HoSS notes

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Mathematics, while deeply technical and logical is and will always be a love letter. A language that estranges most and can be deciphered by few, to write encrypted letters back and forth with the cosmos. I hope that she wants to dance with me one day again, even though I have two left feet. I sometimes think about category theory as the schematics for human dancing, the diagram chase on a commutative diagram becoming the instructions for steps. An infinite repetition of the same steps with slide change to give birth to a beautiful infinite dance. A spiraling helicial movement of lovers moving around one another, like planets orbiting one another. Simple rules birthing the dance of reality. Similarly the upmost simple rules for some signs birth a story, but one needs to bring alot to the table, long breath and more faith than I had so far to dance this dance, to make it infinite dance. Rigid repetition until one can dance all the standards by heart, then one is as well permitted to move freely within the structure, take the lead, discover new moves, teach the art. The prolific mathematicians have grasped alot of intuition for this process by playing piano I feel, it brings the benefit of immediate response, a harmony is for the human ear easy to discern from dissonance, hit the right keys to unravel the tones of the masters of old and anything beyond. These notes are based on a lecture by Gustavo Jasso held at the university of cologne in the winter semester 24/25.

1 Motivation

Let $\mathcal{C}$ be a category and W a class of morphisms.

Definition 1.1. A localisation of $\mathcal{C}$ at W is a category $\mathcal{C}[W^{-1}]$ together with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that $\forall f \in W$, we get that $\gamma(f)$ is an isomorphism in $\mathcal{C}[W^{-1}]$.

Example 1.2.
- A ring considered as category and the localisation at an ideal. Can this be extended to rings with more than one object?
- The derived category of an abelian category is the localisation with respect to the quasi isomorphisms.

Proposition 1.3. *Let $\mathcal{C}[W^{-1}]$ as in Definition 1.1. For any category $\mathcal{D}$ the functor*

$$\begin{aligned} j^*\colon \ \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) &\to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\ (\mathcal{C}[W^{-1}] \to \mathcal{D}) &\longmapsto (\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[\mathcal{W}^{-\infty}] \xrightarrow{F} \mathcal{D}) \end{aligned}$$

is an equivalence.

Theorem 1.4 ([1])**.** *Set-theoretic issues aside, loacalisations always exist.*

where

Example 1.5.
- Let Top be the category with objects given by topological spaces and morphisms given by continuous maps. Let W be the weak homotopy equivalences, that is morphisms $f : X \to Y$ such that the induced maps on path components

  $$\pi_0(f)\colon \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$$

  and for all points $x \in X$ and for all $n \geqslant 1$ with $n \in \mathbb{N}$ the morphism f induces an an isomorphism on homotopy groups

  $$\pi_n(f, x)\colon \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x))$$

  in Grp. The result of the localisation is called the homotopy category $\mathcal{H} : \mathrm{Top}[W^{-1}]$.
- The localisation at all morphism of the category $\mathcal{C}[\mathcal{C}^{-1}]$ is a groupoid.

Remark 1.6. The takeaway is the general paradigm, that the localisation is the <u>truncation</u> of a richer mathematical structure.

2 Presheaves and the Yoneda lemma

Throughout we will fix a small category $\mathcal{A}$.

Definition 2.1. Let $\widehat{\mathcal{A}} := \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$ be the category of contravariant functors from $\mathcal{A}$ to the category of sets. This category will be called the category of presheaves on $\mathcal{A}$. By definition $X \in \widehat{\mathcal{A}}$ consists of the following data:

- $\forall a \in \mathcal{A}$ a set $X_a := X(a) \in \mathrm{Set}$ This set will be called the fibre of X at a.
- $\forall u\colon b \to a \in \widehat{\mathcal{A}}(b,a)$ a map of sets $u^* = X(u)\colon X_a \to X_b$ such that functoriality constraints are satisfied:
- (Unitality) For all objects $a \in \mathcal{A}$ a morphism $(\mathrm{id}_a)^* = X(\mathrm{id}_a)\colon X_a \to X_a$ such that $(\mathrm{id}_a)^* = \mathrm{id}_{X_a}$.
  (Composition) For all composition of morphisms in $\mathcal{A}$ a composition of the induced morphisms on the fibres:

$$\begin{tikzcd} a \arrow[r, "u"] \arrow[rr, bend right, "v\circ u"'] & b \arrow[r, "v"] & c \end{tikzcd} \qquad \begin{tikzcd} & X_b \arrow[dl, "u^*"'] & \\ X_a & & X_c \arrow[ul, "v^*"'] \arrow[ll, "(v\circ u)^*"] \end{tikzcd}$$

  which is equivalent to $u^* \circ v^* = (v \circ u)^*$.

Remark 2.2. Throughout we are going to talk alot alot about presheaves, especially representable ones, what happens when we introduce sheaves and maybe use sheaves in the later theory.

Example 2.3. Let M be a monoid, then BM is the category with a single object $\mathrm{Ob}(BM) := \{*\}$ and morphisms $BM(*,*) \times BM(*,*) \to BM(*,*) = M$. A presheaf $X \in \widehat{BM}$ consists of a set $X = X_* \in \mathrm{Set}$ and for each $m \in M$ a morphism $m^* : X \to X$ that we denote on elements by left multiplication $m^*(x) = x \cdot m$. Moreover a morphism $e_m^*\colon X \to X$ such that $e_m^* = \mathrm{id}_{X^*}$, that is $x \cdot e_M = x$ for all $x \in X$. At last for any diagram of morphisms in BM, a diagram in Set

$$\begin{tikzcd} & * \arrow[dr, "u"] & \\ * \arrow[ur, "m"] \arrow[rr, "um"'] & & * \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\cdot m"'] & \\ X & & X \arrow[ul, "\cdot u"'] \arrow[ll, "\cdot(um)"] \end{tikzcd}$$

which means that $x \cdot (nm) = (xn) \cdot m$.

Definition 2.4. For every $a \in \mathcal{A}$, let $\mathcal{A}$ be the functor,

$$\mathcal{A}(-,a)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd} b \arrow[r, mapsto] & \mathcal{A}(b,a) \arrow[d, "u^*"] & & b \xrightarrow{v} a \\ c \arrow[u, "u"] \arrow[r, mapsto] & \mathcal{A}(c,a) & & c \xrightarrow{u} b \xrightarrow{v} a \end{tikzcd}$$

is the presheaf represented by a $\in \mathcal{A}$.

Let $X, Y \in \widehat{\mathcal{A}}$ be presheaves. By definition a morphism $f\colon X \to Y$ is a natural transformation of functors $\eta\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}$, that is for every $a \in \mathcal{A}$ a morphism of sets $\eta_a : X_a \to Y_a$ such that the usual naturality constraint holds, that is

$$\begin{tikzcd} a \arrow[d, "u"] & X_a \arrow[r, "\eta_a"] & Y_a \\ b & X_b \arrow[u, "u^*"] \arrow[r, "\eta_b"] & Y_b \arrow[u, "u^*"'] \end{tikzcd}$$

commutes for every morphism u in $\mathcal{A}$, so we have $u^* \circ \eta_b = \eta_a \circ u^*$.

Example 2.5. Let M be a monoid and $X, Y \in \widehat{BM}$. A morphism $f\colon X \to Y$ consists of a function $f = f^*\colon X = X_* \to Y_* = Y$ such that, the following diagram commutes

$$\begin{tikzcd} * \arrow[d, "m"] & X \arrow[r, "f"] & Y \\ * & X \arrow[u, "m"] \arrow[r, "f"] & Y \arrow[u, "m"'] \end{tikzcd}$$

for $m \in BM(*,*)$.

Theorem 2.6. *Yoneda lemma version 1 Let a be an object in $\mathcal{A}$ and $X \in \widehat{\mathcal{A}}$ a presheaf. Then*

$$\phi = \phi_{a,x}\colon \operatorname{Hom}_{\widehat{\mathcal{A}}}(\mathcal{A}(-,a), X) \longrightarrow X_a$$

$$f \longmapsto f_a(\mathrm{id}_a)$$

is bijective.

Proof. We first observe that the following square commutes

$$\begin{tikzcd} b \arrow[d, "u"] & \mathcal{A}(b,a) \arrow[r, "f_b"] & X_b \\ a & \mathcal{A}(a,a) \arrow[u, "u^*"] \arrow[r, "f_a"] & X_a \arrow[u, "u^*"'] \end{tikzcd}$$

which means $f_b(u^*(\mathrm{id}_a)) = f_b(u) = u^*(f_a(\mathrm{id}_a)) = u^*(\phi(f))$. So without evaluating at an object we get $f_b(-) = (-)^*(\phi(f)) = X(-)(\phi(f))$. Let us first show that ϕ is injective, suppose $f, g \colon \mathcal{A}(-, a) \to X$ such that $\phi(f) = \phi(g)$. For $b \in \mathcal{A}$ we get morphisms $f_b, g_b \colon \mathcal{A}(b, a) \to X_b$ and for any morphism $u : b \to a$ in $\mathcal{A}$ we get that $f_b(u) = u^*(\phi(f)) = u^*(\phi(g)) = g_b(u)$ and thus $f = g$. Let us now show that ϕ is surjective. Let $x \in X_a$ and $f^\times := (f_b^\times \colon \mathcal{A}(b, a) \to X_b \mid b \in \mathcal{A})$ given on any morphism $u \colon b \to a$ by $u^*(x)$. We need to prove that these are indeed the components of a natural transformation $f \colon \mathcal{A}(-, a) \to X$.

$$\begin{tikzcd}
b \arrow[d, "u"] & \mathcal{A}(b,a) \arrow[r, "f_b^\times"] & X_b \\
a & \mathcal{A}(c,a) \arrow[u, "u^*"] \arrow[r, "f_a^\times"] & X_c \arrow[u, "u^*"']
\end{tikzcd}$$

The square commutes, which gives for any $c \xrightarrow{\nu} a$ in $\mathcal{A}$, that

$$\begin{gathered}
f_b^\times(u^*(\nu)) = f_b^\times(\nu \circ u) = (\nu \circ u)^*(x) = u^*(f_c^\times(\nu)) = \\
u^*(\nu^*(x)) = (u^* \circ \nu^*)(x) = (\nu \circ u)^*(x).
\end{gathered}$$

∎

Lecture 3 15.10

Theorem 2.7. *The functor $\mu : \mathcal{A} \to \widehat{\mathcal{A}}$*

$$\begin{tikzcd}
a \arrow[d] \arrow[r, mapsto] & \hat{a} = \mathcal{A}(-,a) \arrow[d, "\hat{u}"] & \mathcal{A}(c,a) \arrow[d, "\hat{u}_c"] \\
b \arrow[r, mapsto] & \hat{b} = \mathcal{A}(-,b) & \mathcal{A}(c,b)
\end{tikzcd}$$

that sends an object of a to the the presheaf of morphisms into a is fully faithfull called the Yoneda embedding.

Proof. We first have to show that μ is a functor. So let $u \colon a \to b$ be a morphism. The claim is that $\hat{u} \colon \hat{a} \to \hat{b}$ is a natural transformation.

$$\begin{tikzcd}
c \arrow[d, "v"] & \mathcal{A}(c,a) \arrow[r, "u_*"] & \mathcal{A}(c,b) \\
d & \mathcal{A}(d,a) \arrow[u, "\nu^*"] \arrow[r, "u_*"] & \mathcal{A}(d,b) \arrow[u, "\nu^*"']
\end{tikzcd}$$

The square commutes, which means that morphisms are mapped to natural transformations under μ, thus μ is actually a functor. Next let us show that μ is fully faithfull, which means that the μ is a bijection on hom-sets

$$\begin{tikzcd}
\mu \colon \mathcal{A}(a,b) \arrow[r] & \mathrm{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, \hat{b}). \arrow[l, bend right, dashed, "\phi"']
\end{tikzcd}$$

We claim that
$$\phi \circ \mu = \mathrm{id}\,.$$
Let $u\colon a \to b$, then
$$\phi(\hat{u}) = \hat{u}_a(\mathrm{id}_a) = u \circ \mathrm{id}_a = u$$
which proves the above claim. $\square$

Remark 2.8. There is the contravariant Yoneda embedding as well, given by

$$\begin{aligned} \mu_{\mathcal{A}^{\mathrm{op}}}\colon \mathcal{A}^{\mathrm{op}} &\longrightarrow \hat{\mathcal{A}}^{\mathrm{op}} = \mathrm{Fun}(\mathcal{A}, \mathrm{Set}) \\ a &\longmapsto \mathcal{A}^{\mathrm{op}}(-, a) = \mathcal{A}(a, -). \end{aligned}$$

Proposition 2.9. *Let $X \in \hat{\mathcal{A}}$ consider the presheaf*

$$\begin{aligned} \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,?), \mathrm{Set})\colon \mathcal{A}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ \mathcal{A} \overset{\mu}{\hookrightarrow} \hat{\mathcal{A}} &\xrightarrow{\mathrm{Hom}_{\hat{\mathcal{A}}}(-,X)} \mathrm{Set} \\ a \mapsto \mathcal{A}(-,a) &\longmapsto \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,a), X) \end{aligned}$$

Then

$\phi_{?,X}(\phi_{a,X}\colon \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,a),X) \xrightarrow{\sim} X_a \mid a \in \mathcal{A})\phi_{?,X} = \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,?),X) \to X)$

is a natural isomorphism of presheaves.

