Homotopy Theory of Simplicial Sets

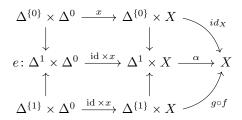
Vincent Siebler

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1 Whitehead's theorem for Kan complexes

Lemma 1.1. Suppose that $f: (X, x) \to (Y, y)$ is a morphism for pointed Kan complexes, such that $f: X \to Y$ admits a left inverse up to homotopy (inverse in hKan). Then f admits a pointed inverse up to homotopy (inverse in hKan*).

Proof. Let $g: Y \to X$ be a homotopy left inverse to f, so there exists a homotopy $h: \mathrm{id}_X \to g \circ f$. Let now $\alpha: \Delta^1 \times X \to X$ be a homotopy and extend the homotopy diagram by the morphism associated to the selected point:



Where $e: x \to g(f(x)) = g(y)$ in X. Take $\Delta^0 \stackrel{y}{\hookrightarrow} Y$ and the lifting diagram

$$\begin{array}{ccc} \Lambda_1^1 = \Delta^{\{1\}} & \xrightarrow{g} & \underline{\operatorname{Hom}}(Y,X) \\ & & & \downarrow^{\operatorname{ev}_y} \\ \Delta^1 & \xrightarrow{e} & \underline{\operatorname{Hom}}(\Delta^0,X) \cong X \end{array}$$

The lifting morphism together with the standard adjunction gives a homotopy $\beta \colon \Delta^1 \times Y \to X$ from g' to g where g'(y) = x. We can concetenate β with f, to obtain $\beta_f \colon \Delta^1 \times X \xrightarrow{\operatorname{id} \times f} \Delta^1 \times Y \xrightarrow{\beta} X$ which is a homotopy of the concetenation $\beta_f \colon g' \circ f \to g \circ f$. We can now consider the homotopy diagram of β extended by the selected point in Y:

The next step is to take the degenerate 2-simplex $s_0(e)$ given by

$$\begin{array}{c}
x \\
\downarrow \\
x \xrightarrow{1_X} g(f(x))
\end{array}$$

and take it as the bottom row map in the following lifting problem

$$\begin{array}{ccc} \Lambda_2^2 & \stackrel{q}{\longrightarrow} & \underline{\operatorname{Hom}}(X,X) \\ \downarrow & & \downarrow^{\operatorname{ev}_x} \\ \Delta^2 & \xrightarrow[s_0(e)]{} & \underline{\operatorname{Hom}}(\Delta^0,X) \end{array}$$

where q is given by the following horn diagram.

$$g'f$$

$$\downarrow \qquad \qquad \qquad \beta_f$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\mathrm{id}_X \xrightarrow{\alpha} gf$$

Since ev_x is a Kan fibration, we obtain a lift and thus a pointed homotopy $\gamma:\operatorname{id}_X\to g'f$, thus g' is a pointed homotopy inverse to f.

Corollary 1.2. Let $f:(X,x) \to (Y,y)$ be a morphism of Kan complexes such that $f:X \to Y$ is a homotopy equivalence then f is also a pointed homotopy equivalence.

Proof. By ?? 1.1 [f] admits a pointed left inverse $g: (Y,y) \to (X,x)$ and $[g] \circ [f] = [\mathrm{id}_X]$ in hKan*. Then $g: Y \to X$ is a homotopy equivalence. There exists a homotopy left inverse $h: (X,x) \to (Y,y), [h] = [h]([g][f]) = [f]$

Corollary 1.3. Let $f: X \to Y$ be a homotopy equivalence. Then for all $x \in X$ and all $n \ge 0$, there is an isomorphism $\pi_n(f): \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x))$.

Proof. By ?? 1.2 $f: (X, x) \to (Y, f(x))$ is a pointed homotopy equivalence. Now $\pi_n(X, x) = \pi_n(\underline{\text{Hom}}((S^n, *), (X, x)), \text{ similarly for } (Y, f(x))$

Definition/Proposition 1.4. Let X be a Kan complex, then the following are equivalent:

- 1. $X \to \Delta^0$ is a homotopy equivalence,
- 2. for all $x \in X$ and all $n \ge 0$, $\pi_n(X, x) = \{*\}$,
- 3. $X \to \Delta^0$ is a trivial Kan fibration.

