Itic notes

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Contents

1 Motivation

Notice that many proofs of statements in the lecture are contained as Exercises, which I still have to add at the current point in time. If you want solutions to any of the Exercises you may contact the author.

Fix $0 \leqslant m \leqslant n \leqslant \infty$. An (n, m) category is a "category-like" structure consisting of a class of objects, notions of 1-morphism, 2-morphism, ... , n-morphism (i.e. k-morphsim $0 < k \leqslant n$) with a "suitable composition law" (satisfying "suitable axioms") and such that $\forall m < k \leqslant n$ the k-morphisms are "invertible".

$$\begin{aligned}
&(0,0)\text{-cat.} = \text{set}\\
&(1,0)\text{-cat.} = \text{groupoid, i.e. 1-groupoid}\\
&(1,1)\text{-cat.} = \text{category, i.e. 1-category}\\
&(2,0)\text{-cat.} = \text{2-groupoids}\\
&(2,1)\text{-cat.}\\
&(2,2)\text{-cat.} = \text{2-categories}\\
&\vdots\\
&(n,0)\text{-cat.} = \text{n-groupoid}\\
&(n,n)\text{-cat.} = \text{n-cat.}\\
&\vdots\\
&(\infty,0) = \infty\text{-groupoids}\\
&(\infty,1) = \infty\text{-categories (see. Boardman-Vogt)}
\end{aligned}$$

Reminder 1.1. *A map $f \colon X \to Y$ between topological spaces is a weak homotopy equivalence if $\forall x \in X, \forall n \in \mathbb{N}$ the map*

$$\pi_n(f) \colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection.

Theorem 1.2 (Grothendieck's Homotopy Hypothesis)**.** *There is an $(\infty, 1)$-category of topological spaces up to weak homotopy equivalence and there is an $(\infty, 1)$-category of ∞-groupoids up to eqivalence. There is furthermore an ∞-functor assigning to each topological space X its Poincare ∞-groupoid $\pi_\infty(X)$, this is an equivalence.*

Remark 1.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category, then for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an ∞-groupoid/ "space". We have the homotopy category of $\mathcal{C}$ denoted by $\mathrm{Ho}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and for all objects $X, Y \in \mathrm{Ho}(\mathcal{C})$ we have that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$.

Warning! 1.4. The passage from $\mathcal{C}$ often results in a tremendous loss of information, that is essential for various purposes.

- Computing co-/limits within $\mathcal{C}$.
- Computing co-/limits with $\mathcal{C}$.
- Define invariants associated to $\mathcal{C}$ (f.e. Hochschild cohomology).

Reminder 1.5. *Many important 1-categories arise as homotopy categories of genuine $(\infty, 1)$-categories, for example derived categories. Recall for a ring R, mod_R its category of right R-modules and $\mathrm{Ch}(\mathrm{mod}_R)$ the category of chain complexes in mod_R, that a morphism of chain complexes $f^\bullet\colon X^\bullet \to Y^\bullet$ in $\mathrm{Ch}(\mathrm{mod}_R)$ is a quasi-isomorphism if $\forall n \in \mathbb{Z}$ we have that $H_n(f^\bullet) = H_n(X^\bullet) \xrightarrow{\sim} H_n(Y^\bullet)$ is an isomorphism. The derived category is defined as follows $D(\mathrm{mod}_R) := \mathrm{Ch}(\mathrm{mod}_R)[qiso^{-1}]$, i.e. the localisation at the quasi-isomorphisms. Furthermore we have that $\mathrm{Ho}(\mathcal{D}(\mathrm{mod}_R)) = D(\mathrm{mod}_R)$. In the first case, that is the right side of the equality above, we obtain the derived category by building it from the ground up so to say and in the second case, the left side of the equation, we obtain it by forgetting information from a higher structure.*

Warning! 1.6. The homotopy theory of $(\infty, 1)$-categories has many equivalent implementations (Quillen):

- Topological categories (Ilias)
- Simplicial categories (Bergner)
- Complete Segal spaces (Rezk)
- Relative categories (Barwick-Kan)
- Pre-derivations
- ∞-categories (Joyal, Lurie)

In the k-linear setting, for k a field we have:

1. Differentially graded k-categories
2. A_∞-categories (Lefèvre-Hasegawa)

The Plan for the lecture is to start of with investigating 2-categories, then give definition and examples of ∞-categories, then do enriched category theory and end on the homotopy theory of ∞-categories.

2 Heuristics: Isomorphism vs. Equivalence

The reference for this section is [2, ch.4, 5, 6].
Makkai's Principle of Isomorphism (1998) says:

"All grammatically correct properties about objects in a fixed category are to be invariant under isomorphism."

Fix a category $\mathcal{C}$ and a small category A and take the functor category $\mathrm{Fun}(A, \mathcal{C})$, that is A-shaped diagrams in $\mathcal{C}$. For $X \in \mathcal{C}$ define:

$$\lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) := \{(p_a \colon X \to D(a))_{a \in A} \mid \begin{tikzcd} X \arrow[d, "f_a"'] \arrow[dr, "f_b"] & \\ D(a) \arrow[r] & D(b) \end{tikzcd} \quad \forall f \colon a \to b \text{ in } A\}$$

For $\phi \colon Y \to X$ define the function

$$\begin{aligned} \phi^* \colon \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) &\to \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \\ p = (p_a \colon X \to D(a))_{a \in A} &\mapsto \phi^*(p) = (p_a \circ \phi \colon Y \to D(a))_{a \in A} \end{aligned}$$

Thus $\lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf of sets on $\mathcal{C}$.

Definition 2.1. A limit of a diagram $D \colon A \to \mathcal{C}$ is a cone $p \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ that is universal in the sense that $\forall Y \in \mathcal{C}, \forall q \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)), \exists! \varphi \colon Y \to X$ such that $\varphi^*(p) = q$. We write $\lim_{a \in A} D(a)$ for any limit of D (which may or may not exist).

Reminder 2.2 (Yoneda Lemma)**.** *Let $X \in \mathcal{C}$ and let*

$$\begin{aligned} \nu \colon \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, X), \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))) &\xrightarrow{\sim} \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) \\ \eta = (\eta_Y \colon \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \lim_{a \in \mathcal{C}}(Y, D(a))_{Y \in \mathcal{C}} &\mapsto \eta_X(\mathrm{id}_X) \\ (\eta^p_Y(p) := \varphi^*(p))_{Y \in \mathcal{C}} &\leftarrow\!\shortmid p \end{aligned}$$

If $p \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ is a limit of $D \colon A \to \mathcal{C}$ then $\eta^p \colon \mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, D(a))$ is a natural isomorphism. We furthermore obtain, that for an isomorphism $\psi \colon Y \to X$ in $\mathcal{C}$ we have

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(-, X) \arrow[dr, "\eta^p", "\sim"'] & \\ \mathrm{Hom}_{\mathcal{C}}(-, Y) \arrow[ur, "\psi^*", "\sim"'] \arrow[rr, "\sim"] & & \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) \end{tikzcd}$$

Example 2.3. Let the following be a diagram in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

then the limit of the diagram (if it exists) is called a pullback.

$$\begin{tikzcd}
W \arrow[dr, "\exists! p"] \arrow[drr, bend left, "\forall f''"] \arrow[ddr, bend right, "\forall g''"'] & & \\
& X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

For example if $\mathcal{C} = \text{Set}$ then $X \times_Z Y = \{(X, Y) \in X \times Y \mid f(x) = g(y)\}$.

The commutativity condition takes place in $\text{Hom}_{\mathcal{C}}(X \times_Z Y, Z) \ni g \circ f' = f \circ g'$

Makkai's Principle of Equivalence: All grammatically correct properties of objects in a fixed 2-category are to be invariant under equivalence.

Remark 2.4. We want Cat to be the strict 2-category of (small) categories with functors as 1-morphisms and natural transformations as 2-morphisms. Now natural transformation allow for a notion of equivalence of morphisms, that is in a 1-category we only knew what it meant for two morphisms to be equal, but now we can talk about two functors being naturally isomorphic giving us a notion of equivalence of 1-morphisms, via 2-morphisms.

Definition/Proposition 2.5 (Godement Product)**.** Consider natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"{name=A}] \arrow[r, bend right=50, "F_2"'{name=B}] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E}
\end{tikzcd}$$

Their Godement product is the natural transformation.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=50, "G_1 \circ F_1"] \arrow[rr, bend right=50, "G_2 \circ F_2"'] \arrow[rr, phantom, "\Downarrow \beta * \alpha"] & & \mathcal{E}
\end{tikzcd}$$

Let $X \in \mathcal{C}$, we obtain the following diagram

$$\begin{tikzcd}
F_1(X) \arrow[d, "\alpha_x"] & G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta * \alpha)_X"] & G_2(F_1(X)) \arrow[d, "G_2(\alpha_X)"] \\
F_2(X) & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"'] & G_2(F_2(X))
\end{tikzcd}$$

in $\mathcal{D}$.

Proof. We show that $\beta * \alpha\colon G_1 \circ F_1 \Rightarrow G_2 \circ F_2$ is indeed a natural transformation. For that we take the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & G_1(F_1(X)) \arrow[r, "G_1(\alpha_X)"] \arrow[d, "G_1(F_1(f))"'] & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(F_2(f))"] & G_2(F_2(X)) \arrow[d, "G_2(F_2(f))"] \\
Y & G_1(F_1(Y)) \arrow[r, "G_1(\alpha_Y)"'] & G_1(F_2(Y)) \arrow[r, "\beta_{F_2(Y)}"'] & G_2(F_2(Y))
\end{tikzcd}$$

the inner squares commute by the naturality of α and β, thus the outer square commutes, meaning it is a natural transformation. $\square$

Proposition 2.6. *Consider natural transformations*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F_1"] \arrow[r, "F_2"'] \arrow[r, bend right=70, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma"] & \mathcal{D} \arrow[r, bend left=70, "G_1"] \arrow[r, "G_2"'] \arrow[r, bend right=70, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & \mathcal{E}
\end{tikzcd}$$

Then $(\delta\beta) * (\gamma\alpha) = (\delta * \gamma) \circ (\beta * \alpha)$.