Proof. We only need to prove naturality since $\phi_{a,X}$ is an isomorphism for every a by Theorem 2.6. So let us look at the following square

$$\begin{tikzcd} a \arrow[d, "u"] & \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,a),X) \arrow[r, "\phi"] & X_a \\ b & \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathcal{A}(-,b),X) \arrow[u, "?\circ\hat{u}"] \arrow[r, "\phi"] & X_b \arrow[u, "u^*"'] \end{tikzcd}$$

For $f\colon \hat{b} \to X$ the commutative square yields

$$\begin{tikzcd} & u^*(f_b(\mathrm{id}_b)) = f_a(u) & (f \circ \hat{u}) \arrow[r, mapsto] & (f \circ \hat{u})_a(\mathrm{id}_a) \\ f \arrow[r, mapsto] & f_b(\mathrm{id}_b) \arrow[u, mapsto] & f \arrow[u, mapsto] & \end{tikzcd}$$

where $f_a\colon \mathcal{A}(a,b) \xrightarrow{f_a} X_a$. Notice that $(f\circ\hat{u})_a(\mathrm{id}_a) = f_a \circ \hat{u}_a(\mathrm{id}_a) = f_a(u \circ \mathrm{id}_a) = f_a(u)$ which means both compositions are the same, so the square commutes. $\square$

there is more in this proof, but idk why it is there tbh.

Definition/Proposition 2.10. Let $X \in \widehat{\mathcal{A}}$ then the following are equivalent:

1. There $\exists a \in \mathcal{A}$ such that $\exists f\colon \hat{a} \to X$ that is an isomorphism in $\widehat{\mathcal{A}}$.
2. $\exists a \in \mathcal{A}$ and $\exists x \in X_a$ such that $\forall b \in \mathcal{A}$, we have that
$$\mathcal{A}(b,a) \to X_b \qquad u \mapsto u^*(x).$$
3. There $\exists a \in \mathcal{A}$ and $\exists x \in X_a$ such that $\forall b \in \mathcal{A}$ and $\forall u \in \mathcal{A}(b,c)$ we have that $\exists! y \in X_b$ such that $u^*(x) = y$.

We call the pair $(a \in \mathcal{A}, x \in X_a)$ a representation of X and $a \in \mathcal{A}$ a representing object and $x \in X_a$ a universal element.

Proof. ∎

not in lecture, probably follows from the statements before and after immediately

Proposition 2.11. *For an element $a \in \mathcal{A}$ the isomorphism $\phi_{a,X}\colon \operatorname{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, X) \xrightarrow{\sim} X$ is natural in X.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"] & \operatorname{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, X) \arrow[r, "\phi"] \arrow[d, "f\circ ?"] & X_a \arrow[d, "f_a"] \\ Y & \operatorname{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, Y) \arrow[r, "\phi"] & Y_b \end{tikzcd}$$

which evaluates on an element $g\colon \hat{a} \to X$ to

$$\begin{tikzcd} g \arrow[r, mapsto] & \phi(g) = g_a(\mathrm{id}_a) \arrow[d, mapsto] \\ & f_a(g_a(\mathrm{id}_a)) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[d, mapsto] & \\ (f \circ g) \arrow[r, mapsto] & \phi(f \circ g) = (f \circ g)_a(\mathrm{id}_a) \end{tikzcd}$$

comparing the two outcomes, we get the following equalities

$$f_a(g_a(\mathrm{id}_a)) = (f_a \circ g_a)(\mathrm{id}_a) = (f \circ g)_a(\mathrm{id}_a)$$

which yields the result. ∎

Theorem 2.12. *The Yoneda lemma Let $\mathcal{A}$ be a small category. The functions*

$$\begin{aligned} \phi_{a,X}\colon \operatorname{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, X) &\xrightarrow{\phi} X_a \\ f &\longmapsto f_a(\mathrm{id}_a) \end{aligned}$$

are natural in $a \in \mathcal{A}^{\mathrm{op}}$ and $X \in \widehat{\mathcal{A}}$ seperately. Hence they yield an isomorphism of functors.

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \times \widehat{\mathcal{A}} \arrow[rr, bend left, "\mathrm{Hom}_{\widehat{\mathcal{A}}}(\mu(?), -)"] \arrow[rr, bend right, "\mathrm{ev}"'] \arrow[rr, phantom, "\Downarrow"] & & \mathrm{Set}
\end{tikzcd}$$

Given on and object (a, X) as follows.

this implies is fucked up, fix it

$$\begin{tikzcd}
& \mathrm{Hom}_{\widehat{\mathcal{A}}}(\hat{a}, X) \arrow[dd, "\phi_{(a,X)}"] \\
(a, X) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
& X_a
\end{tikzcd}$$

3 Limits and Colimits

Let D be a small category, $\mathcal{C}$ be a category and $F\colon D \to \mathcal{C}$ a functor (a D-shaped diagram in $\mathcal{C}$). For example let D be given by

$$10 \xrightarrow{g} 11 \xleftarrow{f} 01$$

and let $F\colon D \to \mathcal{C}$ be a functor. We get a diagram

$$F(10) \xrightarrow{F(g)} F(11) \xleftarrow{F(f)} F(01).$$

Definition 3.1. A cone over F is a pair $(X, (\phi_a)_{a\in A})$ consisting of

1. $X \in \mathcal{C}$,
2. $(\phi_a\colon X \to F(a) \mid a \in A)$

such that $\forall u\colon a \to b$ in $\mathcal{A}$
$$\begin{tikzcd} & X \arrow[dl, "\phi_a"'] \arrow[dr, "\phi_b"] & \\ F(a) \arrow[rr, "F(u)"'] & & F(b) \end{tikzcd}$$
And $\phi_b = F(u) \circ \phi_a$.

Cones form a category $\mathcal{C}/F$ with morphisms given by

$$f\colon (X, (\phi_a)_{a\in A}) \to (Y, (\sigma_a)_{a\in A})$$

given by $f\colon X \to Y$ such that for all $a \in A$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_a"'] & & Y \arrow[dl, "\sigma_a"] \\ & F(a) & \end{tikzcd}$$

A limit (cone) of F is a final object in $\mathcal{C}/F$. Explicitely $(\lim F, (\sigma_a)_{a\in A})$ is a limit of F if for all cones $(X, (\phi_a)_{a\in A})$, there exists a unique $f\colon X \to \lim F$ in $\mathcal{C}$ such that for all $a \in A$ the following diagram commutes:

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "\phi_a"'] & & \lim F \arrow[dl, "\phi_a"] \\ & F(a) & \end{tikzcd}$$

Example 3.2. Let $A = \phi$ and $F\colon \phi \to \mathcal{C}$. A limit is an object $\mathbb{1} \in \mathcal{C}$ such that for all $X \in \mathcal{C}$ there exists a unique $f\colon X \to \mathbb{1}$ that is $\mathbb{1}$ is a final object in $\mathcal{C}$.

Example 3.3. Let $A = \{①, ②\} \xrightarrow{F} \mathcal{C}$ be a functor the limit cone in $\mathcal{C}$ is given by the product, that is a cone (Y, π_i) in $\mathcal{C}$.

$$\begin{tikzcd} & & F(2) \\ X \arrow[rru, bend left, "\phi_2"] \arrow[rrd, bend right, "\phi_1"'] \arrow[r, "\exists!"] & Y \arrow[ru, "\pi_2"] \arrow[rd, "\pi_1"] & \\ & & F(1) \end{tikzcd}$$

Lecture 17.10

If $(X, \overline{\rho}) = \overline{X}$ and $(Y, \overline{\sigma}) = \overline{Y}$ are limits of $F\colon A \to \mathcal{C}$. Then there is a unique isomorphism of cones $\overline{X} \to \overline{Y}$. It is enough to prove the statement for final objects, by definition of the limit cone. Let $X, Y \in \mathcal{D}$ be final objects. Then there exists a unique morphism $f\colon X \to Y$ in $\mathcal{D}$ since Y is final. Then there exists a unique morphism $f\colon Y \to X$ in $\mathcal{D}$ since X is final. But then $g \circ f(X) = X$ must be $g \circ f = \mathrm{id}_X$ since X is <u>final</u>.

Proposition 3.4. *Suppose that* $(\lim F, \overline{\sigma})$ *is a limit of* $F\colon \mathcal{A} \to \mathcal{C}$ *and* $f\colon X \to \lim F$ *is an isomorphism. Then* $(X, (\sigma_a f\colon X \to F(a))a \in \mathcal{A})$ *is a limit cone.*

Proof. For all $u\colon a \to b$ in A, we have get the following commutative diagram:

$$\begin{tikzcd} & X \arrow[ld, "\sigma_a \circ f"'] \arrow[rd, "\sigma_b \circ f"] & \\ F(a) \arrow[rr, "F(u)"'] & & F(b) \end{tikzcd}$$

which means that

$$F(u) \circ (\sigma_a \circ f) = \sigma_b \circ f.$$

Thus $(X, (\sigma_a \circ f)_{a \in A})$ is indeed a cone and $f\colon (X, (\sigma \circ f)_{a \in A}) \to (\lim_A F, \overline{\sigma})$ is an isomorphism of cones since f is an isomorphism and since

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[rd, "\sigma_a \circ f"'] & & \lim_A F \arrow[ld, "\sigma_a"] \\ & F(a) & \end{tikzcd}$$

commutes for all a in A. $\square$

Definition/Proposition 3.5. A category $\mathcal{C}$ is complete if for all small categories A and functors $F : A \to \mathcal{C}$ a limit of F exists. The category Set is complete.

Proof. Let A be a small category and $F\colon A \to \mathrm{Set}$ a diagram. Let

$$\lim_A F := \{\bar{X} = (X_a)_{a \in A} \in \prod_{a \in A} F(a) \mid \forall u\colon a \to b \text{ in } A, F(u)(X_a) = X_b\}$$

This this is a subset of a product, it comes with projections. We get that $(\lim_A F, (\pi_a : \bar{X} \to X_a)_{a \in A})$ is a cone over F since for all morphisms $u\colon a \to b$ in A we get that the following diagram

$$\begin{tikzcd}
 & \lim_A F \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
F(a) \arrow[rr, "F(a)"'] & & F(b)
\end{tikzcd}$$

equates to

$$\begin{tikzcd}
 & \bar{X} \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\
X_a \arrow[rr, "F(u)(X_a)"'] & & X_b
\end{tikzcd}$$

Now let $(X, (\rho_a\colon X \to F(a) \mid a \in A))$ be another cone over F. Define $\bar{\rho} : X \to \prod_{a \in A} F(a)$ by $x \mapsto (\rho_a(x))_{a \in A}$. Notice that $\bar{\rho}$ factors through $\lim_A F \subseteq \prod_{a \in A} F(a)$ since for all $x \in X$ and for all morphisms $u\colon a \to b$, we have that $F(u)(\rho_a(x)) = \rho_b(x)$ since the following diagram commutes

$$\begin{tikzcd}
 & X \arrow[dl, "\rho_a"'] \arrow[dr, "\rho_b"] & \\
F(a) \arrow[rr, "F(u)"'] & & F(b)
\end{tikzcd}$$

Thus $\bar{\rho} \to \lim_A F$ is well defined. Observe that $\bar{\rho}$ is actually a morphism of cones, since

$$\begin{tikzcd}
X \arrow[rr, "\bar{\rho}"] \arrow[dr, "\rho_b"'] & & \lim F \arrow[dl, "\pi_b"] \\
 & F(b) &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[rr, mapsto, "\bar{\rho}"] \arrow[dr, mapsto] & & (\rho_a(x))_{a \in A} \arrow[dl, mapsto, "\pi_b"] \\
 & \rho_b(x) &
\end{tikzcd}$$

Finally if $f\colon (X, (\rho_a)_{a \in A}) \to (\lim_A F, (\pi_a)_{a \in A})$ is a morphism of cones, then (by definition) we get for all $a \in A$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\rho_a"'] & & \lim_A F \arrow[dl, "\pi_a"] \\
 & F(a) &
\end{tikzcd}$$

that is for all $x \in X$ we get $\pi_a(f(x)) = \rho_a(x)$, so $f = \bar{\rho}$. $\square$

Definition 3.6. A functor G preserves limits of shape A if for all functors $F\colon A \to \mathcal{C}$, G sends limit cones of F to limit cones of $G \circ F$. A functor G preserves limits if for all small categories A we have that G preserves limits of shape A.

Remark 3.7. Let $F\colon A \to \mathcal{C}$ and $X \in \mathcal{C}$. Consider the covariant functor $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Set}$ and a limit $\sigma_a\colon \lim_A F \to F(a)$

toooo tired, dont get what the rmk wants from me miau miau

Theorem 3.8. *The functor* $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathrm{Set}$ *preserves limits.*

Proof. Consider the map

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_A F) \xrightarrow{\quad\phi\quad} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, F(a))$$

$$(X \xrightarrow{f} \lim_A F) \longmapsto (\sigma_a \circ f\colon X \xrightarrow{f} \lim_A F \xrightarrow{\sigma_a} F(a))_{a\in A}$$

This is a morphism of cones, now we need to show it is bijective. For injectivity, assume there are two morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} \lim_A F$ such that $\phi(f) = \phi(g)$. Then for all $a \in A$ we get that $\sigma_a \circ f = \sigma_a \circ g$ and for all morphisms $a \to b$

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[rrr, shift left, "f"] \arrow[rrr, shift right, "g"'] & & & \lim_A F \arrow[dl, "\sigma_a"'] \arrow[dr, "\sigma_b"] & \\
F(a) \arrow[rr, "F(u)"'] & & F(b) & F(a) \arrow[rr, "F(u)"'] & & F(b)
\end{tikzcd}$$

which means f and g are morphisms of cones, but by the uniqueness of a morphism into a limit, such that the above commutes, we get that $f = g$. For the surjectivity, let $(f_a\colon X \to F(a))_{a\in A}$

finish this proof you tired phoq

$\square$

Theorem 3.9. *The Yoneda embedding* $\mu\colon \mathcal{A} \to \hat{\mathcal{A}} = \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Set})$ *preserves limits.*

4 Adjunctions

Lecture 22.10

Everytime you encounter some free object, in the sense that it is freely generated from some other object of some other category, like a free group on some set, you are most likely going to use the properties of the object stemming from an adjunction.

