Itic notes

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Contents

1 Motivation

Notice that many proofs of statements in the lecture are contained as Exercises, which I still have to add at the current point in time. If you want solutions to any of the Exercises you may contact the author.

Fix $0 \leqslant m \leqslant n \leqslant \infty$. An (n, m) category is a "category-like" structure consisting of a class of objects, notions of 1-morphism, 2-morphism, ... , n-morphism (i.e. k-morphsim $0 < k \leqslant n$) with a "suitable composition law" (satisfying "suitable axioms") and such that $\forall m < k \leqslant n$ the k-morphisms are "invertible".

$$\begin{aligned}
&(0,0)\text{-cat.} = \text{set}\\
&(1,0)\text{-cat.} = \text{groupoid, i.e. 1-groupoid}\\
&(1,1)\text{-cat.} = \text{category, i.e. 1-category}\\
&(2,0)\text{-cat.} = \text{2-groupoids}\\
&(2,1)\text{-cat.}\\
&(2,2)\text{-cat.} = \text{2-categories}\\
&\vdots\\
&(n,0)\text{-cat.} = \text{n-groupoid}\\
&(n,n)\text{-cat.} = \text{n-cat.}\\
&\vdots\\
&(\infty,0) = \infty\text{-groupoids}\\
&(\infty,1) = \infty\text{-categories (see. Boardman-Vogt)}
\end{aligned}$$

Reminder 1.1. *A map $f\colon X \to Y$ between topological spaces is a weak homotopy equivalence if $\forall x \in X, \forall n \in \mathbb{N}$ the map*

$$\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection.

Theorem 1.2 (Grothendieck's Homotopy Hypothesis)**.** *There is an $(\infty, 1)$-category of topological spaces up to weak homotopy equivalence and there is an $(\infty, 1)$-category of ∞-groupoids up to eqivalence. There is furthermore an ∞-functor assigning to each topological space X its Poincare ∞-groupoid $\pi_\infty(X)$, this is an equivalence.*

Remark 1.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category, then for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an ∞-groupoid/ "space". We have the homotopy category of $\mathcal{C}$ denoted by $\mathrm{Ho}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and for all objects $X, Y \in \mathrm{Ho}(\mathcal{C})$ we have that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$.

Warning! 1.4. The passage from $\mathcal{C}$ often results in a tremendous loss of information, that is essential for various purposes.

- Computing co-/limits within $\mathcal{C}$.
- Computing co-/limits with $\mathcal{C}$.
- Define invariants associated to $\mathcal{C}$ (f.e. Hochschild cohomology).

Reminder 1.5. *Many important 1-categories arise as homotopy categories of genuine $(\infty, 1)$-categories, for example derived categories. Recall for a ring R, mod_R its category of right R-modules and $\mathrm{Ch}(\mathrm{mod}_R)$ the category of chain complexes in mod_R, that a morphism of chain complexes $f^\bullet \colon X^\bullet \to Y^\bullet$ in $\mathrm{Ch}(\mathrm{mod}_R)$ is a quasi-isomorphism if $\forall n \in \mathbb{Z}$ we have that $H_n(f^\bullet) = H_n(X^\bullet) \xrightarrow{\sim} H_n(Y^\bullet)$ is an isomorphism. The derived category is defined as follows $D(\mathrm{mod}_R) := \mathrm{Ch}(\mathrm{mod}_R)[qiso^{-1}]$, i.e. the localisation at the quasi-isomorphisms. Furthermore we have that $\mathrm{Ho}(\mathcal{D}(\mathrm{mod}_R)) = D(\mathrm{mod}_R)$. In the first case, that is the right side of the equality above, we obtain the derived category by building it from the ground up so to say and in the second case, the left side of the equation, we obtain it by forgetting information from a higher structure.*

Warning! 1.6. The homotopy theory of $(\infty, 1)$-categories has many equivalent implementations (Quillen):

- Topological categories (Ilias)
- Simplicial categories (Bergner)
- Complete Segal spaces (Rezk)
- Relative categories (Barwick-Kan)
- Pre-derivations
- ∞-categories (Joyal, Lurie)

In the k-linear setting, for k a field we have:

1. Differentially graded k-categories
2. A_∞-categories (Lefèvre-Hasegawa)

The Plan for the lecture is to start of with investigating 2-categories, then give definition and examples of ∞-categories, then do enriched category theory and end on the homotopy theory of ∞-categories.

2 Heuristics: Isomorphism vs. Equivalence

The reference for this section is [2, ch.4, 5, 6].
Makkai's Principle of Isomorphism (1998) says:

"All grammatically correct properties about objects in a fixed category are to be invariant under isomorphism."

Fix a category $\mathcal{C}$ and a small category A and take the functor category $\mathrm{Fun}(A, \mathcal{C})$, that is A-shaped diagrams in $\mathcal{C}$. For $X \in \mathcal{C}$ define:

$$\lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) := \{(p_a\colon X \to D(a))_{a\in A} \mid \begin{tikzcd} & X \arrow[d, "f_a"'] \arrow[dr, "f_b"] & \\ & D(a) \arrow[r] & D(b) \end{tikzcd} \quad \forall f\colon a \to b \text{ in } A\}$$

For $\phi\colon Y \to X$ define the function

$$\begin{aligned} \phi^*\colon \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) &\to \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \\ p = (p_a\colon X \to D(a))_{a\in A} &\mapsto \phi^*(p) = (p_a \circ \phi\colon Y \to D(a))_{a\in A} \end{aligned}$$

Thus $\lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf of sets on $\mathcal{C}$.

Definition 2.1. A limit of a diagram $D\colon A \to \mathcal{C}$ is a cone $p \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ that is universal in the sense that $\forall Y \in \mathcal{C}, \forall q \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)), \exists! \varphi\colon Y \to X$ such that $\varphi^*(p) = q$. We write $\lim_{a\in A} D(a)$ for any limit of D (which may or may not exist).

Reminder 2.2 (Yoneda Lemma)**.** *Let $X \in \mathcal{C}$ and let*

$$\begin{aligned} \nu\colon \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, X), \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))) &\xrightarrow{\sim} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) \\ \eta = (\eta_Y\colon \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \lim_{a\in\mathcal{C}}(Y, D(a))_{Y\in\mathcal{C}} &\mapsto \eta_X(\mathrm{id}_X) \\ (\eta^p_Y(p) := \varphi^*(p))_{Y\in\mathcal{C}} &\leftarrow\!\shortmid p \end{aligned}$$

If $p \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ is a limit of $D\colon A \to \mathcal{C}$ then
$\eta^p\colon \mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, D(a))$ is a natural isomorphism. We furthermore obtain, that for an isomorphism $\psi\colon Y \to X$ in $\mathcal{C}$ we have

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(-, X) \arrow[dr, "\eta^p", "\sim"'] & \\ \mathrm{Hom}_{\mathcal{C}}(-, Y) \arrow[ur, "\psi^*", "\sim"'] \arrow[rr, "\sim"] & & \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) \end{tikzcd}$$

Example 2.3. Let the following be a diagram in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

then the limit of the diagram (if it exists) is called a pullback.

$$\begin{tikzcd}
W \arrow[dr, "\exists! p"] \arrow[drr, bend left, "\forall f''"] \arrow[ddr, bend right, "\forall g''"'] & & \\
& X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

For example if $\mathcal{C} = \text{Set}$ then $X \times_Z Y = \{(X, Y) \in X \times Y \mid f(x) = g(y)\}$.

The commutativity condition takes place in $\mathrm{Hom}_{\mathcal{C}}(X \times_Z Y, Z) \ni g \circ f' = f \circ g'$

Makkai's Principle of Equivalence: All grammatically correct properties of objects in a fixed 2-category are to be invariant under equivalence.

Remark 2.4. We want Cat to be the strict 2-category of (small) categories with functors as 1-morphisms and natural transformations as 2-morphisms. Now natural transformation allow for a notion of equivalence of morphisms, that is in a 1-category we only knew what it meant for two morphisms to be equal, but now we can talk about two functors being naturally isomorphic giving us a notion of equivalence of 1-morphisms, via 2-morphisms.

Definition/Proposition 2.5 (Godement Product)**.** Consider natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"{name=A}] \arrow[r, bend right, "F_2"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, bend left, "G_1"{name=C}] \arrow[r, bend right, "G_2"'{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow \beta"] & \mathcal{E}
\end{tikzcd}$$

Their Godement product is the natural transformation.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "G_1 \circ F_1"{name=A}] \arrow[rr, bend right, "G_2 \circ F_2"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow \beta * \alpha"] & & \mathcal{E}
\end{tikzcd}$$

Let $X \in \mathcal{C}$, we obtain the following diagram

$$\begin{tikzcd}
F_1(X) \arrow[d, "\alpha_x"] & G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta*\alpha)_X"] & G_2(F_1(X)) \arrow[d, "G_2(\alpha_X)"] \\
F_2(X) & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] & G_2(F_2(X))
\end{tikzcd}$$

in $\mathcal{D}$.

Proof. We show that $\beta * \alpha \colon G_1 \circ F_1 \Rightarrow G_2 \circ F_2$ is indeed a natural transformation. For that we take the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & G_1(F_1(X)) \arrow[r, "G_1(\alpha_X)"] \arrow[d, "G_1(F_1(f))"'] & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(F_2(f))"] & G_2(F_2(X)) \arrow[d, "G_2(F_2(f))"] \\
Y & G_1(F_1(Y)) \arrow[r, "G_1(\alpha_Y)"'] & G_1(F_2(Y)) \arrow[r, "\beta_{F_2(Y)}"'] & G_2(F_2(Y))
\end{tikzcd}$$

the inner squares commute by the naturality of α and β, thus the outer square commutes, meaning it is a natural transformation. $\square$

Proposition 2.6. *Consider natural transformations*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=80, "F_1"] \arrow[r, "F_2"'] \arrow[r, bend right=80, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \gamma"] & \mathcal{D} \arrow[r, bend left=80, "G_1"] \arrow[r, "G_2"'] \arrow[r, bend right=80, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \beta"] \arrow[r, phantom, shift right=3, "\Downarrow \delta"] & \mathcal{E}
\end{tikzcd}$$

Then $(\delta\beta) * (\gamma\alpha) = (\delta * \gamma) \circ (\beta * \alpha)$.

Proof. Let $X \in \mathcal{C}$

$$\begin{tikzcd}
G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta*\alpha)_X"] \arrow[dd, bend right=60, "G_1(\gamma_X \alpha_X)"'] & G_2(F_1(X)) \arrow[r, "\delta_{F_1(X)}"] \arrow[d, "G_2(\alpha_X)"] & G_3(F_1(X)) \arrow[d, "G_3(\alpha_X)"] \arrow[dd, bend left=60, "G_3(\gamma_X \alpha_X)"] \\
G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(\gamma_X)"'] & G_2(F_2(X)) \arrow[r, "\delta_{F_2(X)}"] \arrow[d, "G_2(\gamma_X)"] \arrow[dr, "(\delta*\gamma)_X"] & G_3(F_2(X)) \arrow[d, "G_3(\gamma_X)"] \\
G_1(F_3(X)) \arrow[r, "\beta_{F_3(X)}"] & G_2(F_3(X)) \arrow[r, "\delta_{F_2(X)}"] & G_3(F_3(X))
\end{tikzcd}$$

Now the long diagonal of the diagram corresponds to $(\delta * \gamma) \circ (\beta * \alpha)$ and the outer large square to $(\delta \circ \beta) * (\gamma \circ \alpha)$. $\square$

Definition 2.7. The Godement products bellow are called whiskerings

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E} & \mapsto & \mathcal{C} \arrow[r, bend left=50, "G_1 \circ F"] \arrow[r, bend right=50, "G_2 \circ F"'] \arrow[r, phantom, "\Downarrow \beta * \mathrm{id}_F"] & \mathcal{E}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "G_1"{name=U}] \arrow[r, bend right, "G_2"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd} \mapsto \begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "G \circ F_1"{name=U}] \arrow[rr, bend right, "G \circ F_2"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\mathrm{id}_G * \alpha"] & & \mathcal{E}
\end{tikzcd}$$

Construction 2.8. Given a cospan of groupoids

$$\begin{tikzcd}
 & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

its 2-pullback is the diagonal of groupoids along on $\mathcal{C}$ inside the product.

$$\begin{tikzcd}
\mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

The objects are given as $\mathrm{Ob}(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}) = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi\colon F(a) \xrightarrow{\sim} G(b) \text{ in } \mathcal{C})$ and morphisms are given by tuples of morphisms $(u, v)\colon (a, b, \varphi) \to (a', b', \varphi')$, where $u\colon a \to a'$ and $v\colon b \to b'$ are morphisms in the respective groupoids, such that the following square commutes:

$$\begin{tikzcd}
F(a) \arrow[r, "\varphi", "\sim"'] \arrow[d, "F(a)"'] & G(b) \arrow[d, "G(v)"] \\
F(a') \arrow[r, "\varphi'", "\sim"'] & G(b')
\end{tikzcd}$$

Lecture 15.04

Let $X, Y, Z \in \mathrm{Set}$ and consider a pullback diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Where the fiber product is given by $X \times_Z Y := \{(x, y) \in X \times Y, f(x) = g(x)\}$ such that the following isomorphism holds for all W.

$$\mathrm{Hom}_{\mathrm{Set}}(^{\forall}W, X \times_Z Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(W, X) \times_{\mathrm{Hom}_{\mathrm{Set}}(W,Z)} \mathrm{Hom}_{\mathrm{Set}}(W, Y)$$

$$(\varphi\colon X \times_Z Y) \mapsto \begin{tikzcd}
W \arrow[r, "\pi_Y \circ \varphi"] \arrow[d, "\pi_X \circ \varphi"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Thus the case of a pullback of objects is clear, but what does the pullback of morphism-sets look like?

Construction 2.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be groupoids and

$$\begin{tikzcd} \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{A}}"] \arrow[d, "\pi_{\mathcal{B}}"'] \arrow[r, phantom, shift right=4ex, "\overset{\phi}{\underset{\sim}{\Longrightarrow}}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

be the 2-pullback of $\mathcal{A} \xrightarrow{F} \mathcal{C}$. Its objects are triples $X = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi_X \colon F(a) \xrightarrow{\sim} G(b))$ and for another triple $X' = (a' \in \mathcal{A}, b' \in \mathcal{B}, \varphi_{X'} \colon F(a') \xrightarrow{\sim} G(b'))$ the morphisms are given by tuples $(\mathcal{A} \ni u \colon a \to a', \mathcal{B} \ni v \colon b \to b')$ such that

$$\begin{tikzcd} F(a) \arrow[r, "\sim"] \arrow[d, "F(u)"] & G(b) \arrow[d, "G(v)"] \\ F(a') \arrow[r, "\sim"] & G(b') \end{tikzcd}$$

commutes, that is $\varphi_{X'} \circ F(u) = G(v) \circ \varphi_X$. For a groupoid $\mathcal{D}$ we may consider the induced cospan of groupoids:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\overset{\sim}{\underset{2-PB}{\Longrightarrow}}"] & \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[d, "G \circ ?"] \\ \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \arrow[r, "F \circ ?"'] & \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \end{tikzcd}$$

Since functors into a groupoid are again a groupoid we can apply the construction above, i.e. the 2-pullback of groupoids.