Proof. Let $X \in \mathcal{C}$

$$\begin{tikzcd}
G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta*\alpha)_X"] \arrow[dd, bend right=60, "G_1(\gamma_X\alpha_X)"'] & G_2(F_1(X)) \arrow[r, "\delta_{F_1(X)}"] \arrow[d, "G_2(\alpha_X)"] & G_3(F_1(X)) \arrow[d, "G_3(\alpha_X)"] \arrow[dd, bend left=60, "G_3(\gamma_X\alpha_X)"] \\
G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(\gamma_X)"'] & G_2(F_2(X)) \arrow[r, "\delta_{F_2(X)}"] \arrow[d, "G_2(\gamma_X)"] \arrow[dr, "(\delta*\gamma)_X"] & G_3(F_2(X)) \arrow[d, "G_3(\gamma_X)"] \\
G_1(F_3(X)) \arrow[r, "\beta_{F_3(X)}"] & G_2(F_3(X)) \arrow[r, "\delta_{F_2(X)}"] & G_3(F_3(X))
\end{tikzcd}$$

Now the long diagonal of the diagram corresponds to $(\delta * \gamma) \circ (\beta * \alpha)$ and the outer large square to $(\delta \circ \beta) * (\gamma \circ \alpha)$. $\square$

Definition 2.7. The Godement products bellow are called whiskerings

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} & \mapsto & \mathcal{C} \arrow[r, bend left=50, "G_1 \circ F"] \arrow[r, bend right=50, "G_2 \circ F"'] \arrow[r, phantom, "\Downarrow\beta * \mathrm{id}_F"] & \mathcal{E}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd}
\mapsto
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "G\circ F_1"] \arrow[rr, bend right, "G\circ F_2"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_G * \alpha"] & & \mathcal{E}
\end{tikzcd}
$$

Construction 2.8. Given a cospan of groupoids

$$
\begin{tikzcd}
& \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
$$

its 2-pullback is the diagonal of groupoids along on $\mathcal{C}$ inside the product.

$$
\begin{tikzcd}
\mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
$$

The objects are given as $\mathrm{Ob}(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}) = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi\colon F(a) \xrightarrow{\sim} G(b) \text{ in } \mathcal{C})$ and morphisms are given by tuples of morphisms $(u, v)\colon (a, b, \varphi) \to (a', b', \varphi')$, where $u\colon a \to a'$ and $v\colon b \to b'$ are morphisms in the respective groupoids, such that the following square commutes:

$$
\begin{tikzcd}
F(a) \arrow[r, "\varphi", "\sim"'] \arrow[d, "F(a)"'] & G(b) \arrow[d, "G(v)"] \\
F(a') \arrow[r, "\varphi'", "\sim"'] & G(b')
\end{tikzcd}
$$

Lecture 15.04

Let $X, Y, Z \in \mathrm{Set}$ and consider a pullback diagram:

$$
\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}
$$

Where the fiber product is given by $X \times_Z Y := \{(x, y) \in X \times Y, f(x) = g(x)\}$ such that the following isomorphism holds for all W.

$$
\mathrm{Hom}_{\mathrm{Set}}(^\forall W, X \times_Z Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(W, X) \times_{\mathrm{Hom}_{\mathrm{Set}}(W,Z)} \mathrm{Hom}_{\mathrm{Set}}(W, Y)
$$

$$
(\varphi\colon X \times_Z Y) \mapsto
\begin{tikzcd}
W \arrow[r, "\pi_Y \circ \varphi"] \arrow[d, "\pi_X \circ \varphi"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

Thus the case of a pullback of objects is clear, but what does the pullback of morphism-sets look like?

Construction 2.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be groupoids and

$$\begin{tikzcd}
\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{A}}"] \arrow[d, "\pi_{\mathcal{B}}"'] \arrow[dr, phantom, "\overset{\phi}{\underset{\sim}{\Longrightarrow}}"] & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

be the 2-pullback of $\mathcal{A} \xrightarrow{F} \mathcal{C}$. Its objects are triples $X = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi_X : F(a) \xrightarrow{\sim} G(b))$ and for another triple $X' = (a' \in \mathcal{A}, b' \in \mathcal{B}, \varphi_{X'} \colon F(a') \xrightarrow{\sim} G(b'))$ the morphisms are given by tuples $(\mathcal{A} \ni u \colon a \to a', \mathcal{B} \ni v \colon b \to b')$ such that

$$\begin{tikzcd}
F(a) \arrow[r, "\sim"] \arrow[d, "F(u)"] & G(b) \arrow[d, "G(v)"] \\
F(a') \arrow[r, "\sim"] & G(b')
\end{tikzcd}$$

commutes, that is $\varphi_{X'} \circ F(u) = G(v) \circ \varphi_X$. For a groupoid $\mathcal{D}$ we may consider the induced cospan of groupoids:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{D}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\sim}{\underset{2-PB}{\Longrightarrow}}"] & \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[d, "G\circ ?"] \\
\mathrm{Fun}(\mathcal{D}, \mathcal{A}) \arrow[r, "F\circ ?"'] & \mathrm{Fun}(\mathcal{D}, \mathcal{C})
\end{tikzcd}$$

Since functors into a groupoid are again a groupoid we can apply the construction above, i.e. the 2-pullback of groupoids.

Interlude 2.10. Fix a groupoid G. Then, the construction $\mathcal{D} \in \mathrm{Gpd} \mapsto \mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is suitably functorial, which means

- For all $D \in \mathrm{Gpd}$ it holds that $\mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is a groupoid.
- For all $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ it holds that $? \circ F \colon \mathrm{Fun}(\mathcal{D}_2, \mathcal{G}) \to \mathrm{Fun}(\mathcal{D}_1, \mathcal{G})$ is a functor, given on morphisms as:

$$\begin{tikzcd}
\mathcal{D}_2 \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{G} & \mapsto & \mathcal{D}_1 \arrow[r, bend left, "G_1 \circ F"] \arrow[r, bend right, "G_2 \circ F"'] \arrow[r, phantom, "\Downarrow \beta * \mathrm{id}_F"] & \mathcal{G}
\end{tikzcd}$$

- For a natural transformation α of functors between groupoids,

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D}_2 \end{tikzcd}$$

  there is a natural transformation:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}_2, \mathcal{G}) \arrow[r, bend left, "? \circ F_1"] \arrow[r, bend right, "? \circ F_2"'] \arrow[r, phantom, "\Downarrow ? * \alpha"] & \mathrm{Fun}(\mathcal{D}_1, \mathcal{G}). \end{tikzcd}$$

Let us take the 2-pullback $\mathbb{F}\mathcal{D} := \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{B})$, see 2.9 and analyze the map $\mathcal{D} \to \mathbb{F}\mathcal{D}$.

- For all groupoids $\mathcal{D}$ the category $\mathbb{F}\mathcal{D}$ is a groupoid.
- For all morphisms of groupoids $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ the attribution $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_1$ is a functor.
- The objects of $\mathbb{F}\mathcal{D}_1$ are given as $(p_{\mathcal{A}}\colon \mathcal{D}_1 \to \mathcal{A}, p_{\mathcal{B}}\colon \mathcal{D}_1 \to \mathcal{B}, \phi\colon F \circ p_{\mathcal{A}} \xrightarrow{\sim} G \circ p_{\mathcal{B}})$, that is the 2-pullback, to be more specific they are given by the datum of a pullback diagram:

$$\begin{tikzcd} \mathcal{D}_1 = \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"'] \arrow[dr, phantom, "\xRightarrow{\phi}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

- For all functors $H\colon \mathcal{D}_1 \to \mathcal{D}_1$ we get $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_2$, that is for a second pullback square

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"'] \arrow[dr, phantom, "\xRightarrow{\psi}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

we obtain a commutative diagram as follows:

$$\begin{tikzcd}
\mathcal{D}_1 \arrow[dr, "H"] \arrow[drr, bend left, "q_{\mathcal{B}} \circ H"] \arrow[ddr, bend right, "q_{\mathcal{A}} \circ H"'] & & \\
& \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

The associated natural transformation is given by $\psi_H \colon F \circ q_{\mathcal{A}} \circ H \implies G \circ q_{\mathcal{B}} \circ H$.

The next question we can naturally ask is, what do morphisms in $\mathbb{F}\mathcal{D}$ look like? They are given by quadruples natural transformations $(\alpha, \beta, F_\alpha, G_\beta)$, such that:

$$\begin{tikzcd}
(p_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{A}, \arrow[d, Rightarrow, "\alpha"] & p_{\mathcal{B}} \colon \mathcal{D}_2 \longrightarrow \mathcal{B}, \arrow[d, Rightarrow, "\beta"] & F \circ p_{\mathcal{A}} \overset{\phi}{\underset{\sim}{\Longrightarrow}} \arrow[d, Rightarrow, "F_\alpha"] & G \circ p_{\mathcal{B}}) \arrow[d, Rightarrow, "G_\beta"] \\
(q_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{A}, & q_{\mathcal{B}} \colon \mathcal{D}_2 \longrightarrow \mathcal{B}, & F \circ q_{\mathcal{A}} \overset{\sim}{\underset{\psi}{\Longrightarrow}} & G \circ q_{\mathcal{B}})
\end{tikzcd}$$

For every $\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left=50, "H_1"{name=U}] \arrow[r, bend right=50, "H_2"'{name=D}] & \mathcal{D}_2 \arrow[from=U, to=D, Rightarrow, "\gamma"] \end{tikzcd}$ there is a diagram $\begin{tikzcd} \mathbb{F}\mathcal{D}_2 \arrow[r, bend left=50, "\mathbb{F}H_1"{name=U}] \arrow[r, bend right=50, "\mathbb{F}H_2"'{name=D}] & \mathbb{F}\mathcal{D}_1 \arrow[from=U, to=D, Rightarrow, "\mathbb{F}\gamma"] \end{tikzcd}$.

Proposition 2.11. *Let $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ be a cospan of groupoids. Then for all groupoids $\mathcal{D}$, it holds that*

$$\mathbb{X} \colon \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \to \mathbb{F}(\mathcal{D} \xleftarrow{\mathbb{F}H} \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}))$$

$$(\mathcal{D} \xrightarrow{H} \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \mapsto \begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[rr, dashed, bend left] \arrow[dr, dashed, bend right] & \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd} = H^*(\mathrm{can})$$

is an isomorphism.

Example 2.12. Let A, B, C be groups and $\mathbb{B}A, \mathbb{B}B, \mathbb{B}C$ their associated groupoids, then for group homomorphisms $A \xrightarrow{f} C \xleftarrow{g} B$, we get that the objects of $\mathbb{B}A \times^{(2)}_{\mathbb{B}C} \mathbb{B}B$ correspond to triples $c_1 = (*_A, *_B, *_C \xrightarrow{c \in C} *_C)$ and a morphism from c_1 to $c_2 = (*_A, *_B, *_C \xrightarrow{c' \in C} *_C)$ from correspond to choices $a \in A, b \in B$ such that $c'a = bc$.

Lecture 17.04

3 2-categories

Reference for this section is [3, ch. 2.2.1- 2.2.3].

Definition 3.1. A **strict 2-category** $\mathcal{C}$ consists of:

1. A class $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$
2. For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose objects $f\colon X \to Y$ are called 1-morphisms and whose morphisms $\alpha\colon f \Rightarrow g$ are called 2-morphisms, with a vertical composition of 2-morphisms, that is associative, and unital.
3. For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a horizontal composition functor
$$- \circ -\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
that is compatible with the vertical composition, in the following way
$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z & \mapsto & X \arrow[r, bend left=50, "g\circ f"] \arrow[r, bend right=50, "g'\circ f'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & Z \end{tikzcd}$$
4. Functoriality of horizontal composition:
$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_g"] & Z & \mapsto & X \arrow[r, bend left=50, "g\circ f"] \arrow[r, bend right=50, "g\circ f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{g\circ f}"] & Z \end{tikzcd}$$
5. There is a composition
$$\begin{tikzcd} X \arrow[r, bend left=80, "f_1"] \arrow[r, "f_2"'] \arrow[r, bend right=80, "f_3"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma"] & Y \arrow[r, bend left=80, "g_1"] \arrow[r, "g_2"'] \arrow[r, bend right=80, "g_3"'] \arrow[r, phantom, shift left=4, "\Downarrow\beta"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & Z & \mapsto & X \arrow[r, bend left=50, "g_1\circ f_1"] \arrow[r, bend right=50, "g_3\circ f_3"'] \arrow[r, phantom, "\Downarrow\eta"] & Z \end{tikzcd}$$
where η is the Godement product $(\delta * \gamma) \circ (\beta * \alpha)$.

The above data should satisfy the following axioms:

- (Unitality) For all $X \in \mathrm{Ob}(\mathcal{C})$ there exists $\mathrm{id}_X \in \mathrm{Ob}(\mathrm{Hom}_{\mathcal{C}}(X, X))$ an identity 1-morphism such that for all $Y \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(X,X) \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f \mapsto (f, \mathrm{id}_X) \mapsto f \circ \mathrm{id}_X \end{aligned}$$

  is equal to the identity functor $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Similarly the functor

$$\begin{aligned} \mathbb{1} \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(X,X) \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X) \\ g \mapsto \mathrm{id}_X \circ g = g \end{aligned}$$

  is equal to the identity functor $\mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ the following square of functors commutes <u>strictly</u>.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(-\circ-)"] \arrow[d, "(-\circ-)\times\mathrm{Hom}_{\mathcal{C}}(W,X)"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[d, "(-\circ-)"] \\ \mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(-\circ-)"] & \mathrm{Hom}_{\mathcal{C}}(W,Z) \end{tikzcd}$$

Definition 3.2. A **strict monoidal category** is a strict 2-category with a single object, that is the following data:

1. A strict 2-category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$.
2. A category $\mathcal{M} := \mathrm{Hom}_{B\mathcal{M}}(*, *)$.
3. A monoidal composition $-\otimes-\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, that fullfills the following axioms.

