,\beta)$ so that $[\alpha][\beta] = [\gamma]$. But then
$$f_*(w)\colon \Delta^{n+1} \xrightarrow{w} X \xrightarrow{f} Y$$
has boundary $\partial(f_*(w)) = (y,\dots,y,f_*(\alpha),f_*(\gamma),f_*(\beta))$ $\square$

Proposition 12.14. *Let $f_1, f_2\colon X \to Y$ and $g_1, g_2\colon Y \to Z$ be morphisms between Kan complexes. Supose that $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1 \circ f_1 \sim g_2 \circ f_2$*

Proof. Enough to treat the case $(f_1 = f_2, g_1 \sim g_2)$ and $(f_1 \sim f_2, g_1 = g_2)$. The first case is $f \coloneqq f_1 = f_2$ and $g_1 \sim g_2$, which gives
$$\begin{tikzcd}
X \cong \Delta^{\{0\}} \times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "i_0\times\mathrm{id}"'] & \Delta^{\{0\}} \times Y \arrow[d] \arrow[dr, bend left, "g_1"] & \\
\Delta^1 \times X \arrow[r, "\mathrm{id}\times f"] & \Delta^1 \times X \arrow[r, "h"] & Z \\
X \cong \Delta^{\{1\}} \times X \arrow[r, "\mathrm{id}\times f"] \arrow[u, "i_1\times\mathrm{id}"] & \Delta^{\{1\}} \times Y \arrow[u] \arrow[ur, bend right, "g_2"'] &
\end{tikzcd}$$
$h_f\colon g_1 f \to g_2 f$ is a homotopy of of the concatenation.

The second case is $g \coloneqq g_1 = g_2$
$$\begin{tikzcd}
\Delta^{\{0\}} \times X \arrow[d] \arrow[dr, bend left, "f_1"] \arrow[drr, bend left, "gf_1"] & & \\
\Delta^1 \times X \arrow[r, "h"] & Y \arrow[r] & Z \\
\Delta^{\{1\}} \times X \arrow[u] \arrow[ur, bend right, "f_2"'] \arrow[urr, bend right, "gf_2"'] & &
\end{tikzcd}$$
The horizontal composition then gives a homotopy from gf_1 to gf_2. $\square$

12.3 Exercises

Exercise 1.

Consider a set G with two unital binary operations $\otimes : G \times G \to G$. Suppose that
$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d)$$
for all $a, b, c, d \in G$.

1. Show that both units e and $e_\otimes$ agree.
2. Deduce that $\cdot = \otimes$.
3. Conclude that $(G, \cdot, e)$ is an abelian monoid.

Exercise 2.

Fix a Kan complex X and $n \in \mathbb{N}_0$.

(a) Let $\alpha : \Delta^n \to X$ represent an element of $\pi_n(X, x)$ for some $x \in X_0$ and $n \in \mathbb{N}_0$. Show that α is homotopic to the neutral element $x : \Delta^n \to \Delta^0 \xrightarrow{x} X$ if and only if there exists some $\sigma \in \Delta^{n+1}$ such that $d_i(\sigma) = x$ for $0 \leqslant i \leqslant n$ and $d_{n+1}(\sigma) = \alpha$.

(b) Deduce that if $p : X \to \Delta^0$ is a trivial Kan fibration, i.e. $p \in r(\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}_0\})$, then $\pi_n(X, x) = 0$ for all $x \in X_0$.

(c) Deduce that for X n-skeletal we have that $\pi_m(X, x) \cong 0$ for any $x \in X_0$ and $m \geqslant n$.

(d) Show that for a group G we have that

$$\pi_n(N(BG), \star) \cong \begin{cases} G & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

where $\star$ is the unique object of BG.

Lecture 17.12

13 Trivial Kan fibrations, loop spaces and the Serre long exact sequence

<u>Construction</u>: Let X be a Kan complex and $x \in X_0$. The <u>(based) loop space</u> $\Omega(X, x)$ is given by the following pullback:

$$\begin{tikzcd}
\Omega(X,x) \arrow[rr] \arrow[d] & & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d] \arrow[d, shift left=6, "i^*"] \\
\Delta^0 \arrow[r, "{(x,x)}"] \arrow[rr, bend right, "c(x)"'] & {} & X \times X \cong \underline{\mathrm{Hom}}(\partial\Delta^1, X)
\end{tikzcd}$$

where i^* is a Kan fibration and thus $\Omega(X, x)$ a Kan complex. The <u>path space</u> $\mathrm{P}\,X$ is given by the following diagram

$$\begin{tikzcd}
\Omega(X,x) \arrow[r] \arrow[d] & \mathrm{P}\,X \arrow[r] \arrow[d, "\pi"] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "i^*"] \\
\Delta^0 \arrow[r] \arrow[rr, bend right, "{(x,x)}"'] & X \arrow[r, "{(i(x), \mathrm{id}_X)}"] & X \times X \cong \underline{\mathrm{Hom}}(\partial\Delta^1, X)
\end{tikzcd}$$

<u>Aim</u>: Prove that for all $n \geqslant 0$ there is an isomorphism $\pi_{n+1}(X, x) \xrightarrow{\sim} \pi_n(\Omega(X, x), 1_X)$ and that for all $n \geqslant 1$ the group $\pi_n(\Omega(X, x), 1_X)$ is abelian and hence for all $n \geqslant 2$ the group $\pi_n(X, x)$ is abelian.

Definition/Proposition 13.1. Let $p\colon X \to Y$ be a morphism in Set_Δ. The following are equivalent

1. $\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\} \boxslash p$
2. $\overline{\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\}} = \{\text{monomorphisms}\} \boxslash p$

We call such a map a <u>trivial Kan fibration</u>. Since $\Lambda^n_k \hookrightarrow \Delta^n$ is a monomorphism, all trivial Kan fibrations are especially Kan fibrations.

Proposition 13.2. *Let $X \xrightarrow{p} \Delta^0$ be a trivial Kan fibration, then for all $x \in X$ and for all $n \geqslant 0$ the homotopy group is trivial, that is $\pi_n(X, x) = \{*\}$.*

Proof. Exercise. $\square$

Proposition 13.3. *Let $X \xrightarrow{p} Y$ be a Kan fibration and $L \xrightarrow{i} K$ be an anodyne extension, then $\underline{\mathrm{Hom}}(K, X) \to \underline{\mathrm{Hom}}(L, X) \times_{\underline{\mathrm{Hom}}(L,Y)} \underline{\mathrm{Hom}}(K, Y)$ is a trivial Kan fibration.*

Proof. The idea of the proof is the same as for Exercise 10.4.

$$\begin{tikzcd}
(\Lambda^n_k \times L) \cup (\Delta^n \times K) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times L \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(K, X) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(L, X) \times_{\underline{\mathrm{Hom}}(L,Y)} \underline{\mathrm{Hom}}(K, Y)
\end{tikzcd}$$

□

Corollary 13.4. *Let X be a Kan complex then there is a pullback square:*

$$\begin{tikzcd}
\mathrm{P}\, X \arrow[r] \arrow[d, "p"] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "i^*"] \\
\Delta^0 \arrow[r, "x"] & \underline{\mathrm{Hom}}(\Delta^{\{0\}}, X) \cong X
\end{tikzcd}$$

where i^* is a trivial Kan fibration by **??** *13.1 and we claim that p is one as well.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{P}\, X \arrow[r] \arrow[d, "\pi"'] & \underline{\mathrm{Hom}(\Delta^1, X)} \arrow[d] \arrow[dd, bend left, "i^*"] \\
X \arrow[r, "{(c(x), \mathrm{id}_X)}"] \arrow[d] & X \times X \cong \underline{\mathrm{Hom}(\partial\Delta^1, X)} \arrow[d, "\pi_0"] \\
\Delta^0 \arrow[r, "x"] & X \cong \underline{\mathrm{Hom}(\Delta^{\{0\}}, X)}
\end{tikzcd}$$

We know the top square is a pullback, thus if the bottom one were one as well, then we would be done by the pasting lemma. So let us take a closer look here. So the following diagram gives a pullback, where the first component of the map into the product is determined to be $c(x)$ by the commutativity of the square

$$\begin{tikzcd}
W \arrow[dr, dashed, "g"] \arrow[drr, bend left, "{(f,g)=(c(x),g)}"] \arrow[ddr, bend right] & & \\
& X \arrow[r, "{(c(x), \mathrm{id}_X)}"] \arrow[d] & X \times X \arrow[d, "\pi_0"] \\
& \Delta^0 \arrow[r] & X
\end{tikzcd}$$

Corollary 13.5. *For all $x \in \mathrm{P}\, X_0$ and for all $n \geqslant 0$ we have that $\pi_n(\mathrm{P}\, X, x) = \{*\}$.*

Let $X \xrightarrow{p} Y$ be a Kan fibration, with X as well as Y Kan complexes and $x \in X_0$ as well as $y = p(x) \in Y_0$. Define

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

Notice that $x \in F_0$ by construction. By the functoriality of the homotopy groups we get a sequence

$$\pi_n(F, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{p_*} \pi_n(Y, y)$$

for all $n \geqslant 0$. $\square$

Construction 13.6. Let $n \geqslant 1$ and $\alpha\colon \Delta^n \to Y$ be a representative of a class in $\pi_n(Y, y)$. Consider the following lifting diagram:

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & \Delta^0 \arrow[r, "x"] & X \arrow[d, "p"] \\ \Delta^n \arrow[rru, "\exists v"] \arrow[rr, "\alpha"] & & Y \end{tikzcd}$$

Where the left horn-inclusion is an anodyne extension and the morphism p is a Kan fibration. Since Kan fibrations are exactly the morphisms with the right lifting property with respect to the anodyne extensions we get a lift v. We define $\partial([\alpha]) = [d_0(v)]$ for $d_0(v)\colon \Delta^{n-1} \to F$, notice that since F is a pullback there is a unique morphism into F given by the morphism $d_0(v)$ that goes to X and the unique morphism into the terminal object Δ^0. So it makes sense to take $d_0(v)$ as a morphism into F. This defines a map $\partial\colon \pi_n(Y, y) \to \pi_n(F, x)$.