Interlude 2.10. Fix a groupoid G. Then, the construction $\mathcal{D} \in \mathrm{Gpd} \mapsto \mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is suitably functorial, which means

- For all $D \in \mathrm{Gpd}$ it holds that $\mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is a groupoid.
- For all $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ it holds that $? \circ F \colon \mathrm{Fun}(\mathcal{D}_2, \mathcal{G}) \to \mathrm{Fun}(\mathcal{D}_1, \mathcal{G})$ is a functor, given on morphisms as:

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{G} & \mapsto & \mathcal{D}_1 \arrow[r, bend left, "G_1 \circ F"] \arrow[r, bend right, "G_2 \circ F"'] \arrow[r, phantom, "\Downarrow \beta * \mathrm{id}_F"] & \mathcal{G} \end{tikzcd}$$

- For a natural transformation α of functors between groupoids,

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D}_2 \end{tikzcd}$$

  there is a natural transformation:

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}_2, \mathcal{G}) \arrow[rr, bend left, "? \circ F_1"] \arrow[rr, bend right, "? \circ F_2"'] \arrow[rr, phantom, "\Downarrow ? * \alpha"] & & \mathrm{Fun}(\mathcal{D}_1, \mathcal{G}). \end{tikzcd}$$

Let us take the 2-pullback $\mathbb{F}\mathcal{D} := \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{B})$, see 2.9 and analyze the map $\mathcal{D} \to \mathbb{F}\mathcal{D}$.

- For all groupoids $\mathcal{D}$ the category $\mathbb{F}\mathcal{D}$ is a groupoid.
- For all morphisms of groupoids $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ the attribution $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_1$ is a functor.
- The objects of $\mathbb{F}\mathcal{D}_1$ are given as $(p_{\mathcal{A}}\colon \mathcal{D}_1 \to \mathcal{A}, p_{\mathcal{B}}\colon \mathcal{D}_1 \to \mathcal{B}, \phi\colon F \circ p_{\mathcal{A}} \xrightarrow{\sim} G \circ p_{\mathcal{B}})$, that is the 2-pullback, to be more specific they are given by the datum of a pullback diagram:

$$\begin{tikzcd} \mathcal{D}_1 = \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

- For all functors $H\colon \mathcal{D}_1 \to \mathcal{D}_1$ we get $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_2$, that is for a second pullback square

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"'] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

we obtain a commutative diagram as follows:

$$
\begin{tikzcd}
\mathcal{D}_1 \arrow[dr, "H"] \arrow[drr, bend left, "q_{\mathcal{B}}\circ H"] \arrow[ddr, bend right, "q_{\mathcal{A}}\circ H"'] & & \\
& \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"'] \arrow[dr, phantom, "\xRightarrow{\psi}"] & \mathcal{B} \arrow[d, "G"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
$$

The associated natural transformation is given by $\psi_H \colon F \circ q_{\mathcal{A}} \circ H \implies G \circ q_{\mathcal{B}} \circ H$.

The next question we can naturally ask is, what do morphisms in $\mathbb{F}\mathcal{D}$ look like? They are given by quadruples natural transformations $(\alpha, \beta, F_\alpha, G_\beta)$, such that:

$$
\begin{array}{cccc}
(p_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{A}, & p_{\mathcal{B}} \colon \mathcal{D}_2 \longrightarrow \mathcal{B}, & F \circ p_{\mathcal{A}} \xRightarrow[\sim]{\phi} & G \circ p_{\mathcal{B}}) \\
\Downarrow{\scriptstyle\alpha} & \Downarrow{\scriptstyle\beta} & \Downarrow{\scriptstyle F_\alpha} & \Downarrow{\scriptstyle G_\beta} \\
(q_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{A}, & q_{\mathcal{B}} \colon \mathcal{D}_2 \longrightarrow \mathcal{B}, & F \circ q_{\mathcal{A}} \xRightarrow[\psi]{\sim} & G \circ q_{\mathcal{B}})
\end{array}
$$

For every $\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "H_1"] \arrow[r, bend right, "H_2"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{D}_2 \end{tikzcd}$ there is a diagram $\begin{tikzcd} \mathbb{F}\mathcal{D}_2 \arrow[r, bend left, "\mathbb{F}H_1"] \arrow[r, bend right, "\mathbb{F}H_2"'] \arrow[r, phantom, "\Downarrow\mathbb{F}\gamma"] & \mathbb{F}\mathcal{D}_1 \end{tikzcd}$.

Proposition 2.11. *Let $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ be a cospan of groupoids. Then for all groupoids $\mathcal{D}$, it holds that*

$$
\mathbb{X} \colon \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \to \mathbb{F}(\mathcal{D} \xleftarrow{\mathbb{F}H} \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}))
$$

$$
(\mathcal{D} \xrightarrow{H} \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \mapsto
\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[rr, bend left, dashed] \arrow[dr, bend right, dashed] & \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] \arrow[dr, phantom, "\xRightarrow{\phi}"] & \mathcal{B} \arrow[d, "G"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
= H^*(\mathrm{can})
$$

is an isomorphism.

Example 2.12. Let A, B, C be groups and $\mathbb{B}A, \mathbb{B}B, \mathbb{B}C$ their associated groupoids, then for group homomorphisms $A \xrightarrow{f} C \xleftarrow{g} B$, we get that the objects of $\mathbb{B}A \times^{(2)}_{\mathbb{B}C} \mathbb{B}B$ correspond to triples $c_1 = (*_A, *_B, *_C \xrightarrow{c \in C} *_C)$ and a morphism from c_1 to $c_2 = (*_A, *_B, *_C \xrightarrow{c' \in C} *_C)$ from correspond to choices $a \in A, b \in B$ such that $c'a = bc$.

2.1 Exercises

Exercise 1. Let $\mathcal{C}$ be a small category and denote by $\widehat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ its category of presheaves.

(a) Show that for any object $c \in \mathcal{C}$ there is a functor $\widehat{c} := \mathrm{Hom}_{\mathcal{C}}(-, c) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

(b) Show that for any morphism $f \colon c \to d$ in $\mathcal{C}$ there is a natural transformation $f_* \colon \widehat{c} \implies \widehat{d}$ defined by postcomposition.

(c) Conclude that there is a functor $\mathcal{Y} \colon \mathcal{C} \to \widehat{\mathcal{C}}$.

Exercise 2. Observe that we can use the Yoneda functor to construct a functor $(\mathcal{Y}_{\mathcal{C}})^* \circ \mathcal{Y}_{\widehat{\mathcal{C}}} : \widehat{\mathcal{C}} \to \widehat{\widehat{\mathcal{C}}} \to \widehat{\mathcal{C}}$ which associates to a presheaf $X \in \widehat{\mathcal{C}}$ the presheaf $(c \mapsto \mathrm{Nat}(\mathcal{Y}_{\mathcal{C}}(c), X))$.

(a) Show that there is a natural transformation $\chi^X \colon (\mathcal{Y}_{\mathcal{C}})^* \circ \mathcal{Y}_{\widehat{\mathcal{C}}}(X) \implies X$ defined by

$$\begin{aligned} \chi^X_c : (\mathcal{Y}_{\mathcal{C}})^* \circ \mathcal{Y}_{\widehat{\mathcal{C}}}(X)(c) = \mathrm{Nat}(\widehat{c}, X) &\to X(c) \\ \eta &\mapsto \eta_c(\mathrm{id}_c) \end{aligned}$$

for each $c \in \mathcal{C}$.

(b) Show that $\{\chi^X\}_{X \in \widehat{\mathcal{C}}}$ assemble into a natural transformation $\chi \colon (\mathcal{Y}_{\mathcal{C}})^* \circ \mathcal{Y}_{\widehat{\mathcal{C}}} \Rightarrow \mathrm{id}_{\widehat{\mathcal{C}}}$.

(c) Show that for a presheaf $X \in \widehat{\mathcal{C}}$ and $c \in \mathcal{C}$, any element $x \in X(c)$ defines a natural transformation $\eta^x \colon \mathcal{Y}(c) \Rightarrow X$ defined by

$$\begin{aligned} \eta^x_d \colon \widehat{c}(d) = \mathrm{Hom}_{\mathcal{C}}(d, c) &\to X(d) \\ f &\mapsto X(f)(x) \end{aligned}$$

for every $d \in \mathcal{C}$.

(d) Deduce that the natural transformation χ^X is a natural isomorphism for every presheaf X. Moreover, conclude that χ is a natural isomorphism itself.

Exercise 3. Fix a small category $\mathcal{C}$.

(a) Show that the category of small categories admits products.

(b) Deduce that the functor $\mathrm{Fun}(\mathcal{C}, -) \colon \mathrm{Cat} \to \mathrm{Cat}$ which associates to a small category $\mathcal{D}$ the category of functors $\mathcal{C} \to \mathcal{D}$ admits $- \times \mathcal{C} \colon \mathrm{Cat} \to \mathrm{Cat}$ as a left adjoint.

(c) Show that this adjunction upgrades to an isomorphism of categories

$$\Phi\colon \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

in the category of large categories.

Exercise 4. Fix a small category $\mathcal{C}$

(a) Deduce that $\phi((\mathcal{Y}_{\mathcal{C}})^* \circ \mathcal{Y}_{\widehat{\mathcal{C}}})$ is isomorphic to the evaluation functor in $\mathrm{Fun}(\widehat{\mathcal{C}} \times \mathcal{C}, \mathrm{Set})$.

(b) Deduce that $\mathrm{Nat}(\mathcal{Y}(c), X) \cong X(c)$ naturally in both X and c.

(c) Conclude that $\mathcal{Y}\colon \mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful.

Exercise 5. Consider two small categories A and B and a functor $F\colon B \to \mathrm{Fun}(A, \mathcal{C})$. Assume further that $\lim_B(\mathrm{ev}_a \circ F)$ exists in $\mathcal{C}$ for every $a \in A$.

(a) Show that a cone C of F is a limit cone if and only if for every $a \in A$ the evaluation $C(a)$ is a limit cone of $\mathrm{ev}_a \circ F$.

(b) Show that $\mathrm{Fun}(A, \mathcal{C})^{\mathrm{op}} \cong \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ and give the dual statement of (a).

(c) Deduce that for any small category A the category of presheaves $\widehat{A}$ is cocomplete.

Let us note that the existence of $\lim_B(\mathrm{ev}_a \circ F)$ in (a) is a necessary condition as monomorphisms in functor categories are not necessarily pointwise monomorphisms.

Exercise 6. We have seen that a category A fully faithfully embeds into its presheaf category $\widehat{A}$ via the Yoneda embedding. Conversely, we will now show that any presheaf comes with a diagram in $\mathcal{Y}(A)$ whose colimit is X itself.

(a) Show that for a presheaf $X \in \widehat{A}$ there is an associate category $\int^A X$ with objects (a, x) where $a \in A$ and $x \in X(a)$ and morphisms $f\colon (a, x) \to (b, y)$ given by morphisms $f\colon a \to b$ in A such that $X(f)(y) = x$.

The category $\int^A X$ is called the category of elements of X.

(b) Show that the forgetful functor together with the Yoneda embedding

$$\begin{aligned} \mathcal{Y} \circ \pi_X\colon \int^A X &\xrightarrow{\pi_X} A \xrightarrow{\mathcal{Y}} \widehat{A} \\ (a, x) &\mapsto a \mapsto \widehat{a} \end{aligned}$$

admits X as its colimit.

(c) Show that a morphism $f\colon X \to Y$ in $\widehat{A}$ induces a functor $\int^A f\colon \int^A X \to \int^A Y$, upgrading the assignment to a functor $\int^A\colon \widehat{A} \to \mathrm{Cat}$.

Exercise 7. Let $u\colon A \to \mathcal{C}$ be a functor between small categories. By precomposing we obtain a functor $u^*\colon \widehat{\mathcal{C}} \to \widehat{A}$ and precomposing this with the Yoneda embedding we obtain a functor $\tilde{u}^* := u^* \circ \mathcal{Y}\colon \mathcal{C} \to \widehat{\mathcal{C}} \to \widehat{A}$. Assuming that $\mathcal{C}$ is cocomplete, we aim to show $\tilde{u}^*$ admits a left adjoint $u_!\colon \widehat{A} \to \mathcal{C}$.

(a) Show that if $u_!$ exists, then it preserves colimits.

(b) Show that if $F\colon \widehat{A} \to \mathcal{C}$ preserves colimits, then $F \dashv (F \circ \mathcal{Y})^*$.

Hence, it is sufficient to show that there exists a colimit preserving functor $u_! := U\colon \widehat{A} \to \mathcal{C}$ such that $U \circ \mathcal{C}$ and u are isomorphic in $\mathrm{Fun}(A, \mathcal{C})$.

(c) Show that the assignment

$$U(X) := \operatorname{colim} u\left(\int^A X\right) := \operatorname{colim}_{\int^A X} u \circ \pi_X$$

assembles into a functor where a morphism $f\colon X \to Y$ is send to the morphism induced by the diagram $u(\int^A f)$ with the notation from Exercise 1.2.

(d) Argue that U is a colimit preserving by showing that $\int^A \operatorname{colim}_{i\in I} X_i$ is a colimit for $\int^A X_i$ in Cat.

(e) Conclude by showing that $U \circ \mathcal{Y} \cong u$ in $\mathrm{Fun}(A, \mathcal{C})$.

Exercise 8. Let $u\colon A \to B$ be a functor between small categories. Let $u^*\colon \widehat{B} \to \widehat{A}$ denote the functor obtained by precomposition with u.

(a) Deduce from Exercise 1.3 that a functor $F\colon \widehat{B} \to \mathcal{C}$ admits a right adjoint if $\mathcal{C}$ is a cocomplete category.

(b) Show that u^* preserves colimits.

(c) Conclude that there exists a right adjoint $u^* \dashv u_*$. Give an explicit description of u_*.

Exercise 9. Recall that for a presheaf $X \in \widehat{\mathcal{C}}$ and $c \in \mathcal{C}$ and element $u \in X_c := X(c)$ is called universal if the natural transformation $\eta^u\colon \widehat{c} \to X$ from Exercise 0.2 is an isomorphism.

(a) Show that $u \in X_c$ is a universal element if and only if (c, u) is a final element if $\int^A X$, i.e. any other object (d, v) admits a unique morphism $f\colon (d, v) \to (c, u)$.

(b) Deduce that a presheaf is representable, i.e. isomorphic to $\widehat{d}$ for some $d \in \mathcal{C}$ if and only if $\int^A X$ has a final element.