   - (Unitality) There exists $\mathbb{1}_{\mathcal{M}} \in \mathrm{Ob}(\mathcal{M}) = \mathrm{Ob}(Hom_{B\mathcal{M}}(*, *))$ such that the functor

$$\begin{aligned} \mathcal{M} \to \mathcal{M} \\ M \mapsto M \otimes \mathbb{1}_{\mathcal{M}} \end{aligned}$$

     is the identity, meaning there are equalities $M \otimes \mathbb{1}_{\mathcal{M}} = M = \mathbb{1}_{\mathcal{M}} \otimes M$.
   - (Associativity) The following square commutes

$$\begin{tikzcd} \mathcal{M} \times \mathcal{M} \times \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{M}} \times(-\circ-)"] \arrow[d, "(-\otimes-)\times\mathrm{id}_{\mathcal{M}}"'] & \mathcal{M} \times \mathcal{M} \arrow[d, "\otimes"] \\ \mathcal{M} \times \mathcal{M} \arrow[r, "-\otimes-"] & \mathcal{M} \end{tikzcd}$$

     that is for all $M_1, M_2, M_3 \in \mathcal{M}$ it holds that $M_1 \otimes (M_2 \otimes M_3) = (M_1 \otimes M_2) \otimes M_3$.

Definition 3.3. Let $\mathcal{C}, \mathcal{D}$ be strict 2-categories. A **strict 2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of

1. A map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$ sending an object X to $F(X)$.
2. For all $X, Y \in \operatorname{Ob}(\mathcal{C})$ a functor $F_{XY}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(FX, FY)$ such that the following hold:
   - (Unitality) $\forall X \in \operatorname{Ob}(\mathcal{C}), F(\mathrm{id}_X) = \mathrm{id}_{FX} \in \operatorname{Hom}_{\mathcal{D}}(FX, FX)$
   - (Composition) Let $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "- \circ^{\mathcal{C}} -"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"] & \operatorname{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\ \underline{\operatorname{Hom}}_{\mathcal{D}}(FY,FZ) \times \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FY) \arrow[r, "- \circ^{\mathcal{D}} -"] & \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FZ) \end{tikzcd}$$

Let $\mathcal{C}$ be a strict 2-category then the opposite category $\mathcal{C}^{\mathrm{op}}$ is also a strict 2-category.

Lecture 22.04

Remark 3.4. Every ordinary 1-category $\mathcal{C}$ can be viewed as a strict 2-category as follows:

- for all $X, Y \in \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(X,Y)$,
- horizontal composition = composition in $\mathcal{C}$,
- for all $X \in \operatorname{Ob}(\mathcal{C})$ it holds that $\mathrm{id}_X \in \operatorname{Hom}_{\mathcal{C}}(X,X) = \operatorname{Hom}_{\mathcal{C}}(X,X)$ is the identity morphism in the original catgory $\mathcal{C}$,
- conversely every strict 2-catgory $\mathcal{C}$ has an underlying ordinary category $\mathcal{C}_0$ with $\operatorname{Ob}(\mathcal{C}_0) := \operatorname{Ob}(\mathcal{C})$ and $\forall X, Y \in \operatorname{Ob}(\mathcal{C}_0) = \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}_0}(X,Y) := \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,Y))$. The composition law in $\mathcal{C}_0$ is horizontal composition of 1-morphisms in $\mathcal{C}$, this composition is associative since $\mathcal{C}$ is a strict 2-category.

Definition 3.5. A **2-category** (bicategory) $\mathcal{C}$ is given as follows:

- a class of objects of $\mathcal{C}$ denoted $\operatorname{Ob}(\mathcal{C})$,
- for all $X, Y \in \operatorname{Ob}\mathcal{C}, \operatorname{Hom}_{\mathcal{C}}(X,Y)$ a category of 1-morphisms,
- for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ a composition functor

$$(\text{-} \circ \text{-})\colon \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Z),$$

- for all $X \in \operatorname{Ob}(\mathcal{C})$ an object $\mathrm{id}_X \in \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,X))$, called identity of X, together with an invertible 2-morphism $\upsilon_X = \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$ in $\operatorname{Hom}_{\mathcal{C}}(X,X)$ called **unit constraint**,

- for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a natural isomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(\text{-}\circ\text{-})"] \arrow[d, "(\text{-}\circ\text{-})\times \mathrm{Hom}_{\mathcal{C}}(W,X)"'] \arrow[dr, phantom, "\sim\Downarrow\alpha=\alpha_{W,X,Y,Z}"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,Y) \arrow[d, "(\text{-}\circ\text{-})"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(\text{-}\circ\text{-})"'] & \mathrm{Hom}_{\mathcal{C}}(W,Z)
\end{tikzcd}$$

  where

$$\alpha_{f,g,h} \colon h \circ (g \circ f) \overset{\sim}{\Longrightarrow} (h \circ g) \circ f \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$$

  is an isomorphism called **associativity constraint**,

- for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the following functors are fully faithful

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f &\mapsto \mathrm{id}_Y \circ f \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f &\mapsto f \circ \mathrm{id}_X \end{aligned}$$

- and for all $V, W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all composable 1-morphisms $V \xrightarrow{e} W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ the following diagram commutes

$$\begin{tikzcd}
 & h \circ ((g \circ f) \circ e) \arrow[rr, Rightarrow, "\alpha_{h,g\circ f,e}"] & & (h \circ (g \circ f)) \circ e \arrow[dr, Rightarrow, "\alpha_{h,g,f} * \mathrm{id}_e"] & \\
h \circ (g \circ (f \circ e)) \arrow[ur, Rightarrow, "\mathrm{id}_h * \alpha_{g,f,e}"] \arrow[drr, Rightarrow, "\alpha_{h,g,f\circ e}"] & & & & ((h \circ g) \circ f) \circ e \\
 & & (h \circ g) \circ (f \circ e) \arrow[urr, Rightarrow, "\alpha_{h\circ g,f,e}"] & &
\end{tikzcd}$$

  which is called the **Pentagon identity**.

Example 3.6. Every strict 2-category can be viewed as a 2-category with unit constraints and associativity constraints given by identities.

Definition 3.7. Monoidal categories are 2-categories with a single object. That is a 2 category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$ and $\mathcal{M} := \underline{\mathrm{Hom}}_{B\mathcal{M}}(*,*)$. The horizontal composition defines the monoidal composition

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

and there is an associativity constraint $\alpha_{M_1,M_2,M_3} \colon M_1 \otimes (M_2 \otimes M_3) \xrightarrow{\sim} (M_1 \otimes M_2) \otimes M_3$ in $\mathcal{M}$.

Example 3.8. Let k be a field and $(\mathrm{Vect}_k, \otimes_k, k)$ a monoidal category, the associator is given as:

$$\begin{aligned} \mathrm{can} \colon V_1 \otimes (V_2 \otimes V_3) &\xrightarrow{\sim} (V_1 \otimes V_2) \otimes V_3 \\ v_1 \otimes (v_2 \otimes v_3) &\mapsto (v_1 \otimes v_2) \otimes v_3 \end{aligned}$$

Example 3.9. Let V be a category with finite products, then $(V, x, *)$ is a monoidal category, with $*$ its terminal object and a functor $- \times -\colon V \times V \to V$, such that $V_1 \times (V_2 \times V_3) \xrightarrow{\sim} (V_1 \times V_2) \times V_3$.

Example 3.10. The 2-category Bim of all bimodules has

- Objects Ob(Bim) given by all associative unital rings,
- for $R, S \in \mathrm{Ob}(\mathrm{Bim}), \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R, S) := {}_S\mathrm{mod}_R \simeq L\,\mathrm{Fun}({}_R\mathrm{mod}, {}_S\mathrm{mod})$,
- for all $R, S, T \in \mathrm{Ob}(\mathrm{Bim})$ the horizontal composition is given by the functor

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathrm{Bim}}(S,T) \times \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,S) \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,T) \arrow[d, equal] \\
{}_T\mathrm{mod}_S \times {}_S\mathrm{mod}_R \arrow[r] & {}_T\mathrm{mod}_R
\end{tikzcd}$$

  $({}_TM_S, {}_SN_R) \mapsto ({}_TM \otimes_S N_R)$

- For all $R \in \mathrm{Ob}(\mathrm{Bim}), \mathrm{id}_R = {}_RR_R \in {}_R\mathrm{mod}_R = \underline{\mathrm{Hom}}\,\mathrm{Bim}(R, R)$.

$$\begin{gathered}
{}_RR \otimes_R R_R \xrightarrow{\sim} {}_RR_R \\
{}_UL \otimes_T (M \otimes_S N) \xrightarrow{\sim}_{\mathrm{can}} ({}_UL \otimes_T M_S) \otimes_S N_R
\end{gathered}$$

Construction 3.11. Let $\mathcal{C}$ be a 2-category, there is a fully faithful functor

$$\begin{gathered}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \\
f \mapsto \mathrm{id}_Y \circ f
\end{gathered}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and there is a bijection of morphisms called the **left unit constraint**:

$$\mathrm{id}_Y \circ ?\colon \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{id}_Y \circ f, f) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)(\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f), \mathrm{id}_Y \circ f)$$

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\exists! \lambda_f"] & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim"] & & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\sim"', "v_Y * \mathrm{id}_f"] \\
f \arrow[u, phantom, "\mapsto", shift left=8ex] & & \mathrm{id}_Y \circ f &
\end{tikzcd}$$

furthermore there is a fully faithful functor:

$$\begin{gathered}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \\
f \mapsto f \circ \mathrm{id}_X
\end{gathered}$$

as well as a bijection of morphism called the **right unit constraint**:

$$\begin{tikzcd}
f \circ \mathrm{id}_X \arrow[d, Rightarrow, "\exists! \rho_f"] & f \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_f * v_X"'] & & (f \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "\sim"'] \\
f \arrow[u, phantom, "\mapsto", shift left=8ex] & & f \circ \mathrm{id}_X &
\end{tikzcd}$$

Proposition 3.12. *Let $\mathcal{C}$ be a 2-category. The left and right unit constraints determine natural isomorphisms.*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[rr, bend left=50, "f \mapsto \mathrm{id}_Y \circ f"] \arrow[rr, bend right=50, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow \lambda"] & & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[rr, bend left=50, "f \mapsto f \circ \mathrm{id}_X"] \arrow[rr, bend right=50, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow \rho"] & & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}$$

Lecture 24.4

Proof. Exercise. Let $\forall\colon X \to Y, \lambda_f$ is an isomorphism. We only prove $\lambda = (\lambda_f\colon \mathrm{id}_Y \circ f \Rightarrow f)_{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)}$ is a natural transformation. Let

$$f \overset{\eta}{\Longrightarrow} g$$

be a morphism in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"'] & f \arrow[d, Rightarrow, "\eta"] & & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[r, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & \mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"] \\
\mathrm{id}_Y \circ g \arrow[r, Rightarrow, "\lambda_g"] & g \arrow[r, phantom, ""] \arrow[ur, phantom, "\xrightarrow{\mathrm{id}_Y \circ -}"] & & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[r, Rightarrow, "\mathrm{id}_Y \circ \lambda_g"] & \mathrm{id}_Y \circ g
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[dr, Rightarrow, "\alpha", "\sim"'] \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & & \mathrm{id} \circ f \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ \eta = \mathrm{id}_{\mathrm{id}_Y} * \eta"] \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[ur, Rightarrow, "v_y * \mathrm{id}_f", "\sim"'] \arrow[d, Rightarrow, "(\mathrm{id}_Y \circ \mathrm{id}_Y) \circ \eta"] & \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ g \arrow[dr, Rightarrow, "v_Y * \mathrm{id}_g", "\sim"'] & \\
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[ur, Rightarrow, "\sim", "\alpha"'] \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_g"] & & \mathrm{id}_Y \circ g
\end{tikzcd}$$

where the left square commutes by the naturality of the associator constraint, and the top and bottom triangle commute by the left unit constraint. For the right square we use the interchange law for composition and the Godement product to obtain $(\mathrm{id}_{\mathrm{id}_Y} * \eta) \circ (v_Y * \mathrm{id}_f) = (v_\eta * \eta) = (v_Y * \mathrm{id}_g) \circ (\mathrm{id}_Y \circ \eta)$ $\square$