Definition 4.1. $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ An adjunction $L \dashv R$ is a natural isomorphism $\phi_{c,d}\colon \operatorname{Hom}_{\mathcal{D}}(Lc, d) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(c, Rd)$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\operatorname{Hom}_{\mathcal{D}}(L(-),-)"] \arrow[r, bend right, "\operatorname{Hom}_{\mathcal{C}}(-,R(-))"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathrm{Set}
\end{tikzcd}$$

That is explicitely, we have commutative squares for all pairs of morphisms:

$$\begin{tikzcd}
(c,d) \arrow[d, shift left, "\mathrm{id}_d"] & \operatorname{Hom}_{\mathcal{D}}(Lc,d) \arrow[r, "\phi_{c,d}"] \arrow[d, "g\circ ?"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd) \arrow[d, "R(g)\circ ?"] \\
(c',d) \arrow[u, shift left, "f"] & \operatorname{Hom}(Lc,d') \arrow[r, "\phi_{c,d'}"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd')
\end{tikzcd}$$

which means that for all $f\colon Lc \to d$ and all $g\colon d \to d'$ we have that $R(g) \circ \bar{f} = \overline{g \circ f}$, where the closure operator denotes the image of an element under ϕ

$$\begin{tikzcd}
(c,d) \arrow[d, shift left, "\mathrm{id}_g"] & \operatorname{Hom}_{\mathcal{D}}(Lc,d) \arrow[r, leftrightarrow, "\phi_{c,d}", "\bar{(-)}"'] \arrow[d, "?\circ Lf"'] & \operatorname{Hom}_{\mathcal{C}}(c,Rd) \arrow[d, "?\circ f?"] \\
(c',d) \arrow[u, shift left, "f"] & \operatorname{Hom}(Lc',d) & \operatorname{Hom}_{\mathcal{C}}(c',Rd) \arrow[l, "\bar{(-)}"]
\end{tikzcd}$$

which means that for all $k\colon c \to Rd$ and all $f\colon c' \to c$ we have that $\bar{k} \circ Lf = \overline{k \circ f}$.

Remark 4.2. For $c \in \mathcal{C}$ consider η_c given as follows

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}}(Lc, Lc) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(c, RLc) \\
\mathrm{id}_{Lc} &\longmapsto \eta_c := \overline{\mathrm{id}_{Lc}}
\end{aligned}$$

as well as for any $d \in \mathcal{D}$ consider ϵ_d given as follows

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}(Rc, Rc) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(LRd, d) \\
\mathrm{id}_{Rc} &\longmapsto \epsilon_d := \overline{\mathrm{id}_{Rd}}
\end{aligned}$$

Proposition 4.3. *Notice that $\eta = (\eta_c\colon c \to RLc \mid c \in \mathcal{C})$ is a natural transformation $\eta\colon \mathbb{1}_\mathcal{C}\ RL$ we call this the unit of the adjunction and similarly $\epsilon = (\epsilon_c\colon c \to LRd \mid d \in \mathcal{C})$ is a natural transformation $\epsilon\colon LR \to \mathbb{1}_\mathcal{D}$ and is a called the counit of the adjunction.*

Proof. Consider the following square:

$$
\begin{tikzcd}
c \arrow[d, "f"] & c \arrow[r, "\eta c"] \arrow[d, "f"'] & RLc \arrow[d, "RLf"] \\
c' & c' \arrow[r, "\eta_{c'}"'] & c'
\end{tikzcd}
$$

and the resulting equation

$$
\begin{aligned}
& RLf \circ \eta_c = \eta'_c \circ f && (1)\\
\iff\ & \overline{RLf \circ \eta_c} = \overline{\eta_{c'} \circ f} && (2)\\
\iff\ & \overline{\overline{Lf}} = \overline{\mathrm{id}_{c'} \circ Lf} && (3)\\
\iff\ & Lf = \mathrm{id}_f \circ Lf && (4)
\end{aligned}
$$

□

Proposition 4.4. *The unit $\eta\colon \mathbb{1}_\mathcal{C} \Rightarrow RL$ aand the counit $\epsilon\colon LR \Rightarrow \mathbb{1}_\mathcal{D}$ satisfy the triangle identities:*

$$
\begin{tikzcd}
& LRLc \arrow[dr, "\epsilon_{Lc}"] & \\
Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc
\end{tikzcd}
\qquad
\begin{tikzcd}
& RLRd \arrow[dr, "R(\epsilon_d)"] & \\
Rd \arrow[ur, "\eta_{Rd}"] \arrow[rr, "\mathrm{id}_{Rd}"'] & & Rd
\end{tikzcd}
$$

For all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Proof. If we take the triangle for the unit above and apply the bar-operator we obtain the following

$$
\begin{tikzcd}
& LRLc \arrow[dr, "\epsilon_{Lc}"] & \\
Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc
\end{tikzcd}
\quad \overset{\overline{(-)}}{\longleftrightarrow} \quad
\begin{tikzcd}
& RLc \arrow[dr, "\mathrm{id}_{RLc}"] & \\
c \arrow[ur, "\eta_c"] \arrow[rr, "\overline{\mathrm{id}_c} = \eta_c"'] & & RLc
\end{tikzcd}
$$

The second triangle clearly commutes, the argument for the counit is analogous.

□

Proposition 4.5. *Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be two functors between categories and suppose there exist natural transformations $\eta\colon \mathbb{1}_\mathcal{C} \Rightarrow RL$ and $\eta\colon LR \Rightarrow \mathbb{1}_\mathcal{D}$ that satisfy the triangle identities. Then the following defines an adjunction $L \dashv R$*

$$\phi\colon \operatorname{Hom}_\mathcal{D}(Lc, d) \rightleftarrows \operatorname{Hom}_\mathcal{C}(c, Rd)\colon \psi$$

$$Lc \xrightarrow{g} d \ \longmapsto\!\!\!\!\longleftarrow\ R(g) \circ \eta_c$$

Proof. For $g\colon Lc \to d$ we have that

$$g = (\psi \circ \phi)(g) = \psi(R(g) \circ \eta_c) = \epsilon_d \circ LR(g) \circ L(\eta_c)$$

Now we have the following the diagram

$$\begin{tikzcd}
 & LRLc \arrow[rr, "LRg"] \arrow[dr, "\epsilon_{Lc}"] & & LRd \arrow[dr, "\epsilon_d"] & \\
Lc \arrow[ur, "L(\eta_c)"] \arrow[rr, "\mathrm{id}_{Lc}"'] & & Lc \arrow[rr, "g"'] & & d
\end{tikzcd}$$

The triangle commutes due to the triangle identities and the square commutes by the naturality of the counit. thus the whole diagram commutes, which means that ψ is an inverse to ϕ which yields the statement. ☐

Definition 4.6. For $c \in \mathcal{C}$ and $R\colon \mathcal{D} \to \mathcal{C}$ a functor, define the category c/R with objects given by tuples (c, f) where $f\colon c \to R(d)$ is a morphism for some $d \in \mathcal{D}$ and morphisms are given by

$$\begin{tikzcd}
d \in \mathcal{D} \arrow[d, "g"] & c \arrow[r, "f"] \arrow[d, equal] & Rd \arrow[d, "R(g)"] \\
d' \in \mathcal{D} & c \arrow[r, "f'"'] & Rd'
\end{tikzcd}$$

Dually for $L\colon \mathcal{C} \to \mathcal{D}$ and $d \in \mathcal{D}$ we define L/d as tuples (d, g) where $g\colon Lc \to g$ is a morphism and morphisms are given by

$$\begin{tikzcd}
c \in \mathcal{C} \arrow[d, "f"] & Lc \arrow[r, "g"] \arrow[d, "Lf"'] & d \arrow[d, equal] \\
c' \in \mathcal{C} & Lc' \arrow[r, "g'"'] & d
\end{tikzcd}$$

Notice that given an adjunction $L \dashv R$ we have that $\forall c \in \mathcal{C}(c, c \xrightarrow{\eta_c} Rc)$ is in c/R and $\forall \in \mathcal{D}(d, LR \xrightarrow{\epsilon_d} d)$ is in L/d.

Proposition 4.7. *The object $(c, c \xrightarrow{\eta_c} Rc)$ is initial in c/R and $(d, LR \xrightarrow{\epsilon_d} d)$ is final in L/d.*

Proof. Consider the following commutative triangle

$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[dr, "\overline{\overline{f}}=f"'] & RLc \arrow[d, "R(\overline{f})"] \\
 & Rd
\end{tikzcd}$$

where $\overline{\overline{f}} = \overline{R(\overline{f}) \circ \eta_c}$. Thus the morphism f of the object (c, f) uniquely determines the morphism $R(\overline{f})$. The argument for final object is dual ☐

Lecture 24.10

Proposition 4.8. *Let*

$$\mathcal{C} \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathcal{D} \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathcal{E}$$

be adjunctions then their composition

$$\mathcal{C} \underset{R_1 \circ R_2}{\overset{L_1 \circ L_2}{\rightleftarrows}} \mathcal{D}$$

is an adjunction as well.

Proof. Consider the following transformations, given by the adjunction isomorphisms

$$\mathrm{Hom}_{\mathcal{E}}(L_2 L_1 c, e) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(L_1 c, R_2 e) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, R_1 R_2 e)$$

□

Remark 4.9. Given an adjunction

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

then the following is an adjunction as well

$$\mathcal{C}^{\mathrm{op}} \underset{R^{\mathrm{op}}}{\overset{L^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{op}}$$

and the unit $\eta_c \colon c \to RLc$ in $\mathcal{C}$ corresponds to the counit $c \leftarrow RLc$ in $\mathcal{C}^{\mathrm{op}}$

Proposition 4.10. *Let $L_1; L_2 \colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \leftarrow \mathcal{D} : R$ be functors. Suppose that $L_1 \dashv R$ and $L_2 \dashv R$ are adjunctions, then it follows that $L_1 \cong L_2$.*

Proof. We need to construct a natural isomorphism $\phi \colon L_1 \xrightarrow{\sim} L_2$. Let $\eta^{(1)} \colon \mathbb{1} \to RL_1$ and $\eta^{(2)} \colon \mathbb{1}_{\mathcal{C}} \to RL_2$. By the uniqueness of initial objects in c/R, for $c \in \mathcal{C}$ Proposition 4.7 we obtain

$$\begin{tikzcd} & c \arrow[dl, "\eta_c^{(1)}"'] \arrow[d, "\eta_2^{(2)}"] \arrow[dr, "\eta_c^{(1)}"] & \\ RL_1 c \arrow[r, dashed, "Rg"] & RL_2 c \arrow[r, dashed, "Rh"] & RL_1 c \end{tikzcd}$$

By the uniqueness of a morphism out of an initial object we obtain for the composition that $L_1 c \xrightarrow{g} L_2 c \xrightarrow{h} L_1 c$ is given by $h \circ g = \mathrm{id}_{L_1 c}$. Similarly one obtains $g \circ h = \mathrm{id}_{L_2 c}$. Thus $g =: U_c \colon L_1 c \xrightarrow{\sim} L_2 c$. We now have to check that U_c is actually a natural transformation of functors. Consider the following diagram

$$
\begin{tikzcd}
c \arrow[d, "f"] & L_1 c \arrow[r, "U_c"] \arrow[d, "L_1 f"'] & L_2 c \arrow[d, "L_2 f"] \\
c' & L_1 c' \arrow[r, "U_c'"'] & L_2 c'
\end{tikzcd}
$$

apply R to it

$$
\begin{tikzcd}
& c \arrow[dl, "\eta_c^{(1)}"'] \arrow[dr, "\eta_c^{(2)}"] \arrow[dd] & \\
RL_1 c \arrow[rr, "R(U_c)"] \arrow[dd, "RL_1(f)"'] & & RL_2 c \arrow[dd, "RL_2(f)"] \\
& c' \arrow[dl, "\eta_{c'}^{(1)}"'] \arrow[dr, "\eta_{c'}^{(2)}"] & \\
RL_1 c' \arrow[rr, "R(U_{c'})"] & & RL_2(c')
\end{tikzcd}
$$

This yields the following equations

$$
\begin{aligned}
R(U_{c'} \circ L_1(f)) \circ \eta_c^{(1)} &= R(U_{c'}) \circ \eta_c^{(1)} \circ f && (5)\\
&= \eta_c^{(2)} \circ f && (6)\\
&= RL_2(f) \circ \eta_c^{(2)} && (7)\\
&= RL_2(f) \circ R(U_c) \circ \eta_c^{(1)} && (8)\\
&= R(L_2(f) \circ U_c) \circ \eta_c^{(1)} && (9)
\end{aligned}
$$

And thus results in the following commutative triangle

$$
c \xrightarrow{\eta_c^{(1)}} RL_1 c
$$

□

why are we done here ??

Proposition 4.11. *Let* $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ *be an adjunction then* L *preserves colimits that exist in* $\mathcal{C}$ *and* R *preserves limits that exist in* $\mathcal{D}$.

Proof. Let $X\colon A \to \mathcal{C}$ be a diagram that admits a colimit in $\mathcal{C}$, $\operatorname{colim} X_a \in \mathcal{C}$ and $a \in A$.