(c) Show that for any isomorphism $f\colon d \xrightarrow{\sim} c$ in $\mathcal{C}$, $X(f)$ induces a bijection between the universal elements in X_c and X_d.

$$f^* := X(f)\colon \{u \in X_c \text{ universal}\} \xrightarrow{\sim} \{v \in X_d \text{ universal}\}$$

Exercise 10. Let $D\colon I \to \mathcal{C}$ be a functor between small cetegories and consider $\tilde{D} := \mathcal{Y} \circ D\colon I \to \mathcal{C} \to \widehat{\mathcal{C}}$. Recall that a cone of D is an object $a \in \mathcal{C}$ togehter with a morphism $(a_i\colon a \to D(a))_{i \in A}$ such that for each $f\colon i \to j$ in I we have that $a_j = D(f) \circ a_i$. Furthermore, the cones form a category with morphisms $g\colon (a, (a_i)_{i\in I}) \to (b, (b_i)_{i \in I})$ by $g\colon a \to b$ with $a_i = b_i \circ g$ for all $i \in I$.

1. Show that the category of cones of D in $\mathcal{C}$ is isomorphic to the category $\int^{\mathcal{C}} \lim \tilde{D}$ which exists by the completeness of $\widehat{\mathcal{C}}$.

2. Deduce that D admits a limit if and only if $\lim \tilde{D}$ admits a universal element if and only if $\lim \tilde{D}$ is representable.

3. Conclude that the Yoneda embedding preserves limits.

4. Conclude that any isomorphism $f\colon d \xrightarrow{\sim} c$ in $\mathcal{C}$ induces an isomorphism

   $$f^*\colon \{u \in \lim_{i \in I} \operatorname{Hom}(c, D(i)) \text{ universal}\} \xrightarrow{\sim} \{v \in \lim_{i\in I} \operatorname{Hom}(d, D(i)) \text{ universal}\}$$

   of universal elements.

Lecture 17.04

3 2-categories

Reference for this section is [3, ch. 2.2.1- 2.2.3].

Definition 3.1. A **strict 2-category** $\mathcal{C}$ consists of:

1. A class $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$
2. For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose objects $f\colon X \to Y$ are called 1-morphisms and whose morphisms $\alpha\colon f \Rightarrow g$ are called 2-morphisms, with a vertical composition of 2-morphisms, that is associative, and unital.
3. For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a horizontal composition functor

$$- \circ -\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

   that is compatible with the vertical composition, in the following way

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[rr, bend left=50, "g \circ f"] \arrow[rr, bend right=50, "g' \circ f'"'] \arrow[rr, phantom, "\Downarrow \beta * \alpha"] & & Z
\end{tikzcd}$$

4. Functoriality of horizontal composition:

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_f"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_g"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[rr, bend left=50, "g \circ f"] \arrow[rr, bend right=50, "g \circ f"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_{g \circ f}"] & & Z
\end{tikzcd}$$

5. There is a composition

$$\begin{tikzcd}
X \arrow[r, bend left=80, "f_1"] \arrow[r, "f_2"'] \arrow[r, bend right=80, "f_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow \gamma"] & Y \arrow[r, bend left=80, "g_1"] \arrow[r, "g_2"'] \arrow[r, bend right=80, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \beta"] \arrow[r, phantom, shift right=3ex, "\Downarrow \delta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left=50, "g_1 \circ f_1"] \arrow[r, bend right=50, "g_3 \circ f_3"'] \arrow[r, phantom, "\Downarrow \eta"] & Z
\end{tikzcd}$$

   where η is the Godement product $(\delta * \gamma) \circ (\beta * \alpha)$.

The above data should satisfy the following axioms:

- (Unitality) For all $X \in \mathrm{Ob}(\mathcal{C})$ there exists $\mathrm{id}_X \in \mathrm{Ob}(\mathrm{Hom}_{\mathcal{C}}(X,X))$ an identity 1-morphism such that for all $Y \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(X,X) \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f \mapsto (f, \mathrm{id}_X) \mapsto f \circ \mathrm{id}_X \end{aligned}$$

  is equal to the identity functor $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Similarly the functor

$$\begin{aligned} \mathbb{1} \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(X,X) \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X) \\ g \mapsto \mathrm{id}_X \circ g = g \end{aligned}$$

  is equal to the identity functor $\mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ the following square of functors commutes strictly.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(-\circ-)"] \arrow[d, "(-\circ-)\times\mathrm{Hom}_{\mathcal{C}}(W,X)"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[d, "(-\circ-)"] \\ \mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(-\circ-)"] & \mathrm{Hom}_{\mathcal{C}}(W,Z) \end{tikzcd}$$

Definition 3.2. A **strict monoidal category** is a strict 2-category with a single object, that is the following data:

1. A strict 2-category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$.
2. A category $\mathcal{M} := \mathrm{Hom}_{B\mathcal{M}}(*,*)$.
3. A monoidal composition $- \otimes -\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, that fullfills the following axioms.

   - (Unitality) There exists $\mathbb{1}_{\mathcal{M}} \in \mathrm{Ob}(\mathcal{M}) = \mathrm{Ob}(Hom_{B\mathcal{M}}(*,*))$ such that the functor

$$\begin{aligned} \mathcal{M} \to \mathcal{M} \\ M \mapsto M \otimes \mathbb{1}_{\mathcal{M}} \end{aligned}$$

     is the identity, meaning there are equalities $M \otimes \mathbb{1}_{\mathcal{M}} = M = \mathbb{1}_{\mathcal{M}} \otimes M$.
   - (Associativity) The following square commutes

$$\begin{tikzcd} \mathcal{M} \times \mathcal{M} \times \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{M}} \times (-\circ-)"] \arrow[d, "(-\otimes-)\times\mathrm{id}_{\mathcal{M}}"'] & \mathcal{M} \times \mathcal{M} \arrow[d, "\otimes"] \\ \mathcal{M} \times \mathcal{M} \arrow[r, "-\otimes-"] & \mathcal{M} \end{tikzcd}$$

     that is for all $M_1, M_2, M_3 \in \mathcal{M}$ it holds that $M_1 \otimes (M_2 \otimes M_3) = (M_1 \otimes M_2) \otimes M_3$.

Definition 3.3. Let $\mathcal{C}, \mathcal{D}$ be strict 2-categories. A **strict 2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of

1. A map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$ sending an object X to $F(X)$.
2. For all $X, Y \in \operatorname{Ob}(\mathcal{C})$ a functor $F_{XY}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(FX, FY)$ such that the following hold:
   - (Unitality) $\forall X \in \operatorname{Ob}(\mathcal{C}), F(\mathrm{id}_X) = \mathrm{id}_{FX} \in \operatorname{Hom}_{\mathcal{D}}(FX, FX)$
   - (Composition) Let $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "-\circ^{\mathcal{C}}-"] \arrow[d, "F_{Y,Z}\times F_{X,Y}"] & \operatorname{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\ \underline{\operatorname{Hom}}_{\mathcal{D}}(FY,FZ) \times \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FY) \arrow[r, "-\circ^{\mathcal{D}}-"] & \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FZ) \end{tikzcd}$$

Let $\mathcal{C}$ be a strict 2-category then the opposite category $\mathcal{C}^{\mathrm{op}}$ is also a strict 2-category.

Lecture 22.04

Remark 3.4. Every ordinary 1-category $\mathcal{C}$ can be viewed as a strict 2-category as follows:

- for all $X, Y \in \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(X,Y)$,
- horizontal composition = composition in $\mathcal{C}$,
- for all $X \in \operatorname{Ob}(\mathcal{C})$ it holds that $\mathrm{id}_X \in \operatorname{Hom}_{\mathcal{C}}(X,X) = \operatorname{Hom}_{\mathcal{C}}(X,X)$ is the identity morphism in the original catgory $\mathcal{C}$,
- conversely every strict 2-catgory $\mathcal{C}$ has an underlying ordinary category $\mathcal{C}_0$ with $\operatorname{Ob}(\mathcal{C}_0) := \operatorname{Ob}(\mathcal{C})$ and $\forall X, Y \in \operatorname{Ob}(\mathcal{C}_0) = \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}_0}(X,Y) := \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,Y))$. The composition law in $\mathcal{C}_0$ is horizontal composition of 1-morphisms in $\mathcal{C}$, this composition is associative since $\mathcal{C}$ is a strict 2-category.

Definition 3.5. A **2-category** (bicategory) $\mathcal{C}$ is given as follows:

- a class of objects of $\mathcal{C}$ denoted $\operatorname{Ob}(\mathcal{C})$,
- for all $X, Y \in \operatorname{Ob}\mathcal{C}, \operatorname{Hom}_{\mathcal{C}}(X,Y)$ a category of 1-morphisms,
- for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ a composition functor

$$(\text{-} \circ \text{-})\colon \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Z),$$

- for all $X \in \operatorname{Ob}(\mathcal{C})$ an object $\mathrm{id}_X \in \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,X))$, called identity of X, together with an invertible 2-morphism $v_X = \mathrm{id}_X \circ \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}_X$ in $\operatorname{Hom}_{\mathcal{C}}(X,X)$ called **unit constraint**,

- for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a natural isomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(\text{-}\circ\text{-})"] \arrow[d, "(\text{-}\circ\text{-})\times \mathrm{Hom}_{\mathcal{C}}(W,X)"'] \arrow[dr, phantom, "\sim \Downarrow \alpha=\alpha_{W,X,Y,Z}"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,Y) \arrow[d, "(\text{-}\circ\text{-})"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(\text{-}\circ\text{-})"'] & \mathrm{Hom}_{\mathcal{C}}(W,Z)
\end{tikzcd}$$

  where

  $$\alpha_{f,g,h} \colon h \circ (g \circ f) \overset{\sim}{\Longrightarrow} (h \circ g) \circ f \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$$

  is an isomorphism called **associativity constraint**,

- for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the following functors are fully faithful

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f &\mapsto \mathrm{id}_Y \circ f \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ f &\mapsto f \circ \mathrm{id}_X \end{aligned}$$

- and for all $V, W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all composable 1-morphisms $V \xrightarrow{e} W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ the following diagram commutes

$$\begin{tikzcd}
 & h \circ ((g \circ f) \circ e) \arrow[rr, Rightarrow, "\alpha_{h,g\circ f,e}"] & & (h \circ (g \circ f)) \circ e \arrow[dr, Rightarrow, "\alpha_{h,g,f} * \mathrm{id}_e"] & \\
h \circ (g \circ (f \circ e)) \arrow[ur, Rightarrow, "\mathrm{id}_h * \alpha_{g,f,e}"] \arrow[drr, Rightarrow, "\alpha_{h,g,f\circ e}"] & & & & ((h \circ g) \circ f) \circ e \\
 & & (h \circ g) \circ (f \circ e) \arrow[urr, Rightarrow, "\alpha_{h\circ g,f,e}"] & &
\end{tikzcd}$$

  which is called the **Pentagon identity**.

Example 3.6. Every strict 2-category can be viewed as a 2-category with unit constraints and associativity constraints given by identities.

Definition 3.7. Monoidal categories are 2-categories with a single object. That is a 2 category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$ and $\mathcal{M} := \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *)$. The horizontal composition defines the monoidal composition

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

and there is an associativity constraint $\alpha_{M_1,M_2,M_3} \colon M_1 \otimes (M_2 \otimes M_3) \xrightarrow{\sim} (M_1 \otimes M_2) \otimes M_3$ in $\mathcal{M}$.

Example 3.8. Let k be a field and $(\mathrm{Vect}_k, \otimes_k, k)$ a monoidal category, the associator is given as:

$$\begin{aligned} \mathrm{can} \colon V_1 \otimes (V_2 \otimes V_3) &\xrightarrow{\sim} (V_1 \otimes V_2) \otimes V_3 \\ v_1 \otimes (v_2 \otimes v_3) &\mapsto (v_1 \otimes v_2) \otimes v_3 \end{aligned}$$

Example 3.9. Let V be a category with finite products, then $(V, x, *)$ is a monoidal category, with $*$ its terminal object and a functor $- \times -\colon V \times V \to V$, such that $V_1 \times (V_2 \times V_3) \xrightarrow{\sim} (V_1 \times V_2) \times V_3$.

Example 3.10. The 2-category Bim of all bimodules has

- Objects Ob(Bim) given by all associative unital rings,
- for $R, S \in \mathrm{Ob}(\mathrm{Bim}), \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R, S) := {}_S\mathrm{mod}_R \simeq L\,\mathrm{Fun}({}_R\mathrm{mod}, {}_S\mathrm{mod})$,
- for all $R, S, T \in \mathrm{Ob}(\mathrm{Bim})$ the horizontal composition is given by the functor

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathrm{Bim}}(S,T) \times \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,S) \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,T) \arrow[d, equal] \\
{}_T\mathrm{mod}_S \times {}_S\mathrm{mod}_R \arrow[r] & {}_T\mathrm{mod}_R
\end{tikzcd}$$

$({}_TM_S, {}_SN_R) \mapsto ({}_TM \otimes_S N_R)$

- For all $R \in \mathrm{Ob}(\mathrm{Bim}), \mathrm{id}_R = {}_RR_R \in {}_R\mathrm{mod}_R = \underline{\mathrm{Hom}}\,\mathrm{Bim}(R, R)$.

$$\begin{gathered}
{}_RR \otimes_R R_R \xrightarrow{\sim} {}_RR_R \\
{}_UL \otimes_T (M \otimes_S N) \xrightarrow{\sim}_{\mathrm{can}} ({}_UL \otimes_T M_S) \otimes_S N_R
\end{gathered}$$

Construction 3.11. Let $\mathcal{C}$ be a 2-category, there is a fully faithful functor

$$\begin{gathered}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \\
f \mapsto \mathrm{id}_Y \circ f
\end{gathered}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and there is a bijection of morphisms called the **left unit constraint**:

$$\mathrm{id}_Y \circ ?\colon\ \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{id}_Y \circ f, f) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)(\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f), \mathrm{id}_Y \circ f)$$

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\exists ! \lambda_f"] & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim"] & & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\sim"', "v_Y * \mathrm{id}_f"] \\
f \arrow[u, phantom, "\mapsto"] & & \mathrm{id}_Y \circ f &
\end{tikzcd}$$

furthermore there is a fully faithful functor:

$$\begin{gathered}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \\
f \mapsto f \circ \mathrm{id}_X
\end{gathered}$$

as well as a bijection of morphism called the **right unit constraint**:

$$\begin{tikzcd}
f \circ \mathrm{id}_X \arrow[d, Rightarrow, "\exists ! \rho_f"] & f \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_f * v_X"'] & & (f \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "\sim"'] \\
f & & f \circ \mathrm{id}_X &
\end{tikzcd}$$

Proposition 3.12. *Let $\mathcal{C}$ be a 2-category. The left and right unit constraints determine natural isomorphisms.*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[rr, bend left=50, "f \mapsto \mathrm{id}_Y \circ f"] \arrow[rr, bend right=50, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow \lambda"] & & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[rr, bend left=50, "f \mapsto f \circ \mathrm{id}_X"] \arrow[rr, bend right=50, "\mathbb{1}"'] \arrow[rr, phantom, "\Downarrow \rho"] & & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}$$