Proposition 3.13. *Let $\mathcal{C}$ be a 2-category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ two composable 1-morphisms in $\mathcal{C}$. Then the following triangle*

$$\begin{tikzcd}
g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_g \times \lambda_f"'] & & (g \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\rho_g \times \mathrm{id}_f"] \\
& g \circ f &
\end{tikzcd}$$

commutes.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& g \circ ((\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "v_Y"] & & (g \circ (\mathrm{id}_Y \circ \mathrm{id}_Y)) \circ f \arrow[d, Rightarrow, "u_Y"] \arrow[ddr, Rightarrow, "\alpha"] & \\
& g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "\alpha", "\sim"'] & & (g \circ id_Y) \circ f & \\
g \circ (\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f)) \arrow[uur, Rightarrow, "\alpha"] \arrow[ur, Rightarrow, "\lambda_f"'] \arrow[drr, Rightarrow, "\alpha"'] & (g \circ \mathrm{id}_Y) \circ f \arrow[r, Rightarrow, "\rho_g", "\sim"'] \arrow[urr, equal] & g \circ f & g \circ (\mathrm{id}_Y \circ f) \arrow[l, Rightarrow, "\lambda_f"', "\sim"] \arrow[u, Rightarrow, "\alpha", "\sim"'] & ((g \circ \mathrm{id}_Y) \circ \mathrm{id}_Y) \circ f \arrow[ul, Rightarrow, "\rho_g"'] \\
& & (g \circ \mathrm{id}_Y) \circ (\mathrm{id}_Y \circ f) \arrow[ul, Rightarrow, "\lambda_f"] \arrow[ur, Rightarrow, "\rho_g"'] \arrow[urr, Rightarrow, "\alpha"'] \arrow[u, phantom, "*)"] & &
\end{tikzcd}$$

The triangles commute by applying unit constraints 3.11 and the square commute and the squares that include an alpha commute by the associator constraints. The only square that remains is $*$), here we use the interchange law for the Godement product to obtain $(\rho_g * \mathrm{id}_f)(\mathrm{id}_g * \lambda_f) = \rho_g * \lambda_f = (\mathrm{id}_g * \lambda_f) \circ (\rho_g * \mathrm{id}_f)$. $\square$

Corollary 3.14. *Let $\mathcal{C}$ be a 2-category and $X \in \mathcal{C}$, consider $\mathrm{id}_X \colon X \to X$. Then*

$$\lambda_{\mathrm{id}_X} \colon \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

$$\rho_{\mathrm{id}_X} \colon \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

are both equal to $v_X \colon \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$.

Proof. We only do the case $\lambda_{\mathrm{id}_X} = v_X$. By the triangle identity and definition of λ_{id_X} we get that

$$\begin{tikzcd}
\mathrm{id}_X \circ(\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_X \circ\lambda_{\mathrm{id}_X}"'] & & (\mathrm{id}_X \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "v_X * \mathrm{id}_X"] \arrow[dl, Rightarrow, bend left, "\rho_{\mathrm{id}_X} * \mathrm{id}_X"] \\
& \mathrm{id}_X \circ \mathrm{id}_X &
\end{tikzcd}$$

and thus $v_X * \mathrm{id}_X = \rho_{\mathrm{id}_X} * \mathrm{id}_X$ which implies that $v_X = \rho_{\mathrm{id}_X}$ since the composition with the identity is fully faithful. $\square$

Definition 3.15. Let $\mathcal{C}$ be a 2-category. The conjugate of $\mathcal{C}$ is the 2-category $\mathcal{C}^c = \mathcal{C}^{co}$ with $\mathrm{Ob}(\mathcal{C}^c) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^c}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)^{\mathrm{op}}$.

Definition 3.16. A $(2, 1)$-category is a 2-category such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a groupoid.

Definition 3.17. Let $\mathcal{C}$ be a 2-category. The coarse homotopy category of $\mathcal{C}$ is the 1-ccategory $h\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(\mathcal{C})$ and with sets of morphisms, $\mathrm{Hom}_{h\mathcal{C}}(X, Y) = \pi_0(L\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) = \pi_0(N\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y))$ with the induced composition law, where L is the localisation functor from Cat to Gpd.

Definition 3.18. Let $\mathcal{C}$ be a 2-category. The pith of $\mathcal{C}$ is the 2-category $\mathrm{Pith}(\mathcal{C})$ with objects $\mathrm{Ob}(\mathrm{Pith}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ with $\underline{\mathrm{Hom}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) := \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\cong}$, where $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\cong}$ is the maximal subgroupoid of $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$.

Definition 3.19. The homotopy category of $\mathcal{C}$ is $\mathrm{hPith}(\mathcal{C})$.

4 Functors in 2-category theory

The reference for this section is [3, **pages** 2.2.4–2.2.8].

Definition 4.1. Let $\mathcal{C}, \mathcal{D}$ be 2-categories. A **lax 2-functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$ where $X \mapsto F(X)$,
- for all $X, Y \in \operatorname{Ob}(\mathcal{C})$ a functor $F = F_{X,Y} : \underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\operatorname{Hom}}_{\mathcal{D}}(FX, FY)$,
- for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ a morphism $\epsilon_X\colon \operatorname{id}_{FX} \Rightarrow F(\operatorname{id}_X)$ in $\underline{\operatorname{Hom}}_{\mathcal{D}}(FX, FX)$ called **unit constraint**,
- for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ a commutative diagram

$$\begin{tikzcd}
\underline{\operatorname{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "- \circ -"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"'] \arrow[dr, phantom, "\Rightarrow \mu"] & \operatorname{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\
\underline{\operatorname{Hom}}_{\mathcal{D}}(FY,FZ) \times \operatorname{Hom}_{\mathcal{D}}(FX,FY) \arrow[r, "- \circ -"] & \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FZ)
\end{tikzcd}$$

where $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ is called the **composition constraint**. The above data is required to satisfy the following:

(a) For all $f : X \to Y$ 1-morphisms in $\mathcal{C}$ the following diagrams commute

$$\begin{tikzcd}
F(\operatorname{id}_Y) \circ F(f) \arrow[r, Rightarrow, "\mu"] & F(\operatorname{id}_Y \circ f) \arrow[d, Rightarrow, "F(\lambda_f)"] \\
\operatorname{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\epsilon_Y * \operatorname{id}_{F(f)}"] \arrow[r, Rightarrow, "\lambda_{F(f)}"] & F(f)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f) \circ F(\operatorname{id}_X) \arrow[r, Rightarrow, "\mu"] & F(f \circ \operatorname{id}_X) \arrow[d, Rightarrow, "F(\rho_f)"] \\
\operatorname{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\operatorname{id}_{F(f)} * \epsilon_X"] \arrow[r, Rightarrow, "\rho_{F(f)}"] & F(f)
\end{tikzcd}$$

in $\underline{\operatorname{Hom}}(FX, FY)$,

(b) and for all $W, X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and for all $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ 1-morphisms in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
F(h) \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\alpha^{\mathcal{D}}"] \arrow[d, Rightarrow, "\operatorname{id}_{Fh} * \mu"] & (F(h) \circ F(g)) \circ F(f) \arrow[d, Rightarrow, "\mu * \operatorname{id}_{F(f)}"] \\
F(h) \circ F(g \circ f) \arrow[d, Rightarrow, "\mu"] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\mu"] \\
F(h \circ (g \circ f)) \arrow[r, Rightarrow, "F(\alpha^{\mathcal{C}})"] & F((h \circ g) \circ f)
\end{tikzcd}$$

in $\underline{\operatorname{Hom}}_{\mathcal{D}}(FW, FZ)$.

Definition 4.2. A **2-functor** $F : \mathcal{C} \to \mathcal{D}$ is a lax 2-functor such that for all $X \in \operatorname{Ob}(\mathcal{C})$ the morphism $\epsilon_X\colon \operatorname{id}_{FX} \overset{\sim}{\Longrightarrow} F(\operatorname{id}_X)$ is invertible and such that $\forall X, Y \in \operatorname{Ob}(\mathcal{C}), \forall X \xrightarrow{f} Y \xrightarrow{g} Z$ the morphism $\mu_{g,f} : F(g) \circ F(f) \overset{\sim}{\Longrightarrow} F(g \circ f)$ is invertible.

Definition 4.3. A **strict 2-functor** is a 2-functor, such that for all $X \in \mathrm{Ob}(\mathcal{C})$ the following hold $\epsilon_X = \mathrm{id} \colon \mathrm{id}_{FX} \Rightarrow F(\mathrm{id}_X), \forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\mu_{g,f} = \mathrm{id} = F(g) \circ F(f) \Rightarrow F(g \circ f)$.

Example 4.4. Lax monoidal functors $\mathcal{M} \to \mathcal{N}$ for $\mathcal{M}$ and $\mathcal{N}$ monoidal categories correspond to lax 2-functors $B\mathcal{M} \to B\mathcal{N}$.

Example 4.5. Let S be a set and $\mathcal{E}_S$ be a category with $\mathrm{Ob}(\mathcal{E}_S) = S$ and for all $x, y \in S, \underline{\mathrm{Hom}}_{\mathcal{E}_S}(x, y) := \{*\}$.

- Fix $\mathcal{M}$ a monoidal category and let $B\mathcal{M}$ be its delooping and $\underline{\mathcal{C}} \colon \mathcal{E}_S \to B\mathcal{M}$ a lax monoidal functor.
- Fix a map $\underline{\mathcal{C}} \colon \mathrm{Ob}(\mathcal{E}_S) = S \to \mathrm{Ob}(B\mathcal{M}) = \{*\}$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$ a functor:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) &\to \mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \\ * &\mapsto \underline{\mathcal{C}}(X, Y) \end{aligned}$$

- For all $X \in S$, $\epsilon_X \colon \mathrm{id}_{\underline{\mathcal{C}}(X)} = \mathrm{id}_*$, such that

$$\begin{tikzcd} \mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \arrow[loop below, "\mathbb{1}_{\mathcal{M}} = \mathrm{id}_*"] \end{tikzcd} \qquad \mathbb{1}_{\mathcal{M}} \otimes M \xrightarrow[\lambda_M]{\sim} M$$

  This is called the identity constraint.