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{D}}(L(\operatorname{colim}_{a\in A} X_a), d) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_{a\in A} X_a, Rd) && (10)\\
&\xrightarrow{\sim} \lim_{a\in A} \operatorname{Hom}_{\mathcal{C}}(X_a, Rd) && (11)\\
&\xrightarrow{\sim} \lim_{a\in A} \operatorname{Hom}_{\mathcal{C}}(LX_a, d) && (12)\\
& && (13)
\end{aligned}
$$

This exhibits $L(\operatorname{colim}_{a\in A} X)$ as a colimit of $X\colon A \xrightarrow{X} \mathcal{C} \xrightarrow{L} \mathcal{D}$. □

Let A be a small category and $\mathcal{C}$ a category. Consider the functor $\mathrm{const}_A \colon \mathcal{C} \to \mathrm{Fun}(A, \mathcal{C})$ that maps each object of $\mathcal{C}$ to the functor $F_c(a) = c$ for all and $a \in A$ and each morphism to id_c.

Proposition 4.12. *Suppose that there exists $L\colon \mathrm{Fun}(A, \mathcal{C}) \to \mathcal{C}$ a left adjoint to const_A. Then for all $X \colon A \to \mathcal{C}$ the unit $\eta_X \colon X \to \mathrm{const}_A(LX)$ exhibits LX as a colimit of X.*

Proof. We know that $\eta_X \colon X \to \mathrm{const}_A(LX)$ is initial in X/const_A. Notice that the objects of X/const_A are pairs $(c \in \mathcal{C}, \rho\colon X \Rightarrow \mathrm{const}_A(c))$. Thus $\bar{\rho} = (\rho_a \colon X_a \to c \mid a \in A)$ is such that

$$\begin{tikzcd}
a \arrow[d, "u"] & X_a \arrow[r, "\rho_a"] \arrow[d, "X_u"'] & c \arrow[d, "\mathrm{id}_c"] \\
b & X_b \arrow[r, "\rho_b"] & c
\end{tikzcd}$$

Let furthermore $f\colon c \to c'$ be a morphism inducing a morphism in X/const_A

$$\begin{tikzcd}
X \arrow[r, Rightarrow, "\bar{\rho}"] \arrow[d, equal] & \mathrm{const}_A(c) \arrow[d, equal, "\mathrm{const}_A(f)"] \\
X \arrow[r, Rightarrow, "\bar{\sigma}"] & \mathrm{const}_A(c')
\end{tikzcd}$$

given evaluated on objects $a \in A$ by

$$\begin{tikzcd}
X_a \arrow[r, "\phi_a"] \arrow[d, equal] \arrow[dr, "\sigma"] & c \arrow[d, "f"] \\
X_a \arrow[r, "\sigma"] & c'
\end{tikzcd}$$

$\square$

why are we done here ? Don't see it.

Proposition 4.13. *Let $\mathcal{C} \leftarrow \mathcal{D} : R$ be such that for all $c \in \mathcal{C}$ the functor $\mathrm{Hom}_{\mathcal{C}}(c, R(-))\colon \mathcal{D} \to \mathrm{Set}$ is corepresentable by an object $L(c) \in \mathcal{D}$ via $\phi_c \colon \mathrm{Hom}_{\mathcal{D}}(L(c), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(c, R(-))$. Then the association $c \mapsto L(c)$ can be promoted to a functor $L\colon \mathcal{C} \to \mathcal{D}$ that is adjoint to R via ϕ.*

Proof. We need to define L on morphisms. For $c \xrightarrow{f} c'$ in $\mathcal{C}$ consider the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(c), -) \arrow[r, "\phi_c"] & \mathrm{Hom}_{\mathcal{C}}(c, R(-)) \\
\mathrm{Hom}_{\mathcal{D}}(L(c'), -) \arrow[r, "\phi_{c'}"] \arrow[u, dashed, "L(f)^*"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "f^*"']
\end{tikzcd}$$

By Yoneda we obtain an object $d \in \mathcal{D}$ such that

$$\begin{tikzcd}
L(c) \arrow[d, "L(f)"'] \arrow[dr] & \\
L(c') \arrow[r] & d
\end{tikzcd}$$

Now we need to prove that this actually defines a functor $L\colon \mathcal{C} \to \mathcal{D}$. For $c \in \mathcal{C}$ we have that $L(\mathrm{id}_c) = \mathrm{id}_{L(c)}$ by construction. Let $c \xrightarrow{f} c' \xrightarrow{g} c''$ be in $\mathcal{C}$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(c), -) \arrow[r, "\phi_c"] & \mathrm{Hom}_{\mathcal{C}}(c, R(-)) \\
\mathrm{Hom}_{\mathcal{D}}(L(c'), -) \arrow[u, "L(f)*"] \arrow[r, "\phi'_c"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "f*"'] \\
\mathrm{Hom}_{\mathcal{D}}(L(c''), -) \arrow[u, "L(g)*"] \arrow[r, "\phi''_c"] \arrow[uu, bend left=60, "L(g\circ f)"] & \mathrm{Hom}_{\mathcal{C}}(c', R(-)) \arrow[u, "g*"'] \arrow[uu, bend right=60, "g\circ f"']
\end{tikzcd}$$

The uniqueness given by Yoneda, implies

$$L(g \circ f) = L(g) \circ L(f)$$

Let $\phi_{c,d} = (\phi_c)_d$ $\square$

5 Extending functors by colimits

Consider $X \in \mathrm{Set} = \hat{\mathbb{1}} = \mathrm{Fun}(\mathbb{1}^{op}, \mathrm{Set})$ with $\mathbb{1}$ the category $\{* \circlearrowleft \mathrm{id}\}$. The following diagram

$$\begin{tikzcd}
\{(*, y)\} \arrow[dr, hook] & \cdots & \{(*, z)\} \arrow[dl, hook'] \\
& X \cong \coprod\limits_{x \in X} \{x\} &
\end{tikzcd}$$

exhibits X as colimit. Now consider a small category A, a presheaf $X \in \hat{A}$, a morphism $u \colon a \to b$ in A and elements $s \in X_a, t \in X_b$ with

$$\begin{tikzcd}
X_a & X_b \arrow[l, "u^*"'] \\
s & t \arrow[l, mapsto]
\end{tikzcd}$$

Owing to Yoneda, we have a commutative diagram in $\hat{A}$:

$$\begin{tikzcd}
\hat{a} \arrow[rr, "\hat{u}"] \arrow[dr, "s"'] & & \hat{b} \arrow[dl, "t"] \\
& X &
\end{tikzcd}$$

Replacing $\mathbb{1}$ with a small category A we can generalize the construction from the beginning.

Definition 5.1. The category of elements of X, denoted $\int^A X$ has as objects the pairs $(a \in A, s \in X_a) = (a \in A, \hat{a} \xrightarrow{s} X)$ with morphisms

$$\begin{tikzcd}
a \arrow[d, "\forall u \in A"'] & \hat{a} \arrow[r, "s"] \arrow[d, "\hat{u}"'] & X \arrow[d, equal] \\
b & \hat{b} \arrow[r, "t"'] & X
\end{tikzcd}$$

given that $u^*(t) = s$. Note that there is a canonical projection $can \colon \int^A X \to A$.

We will see that the presheaf $X \in \hat{A}$ acts as a colimit with $\int^A X$ as the indexing category.

Example 5.2.
- $A = \mathbb{1}, X \in \mathrm{Set}$. Then $\int^{\mathbb{1}} X = \{(*, s \in X) | s \in X\}$. A morphism $(*, s \in X) \xrightarrow{\mathrm{id}^*} (*, t \in X)$ requires $s = t$.
- $M \colon$ monoid $\rightsquigarrow \widehat{BM} = \mathrm{Fun}(* \circlearrowleft M^{op}, \mathrm{Set})$. We have the following morphisms in $\int^{BM} X$:

$$(*, x \in X) \xrightarrow{m \in M} (*, y \in X)$$

with $m^*(y) = y \cdot m = x$, i.e. morphisms exist precisely within orbits.

- $b \in A \rightsquigarrow \int^A A(-, b)$. The morphisms are given by

$$\begin{tikzcd} a \arrow[d, "\forall u \in A"'] \\ a' \end{tikzcd} \qquad \begin{tikzcd} (a \in A, f \in A(a,b)) \arrow[r, equal] \arrow[d, "u"'] & (a \in A, f \colon a \to b) \arrow[d, "u"] \\ (a' \in A, g \in A(a',b)) \arrow[r, equal] & (a' \in A, g \colon a' \to b) \end{tikzcd}$$

i.e. $u^*(g) = g \circ u = f$.

Consider the composite

$$\begin{tikzcd} \int^A X \arrow[r] & A \arrow[r, hook, "\mu"] & \hat{A} \ni X \\ (a, s) \arrow[r, mapsto] & a \arrow[r, mapsto] & \hat{a} \end{tikzcd}$$

The presheaf X has a canonical cone structure under this diagram, which is what we alluded to before:

$$\begin{tikzcd} (a, s) \arrow[d, "\substack{\forall u \in \int^A X \\ (u^*(t) = s)}"'] \\ (b, t) \end{tikzcd} \qquad \begin{tikzcd} \hat{a} \arrow[rr, "\hat{u}"] \arrow[dr, "s"'] & & \hat{b} \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

6 Simplicial sets

Definition 6.1. The simplicial category Δ has objects $[n] := \{0 < 1 < 2 < \ldots < n\}$ for $n \geqslant 0$ and morphisms $\Delta(m,n) := \mathrm{Hom}_\Delta([m],[n]) := \{f\colon [m] \to [n], \text{order preserving}\}$.

Definition 6.2. The <u>category of simplicial sets</u> is given by $\mathrm{Set}_\Delta = \mathrm{sSet} = \widehat{\Delta} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Remark 6.3. An alternative definiton of a simplicial set X can be given as follows:

- For all $n \geqslant m$ a set X_n called the <u>n-simplices</u> of X.
- For all $0 \leqslant i \leqslant n$ morphisms $d_i\colon X_n \to X_{n+1}$ called the <u>face maps</u>.
- For all $0 \leqslant i \leqslant n$ morphisms $s_i\colon X_n \to X_{n+1}$ called the <u>degeneracy maps</u>.
- The face and degeneracy maps satisfy the following identities:

$$\begin{array}{ccc} d_i d_j = d_{j-1} d_i & d_i s_j = s_{j-1} d_i & d_j s_j = \mathrm{id} = d_{j+1} s_j \\ i < j & i < j & \\ d_i s_j = s_j d_{i-1} & s_i s_j = s_{j+1} s_i & \\ i > j+1 & i \leqslant j & \end{array}$$

Example 6.4. 1. For an arbitrary simplicial set we often write

$$X\colon \ldots \qquad X_3 \;\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}\; X_1 \;\substack{\longrightarrow\\[-1ex]\longleftarrow\\[-1ex]\longrightarrow}\; X_0$$

where the arrows correspond to the face and boundary maps.

2. $[0]=\{0\}$

$[1]=\{ \; id \circlearrowright 0 \xrightarrow{10} 1 \circlearrowleft id \; \}$

$$[2] = \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\}$$

dis all fucked up, ... dont like how this looks

By Yoned we obtain an isomorphism $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta(-,n), X) \xrightarrow{\sim} X_n$. Let $x \in X_n$ and let $\Delta^n := \Delta(-,n)$ we obtain the following diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "x"] \arrow[dr, "\sigma_x"'] & & X \\ & \Delta^m \arrow[ur, "\mu"'] & \end{tikzcd}$$

unsure what all is happening here

Let $d^i\colon [n-1] \to [n]$ be the unique order preserving injective map not having $i \in [n]$ in its image for all $0 \leqslant i \leqslant n$.

$$[0] = \{0\} \xrightarrow{d^0} \{0 \to 1\} = [1]$$

$$[0] = \{0\} \xrightarrow{d^0} \{0 \to 1\} = [1]$$

$$[1] = \{0 \to 1\} \xrightarrow{d^0} \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\} = [2]$$

more stuff missing cause dk yet how to tex this

Definition 6.5. The category Δ_{big} has as objects the finite non-empty total orders with order preserving maps between them

$$\Delta \leftrightarrows \Delta_{\text{big}} \ni I = \{i_0 < i_1 < \ldots < i_n\}$$

alotta diagrams i am too lazy for currently

Example 6.6. the nerve of a category Let $\mathcal{C}$ be a small category. We define $N(\mathcal{C}) \in \mathrm{Set}_\Delta$ as follows. Let $N(\mathcal{C})_0 := \mathrm{Ob}(\mathcal{C})$, $N(\mathcal{C})_1 = \mathrm{Mor}(\mathcal{C})$: set of morphisms in $\mathcal{C}$ with the face and boundary maps given as follows

again to many arrows that I am too tired for right now

Theorem 6.7. *For all $X \in \mathrm{Set}_\Delta$ $|X| = \operatorname{colim}_{\substack{([n],x)\in \int_X^\Delta \\ X\colon \Delta^n \dashv X}} |\Delta^n|$ is a CW complex.*

Definition 6.8. An element x of simplicial set X is degenerate if $x \in \operatorname{im} s_i$ for some i.

Theorem 6.9. *(Dold-Kan Correspondences)*

7 Kan complexes

lecture 19.11

The aim of this chapter is to introduce a calss of simplicaial sets that behave simultaneously as nerves of groupoids and singular sets of spaces.

Definition 7.1. Let $n \geqslant 1$ and $0 \leqslant k \leqslant n$. The k-th horn $\Lambda^n_k \subseteq \Delta^n$ is the simplicial subset generated by $\{d^i \colon [n-1] \to [n] \mid 0 \leqslant i \leqslant n; i \neq n\} = \Delta(n-1, n) = (\Delta^n)_{n-1}$, or equivalently $\Lambda^n_k = \operatorname{colim}_{\varnothing \neq k \in I \subsetneq [n]} \Delta^I$ where $\Delta^I \cong \Delta^{|I|-1}$.

Remark 7.2. We will sometimes refer to Λ^n_k as the n-horn at position k.