Lecture 24.4

Proof. Exercise. Let $\forall\colon X \to Y, \lambda_f$ is an isomorphism. We only prove $\lambda = (\lambda_f\colon \mathrm{id}_Y \circ f \Rightarrow f)_{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)}$ is a natural transformation. Let

$$f \overset{\eta}{\Longrightarrow} g$$

be a morphism in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"'] & f \arrow[d, Rightarrow, "\eta"] & & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[r, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & \mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"] \\
\mathrm{id}_Y \circ g \arrow[r, Rightarrow, "\lambda_g"] & g \arrow[r, phantom, "\xrightarrow{\mathrm{id}_Y \circ -}"] & & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[r, Rightarrow, "\mathrm{id}_Y \circ \lambda_g"] & \mathrm{id}_Y \circ g
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[dr, Rightarrow, "\alpha", "\sim"'] \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & & \mathrm{id} \circ f \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ \eta = \mathrm{id}_{\mathrm{id}_Y} * \eta"] \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[ur, Rightarrow, "v_y * \mathrm{id}_f", "\sim"'] \arrow[d, Rightarrow, "(\mathrm{id}_Y \circ \mathrm{id}_Y) \circ \eta"] & \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ g \arrow[dr, Rightarrow, "v_Y * \mathrm{id}_g", "\sim"'] & \\
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[ur, Rightarrow, "\sim", "\alpha"'] \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_g"] & & \mathrm{id}_Y \circ g
\end{tikzcd}$$

where the left square commutes by the naturality of the associator constraint, and the top and bottom triangle commute by the left unit constraint. For the right square we use the interchange law for composition and the Godement product to obtain $(\mathrm{id}_{\mathrm{id}_Y} * \eta) \circ (v_Y * \mathrm{id}_f) = (v_\eta * \eta) = (v_Y * \mathrm{id}_g) \circ (\mathrm{id}_Y \circ \eta)$ $\square$

Proposition 3.13. *Let $\mathcal{C}$ be a 2-category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ two composable 1-morphisms in $\mathcal{C}$. Then the following triangle*

$$\begin{tikzcd}
g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_g \times \lambda_f"'] & & (g \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\rho_g \times \mathrm{id}_f"] \\
& g \circ f &
\end{tikzcd}$$

commutes.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& & g \circ ((\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f) \arrow[rrr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "v_Y"] & & & (g \circ (\mathrm{id}_Y \circ \mathrm{id}_Y)) \circ f \arrow[d, Rightarrow, "u_Y"'] \arrow[ddr, Rightarrow, "\alpha"] & \\
& & g \circ (\mathrm{id}_Y \circ f) \arrow[rrr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "\alpha", "\sim"'] & & & (g \circ id_Y) \circ f \arrow[dlll, equal] & \\
g \circ (\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f)) \arrow[uurr, Rightarrow, "\alpha"] \arrow[urr, Rightarrow, "\lambda_f"'] \arrow[drrr, Rightarrow, "\alpha"] & & (g \circ \mathrm{id}_Y) \circ f \arrow[r, Rightarrow, "\rho_g", "\sim"'] & g \circ f & & g \circ (\mathrm{id}_Y \circ f) \arrow[ll, Rightarrow, "\lambda_f"', "\sim"] \arrow[u, Rightarrow, "\alpha", "\sim"'] & ((g \circ \mathrm{id}_Y) \circ \mathrm{id}_Y) \circ f \arrow[ul, Rightarrow, "\rho_g"'] \\
& & & (g \circ \mathrm{id}_Y) \circ (\mathrm{id}_Y \circ f) \arrow[ul, Rightarrow, "\lambda_f"] \arrow[urr, Rightarrow, "\rho_g"] \arrow[urrr, Rightarrow, "\alpha"] \arrow[u, phantom, "*)"] & & &
\end{tikzcd}$$

The triangles commute by applying unit constraints 3.11 and the square commute and the squares that include an alpha commute by the associator constraints. The only square that remains is $*)$, here we use the interchange law for the Godement product to obtain $(\rho_g * \mathrm{id}_f)(\mathrm{id}_g * \lambda_f) = \rho_g * \lambda_f = (\mathrm{id}_g * \lambda_f) \circ (\rho_g * \mathrm{id}_f)$. $\square$

Corollary 3.14. *Let $\mathcal{C}$ be a 2-category and $X \in \mathcal{C}$, consider $\mathrm{id}_X : X \to X$. Then*

$$\lambda_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

$$\rho_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

are both equal to $v_X : \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$.

Proof. We only do the case $\lambda_{\mathrm{id}_X} = v_X$. By the triangle identity and definition of λ_{id_X} we get that

$$\begin{tikzcd}
\mathrm{id}_X \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_X \circ \lambda_{\mathrm{id}_X}"'] & & (\mathrm{id}_X \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "v_X * \mathrm{id}_X"] \arrow[dl, Rightarrow, bend left, "\rho_{\mathrm{id}_X} * \mathrm{id}_X"] \\
& \mathrm{id}_X \circ \mathrm{id}_X &
\end{tikzcd}$$

and thus $v_X * \mathrm{id}_X = \rho_{\mathrm{id}_X} * \mathrm{id}_X$ which implies that $v_X = \rho_{\mathrm{id}_X}$ since the composition with the identity is fully faithful. $\square$

Definition 3.15. Let $\mathcal{C}$ be a 2-category. The conjugate of $\mathcal{C}$ is the 2-category $\mathcal{C}^c = \mathcal{C}^{co}$ with $\mathrm{Ob}(\mathcal{C}^c) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^c}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)^{\mathrm{op}}$.

Definition 3.16. A $(2,1)$-category is a 2-category such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a groupoid.

Definition 3.17. Let $\mathcal{C}$ be a 2-category. The coarse homotopy category of $\mathcal{C}$ is the 1-ccategory $h\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(\mathcal{C})$ and with sets of morphisms, $\mathrm{Hom}_{h\mathcal{C}}(X, Y) = \pi_0(L\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) = \pi_0(N\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y))$ with the induced composition law, where L is the localisation functor from Cat to Gpd.

Definition 3.18. Let $\mathcal{C}$ be a 2-category. The pith of $\mathcal{C}$ is the 2-category $\mathrm{Pith}(\mathcal{C})$ with objects $\mathrm{Ob}(\mathrm{Pith}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ with $\underline{\mathrm{Hom}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) := \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\cong}$, where $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\cong}$ is the maximal subgroupoid of $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$.

Definition 3.19. The homotopy category of $\mathcal{C}$ is $\mathrm{hPith}(\mathcal{C})$.

3.1 Exercises

Exercise 1. In the lecture, we defined the Godement product $\beta * \alpha\colon G_1 \circ F_1 \Rightarrow G_2 \circ F_1$ of two natural transformations $\alpha\colon F_1 \Rightarrow F_2$ and $\beta\colon G_1 \Rightarrow G_2$ for functors $F_i\colon \mathcal{A} \to \mathcal{B}$ and $G_i\colon \mathcal{B} \to \mathcal{C}$ for $i \in \{1, 2\}$, pointwise given by $(\beta * \alpha)_c := G_2(\alpha_c) \circ \beta_{F_1(c)} = \beta_{F_2(c)} \circ G_1(\alpha_c)$.

(a) Show that there is a strict 2-category of categories Cat with functor categories as homomorphisms and horizontal composition given by the Godement product, in other words, show that

- the Godement product is unital, i.e. $\mathrm{id}_{G_1} * \alpha = \alpha$ and $\beta * \mathrm{id}_{F_1} = \beta$,
- the Godement product is associative, i.e. $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$ for $\gamma\colon H_1 \Rightarrow H_2\colon \mathcal{C} \to \mathcal{D}$,
- for any two categories $\mathcal{C}$ and $\mathcal{D}$, we have that both functors

$$\begin{aligned} \mathrm{id}_{\mathcal{C}} \circ (-), (-) \circ \mathrm{id}_{\mathcal{D}}\colon\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) &\to \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \\ \eta &\mapsto \mathrm{id}_{\mathrm{id}_{\mathcal{C}}} * \eta, \eta * \mathrm{id}_{\mathrm{id}_{\mathcal{C}}} \end{aligned}$$

are the identity functor $\mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}$ and

- vertical and horizontal composition are compatible with each other, i.e. for natural transformations $\eta\colon F \to F', \eta'\colon F' \to F'', \xi\colon G \to G'$ and $G' \to G''$ of functors $F, F', F''\colon \mathcal{A} \to \mathcal{B}$ and $G, G', G''\colon \mathcal{B} \to \mathcal{C}$ we have that
$$(\xi' \cdot \xi) * (\eta' \cdot \eta) = (\xi' * \eta') \cdot (\xi * \eta)$$
for $\cdot$ the pointwise composition of natural transformations.

The centre of a category $\mathcal{C}$ is defined to be the natural endomorphisms of the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$. Observe that the centre $Z(\mathcal{C}) := \mathrm{End}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}})$ is a set that comes both with the pointwise composition $\cdot$ as well as the Godement product $*$.

2. Let M be a set with two unital binary operations $\cdot, *\colon M \times M \to M$ such that
$$(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d).$$
The Eckmann-Hilton argument states that then both products agree and $(M, \cdot, e)$ is an abelian monoid. Prove that this assertion is true.

3. Conclude that the centre of a category is an abelian monoid with either product.

Exercise 2. Fix three groupoids $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ together with functors $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$. Recall that we defined the 2-pullback $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ with objects $(A \in \mathcal{A}, B \in \mathcal{B}, \varphi\colon FA \to GB)$.

(a) Confirm that $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ is a groupoid.

(b) Show that $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is a groupoid for any groupoid $\mathcal{D}$.

(c) Show that the canonical morphism
$$\mathrm{Fun}(-, \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \to \mathrm{Fun}(-, \mathcal{A}) \times^{(2)}_{\mathrm{Fun}(-,\mathcal{C})} \mathrm{Fun}(-, \mathcal{B})$$
is an isomorphism of functors $\mathrm{Gpd}^{\mathrm{op}} \to \mathrm{Gpd}$.

Exercise 3. Fix a monoid $M = (M, \otimes, e)$ and view it as a category $\mathcal{M}$ whose objects are the elements of M with only identity morphisms. Define the monoidal category of M to be the 2-category $\underline{BM}$ with one object $\star$ and $\mathrm{End}_{\underline{BM}}(\star) := \mathcal{M}$. Show that $\underline{BM}$ is indeed a 2-category with horizontal composition functor
$$\otimes\colon \mathcal{M} \times \mathcal{M} = M \times M \to M = \mathcal{M}$$
which is defined on morphisms in the unique way $\mathrm{id}_m \otimes \mathrm{id}_n = \mathrm{id}_{m \times n}$. How does this construction compare to an ordinary 1-category with one object viewed as a 2-category with only identity 2-morphisms?

Exercise 4. A strict monoidal category is a category $\mathcal{C}$ together with a functor $- \times -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and distinguished element $\mathbb{1} \in \mathcal{C}$ such that its delooping $B\mathcal{C}$ with single object $\star$ and $\mathrm{End}_{B\mathcal{C}}$ defines a strict 2-category with horizontal composition $\otimes$ and $\mathrm{id}_* := \mathbb{1}$.

(a) Spell out the explicit axioms of a strict monoidal category without reference to a 2-category.

Consider the category $\mathcal{V} := \mathcal{V}(\mathbb{K})$ which has as objects the natural numbers $\mathbb{N}_0$ and homomorphisms from n to m given by $n \times m$-matrices over a fixed field $\mathbb{K}$. Equip it with a monoidal structure via

$$\begin{aligned} - \otimes -\colon \mathcal{V} \otimes \mathcal{V} &\to \mathcal{V} \\ (n, m) &\mapsto mn \\ (A, B) &\mapsto A \otimes B \end{aligned}$$

where for matrices A and B we denote by $A \otimes B$ their Kronecker product, i.e. the matrix $(a_{ij}B)_{ij}$ for $A = (a_{ij})_{ij}$.

(b) Confirm that $\mathcal{V}$ is a strict monoidal category with unit element $\mathbb{1} := 1$.

(c) What is the difference between $\mathcal{V}$ and the category $\mathrm{vect}_{fd}(\mathbb{K})$ of finite dimensional $\mathbb{K}$-vector spaces equipped with the ordinary tensor product? Is $\mathrm{vect}_{fd}(\mathbb{K})$ with the ordinary tensor product a strict monoidal category.

Exercise 5. Fix a strict 2-category $\mathcal{C} = (\mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(\circ, *))$.

(a) Show that any object $c \in \mathcal{C}$ induces a strict 2-functor

$$\mathrm{Hom}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \underline{\mathrm{Cat}}$$

from $\mathcal{C}$ to the 2-category of categories $\underline{\mathrm{Cat}}$.

(b) Define the composition law of $\mathcal{C}^{\mathrm{op}}$ with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c, d) := \mathrm{Hom}_{\mathcal{C}}(d, c)$ and confirm that your definition yields a strict 2-category.

(c) Show that $\mathrm{Hom}_{\mathcal{C}}(-, c)\colon \mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ is also a 2-functor by arguing that

$$\mathrm{Hom}_{\mathcal{C}}(-, c) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c, -)$$

for all $c \in \mathcal{C}$.

Exercise 6. Fix a group $G = (G, \cdot, 1)$ and an abelian group $\Gamma = (\Gamma, +, 0)$ together with a group homomorphism $\chi\colon G \to \mathrm{Aut}_{\mathrm{Grp}}(\Gamma)$. Instead of $\chi(g)(\gamma)$ we will also write $g(\gamma)$.

(a) Show that there is a cochain complex of abelian groups $C^\bullet(G,\Gamma)$ with $C^n(G,\Gamma) := \mathrm{Hom}_{\mathrm{Set}}(G^n,\Gamma)$ with $G^0 := \{1\}$ and differential

$$\begin{aligned} d^n(\varphi)(g_1,\dots,g_{n+1}) :=& g_1(\varphi(g_2,\dots,g_{n+1})) + (-1)^{n+1}\varphi(g_1,\dots,g_n) \\ &+ \sum_{i=1}^{n}(-1)^i\varphi(g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_{n+1}) \end{aligned}$$

for $n \in \mathbb{N}_0$ and 0 otherwise. In other words, show that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

We call a morphism $\varphi\colon G^n \to \Gamma$ an n-cocycle if $d^n(\varphi) = 0$. The n-th cohomology $H^n(G,\Gamma)$ of G with coefficients in Γ is defined as the quotient of $\ker(d^n)$ by the image of d^{n-1}.

(b) Describe the 0-th cohomology $H^0(G,\Gamma)$.

Consider now a category $\mathcal{C} = \mathcal{C}(G,\Gamma)$ whose objects are the elements of G, any object has endomorphisms Γ and no other morphisms exist.

(c) Show that there is a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which on objects is just the multiplication of G and for morphisms $\gamma\otimes\delta := \gamma + x(\delta)$ for $(\gamma,\delta)\colon (x,y) \to (x,y)$.

(d) Argue that upgrading $\mathcal{C}$ to a monoidal category with tensor product (horizontal multiplication) $\otimes$ and unit constraint $0\colon 1\times 1 \to 1$ is equivalent to choosing a 3-cocycle $\alpha\colon G\times G\times G \to \Gamma$.

Exercise 7. Let $\mathcal{C}$ be a small category with pullbacks.