We thus get a map $\partial\colon \pi_n(Y, y) \to \pi_{n-1}(F, x)$. To see this map is well defined consider $h\colon \Delta^1 \times \Delta^n \to Y$ a homtopy of α to α' (rel $\partial\Delta^n$). Then we choose lifts

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "c(x)"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha"] \arrow[ru, dashed, "v"] & Y \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_0 \arrow[r, "c(x)"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha'"] \arrow[ru, dashed, "v'"] & Y \end{tikzcd}$$

to obtain for the tuple $\omega = (v, v', (\bullet, x \ldots, x))$ a lift

$$\begin{tikzcd} (\partial\Lambda^n_k \times L) \cup (\Delta^n \times K) \arrow[r, "\omega"] \arrow[d, "\mathrm{An}\ni"'] & X \arrow[d, "p"] \\ \Delta^n \times L \arrow[r] \arrow[ru, dashed, "\exists \tilde{h}"] & Y \end{tikzcd}$$

which again exists since the left vertical map is anodyne and the morphism p a Kan fibration. This results in

$$\Delta^1 \times \Delta^{n-1} \xrightarrow{\mathrm{id} \times d^0} \Delta^1 \times \Delta^n \xrightarrow{\tilde{h}} X$$

Now this is a homotopy of $d_0 v$ to $d_0 v'$ $(\operatorname{rel}\partial)\Delta^{n+1}$ and thus $[d_0 v] = [d_0 v']$. For $n = 2$ this amounts to the following picture.

$$\begin{tikzcd}
& 01 \arrow[dl, equal] \arrow[dd, equal] \arrow[dr, "d_v"] & & & & 01 \arrow[dl, equal] \arrow[dd, equal] \arrow[dr, equal] & \\
00 \arrow[rr, equal] \arrow[dd, equal] & & 02 \arrow[dd, equal] & \xrightarrow{p} & 00 \arrow[rr, equal] \arrow[dd, equal] & & 02 \arrow[dd, equal] \\
& 11 \arrow[dl, equal] \arrow[dr, "d_0 v'"] & & & & 11 \arrow[dl, equal] \arrow[dr, equal] & \\
10 \arrow[rr, equal] & & 12 & & 10 \arrow[rr, equal] & & 12
\end{tikzcd}$$

Theorem 13.7. *Serre's Long exact sequence Let $X \xrightarrow{p} Y$ be a Kan fibration such that Y is a Kan complex. Let $x \in X_0$ and $y := p(x) \in Y_0$. Then there is a long exact sequence of pointed sets.*

$$\begin{tikzcd}
& \dots \arrow[r] & \pi_2(Y,y) \arrow[dll, "\partial"] \\
\pi_1(F,x) \arrow[r] & \pi_1(X,x) \arrow[r] & \pi_1(Y,y) \arrow[dll, "\partial"] \\
\pi_0(F,x) \arrow[r] & \pi_0(X,x) \arrow[r] & \pi_0(Y,y)
\end{tikzcd}$$

for a given pullback square

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "x"] & Y
\end{tikzcd}$$

We call a sequence of pointed sets $((A,a) \xrightarrow{f} (B,b) \xrightarrow{g} (C,c))$ exact if $\ker g := \{b \in B \mid g(b) = c\} = \operatorname{im}(f)$. Moreover there is a natural action of $\pi_1(Y,y)$ on $\pi_0(F,x)$ such that the stabilizer of $[x] \in \pi_1(Y,y)$ is precisely $\operatorname{im}(p_*)$, this means that $i_*([x']) = i_*([x''])$ if and only if $[x']$ and $[x'']$ are in some $\pi_1(Y,y)$-orbit.

Corollary 13.8. *Let X be a Kan complex and $x \in X_0$. Take the pullback*

$$\begin{tikzcd}
\Omega(X,x) \arrow[r] \arrow[d] & \mathrm{P}\,X \arrow[d, "\pi"] \\
\Delta^0 \arrow[r] & X
\end{tikzcd}$$

Then for all $n \geqslant 0$ the morphism $\pi_{n+1}(X,x) \xrightarrow{\partial} \pi_n(\Omega(X,x), 1_x)$ is bijective.

Proof. By the Serre long exact sequence we have for all $n \geqslant 1$

$$\{*\} = \pi_{n+1}(\mathrm{P}\,X, 1_x) \to \pi_{n+1}(X,x) \underset{\sim}{\xrightarrow{\partial}} \pi_n(F,x) \to \pi_n(\mathrm{P}\,X, 1_x) = \{*\}$$

since ∂ is a group homomorphism here, we get the isomorphism. For $n = 0$ we have

$$\{*\} = \pi_1(\mathrm{P}\,X, 1_x) \to \pi_1(X, x) \xrightarrow[\sim]{\partial} \pi_0(F, x) \to \pi_0(\mathrm{P}\,X, 1_x) = \{*\}$$

notice that $\pi_0(F, 1_x)$ and $\pi_(\mathrm{P}\,X, 1_x)$ are not groups, thus we need to use the orbit stabilizer theorem and the last part of **??** 13.7 to obtain that ∂ is a bijection. $\square$

Proposition 13.9. *The map $\partial\colon \pi_n(Y, y) \to \pi_{n-1}(F, x)$ is a group homomorphism for $n \geqslant 2$.*

Proof. Let $\omega\colon \Delta^{n+1} \to Y$ be an (n+1)-simplex, such that $\partial\omega = (y, \ldots, y, \alpha_{n-1}, \alpha_n, \alpha_{n+1})$. Furthermore ω is a witness of $[\alpha_{n-1}][\alpha_{n+1}] = [\alpha_n]$ in $\pi_n(Y, y)$. Choose (for $i = n-1, n, n+1$) witnesses of $\partial([\alpha_i]) = [d_0 v_i]$

$$\begin{tikzcd}
\Lambda_0^n \arrow[r, "c(x)"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\alpha_i"'] \arrow[ur, dashed, "v_i"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda_0^{n+1} \arrow[r, "{(x,\ldots,x,v_{n-1},v_n,v_{n+1})}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^{n+1} \arrow[r, "\omega"'] \arrow[ur, dashed, "\exists\gamma"] & Y
\end{tikzcd}$$

We get the folowing equation $\partial([\alpha_{n-1}])\partial([\alpha_{n+1}]) = [d_0 v_{n-1}][d_0 v_{n+1}] = [d_0 v_n] = \partial([\alpha_n]) = \partial([\alpha_{n-1}][\alpha_{n+1}])$. Then $d_0\gamma$ is a witness of the above composition since $\partial(d_0\gamma) = (x, \ldots, x, d_0 v_{n-1}, d_0 v_n, d_0 v_{n+1})$. We now want to show (parts of) the exactness of Serre's long exact sequence. Let

$$[\alpha] \in \pi_n(F, x) \xrightarrow{i_*} \pi_n(X, x) \xrightarrow{p_*} \pi_n(Y, y)$$

be the sequence of homtopygroups. Then we have the following square

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "y"] & Y
\end{tikzcd}$$

This shows that $\mathrm{im}(i_*) \subseteq \ker(p_*)$. Conversely, if we have a commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "p_*\alpha"] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "y"] & Y
\end{tikzcd}$$

Then $\alpha\colon \Delta^n \to X$ is an n-simplex of the fibre $\pi_n(X,x) \xrightarrow{p_*} \pi_n(Y,y) \xrightarrow{\partial} \pi_{n-1}(F,x)$. To show $\mathrm{im}(p_*) \subseteq \ker(\partial)$, take an n-simplex and extend it along p

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] \arrow[dr, "p_\alpha"'] & X \arrow[d, "p"] \\
 & Y
\end{tikzcd}$$

This can be completed to a commutative square

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "c(x)"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "p_\alpha"] \arrow[ur, "\alpha"] & Y
\end{tikzcd}$$

where $\partial[p_*\alpha] = [d_0\alpha] = c(x)$. To show that $\mathrm{im}(p_*) \supseteq \ker\partial$, let $[\alpha] \in \pi_n(Y,y)$ such that $\partial([\alpha]) = [d_0 v] = [c(x)]$, that is $[\alpha] \in \ker\partial$. This means we have a homotopy from $[d_0 v]$ to $[c(x)]$

$$\begin{tikzcd}
\Delta^{\{0\}} \times \Delta^{n-1} \arrow[d, hook] \arrow[dr, bend left, "d_0 v"] & \\
\Delta^1 \times \Delta^{n-1} \arrow[r, "\exists h_0"] & X \\
\Delta^{\{1\}} \times \Delta^n \arrow[u, hook] \arrow[ur, bend right, "c(x)"'] &
\end{tikzcd}$$

We thus obtain a commutative square

$$\begin{tikzcd}
 & (\Delta^{\{1\}} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) \arrow[r, "(v(h_0,x,\ldots,x))"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^{\{1\}} \times \Delta^n \arrow[r] & \Delta^1 \times \Delta^n \arrow[r, "h:=p\tilde{h}"] \arrow[ur, "\exists\tilde{h}"] & Y
\end{tikzcd}$$

Then h is a homotopy $\alpha \to p(\tilde{h}d^1)$ rel $\partial\Delta^n$ The rest of the proof is given as exercise. $\square$

13.1 Interlude

As we know, $\mathrm{Kan} \subseteq \mathrm{Set}_\Delta$ and we can pass to hKan the homotopy category of Kan complexes. Since we wish to interpret Kan complexes as models for ∞-groupoids, we have been studying "the homotopy category of ∞-groupoids."

13.2 Challenge

We wish to regard homotopy equivalent Kan complexes as being isomorphic, while having access to universal properties (limit/colimit constructions).

Aim: Understand how to use Set_Δ to study the $(\infty,1)$-category of Kan complexes in which instead of Hom sets we have "Hom Kan complexes" (="Hom ∞-groupoids").

Definition 13.10. Let $X \xrightarrow{f} Y$ be a map of Kan complexes. Then f is a weak homotopy equivalence if $\forall x \in X$ and $\forall n \geqslant 0$, $\pi_n(f) \colon \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y, f(x))$.