Example 7.3.

decide if you want the actual picture, i.e. do simplices in tikz, or just make a reduced version

Remark 7.4. The horn $\Lambda^n_k \subseteq \Delta^n$ enjoys the following universal property: For $X \in \mathrm{Set}_\Delta$ the map

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_k, X) \arrow[r, hook] & \prod_{\substack{0\leqslant i \leqslant \\ i \neq k}} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n-1}, X) \\
\sigma \arrow[r, mapsto] & (\sigma \circ d^i)_{\substack{0\leqslant i \leqslant n \\ i \neq k}}
\end{tikzcd}$$

is injective, with image the subset of tuples $(\sigma_0, \sigma_1, \dots, \sigma_{k-1}, \cdot, \sigma_{k+1}, \dots, \sigma_n) \in (X_{n-1})^n$ such that for all $0 \leqslant i < j \leqslant n$, $i \neq k$ $d_(\sigma_j) = d_{j-1}(\sigma_i)$ with $I = [n] \backslash \{i\}$ and $J = [n] \backslash \{j\}$ the following diagrams commute

$$\begin{tikzcd}
& \Delta^{I \cap J} \arrow[dl, hook] \arrow[dr, hook] & \\
\Delta^I \arrow[dr] & & \Delta^J \arrow[dl] \\
& \Lambda^n_k &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Delta^{n-2} \arrow[dl, "d^{j-1}"'] \arrow[dr, "d^i"] & \\
\Delta^{n-1} \arrow[dr, "d^i"'] & & \Delta^{n-1} \arrow[dl, "d^j"] \\
& \Delta^n &
\end{tikzcd}$$

Definition 7.5. Let $X \in \mathrm{Set}_\Delta$, X is a Kan complex (= ∞-groupoid) if for all $n \geqslant 1$ and all morphisms $\sigma \colon \Lambda^n_k \to X$ we have the following diagram:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{can}"'] & X \\
\Delta^n \arrow[ur, dashed, "\exists \hat{\sigma}"'] &
\end{tikzcd}$$

Remark 7.6. Notice that the morphism $\hat{\sigma}$ need not be unique.

Recall the following adjunctions

$$\begin{tikzcd}
\mathrm{Gpd} \arrow[r, bend left, "j"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "L"] \arrow[r, bend left, "N"] \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend left, "\tau"] \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, bend left, "\mathrm{Sing}"]
\end{tikzcd}$$

Proposition 7.7. *Let* $X \in \mathrm{Top}$. *Then* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. For all $n \geqslant 1$ and $0 \leqslant k \leqslant n$ we want the following diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\iota"'] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed, "\exists\tilde{\sigma}"'] & \end{tikzcd}$$

Now after applying the geometric realization functor $|\cdot|$ to the above diagram, we obtain a diagram

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "\overline{\sigma}"] \arrow[d, hook, shift right, "|\iota|"'] & X \\ {|\Delta^n|} \arrow[u, shift right, "\exists r"'] \arrow[ur, "\alpha = \overline{\sigma} \circ r"'] & \end{tikzcd}$$

where r is a continuous retraction of Δ^n onto $|\Lambda^n_k|$. Then we apply the adjunction to the following composition

$$|\Lambda^n_k| \xrightarrow{|\iota|} |\Delta^n| \xrightarrow{\alpha} X = |\Lambda^n_k| \xrightarrow{|\sigma|} X$$

to obtain

$$\Lambda^n_k \xrightarrow{\iota} \Delta^n \xrightarrow{\overline{\alpha}} \mathrm{Sing}(X) = \Lambda^n_k \xrightarrow{\overline{\overline{\sigma}} = \sigma} \mathrm{Sing}(X)$$

which then gives the desired horn extension:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, "\overline{\alpha} =: \tilde{\sigma}"'] & \end{tikzcd}$$

□

Definition 7.8. Let $X \in \mathrm{Set}_\Delta$, it is called an <u>inner Kan complex</u> (= ∞-category) if for all $n \geqslant 2$ and $0 < k < n$ we have the following diagram:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{can}"'] & X \\ \Delta^n \arrow[ur, dashed, "\exists\hat{\sigma}"'] & \end{tikzcd}$$

Remark 7.9. Let $n \geqslant 0$ and $\Delta_{\leqslant n} := \{[m] \in \Delta \mid 0 \leqslant m \leqslant n\} \overset{\iota}{\hookrightarrow} \Delta$ as well as $\mathrm{tr}_n = \iota^*$. We have the following adjunction

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, "\mathrm{tr}_n"] & \mathrm{Set}_{\Delta_{\leqslant n}} = \mathrm{Fun}(\Delta^{\mathrm{op}}_{\leqslant n}, \mathrm{Set}) \arrow[l, bend right, "\mathrm{sk}_n"'] \arrow[l, bend left, "\mathrm{cosk}_n"] \end{tikzcd}$$

and call $X \in \mathrm{Set}_\Delta$ n-coskeletal if the following equivalent conditions hold:

1. $X \in \mathrm{Im}(\mathrm{cosk}_n)$
2. $X \xrightarrow{\sim} \mathrm{cosk}_n(\mathrm{tr}_n)$
3. $\forall Y \in \mathrm{Set}_\Delta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta_{\leqslant n}}}(\mathrm{tr}_n Y, \mathrm{tr}_n X)$

Also note that for $\mathcal{C} \in \mathrm{Cat}$ we have that $N(\mathcal{C}) \in \mathrm{Set}_\Delta$ is 2-coskeletal.

Proposition 7.10. *Let $\mathcal{C} \in \mathrm{Cat}$. The nerve of $\mathcal{C}$ is an inner Kan complex.*

Proof. Let $n \geqslant 2$ and $0 < k < n$, consider the horn extension diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook] & N(\mathcal{C}) \\
\Delta^n \arrow[ur, dashed, "\exists\hat{\sigma}?"'] &
\end{tikzcd}$$

By coskeletality of $N(\mathcal{C})$ it is enough to solve the 2-truncated extension

$$\begin{tikzcd}
\mathrm{tr}_n(\Lambda^n_k) \arrow[r, "\mathrm{tr}_2\,\sigma"] \arrow[d, "\mathrm{tr}_2(\iota)"] & \mathrm{tr}_2(N(\mathcal{C})) \\
\mathrm{tr}_n(\Delta^n) \arrow[ur, "\mathrm{tr}_2 \circ \exists\tilde{\sigma}?"'] &
\end{tikzcd}$$

For $n \geqslant 4$ we have that $\mathrm{tr}_2(\Lambda^n_k) = \mathrm{tr}_2(\Delta^n)$ and hence the problem is trivial in that case. For $n = 2$ we have that $0 < k < 2$, thus $k = 1$, so we consider the horn at 1 and the corresponding horn extension problem

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
0 \arrow[ur] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2_1 \arrow[r, "\sigma"] \arrow[d] & N(\mathcal{C}) \\
\Delta^n \arrow[ur, dashed, "\tilde{\sigma}"'] &
\end{tikzcd}$$

So σ is explicitely given as

$$\begin{tikzcd}
& X_1 \arrow[dr, "\sigma_{21}"] & \\
X_0 \arrow[ur, "\sigma_{10}"] & & X_2
\end{tikzcd}$$

so we can choose $\sigma_{21} \circ \sigma_{10}$ to complete the horn to a full simplex giving the desired horn extension. For $n = 3$ we have the $k = 1, 2$, we are going to consider the case $k = 1$ explicitly. We get the following simplex diagramm

$$\begin{tikzcd}
& X_1 \arrow[dr, "\sigma_{21}"] \arrow[dd, "\sigma_{31}"] & \\
X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"] \arrow[dr, "\sigma_{30}"'] & & X_2 \arrow[dl, "\sigma_{32}"] \\
& X_3 &
\end{tikzcd}$$

where the red triangle is not commuting a priori. The simplex gives the following identities

$$\sigma_{30} = \sigma_{32} \circ \sigma_{20} \iff \sigma_{30} = \sigma_{32} \circ (\sigma_{21} \circ \sigma_{10}) = (\sigma_{32} \circ \sigma_{21}) \circ \sigma_{10} = \sigma_{31} \circ \sigma_{10} = \sigma_{30}$$

which yields the commutativity of the bottom simplex. □

Proposition 7.11. *Let $\mathcal{C} \in \mathrm{Cat}$. Then the following are equivalent:*

1. *$\mathcal{C}$ is a groupoid,*
2. *$N(\mathcal{C})$ is a Kan complex.*

Proof. "2) $\implies$ 1)" Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. The horns

$$\begin{tikzcd} & Y \arrow[dr, dashed, "\exists g"] & \\ X \arrow[ur, "f"] \arrow[rr, "1_X"'] & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, dashed, "\exists h"] \arrow[rr, "1_Y"'] & & Y \end{tikzcd}$$

extend to 2-simplices of $N(\mathcal{C})$ since $N(\mathcal{C})$ is a Kan complex. So we get that $gf = \mathrm{id}_X$ and $fh = \mathrm{id}_Y$. Thus $\mathcal{C}$ is a groupoid. "1) $\implies$ 2)" We already know that $N(\mathcal{C})$ is an inner Kan complex. "1) $\implies$ 2)" We already know that $N(\mathcal{C})$ is an inner Kan complex. It is enough to consider the cases $n = 2$, $k = 0, 2$ and $n = 3$ $k = 0, 3$ by 2-coskeletality of $N(\mathcal{C})$. For $n = 2$ and $k = 0$ we have that the diagram

why is the inverse unique here? and why again does 1_X correspond to the identity

$$\begin{tikzcd} & X_1 & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"'] & & X_2 \end{tikzcd}$$

in $\mathcal{C}$, extends to

$$\begin{tikzcd} & X_1 \arrow[dr, dashed, "\sigma_{20} \circ \sigma_{10}^{-1}"] & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"'] & & X_2 \end{tikzcd}$$

where $\sigma_{20} \circ \sigma_{10}^{-1}$ exists since $\mathcal{C}$ is a groupoid. The case for $k = 2$ is done analogously. For $n = 3$ and $k = 2$, consider the following 2-simplex

$$\begin{tikzcd} & X_1 \arrow[dr, "\sigma_{21}"] \arrow[dd, "\sigma_{31}"] & \\ X_0 \arrow[ur, "\sigma_{10}"] \arrow[rr, "\sigma_{20}"] \arrow[dr, "\sigma_{30}"'] & & X_2 \arrow[dl, "\sigma_{32}"] \\ & X_3 & \end{tikzcd}$$

which gives the following chain of identities:

$$\sigma_{31} \circ \sigma_{10} = \sigma_{32} \circ \sigma_{21} \circ \sigma_{10} = \sigma_{32} \circ \sigma_{20} = \sigma_{30}$$

$\square$

Lecture 21.11

Example 7.12. 1. For all $X \in \mathrm{Top}$, $\mathrm{Sing}(X) \in \mathrm{Set}_\Delta$ is a Kan complex.

2. For all $\mathcal{C} \in \mathrm{Cat}$, $N(\mathcal{C})$ is an inner Kan complex.

3. $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

4. For all $n \geqslant 0$, $\Delta^n = N([n])$ is an inner Kan complex and furthermore Δ^n is a Kan complex if and only if $n = 0$.

5. If M is a monoid then $N(BM)$ is an inner Kan complex and furthermore $N(BM)$ is a Kan complex if and only if M

Definition 7.13. The category of simplicial groups is $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grp})$. Thus $X \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Grp})$ consists of the following data:

- For all $n \geqslant 0$ a group X_n with a neutral element $e_n \in X_n$.
- For all $0 \leqslant i \leqslant n$ $\begin{array}{r} d_i \colon X_n \to X_{n-1} \text{ face map} \\ s_i \colon X_n \to X_{n+1} \text{ degeneracy map,} \end{array}$ satisfy the simplicial identities and are group homomorphisms.

Proposition 7.14. *Let X be a simplicial group. Then the underlying simplicial set of X is a Kan complex.*

Proof. The case $n = 1$ is trivial. We illustrate the argument for $n = 2$ and $k = 1$. Suppose given a horn

$$\begin{tikzcd} & X_1 \arrow[dr, "x_{21}"] & \\ X_0 \arrow[ur, "x_{10}"] \arrow[rr, dashed, "\exists w"'] & & X_2 \end{tikzcd}$$

Form the degenerate 2-simplex $y =: s_0(x_{21})$

I think there is a notational flaw here, denoting the simplicial sets as well as the objects in the diagram by X_n

$$\begin{tikzcd} X_1 \arrow[dr, "x_{21}"] & \\ X_1 \arrow[u, dashed, "1_{X_1}"] \arrow[r, "x_{21}"'] & X_2 \end{tikzcd}$$

Let $z := d_2(y) = 1_{X_1} \in X_1$. Consider now $s_1(x_{10}z^{-1}) = s_1(x_{10}d_2(y)^{-1}) = d_2(y^{-1})$.

$$
s_1(x_{10}z^{-1}): \quad
\begin{tikzcd}
& X_1X_1^{-1} = e_0 \arrow[dr, "1_{e_0} = s_0(e_0) = e_1"] & \\
X_0X_1 \arrow[ur, "x_{10}1_{X_1}^{-1}"] \arrow[rr, "x_{10}1_{X_1}^{-1}"] & & X_1X_1^{-1} = e_0
\end{tikzcd}
$$

Let $w := s_1(x_{10}z^{-1})y \in X_2$ and we get the chain of equalities

$$d_0(w) = d_0((s_1(x_{10}z^{-1}))y) = d_0(s_1(x_{10}z^{-1}))d_0(y) = e_1x_{21} = x_{21}$$

as well as

$$d_2(w) = d_2(s_1(x_0z^{-1}))d_2(y) = (x_{10}1_{X_1}^{-1})1_{X_1} = x_{10}e_1 = x_{10}$$

For the general case, consider a horn

$$(x_0, x_1, \ldots, x_{k-1}, \bullet, x_{k+1}, \ldots, x_n)$$

in X. Where $x_i \in X_{n-1}$ and $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ and $i, j \neq k$. Suppose that there exists a $y \in X_n$ such that for all $0 \leqslant i < k$ and for all $l \leqslant i \leqslant n$. Then $w := s_{l-2}(x_{l-1} \cdots d_{l-1}(y)^{-1})y$ satisfies $d_i(w) = x_i$ for all $0 \leqslant i < k$ and all $l - 1 \leqslant i \leqslant n$. $\square$

Recall the Dold-Kan correspondence.

$$
\begin{tikzcd}
\tau\colon \mathrm{Ab}_\Delta \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\text{forgetfull}"'] & \mathrm{Ch}_{\leqslant}(\mathrm{Ab})\colon Dk \arrow[d, hook] & \\
\mathrm{Set}_\Delta & \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\gamma"] & D(\mathrm{Ab})
\end{tikzcd}
$$

Proposition 7.15. *Let X be a Kan complex ($X \in \mathrm{Set}_\Delta$). Then $x, y \in X_0$ satisfy $[x] = [y]$ in $\pi_0(X)$ if and only if there exists a $\sigma \in X_1$ such that $d_0(\sigma) = y$ and $d_1(\sigma) = x$.*

Proof. $\square$

tooooo tired and idk, emotionally hurt, struck ...smth. like that to get the proof

8 Fundamental groupoid revisited

Recall the adjunction $\mathrm{Set}_\Delta \underset{N}{\overset{\tau}{\rightleftarrows}} \mathrm{Gpd}$ and suppose that $X \in \mathrm{Set}_\Delta$ is a Kan complex.