(a) Show that for objects $x, y \in \mathcal{C}$ there is a category $\mathrm{Span}(\mathcal{C})(x,y)$ with objects

$$\coprod_{z\in\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(z,x)\times\mathrm{Hom}_{\mathcal{C}}(z,y))$$

and morphism sets

$$\mathrm{Hom}_{\mathrm{Span}(\mathcal{C})(x,y)}(x \xleftarrow{f} z \xrightarrow{g} y, x \xleftarrow{f'} z' \xrightarrow{g'} y) := \{\eta \in \mathrm{Hom}_{\mathcal{C}}(z,z') \mid f = f'\circ\eta \wedge g = g'\circ\eta\}$$

with the induced comosition law from $\mathcal{C}$.

(b) Show that the pullbacks in $\mathcal{C}$ induce functors

$$\mu_{x,y,z}\colon \mathrm{Span}(\mathcal{C})(y,z)\times\mathrm{Span}(\mathcal{C})(x,y) \to \mathrm{Span}(\mathcal{C})(x,z)$$

for every $x, y, z \in \mathcal{C}$.

(c) Explain how the above data assembles into a 2-category $\underline{\mathrm{Span}}(\mathcal{C})$. Justify that it is not a strict 2-category.

Exercise 8. Consider the final category $[0]$ which has one object whose only endomorphism is its identity. We may canonically view it as a (strict) 2-category $\underline{[0]}$ with only an identity 2-morphism. Let $\mathcal{M} := (\mathcal{M}, \otimes, \mathbb{1})$ be a monoidal category with delooping 2-category $B\mathcal{M}$. An algebra object of $\mathcal{M}$ is defined as a lax functor $A\colon \underline{[0]} \to B\mathcal{M}$.

(a) Give the definition of an algebra object of $\mathcal{M}$ without reference to 2-categories.

(b) Justify the name algebra object by computing algebra objects of the monoidal category $\mathrm{vect}_{\mathbb{K}} := (\mathrm{vect}_{\mathbb{K}}, \otimes, \mathbb{K})$ for a field $\mathbb{K}$.

(c) Describe algebra objects in the following monoidal categories.

$$1.(\mathrm{Set}, \times, \{\star\}) \qquad 2.(\mathrm{Ab}, \otimes, \mathbb{Z}) \qquad 3.(\mathrm{Cat}, \times, [0])$$

(d) Sketch that the algebra objects in $\mathcal{M}$ form themselves a monoidal category $\mathrm{Alg}(\mathcal{M})$.

(e) Argue that the algebra objects of $\mathrm{Alg}(\mathcal{M})$ are 'commutative algebra objects'.

(f) Give the definition of a coalgebra object.

(g) Show that for a group G, the group algebra $\mathbb{K}G$ carries the structure of a coalgebra with comultiplication $g \mapsto g \otimes g$ and counit $g \mapsto 1_{\mathbb{K}}$ in the monoidal category $\mathrm{vect}_{\mathbb{K}}$.

(h) What kind of additional structure on mod $(\mathbb{K}G)$ could the coalgebra structure above yield?

4 Functors in 2-category theory

The reference for this section is [3, **pages** 2.2.4–2.2.8].

Definition 4.1. Let $\mathcal{C}, \mathcal{D}$ be 2-categories. A **lax 2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ where $X \mapsto F(X)$,
- for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a functor $F = F_{X,Y}\colon \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FX, FY)$,
- for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a morphism $\epsilon_X\colon \mathrm{id}_{FX} \Rightarrow F(\mathrm{id}_X)$ in $\underline{\mathrm{Hom}}_{\mathcal{D}}(FX, FX)$ called **unit constraint**,
- for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "- \circ -"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"] \arrow[dr, phantom, "\overset{\mu}{\Longrightarrow}"] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(FY,FZ) \times \mathrm{Hom}_{\mathcal{D}}(FX,FY) \arrow[r, "- \circ -"] & \underline{\mathrm{Hom}}_{\mathcal{D}}(FX,FZ)
\end{tikzcd}$$

  where $\mu_{g,f}\colon F(g) \circ F(f) \Rightarrow F(g \circ f)$ is called the **composition constraint**. The above data is required to satisfy the following:

  (a) For all $f\colon X \to Y$ 1-morphisms in $\mathcal{C}$ the following diagrams commute

$$\begin{tikzcd}
F(\mathrm{id}_Y) \circ F(f) \arrow[r, Rightarrow, "\mu"] & F(\mathrm{id}_Y \circ f) \arrow[d, Rightarrow, "F(\lambda_f)"] \\
\mathrm{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\epsilon_Y * \mathrm{id}_{F(f)}"] \arrow[r, Rightarrow, "\lambda_{F(f)}"] & F(f)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f) \circ F(\mathrm{id}_X) \arrow[r, Rightarrow, "\mu"] & F(f \circ \mathrm{id}_X) \arrow[d, Rightarrow, "F(\rho_f)"] \\
\mathrm{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\mathrm{id}_{F(f)} * \epsilon_X"] \arrow[r, Rightarrow, "\rho_{F(f)}"] & F(f)
\end{tikzcd}$$

  in $\underline{\mathrm{Hom}}(FX, FY)$,

  (b) and for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and for all $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ 1-morphisms in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
F(h) \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\alpha^{\mathcal{D}}"] \arrow[d, Rightarrow, "\mathrm{id}_{Fh} * \mu"] & (F(h) \circ F(g)) \circ F(f) \arrow[d, Rightarrow, "\mu * \mathrm{id}_{F(f)}"] \\
F(h) \circ F(g \circ f) \arrow[d, Rightarrow, "\mu"] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\mu"] \\
F(h \circ (g \circ f)) \arrow[r, Rightarrow, "F(\alpha^{\mathcal{C}})"] & F((h \circ g) \circ f)
\end{tikzcd}$$

  in $\underline{\mathrm{Hom}}_{\mathcal{D}}(FW, FZ)$.

Definition 4.2. A **2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a lax 2-functor such that for all $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\epsilon_X\colon \mathrm{id}_{FX} \overset{\sim}{\Longrightarrow} F(\mathrm{id}_X)$ is invertible and such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C}), \forall X \xrightarrow{f} Y \xrightarrow{g} Z$ the morphism $\mu_{g,f}\colon F(g) \circ F(f) \overset{\sim}{\Longrightarrow} F(g \circ f)$ is invertible.

Definition 4.3. A **strict 2-functor** is a 2-functor, such that for all $X \in \mathrm{Ob}(\mathcal{C})$ the following hold $\epsilon_X = \mathrm{id}\colon \mathrm{id}_{FX} \Rightarrow F(\mathrm{id}_X), \forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\mu_{g,f} = \mathrm{id} = F(g) \circ F(f) \Rightarrow F(g \circ f)$.

Example 4.4. Lax monoidal functors $\mathcal{M} \to \mathcal{N}$ for $\mathcal{M}$ and $\mathcal{N}$ monoidal categories correspond to lax 2-functors $B\mathcal{M} \to B\mathcal{N}$.

Example 4.5. Let S be a set and $\mathcal{E}_S$ be a category with $\mathrm{Ob}(\mathcal{E}_S) = S$ and for all $x, y \in S, \underline{\mathrm{Hom}}_{\mathcal{E}_S}(x, y) := \{*\}$.

- Fix $\mathcal{M}$ a monoidal category and let $B\mathcal{M}$ be its delooping and $\underline{\mathcal{C}}\colon \mathcal{E}_S \to B\mathcal{M}$ a lax monoidal functor.
- Fix a map $\underline{\mathcal{C}}\colon\ \mathrm{Ob}(\mathcal{E}_S) = S \to \mathrm{Ob}(B\mathcal{M}) = \{*\}$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$ a functor:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) &\to \mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \\ * &\mapsto \underline{\mathcal{C}}(X, Y) \end{aligned}$$

- For all $X \in S$, $\epsilon_X\colon\ \mathrm{id}_{\underline{\mathcal{C}}(X)} = \mathrm{id}_*$, such that

$$\begin{tikzcd} \mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \arrow[loop below, "\mathbb{1}_{\mathcal{M}} = \mathrm{id}_*"] \end{tikzcd} \qquad \mathbb{1}_{\mathcal{M}} \otimes M \xrightarrow[\lambda_M]{\sim} M$$

  This is called the identity constraint.

- For all $X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S = S)$

$$\begin{tikzcd} \{*\} \times \{*\} = \underline{\mathrm{Hom}}_{\mathcal{E}_S}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) \arrow[r, "\circ"', "\sim"] \arrow[d, "\underline{\mathcal{C}} \times \underline{\mathcal{C}}"] \arrow[dr, phantom, "\xRightarrow{\mu}"] & \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Z) = \{*\} \arrow[d, "\underline{\mathcal{C}}"] \\ \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \times \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \arrow[r, "\otimes"] & \mathrm{Hom}_{B\mathcal{M}}(*, *) \end{tikzcd}$$

  where $\mu\colon \underline{\mathcal{C}}(Y, Z) \otimes \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, Z)$. The above data should satisfy the following: $\forall X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$.

$$\begin{tikzcd} \underline{\mathcal{C}}(X, Y) \otimes \underline{\mathcal{C}}(X, Y) \arrow[r, "\mu"] & \underline{\mathcal{C}}(X, Y) \arrow[d, "\mathrm{id}"] \\ \mathbb{1}_{\mathcal{M}} \otimes \underline{\mathcal{C}}(X, Y) \arrow[u, "\mathcal{E}_Y \otimes \mathrm{id}"] \arrow[r, "\lambda_{\underline{\mathcal{C}}(X,Y)}"] & \underline{\mathcal{C}}(X, Y) \end{tikzcd}$$

- Let $\mathcal{M} = \mathrm{Set}$

$$\begin{tikzcd} (\mathrm{id}_X, f) \arrow[r, mapsto] & \mathrm{id}_X \circ f \arrow[d, equal] \\ (x, f) \arrow[u] \arrow[r, mapsto] & f\colon X \to Y \end{tikzcd}$$

  similarly for the right constraint.

- For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S)$ the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathcal{C}}(Y,Z)\otimes(\underline{\mathcal{C}}(X,Z)\otimes\underline{\mathcal{C}}(W,X)) \arrow[r, "\alpha^M"] \arrow[d] & (\underline{\mathcal{C}}(Y,Z)\otimes\underline{\mathcal{C}}(X,Z))\otimes\underline{\mathcal{C}}(W,X) \arrow[d] \\
\underline{\mathcal{C}}(Y,Z)\otimes\underline{\mathcal{C}}(W,Y) \arrow[d, "\mu"] & \underline{\mathcal{C}}(X,Z)\otimes\underline{\mathcal{C}}(w,Z) \arrow[d] \\
\underline{\mathcal{C}}(W,Z) \arrow[r, "\underline{\mathcal{C}}(\alpha)=\mathrm{id}"] & \underline{\mathcal{C}}(W,Z)
\end{tikzcd}$$

Remark 4.6. Dg-categories are Lax 2-functors.

Exercise 1. Let $\mathcal{C}$ be a 2-category and $\mathcal{D}$ a 1-category, then every lax 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ is strict.

Definition 4.7. A $F\colon \mathcal{C} \to \mathcal{D}$ lax functor between 2-categories is

- **unitary** if $\forall X \in \mathcal{C}, \epsilon_X$ is an isomorphism,
- **strictly unitary** $\forall X \in \mathcal{C}, \epsilon_X = \mathrm{id}_X$,
- composition of lax functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

between 2-categories is the lax functor $G \circ F$ defined as follows:

  - on objects $(G \circ F)(X) = G(F(X))$,
  - on morphisms the composition is given by the dashed arrow,

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, dashed, "GF_{X,Y}"] \arrow[d, "F"] & \underline{\mathrm{Hom}}_{\mathcal{E}}((GF)X,(GF)Y) \arrow[d, equal] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(FX,FY) \arrow[r, "G_{FX,FY}"] & \underline{\mathrm{Hom}}_{\mathcal{E}}(G(FX),G(FY))
\end{tikzcd}$$

  - identity constraints are given as,

$$\begin{tikzcd}
& G(\mathrm{id}_{FX}) \arrow[dr, Rightarrow, "G(\epsilon_X^F)"] & & \mathrm{id}_{FX} \arrow[d, Rightarrow, "\epsilon_X^F"] \\
\mathrm{id}_{GFX} \arrow[ur, Rightarrow, "\epsilon_{FX}^G"] \arrow[rr, equal, "\epsilon_X^{GF}"'] & & GF(id_X) & F(\mathrm{id}_X)
\end{tikzcd}$$

  - $\forall X \xrightarrow{f} Y \xrightarrow{g} Z$ 1-morphisms in $\mathcal{C}$ the bottom arrow in the following diagram gives the composition constraint:

$$\begin{tikzcd}
& G(F(g)\circ F(f)) \arrow[dr, Rightarrow, "G(\mu_{g,f}^F)"] & \\
GF(g)\circ GF(f) \arrow[ur, Rightarrow, "\mu_{Fg,Ff}^G"] \arrow[rr, equal, "\mu_{g,f}^{GF}"'] & & GF(g\circ f)
\end{tikzcd}$$

4.1 Exercises

Exercise 1. Recall that in the lecture we have constructed for a 2-category $\mathcal{C}$ its left unit constraint $\lambda = \lambda_{(x,y)} = (\lambda_f)_{f \in \mathrm{Hom}_{\mathcal{C}}(x,y)}$ as the preimage of

$$\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \overset{\alpha}{\Longrightarrow} (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \xRightarrow{u_Y * \mathrm{id}_f} \mathrm{id}_Y \circ f$$

under the fully faithful functor $\mathrm{id}_Y \circ -\colon \mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y)$ respectively for each 1-morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$. Furthermore, we have shown in the lecture that $\lambda\colon \mathrm{id}_y \circ - \Rightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ assembles into a natural transformation.

(a) Argue that λ is in fact a natural isomorphism.

Similarly ,we have defined the right unit constrained $\rho := \rho_{(x,y)}\colon - \circ \mathrm{id}_x \Longrightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ and then showed that the left and right unit constrained satisfy the triangle identity, $\mathrm{id}_y * \lambda_f = (\rho_g * \mathrm{id}_f) \cdot \alpha$ in $\mathrm{Hom}_{\mathcal{C}}(x,z)$ for $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(y,z)$.

(b) Show that the following may be equivalently used in the definition of a 2-category.

   (1) The functors $\mathrm{id} \circ -$ and $- \circ \mathrm{id}_x$ are fully faithful for every $x, y \in \mathcal{C}$ and there are isomorphisms $u_x\colon \mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$ for every $x \in \mathcal{C}$.

   (2) There are natural isomorphisms $\lambda\colon \mathrm{id}_y \circ - \Longrightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}(x,y)}}$ and $\rho\colon - \circ \mathrm{id}_x \Longrightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ satisfying the triangle identity for every $x, y, z \in \mathcal{C}$.

   (3) The functors $\mathrm{id}_y \circ -$ and $- \circ \mathrm{id}_x$ are naturally isomorphic to the identity functor $\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ for every $x, y \in \mathcal{C}$.

   (4) The functors $\mathrm{id}_y \circ -$ and $- \circ \mathrm{id}_x$ are equivalneces of categories for every $x, y \in \mathcal{C}$ and there are isomorphisms $u_x\colon \mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$ for every $x \in \mathcal{C}$.