- For all $X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S = S)$

$$\begin{tikzcd} \{*\} \times \{*\} = \underline{\mathrm{Hom}}_{\mathcal{E}_S}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) \arrow[r, "\circ", "\sim"'] \arrow[d, "\underline{\mathcal{C}} \times \underline{\mathcal{C}}"] \arrow[dr, phantom, "\overset{\mu}{\Longrightarrow}"] & \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Z) = \{*\} \arrow[d, "\underline{\mathcal{C}}"] \\ \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \times \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \arrow[r, "\otimes"] & \mathrm{Hom}_{B\mathcal{M}}(*, *) \end{tikzcd}$$

  where $\mu \colon \underline{\mathcal{C}}(Y, Z) \otimes \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, Z)$. The above data should satisfy the following: $\forall X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$.

$$\begin{tikzcd} \underline{\mathcal{C}}(X, Y) \otimes \underline{\mathcal{C}}(X, Y) \arrow[r, "\mu"] & \underline{\mathcal{C}}(X, Y) \arrow[d, "\mathrm{id}"] \\ \mathbb{1}_{\mathcal{M}} \otimes \underline{\mathcal{C}}(X, Y) \arrow[u, "\mathcal{E}_Y \otimes \mathrm{id}"] \arrow[r, "\lambda_{\underline{\mathcal{C}}(X, Y)}"] & \underline{\mathcal{C}}(X, Y) \end{tikzcd}$$

- Let $\mathcal{M} = \mathrm{Set}$

$$\begin{tikzcd} (\mathrm{id}_X, f) \arrow[r, mapsto] & \mathrm{id}_X \circ f \arrow[d, equal] \\ (x, f) \arrow[u] \arrow[r, mapsto] & f \colon X \to Y \end{tikzcd}$$

  similarly for the right constraint.

- For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S)$ the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathcal{C}}(Y,Z) \otimes (\underline{\mathcal{C}}(X,Z) \otimes \underline{\mathcal{C}}(W,X)) \arrow[r, "\alpha^M"] \arrow[d] & (\underline{\mathcal{C}}(Y,Z) \otimes \underline{\mathcal{C}}(X,Z)) \otimes \underline{\mathcal{C}}(W,X) \arrow[d] \\
\underline{\mathcal{C}}(Y,Z) \otimes \underline{\mathcal{C}}(W,Y) \arrow[d, "\mu"] & \underline{\mathcal{C}}(X,Z) \otimes \underline{\mathcal{C}}(w,Z) \arrow[d] \\
\underline{\mathcal{C}}(W,Z) \arrow[r, "\underline{\mathcal{C}}(\alpha)=\mathrm{id}"] & \underline{\mathcal{C}}(W,Z)
\end{tikzcd}$$

Remark 4.6. Dg-categories are Lax 2-functors.

Exercise 1. Let $\mathcal{C}$ be a 2-category and $\mathcal{D}$ a 1-category, then every lax 2-functor $F : \mathcal{C} \to \mathcal{D}$ is strict.

Definition 4.7. A $F : \mathcal{C} \to \mathcal{D}$ lax functor between 2-categories is

- **unitary** if $\forall X \in \mathcal{C}, \epsilon_X$ is an isomorphism,
- **strictly unitary** $\forall X \in \mathcal{C}, \epsilon_X = \mathrm{id}_X$,
- composition of lax functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

between 2-categories is the lax functor $G \circ F$ defined as follows:

  - on objects $(G \circ F)(X) = G(F(X))$,
  - on morphisms the composition is given by the dashed arrow,

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, dashed, "GF_{X,Y}"] \arrow[d, "F"] & \underline{\mathrm{Hom}}_{\mathcal{E}}((GF)X,(GF)Y) \arrow[d, equal] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(FX,FY) \arrow[r, "G_{FX,FY}"] & \underline{\mathrm{Hom}}_{\mathcal{E}}(G(FX),G(FY))
\end{tikzcd}$$

  - identity constraints are given as,

$$\begin{tikzcd}
& G(\mathrm{id}_{FX}) \arrow[dr, Rightarrow, "G(\epsilon^F_X)"] & & \mathrm{id}_{FX} \arrow[d, Rightarrow, "\epsilon^F_X"] \\
\mathrm{id}_{GFX} \arrow[ur, Rightarrow, "\epsilon^G_{FX}"] \arrow[rr, equal, "\epsilon^{GF}_X"'] & & GF(id_X) & F(\mathrm{id}_X)
\end{tikzcd}$$

  - $\forall X \xrightarrow{f} Y \xrightarrow{g} Z$ 1-morphisms in $\mathcal{C}$ the bottom arrow in the following diagram gives the composition constraint:

$$\begin{tikzcd}
& G(F(g) \circ F(f)) \arrow[dr, Rightarrow, "G(\mu^F_{g,f})"] & \\
GF(g) \circ GF(f) \arrow[ur, Rightarrow, "\mu^G_{Fg,Ff}"] \arrow[rr, equal, "\mu^{GF}_{g,f}"'] & & GF(g \circ f)
\end{tikzcd}$$

5 Grothendieck Construction

The reference for this section is [1, B.1.2, B.1.3].

Let A be a small 1-category, $X\colon A^{\mathrm{op}} \to \mathrm{Set}$ a presheaf.

Recall: The category of elements of X is the category $\int^A X$ with

- objects given by tuples $(a \in A, x \in X(a))$,
- morphisms given for another tuple $(b \in A, y \in X(b))$ by a morphism $f\colon a \to b$ such that $f^*(y) = x$.

Example 5.1. Consider $\int^A \mathrm{Hom}_A(-, b)$. The objects are given as $(a \in A, x\colon a \to b)$ and for a second tuple $(a' \in A, y\colon a' \to b)$ we have morphisms given $f\colon a \to a'$ such that $f^*(y) = x = y \circ f$.

Example 5.2. Let $\int^{A \times A^{\mathrm{op}}} \mathrm{Hom}_A(-,-)$ be the twisted arrow category of A.

- The objects are given as $((a, b) \in A \times A^{\mathrm{op}}, x\colon a \to b)$,
- the morphisms are given by tuples (f, g), where $f\colon a \to a', g\colon b' \to b$ such that the following square commutes

$$\begin{tikzcd} a' \arrow[r, "y"] & b' \arrow[d, "g"] \\ a \arrow[u, "f"] \arrow[r, "x"] & b \end{tikzcd}$$

for all objects $((a', b') \in A \times A^{\mathrm{op}}, y\colon a' \to b'), ((a, b) \in A \times A^{\mathrm{op}}, x\colon a \to b)$.

Consider the functor $\mathrm{Set}_* \xrightarrow{v} \mathrm{Set}$ where Set_* is the category of pointed sets. We can also construct the category of elements of X as the following pullback

$$\begin{tikzcd} A \arrow[r, "X^{\mathrm{op}}"] & \mathrm{Set}^{\mathrm{op}} \\ \int^A X \cong \mathcal{X} \arrow[u, "p"] \arrow[r] & (Set_*)^{\mathrm{op}} \arrow[u, "v^{\mathrm{op}}"'] \end{tikzcd}$$

the objects of the pullback are given as $\mathcal{X} = (a \in A, (X_a, x \in X_a))$ with morphisms $f\colon a \to b$ such that $f^*(y) = x$ for another object $(b \in B, (X_b, y \in X_b))$, thus we have the same objects as in the Grothendieck construction, just with the additional, but redundant data, of X_a.

Theorem 5.3. *The functor $\int^A\colon \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}/A$ is fully faithful, its essential image consists of the discrete Grothendieck fibrations, i.e. those functors $\mathcal{X} \xrightarrow{p} A$ such that $\forall x \in \mathcal{X}$ and $\forall f\colon a \to b = p(y)$ there exists a unique $\varphi\colon x \to y$ such that $p(\varphi) = f$.*

Idea 5.4. Let $\mathcal{X} \xrightarrow{p} Y$ be a discrete Grothendieck fibration of some presheaf $X\colon A^{\mathrm{op}} \to \mathrm{Set}$, then we have a pullback square

$$\begin{tikzcd} \mathcal{X}_a \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ \mathbb{1} \arrow[r] & A \end{tikzcd}$$

where the objects of $\mathcal{X}_a$ are given by $x \in \mathcal{X}$ such that $p(x) = a$ and morphisms φ such that $p(\varphi) = \mathrm{id}_a$.

Uniqueness of lifts yields that $X\colon A^{\mathrm{op}} \to \mathrm{Set}$ is well defined.

Definition 5.5. The **Grothendieck construction** of a 2-functor $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ is the 1-category $\int^A X$ with objects $(a \in A, x \in X_a)$ and for another object $(b \in B, y \in X_b)$ morphisms are given by tuples (f, φ) such that $f\colon a \to b$ and $\varphi\colon x \to f^*(y)$.

The composition law in $\int^A X$ is given as follows

$$\begin{tikzcd} & (b \in A, y \in X_b) \arrow[dr, "{(g,\psi)}"] & \\ (a \in A, x \in X_a) \arrow[ur, "{(f,\varphi)}"] \arrow[rr, "{(gf,\alpha)}"'] & & (c \in A, z \in X_c) \end{tikzcd}$$

where

$$X \xrightarrow{\varphi} f^*(y) \xrightarrow{f^*(\psi)} f^*(g^*(z)) \xrightarrow{\mu} (gf)^*(z)$$

and $\alpha = \mu \circ f^*(\psi) \circ \varphi$.

We need to verify that the composition law is associative and unital. For the identity morphisms it holds that

$$(a \in A, x \in X_a) \xrightarrow{(\mathrm{id}_a, x \xrightarrow{\epsilon} \mathrm{id}^*(x))} (a \in A, x \in X_a)$$

where

$$\epsilon_a\colon \mathrm{id}_{X_a} \Rightarrow X(\mathrm{id}_a) = (\mathrm{id}_a)^*.$$

Theorem 5.6. *The functor* $\int^A\colon$ 2-Fun$(A^{\mathrm{op}}, \underline{\mathrm{Cat}}) \to \mathrm{Cat}/A$ *is fully faithful, with essential image the Grothendieck fibrations.*

Example 5.7. Take the functor from Ring to Cat that takes a ring R to its module category, we have the following commutative diagram

$$\begin{tikzcd} R \arrow[r, mapsto] \arrow[d, "f"] & \mathrm{mod}_R \arrow[d, dashed, bend right, "f_*"'] \arrow[d, dashed, bend left, "f_! = - \otimes_k S"] \\ S \arrow[r, mapsto] & \mathrm{mod}_S \arrow[u, "\vdash f^* \vdash"] \end{tikzcd}$$

Let $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ be a 2-functor and A a small 1-category, as well as $\underline{\mathrm{Cat}}$ the strict 2-category of categories.

Proposition 5.8. *The Grothendieck Construction $\int^A X$ is a 1-category.*

Proof. Consider for the Unitality

$$\begin{tikzcd}
(a \in A, x \in X_a) \arrow[r, "(\mathrm{id}_a, x \xrightarrow{\epsilon_a} (\mathrm{id}_a)^*(x))"] \arrow[dr, "h"'] & (a \in A, x \in X_a) \arrow[d, "(a \xrightarrow{f} c, x \xrightarrow{\varphi} f^*(y))"] \\
& (c \in A, z \in X_c)
\end{tikzcd}$$

where $\begin{tikzcd} a \arrow[r, "\mathrm{id}_*"] \arrow[rr, bend right, "f"'] & a \arrow[r, "f"] & b \end{tikzcd}, x \xrightarrow{\alpha} f^*(y)$.

$$\begin{tikzcd}
(\mathrm{id}_a)^* \\
\mathbb{1}_{X_a} \arrow[u, Rightarrow, "\epsilon_a"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{id}_a)^*(x) \arrow[r, "(\mathrm{id}_a)^*(\varphi)"] & (\mathrm{id}_a)^*(f^*(y)) \arrow[r, "\mu", "\sim"'] & (f \circ \mathrm{id}_a)^*(y) = f^*(y) \\
X \arrow[u, "(\epsilon_a)_*"] \arrow[r, "\varphi"] & f^*(y) \arrow[u, "(\epsilon_a)_{f^*y}"] \arrow[ur, "\mathrm{id}_{f^*(y)}"']
\end{tikzcd}$$

The right triangle commutes by the definition of ϵ and the square by the naturality of ϵ_a . Now for the associativity take the following diagram:

$$\begin{tikzcd}
(a \in A, s \in X_a) \arrow[r, "(f,u)"] \arrow[dr, "(g \circ f, \mu \circ f^*(v) \circ u)"'] & (b \in A, x \in X_b) \arrow[d, "(g,v)"] \arrow[dr, "(hg, \mu \cdot g^*(v) \circ u)"] \\
& (c \in A, y \in X_c) \arrow[r, "(h,w)"] & (d \in A, z \in X_d)
\end{tikzcd}$$