Construction 8.1. The homotopy category of X is called $\mathrm{H}_0(X)$ is defined as follows:

- $\mathrm{Ob}(\mathrm{H}_0(X)) = X_0$
- $\hom_{\mathrm{H}_0(X)}(X, Y) = \{f \in X_1 \mid d_1(f) = X, d_0(f) = y\}/\sim$

Recall that a 2-simplex of X

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

should be a "witness" of the fact the h is a composition of f and g.

Problem 8.2. Why are there different choices of relation $\sim$? Given as follows

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "g"'] & & Y \end{tikzcd} = f \sim_1 g \qquad \begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Y \end{tikzcd} = f \sim_2 g$$

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd} = f \sim_3 g \qquad \begin{tikzcd} & Y \arrow[dr, "1_Y"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \end{tikzcd} = f \sim_4 g$$

where $1_X := s_0(X)$. What remains to be shown is that all these are equivalence relations and are in fact the same.

Lecture 26.11

Proposition 8.3. *The four relations above are the same and are equivalence relations.*

Proof. We show ($f \sim_3 g \implies f \sim_1 g$), thus let

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd} = \sigma \in X_2$$

We can glue the following 2-simplices

$$\begin{tikzcd} & X \arrow[dr, "1_X"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "1_X"'] & & X \end{tikzcd} = c^2(X) \qquad \begin{tikzcd} X \arrow[rr, "1_X"] \arrow[dr, "g"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd} = s_0(g)$$

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "1_X"] & \\ & X \arrow[dl, "g"] \\ Y & \end{tikzcd} = \sigma$$

to obtain a 3-horn

$$\begin{tikzcd} & X \arrow[dr, "1_X"] \arrow[dd, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, dashed, "1_X"] \arrow[dr, "g"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd} = \Lambda^3_2$$

The red 2-simplex is exactly the one of the equivalence relation $f \sim_1 g$. Since X is an inner Kan complex by assumption, we have the desired horn extension. The other direction were part of an exercise and will be included here eventually. What remains to be shown, is that it is an equivalence relation.

- (Reflexivity) Let $X \xrightarrow{f} Y$ be in X_1, then we have the 2-simplex

  $$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, "1_X"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

  which means that $\sim$ is associative.

- (Symmetry) We have the following chain of equivalences

  $$f \sim g \iff f \sim_1 g \iff f \sim_3 g \iff g \sim_1 f \iff g \sim f$$

- (Transitivity) Let $f \sim g$ and $g \sim h$. Consider the following diagrams

$$
\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "1_Y"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "f"] \arrow[dd, "f"'] & \\
& Y \arrow[dl, "1_Y"] \\
Y &
\end{tikzcd}
$$

which glued at 1_Y and f result in the following horn

$$
\begin{tikzcd}
& X \arrow[dr, "f"] \arrow[dd, "f"] & \\
X \arrow[ur, "1_X"] \arrow[rr, dashed, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "1_Y"] \\
& Y &
\end{tikzcd}
$$

By the horn filling property of the inner Kan complex for a 3-horn at position 2 we get the desired $f \sim g$.

$\square$

Proposition 8.4. *The composition law in* $\mathrm{H}_0(X)$ *is well defined, unital and associative.*

Proof. Suppose that $f \sim f'$. From now on we write $gf \sim h$ to mean that there exists a 2-simplex

$$
\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}
\in X_2
$$

We prove that if $gf \sim h$, $gf' \sim h'$ and $gf' \sim h$ it follows that $h \sim h'$. So consider the corresponding 2-simplices $\square$

i am too lazy to do the "assemble the obvious simplex proves" right now, soooo fuck you future Vincent suuuckkaaaaa

Remark 8.5. Z $\mathrm{H}_0(X)$ is a well defined category when X is an inner Kan complex.

what does this remark mean?

Proposition 8.6. *Let* $X \in \mathrm{Set}_\Delta$ *be* X *a Kan complex, then* $\mathrm{H}_0(X)$ *is a groupoid.*

Proof. Let $f \in \mathrm{H}_0(X)$ be a morphism. Consider the simplex

$$
\begin{tikzcd}
& X \arrow[dr, "f"] & \\
Y \arrow[ur, dashed, "\exists g"] \arrow[rr, "1_Y"'] & & Y
\end{tikzcd}
$$

$\square$

Corollary 8.7. *Let $X \in \mathrm{Set}_\Delta$ be a Kan complex, then $L\,\mathrm{H}_0(X) \cong \mathrm{H}_0(X)$.*

Proof. Consider the adjunction

$$\mathrm{Cat} \underset{y'}{\overset{L}{\rightleftarrows}} \mathrm{Gpd}$$

Then the counit η of the adjunction is the desired isomorphism. $\square$ details

Proposition 8.8. *Let $X \in \mathrm{Set}_\Delta$ be an inner Kan complex, then $H_0(X) \cong \tau X$.*

Proof. Let $\mathcal{D} \in \mathrm{Cat}$, since $N(\mathcal{D}) \in \mathrm{Set}_\Delta$ is 2-coskeletal, we have a natural bijection.

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(X, N(\mathcal{D})) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{tr}_2\, X, \mathrm{tr}_2(N(\mathcal{D}))$$

Consider a morphism $\mathrm{tr}_2(X) \xrightarrow{f} \mathrm{tr}_2(N(\mathcal{D}))$ given by

$$\begin{tikzcd}
\mathrm{tr}_2 \arrow[d, "f"] & X\colon \ \dots & X_2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "f_2"] & X_1 \arrow[r, shift left] \arrow[l] \arrow[d, "f_1"] & X_0 = \mathrm{Ob}(\mathrm{Ho}(X)) \arrow[l, shift left] \arrow[d, "f_0"] \\
\mathrm{tr}_2(N(\mathcal{D})) & N(\mathcal{D})\colon \ \dots & N(\mathcal{D})_2 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] & N(\mathcal{D})_1 \arrow[r, shift left] \arrow[l] & N(\mathcal{D})_0 = \mathrm{Ob}(\mathcal{D}) \arrow[l, shift left]
\end{tikzcd}$$

Note that $N(\mathcal{D})_1 = \mathrm{Mor}(\mathcal{D})$. We have for any $\alpha \in X_1$ that $f_0(d_0(\alpha)) = \mathrm{target}(f_1(\alpha))$ and that for any $d_1(\alpha) \xrightarrow{\alpha} d_0(\alpha)$ that $f_0(d_1(\alpha)) = \mathrm{source}(\alpha)$. Now for 2-simplices we have

$$\alpha \sim \beta \quad \begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ x \arrow[ur, "1_x"] \arrow[rr, "\beta"'] & & y \end{tikzcd} \quad \overset{f_2}{\longmapsto} \quad \begin{tikzcd} & f_0(x) \arrow[dr, "f_1(\alpha)"] & \\ f_0(x) \arrow[ur, "\mathrm{id}_{f_0(x)}"] \arrow[rr, "f_1(\beta)"'] & & f_0(y) \end{tikzcd}$$

Thus $f_1(\alpha) = f_1(\beta)$ which results in

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Tr}_2(X), \mathrm{Tr}_2(N(\mathcal{D})) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Ho}(X), \mathcal{D})$$

$\square$

Corollary 8.9. *Let $X \in \mathrm{Set}_\Delta$ be a Kan complex. Then $\pi_1(X) = \mathrm{Ho}(X)$ and for all $x \in X_0$ it holds that $\pi_1(X, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(X)}(x, x)$.*

Proof. We know that

$$\pi_1(X, x) = \mathrm{Hom}_{\pi_1(X)}(x, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(X)}(x, x)$$

as well as

$$\pi_1(X) = L(\tau X) \xrightarrow{\sim} L(\mathrm{Ho}(X)) \xrightarrow{\sim} \mathrm{Ho}(X).$$

$\square$ check this proof

9 Kan Fibrations

Aim: Formalize the idea of a "family of Kan complexes parametrized by a simplicial set", i.e. suitable morphisms $X \xrightarrow{p} Y$ in Set_Δ whose fibres are Kan complexes. That is we have a pullback diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

Terminology: Let $L \xrightarrow{i} K$ and $X \xrightarrow{p} Y$ be morphisms. We say that $(i \boxslash p)$ i has the left lifting property with respect to p (equivalently p has the right lifting property with respect to i). If there exists for every $L \xrightarrow{f} X$ and $K \xrightarrow{g} Y$ a morphism $h : K \to X$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ K \arrow[r, "g"] \arrow[ur, "h"] & Y \end{tikzcd}$$

Let $X \in \mathrm{Set}_\Delta$ be a Kan complex. Then

$$\{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\} \boxslash (X \to \Delta^0)$$

is given by

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & X \arrow[d, "!"] \\ \Delta^n \arrow[r, "!"'] \arrow[ur, "\tilde{\sigma}"] & \Delta^0 \end{tikzcd}$$

Definition 9.1. A morphism $X \xrightarrow{p} Y$ is a Kan fibration if

$$\{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\} \boxslash (X \xrightarrow{p} Y)$$

Remark 9.2. We may as well write $X \xrightarrow{p} Y \in \{\Lambda^n_k \to \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}^{\boxslash}$.

Lemma 9.3. *Let $i \boxslash p$ and*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

be a pullbacksquare, then $i \boxslash p'$.

Proof. Consider the diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"'] & X' \arrow[r, "u"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ K \arrow[r, "g"'] \arrow[ur, dashed, "h"] \arrow[urr, dashed, "r"] & Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

where r exists since i has the left lifting property with respect to p and h exists by the universal property of the pullback. Now $u \circ h \circ i = r \circ i = u \circ f$ and $p' \circ h \circ i = q \circ i = p' \circ f$, since the morphism given by the pullback is unique we get that $h \circ i = f$. □

Lecture 3.12

Formalise the notion of a locally constant family of Kan complexes relative to a base simplicial set.

Definition 9.4. Let $X \xrightarrow{p} Y$ be a Kan fibration. By the preceding theorem, we get that for every $y \in Y_0$ the fibre X_y is a Kan complex. Now consider Δ^0 as the simplicial set where $\Delta^0_n = \mathrm{Hom}_{\mathrm{Set}_\Delta}([n], [0]) = \{[n] \to [0]\}$ is the unique map into the final object. We have the constant map

alot missin here

Proposition 9.5. *Let $L \xrightarrow{i} K$ be a morphism and suppose that $X \xrightarrow{p} Y$, $Y \xrightarrow{g} Z$ have the right lifting property with respect to i, then the composition $g \circ p$ has the right lifting property with respect to i.*

Proof. This is just a simple matter of writing down the square for g and using the obtained morphism to write down the square p obtaining a morphism that fllfills the desired property. □

Corollary 9.6. *Suppose that $X \xrightarrow{p} Y$ is a Kan fibration and Y is a Kan complex, then X is a Kan complex.*

Proof. Since $X \xrightarrow{p} Y$ is a Kan fibration and $Y \to \Delta^0$ is a Kan fibration, we get by Proposition 9.5 that $X \to \Delta^0$ is a Kan fibration, thus we get that X is a Kan complex. □

Proposition 9.7. *Let $X^{(i)} \xrightarrow{p^i} Y^{(i)}$ with $i \in I$ be a set indexed family of Kan fibrations. Then $\prod X^{(i)} \xrightarrow{\prod p^i} \prod Y^{(i)}$ is a Kan fibration.*

Proof. Consider the diagram arising from the assumptions

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"] & \prod X^{(i)} \arrow[r, "\prod \pi_j"] \arrow[d, "\prod p^i"] & X^{(j)} \arrow[d, "p^j"] \\
\Delta^n \arrow[r, "\tau"'] \arrow[ur, dashed, "\exists ?\ h_j"'] \arrow[urr, dashed] & \prod Y^{(i)} \arrow[r, "\pi_j"'] & Y^{(j)}
\end{tikzcd}$$

Since p_j is a Kan fibration, there exists $h_j \colon \Delta^n \to X^{(j)}$ such that $\begin{cases} p_j h_j = \pi_j \tau \\ h_j \iota = \pi_j \sigma_j. \end{cases}$