Remark 13.11. By Whitehead's theorem for a category $\mathcal{C}$ and $W \subseteq \mathrm{Mor}(\mathcal{C})$ a class of maps, we obtain:

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r] \arrow[dr] & \mathrm{Gpd}_\infty = \mathrm{Set}_\Delta[Weq^{-1}]_{(\infty,1)cat} \arrow[d] \\ & \mathrm{hKan} \end{tikzcd}$$

We are not gonna detail the motivation here, since it is going to be extended in its formality in the nexte lectures.

Corollary 13.12. *Let $X \xrightarrow{p} Y$ be a Kan fibration and $f \colon y_0 \to y_1$ be an edge in Y. Then there exist $X_{y_0} \leftarrow \bullet \to X_{y_1}$ trivial Kan fibrations.*

Proof. For

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "r"'] & Y \\ X \arrow[ur, "p"'] & \end{tikzcd}$$

define $\underline{\mathrm{Hom}}_Y((W,q)(X,p))$ by and take the pullback

$$\begin{tikzcd} \underline{\mathrm{Hom}}_Y((W,q),(X,p)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(W,X) \arrow[d, "p_*"] \\ \Delta^0 \arrow[r, "q"] & \underline{\mathrm{Hom}}(W,Y) \end{tikzcd}$$

Where both vertical maps are Kan fibrations. Consider for example a morphism from the trivial simplicial set $\Delta^0 \xrightarrow{y_e} Y$ then

$$\begin{tikzcd} X_{y_e} = \underline{\mathrm{Hom}}_Y((\Delta^0, y_e)(X, \circ)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^0, X) \arrow[r, "\sim"] \arrow[d, "p_*"] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y_e"] & \underline{\mathrm{Hom}}(\Delta^0, Y) \arrow[r, "\sim"] & Y \end{tikzcd}$$

Consider now the pullback-square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_Y((\Delta^1, f),(X,p)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d] \\ \underline{\mathrm{Hom}}_Y((\Delta^{\{l\}}, y_e),(X,p)) \arrow[r] & \underline{\mathrm{Hom}}(\Delta^1, Y) \times_{\underline{\mathrm{Hom}}} (\Delta^{\{l\}}, Y)\underline{\mathrm{Hom}}(\Delta^{\{l\}}, X) \end{tikzcd}$$

$\square$

Lecture 7.1

Proposition 13.13. *Let X be a Kan complex. Then $\pi_1(\Omega(X,x),1_X)$ is abelian.*

Proof. We have $\pi_1(\Omega(X,x),1_X)$ has 2 binary operations. By the Eckmann Hilton argument the result follows. The details of that are to be worked out in the exercises.

Let now $\alpha\colon \Delta^1 \to \Omega(X,x)$ be a 1-simplex of the loop space. Take the pullback diagram:

$$\begin{tikzcd}
\Delta^1 \arrow[dr, "\alpha"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega(X,x) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^1,X) \arrow[d, "{(s,t)}"] \\
& \Delta^0 \arrow[r, "{(X,x)}"] & X\times X \cong \underline{\mathrm{Hom}}(\partial\Delta^1,X)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "1_x"] \arrow[d, "1_x"'] & x \arrow[d, "1_x"] \\
x \arrow[r, "1_x"'] & x
\end{tikzcd}$$

We see that binary operations correspond to "vertical stacking" and "horizontal stacking".

$$\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dd, bend right] & x \arrow[r] \arrow[d] & x \arrow[d] \arrow[dd, bend left] \\
x \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[ur, phantom, "\beta"] & x \arrow[r] \arrow[d] \arrow[ur, phantom, "\partial"] & x \arrow[d] \\
x \arrow[r] \arrow[rr, bend right] \arrow[ur, phantom, "\alpha"] & x \arrow[r] \arrow[ur, phantom, "\gamma"] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
\arrow[r, "\begin{pmatrix}\beta & \partial \\ \alpha & \gamma\end{pmatrix}"] & X & \\
\arrow[r, hook, "\in\mathrm{An}"'] & (\Lambda^2_1\times\Delta^2)\cup(\Delta^2\times\Lambda^2_1) \arrow[r, hook, "\in\mathrm{An}"'] & \Delta^2\times\Delta^2 \arrow[ul, "w"']
\end{tikzcd}$$

$$\begin{aligned}
&([\alpha]\circ[\beta])\bullet([\gamma]\circ[\partial]) \\
&= \left(\left[\begin{tikzcd} 00 \arrow[r] \arrow[d] & 01 \arrow[d] \\ 10 \arrow[r] & 21 \end{tikzcd}\right]\right)\circ\left(\left[\begin{tikzcd} 01 \arrow[r] \arrow[d] & 02 \arrow[d] \\ 21 \arrow[r] & 22 \end{tikzcd}\right]\right) = \left(\left[\begin{tikzcd} 00 \arrow[r] \arrow[d] & 02 \arrow[d] \\ 20 \arrow[r] & 22 \end{tikzcd}\right]\right) \\
&= \left(\left[\begin{tikzcd} 10 \arrow[r] \arrow[d] & 12 \arrow[d] \\ 20 \arrow[r] & 22 \end{tikzcd}\right]\right)\circ\left(\left[\begin{tikzcd} 00 \arrow[r] \arrow[d] & 02 \arrow[d] \\ 10 \arrow[r] & 12 \end{tikzcd}\right]\right)
\end{aligned}$$

□

13.3 Exercises

Exercise 1.

Let $p : X \to Y$ be a Kan fibration with Y a Kan complex. Recall that for $x \in X_0$ we may construct the fibre F of p at $p(x)$ via the following pullback diagram.

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "p(x)"] & Y \end{tikzcd}$$

1. Show that both F and X are Kan complexes and that x may be naturally be viewed as an object F.

In the lecture we have constructed a morphism $\partial : \pi_{n+1}(Y, p(x)) \to \pi_n(F, x)$ which assigns to $\alpha : \Delta^{n+1} \to Y$ the 0th face of θ where θ is a solution to the 0-horn lifting problem induced by α and the constant map $x : \Lambda_0^{n+1} \to \Delta^0 \to X$ for each $n \in \mathbb{N}_0$. Furthermore, we have established in the lecture that ∂ is a group homomorphism for $n \in \mathbb{N}_+$. Our goal in this exercise is to show that the ensuing long sequence

$$\ldots \xrightarrow{\partial} \pi_1(F, x) \xrightarrow{\pi_1(i)} \pi_1(X, x) \xrightarrow{\pi_1(p)} \pi_1(Y, p(x)) \xrightarrow{\partial} \pi_0(F, x) \xrightarrow{\pi_0(i)} \pi_0(X, x) \xrightarrow{\pi_0(p)} \pi_0(Y, p(x))$$

is in fact a long exact sequence of groups respective pointed sets. (Recall that a group is canonically a pointed set by taking the neutral element as base point and that the kernel of a morphism of pointed sets is defined as the preimage of the distinguished element.) To this end, we have shown already in the lecture that

$$\pi_n(F, x) \xrightarrow{\pi_n(i)} \pi_n(X, x) \xrightarrow{\pi_n(p)} \pi_n(Y, p(x))$$

are exact segments for all $n \in \mathbb{N}_0$ and that the segments

$$\pi_{n+1}(X, x) \xrightarrow{\pi_{n+1}(p)} \pi_{n+1}(Y, p(x)) \xrightarrow{\partial} \pi_n(F, x)$$

are exact for $n \in \mathbb{N}_+$

(b) Show that the segment

$$\pi_{n+1}(Y, p(x)) \xrightarrow{\partial} \pi_n(F, x) \xrightarrow{\pi_n(i)} \pi_n(X, x)$$

is exact for every $n \in \mathbb{N}_+$, i.e. $\mathrm{im}(\partial) = \ker(\pi_n(i))$.

For $y \in F_0$ we obtain a lifting problem

$$\begin{tikzcd} \Lambda_0^1 \arrow[r, "v"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r, "\alpha"] \arrow[ur, dashed, "\exists\theta"] & Y \end{tikzcd}$$

for any α representing a class of $\pi_1(Y, p(x))$. Let θ be a solution to this lifting problem and define $[\alpha] \cdot [v] := [d_0(\theta)]$.

(c) Argue that the above is a well defined action of $\pi_1(Y, p(x))$ on $\pi_0(F, x)$ and describe ∂ in terms of the action for $n = 0$.

(d) Show that $[\alpha] \cdot [x] = [x]$ for $[\alpha] \in \pi_1(Y, p(x))$ if and only if $[\alpha] \in \mathrm{im}(\pi_1(p))$, i.e. that the image of $\pi_1(p)$ is the stabilizer of the distinguished point $[x]$. In other words, the segment

$$\pi_1(X, x) \xrightarrow{\pi_1(p)} \pi_1(Y, p(x)) \xrightarrow{\partial} \pi_0(F, x)$$

is exact.

(e) Show that $\pi_0(i)([v]) = \pi_0(i)([w])$ if and only if there exists some $[a] \in \pi_1(Y, p(x))$ such that $[\alpha] \cdot [v] = [w]$. Deduce from this the exactness for $n = 0$ in (b).

(f) Finally, conclude from Exercise 11.2 that, if $p : X \to Y$ is a trivial Kan fibration, then p is a weak homotopy equivalence, i.e. $\pi_n(p) = \pi_n(p, x)$ is an isomorphism for all $n \in \mathbb{N}_0$ and $x \in X_0$.

We will show later that a Kan fibration which is a weak homotopy equivalence is itself a trivial Kan fibration.

Exercise 2. Recall from Exercise 10.1 the class of trivial Kan fibrations, $r($ Monomorphsisms $)$.

(a) Show that any trivial Kan fibration $p\colon X \to Y$ admits a section $s\colon Y \to X$ such that $p \circ s = \mathrm{id}_Y$.

A morphism $s\colon Y \to X$ of simplicial sets is a deformation retract if there exists a retraction $r\colon X \to Y$ with $r \circ s = \mathrm{id}_Y$ and a homotopy $h\colon \Delta^1 \times X \to X$ such that $\partial_0(h) = \mathrm{id}_X$ and $\partial_1(h) = s \circ r$. Here we write $\partial_\epsilon = (\{\epsilon\} \times \mathrm{id}_Y)^*$ for $\epsilon \in \Delta^1_0$. We say that s is a strong deformation retract if we have additionally that $h \circ (\mathrm{id}_{\Delta^1} \times s) = (s_0)_* \times s$

(b) Show that any section of a trivial Kan fibration is in fact a strong deformation retract.

