Homotopy Theory of Simplicial Sets

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1 Whitehead's theorem for Kan complexes

Lemma 1.1. *Suppose that $f\colon (X,x)\to(Y,y)$ is a morphism for pointed Kan complexes, such that $f\colon X\to Y$ admits a left inverse up to homotopy (inverse in* hKan*). Then f admits a pointed inverse up to homotopy (inverse in* hKan_**).*

Proof. Let $g\colon Y\to X$ be a homotopy left inverse to f, so there exists a homotopy $h\colon \mathrm{id}_X\to g\circ f$. Let now $\alpha\colon \Delta^1\times X\to X$ be a homtopy and extend the homotopy diagram by the morphism associated to the selected point:

$$\begin{tikzcd}
\Delta^{\{0\}}\times\Delta^0 \arrow[r, "x"] \arrow[d] & \Delta^{\{0\}}\times X \arrow[d] \arrow[dr, bend left, "id_X"] & \\
e\colon \Delta^1\times\Delta^0 \arrow[r, "\mathrm{id}\times x"] & \Delta^1\times X \arrow[r, "\alpha"] & X \\
\Delta^{\{1\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times x"] \arrow[u] & \Delta^{\{1\}}\times X \arrow[u] \arrow[ur, bend right, "g\circ f"'] &
\end{tikzcd}$$

Where $e\colon x\to g(f(x))=g(y)$ in X. Take $\Delta^0 \xhookrightarrow{y} Y$ and the lifting diagram

$$\begin{tikzcd}
\Lambda^1_1=\Delta^{\{1\}} \arrow[r, "g"] \arrow[d, hook] & \underline{\mathrm{Hom}}(Y,X) \arrow[d, "\mathrm{ev}_y"] \\
\Delta^1 \arrow[r, "e"] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(\Delta^0,X)\cong X
\end{tikzcd}$$

The lifting morphism together with the standard adjunction gives a homotopy $\beta\colon \Delta^1\times Y\to X$ from g' to g where $g'(y)=x$. We can concetenate β with f, to obtain $\beta_f\colon \Delta^1\times X\xrightarrow{\mathrm{id}\times f}\Delta^1\times Y\xrightarrow{\beta}X$ which is a homotopy of the concetenation $\beta_f\colon g'\circ f\to g\circ f$. We can now consider the homotopy diagram of β extended by the selected point in Y:

$$\begin{tikzcd}
\Delta^{\{0\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] \arrow[d] & \Delta^{\{0\}}\times Y \arrow[d] \arrow[dr, bend left, "g'"] & \\
e\colon \Delta^1\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] & \Delta^1\times Y \arrow[r, "\beta"] & X \\
\Delta^{\{1\}}\times\Delta^0 \arrow[r, "\mathrm{id}\times y"] \arrow[u] & \Delta^{\{1\}}\times Y \arrow[u] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

The next step is to take the degenerate 2-simplex $s_0(e)$ given by

$$\begin{tikzcd}
x \arrow[dr, "e"] & \\
x \arrow[u, "1_X"] \arrow[r, "1_X"'] & g(f(x))
\end{tikzcd}$$

and take it as the bottom row map in the following lifting problem

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "q"] \arrow[d] & \underline{\mathrm{Hom}}(X,X) \arrow[d, "\mathrm{ev}_x"] \\
\Delta^2 \arrow[r, "s_0(e)"'] & \underline{\mathrm{Hom}}(\Delta^0,X)
\end{tikzcd}$$

where q is given by the following horn diagram.

$$\begin{tikzcd}
g'f \arrow[dr, "\beta_f"] & \\
\mathrm{id}_X \arrow[u, dashed] \arrow[r, "\alpha"'] & gf
\end{tikzcd}$$

Since ev_x is a Kan fibration, we obtain a lift and thus a pointed homotopy $\gamma : \mathrm{id}_X \to g'f$, thus g' is a pointed homotopy inverse to f. $\square$

Corollary 1.2. *Let $f\colon (X,x) \to (Y,y)$ be a morphism of Kan complexes such that $f\colon X \to Y$ is a homotopy equivalence then f is also a pointed homotopy equivalence.*

Proof. By **??** 1.1 $[f]$ admits a pointed left inverse $g\colon (Y,y) \to (X,x)$ and $[g] \circ [f] = [\mathrm{id}_X]$ in hKan_*. Then $g\colon Y \to X$ is a homotopy equivalence. There exists a homotopy left inverse $h\colon (X,x) \to (Y,y)$, $[h] = [h]([g][f]) = [f]$ $\square$

Corollary 1.3. *Let $f\colon X \to Y$ be a homotopy equivalence. Then for all $x \in X$ and all $n \geqslant 0$, there is an isomorphism $\pi_n(f)\colon \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y,f(x))$.*

Proof. By **??** 1.2 $f\colon (X,x) \to (Y,f(x))$ is a pointed homotopy equivalence. Now $\pi_n(X,x) = \pi_n(\underline{\mathrm{Hom}}((S^n,*),(X,x))$, similarly for $(Y,f(x))$ $\square$

Definition/Proposition 1.4. Let X be a Kan complex, then the following are equivalent:

1. $X \to \Delta^0$ is a homotopy equivalence,
2. for all $x \in X$ and all $n \geqslant 0$, $\pi_n(X,x) = \{*\}$,
3. $X \to \Delta^0$ is a trivial Kan fibration.