(c) Show that two families of natural transformations $\lambda' := \lambda'_{(x,y)}\colon \mathrm{id}_y \circ - \Longrightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ and $\rho' := \rho'_{(x,y)}\colon - \circ \mathrm{id}_x \Longrightarrow \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)}$ satisfying the triangle identity are the left and right unit constraint if and only if both families are the unitors at the identity 1-morphisms, $\lambda'_{\mathrm{id}_x} = u_x = \rho'_{\mathrm{id}_x}$.

Recall that in the definition of a lax functor $F\colon \mathcal{C} \to \mathcal{D}$ we required that the functor preserves the left and right unit constraint in the sense that

$$\lambda_{F(f)} = F(\lambda_f) \cdot \mu_{\mathrm{id}_y, f} \cdot (\epsilon_y * \mathrm{id}_{F(f)})\colon \mathrm{id}_{F(y)} \circ F(f) \xRightarrow{\epsilon_y * \mathrm{id}_{F(f)}} F(\mathrm{id}_y) \circ F(f) \xRightarrow{\mu_{\mathrm{id}_y, f}} F(\mathrm{id}_y \circ f) \overset{\lambda_f}{\Longrightarrow} F(f)$$

$$\rho_{F(f)} = F(\rho_f) \cdot \mu_{f, \mathrm{id}_x} \cdot (\mathrm{id}_{F(f)} * \epsilon_x)\colon F(f) \circ \mathrm{id}_{F(y)} \xRightarrow{\mathrm{id}_{F(x)} * \epsilon_x} F(f) \circ F(\mathrm{id}_x) \xRightarrow{\mu_{f, \mathrm{id}_x}} F(f \circ \mathrm{id}_x) \overset{\rho_f}{\Longrightarrow} F(f)$$

for all $x, y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$.

(d) Show that equivalently, in the definition of a Lax functor we may require that

$$\epsilon_x \cdot u^{\mathcal{D}}_{F(x)} = F(u^{\mathcal{C}}_x) \cdot \mu_{\mathrm{id}_x,\mathrm{id}_x} \cdot (\epsilon_x * \epsilon_x)$$

for every $x \in \mathcal{C}$ and additionally for every 1-morphism $f \colon x \to y$ in $\mathcal{C}$ the canonical 2-morphisms

$$\mu_{\mathrm{id}_y,f} \cdot (\epsilon_y * \mathrm{id}_{F(f)}) \colon \mathrm{id}_{F(y)} \circ F(f) \xRightarrow{\epsilon_y * \mathrm{id}_{F(f)}} F(\mathrm{id}_y) \circ F(f) \xRightarrow{\mu_{\mathrm{id}_y,f}} F(\mathrm{id}_y \circ f)$$

$$\mu_{f,\mathrm{id}_x} \cdot (\mathrm{id}_{F(f)} * \epsilon_x) \colon F(f) \circ \mathrm{id}_{F(x)} \xRightarrow{\mathrm{id}_{F(f)} * \epsilon_x} F(f) \circ F(\mathrm{id}_X) \xRightarrow{\mu_{f,\mathrm{id}_x}} F(f \circ \mathrm{id}_x)$$

are monomorphisms.

Exercise 2. Recall that there is an embedding of the category of small categories into the category of small 2-categories, $\iota \colon$ Cat $\to$ 2-Cat, by adding to a category $\mathcal{D}$ only identity 2-morphisms. The goal of the exercise is to show that ι admits a left adjoint.

(a) Confirm that ι is fully faithful by arguing that any lax functor $F \colon \mathcal{C} \to \iota(\mathcal{D})$ is in fact a strict functor.

By defintion the connected components $\pi_0(\mathcal{D})$ of a small category $\mathcal{C}$ are defined as the coequalizer

$$\mathrm{Mor}(\mathcal{D}) \underset{\text{source}}{\overset{\text{target}}{\rightrightarrows}} \mathrm{Ob}(\mathcal{D}),$$

i.e. the set of all objects modulo the equivalence relation that identifies objects which have a morphism between them. The coarse homotopy category $h\mathcal{C}$ of a 2-category $\mathcal{C}$ is defined to be the category with the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{h\rfloor}(x,y) := \pi_0(\mathrm{Hom}_{\mathcal{C}}(x,y))$ for objects $x, y \in h\mathcal{C}$.

(b) Define the composition law of the coarse homotopy category $h\mathcal{C}$ and confirm that it is indeed a category.

(c) Show that h upgrades to a functor $h \colon$ 2- Cat $\to$ Cat which is a left adjoint to ι. Describe the universal property of the coarse homotopy category.

Recall that the pith $\mathrm{Pith}(\mathcal{C})$ of a 2-category $\mathcal{C}$ is the largest contained (2,1)-category. The homotopy category of a 2-category $\mathcal{C}$ is defined to be the coarse homotopy category of its pith, that is $h\,\mathrm{Pith}(\mathcal{C}$.

(d) Construct a comparison between the homotopy category of a 2-catgeory and its coarse homotopy category. Which way does the comparison map go ? When is it an equivalence?

Exercise 3. Recall from Exercise 3.4 that we defined for a group G acting on an abelian group Γ a monoidal category $\mathcal{C}_\alpha := (\mathcal{C}(G,\Gamma), \otimes, \alpha, 0)$ for a 3-cocycle $\alpha \colon G \times G \times G \to \Gamma$.

(a) Show that for a monoidal functor $F = (F, \mu, \epsilon)\colon \mathcal{C}_\alpha \to \mathcal{C}_\beta$, the unit constraint $\epsilon \in \Gamma$ is already determined by the tensor constraint $\mu\colon G \times G \to \Gamma$.

(b) Show that the identity functor $\mathrm{id}_{\mathcal{C}(G,\Gamma)}$ upgrades to a monoidal functor $\iota_{\beta,\alpha}\colon \mathcal{C}_\alpha \to \mathcal{C}_\beta$ if and only if $\beta - \alpha$ is in the image of $d^2\colon \operatorname{Hom}_{\mathrm{Set}}(G \times G, \Gamma) \to \operatorname{Hom}_{\mathrm{Set}}(G \times G \times G, \Gamma)$.

(c) Give an interpretation of the third cohomology $H^3(G, \Gamma)$ of G valued in Γ.

5 Grothendieck Construction

The reference for this section is [1, B.1.2, B.1.3].

Let A be a small 1-category, $X\colon A^{\mathrm{op}} \to \mathrm{Set}$ a presheaf.

Recall: The category of elements of X is the category $\int^A X$ with

- objects given by tuples $(a \in A, x \in X(a))$,
- morphisms given for another tuple $(b \in A, y \in X(b))$ by a morphism $f\colon a \to b$ such that $f^*(y) = x$.

Example 5.1. Consider $\int^A \mathrm{Hom}_A(-, b)$. The objects are given as $(a \in A, x\colon a \to b)$ and for a second tuple $(a' \in A, y\colon a' \to b)$ we have morphisms given $f\colon a \to a'$ such that $f^*(y) = x = y \circ f$.

Example 5.2. Let $\int^{A \times A^{\mathrm{op}}} \mathrm{Hom}_A(-, -)$ be the twisted arrow category of A.

- The objects are given as $((a, b) \in A \times A^{\mathrm{op}}, x\colon a \to b)$,
- the morphisms are given by tuples (f, g), where $f\colon a \to a', g\colon b' \to b$ such that the following square commutes

$$\begin{tikzcd} a' \arrow[r, "y"] & b' \arrow[d, "g"] \\ a \arrow[u, "f"] \arrow[r, "x"] & b \end{tikzcd}$$

for all objects $((a', b') \in A \times A^{\mathrm{op}}, y\colon a' \to b'), ((a, b) \in A \times A^{\mathrm{op}}, x\colon a \to b)$.

Consider the functor $\mathrm{Set}_* \xrightarrow{v} \mathrm{Set}$ where Set_* is the category of pointed sets. We can also construct the category of elements of X as the following pullback

$$\begin{tikzcd} A \arrow[r, "X^{\mathrm{op}}"] & \mathrm{Set}^{\mathrm{op}} \\ \int^A X \cong \mathcal{X} \arrow[u, "p"] \arrow[r] & (Set_*)^{\mathrm{op}} \arrow[u, "v^{\mathrm{op}}"] \end{tikzcd}$$

the objects of the pullback are given as $\mathcal{X} = (a \in A, (X_a, x \in X_a))$ with morphisms $f\colon a \to b$ such that $f^*(y) = x$ for another object $(b \in B, (X_b, y \in X_b))$, thus we have the same objects as in the Grothendieck construction, just with the additional, but redundant data, of X_a.

Theorem 5.3. *The functor $\int^A\colon \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}/A$ is fully faithful, its essential image consists of the discrete Grothendieck fibrations, i.e. those functors $\mathcal{X} \xrightarrow{p} A$ such that $\forall x \in \mathcal{X}$ and $\forall f\colon a \to b = p(y)$ there exists a unique $\varphi\colon x \to y$ such that $p(\varphi) = f$.*

Idea 5.4. Let $\mathcal{X} \xrightarrow{p} Y$ be a discrete Grothendieck fibration of some presheaf $X\colon A^{\mathrm{op}} \to \mathrm{Set}$, then we have a pullback square

$$\begin{tikzcd}
\mathcal{X}_a \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\
\mathbb{1} \arrow[r] & A
\end{tikzcd}$$

where the objects of $\mathcal{X}_a$ are given by $x \in \mathcal{X}$ such that $p(x) = a$ and morphisms φ such that $p(\varphi) = \mathrm{id}_a$.

Uniqueness of lifts yields that $X\colon A^{\mathrm{op}} \to \mathrm{Set}$ is well defined.

Definition 5.5. The **Grothendieck construction** of a 2-functor $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ is the 1-category $\int^A X$ with objects $(a \in A, x \in X_a)$ and for another object $(b \in B, y \in X_b)$ morphisms are given by tuples (f, φ) such that $f\colon a \to b$ and $\varphi\colon x \to f^*(y)$.

The composition law in $\int^A X$ is given as follows

$$\begin{tikzcd}
& (b \in A, y \in X_b) \arrow[dr, "{(g,\psi)}"] & \\
(a \in A, x \in X_a) \arrow[ur, "{(f,\varphi)}"] \arrow[rr, "{(gf,\alpha)}"'] & & (c \in A, z \in X_c)
\end{tikzcd}$$

where

$$X \xrightarrow{\varphi} f^*(y) \xrightarrow{f^*(\psi)} f^*(g^*(z)) \xrightarrow{\mu} (gf)^*(z)$$

and $\alpha = \mu \circ f^*(\psi) \circ \varphi$.

We need to verify that the composition law is associative and unital. For the identity morphisms it holds that

$$(a \in A, x \in X_a) \xrightarrow{(\mathrm{id}_a, x \xrightarrow{\epsilon} \mathrm{id}^*(x))} (a \in A, x \in X_a)$$

where

$$\epsilon_a\colon \mathrm{id}_{X_a} \Rightarrow X(\mathrm{id}_a) = (\mathrm{id}_a)^*.$$

Theorem 5.6. *The functor* $\int^A\colon$ 2-Fun$(A^{\mathrm{op}}, \underline{\mathrm{Cat}}) \to \mathrm{Cat}/A$ *is fully faithful, with essential image the Grothendieck fibrations.*

Example 5.7. Take the functor from Ring to Cat that takes a ring R to its module category, we have the following commutative diagram

$$\begin{tikzcd}
R \arrow[r, mapsto] \arrow[d, "f"] & \mathrm{mod}_R \arrow[d, dashed, bend right=60, "f_*"'] \arrow[d, dashed, bend left=60, "f_! = - \otimes_k S"] \arrow[d, phantom, shift right=3, "\vdash"] \arrow[d, phantom, shift left=3, "\vdash"] \\
S \arrow[r, mapsto] & \mathrm{mod}_S \arrow[u, "f^*"']
\end{tikzcd}$$

Let $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ be a 2-functor and A a small 1-category, as well as $\underline{\mathrm{Cat}}$ the strict 2-category of categories.

Proposition 5.8. *The Grothendieck Construction $\int^A X$ is a 1-category.*

Proof. Consider for the Unitality

$$\begin{tikzcd}
(a \in A, x \in X_a) \arrow[r, "{(\mathrm{id}_a, x \xrightarrow{\epsilon_a} (\mathrm{id}_a)^*(x))}"] \arrow[dr, "h"'] & (a \in A, x \in X_a) \arrow[d, "{(a \xrightarrow{f} c, x \xrightarrow{\varphi} f^*(y))}"] \\
& (c \in A, z \in X_c)
\end{tikzcd}$$

where $\begin{tikzcd} a \arrow[r, "\mathrm{id}_*"] \arrow[rr, bend right, "f"'] & a \arrow[r, "f"] & b \end{tikzcd}, x \xrightarrow{\alpha} f^*(y)$.

$$\begin{tikzcd}
(\mathrm{id}_a)^* & (\mathrm{id}_a)^*(x) \arrow[r, "(\mathrm{id}_a)^*(\varphi)"] & (\mathrm{id}_a)^*(f^*(y)) \arrow[r, "\mu", "\sim"'] & (f \circ \mathrm{id}_a)^*(y) = f^*(y) \\
\mathbb{1}_{X_a} \arrow[u, Rightarrow, "\epsilon_a"] & X \arrow[u, "(\epsilon_a)_*"] \arrow[r, "\varphi"] & f^*(y) \arrow[u, "(\epsilon_a)_{f^*y}"] \arrow[ur, "\mathrm{id}_{f^*(y)}"']
\end{tikzcd}$$

The right triangle commutes by the definition of ϵ and the square by the naturality of ϵ_a . Now for the associativity take the following diagram:

$$\begin{tikzcd}
(a \in A, s \in X_a) \arrow[r, "{(f,u)}"] \arrow[dr, "{(g \circ f, \mu \circ f^*(v) \circ u)}"'] & (b \in A, x \in X_b) \arrow[d, "{(g,v)}"] \arrow[dr, "{(hg, \mu \cdot g^*(v) \circ u)}"] & \\
& (c \in A, y \in X_c) \arrow[r, "{(h,w)}"] & (d \in A, z \in X_d)
\end{tikzcd}$$

The outer two compositions are given by

$$\begin{tikzcd}
(a \in A, s \in X_a) \arrow[r, bend left, "{(hgf, \mu \circ f^*(\mu g^*(w) \circ v) \circ u)}"] \arrow[r, bend right, "{(hgf, \mu \circ (gf)^*(w) \circ \mu f^*(v) \circ u)}"'] & (d \in A, z \in X_d)
\end{tikzcd}$$

Let us examine these compositions in detail, we have

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

in A and morphisms $s \xrightarrow{u} f^*(x), x \xrightarrow{v} g^*(y), y \xrightarrow{w} h^*(z)$, put together we get the commutative diagram

$$\begin{tikzcd}
s \arrow[r, "u"] & f^*(x) \arrow[r, "f^*(v)"] & f^*(g^*(y)) \arrow[r, "\mu"] \arrow[d, "f^*(g^*(w))"] & (g \circ f)^*(y) \arrow[d, "(g \circ f)*(w)"] \\
& & f^*(g^*(h^*(z))) \arrow[r, "\mu"] \arrow[d, "f^*(\mu)"] & (gf)^*(h^*(z)) \arrow[d, "\mu"] \\
& & f^*((hg)^*(z)) \arrow[r, "\mu"] & (gfh)^*(z)
\end{tikzcd}$$

Since the squares commute by the associativity constraint, we get that both compositions are equal and thus the associativity constraint holds. $\square$

Definition 5.9. A morphism $\varphi\colon Y \to X$ is called p-cartesian if $\forall g, \exists!\rho$ such that

$$\begin{tikzcd} & Y \arrow[dr, "\varphi"] & \\ Z \arrow[ur, dashed, "\exists!\rho"] \arrow[rr, "\forall\psi"] & & X \end{tikzcd} \quad \overset{p}{\mapsto} \quad \begin{tikzcd} & p(Y)=a \arrow[dr, "f=p(\rho)"] & \\ p(Z)=a' \arrow[ur, "\forall g"] \arrow[rr, "h=p(X)"'] & & b=p(X) \end{tikzcd}$$

or equivalently if the following lift exists

$$\begin{tikzcd} \{1 \to 2\} \arrow[r, hook, "i"] & \lambda^2_2 \arrow[r, "F"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ & \Delta^2 \arrow[r, "\forall G"] \arrow[ur, dashed, "\exists!"] & A \end{tikzcd}$$

where $F(i(1 \to 2)) = \varphi$.