The universal property of the product gives $h \colon \Delta^n \to \prod X^{(i)}$ such that $\pi_j \circ h = h_j$. Now we have that $\pi_j(h \circ \iota) = h_j \circ \iota = \pi_i \circ \sigma$ and thus $h \circ \iota = \sigma$ as well as $\pi_j(\prod p_i \circ h) = p_j \circ \pi_j \circ h = p_j h_j = \pi_j \tau$ and thus $\prod p_i \circ h = \tau$, where the equalities follow from the uniqueness of the morphism given by the pullback. □

Proposition 9.8. *Let* $\ldots \to X^{(n)} \xrightarrow{p_n} \ldots \to X^{(1)} \xrightarrow{p_1} X^{(0)}$ *be a "tower" of Kan fibrations, that is for all* i, p_i *is a Kan fibration. Then* $X^{\infty} := \lim X^{(n)} \xrightarrow{\pi_0} X^{(0)}$ *is a Kan fibration.*

Proof. Consider the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, "\iota"'] & X^{(\infty)} \arrow[d, "\pi_0"] \\ \Delta^n \arrow[r, "\tau"'] \arrow[ur, dashed, "\exists?"] & X^{(0)} \end{tikzcd}$$

The idea is to construct a cone of the tower with apex Δ^n and then use the universal property X^{∞}. □

finish this ya lazy bombaclat

10 Anodyne extensions

why dis observation, how does dis motivate anodyne extensions

Definition 10.1. We say that a class $\mathcal{M}$ of morphisms in $\mathcal{C}$ is a saturated class if

1. $\mathcal{M}$ contains all isomorphisms,
2. $\mathcal{M}$ is closed under compositions,
3. if $i : L \to K$ is in $\mathcal{M}$ then so is the pushout i' given by the pushout diagram

$$\begin{tikzcd} L \arrow[r, "\sigma"] \arrow[d, "i"] & L' \arrow[d, "i'"] \\ K \arrow[r] & K' \end{tikzcd}$$

4. $\mathcal{M}$ is closed under coproducts and
5. for any sequence of objects and morphism

$$L^{(0)} \to L^{(1)} \to L^{(2)} \to \dots$$

in $\mathcal{M}$, the morphism $L^{(0)} \to L^{(\infty)} = \operatorname{colim} L^{(n)}$ is in $\mathcal{M}$.

We say that $\mathcal{M}$ is a saturated class of monomorphisms if all $i \in \mathcal{M}$ are monomorphisms.

Definition 10.2. The saturation of a class of monos is the intersection of all such containing it. We write $\overline{\mathcal{M}}$ for its saturation.

Definition 10.3. The class of anodyne extensions is $\overline{\{\Lambda^n_k \xrightarrow{i^n_k} \Delta^n \mid 1 \leqslant n, 0 \leqslant k \leqslant n\}} =$ An.

Remark 10.4. The term anodyne extensions translates to something like harmless extensions.

Proposition 10.5. *The class* $\mathrm{An}^{\boxslash}$ *is equal to the Kan fibrations.*

Proof. ($\subseteq$) Let $p \in \mathrm{An}^{\boxslash}$, then for all $n \geqslant 0$ there exists $0 \leqslant k \leqslant n$ such that $i_{n,k} \boxslash p : p \in \mathrm{KanFib}$

($\supseteq$) Consider ${}^{\boxslash}\mathrm{KanFib}$, this is a saturated class. By unitality of $\mathrm{An} \subseteq {}^{\boxslash}\mathrm{KanFib}$ it holds that for all $i \in \mathrm{An}$ and all $p \in \mathrm{KanFib}$ $i \boxslash p$ which means that for all $p \in \mathrm{KanFib}$ we have that $p \in \mathrm{An}^{\boxslash}$ $\square$

11 Quillens small object argument

Let $\mathcal{C}$ be a category that has all small colimits, let furthermore $\mathcal{C}$ be a cocomplete category (e.g. Set_Δ) and J a set of morphisms in $\mathcal{C}$

Aim: For all f in $\mathcal{C}$ construct a factorisation under some assumption in J.

$$\begin{tikzcd}
 & \bullet \arrow[dr, "\in J^{\boxslash}"] & \\
\bullet \arrow[ur, "{}^{\boxslash}(J^{\boxslash})\ni"] \arrow[rr, "f"'] & & \bullet
\end{tikzcd}$$

That is $J = \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geqslant 1, 0 \leqslant k \leqslant n\}$ and then $J^{\boxslash}$ =Kan fibrations, or $J = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geqslant 0\}$ and then $J^{\boxslash}$ =Trivial Kan fibrations. But we also want this factorisation to be functorial.

Definition 11.1. A functorial factorisation in $\mathcal{C}$ is a section $\mathrm{Fun}([1],\mathcal{C}) \to \mathrm{Fun}([2],\mathcal{C})$ of the composition functor $\mathrm{Fun}([2],\mathcal{C}) \xrightarrow{?\circ d_1} \mathrm{Fun}([1],\mathcal{C})$ where $([1] \xrightarrow{d_1} [2] \to \mathcal{C})$.

Let us unravel the definition: For all morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$ we get a 2-simplex

$$\begin{tikzcd}
 & U(f) \arrow[dr, "Rf"] & \\
x \arrow[ur, "Lf"] \arrow[rr, "f"'] & & y
\end{tikzcd}$$

in $\mathcal{C}$. For all commutative squares

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "a"'] & y \arrow[d, "b"] \\
x' \arrow[r, "f'"'] & y'
\end{tikzcd}$$

in $\mathcal{C}$, we get a diagram

$$\begin{tikzcd}
 & U(f) \arrow[dr, "R(f)"] \arrow[dd, dashed, "U(a,b)"] & \\
x \arrow[ur, "Lf"] \arrow[rr, "f"] \arrow[dd, "a"'] & & y \arrow[dd, "b"] \\
 & U(f') \arrow[dr, "Rf'"] & \\
x' \arrow[ur, "Lf'"] \arrow[rr, "f'"'] & & y'
\end{tikzcd}$$

Definition 11.2. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ is a pair of classes of morphisms such that the following properties hold:

1. (Factorisation) For all morphism $f : x \to y$ in $\mathcal{C}$ there exists a 2-simplex

$$\begin{tikzcd} & z \arrow[dr, "\in\mathcal{R}"] & \\ x \arrow[ur, "\mathcal{L}\ni"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

,

2. (Lifting) $\mathcal{L} \boxslash \mathcal{R}$,
3. (Closure) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{L}^{\boxslash} = \mathcal{R}$.

Lemma 11.3. *The retract argument Suppose* $\begin{tikzcd} \bullet \arrow[r, "l"] \arrow[d, "f"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$ *and* $f \boxslash r$. *Then* f *is a retract of* l *as objects in the arrow category .*

Proof. Since f has the left lifting property with respect to r, we get a lift $\begin{tikzcd} \bullet \arrow[r, "l"] \arrow[d, "f"] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, equal] \arrow[ur, "w"] & \bullet \end{tikzcd}$ We can rewrite this diagram:

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "f"] & \bullet \arrow[r, equal] \arrow[d, "l"] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, dashed, "w"'] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "r"] & \bullet \end{tikzcd}$$

□

Lemma 11.4. *Suppose* $(\mathcal{L}, \mathcal{R})$ *satisfy the properties Factorisation and Lifting from Definition 11.2. Then the property Closure holds if and only if* $\mathcal{L}, \mathcal{R}$ *are closed under retracts.*

Proof. □

I am not convinced here, are we using the wrong idea of retract here?

Theorem 11.5. *Quillen's small object argument Let* $\mathcal{C}$ *be a cocomplete category,* J *a set of morphisms in* $\mathcal{C}$ *suppose that for all* $j \in J$ *we have that* $\mathrm{Hom}_{\mathcal{C}}(\mathrm{dom}\, j, -) \colon \mathcal{C} \to \mathrm{Set}$ *preserve (countable) sequential colimits (shape* $(\mathcal{N}, \leqslant)$*). Then there exists a functorial factorisation in* $\mathcal{C}$ *turning* $({}^{\boxslash}(J^{\boxslash}), J^{\boxslash})$ *into a weak factorisation system. Moreover* ${}^{\boxslash}(J^{\boxslash})$ *is the saturation of* J*.*

Proof. Let f be a morphism in $\mathcal{C}$. For $j \in J$, let $S_q(j, f) = \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right\}$

$$\coprod_{j \in J} \coprod_{S_q(j,f)} \begin{tikzcd} \bullet \arrow[r, "d_f"] \arrow[d, "j"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "c_f"'] & \bullet \end{tikzcd}$$

Consider now the pushout:

$$\begin{tikzcd} \coprod_{j\in J}\coprod_{S_q(j,f)} \bullet \arrow[r, "d_f"] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & \bullet \arrow[r, equal] \arrow[d, "Lf"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "b_f"'] & x_1 \arrow[r, "Rf"'] & \bullet \end{tikzcd}$$

By construction $Lf \in {}^{\boxslash}(J^{\boxslash})$, but we have no guarantee that $Rf \in J^{\boxslash}$. Apply now the above construction to Rf and thus we obtain a square:

$$\begin{tikzcd} x_1 \arrow[r, equal] \arrow[d, "LRf"] & x_1 \arrow[d, "Rf"] \\ x_2 \arrow[r, "R^2 f"] & x_2 \end{tikzcd}$$

Repeating the construction iteratively we obtain a diagram:

$$\begin{tikzcd} x_0 \arrow[r, "Lf"] \arrow[drr, bend right, "f"'] \arrow[rrrrr, bend left, "L^w f \in {}^{\boxslash}(J^{\boxslash})"] & x_1 \arrow[r, "LRf"] \arrow[dr, bend right, "Rf"] & x_2 \arrow[r, "LR^2 f"] \arrow[d, "R^2 f"] & x_3 \arrow[r] \arrow[dl, bend left, "R^3 f"] & \dots \arrow[r] & x_w = \mathrm{colim}_n\, x_n \arrow[dlll, bend left, "\exists! R^w f"] \\ & & \bullet & & & \end{tikzcd}$$

Where $L^w f$ is the transfinite composition of the $LR^i f$ for $i \in \mathbb{N}$ and $R^{i+1} f \circ LR^i f = R^i f$, with $R^0 f = f$.

We claim that $R^w f \in J^{\boxslash}$. Consider the square

$$\begin{tikzcd} j_w \arrow[r, "u"] \arrow[d, "j"] & x_w \arrow[d, "R^w f"] \\ \bullet \arrow[r, "v"] & \bullet \end{tikzcd}$$

□

we are going to skip the rest for now it can be found in the literature (Emily Riehl)

12 Weak equivalences of simplicial sets

Recall that for $X \in \mathrm{Set}_\Delta$ and $K \in \mathrm{Set}_\Delta$ a Kan complex then $\underline{\mathrm{Hom}}(X, K)$ is a Kan complex. Furthermore $f\colon X \to Y$ in Set_Δ is a homtopy equivalence if there exists $g\colon Y \to X$ and homotopies $g \circ f \to \mathrm{id}_X$ and $f \circ g \to \mathrm{id}_Y$.

Example 12.1. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be an adjunction $L \dashv R$. Then $N(L)\colon N(\mathcal{C} \to \mathcal{D})$ is a homotopy equivalence. Let $\eta\colon \mathbb{1}_{\mathcal{C}} \to RL$ be the unit and $\epsilon\colon LR \to \mathbb{1}_{\mathcal{D}}$ the counit of the adjunction, thus η is a morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and ϵ on $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. Thus we can also consider $\eta\colon [1] \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\epsilon\colon [1] \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$. This can again be rephrased as $\eta \in \mathrm{Fun}([1], \mathrm{Fun}(\mathcal{C}, \mathcal{C})) \cong \mathrm{Fun}([1] \times \mathcal{C}, \mathcal{C}) \ni \overline{\eta}$ and $\epsilon \in \mathrm{Fun}([1], \mathrm{Fun}(\mathcal{D}, \mathcal{D})) \cong \mathrm{Fun}([1] \times \mathcal{D}, \mathcal{D}) \ni \overline{\epsilon}$. Now $N(\overline{\eta})\colon N([1] \times \mathcal{C}) \to N(\mathcal{C})$ and we have isomorphisms $N([1] \times \mathcal{C}) \cong N([1]) \times N(\mathcal{C}) \cong \Delta^1 \times N(\mathcal{C})$. Take the nerve $N(\overline{\eta})\colon N(\mathbb{1}_{\mathcal{C}}) \to N(RL) = N(R) \circ N(L)$

$$\begin{tikzcd}
\mathcal{C} \cong [0] \times \mathcal{C} \arrow[d, "L_0 \times \mathbb{1}_{\mathcal{C}}"'] \arrow[dr, bend left, "\mathbb{1}_{\mathcal{C}}"] & \\
\times\mathcal{C} \arrow[r, "\overline{\eta}"] & \mathcal{C} \\
\overline{C} \cong [0] \times \mathcal{C} \arrow[u, "L_1 \times \mathbb{1}_{\mathcal{C}}"] \arrow[ur, bend right, "RL"'] &
\end{tikzcd}$$

Proposition 12.2. *Let $f\colon X \to Y$ be a morphism between Kan complexes, then the following are equivalent:*

1. *f is a homtopy equivalence,*
2. *for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $\pi_0(f^*)\colon \pi_0(\underline{\mathrm{Hom}}(Y, K)) \to \pi_0(\underline{\mathrm{Hom}}(X, K))$ is bijective.*

Definition 12.3. A morphism of simplicial sets $f\colon X \to Y$ is a <u>weak equivalence</u> if for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $\pi_0(f^*)\colon \pi_0(\underline{\mathrm{Hom}}(Y, K)) \to \pi_0(\underline{\mathrm{Hom}}(X, K))$ is bijective.