In this case we say that X is contractible.

Proof. Exercise $\square$

Proposition 1.5. *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes, then the following are equivalent:*

1. *p is a trivial Kan fibration,*
2. *p is a homotopy equivalence,*

3. *for all $x \in X$ and for all $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$ is bijective,*

4. *for all $y \in Y$ the fibre X_y given by the pullback*

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

is contractible.

Proof.

1. $\implies$ 2. Exercise

2. $\implies$ 3. homtopy inverse

3. $\implies$ 4. Serre long exact sequence

4. $\implies$ 1. Suppose that for all $y \in Y$ we have that $X_y \to \Delta^0$ is a trivial Kan fibration, which is by **??** 1.4 equivalent to contractibility of X_y. Take a boundary inclusion

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"] & Y \end{tikzcd}$$

and consider a homotopy H from the constant map to the identity on Δ^n, that is $H\colon \Delta^1 \times \Delta^n \to \Delta^n$, where $H\colon c(0) \to \mathrm{id}_{\Delta^n}$. Putting together the diagram for the homotopy H and the n-simplex β we obtain:

$$\begin{tikzcd} \Delta^{\{0\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[d] & \Delta^{\{0\}} \times \Delta^n \arrow[d] \arrow[dr, bend left, "c(0)"] \arrow[drr, bend left, "c(y)"] & & \\ e\colon \Delta^1 \times \partial\Delta^n \arrow[r, "\mathrm{id} \times i"] & \Delta^1 \times \Delta^n \arrow[r, "H"] & \Delta^n \arrow[r, "\beta"] & Y \\ \Delta^{\{1\}} \times \partial\Delta^n \arrow[r, "i"] \arrow[u] & \Delta^{\{1\}} \times \Delta^n \arrow[u] \arrow[ur, bend right, "\mathrm{id}_{\Delta^n}"'] \arrow[urr, bend right, "\beta"'] & & \end{tikzcd}$$

$$\begin{tikzcd} \overline{\alpha} \arrow[r, "\tilde{h}"] \arrow[dr, phantom, "\downarrow p_*"] & \alpha \\ c(y) \arrow[r, "h"'] & p_\alpha \end{tikzcd} \qquad \begin{tikzcd} \Delta^{\{1\}} \arrow[r, "\alpha"] \arrow[d] & \underline{\mathrm{Hom}}(\partial\Delta^n, X) \arrow[d, "p_*"] \\ \Delta^1 \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & \underline{\mathrm{Hom}}(\partial\Delta^n, Y) \end{tikzcd}$$

where $\tilde{h}\colon \Delta^1 \times \partial\Delta^n \to X, h\colon \overline{\alpha} \to \alpha, \overline{\alpha}\colon \Delta^n \to X$ and $p \circ \overline{\alpha} = c(y)$.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, dashed, "\exists"] \arrow[d, "i"] \arrow[rr, bend left, "\overline{\alpha}"] & X_y \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\mathrm{PB}"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "!"] \arrow[ur, dashed] \arrow[urr] \arrow[rr, bend right, "c(y)"'] & \Delta^0 \arrow[r, "y"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^{\{0\}} \times \partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "\alpha"] & (\Delta^{\{0\}} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) \arrow[r, "{(v,\tilde{h})}"] \arrow[d, hook, "\in \mathrm{An}"] & X \arrow[d, "p"] \\
\Delta^{\{0\}} \times \Delta^n \arrow[r] \arrow[rr, bend right, "\beta"'] & \Delta^1 \times \Delta^n \arrow[r, "\beta\circ H"] \arrow[ur] & Y
\end{tikzcd}$$

□

Theorem 1.6. *Whitehead's theorem Let $f\colon X \to Y$ be a morphsim between Kan complexes. Then the following are equivalent:*

1. *f is a homotopy equivalence,*
2. *for all $x \in X$ and all $n \geqslant 0$, $\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y,y)$ is a bijection.*

Proof.

1. $\Longrightarrow$ 2. This is known.

2. $\Longrightarrow$ 1. By the use of Quillen's small object argument **??**, we obtain a factorisation of f

$$\begin{tikzcd}
& \tilde{X} \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

where $i \in \mathrm{An}$ and $p \in \mathrm{KanFib}$. Since $i\colon X \to \tilde{X}$ is anodyne, i is a weak equivalence and by **??**, using that X and Y are Kan complexes, i is also a homotopy equivalence and thus satisfies 2. By **??** p also satisfies the property 2. and by the previous proposition p is a homotopy equivalence.

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