Definition 5.10. A functor of 1-categories $\mathcal{X} \xrightarrow{p} A$ is a Grothendieck fibration. If $\forall X \in \mathcal{X}, \forall a \xrightarrow{f} b = p(X)$ there exists a morphism $Y \xrightarrow{\varphi} X$ in $\mathcal{X}$ that is p-cartesian such that $p(\varphi) = f$.

Definition 5.11. Let $\mathcal{C}$ be a category. A lifitng problem in $\mathcal{C}$ is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathcal{C}$. A solution to the lifting problem (a lift) is a morphism $h\colon B \to X$ such that $ph = q, h \circ i = f$.

Proposition 5.12. *The functor $\int^A X \xrightarrow{p} A$ is a Grothendieck fibration.*

Proof. Let $Z = (c \in A, z \in X_c) \in \int^A_X$ be an object and $g\colon b \to p(z) = c$ in A be a morphism in A, then $g^*\colon X_c \to X_b$ and $z \mapsto g^*(z)$. Consider

$$\psi := (b \in A, g^*(z) \in X_b) \xrightarrow{(g, g^*(z) \xrightarrow{\mathrm{id}} g^*(z))} (c \in A, z \in X_c) = X$$

<u>Claim</u>: ψ is p-cartesian. Take a diagram

$$\begin{tikzcd} & b \arrow[dr, "g=p(\psi)"] & \\ a \arrow[ur, "\forall f"] \arrow[rr, "h=p(\alpha)"'] & & c \end{tikzcd}$$

the associated lifting problem is

$$\begin{tikzcd}
 & (b \in A, g^*(z) \in X_b) \arrow[dr, "{(g, \mathrm{id}_{g^*(z)} = \psi)}"] & \\
(a \in A, x \in X_a) \arrow[ur, dashed, "{(f, x \xrightarrow{u} f^*(g^*(z)))}"] \arrow[rr, "{\forall (h, x \xrightarrow{\sim} h^*(z)) = \alpha}"'] & & (c \in A, z \in X_c)
\end{tikzcd}$$

In order for the lifting diagram to commute, we need to construct u as follows. We have the composition

$$\begin{aligned}
x \xrightarrow{u} f^*(g^*(z)) \xrightarrow{f^*(\mathrm{id}_{g^*})} f^*(g^*(z)) \xrightarrow{\mu} (gf)^*(z) \\
= x \xrightarrow{v} (gf)^*(z)
\end{aligned}$$

thus taking $u = \mu^{-1} \circ v$ works. $\square$

Definition 5.13. The functor $\mathcal{X} \xrightarrow{p} A$ is called an **isofibration** if $\forall f\colon a \xrightarrow{\sim} p(x) = b$ there exists $Y \xrightarrow{\varphi} X$ an isomorphism in $\mathcal{X}$ such that $p(\varphi) = f$. That is if there exists a solution to the following lifting problem:

$$\begin{tikzcd}
\Lambda^0_0 \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\
\Delta^1 \arrow[r, "{a \xrightarrow{f} p(x)}"'] \arrow[ur, dashed, "\exists \varphi"] & A
\end{tikzcd}$$

Lemma 5.14. *If $p\colon \mathcal{X} \to A$ is a Grothendieck fibration then p is an isofibration.*

Proof. Let $x \in \mathcal{X}$ and $f\colon a \xrightarrow{\sim} p(x) = b$ an isomorphism in A. Choose a p-cartesian lift of f, $\varphi\colon Y \to X$ in $\mathcal{X}$ take a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "\varphi"] & \\
X \arrow[ur, dashed, "\exists ! \psi"] \arrow[rr, "\mathrm{id}_*"'] & & X
\end{tikzcd}$$

where $\varphi \circ \psi = \mathrm{id}_x$ and $p(\psi) = g = f^{-1}$, which means we have the following diagram:

$$\begin{tikzcd}
 & a \arrow[dr, "f"] & \\
b \arrow[ur, "f^{-1} = g"] \arrow[rr, "\mathrm{id}_b"'] & & b = p(x)
\end{tikzcd}$$

The following equalities hold

$$\varphi \circ (\psi \circ \varphi) = (\varphi \circ \psi) \circ \varphi = \varphi \circ \mathrm{id}_x = \varphi$$

now since $p(\psi) \circ p(\varphi) = f^{-1} \circ f = \mathrm{id}_a$ we get by the uniqueness of lifts that $\psi \circ \varphi = \mathrm{id}_Y$. Thus we found a lift that is an isomorphism, which proves the claim. $\square$

Let now $\mathcal{X} \xrightarrow{p} A$ be a Grothendieck fibration and $X\colon A^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ a 2-functor defined by sending each object a of A to the two-pullback $\mathcal{X}_a$ along the object a and $p\colon \mathcal{X} \to A$, that is,

$$\begin{tikzcd} \mathcal{X}_a \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "a"] & A \end{tikzcd}$$

now for any morphism $f\colon a \to b$ we obtain a functor $f^*\colon \mathcal{X}_b \to \mathcal{X}_a$. Let $x \in \mathcal{X}_b \xrightarrow{f^*} \mathcal{X}_a$ and $\varphi\colon f^*(x) \to x$ in $\mathcal{X}$ (notice the slight abuse of notation when we identify an object in the fiber with its image in $\mathcal{X}$) so that $p(\varphi) = f$.

5.1 Exercises

Exercise 1. A functor $G\colon \mathcal{B} \to \mathcal{C}$ is called an isofibration if for any $b \in \mathcal{B}$ and any isomorphism $f\colon G(b) \xrightarrow{\sim} c'$ there is an isomorphism $f'\colon b \xrightarrow{\sim} b'$ with $G(f') = f$.

(a) Show that the opposite $G^{\mathrm{op}}\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ of an isofibration $G\colon \mathcal{B} \to \mathcal{C}$ is an isofibration.

(b) Show that for a functor $F\colon \mathcal{A} \to \mathcal{C}$ and an isofibration $G\colon \mathcal{B} \to \mathcal{C}$ the canonical inclusion $\iota\colon \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$ of the pullback in Cat to the 2-pullback in $\underline{\mathrm{Cat}}$ is an equivalence of categories.

Recall that in the lecture, we defined a Grothendieck fibration to be the functor $p\colon \mathcal{X} \to A$ such that for any $x \in \mathcal{X}$ and any morphism $f\colon a \to p(x)$ admits a (p-)cartesian lift $f'\colon y \to x$ with $p(f') = f$. Here we call a morphism $f'\colon y \to x$ (p-)cartesian if for any morphism $g'\colon z \to x$, whenever $p(g') = p(f') \circ h$ for some $h\colon p(z) \to p(y)$, there exists a unique lift $h'\colon z \to y$ such that $p(h') = h$ and $g' = f' \circ h'$.

(c) Show that a (p-)cartesian lift of an isomorphism is an isomorphism. Conclude that any Grothendieck fibration $p\colon \mathcal{X} \to A$ is an isofibration.

(d) Conclude that the 2-pullback of a Grothendieck fibration $p\colon \mathcal{X} \to A$ along a functor $a\colon [0] \to \mathcal{A}$ is the fiber $\mathcal{X}_a$ of p at a, i.e. the ordinary pullback along a.

Exercise 2. Consider a 2-functor $F\colon A \to \underline{\mathrm{Cat}}$ from an ordinary category A to the 2-category of categories. The Grothendieck construction $\in_A F$ of F is an ordinary category whose objects are tuples (a, x) of an object $a \in A$ and objects of the category $x \in F(a)$. A morphisms in $\int_A F$ is a tuple $(f, u)\colon (a, x) \to (b, y)$ consisting of a morphism $f\colon a \to b$ in A and a morphism $u\colon F(f)(x) \to y$ in $F(b)$ where we used that $F(f)\colon F(a) \to F(b)$ is a functor.

(a) Use the additional structure of the 2-functor F to define a composition law on $\int_A F$ extending the composition of morphisms in A. Specify the identity morphisms and confirm that $\int_A F$ is a category.

(b) Explain why the Grothendieck construction $\int_A F$ and $\int^{A^{\mathrm{op}}} F$ yield in general different categories.

A functor $p\colon \mathcal{X} \to A$ is called a Grothendieck opfibration if its opposite $p^{\mathrm{op}}\colon \mathcal{X}^{\mathrm{op}} \to A^{\mathrm{op}}$ is a Grothendieck fibration.

(c) Give the definition of a Grothendieck opfibration without reference to a Grothendieck fibration and cartesian morphisms. Include also a defintion of (p-)cocartesian morphisms.

(d) Show that the composition of two (p-)cocartesian morphisms is (p-)cocartesian for some functor $p\colon \mathcal{X} \to A$.

(e) Show that the canonical forgetful functor $\int_A F \to A$ is a Grothendieck opfibration.

Exercise 3. Fix a Grothendieck opfibration $p\colon \mathcal{X} \to A$. We aim to construct a 2-functor $f_p\colon A \to \underline{\mathrm{Cat}}$.

(a) Describe the category $F(a) := \mathcal{X}_a$ which is defined as the fibre of p at a, i.e. the pullback of p along $a\colon [0] \to A$.

Using the axiom of choice, we fix for every $x \in \mathcal{X}$ and $f\colon p(x) \to b$ a (p-)cocartesian lift $f_x\colon x \to x_f$.

(b) Show that for a fixed morphism $f\colon a \to b$ in A, the above choices induce a functor $F(f)\colon F(a) \to F(b)$ with $F(f)(x) := x_f$ for $x \in F(a)$ and $F(f)(u) := u_f$ for u a morphism in A. Here u_f for $u\colon x \to y$ in $F(a)$ is defined as the unique lift of $f_y \circ u$ along f_x induced by id_b.

$$\begin{tikzcd} & x_f \arrow[dr, dashed, "\exists ! u_f"] & & & & b \arrow[dr, equal] & \\ x \arrow[ur, "f_x"] \arrow[r, "u"'] & y \arrow[r, "f_y"'] & y_f & \xrightarrow{p} & a \arrow[ur, "f"] \arrow[r, equal] & a \arrow[r, "f"'] & b \end{tikzcd}$$

(c) Deduce from the dual of Exercise 5.1(c) that the collection $\epsilon_a := ((\mathrm{id}_a)_x)_{x \in F(a)}$ is a natural isomorphism

$$\epsilon_a\colon \mathrm{id}_{F(a)} \implies F(\mathrm{id}_a)$$

for every $a \in A$.

(d) Use Exercise 5.2(d) to construct a natural isomorphism

$$\mu_{g,f} \colon F(g) \circ F(f) \implies F(g \circ f)$$

for any two composable morphisms $f \colon a \to b$ and $g \colon b \to c$ in A.

(e) Conclude that $F = (F, \mu, \epsilon) \colon A \to \underline{\mathrm{Cat}}$ is a 2-functor. (Hint: Deduce from Exercise 4.3(d) that it suffices to show that $\epsilon_a = \mu_{\mathrm{id}_a, \mathrm{id}_a} \cdot (\epsilon_a * \epsilon_a)$ for all $a \in A$ and $\mu_{hg,f} \cdot (\mu_{h,g} * \mathrm{id}_f) = \mu_{h,gf} \cdot (\mathrm{id}_h * \mu_{g,f})$ for three composable morphisms f, g, h in A.

Exercise 4. Consider a pair of functors $L \colon \mathcal{C}_0 \to \mathcal{C}_1$ and $R \colon \mathcal{C}_1 \to \mathcal{C}_0$ and view them as strict 2-functors $F := F_L \colon [1] \to \underline{\mathrm{Cat}}$ and $G := G_R \colon [1]^{\mathrm{op}} \to \underline{\mathrm{Cat}}$ where $[1] := \{0 < 1\}$ is the category with two objects and one morphism between them.

(a) Show that the functor $H \colon \int_{[1]} F \to \int^{[1]} G$ with constant fibres over $[1]$, i.e. $H(i, c) = (i, c)$ and $H(\mathrm{id}_i, u) = (\mathrm{id}_i, u)$ for $i \in [1]$, are in bijection to natural transformations

$$\eta \colon \mathrm{Hom}_{\mathcal{C}_1}(L(-), ?) \to \mathrm{Hom}_{\mathcal{C}_0}(-, R(?))$$

of functors $\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{Set}$.

(b) Conclude that we have an adjunction $L \dashv R$ if and only if $\int_{[1]} F$ and $\int^{[1]} G$ are isomorphic via a functor with constant fibres. Deduce in particular that in this case $\int_{[1]} F$ and $\int^{[1]} G$ together with the forgetful functors are isomorphic categories over $[1]$.

(c) Show that if $L \dashv R$, then $p_G \colon \int^{[1]} G \to [1]$ is also a Grothendieck opfibration.

A functor $p \colon \mathcal{X} \to A$ that is both a Grothendieck fibration and a Grothendieck opfibration is called a Grothendieck bifibration.

(d) Explain how a Grothendieck bifibration $p \colon \mathcal{X} \to [1]$ gives rise to a pair of functors $L \colon \mathcal{C}_0 \to \mathcal{C}_1$ and $R \colon \mathcal{C}_1 \to \mathcal{C}_0$. Furthermore, show that they form an adjoint pair, $L \dashv R$, by constructing an explicit natural isomorphism $\eta \colon \mathrm{Hom}_{\mathcal{C}_1}(L(-), ?) \to \mathrm{Hom}_{\mathcal{C}_2}(-, R(?))$ for any choice of cleavage.

Exercise 5. Let $\mathcal{C}$ be a complete category and consider the arrow category $\mathrm{Fun}([1], \mathcal{C})$ together with the projection

$$p := \mathrm{ev}_1 \colon \mathrm{Fun}([1], \mathcal{C}) \to \mathcal{C}$$

given by evaluating at $1 \in [1]$. Describe the (p-)cartesian morphisms in $\mathrm{Fun}([1], \mathcal{C})$. (Recall that pullback squares are also called cartesian squares.) Is p a Grothendieck fibration?