The outer two compositions are given by

$$\begin{tikzcd}
(a \in A, s \in X_a) \arrow[r, bend left, "(hgf, \mu \circ f^*(\mu g^*(w) \circ v) \circ u)"] \arrow[r, bend right, "(hgf, \mu \circ (gf)^*(w) \circ \mu f^*(v) \circ u)"'] & (d \in A, z \in X_d)
\end{tikzcd}$$

Let us examine these compositions in detail, we have

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

in A and morphisms $s \xrightarrow{u} f^*(x), x \xrightarrow{v} g^*(y), y \xrightarrow{w} h^*(z)$, put together we get the commutative diagram

$$\begin{tikzcd}
s \arrow[r, "u"] & f^*(x) \arrow[r, "f^*(v)"] & f^*(g^*(y)) \arrow[r, "\mu"] \arrow[d, "f^*(g^*(w))"] & (g \circ f)^*(y) \arrow[d, "(g \circ f)*(w)"] \\
& & f^*(g^*(h^*(z))) \arrow[r, "\mu"] \arrow[d, "f^*(\mu)"] & (gf)^*(h^*(z)) \arrow[d, "\mu"] \\
& & f^*((hg)^*(z)) \arrow[r, "\mu"] & (gfh)^*(z)
\end{tikzcd}$$

Since the squares commute by the associativity constraint, we get that both compositions are equal and thus the associativity constraint holds. $\square$

Definition 5.9. A morphism $\varphi\colon Y \to X$ is called p-cartesian if $\forall g, \exists! \rho$ such that

$$\begin{tikzcd} & Y \arrow[dr, "\varphi"] & \\ Z \arrow[ur, dashed, "\exists ! \rho"] \arrow[rr, "\forall \psi"] & & X \end{tikzcd} \quad \overset{p}{\mapsto} \quad \begin{tikzcd} & p(Y) = a \arrow[dr, "f = p(\rho)"] & \\ p(Z) = a' \arrow[ur, "\forall g"] \arrow[rr, "h = p(X)"'] & & b = p(X) \end{tikzcd}$$

or equivalently if the following lift exists

$$\begin{tikzcd} \{1 \to 2\} \arrow[r, hook, "i"] & \lambda^2_2 \arrow[r, "F"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ & \Delta^2 \arrow[r, "\forall G"] \arrow[ur, dashed, "\exists !"] & A \end{tikzcd}$$

where $F(i(1 \to 2)) = \varphi$.

Definition 5.10. A functor of 1-categories $\mathcal{X} \xrightarrow{p} A$ is a Grothendieck fibration. If $\forall X \in \mathcal{X}, \forall a \xrightarrow{f} b = p(X)$ there exists a morphism $Y \xrightarrow{\varphi} X$ in $\mathcal{X}$ that is p-cartesian such that $p(\varphi) = f$.

Definition 5.11. Let $\mathcal{C}$ be a category. A lifitng problem in $\mathcal{C}$ is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathcal{C}$. A solution to the lifting problem (a lift) is a morphism $h\colon B \to X$ such that $ph = q, h \circ i = f$.

Proposition 5.12. *The functor* $\int^A X \xrightarrow{p} A$ *is a Grothendieck fibration.*

Proof. Let $Z = (c \in A, z \in X_c) \in \int^A_X$ be an object and $g\colon b \to p(z) = c$ in A be a morphism in A, then $g^*\colon X_c \to X_b$ and $z \mapsto g^*(z)$. Consider

$$\psi := (b \in A, g^*(z) \in X_b) \xrightarrow{(g, g^*(z) \xrightarrow{\mathrm{id}} g^*(z))} (c \in A, z \in X_c) = X$$

<u>Claim</u>: ψ is p-cartesian. Take a diagram

$$\begin{tikzcd} & b \arrow[dr, "g = p(\psi)"] & \\ a \arrow[ur, "\forall f"] \arrow[rr, "h = p(\alpha)"'] & & c \end{tikzcd}$$

the associated lifting problem is

$$\begin{tikzcd}
 & (b \in A, g^*(z) \in X_b) \arrow[dr, "{(g, \mathrm{id}_{g^*(z)} = \psi)}"] & \\
(a \in A, x \in X_a) \arrow[ur, dashed, "{(f, x \xrightarrow{u} f^*(g^*(z)))}"] \arrow[rr, "{\forall (h, x \xrightarrow{\sim} h^*(z)) = \alpha}"'] & & (c \in A, z \in X_c)
\end{tikzcd}$$

In order for the lifting diagram to commute, we need to construct u as follows. We have the composition

$$\begin{aligned} x \xrightarrow{u} f^*(g^*(z)) \xrightarrow{f^*(\mathrm{id}_{g^*})} f^*(g^*(z)) \xrightarrow{\mu} (gf)^*(z) \\ = x \xrightarrow{v} (gf)^*(z) \end{aligned}$$

thus taking $u = \mu^{-1} \circ v$ works. $\square$

Definition 5.13. The functor $\mathcal{X} \xrightarrow{p} A$ is called an **isofibration** if $\forall f \colon a \xrightarrow{\sim} p(x) = b$ there exists $Y \xrightarrow{\varphi} X$ an isomorphism in $\mathcal{X}$ such that $p(\varphi) = f$. That is if there exists a solution to the following lifting problem:

$$\begin{tikzcd}
\Lambda^0_0 \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\
\Delta^1 \arrow[r, "{a \xrightarrow{f} p(x)}"'] \arrow[ur, dashed, "\exists \varphi"] & A
\end{tikzcd}$$

Lemma 5.14. *If $p \colon \mathcal{X} \to A$ is a Grothendieck fibration then p is an isofibration.*

Proof. Let $x \in \mathcal{X}$ and $f \colon a \xrightarrow{\sim} p(x) = b$ an isomorphism in A. Choose a p-cartesian lift of f, $\varphi \colon Y \to X$ in $\mathcal{X}$ take a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "\varphi"] & \\
X \arrow[ur, dashed, "\exists ! \psi"] \arrow[rr, "\mathrm{id}_*"'] & & X
\end{tikzcd}$$

where $\varphi \circ \psi = \mathrm{id}_x$ and $p(\psi) = g = f^{-1}$, which means we have the following diagram:

$$\begin{tikzcd}
 & a \arrow[dr, "f"] & \\
b \arrow[ur, "f^{-1} = g"] \arrow[rr, "\mathrm{id}_b"'] & & b = p(x)
\end{tikzcd}$$

The following equalities hold

$$\varphi \circ (\psi \circ \varphi) = (\varphi \circ \psi) \circ \varphi = \varphi \circ \mathrm{id}_x = \varphi$$

now since $p(\psi) \circ p(\varphi) = f^{-1} \circ f = \mathrm{id}_a$ we get by the uniqueness of lifts that $\psi \circ \varphi = \mathrm{id}_Y$. Thus we found a lift that is an isomorphism, which proves the claim. $\square$

Let now $\mathcal{X} \xrightarrow{p} A$ be a Grothendieck fibration and $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ a 2-functor defined by sending each object a of A to the two-pullback $\mathcal{X}_a$ along the object a and $p\colon \mathcal{X} \to A$, that is,

$$\begin{tikzcd} \mathcal{X}_a \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "a"] & A \end{tikzcd}$$

now for any morphism $f\colon a \to b$ we obtain a functor $f^*\colon \mathcal{X}_b \to \mathcal{X}_a$. Let $x \in \mathcal{X}_b \xrightarrow{f^*} \mathcal{X}_a$ and $\varphi\colon f^*(x) \to x$ in $\mathcal{X}$ (notice the slight abuse of notation when we identify an object in the fiber with its image in $\mathcal{X}$) so that $p(\varphi) = f$.

6 Homotopy theory of simplicial sets

This chapter is a revision of material covered in the preceeding lecture, Homotopy theory of simplicial sets held by Professor Dr. Jasso in the WS 24/25 at the university of cologne. For reference see section 6 in the script of the lecture.

7 Duskin nerve

The reference for this section is [3, ch. 2.3].

We fix a 2-category $\mathcal{C}$ that we want to construct a simplicial set from.

Definition 7.1. The Duskin nerve is

$$\mathrm{Set}_\Delta \leftarrow \mathrm{twocat}_{\mathrm{lax}} : N^D = u^*$$

where $\mathrm{twocat}_{\mathrm{lax}}$ is the category of 2-categories with morphisms given by strictly unital lax 2-functors. Then $N^D(\mathcal{C})_n := \{\text{strictly unital lax 2-functors } [n] \xrightarrow{F} \mathcal{C}\}$.

Remark 7.2. [3] If $\mathcal{C}$ is a 1-category, then $N^D(\mathcal{C}) = N(\mathcal{C})$. We analyze the n-simplices of $N^D(\mathcal{C})$.

- (n=0) $[0] = \{0\} \xrightarrow{F} \mathcal{C}$ given $0 \mapsto X_0 = F(0) \in \mathrm{Ob}(\mathcal{C})$
- (n=1) $[1] = \{0 \to 1\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} 0 \arrow[d] \arrow[r, phantom, "\longmapsto"] & X_0 \arrow[d] \arrow[loop right, "f_{00}=\mathrm{id}_{X_0}"] \\ 1 & X_1 \arrow[loop right, "f_{11}=\mathrm{id}_{X_1}"] \end{tikzcd}$$

- (n=2) $[2] = \left\{ \begin{tikzcd}[cramped] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd}$$

- (n=3) $[3] = \left\{ \begin{tikzcd}[cramped] & 1 \arrow[dr] \arrow[dd] & \\ 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \right\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{13}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] \arrow[dr, "f_{30}"'] & & X_2 \arrow[dl, "f_{32}"] \\ & X_3 & \end{tikzcd}$$

with a 2-morphism $\xRightarrow{\gamma}$ for every 2-simplex in the boundary.

More precisely: A strictly unital lax 2-functor $F\colon [n] \to \mathcal{C}$ consist of the following data

- $\forall 0 \leqslant i \leqslant n$ an object $X_i := F(i)$,
- $\forall 0 \leqslant i \leqslant j \leqslant n$ a 1-morphism $f_{ji} = X(i \to j)$,
- and $\forall 0 \leqslant i \leqslant j \leqslant k \leqslant n$ a 2-morphism $\mu_{kji} = f_{kj} \cdot f_{ji} \Rightarrow f_{ki}$.