In this case we say that X is contractible.

Proof. Exercise
$$\Box$$

Proposition 1.5. Let $p: X \to Y$ be a Kan fibration between Kan complexes, then the following are equivalent:

- 1. p is a trivial Kan fibration,
- 2. p is a homotopy equivalence,

- 3. for all $x \in X$ and for all $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is bijective,
- 4. for all $y \in Y$ the fibre X_y given by the pullback

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow^p \\ \Delta^0 & \xrightarrow{y} & Y \end{array}$$

is contractible.

Proof.

 $1. \implies 2.$ Exercise

 $2. \implies 3.$ homtopy inverse

 $3. \implies 4.$ Serre long exact sequence

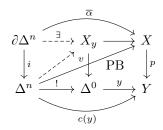
4. \Longrightarrow 1. Suppose that for all $y \in Y$ we have that $X_y \to \Delta^0$ is a trivial Kan fibration, which is by ?? 1.4 equivalent to contractibility of X_y . Take a boundary inclusion

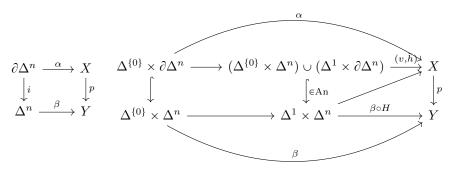
$$\begin{array}{ccc}
\partial \Delta^n & \stackrel{\alpha}{\longrightarrow} X \\
\downarrow & & \downarrow^p \\
\Delta^n & \stackrel{\beta}{\longrightarrow} Y
\end{array}$$

and consider a homotopy H from the constant map to the identity on Δ^n , that is $H: \Delta^1 \times \Delta^n \to \Delta^n$, where $H: c(0) \to \mathrm{id}_{\Delta^n}$. Putting together the diagram for the homotopy H and the n-simplex β we obtain:

$$\begin{array}{c} \Delta^{\{0\}} \times \partial \Delta^{n} \stackrel{i}{\longrightarrow} \Delta^{\{0\}} \times \Delta^{n} & \overbrace{^{c(0)}}^{c(y)} \\ \downarrow & \downarrow & \downarrow \\ e \colon \Delta^{1} \times \partial \Delta^{n} \stackrel{\operatorname{id} \times i}{\longrightarrow} \Delta^{1} \times \Delta^{n} & \stackrel{H}{\longrightarrow} \Delta^{n} \stackrel{\beta}{\longrightarrow} Y \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \Delta^{\{1\}} \times \partial \Delta^{n} \stackrel{i}{\longrightarrow} \Delta^{\{1\}} \times \Delta^{n} & \underbrace{^{\{1\}}}^{id_{\Delta^{n}}} & \underbrace{^{\{1\}}}^{id_{\Delta^{n$$

where $\tilde{h}: \Delta^1 \times \partial \Delta^n \to X, h: \overline{\alpha} \to \alpha, \overline{\alpha}: \Delta^n \to X$ and $p \circ \overline{\alpha} = c(y)$.



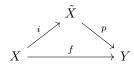


Theorem 1.6. Whitehead's theorem Let $f: X \to Y$ be a morphsim between Kan complexes. Then the following are equivalent:

- 1. f is a homotopy equivalence,
- 2. for all $x \in X$ and all $n \ge 0$, $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$ is a bijection.

Proof.

- 1. \implies 2. This is known.
- 2. \Longrightarrow 1. By the use of Quillen's small object argument $\ref{eq:quillen}$, we obtain a factorisation of f



where $i \in An$ and $p \in KanFib$. Since $i: X \to \tilde{X}$ is anodyne, i is a weak equivalence and by ??, using that X and Y are Kan complexes, i is also a homotopy equivalence and thus satisfies 2. By ?? p also satisfies the property 2. and by the previous proposition p is a homotopy equivalence.