(c) Deduce that a trivial Kan fibration is a homotopy equivalence, i.e. invertible up to homotopy.

14 Quillens small object argument

Let $\mathcal{C}$ be a category that has all small colimits, let furthermore $\mathcal{C}$ be a cocomplete category (e.g. Set_Δ) and J a set of morphisms in $\mathcal{C}$

<u>Aim</u>: For all f in $\mathcal{C}$ construct a factorisation under some assumption in J.

$$\begin{tikzcd} & \bullet \arrow[dr, "\in J^{\boxslash}"] & \\ \bullet \arrow[ur, "{}^{\boxslash}(J^{\boxslash})\ni"] \arrow[rr, "f"'] & & \bullet \end{tikzcd}$$

That is $J = \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}$ and then $J^{\boxslash}$ =Kan fibrations, or $J = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\}$ and then $J^{\boxslash}$ =Trivial Kan fibrations. But we also want this factorisation to be functorial.

Definition 14.1. A functorial factorisation in $\mathcal{C}$ is a section $\mathrm{Fun}([1], \mathcal{C}) \to \mathrm{Fun}([2], \mathcal{C})$ of the composition functor $\mathrm{Fun}([2], \mathcal{C}) \xrightarrow{? \circ d_1} \mathrm{Fun}([1], \mathcal{C})$ where $([1] \xrightarrow{d_1} [2] \to \mathcal{C})$.

Let us unravel the definition: For all morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$ we get a 2-simplex

$$\begin{tikzcd} & U(f) \arrow[dr, "Rf"] & \\ x \arrow[ur, "Lf"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

in $\mathcal{C}$. For all commutative squares

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "a"'] & y \arrow[d, "b"] \\ x' \arrow[r, "f'"'] & y' \end{tikzcd}$$

in $\mathcal{C}$, we get a diagram

$$\begin{tikzcd} & U(f) \arrow[dr, "R(f)"] \arrow[dd, dashed, "U(a,b)"] & \\ x \arrow[ur, "Lf"] \arrow[rr, "f"] \arrow[dd, "a"'] & & y \arrow[dd, "b"] \\ & U(f') \arrow[dr, "Rf'"] & \\ x' \arrow[ur, "Lf'"] \arrow[rr, "f'"'] & & y' \end{tikzcd}$$

Definition 14.2. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ is a pair of classes of morphisms such that the following properties hold:

1. (Factorisation) For all morphism $f : x \to y$ in $\mathcal{C}$ there exists a 2-simplex

$$\begin{tikzcd} & z \arrow[dr, "\in\mathcal{R}"] & \\ x \arrow[ur, "\mathcal{L}\ni"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

,

2. (Lifting) $\mathcal{L} \boxslash \mathcal{R}$,

3. (Closure) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{L}^{\boxslash} = \mathcal{R}$.

Lemma 14.3. *Suppose* $\begin{tikzcd} \bullet \arrow[r, "l"] \arrow[d, "f"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$ *and* $f \boxslash r$*. Then* f *is a retract of* l *as objects in the arrow category, that is* $\mathrm{Fun}([1], \mathcal{C})$*.*

Proof. Since f has the left lifting property with respect to r, we get a lift

$$\begin{tikzcd} \bullet \arrow[r, "l"] \arrow[d, "f"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, equal] \arrow[ur, dashed, "w"] & \bullet \end{tikzcd}$$

We can rewrite this diagram

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "f"] & \bullet \arrow[r, equal] \arrow[d, "l"] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, dashed, "w"'] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "r"] & \bullet \end{tikzcd}$$

to obtain the result. $\square$

Lemma 14.4. *Retract argument Suppose* $(\mathcal{L}, \mathcal{R})$ *satisfy the properties Factorisation and Lifting from* **??** *14.2. Then the property Closure holds if and only if* $\mathcal{L}, \mathcal{R}$ *are closed under retracts.*

Proof. ($\implies$) Exercise ($\impliedby$) The inclusion $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ holds by assumption. Let $k \in {}^{\boxslash}\mathcal{R}$, by factorisation there exists a square

$$\begin{tikzcd} \bullet \arrow[r, "l\in\mathcal{L}"] \arrow[d, "k"] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

By the retract argument **??** 14.4 k is a retract of $l \in \mathcal{L}$, hence $k \in \mathcal{L}$ since $\mathcal{L}$ is closed under retracts ${}^{\boxslash}\mathcal{R}$. Dually $\mathcal{L}^{\boxslash} \subseteq \mathcal{R}$. $\square$

Theorem 14.5. *Quillen's small object argument* **??** *Let $\mathcal{C}$ be a cocomplete category, J a set of morphisms in $\mathcal{C}$ suppose that for all $j \in J$ we have that* $\mathrm{Hom}_{\mathcal{C}}(\mathrm{dom}\, j, -) \colon \mathcal{C} \to \mathrm{Set}$ *preserve (countable) sequential colimits (that is colimits of shape* $(\mathbb{N}, \leqslant)$*). Then there exists a functorial factorisation in $\mathcal{C}$ turning $({}^{\boxslash}(J^{\boxslash}), J^{\boxslash})$ into a weak factorisation system. Moreover ${}^{\boxslash}(J^{\boxslash})$ is the saturation of J.*

Proof. Let f be a morphism in $\mathcal{C}$. For $j \in J$, let $S_q(j, f) = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d, "j"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right\}$

Take the following coproduct in the category of arrows

$$\coprod_{j \in J} \coprod_{S_q(j,f)} \begin{tikzcd} \bullet \arrow[r, "d_f"] \arrow[d, "\coprod_{j\in J}\coprod_{S_q(j,f)} j"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "c_f"'] & \bullet \end{tikzcd}$$

Consider now the pushout:

$$\begin{tikzcd} \bullet \arrow[r, "d_f"] \arrow[d, "\coprod_{j\in J}\coprod_{S_q(j,f)} j"'] \arrow[dr, phantom, "\mathrm{PO}"] & \bullet \arrow[r, equal] \arrow[d, "Lf"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "b_f"'] \arrow[rr, bend right, "c_f"'] & x_1 \arrow[r, "Rf"'] & \bullet \end{tikzcd}$$

By construction $Lf \in {}^{\boxslash}(J^{\boxslash})$, since all of this is in the saturation of J. but we have no guarantee that $Rf \in J^{\boxslash}$. Apply now the above construction to Rf and thus we obtain a square:

$$\begin{tikzcd} x_1 \arrow[r, equal] \arrow[d, "LRf"'] & x_1 \arrow[d, "Rf"] \\ x_2 \arrow[r, "R^2 f"'] & x_2 \end{tikzcd}$$

Repeating the construction iteratively we obtain a diagram:

$$\begin{tikzcd} x_0 \arrow[r, "Lf"] \arrow[rrrrrr, bend left, "L^w f \in {}^{\boxslash}(J^{\boxslash})"] \arrow[drr, bend right, "f"'] & x_1 \arrow[r, "LRf"] \arrow[dr, bend right, "Rf"] & x_2 \arrow[r, "LR^2 f"] \arrow[d, "R^2 f"] & x_3 \arrow[r] \arrow[dl, bend left, "R^3 f"] & \dots \arrow[r] & x_w = \mathrm{colim}_n\, x_n \arrow[dllll, bend left, "\exists ! R^w f"] \\ & & \bullet & & & \end{tikzcd}$$

Where $L^w f$ is the transfinite composition of the $LR^i f$ for $i \in \mathbb{N}$ and $R^{i+1} f \circ LR^i f = R^i f$, with $R^0 f = f$.

We claim that $R^w f \in J^{\boxslash}$. Consider the following lifting problem

$$\begin{tikzcd}
j_0 \arrow[r, "u"] \arrow[d, "j"] & x_w \arrow[d, "R^w f"] \\
\bullet \arrow[r, "v"] & \bullet
\end{tikzcd}$$

Per definition we have that $\mathrm{Hom}_{\mathcal{C}}(j_0, x_w \cong \mathrm{colim}_n \mathrm{Hom}(j_0, x_n))$ and with the square above we obtain the following commutative diagram

$$\begin{tikzcd}
j_0 \arrow[r, "u"] \arrow[d, "j"] \arrow[rr, bend left, "u_w"] & x_n \arrow[r] \arrow[d, "R^n f"] & x_w \arrow[d, "R^w f"] \\
\bullet \arrow[r, "v"] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

We now toke the leftmost square in the diagram above, then include it into the coproduct of squares $\amalg_{j\in J}\amalg_{S_q(j', R^n f)}$, which leads to the following.

$$\begin{tikzcd}
j_0 \arrow[r] \arrow[d, "j"'] \arrow[rr, bend left, "u"] & \bullet \arrow[r] \arrow[d, "\amalg_{j\in J}\amalg j'"'] \arrow[dr, phantom, "\mathrm{PO}"] & x_n \arrow[r, equal] \arrow[d, "LR^n f"] & \bullet \arrow[d, "R^n f"] \\
\bullet \arrow[r] \arrow[rr, bend right, red] \arrow[rrr, bend right=60, "v"'] & \bullet \arrow[r] & x_{n+1} \arrow[r, "R^{n+1} f"'] & \bullet
\end{tikzcd}$$

We can now construct the diagram below, where the red arrow corresponds to the red arrow above, which is our desired lift.

$$\begin{tikzcd}
j_0 \arrow[r, "u"] \arrow[d] & x_n \arrow[r, "LR^n f"] & x_{n+1} \arrow[r] \arrow[d, "R^{n+1} f"] & x_w \arrow[d, "R^w f"] \\
\bullet \arrow[r, "v"] \arrow[urr, red] & \bullet \arrow[r, equal] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

Remains to show that ${}^{\boxslash}(J^{\boxslash})$ is the saturation of J. Let for that $c \in {}^{\boxslash}(J^{\boxslash})$ and

$$\begin{tikzcd}
\bullet \arrow[r, "L^w c"] \arrow[d, "c"] & \bullet \arrow[d, "R^w c \in J^{\boxslash}"] \\
\bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

be a lifting diagram, that is $c \boxslash R^w c$. Now by the retract argument c is a retract of $L^w c$. $\square$

15 Weak equivalences of simplicial sets

Recall that for $X \in \mathrm{Set}_\Delta$ and $K \in \mathrm{Set}_\Delta$ a Kan complex then $\underline{\mathrm{Hom}}(X, K)$ is a Kan complex. Furthermore $f \colon X \to Y$ in Set_Δ is a homtopy equivalence if there exists $g \colon Y \to X$ and homotopies $g \circ f \to \mathrm{id}_X$ and $f \circ g \to \mathrm{id}_Y$.