Moreover the above data must satisfy

- (strict unitality) $\forall 0 \leqslant i \leqslant n, f_{ii} = \mathrm{id}_{X_i}$ the following diagram commutes

$$\begin{tikzcd} f_{jj} \circ f_{ji} \arrow[r, Rightarrow, "\mu_{jji}"] & f_{ji} \arrow[d, Rightarrow, "F(\lambda_{ji})=\mathrm{id}_{f_{ji}}"] \\ \mathrm{id}_{X_j} \circ f_{ji} \arrow[u, Rightarrow, "\mathrm{id} * \mathrm{id}"] \arrow[r, Rightarrow, "\lambda_{f_{ji}}"] & f_{ji} \end{tikzcd}$$

that is $\forall 0 \leqslant i \leqslant j \leqslant n$

$$\mu_{jji} = \lambda_{f_{ji}} \colon \mathrm{id}_{X_j} \circ f_{ji} \Rightarrow f_{ji}$$
$$\mu_{jii} = \rho_{f_{ji}} \colon f_{ji} \circ \mathrm{id}_{X_i} \Rightarrow f_{ji}.$$

Recall that

$$\mu_{iii} = \lambda_{f_{ii}} = \mathrm{id}_{X_i} = \rho_{\mathrm{id}_{X_i}} = \rho_{f_{ii}} = \mu_{iii}.$$

- (Composition) $\forall 0 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n$

$$\begin{tikzcd} f_{lk} \circ (f_{kj} \circ f_{ji}) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\mathrm{id} * \mu"] & (f_{lk} \circ f_{kj}) \circ f_{ji} \arrow[d, Rightarrow, "\mu * \mathrm{id}"] \\ f_{lk} \circ f_{ki} \arrow[d, Rightarrow, "\mu"] & f_{lj} \circ f_{ji} \arrow[d, Rightarrow, "\mu"] \\ f_{li} \arrow[r, Rightarrow, "F(\alpha)=\mathrm{id}"] & f_{li} \end{tikzcd} \tag{1}$$

Proposition 7.3. *An n-simplex $F \in N^D(\mathcal{C})_n (F\colon [n] \to \mathcal{C})$ is uniquely determined by the following data:*

- $0 \leqslant i \leqslant n, X_i \in \mathrm{Ob}(\mathcal{C}), \forall 0 \leqslant i < j \leqslant n, f_j \colon X_i \to X_j$ *a collection of 1-morphisms in $\mathcal{C}$,*
- $0 \leqslant i < j < k \leqslant n, \mu_{kji} = f_{kj} \circ f_{ji} \Rightarrow f_{ki}$ *such that* $\forall 0 \leqslant i < j < k < l \leqslant n$ (1) *is satisfied.*

Proof. Sketch:

The uniqueness is clear, since we must have

$$\forall 0 \leqslant i \leqslant n, f_{ii} = \mathrm{id}_{X_i}, \forall 0 \leqslant i \leqslant j \leqslant n.$$

Given the data as in the statement, we define

$$\mu_{jji} = \lambda_{f_{ji}}, \mu_{jii} = \rho_{f_{ji}}.$$

We know 1 holds for $i < j < l < k$. To check that 1 holds when some indices are equal is given as an Exercise, it uses the triangle identity and the following lemma. $\square$

Lemma 7.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be 1-morphisms in $\mathcal{C}$. Then the following diagrams commute:*

1.
$$\begin{tikzcd}
\mathrm{id}_Z \circ (g \circ f) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\lambda_{g\circ f}"'] & & (\mathrm{id}_Z \circ g) \circ f \\
& g \circ f \arrow[ur, Rightarrow, "(\lambda_g)^{-1} * \mathrm{id}_f"'] &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
g \circ (f \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\mathrm{id}_g * \rho_f"'] & & (g \circ f)\, \mathrm{id}_X \\
& g \circ f \arrow[ur, Rightarrow, "\rho^{-1}_{g\circ f}"'] &
\end{tikzcd}$$

Proof. We prove 2., consider for that the following diagram, notice that we omitted taking products with identities in the notation of the morphisms.

$$\begin{tikzcd}
& g \circ ((f \circ \mathrm{id}_X) \circ \mathrm{id}_X) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\rho"] & (g \circ (f \circ \mathrm{id}_X)) \circ \mathrm{id}_X \arrow[d, Rightarrow, "\mathrm{id} * \rho"] \arrow[dr, Rightarrow, "\alpha"] & \\
g \circ (f \circ (\mathrm{id}_X \circ \mathrm{id}_X)) \arrow[ur, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "\lambda"] \arrow[drr, Rightarrow, "\alpha"'] & g \circ (f \circ \mathrm{id}_X) \arrow[r, Rightarrow, "\alpha"] & (g \circ f) \circ \mathrm{id}_X \arrow[r, Rightarrow, "\rho^{-1}"] & ((g \circ f) \circ \mathrm{id}_X) \circ \mathrm{id}_X \\
& & (g \circ f) \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[u, Rightarrow, "\lambda"] \arrow[ur, Rightarrow, "\alpha"'] &
\end{tikzcd}$$

By the pentagon axiom the outer composition in the diagram commutes, the bottom, top left and bottom right triangles commute by the triangle identity and the top middle square commutes by naturality of ρ, thus the top right triangle commutes. Lastly the diagram we aim to show commutes is the top right triangle with $- \circ \mathrm{id}_X$ applied to it, since $- \circ \mathrm{id}_X$ is fully faithful the original triangle commutes. $\square$

Corollary 7.5. *The Duskin nerve $N^D(\mathcal{C})$ is 3-coskeletal and*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^3, N^D(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^3, N^D(\mathcal{C}))$$

is injective.

The 3-simplices can be given by 2 boundary quadrilaterals as follows, let $0 \leqslant i < j < k < l \leqslant n$. Then the boundary quadrilaterals are given by

$$\begin{tikzcd}
& X_j \arrow[r, "f_{kj}"] \arrow[d, phantom, "\Downarrow\mu_{kji}"] & X_k \arrow[dr, "f_{lk}"] \arrow[d, phantom, "\Downarrow\mu_{lki}"] & \\
X_i \arrow[ur, "f_{ji}"] \arrow[urr, bend right] \arrow[rrr, "f_{kl}"] & & & X_l
\end{tikzcd}$$

$$\begin{tikzcd}
& X_j \arrow[r, "f_{kj}"] \arrow[drr, bend right] \arrow[d, phantom, "\mu_{lij}\Downarrow"] & X_k \arrow[dr, "f_{lk}"] \arrow[d, phantom, "\Downarrow\mu_{klj}"] & \\
X_i \arrow[ur, "f_{ji}"] \arrow[rrr, "f_{li}"] & & & X_l
\end{tikzcd}$$

Definition 7.6. Let $X \in \mathrm{Set}_\Delta$ be a simplicial set. A 2-simplex $\sigma \in X_2$ is thin if $\forall n \geqslant 3, \forall 0 < i < n$, there exists a morphism $\tilde{\rho}$ for all morphisms τ such that the following commutes

$$\begin{tikzcd}
\Delta^2 = [2] \arrow[r, "j"] \arrow[rr, bend left, "\sigma"] & \Lambda^n_i \arrow[r, "\forall\tau"] \arrow[d, hook] & X \\
& \Delta^n \arrow[ur, dashed, "\exists\tilde{\rho}"'] &
\end{tikzcd}$$

with $j := [2] \to \{i-1, i, i+1\}$.

Remark 7.7. Let $X \in \mathrm{Set}_\Delta$ be an ∞-category, then every 2-simplex of X is thin. Conversely, if every 2-simplex of X is thin, then X is an ∞ category if and only if the following diagram commutes:

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Remark 7.8. Take the following horn filling problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "\sigma"] \arrow[d, hook] & N^D(\mathcal{C}) \\
\Delta^2 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

where the dashed arrow exists since we have the horizontal composition of 1-morphisms. Thus we can extend any 2-horn and would have an infinity category if every 2-simplex were thin.

Theorem 7.9. *Let $\mathcal{C}$ be a 2-category, then $\mathcal{C}$ is a $(2,1)$-category if and only if $N^D(\mathcal{C})$ is an ∞-category.*

Proof. Suppose $\mathcal{C}$ is a (2,1)-category. Assume every 2-simplex were thin by 7.10, then by 7.8 $N^D(\mathcal{C})$ is an ∞-category. Conversely, if $N^D(\mathcal{C})$ is an ∞-category, then by 7.7 every 2-simplex in $N^D(\mathcal{C})$ is thin if the following is satisfied. Let

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \end{tikzcd}$$

be a 2-morphism in $\mathcal{C}$, then we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[dr, Rightarrow, "\rho"'] & f \arrow[d, Rightarrow, "\gamma"] \\ & g \end{tikzcd}$$

where ρ exists since $N^D(\mathcal{C})$ is thin. Now by assumption ρ is invertible, thus ρ is invertible. $\square$

Lecture 27.5

For the next part let $\mathcal{C}$ be a fixed 2-category and let $N^D(\mathcal{C}) \in \mathrm{Set}_\Delta$ be its Duskin nerve, i.e. $N^D(\mathcal{C})_n := \{[n] \to \mathcal{C} \mid \text{strictly unital lax functors}\}$.

Theorem 7.10. *The 2-simplex* $\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "\gamma_{210}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd} \in N^D(\mathcal{C})_2$ *is thin if and only if* μ_{210} *is invertible.*

Proof. We divide the proof into its natural parts. $\square$

Proposition 7.11. *Let* $n \geqslant 3$ *as well as* $0 \leqslant l \leqslant n$,

$$\sigma = \begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "\gamma"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd}$$

be a 2-simplex in $N^D(\mathcal{C})$ *and*

$$\begin{tikzcd} \Lambda^n_l \arrow[r, "u"] & N^D(\mathcal{C}) \\ \Delta^{\{l-1,l,l+1\}} \arrow[u] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commute. If γ *is invertible, then* u *extends uniquely to an* n*-simplex of* $N^D(\mathcal{C})$.

Proof. Recall that $N^D(\mathcal{C})$ is 3-coskeletal, hence we may assume $n = 3, 4$. (Case 1) $n = 3, l = 1$

$$\begin{tikzcd} & X_1 \arrow[r, "f_{21}"] \arrow[d, phantom, "\Downarrow\mu_{210}=\gamma"] & X_2 \arrow[dr, "f_{32}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[urr, bend right] \arrow[rrr, "f_{30}"'] & & & X_3 \end{tikzcd}
\qquad
\begin{tikzcd} & X_1 \arrow[r, "f_{21}"] \arrow[drr, bend right] \arrow[d, phantom, "\Downarrow \mu_{310}"] & X_2 \arrow[dr, "f_{32}"] \arrow[d, phantom, "\Downarrow\mu_{321}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rrr, "f_{30}"'] & & & X_3 \end{tikzcd}$$

Where the dashed green lined simplex corresponds to the missing simplex given by μ_{320}, which we can construct as follows. Observe that since $\mathrm{id}_{f_{32}}$ and $\mu_{210} = \gamma$ are invertible and composition is functorial, the composition is invertible as well. We know from an Exercise that if μ_{320} would exist, the following identity would hold and that it is sufficient that this holds to extend the horn

$$\mu_{320}(\mathrm{id}_{f_{32}} \circ \mu_{210}) = \mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}})\alpha_{f_{32}, f_{21}, f_{10}}.$$

By the above argument we can choose

$$\mu_{320} = (\mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}})\alpha_{f_{32}, f_{21}, f_{10}})(\mathrm{id}_{f_{32}} \circ \mu_{210})^{-1}.$$

(Case 2: n=4 ,l=2) $u\colon \Lambda^n_l \to N^D(\mathcal{C})$
We need to check the compatibility of composition, since all other diagrams for the Duskin nerve require only simplices up dimension three to commute and, thus hold for the boundary 3-simplices of a 4-horn, leaving only the compatibility of composition to be checked. The outer square of the following diagram is exactly the compatibility of composition diagram for the missing 3 simplex, i.e. all morphisms that do not contain a 2 as an index.

$$\begin{tikzcd}[column sep=small]
f_{43} \circ (f_{31} \circ f_{10}) \arrow[rrrr, Rightarrow, "\alpha"] \arrow[ddddd, Rightarrow, "\mu_{310}"] & & & & (f_{43} \circ f_{31}) \circ f_{10} \arrow[ddddd, Rightarrow, "\mu_{431}"] \\
& f_{43} \circ ((f_{32} \circ f_{21}) \circ f_{10}) \arrow[ul, Rightarrow, "\mu_{321}"] \arrow[rr, Rightarrow, "\alpha"] & & (f_{43} \circ (f_{32} \circ f_{23})) \circ f_{10} \arrow[ur, Rightarrow] \arrow[d, Rightarrow, "\alpha"] & \\
& f_{43} \circ (f_{32} \circ (f_{21} \circ f_{10})) \arrow[u, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\mu_{210}"] \arrow[dr, Rightarrow, "\alpha"] & & ((f_{43} \circ f_{32}) f_{21}) f_{10} \arrow[d, Rightarrow, "\mu_{432}"] & \\
& f_{43} \circ (f_{32} \circ f_{20}) \arrow[d, Rightarrow, "\alpha"] \arrow[ddl, Rightarrow, "\mu_{320}"'] & (f_{43} \circ f_{32}) \circ (f_{21} \circ f_{10}) \arrow[ur, Rightarrow, "\alpha"] \arrow[dl, Rightarrow, "\mu_{210}"] \arrow[dr, Rightarrow, "\mu_{432}"] & (f_{42} \circ f_{21}) \circ f_{10} \arrow[d, Rightarrow, "\alpha"] \arrow[ddr, Rightarrow, "\mu_{421}"] & \\
& (f_{43} \circ f_{32}) \circ f_{20} \arrow[r, Rightarrow, "\mu_{432}"'] & f_{42} \circ f_{20} \arrow[d, Rightarrow, "\mu_{420}"] & f_{42} \circ (f_{21} \circ f_{10}) \arrow[l, Rightarrow, "\mu_{210}"] & \\
f_{43} \circ f_{30} \arrow[rr, Rightarrow, "\mu_{430}"] & & f_{40} & & f_{41} \circ f_{10} \arrow[ll, Rightarrow, "\mu_{410}"]
\end{tikzcd}$$