Lemma 12.4. *Let $f\colon X \to Y$ be a homotopy equivalence. Then for all Kan complexes $K \in \mathrm{Set}_\Delta$ the morphism $f^*\colon \underline{\mathrm{Hom}}(Y, K) \to \underline{\mathrm{Hom}}(X, K)$ is a homotopy equivalence.*

Proof. Let $h\colon \mathrm{id}_X \to g \circ f$ be a homotopy ($\Delta^1 \times X \xrightarrow{h} X$). Let $h^*\colon \underline{\mathrm{Hom}}(X, K) \to \underline{\mathrm{Hom}}(\Delta^1 \times X, K) \cong \underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(X, K))$ be the morphism induced by h on the inner Hom simplicial sets, so $h^* \in \underline{\mathrm{Hom}}(\Delta^1 \times \underline{\mathrm{Hom}}(X, K), \underline{\mathrm{Hom}}(X, K))$. Thus h^* gives a homotopy between $(\mathrm{id}_X)^* = \mathrm{id}_{\underline{\mathrm{Hom}}(X,K)}$ and $(g \circ f)^* = f^* \circ g^*$. □

Corollary 12.5. *Let $f\colon X \to Y$ be a homotopy equivalence of simplicial sets, then f is a weak equivalence.*

Proof. It is an application of Lemma 12.4 and applying the definition of the weak equivalence. □

Proposition 12.6. *Let $i\colon X \hookrightarrow K$ be an anodyne extension, then i is a weak equivalence.*

Proof. Let K be a Kan complex, then $i^*\colon \underline{\mathrm{Hom}}(Y,K) \to \mathrm{Hom}(X,K)$ is a trivial Kan fibration, then it is a homtopy equivalence, thus π_0 is an iso. $\square$

find the statements

Proposition 12.7. *2 out of 3 Weak equivalences satisfy the 2 out of 3 property. That is for every commutative diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

if 2 out of the morphisms f, g and $g \circ f$ are weak equivalences, then so is the third.

Proof. Let $K \in \mathrm{Set}_\Delta$ be a Kan complex. $\square$

too tired to understand the argument here, häääähhhhh

Proposition 12.8. *Let $f^{(i)}\colon X^{(i)} \to Y^{(i)}$ be a family of weak equivalences, indexed over the set I. Then $\coprod_{i\in I} f^{(i)}$ is a weak equivalence.*

Proof. Let K be a Kan complex

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\coprod Y^{(i)}, K) \arrow[r, "(\coprod f^{(i)})^*"] \arrow[d, "\wr"] & \underline{\mathrm{Hom}}(\coprod Y^{(i)}, K) \arrow[d, "\wr"] \\ \prod \underline{\mathrm{Hom}}(Y^{(i)}, K) \arrow[r, "(f^{(i)*})_i"'] & \prod \underline{\mathrm{Hom}}(X^{(i)}, K) \end{tikzcd}$$

Then $\underline{\mathrm{Hom}}(Y^{(i)}, K)$ is a Kan complex, since π_0 preserves all small coproducts of Kan complexes, we are done. $\square$

Proposition 12.9. *Let $f^{(i)}\colon X^{(i)} \to Y^{(i)}$ be a family of weak equivalences of Kan complexes, indexed over a set I. Then $\prod f^{(i)}$ is a weak equivalence. By* **??** *all weak equivalences here are homotopy equivalences.*

Proof. Let K be a Kan complex, then $\prod X^{(i)}$ and $\prod Y^{(i)}$ are Kan complexes. Then we have the following diagram

$$\begin{tikzcd} \pi_0(\underline{\mathrm{Hom}}(K, \prod X^{(i)})) \arrow[r] \arrow[d, "\wr"] & \pi_0(\underline{\mathrm{Hom}}(K, \prod Y^{(i)})) \arrow[d, "\wr"] \\ \pi_0(\prod \underline{\mathrm{Hom}}(K, X^{(i)})) \arrow[r] \arrow[d] & \pi_0(\prod \underline{\mathrm{Hom}}(K, Y^{(i)})) \arrow[d] \\ \prod \pi_0(\underline{\mathrm{Hom}}(K, X^{(i)})) \arrow[r, "sim"] & \prod \pi_0(\underline{\mathrm{Hom}}(K, Y^{(i)})) \end{tikzcd}$$

$\square$

Remark 12.10. For a finite product we do not need homtopy equivalence neither Kan complexes for the statement to hold.

Proposition 12.11. *Let $f\colon X \to Y$ be a weak equivalence. Then for all Kan complexes K the morphism $f^*\colon \underline{\mathrm{Hom}}(Y,K) \to \underline{\mathrm{Hom}}(X,K)$ is a homotopy equivalence.*

Proof. Let $W \in \mathrm{Set}_\Delta$ be a Kan complex. Then

$$\begin{tikzcd}
\pi_0(\underline{\mathrm{Hom}}(W,\underline{\mathrm{Hom}}(Y,K))) \arrow[r, "f^*\circ ?"] \arrow[d, "\wr"] & \pi_0(\underline{\mathrm{Hom}}(W,\underline{\mathrm{Hom}}(X,K))) \arrow[d, "\wr"] \\
\pi_0(\underline{\mathrm{Hom}}(Y,\underline{\mathrm{Hom}}(W,K))) \arrow[r, "\pi_0(f^*)"] & \pi_0(\underline{\mathrm{Hom}}(X,\underline{\mathrm{Hom}}(W,K)))
\end{tikzcd}$$

Now all the inner Hom simplicial sets are Kan complexes and we are done by applying **??**. ☐

Corollary 12.12. *Suppose that $f\colon X \to Y$ admits a factorisation*

$$\begin{tikzcd}
 & Z \arrow[dr, "\in \textit{Triv. Kan fibrations}"] & \\
X \arrow[ur, "\mathrm{An}\ni"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Then f is a weak equivalence.

Proof. Anodyne extensions and trivial Kan fibrations are weak equivalences, f is the composition of two weak equivalences hence a weak equivalence by Proposition 12.7. ☐

Let $X \in \mathrm{Set}_\Delta$ such that

$$\begin{tikzcd}
 & \tilde{X} \arrow[dr, "\in\mathrm{KanFib}"] & \\
X \arrow[ur, "\mathrm{An}\ni"] \arrow[rr, "f"'] & & \Delta^0
\end{tikzcd}$$

then X is weakly equivalent to the Kan complex $\tilde{X}$. Can do this functorially by small object argument.

dk, what this means

Lecture 14.1

Correction

13 Whitehead's theorem for Kan complexes

Lemma 13.1. *Suppose that $f\colon (X,x) \to (Y,y)$ is a morphism for pointed Kan complexes, such that $f\colon X \to Y$ admits a left inverse up to homotopy (inverse in* hKan*). Then f admits a pointed inverse up to homotopy (inverse in* hKan_**).*

Proof. Let $g\colon Y \to X$ be a homotopy left inverse to f, so there exists a homotopy $h\colon \mathrm{id}_X \to g \circ f$. Let now $\alpha\colon \Delta^n \times X \to X$ and extend the homotopy diagram by the morphism associated to the selected point:

$$\begin{tikzcd}
\Delta^{\{0\}} \times \Delta^0 \arrow[r, "x"] \arrow[d] & \Delta^{\{0\}} \times X \arrow[d] \arrow[dr, bend left, "id_X"] & \\
e\colon \Delta^1 \times \Delta^0 \arrow[r, "\mathrm{id} \times x"] & \Delta^1 \times X \arrow[r, "\alpha"] & X \\
\Delta^{\{1\}} \times \Delta^0 \arrow[r, "\mathrm{id} \times x"] \arrow[u] & \Delta^{\{1\}} \times X \arrow[u] \arrow[ur, bend right, "g \circ f"'] &
\end{tikzcd}$$

Where $e\colon x \to g(f(x)) = g(y)$ in X. Take $\Delta^0 \xrightarrow{y} Y$ and the lifting diagram

$$\begin{tikzcd}
\Lambda^1_1 = \Delta^{\{1\}} \arrow[r, "g"] \arrow[d, hook] & \underline{\mathrm{Hom}}(Y,X) \arrow[d, "\mathrm{ev}_y"] \\
\Delta^1 \arrow[r, "e"] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(\Delta^0,X) \cong X
\end{tikzcd}$$

The lifting morphism together with the standard adjunction gives a homotopy $\beta\colon \Delta^1 \times Y \to X$ from g' to g where $g'(y) = x$. We can concetenate β with f, to obtain $\beta_f\colon \Delta^1 \times X \xrightarrow{\mathrm{id} \times f} \Delta^1 \times Y \xrightarrow{\beta} X$ which is a homotopy of the concetenation $\beta_f\colon g' \circ f \to g \circ f$. We can now consider the homotopy diagram of β extended by the selected point in Y:

$$\begin{tikzcd}
\Delta^{\{0\}} \times \Delta^0 \arrow[r, "\mathrm{id} \times y"] \arrow[d] & \Delta^{\{0\}} \times Y \arrow[d] \arrow[dr, bend left, "g'"] & \\
e\colon \Delta^1 \times \Delta^0 \arrow[r, "\mathrm{id} \times y"] & \Delta^1 \times Y \arrow[r, "\beta"] & X \\
\Delta^{\{1\}} \times \Delta^0 \arrow[r, "\mathrm{id} \times y"] \arrow[u] & \Delta^{\{1\}} \times Y \arrow[u] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

□

Do not understand the rest of this proof rn

Corollary 13.2. *Let $f\colon (X,x) \to (Y,y)$ be a morphism of Kan complexes such that $f\colon X \to Y$ is a homotopy equivalence then f is also a pointed homotopy equivalence.*

Proof. By Lemma 13.1 $[f]$ admits a pointed left inverse $g\colon (Y,y) \to (X,x)$ and $[g] \circ [f] = [\mathrm{id}_X]$ in hKan_*. Then $g\colon Y \to X$ is a homotopy equivalence. There exists a homotopy left inverse $h\colon (X,x) \to (Y,y)$, $[h] = [h]([g][f]) = [f]$ □

don't see the argument here as well

Corollary 13.3. *Let $f\colon X \to Y$ be a homtopy equivalence. Then for all $x \in X$ and all $n \geqslant 0$, there is an isomorphism $\pi_n(f)\colon \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y, f(x))$.*

Proof. By Corollary 13.2 $f\colon (X,x) \to (Y, f(x))$ is a pointed homotopy equivalence. Now $\pi_n(X,x) = \pi_n(\underline{\mathrm{Hom}}((S^n,*),(X,x)))$, similarly for $(Y, f(x))$ ☐

Definition/Proposition 13.4. Let X be a Kan complex, then the following are equivalent:

1. $X \to \Delta^0$ is a homotopy equivalence,
2. for all $x \in X$ and all $n \geqslant 0$, $\pi_n(X,x) = \{*\}$,
3. $X \to \Delta^0$ is a trivial Kan fibration.

In this case we say that X is <u>contractible</u>.

Proof. Exercise ☐

Proposition 13.5. *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes, then the following are equivalent:*

1. *p is a trivial Kan fibration,*
2. *p is a homotopy equivalence,*
3. *for all $x \in X$ and for all $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ is bijective,*
4. *for all $y \in Y$ the fibre X_y given by the pullback*

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

   is contractible.

Proof.

1. $\implies$ 2. Exercise

2. $\implies$ 3. homtopy inverse

3. $\implies$ 4. Serre long exact sequence

4. $\implies$ 1. Suppose that for all $y \in Y$ we have that $X_y \to \Delta^0$ is a trivial Kan fibration, which is by Definition/Proposition 13.4 equivalent to contractibility of X_y. Take a boundary inclusion

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"] & Y \end{tikzcd}$$

and consider a homotopy H from the constant map to the identity on Δ^n, that is $H\colon \Delta^1 \times \Delta^n \to \Delta^n$, where $H\colon c(0) \to \mathrm{id}_{\Delta^n}$. Putting together the diagram for the homotopy H and the n-simplex β we obtain:

$$
\begin{tikzcd}
\Delta^{\{0\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[d] & \Delta^{\{0\}} \times \Delta^n \arrow[d] \arrow[dr, bend left, "c(0)"] \arrow[drr, bend left, "c(y)"] & & \\
e\colon \Delta^1 \times \partial\Delta^n \arrow[r, "\mathrm{id} \times i"] & \Delta^1 \times \Delta^n \arrow[r, "H"] & \Delta^n \arrow[r, "\beta"] & Y \\
\Delta^{\{1\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[u] & \Delta^{\{1\}} \times \Delta^n \arrow[u] \arrow[ur, bend right, "\mathrm{id}_{\Delta^n}"'] \arrow[urr, bend right, "\beta"'] & &
\end{tikzcd}
$$

consult Goerss Jardine

□

Theorem 13.6. *Whitehead's theorem Let $f\colon X \to Y$ be a morphsim between Kan complexes. Then the following are equivalent:*

1. *f is a homotopy equivalence,*
2. *for all $x \in X$ and all $n \geqslant 0$, $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,y)$ is a bijection.*

Proof.

1. $\implies$ 2. This is known.

2. $\implies$ 1. By the use of Quillen's small object argument Theorem 11.5, we obtain a factorisation of f

$$
\begin{tikzcd}
 & \tilde{X} \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] & & Y
\end{tikzcd}
$$

where $i \in \mathrm{An}$ and $p \in \mathrm{KanFib}$. Since $i\colon X \to \tilde{X}$ is anodyne, i is a weak equivalence and by Proposition 12.2, using that X and Y are Kan complexes, i is also a homotopy equivalence and thus satisfies 2. By Proposition 12.7 p also satisfies the property 2. and by the previous proposition p is a homotopy equivalence.

□