Example 15.1. Let $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ be an adjunction $L \dashv R$. Then $N(L) \colon N(\mathcal{C} \to \mathcal{D})$ is a homotopy equivalence. Let $\eta \colon \mathbb{1}_{\mathcal{C}} \to RL$ be the unit and $\epsilon \colon LR \to \mathbb{1}_{\mathcal{D}}$ the counit of the adjunction, thus η is a morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and ϵ on $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. Thus we can also consider $\eta \colon [1] \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\epsilon \colon [1] \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$. This can again be rephrased as $\eta \in \mathrm{Fun}([1], \mathrm{Fun}(\mathcal{C}, \mathcal{C})) \cong \mathrm{Fun}([1] \times \mathcal{C}, \mathcal{C}) \ni \overline{\eta}$ and $\epsilon \in \mathrm{Fun}([1], \mathrm{Fun}(\mathcal{D}, \mathcal{D})) \cong \mathrm{Fun}([1] \times \mathcal{D}, \mathcal{D}) \ni \overline{\epsilon}$. Now $N(\overline{\eta}) \colon N([1] \times \mathcal{C}) \to N(\mathcal{C})$ and we have isomorphisms $N([1] \times \mathcal{C}) \cong N([1]) \times N(\mathcal{C}) \cong \Delta^1 \times N(\mathcal{C})$. Take the nerve $N(\overline{\eta}) \colon N(\mathbb{1}_{\mathcal{C}}) \to N(RL) = N(R) \circ N(L)$

$$\begin{tikzcd}
\mathcal{C} \cong [0] \times \mathcal{C} \arrow[d, "L_0 \times \mathbb{1}_{\mathcal{C}}"'] \arrow[dr, bend left, "\mathbb{1}_{\mathcal{C}}"] & \\
\times \mathcal{C} \arrow[r, "\overline{\eta}"] & \mathcal{C} \\
\overline{C} \cong [0] \times \mathcal{C} \arrow[u, "L_1 \times \mathbb{1}_{\mathcal{C}}"] \arrow[ur, bend right, "RL"'] &
\end{tikzcd}$$

Proposition 15.2. *Let $f \colon X \to Y$ be a morphism between Kan complexes, then the following are equivalent:*

1. *f is a homtopy equivalence,*
2. *for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $\pi_0(f^*) \colon \pi_0(\underline{\mathrm{Hom}}(Y, K)) \to \pi_0(\underline{\mathrm{Hom}}(X, K))$ is bijective.*

Definition 15.3. A morphism of simplicial sets $f \colon X \to Y$ is a <u>weak equivalence</u> if for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $\pi_0(f^*) \colon \pi_0(\underline{\mathrm{Hom}}(Y, K)) \to \pi_0(\underline{\mathrm{Hom}}(X, K))$ is bijective.

Lemma 15.4. *Let $f \colon X \to Y$ be a homotopy equivalence. Then for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $f^* \colon \underline{\mathrm{Hom}}(Y, K) \to \underline{\mathrm{Hom}}(X, K)$ is a homotopy equivalence.*

Proof. Let $h \colon \mathrm{id}_X \to g \circ f$ be a homotopy ($\Delta^1 \times X \xrightarrow{h} X$). Let $h^* \colon \underline{\mathrm{Hom}}(X, K) \to \underline{\mathrm{Hom}}(\Delta^1 \times X, K) \cong \underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(X, K))$ be the morphism induced by h on the inner Hom simplicial sets, so $h^* \in \underline{\mathrm{Hom}}(\Delta^1 \times \underline{\mathrm{Hom}}(X, K), \underline{\mathrm{Hom}}(X, K))$. Thus h^* gives a homotopy between $(\mathrm{id}_X)^* = \mathrm{id}_{\underline{\mathrm{Hom}}(X,K)}$ and $(g \circ f)^* = f^* \circ g^*$. □

Corollary 15.5. *Let $f \colon X \to Y$ be a homotopy equivalence of simplicial sets, then f is a weak equivalence.*

Proof. It is an application of **??** 15.4 and applying the definition of weak equivalences. □

Proposition 15.6. *Let $i\colon X \hookrightarrow K$ be an anodyne extension, then i is a weak equivalence.*

Proof. Let K be a Kan complex, then $i^*\colon \underline{\mathrm{Hom}}(Y,K) \to \mathrm{Hom}(X,K)$ is a trivial Kan fibration, then it is a homotopy equivalence, thus π_0 is an iso. $\square$

Proposition 15.7. *(2 out of 3) Weak equivalences satisfy the 2 out of 3 property. That is for every commutative diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

if 2 out of the morphisms f, g and $g \circ f$ are weak equivalences, then so is the third.

Proof. Let $K \in \mathrm{Set}_\Delta$ be a Kan complex. Then we get a diagram induced by the morphisms f, g and gf.

$$\begin{tikzcd} & \pi_0(\underline{\mathrm{Hom}}(Y,K)) \arrow[dl, "\pi_0(f^*)"'] & \\ \pi_0(\underline{\mathrm{Hom}}(X,K)) & & \pi_0(\underline{\mathrm{Hom}}(Z,K)) \arrow[ul, "\pi_0(g^*)"'] \arrow[ll, "\pi_0((g\circ f)^*)"] \end{tikzcd}$$

now if two of the morphisms are bijections, then so is the third, since bijections fullfill the 2 out of 3 property. $\square$

Proposition 15.8. *Let $f^{(i)}\colon X^{(i)} \to Y^{(i)}$ be a family of weak equivalences, indexed over the set I. Then $\coprod_{i\in I} f^{(i)}$ is a weak equivalence.*

Proof. Let K be a Kan complex

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\coprod Y^{(i)}, K) \arrow[r, "(\coprod f^{(i)})^*"] \arrow[d, "\wr"] & \underline{\mathrm{Hom}}(\coprod Y^{(i)}, K) \arrow[d, "\wr"] \\ \prod \underline{\mathrm{Hom}}(Y^{(i)}, K) \arrow[r, "(f^{(i)*})_i"'] & \prod \underline{\mathrm{Hom}}(X^{(i)}, K) \end{tikzcd}$$

Then $\underline{\mathrm{Hom}}(Y^{(i)}, K)$ is a Kan complex, since π_0 preserves all small coproducts of Kan complexes, we are done. $\square$

Proposition 15.9. *Let $f^{(i)}\colon X^{(i)} \to Y^{(i)}$ be a family of weak equivalences of Kan complexes, indexed over a set I. Then $\prod f^{(i)}$ is a weak equivalence. By* **??** *all weak equivalences here are homotopy equivalences.*

Proof. Let K be a Kan complex, then $\prod X^{(i)}$ and $\prod Y^{(i)}$ are Kan complexes. Then we have the following diagram

$$\begin{tikzcd}
\pi_0(\underline{\mathrm{Hom}}(K, \prod X^{(i)})) \arrow[r] \arrow[d, "\wr"] & \pi_0(\underline{\mathrm{Hom}}(K, \prod Y^{(i)})) \arrow[d, "\wr"] \\
\pi_0(\prod \underline{\mathrm{Hom}}(K, X^{(i)})) \arrow[r] \arrow[d] & \pi_0(\prod \underline{\mathrm{Hom}}(K, Y^{(i)})) \arrow[d] \\
\prod \pi_0(\underline{\mathrm{Hom}}(K, X^{(i)})) \arrow[r, "sim"] & \prod \pi_0(\underline{\mathrm{Hom}}(K, Y^{(i)}))
\end{tikzcd}$$

□

Remark 15.10. For a finite product we do not need homtopy equivalence neither Kan complexes for the statement to hold.

Proposition 15.11. *Let $f\colon X \to Y$ be a weak equivalence. Then for all Kan complexes K the morphism $f^*\colon \underline{\mathrm{Hom}}(Y, K) \to \underline{\mathrm{Hom}}(X, K)$ is a homotopy equivalence.*

Proof. Let $W \in \mathrm{Set}_\Delta$ be a Kan complex. Then

$$\begin{tikzcd}
\pi_0(\underline{\mathrm{Hom}}(W, \underline{\mathrm{Hom}}(Y, K))) \arrow[r, "f^* \circ ?"] \arrow[d, "\wr"] & \pi_0(\underline{\mathrm{Hom}}(W, \underline{\mathrm{Hom}}(X, K))) \arrow[d, "\wr"] \\
\pi_0(\underline{\mathrm{Hom}}(Y, \underline{\mathrm{Hom}}(W, K))) \arrow[r, "\pi_0(f^*)"] & \pi_0(\underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(W, K)))
\end{tikzcd}$$

Now all the inner Hom simplicial sets are Kan complexes and we are done by applying **??** 15.2. □

Corollary 15.12. *Suppose that $f\colon X \to Y$ admits a factorisation*

$$\begin{tikzcd}
 & Z \arrow[dr, "\in \textit{Triv. Kan fibrations}"] & \\
X \arrow[ur, "\mathrm{An}\ni"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Then f is a weak equivalence.

Proof. Anodyne extensions and trivial Kan fibrations are weak equivalences, f is the composition of two weak equivalences hence a weak equivalence by **??** 15.7. □

Let $X \in \mathrm{Set}_\Delta$ such that

$$\begin{tikzcd}
 & \tilde{X} \arrow[dr, "\in\mathrm{KanFib}"] & \\
X \arrow[ur, "\mathrm{An}\ni"] \arrow[rr, "f"'] & & \Delta^0
\end{tikzcd}$$

then X is weakly equivalent to the Kan complex $\tilde{X}$.

15.1 Exercises

Exercise 1. Consider a Kan complex X such that $f\colon X \to \Delta^0$ is a weak homotopy equivalence. Our aim is to show that f is a trivial Kan fibration and thus in particular a homotopy equivalence.