Note that composition constraint and associativity constraint are isomorphisms, thus if the inner diagrams commute, so does the outer one. The left and right quadrilaterals commute by the compatibility of composition, given by the other boundary 3-simplices of the horn and since composition with a morphism (f_{43} on the left side, f_{42} on the right side) is fully faithful, the upper square commutes by the naturality of α, the pentagon by the Pentagon identity, the triangles below the Pentagon by the naturality of the associativity constraint, the bottom left and right square commute by the composition compatibility. ☐

Proposition 7.12. *Let $\mathcal{C}$ be a 2-category and*

$$\sigma := \begin{tikzcd} & X_1 \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"'] & {} & X_2 \end{tikzcd} \in N^D(\mathcal{C})_2$$

such that $\forall n \in \{3,4\}$ the following diagram commutes

$$\begin{tikzcd} \Delta^{\{0,1,2\}} \arrow[d, hook] \arrow[dr, bend left, "\sigma"] & \\ \Lambda^n_1 \arrow[r, "u"] \arrow[d, hook] & N^D(\mathcal{C}) \\ \Delta^n \arrow[ur, "\exists"] & \end{tikzcd}$$

then γ is invertible.

Proof. Let the following quadrilaterals be the boundary of a 3-simplex.

$$\begin{tikzcd} & X_1 \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\gamma"] & X_2 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[urr, bend left, "h"] \arrow[rrr, "g\circ f"'] & {} & & X_3 = X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_1 \arrow[r, "g"] \arrow[drr, bend right, "g"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g\circ f}"] & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[d, phantom, "\Downarrow\mathrm{id}_g"] & \\ X_0 \arrow[ur, "f"] \arrow[rrr, "g\circ f"'] & {} & {} & X_3 = X_2 \end{tikzcd}$$

by assumption this has a filling and thus there exists a δ such that $\delta\gamma = \mathrm{id}_{g\circ f} \circ \mathrm{id}_g$, we can assume strict unitality of the category (for details on this see [3]) and thus need obtain that $\delta\gamma = \mathrm{id}_{g\circ f}$. Now we need to show it is a right inverse to γ as well. for that we take the 4-horn given by the following boundary data:

$$\begin{tikzcd} & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\ & X_1 \arrow[ur, "g"] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\ X_0 \arrow[ur, "f"] \arrow[uurr, bend left, "h"] \arrow[urrr, bend right, "g\circ f"] \arrow[rrrr, "h"'] & & & & X_4 = X_2 \end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[ddrr, bend right, "\mathrm{id}_{X_2}"'] & & \\
 & X_1 \arrow[ur, "g"] \arrow[drrr, bend right=15, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\
 & X_1 \arrow[ur, "g"] \arrow[rr, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[urrr, "g \circ f"'] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\
 & X_1 \arrow[ur, "g"] \arrow[rr, "g"] \arrow[drrr, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[ddrr, bend right, "\mathrm{id}_{X_2}"'] & & \\
 & X_1 \arrow[ur, "g"] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[uurr, bend right, "h"'] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

Since the horn given by the simplices above extends, we obtain $\gamma\delta = \mathrm{id}_h$. $\square$

8 Nerves of differential graded categories

The reference for this section is [3, ch. 2.5].

Let us begin with a reminder on cochain complexes. Let k be a commutative ring and let mod_k be the category of (right) k-modules and $\mathrm{Ch}(\mathrm{mod}_k)$ the category of cochain complexes of k-modules, so objects are given by

$$(X^\bullet, d^\bullet) = \ldots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots$$

such that $d^2 = 0$. The category $\mathrm{Ch}(\mathrm{mod}_k)$ is a monoidal category with

$$(X^\bullet \otimes_k Y^\bullet)^l := \coprod_{i+j=l} X^i \otimes_k Y^j$$
$$d_{X^\bullet \otimes Y^\bullet}(x \otimes y) := d_X(x) \otimes y + (-1)^i x \otimes d_Y(y)$$

where $|x| = i$ is the degree of x. The unit of the monoidal structure is given by k viewed as a chain complex concentrated in degree 0. There is a preferred symmetry constraint

$$\tau\colon X^\bullet \otimes Y^\bullet \xrightarrow{\sim} Y^\bullet \otimes X^\bullet$$
$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

which is called the Koszul sign rule.

Definition 8.1. A **differential graded category** $\mathcal{A}$ is a category enriched in the monoidal category $\mathrm{Ch}(\mathrm{mod}_k)$. That is:

- A class $\mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$.
- For all $a, b \in \mathrm{Ob}(\mathcal{A})$ a cochain complex $\mathcal{A}(a,b) \in \mathrm{Ch}(\mathrm{mod}_k)$.
- For all $a \in \mathrm{Ob}(\mathcal{A})$ a unit/identity $\mathrm{id}_a\colon k \to \mathcal{A}(a,a)$.
- For all $a, b, c \in \mathrm{Ob}(\mathcal{A})$ a composition law

$$\mathcal{A}(b,c) \otimes \mathcal{A}(a,b) \xrightarrow{-\circ-} \mathcal{A}(a,b)$$

  given by a morphism in $\mathrm{Ch}(\mathrm{mod}_k)$.

Remark 8.2. This means that if $f \in \mathcal{A}(a,b), |f| = i$ and $g \in \mathcal{A}(b,c), |g| = j$, then

- $|g \circ f| = i + j$ since $|g \otimes f| = i + j$,
- $d_\mathcal{A}(g \circ f) = d_\mathcal{A}(g) \circ f + (-1)^j g \circ d_\mathcal{A}(f), |g| = j$ (Graded Leibniz rule).

This composition law must be associative and unital in the usual sense.

Example 8.3. Let $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ be a the dg category given as follows:

- The objects are given by complexes of k-modules.
- For $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ a complex $\mathrm{Hom}_k(X^\bullet, Y^\bullet) \in \mathrm{Ch}(\mathrm{mod}_k)$.
- Let $\mathrm{Hom}_k(X^\bullet, Y^\bullet)^j := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k(X^i, Y^{i+j})$ be the degree j maps of graded k-modules endowed with the following differential

$$\begin{aligned} \partial\colon \mathrm{Hom}_k(X^\bullet, Y^\bullet)^j &\to \mathrm{Hom}_k(X^\bullet, Y^\bullet)^{j+1} \\ f &\mapsto \partial(f) = (d_Y^{i+j} \circ f^i - (-1)^{|f|} f^{i+1} \circ d_X^i)_{i\in\mathbb{Z}}, |f| = j \end{aligned}$$

Note that $|f| = 0$ and $\partial(f) = 0$ is equivalent to f being a cochain map.

Example 8.4. Let now $\mathrm{Ob}(\mathcal{A}) = \{\star\}$, $A := \mathcal{A}(\star, \star)$, this is a dg algebra.

Construction 8.5. Given a dg category $\mathcal{A}$, its underlying category is denoted by $Z^0(\mathcal{A})$, given by

- $\mathrm{Ob}(Z^0(A)) = \mathrm{Ob}(\mathcal{A})$,
- $\forall a, b \in Z^0(\mathcal{A}), Z^0(a, b) = Z^0(\mathcal{A}(a, b)) = \ker(d_{\mathcal{A}^0}) \subseteq \mathcal{A}(a, b)$.

Definition 8.6. Let $\mathcal{A}$ be a dg category. The homotopy category (0-th cohomology category) of $\mathcal{A}$, denoted $H^0(\mathcal{A})$ has

- $\mathrm{Ob}(H^0(\mathcal{A})) = \mathrm{Ob}(Z^0(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$
- $\forall a, b \in H^0(\mathcal{A}), H^0(\mathcal{A})(a, b) = H^0(\mathcal{A}(a, b))$.

In the case of $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$, we have

$$\begin{aligned} &Z^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) := \mathrm{Ch}(\mathrm{mod}_k) \\ &H^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) = K(\mathrm{mod}_k) \text{ homotopy category of cochain complexes} \end{aligned}$$

Construction 8.7. Let $\mathcal{A}$ be a dg category, then the dg nerve $N_{\mathrm{dg}}(\mathcal{A})$ is defined as follows:

- $N_{\mathrm{dg}}(\mathcal{A})_0$ are the objects of $\mathcal{A}$,
- $N_{\mathrm{dg}}(\mathcal{A})_1$ are the degree zero cocycles $f \in Z^0(\mathcal{A}(a, b))$, that is the morphisms f such that $|f| = 0$ and $d_{\mathcal{A}}(f) = 0$,
- $N_{\mathrm{dg}}(A)_2$ is given by 2-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "f_{21}"] \arrow[d, phantom, "\Downarrow f_{210}"] & \\ a_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & a_2 \end{tikzcd}$$

where $|f_{ij}| = 0.|f_{210}| = -1, d_{\mathcal{A}}(f_{ij}) = 0$ and $d_{\mathcal{A}}(f_{210}) = f_{20} - f_{21} \circ f_{10}$ as well as $[f_{20}] = [f_{21} \circ f_{10}] \in H^0(\mathcal{A}(a_0, a_2))$,

- $N_{\mathrm{dg}}(\mathcal{A})_3$ is given by 3-simplices

$$\begin{tikzcd}
 & a_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{31}"] & \\
a_0 \arrow[ur, "f_{10}"] \arrow[rr, dashed, "f_{20}"] \arrow[dr, "f_{30}"'] & & a_2 \arrow[dl, "f_{32}"] \\
 & a_3 &
\end{tikzcd}$$

  with each boundary 2-simplex having its composition given by some morphism $f_{ijk}, i, j, k \in \{0,1,2,3\}$ as well as a morphism f_{3210} such that $d(f_{3210}) = -(f_{321} \circ f_{10} - f_{320}) + (f_{32} \circ f_{210} - f_{310})$.

Definition 8.8. Let $\mathcal{A}$ be a dg-category. The <u>dg nerve of $\mathcal{A}$</u>, $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is the simplicial set where for $n \geqslant 0$

$$\begin{aligned}
N_{\mathrm{dg}}(\mathcal{A}) &:= \{(a_0, a_1, \ldots, a_n) \in \mathrm{Ob}(\mathcal{A}), (f_I = f_{n \geqslant i_k > \ldots > i_0 \geqslant 0} \in \mathcal{A}(a_{i_0}, a_{i_k}))_{\substack{I \subseteq [n] \\ 2 \leqslant |I|}}\} \\
|f_I| &= -(\#(I \backslash \{i_k, i_0\})) \\
d_{\mathcal{A}}(f_I) &= \sum_{l=1}^{k-1} (-1)^l (f_{i_k > \ldots > i_l} \circ f_{i_l > \ldots > i_0} \circ \ldots \circ f_{i_k > \ldots > i_l > \ldots > i_0})
\end{aligned}$$

Theorem 8.9. *The nerve of a dg-category $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is an ∞-category.*

References

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