(a) Construct a homotopy from id_{Δ^n} to a constant map.

(b) Show that any morphism $\Lambda^n_k \to X$ from a horn to a Kan complex is homotopic to a constant map.

We now fix a lifting problem relative to a boundary inclusion

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & \Delta^0 \end{tikzcd}$$

3. Use the homotopy lifting property to construct a morphism $g'\colon \partial\Delta^n \to X$ which is homotopic to g and constant when restricted to Λ^n_n.

4. Deduce from Exercise **??** 2 that g' has a filling and construct a filling for g.

5. Conclude that f is a trivial fibration.

Exercise 2. We say a category K is κ-filtered for an infinite cardinality κ if for any set of objects $\{\alpha_i\}_{i\in I}$ with $|I| < \kappa$ there is a cocone, i.e. there exists some $\beta \in K$ and morphisms $\alpha_i \to \beta$ for every $i \in I$. Assume that K is κ-complete and let J be a small category such that $|\mathrm{Mor}(J)| < \kappa$. Show that for every functor $D\colon K \times J \to \mathrm{Set}$ the canonical morphism

$$\mathrm{colim}_K \lim_{j\in J} D(-, j) \to \lim_J \mathrm{colim}_{\alpha\in K} D(\alpha, ?)$$

indueced by the morphisms $\{D(\gamma, j) \to \mathrm{colim}_{\alpha\in K} D(\alpha, j)\}_{j\in J}$ is a bijection. Fot thi, use the explicit form of limits and colimits in Set. Furthermore, describe the special case where $\kappa = |\mathbb{N}|$.

Exercise 3. Fix a small category A and consider a presheaf $X \in \widehat{A}$. Our gial is to show that X is κ-compact for any sufficiently large regular cardinal κ, i.e. the canonical map

$$\mathrm{colim}_{\alpha\in\kappa} \mathrm{Hom}_{\widehat{A}}(X, D(\alpha)) \to \mathrm{Hom}_{\widehat{A}}(X, \mathrm{colim}_{\alpha\in\kappa} D(\alpha))$$

is an isomorphism for any κ-indexed colimit $D : \kappa \to \widehat{A}$. Here, we view the cardinal κ as a well ordered set representing it. In particular, we may view κ as the category induced from the underlying partial order. Furthermore, a cardinal κ is called regular if the category associated to κ is κ-complete in the sense of **??** 2.

(a) Show that if X is representable, then X is κ-compact for arbitrary κ.

(b) Deduce that to show that X is κ-compact for some cardinal κ, it is sufficient to show that κ-indexed colimits commute with $\int^A X$-indexed limits in Set.

(c) Show that the category of elements $\int^A X$ is small, i.e. $\mathrm{Mor}(\int^A X)$ is a set and thus has a cardinality.

(d) Combine the above with **??** 2 to give a sufficient condition on κ for X to be κ-compact.

Exercise 4. Let A be a small category and $\mathcal{F}$ is a set of morphisms on $\widehat{A}$. Consider the set $\mathrm{dom}(\mathcal{F}) := \{Y \mid \exists f \in \mathcal{F} f : Y \to Z\}$ of domains of morphisms in $\mathcal{F}$.

(a) Deduce from the Exercise above that there exists a cardinality κ such that every $X \in \mathrm{dom}(\mathcal{F})$ is κ-compact. Here, you may freely use the fact from set theory that for any cardinal there exists a larger cardinality which is regular.

(b) Conclude $(l(r(\mathcal{F})), r(\mathcal{F}))$ is a weak factorisation system.

(c) Deduce that $\overline{\mathcal{F}} = l(r(\mathcal{F}))$ where $\overline{\mathcal{F}}$ is the saturated closure of $\mathcal{F}$.

16 Whitehead's theorem for Kan complexes

Lemma 16.1. *Suppose that $f\colon (X,x)\to(Y,y)$ is a morphism for pointed Kan complexes, such that $f\colon X\to Y$ admits a left inverse up to homotopy (inverse in* hKan*). Then f admits a pointed inverse up to homotopy (inverse in* hKan_**).*

Proof. Let $g\colon Y\to X$ be a homotopy left inverse to f, so there exists a homotopy $h\colon \mathrm{id}_X\to g\circ f$. Let now $\alpha\colon \Delta^1\times X\to X$ be a homtopy and extend the homotopy diagram by the morphism associated to the selected point:

$$\begin{tikzcd}
\Delta^{\{0\}}\times\Delta^0 \arrow[r, "x"] \arrow[d] & \Delta^{\{0\}}\times X \arrow[d] \arrow[dr, bend left, "id_X"] & \\
e\colon \Delta^1\times\Delta^0 \arrow[r, "\mathrm{id}\times x"] & \Delta^1\times X \arrow[r, "\alpha"] & X \\
\Delta^{\{1\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times x"] \arrow[u] & \Delta^{\{1\}}\times X \arrow[u] \arrow[ur, bend right, "g\circ f"'] &
\end{tikzcd}$$

Where $e\colon x\to g(f(x))=g(y)$ in X. Take $\Delta^0\xhookrightarrow{y} Y$ and the lifting diagram

$$\begin{tikzcd}
\Lambda^1_1=\Delta^{\{1\}} \arrow[r, "g"] \arrow[d, hook] & \underline{\mathrm{Hom}}(Y,X) \arrow[d, "\mathrm{ev}_y"] \\
\Delta^1 \arrow[r, "e"] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(\Delta^0,X)\cong X
\end{tikzcd}$$

The lifting morphism together with the standard adjunction gives a homotopy $\beta\colon \Delta^1\times Y\to X$ from g' to g where $g'(y)=x$. We can concetenate β with f, to obtain $\beta_f\colon \Delta^1\times X\xrightarrow{\mathrm{id}\times f}\Delta^1\times Y\xrightarrow{\beta} X$ which is a homotopy of the concetenation $\beta_f\colon g'\circ f\to g\circ f$. We can now consider the homotopy diagram of β extended by the selected point in Y:

$$\begin{tikzcd}
\Delta^{\{0\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] \arrow[d] & \Delta^{\{0\}}\times Y \arrow[d] \arrow[dr, bend left, "g'"] & \\
e\colon \Delta^1\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] & \Delta^1\times Y \arrow[r, "\beta"] & X \\
\Delta^{\{1\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] \arrow[u] & \Delta^{\{1\}}\times Y \arrow[u] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

The next step is to take the degenerate 2-simplex $s_0(e)$ given by

$$\begin{tikzcd}
x \arrow[dr, "e"] & \\
x \arrow[u, "1_X"] \arrow[r, "1_X"'] & g(f(x))
\end{tikzcd}$$

and take it as the bottom row map in the following lifting problem

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "q"] \arrow[d] & \underline{\mathrm{Hom}}(X, X) \arrow[d, "\mathrm{ev}_x"] \\
\Delta^2 \arrow[r, "s_0(e)"'] & \underline{\mathrm{Hom}}(\Delta^0, X)
\end{tikzcd}$$

where q is given by the following horn diagram.

$$\begin{tikzcd}
g'f \arrow[dr, "\beta_f"] & \\
\mathrm{id}_X \arrow[u, dashed] \arrow[r, "\alpha"'] & gf
\end{tikzcd}$$

Since ev_x is a Kan fibration, we obtain a lift and thus a pointed homotopy $\gamma : \mathrm{id}_X \to g'f$, thus g' is a pointed homotopy inverse to f. $\square$

Corollary 16.2. *Let $f\colon (X, x) \to (Y, y)$ be a morphism of Kan complexes such that $f\colon X \to Y$ is a homotopy equivalence then f is also a pointed homotopy equivalence.*

Proof. By **??** 16.1 $[f]$ admits a pointed left inverse $g\colon (Y, y) \to (X, x)$ and $[g] \circ [f] = [\mathrm{id}_X]$ in hKan_*. Then $g\colon Y \to X$ is a homotopy equivalence. There exists a homotopy left inverse $h\colon (X, x) \to (Y, y)$, $[h] = [h]([g][f]) = [f]$ $\square$

Corollary 16.3. *Let $f\colon X \to Y$ be a homotopy equivalence. Then for all $x \in X$ and all $n \geqslant 0$, there is an isomorphism $\pi_n(f)\colon \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x))$.*

Proof. By **??** 16.2 $f\colon (X, x) \to (Y, f(x))$ is a pointed homotopy equivalence. Now $\pi_n(X, x) = \pi_n(\underline{\mathrm{Hom}}((S^n, *), (X, x))$, similarly for $(Y, f(x))$ $\square$

Definition/Proposition 16.4. Let X be a Kan complex, then the following are equivalent:

1. $X \to \Delta^0$ is a homotopy equivalence,
2. for all $x \in X$ and all $n \geqslant 0$, $\pi_n(X, x) = \{*\}$,
3. $X \to \Delta^0$ is a trivial Kan fibration.

In this case we say that X is contractible.

Proof. Exercise $\square$

Proposition 16.5. *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes, then the following are equivalent:*

1. *p is a trivial Kan fibration,*
2. *p is a homotopy equivalence,*

3. for all $x \in X$ and for all $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ is bijective,

4. for all $y \in Y$ the fibre X_y given by the pullback

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

is contractible.

Proof.

1. $\implies$ 2. Exercise

2. $\implies$ 3. homtopy inverse

3. $\implies$ 4. Serre long exact sequence

4. $\implies$ 1. Suppose that for all $y \in Y$ we have that $X_y \to \Delta^0$ is a trivial Kan fibration, which is by **??** 16.4 equivalent to contractibility of X_y. Take a boundary inclusion

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"] & Y \end{tikzcd}$$

and consider a homotopy H from the constant map to the identity on Δ^n, that is $H\colon \Delta^1 \times \Delta^n \to \Delta^n$, where $H\colon c(0) \to \mathrm{id}_{\Delta^n}$. Putting together the diagram for the homotopy H and the n-simplex β we obtain:

$$\begin{tikzcd} \Delta^{\{0\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[d] & \Delta^{\{0\}} \times \Delta^n \arrow[d] \arrow[dr, bend left, "c(0)"] \arrow[drr, bend left, "c(y)"] & & \\ e\colon \Delta^1 \times \partial\Delta^n \arrow[r, "\mathrm{id} \times i"] & \Delta^1 \times \Delta^n \arrow[r, "H"] & \Delta^n \arrow[r, "\beta"] & Y \\ \Delta^{\{1\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[u] & \Delta^{\{1\}} \times \Delta^n \arrow[u] \arrow[ur, bend right, "\mathrm{id}_{\Delta^n}"'] \arrow[urr, bend right, "\beta"'] & & \end{tikzcd}$$

$$\begin{tikzcd} \overline{\alpha} \arrow[r, "\tilde{h}"] \arrow[d, maps to, "p_*"] & \alpha \\ c(y) \arrow[r, "h"'] & p_\alpha \end{tikzcd} \qquad \begin{tikzcd} \Delta^{\{1\}} \arrow[r, "\alpha"] \arrow[d] & \underline{\mathrm{Hom}}(\partial\Delta^n, X) \arrow[d, "p_*"] \\ \Delta^1 \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & \underline{\mathrm{Hom}}(\partial\Delta^n, Y) \end{tikzcd}$$

where $\tilde{h}\colon \Delta^1 \times \partial\Delta^n \to X, h\colon \overline{\alpha} \to \alpha, \overline{\alpha}\colon \Delta^n \to X$ and $p \circ \overline{\alpha} = c(y)$.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, dashed, "\exists"] \arrow[d, "i"] \arrow[rr, bend left, "\overline{\alpha}"] & X_y \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\mathrm{PB}"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "!"] \arrow[ur, dashed] \arrow[urr] \arrow[rr, bend right, "c(y)"'] & \Delta^0 \arrow[r, "y"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^{\{0\}} \times \partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "\alpha"] & (\Delta^{\{0\}} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) \arrow[r, "{(v,h)}"] \arrow[d, hook, "\in \mathrm{An}"] & X \arrow[d, "p"] \\
\Delta^{\{0\}} \times \Delta^n \arrow[r] \arrow[rr, bend right, "\beta"'] & \Delta^1 \times \Delta^n \arrow[r, "\beta \circ H"] \arrow[ur] & Y
\end{tikzcd}$$

□

Theorem 16.6. *Whitehead's theorem Let $f\colon X \to Y$ be a morphsim between Kan complexes. Then the following are equivalent:*

1. *f is a homotopy equivalence,*
2. *for all $x \in X$ and all $n \geqslant 0$, $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, y)$ is a bijection.*

Proof.

1. $\implies$ 2. This is known.

2. $\implies$ 1. By the use of Quillen's small object argument **??** 14.5, we obtain a factorisation of f

$$\begin{tikzcd}
& \tilde{X} \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

where $i \in \mathrm{An}$ and $p \in \mathrm{KanFib}$. Since $i\colon X \to \tilde{X}$ is anodyne, i is a weak equivalence and by **??** 15.2, using that X and Y are Kan complexes, i is also a homotopy equivalence and thus satisfies 2. By **??** 15.7 p also satisfies the property 2. and by the previous proposition p is a homotopy equivalence.

□

17 Kan-Quillen Model structure

Lecture 21.1

<u>Aim</u> We prove that (Set_Δ, Weq, Mono's, Kanfib.) is a model structure.

Let us examine what we know so far.

1. Weak equivalences satisfy the 2 out of 3 property.
2. (Mono's, Triv Kan Fib.) and (An, Kanfib.) form a weak factorisation system. Thus for all $f\colon X \to Y$ we have factorisations

$$\begin{tikzcd} & Z \arrow[dr, "\in \text{Triv. KanFib.}"] & \\ X \arrow[ur, "\in \text{Mono's}"] \arrow[rr, "f"] & & Y \end{tikzcd} \qquad \begin{tikzcd} & Z' \arrow[dr, "\in \text{KanFib.}"] & \\ X \arrow[ur, "\in \text{An}"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

Remember that the Triv. KanFib. $= \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\}^{\boxslash} =$ Mono's $^{\boxslash}$, Mono's $=$ ${}^{\boxslash}$ Triv. KanFib. $= {}^{\boxslash}(\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\}^{\boxslash})$ and that KanFib $:= \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}^{\boxslash} = \mathrm{An}^{\boxslash}, \mathrm{An} = {}^{\boxslash}(\mathrm{KanFib})^{\boxslash} = {}^{\boxslash}(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}^{\boxslash})$.

What we need is An = Weq $\cap$ Mono's and TrivKanFib= Weq $\cap$ KanFib and we already know An $\subseteq$ Weq $\cap$ Mono's and Triv.KanFib. $\subseteq$ Weq $\cap$ KanFib.

Proposition 17.1. *Assume that TrivKanFib = Weq $\cap$ KanFib, then* An $\supseteq$ *Weq $\cap$ Mono's.*

Proof. Let $f\colon X \to Y$ be in Weq $\cap$ Mono's. Choose a factorisation

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

where $i \in$ An and $p \in$ KanFib since $i \in$ An $\subseteq$ Weq $\cap$ Mono's and $f \in$ Weq $\cap$ Mono's. By the 2 out of 3 property $p \in$ Weq $\cap$KanFib =TrivKanFib. Take the square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f\in \text{ Mono's}"'] & Z \arrow[d, "p\in \text{ TrivKanFib.}"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

since we have that $f \boxslash p$ we get by the retract argument **??** 14.4 that f is a retract $i \in$ An. □

We are now reduced to prove Triv. KanFib. $\supseteq$ Weq $\cap$ KanFib.

Proposition 17.2. *Let us call the following property, property A. Let now $p\colon X \to Y$ be a* KanFib *and take the square*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p \in \mathrm{KanFib}} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If p is a weak equivalence, then $q \in$ weak eq. $\cap$ KanFib.

Theorem 17.3. *Let $f\colon X \to Y$ be a Kan Fibration. The following are equivalent*

1. *f is a trivial Kan Fibration,*
2. *f is a homotopy equivalence,*
3. *f is a weak equivalence,*
4. *for all $y \in Y$*

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f \in \ Weq \ \cap \ KanFib} \\ \Delta^0 & \xrightarrow{y} & Y \end{array}$$

X_y is a contractible Kan complex, that is $X_y \to \Delta^0 \in$ TrivKanFib..

Proof. The only implication that is missing to be shown is 3. to 4. which relies on **??** 17.2. Assume that $f\colon X \to Y$ is in Weq. $\cap$ KanFib. Then the following is a pullback

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f \in \ \mathrm{Weq} \ \cap \ \mathrm{KanFib.}} \\ \Delta^0 & \longrightarrow & Y \end{array}$$

and with **??** 17.2 it follows that $X_y \to \Delta^0$ in Weq $\cap$ KanFib. is a trivial KanFib.. We are reduced to proving **??** 17.2. ☐

Theorem 17.4. *There is a functor $Ex^\infty\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ with the following properties*

1. *For all $X \in \mathrm{Set}_\Delta$, Ex^∞ is a Kan complex.*
2. *There exists a natural transformation $\mathbb{1} \xrightarrow{\beta} Ex^\infty$ such that for all $X \in \mathrm{Set}_\Delta$ $X \to Ex^\infty$ is a weak equivalence.*
3. *The functor $Ex^\infty\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ preserves finite limits, weak equivalences and Kan Fibrations.*

Proof. **??** 17.2 Take the square

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "q"] & X \arrow[d, "p \in \text{ Weq } \cap \text{ KanFib.}"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Apply Kan's Ex^{∞}-functor to obtain:

$$\begin{tikzcd}
X' \arrow[r, "\beta_{X'}"] \arrow[d, "g"] \arrow[dr, "\in \text{ Triv. KanFib.}"] & \mathrm{Ex}^{\infty}(X') \arrow[r, "\mathrm{Ex}^{\infty}(f)"] \arrow[d, "\mathrm{Ex}^{\infty}(g)"] & \mathrm{Ex}^{\infty}(X) \arrow[d, "\mathrm{Ex}^{\infty}(p) \in \mathrm{Weq} \cap \mathrm{KanFib}"] \\
Y' \arrow[r, "\beta_{Y'}"] & \mathrm{Ex}^{\infty}(Y') \arrow[r, "\mathrm{Ex}^{\infty}(g)"] & \mathrm{Ex}^{\infty}(Y)
\end{tikzcd}$$

Note that the arrows marked with a circle are weak equivalences. By the 2 out of 3 property and **??** 17.3, we get that g is a weak equivalence. So we just have to show the theorem above. $\square$

Lecture 23.1

18 Kan's Ex functor

Lecture 23.1

Consider the adjunction $\mathrm{Sd}\colon \mathrm{Set}_\Delta \rightleftarrows \mathrm{Set}_\Delta\colon \mathrm{Ex}$ induced by the functor

$$\Delta \to \mathrm{Set}_\Delta, [n] \mapsto N(S([n]))$$

where $S[n]$ is the poset of non-empty subsets of $[n]$ and to $\sigma\colon [m] \mapsto [n]$ we associate. $(U \subseteq [m] \mapsto \sigma(U) \subseteq [n])$. For $X \in \mathrm{Set}_\Delta$ we have $\mathrm{Ex}(X)_n :=$ $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), X)$ with $\mathrm{Sd}(\Delta^n) = N(S[n])$

Example 18.1.
- $(n = 0)\, \mathrm{Sd}(\Delta^0) = N(S[0]) = \{0\}$
- $(n = 1)\, \mathrm{Sd}(\Delta^1) = N(S[1])\colon \{0\} \to \{0,1\} \leftarrow \{1\}$
- $(n = 2)\, \mathrm{Sd}(\Delta^1) = N(S[2])$

$$\begin{tikzcd}
& & \{1\} \arrow[dr] \arrow[dd] & & \\
& \{0,1\} \arrow[ur] \arrow[dr] & & \{1,2\} \arrow[dl] & \\
& & \{0,1,2\} & & \\
\{0\} \arrow[uur] \arrow[urr] \arrow[rr] & & \{0,1\} \arrow[u] & & \{2\} \arrow[uul] \arrow[ull] \arrow[ll]
\end{tikzcd}$$

Example 18.2. Let $T \in \mathrm{Top}$. Then we can consider $\mathrm{Ex}(\mathrm{Sing}(X))$ with $\mathrm{Ex}(\mathrm{Sing}(X))_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), \mathrm{Sing}(X))_n \cong \mathrm{Hom}_{\mathrm{Top}}(|\mathrm{Sd}(\Delta^n)|, T)$, where $|\mathrm{Sd}(\Delta^n)|$ is the barycentric subdivision of $|\Delta^n|$.

We wish to define a natural map
$\beta_X\colon X \to \mathrm{Ex}(X)$. For this we define $a_n\colon S[n] \to [n]$ and $U \subseteq [n] \mapsto \max(U)$ and the induced map $\alpha_n\colon\ \mathrm{Sd}(\Delta^n)N(S[n]) \to N([n]) = \Delta^n$ as well as the induced map $(\beta_X)_n\colon X_n \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \xrightarrow{\alpha_n^*} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), X) = \mathrm{Ex}(X)_n$

Lemma 18.3. *Take $\beta_X\colon X \to \mathrm{Ex}(X)$ is a morphism of simplicial sets.*

Proof. The proof is just an exercise in backtracking the definitions of the constructions. Let $\sigma\colon [m] \to [n]$ be a morphism in Δ, we get the following commutative square

$$X_m \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \xrightarrow{(\beta_X)_m = \alpha_m^*} \mathrm{Hom}(\mathrm{Sd}(\Delta^m), X) = \mathrm{Ex}(X)_m$$

$$X_n \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \xrightarrow{(\beta_X)_m = \alpha^*} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), X) = \mathrm{Ex}(X)_n$$

Let $\Delta^n \xrightarrow{x} X$, it is enough to show that the following commutes:

$$\begin{tikzcd}
\Delta^m \arrow[r, "\sigma"] & \Delta^n \arrow[r, "x"] & X \\
\mathrm{Sd}(\Delta^m) \arrow[u, "\alpha_m"] \arrow[r, "\mathrm{Sd}(\sigma)"] & \mathrm{Sd}(\Delta^n) \arrow[u, "\alpha_n"] \arrow[ur] &
\end{tikzcd}$$

which is the case if the following commutes

$$\begin{tikzcd}
{[m]} \arrow[r, "\sigma"] & {[n]} & \max(U) \arrow[r, mapsto] & \sigma(\max U) \\
S([m]) \arrow[u, "a_m"] \arrow[r, "S(\sigma)"] & S([n]) \arrow[u, "a_n"] & U \arrow[u, mapsto] \arrow[r, mapsto] & \sigma(U) \arrow[u, mapsto]
\end{tikzcd}$$

$\square$

Lemma 18.4. *The map $\beta\colon \mathbb{1}_{\mathrm{Set}_\Delta} \to \mathrm{Ex}$, induced by β_X, is a natural transformation.*

Proof. Let $f\colon X \to Y$ be a morphism in Set_Δ, we obtain a square

$$\begin{tikzcd}
X \arrow[r, "\beta_X"] \arrow[d, "f"] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(f)"] & (\Delta^n \xrightarrow{x} X) \arrow[r, mapsto] \arrow[d] & (\mathrm{Sd}(\Delta^n) \xrightarrow{\alpha_n} \Delta^n \xrightarrow{x} X) \arrow[d, mapsto] \\
Y \arrow[r, "\beta_Y"] & \mathrm{Ex}(Y) & (\Delta^n \xhookrightarrow{x} X \xrightarrow{f} Y) \arrow[r, mapsto] & (\mathrm{Sd}(\Delta^n) \xrightarrow{\alpha_n} \Delta^n \to X \xrightarrow{f} Y)
\end{tikzcd}$$

Now one can check with the definitions of the morphisms that objects (right square) are mapped accordingly and everything commutes. $\square$

We now discuss some properties of $X \mapsto \mathrm{Ex}(X)$

Proposition 18.5. *The functor $\mathrm{Ex}\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ preserves filtered colimits.*

Proof. Notice that $\mathrm{Sd}(\Delta^n) = N(S[n])$ is compact,that is $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), -)$ preserves filtered colimits. Let $J \to \mathrm{Set}_\Delta$ be a filtered diagram where $j \to X^{(j)}$. Let

$$\begin{aligned}
\mathrm{Ex}(\mathrm{colim}_{j\in J} X^{(j)})_n &= \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), \mathrm{colim}_{j\in J} X^{(j)}) \\
&\cong \mathrm{colim}_{j\in J} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^n), X^{(j)}) \\
&= \mathrm{colim}_{j\in J} \mathrm{Ex}(X^{(j)})_n
\end{aligned}$$

$\square$

Remark 18.6. Let $\alpha_X = \overline{\beta_X} \in \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(X), X) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, \mathrm{Ex}(X)) \ni \beta_X$

Proposition 18.7. *For all $i\colon K \hookrightarrow L$ anodyne, $\mathrm{Sd}(i)\colon \mathrm{Sd}(K) \hookrightarrow \mathrm{Sd}(L)$ is anodyne.*

Proof. see cisinski 3.1.18 ☐

Corollary 18.8. *Let* $p\colon X \to Y$ *be a Kan fibration and* $\mathrm{Ex}(p)\colon \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$ *is a Kan fibration.*

Proof. Take the square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "\in \mathrm{An}"'] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(p)"] \\ \Delta^n \arrow[r] & \mathrm{Ex}(Y) \end{tikzcd}$$

which gives by applying the definition of Ex

$$\begin{tikzcd} \mathrm{Sd}(\Lambda^n_k) \arrow[r] \arrow[d, hook, "\in \mathrm{An}"'] & X \arrow[d, "\mathrm{Ex}(p)"] \\ \mathrm{Sd}(\Delta^n) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we can find a lift since An and Kan fibrations are a weak factorisation system. ☐

Corollary 18.9. *Let* $X \in \mathrm{Set}_\Delta$ *be a Kan complex, then* $\mathrm{Ex}(X)$ *is a Kan complex.*

Proof. Since X is a Kan complex, $(X \to \Delta^0) \in \mathrm{KanFib}$, thus by the previous corollary $\mathrm{Ex}(Y) \to \mathrm{Ex}(\Delta^0) \in \mathrm{KanFib}$ and $\mathrm{Ex}(\Delta^0) \cong \Delta^0$ since Ex preserves coolimits. ☐

Theorem 18.10. *For all* $X \in \mathrm{Set}_\Delta$ *we have that* $\beta_X\colon X \to \mathrm{Ex}(X)$ *is a weak equivalence.*

Proof. We are only going to give a sketch of the proof. Let X be a Kan complex. Take the square induced by π_0

$$\begin{tikzcd} \pi_0(\underline{\mathrm{Hom}}(\mathrm{Ex}(X), K)) \arrow[r, "?\circ\beta_X"] \arrow[d, "\beta_K\circ ?"] & \pi_0(\underline{\mathrm{Hom}}(X, K)) \arrow[d, "\beta_K\circ ?"] \arrow[dl, "f\mapsto \mathrm{Ex}(f)"'] \\ \pi_0(\underline{\mathrm{Hom}}(\mathrm{Ex}(X), \mathrm{Ex}(K)) \arrow[r, "?\circ\beta_X"] & \pi_0(\underline{\mathrm{Hom}}(X, \mathrm{Ex}(K))) \end{tikzcd}$$

proof can be found in Goerss-Jardine III thm 4.6. ☐

Corollary 18.11. *The morphism* $(f\colon X \to Y)$ *is a weak equivalence if and only if* $(\mathrm{Ex}(f)\colon \mathrm{Ex}(X) \to \mathrm{Ex}(Y))$ *is a weak equivalence.*

Proof. Take the square

$$\begin{tikzcd} X \arrow[r, "\beta_X"] \arrow[d, "f"'] \arrow[dr] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(f)"] \\ Y \arrow[r, "\beta_Y"] & \mathrm{Ex}(Y) \end{tikzcd}$$

Now the assertion follows by applying **??** 15.7. ☐

Definition 18.12. For $X \in \mathrm{Set}_\Delta$ we let $\mathrm{Ex}^\infty = \operatorname{colim}(X \xrightarrow{\beta_X} \mathrm{Ex}(X) \xrightarrow{\beta_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \to \dots)$ this defines an Endofunctor of Set_Δ and a natural transformation $\beta^\infty \colon \mathbb{1} \to \mathrm{Ex}^\infty$.

Proposition 18.13. *The functor $Ex^\infty \colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ preserves filtered colimits since for all $n \geqslant 0$ the functor Ex^n preserves filtered colimits and Ex^∞ is defined as a filtered colimit of these. Furthermore Ex^∞ preserves finite limits since for all $n \geqslant 0$, Ex^n preserves all limits and finite limits commute with filtered colimits in Set_Δ hence also in Set_Δ.*

Proposition 18.14. *Trivial Kan Fibrations are closed under filtered colimits in* $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$.

Proposition 18.15. *Let $f \colon X \to Y$ be a Kan fibration then $Ex^\infty(f)$ is a Kan fibration.*

Proof. This is an application of **??** 18.8 to all factors of the colimit. □

Corollary 18.16. *The morphism $\beta^\infty_X \colon X \to \mathrm{Ex}(X)$ is a weak equivalence.*

Proof. This is an application of **??** 18.11 to all factors of the colimit. □

19 Grothendieck's Homotopy hypothesis

Let Top be the category of topological spaces and Weq be the weak homotopy equivalences (that is morphisms $f\colon X \to Y$ such that for all $x \in X$ and all $n \in \mathbb{N}$ there is an isomorphism $\pi_n(f)\colon \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y,y)$).

Now Grothendiecks homotopy hypothesis goes as follows: There is an equivalence of $(\infty,1)$-categories $\mathrm{Top}[\mathrm{Weq}^{-1}]_\infty \xrightarrow{\sim} \mathrm{Gpd}_\infty$ given by the passage $X \mapsto \pi_\infty(X)$ (the Poincare ∞-groupoid). Now where do we stand with respect to Grothendiecks homotopy hypothesis?

Definition 19.1. A continuous map $f\colon X \to Y$ is a <u>Serre Fibration</u> if $f \in \{|\Lambda^n_k| \xrightarrow{|i|} |\Delta^n| \mid n \geqslant 1, 0 \leqslant k \leqslant n\}^{\boxslash}$