6 Homotopy theory of simplicial sets

This chapter is a revision of material covered in the preceeding lecture, Homotopy theory of simplicial sets held by Professor Dr. Jasso in the WS 24/25 at the university of cologne. For reference see section 6 in the script of the lecture.

7 Duskin nerve

The reference for this section is [3, ch. 2.3].

We fix a 2-category $\mathcal{C}$ that we want to construct a simplicial set from.

Definition 7.1. The Duskin nerve is

$$\mathrm{Set}_\Delta \leftarrow \mathrm{twocat}_{\mathrm{lax}} : N^D = u^*$$

where $\mathrm{twocat}_{\mathrm{lax}}$ is the category of 2-categories with morphisms given by strictly unital lax 2-functors. Then $N^D(\mathcal{C})_n := \{\text{strictly unital lax 2-functors } [n] \xrightarrow{F} \mathcal{C}\}$.

Remark 7.2. [3] If $\mathcal{C}$ is a 1-category, then $N^D(\mathcal{C}) = N(\mathcal{C})$. We analyze the n-simplices of $N^D(\mathcal{C})$.

- (n=0) $[0] = \{0\} \xrightarrow{F} \mathcal{C}$ given $0 \mapsto X_0 = F(0) \in \mathrm{Ob}(\mathcal{C})$
- (n=1) $[1] = \{0 \to 1\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} 0 \arrow[d] & & X_0 \arrow[d] \arrow[loop right, "f_{00}=\mathrm{id}_{X_0}"] \\ 1 \arrow[r, phantom, "\longmapsto"] & & X_1 \arrow[loop right, "f_{11}=\mathrm{id}_{X_1}"] \end{tikzcd}$$

- (n=2) $[2] = \left\{ \begin{tikzcd}[sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \right\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "\gamma"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[r, phantom, ""] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd}$$

- (n=3) $[3] = \left\{ \begin{tikzcd}[sep=small] & 1 \arrow[dr] \arrow[dd] & \\ 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \right\} \xrightarrow{F} \mathcal{C}$

$$\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{13}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] \arrow[dr, "f_{30}"'] & & X_2 \arrow[dl, "f_{32}"] \\ & X_3 & \end{tikzcd}$$

with a 2-morphism $\xRightarrow{\gamma}$ for every 2-simplex in the boundary.

More precisely: A strictly unital lax 2-functor $F\colon [n] \to \mathcal{C}$ consist of the following data

- $\forall 0 \leqslant i \leqslant n$ an object $X_i := F(i)$,
- $\forall 0 \leqslant i \leqslant j \leqslant n$ a 1-morphism $f_{ji} = X(i \to j)$,
- and $\forall 0 \leqslant i \leqslant j \leqslant k \leqslant n$ a 2-morphism $\mu_{kji} = f_{kj} \cdot f_{ji} \Rightarrow f_{ki}$.

Moreover the above data must satisfy

- (strict unitality) $\forall 0 \leqslant i \leqslant n, f_{ii} = \mathrm{id}_{X_i}$ the following diagram commutes

$$\begin{tikzcd} f_{jj} \circ f_{ji} \arrow[r, Rightarrow, "\mu_{jji}"] & f_{ji} \arrow[d, Rightarrow, "F(\lambda_{ji})=\mathrm{id}_{f_{ji}}"] \\ \mathrm{id}_{X_j} \circ f_{ji} \arrow[u, Rightarrow, "\mathrm{id} * \mathrm{id}"] \arrow[r, Rightarrow, "\lambda_{f_{ji}}"] & f_{ji} \end{tikzcd}$$

that is $\forall 0 \leqslant i \leqslant j \leqslant n$

$$\mu_{jji} = \lambda_{f_{ji}} \colon \mathrm{id}_{X_j} \circ f_{ji} \Rightarrow f_{ji}$$
$$\mu_{jii} = \rho_{f_{ji}} \colon f_{ji} \circ \mathrm{id}_{X_i} \Rightarrow f_{ji}.$$

Recall that

$$\mu_{iii} = \lambda_{f_{ii}} = \mathrm{id}_{X_i} = \rho_{\mathrm{id}_{X_i}} = \rho_{f_{ii}} = \mu_{iii}.$$

- (Composition) $\forall 0 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n$

$$\begin{tikzcd} f_{lk} \circ (f_{kj} \circ f_{ji}) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\mathrm{id} * \mu"] & (f_{lk} \circ f_{kj}) \circ f_{ji} \arrow[d, Rightarrow, "\mu * \mathrm{id}"] \\ f_{lk} \circ f_{ki} \arrow[d, Rightarrow, "\mu"] & f_{lj} \circ f_{ji} \arrow[d, Rightarrow, "\mu"] \\ f_{li} \arrow[r, Rightarrow, "F(\alpha)=\mathrm{id}"] & f_{li} \end{tikzcd} \tag{1}$$

Proposition 7.3. *An n-simplex $F \in N^D(\mathcal{C})_n (F\colon [n] \to \mathcal{C})$ is uniquely determined by the following data:*

- $0 \leqslant i \leqslant n, X_i \in \mathrm{Ob}(\mathcal{C}), \forall 0 \leqslant i < j \leqslant n, f_j\colon X_i \to X_j$ *a collection of 1-morphisms in $\mathcal{C}$,*
- $0 \leqslant i < j < k \leqslant n, \mu_{kji} = f_{kj} \circ f_{ji} \Rightarrow f_{ki}$ *such that* $\forall 0 \leqslant i < j < k < l \leqslant n$ (1) *is satisfied.*

Proof. Sketch:

The uniqueness is clear, since we must have

$$\forall 0 \leqslant i \leqslant n, f_{ii} = \mathrm{id}_{X_i}, \forall 0 \leqslant i \leqslant j \leqslant n.$$

Given the data as in the statement, we define

$$\mu_{jji} = \lambda_{f_{ji}}, \mu_{jii} = \rho_{f_{ji}}.$$

We know 1 holds for $i < j < l < k$. To check that 1 holds when some indices are equal is given as an Exercise, it uses the triangle identity and the following lemma. $\square$

Lemma 7.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be 1-morphisms in $\mathcal{C}$. Then the following diagrams commute:*

1.
$$\begin{tikzcd}
\mathrm{id}_Z \circ (g \circ f) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\lambda_{g\circ f}"'] & & (\mathrm{id}_Z \circ g) \circ f \\
& g \circ f \arrow[ur, Rightarrow, "(\lambda_g)^{-1} * \mathrm{id}_f"'] &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
g \circ (f \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\mathrm{id}_g * \rho_f"'] & & (g \circ f)\, \mathrm{id}_X \\
& g \circ f \arrow[ur, Rightarrow, "\rho^{-1}_{g\circ f}"'] &
\end{tikzcd}$$

Proof. We prove 2., consider for that the following diagram, notice that we omitted taking products with identities in the notation of the morphisms.

$$\begin{tikzcd}
& g \circ ((f \circ \mathrm{id}_X) \circ \mathrm{id}_X) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\rho"] & (g \circ (f \circ \mathrm{id}_X)) \circ \mathrm{id}_X \arrow[d, Rightarrow, "\mathrm{id} * \rho"] \arrow[dr, Rightarrow, "\alpha"] & \\
g \circ (f \circ (\mathrm{id}_X \circ \mathrm{id}_X)) \arrow[ur, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "\lambda"] \arrow[drr, Rightarrow, "\alpha"'] & g \circ (f \circ \mathrm{id}_X) \arrow[r, Rightarrow, "\alpha"] & (g \circ f) \circ \mathrm{id}_X \arrow[r, Rightarrow, "\rho^{-1}"] & ((g \circ f) \circ \mathrm{id}_X) \circ \mathrm{id}_X \\
& & (g \circ f) \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[u, Rightarrow, "\lambda"] \arrow[ur, Rightarrow, "\alpha"'] &
\end{tikzcd}$$

By the pentagon axiom the outer composition in the diagram commutes, the bottom, top left and bottom right triangles commute by the triangle identity and the top middle square commutes by naturality of ρ, thus the top right triangle commutes. Lastly the diagram we aim to show commutes is the top right triangle with $- \circ \mathrm{id}_X$ applied to it, since $- \circ \mathrm{id}_X$ is fully faithful the original triangle commutes. $\square$

Corollary 7.5. *The Duskin nerve $N^D(\mathcal{C})$ is 3-coskeletal and*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^3, N^D(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^3, N^D(\mathcal{C}))$$

is injective.

The 3-simplices can be given by 2 boundary quadrilaterals as follows, let $0 \leqslant i < j < k < l \leqslant n$. Then the boundary quadrilaterals are given by

$$\begin{tikzcd}
& X_j \arrow[r, "f_{kj}"] \arrow[d, phantom, "\Downarrow\mu_{kji}"] & X_k \arrow[dr, "f_{lk}"] \arrow[d, phantom, "\Downarrow\mu_{lki}"] & \\
X_i \arrow[ur, "f_{ji}"] \arrow[urr, bend right] \arrow[rrr, "f_{kl}"] & {} & {} & X_l
\end{tikzcd}$$

$$\begin{tikzcd}
& X_j \arrow[r, "f_{kj}"] \arrow[drr, bend right] \arrow[d, phantom, "\mu_{lij}\Downarrow"] & X_k \arrow[dr, "f_{lk}"] \arrow[d, phantom, "\Downarrow\mu_{klj}"] & \\
X_i \arrow[ur, "f_{ji}"] \arrow[rrr, "f_{li}"] & {} & {} & X_l
\end{tikzcd}$$

Definition 7.6. Let $X \in \mathrm{Set}_\Delta$ be a simplicial set. A 2-simplex $\sigma \in X_2$ is thin if $\forall n \geqslant 3, \forall 0 < i < n$, there exists a morphism $\tilde{\rho}$ for all morphisms τ such that the following commutes

$$\begin{tikzcd}
\Delta^2 = [2] \arrow[r, "j"] \arrow[rr, bend left, "\sigma"] & \Lambda_i^n \arrow[r, "\forall\tau"] \arrow[d, hook] & X \\
& \Delta^n \arrow[ur, dashed, "\exists\tilde{\rho}"'] &
\end{tikzcd}$$

with $j := [2] \to \{i-1, i, i+1\}$.

Remark 7.7. Let $X \in \mathrm{Set}_\Delta$ be an ∞-category, then every 2-simplex of X is thin. Conversely, if every 2-simplex of X is thin, then X is an ∞ category if and only if the following diagram commutes:

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Remark 7.8. Take the following horn filling problem

$$\begin{tikzcd}
\Lambda_1^2 \arrow[r, "\sigma"] \arrow[d, hook] & N^D(\mathcal{C}) \\
\Delta^2 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

where the dashed arrow exists since we have the horizontal composition of 1-morphisms. Thus we can extend any 2-horn and would have an infinity category if every 2-simplex were thin.

Theorem 7.9. *Let $\mathcal{C}$ be a 2-category, then $\mathcal{C}$ is a $(2,1)$-category if and only if $N^D(\mathcal{C})$ is an ∞-category.*

Proof. Suppose $\mathcal{C}$ is a (2,1)-category. Assume every 2-simplex were thin by 7.10, then by 7.8 $N^D(\mathcal{C})$ is an ∞-category. Conversely, if $N^D(\mathcal{C})$ is an ∞-category, then by 7.7 every 2-simplex in $N^D(\mathcal{C})$ is thin if the following is satisfied. Let

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \end{tikzcd}$$

be a 2-morphism in $\mathcal{C}$, then we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[dr, Rightarrow, "\rho"'] & f \arrow[d, Rightarrow, "\gamma"] \\ & g \end{tikzcd}$$

where ρ exists since $N^D(\mathcal{C})$ is thin. Now by assumption ρ is invertible, thus ρ is invertible. □

Lecture 27.5

For the next part let $\mathcal{C}$ be a fixed 2-category and let $N^D(\mathcal{C}) \in \mathrm{Set}_\Delta$ be its Duskin nerve, i.e. $N^D(\mathcal{C})_n := \{[n] \to \mathcal{C} \mid \text{strictly unital lax functors}\}$.

Theorem 7.10. *The 2-simplex*
$$\begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "\gamma_{210}"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd}$$
$\in N^D(\mathcal{C})_2$ *is thin if and only if* μ_{210} *is invertible.*

Proof. We divide the proof into its natural parts. □

Proposition 7.11. *Let* $n \geqslant 3$ *as well as* $0 \leqslant l \leqslant n$,

$$\sigma = \begin{tikzcd} & X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "\gamma"] & \\ X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2 \end{tikzcd}$$

be a 2-simplex in $N^D(\mathcal{C})$ *and*

$$\begin{tikzcd} \Lambda^n_l \arrow[r, "u"] & N^D(\mathcal{C}) \\ \Delta^{\{l-1,l,l+1\}} \arrow[u] \arrow[ur, "\sigma"] & \end{tikzcd}$$

commute. If γ *is invertible, then* u *extends uniquely to an* n*-simplex of* $N^D(\mathcal{C})$.

Proof. Recall that $N^D(\mathcal{C})$ is 3-coskeletal, hence we may assume $n = 3, 4$. (Case 1) $n = 3, l = 1$

$$
\begin{tikzcd}
 & X_1 \arrow[r, "f_{21}"] \arrow[d, phantom, "\Downarrow \mu_{210}=\gamma"] & X_2 \arrow[dr, "f_{32}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[urr, bend right] \arrow[rrr, "f_{30}"'] & & & X_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_1 \arrow[r, "f_{21}"] \arrow[drr, bend right] \arrow[d, phantom, "\Downarrow \mu_{310}"] & X_2 \arrow[dr, "f_{32}"] \arrow[d, phantom, "\Downarrow \mu_{321}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rrr, "f_{30}"'] & & & X_3
\end{tikzcd}
$$

Where the dashed green lined simplex corresponds to the missing simplex given by μ_{320}, which we can construct as follows. Observe that since $\mathrm{id}_{f_{32}}$ and $\mu_{210} = \gamma$ are invertible and composition is functorial, the composition is invertible as well. We know from an Exercise that if μ_{320} would exist, the following identity would hold and that it is sufficient that this holds to extend the horn

$$\mu_{320}(\mathrm{id}_{f_{32}} \circ \mu_{210}) = \mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}})\alpha_{f_{32},f_{21},f_{10}}.$$

By the above argument we can choose

$$\mu_{320} = (\mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}})\alpha_{f_{32},f_{21},f_{10}})(\mathrm{id}_{f_{32}} \circ \mu_{210})^{-1}.$$

(Case 2: n=4 ,l=2) $u\colon \Lambda^n_l \to N^D(\mathcal{C})$

We need to check the compatibility of composition, since all other diagrams for the Duskin nerve require only simplices up dimension three to commute and, thus hold for the boundary 3-simplices of a 4-horn, leaving only the compatibility of composition to be checked. The outer square of the following diagram is exactly the compatibility of composition diagram for the missing 3 simplex, i.e. all morphisms that do not contain a 2 as an index.

$$
\begin{tikzcd}
f_{43} \circ (f_{31} \circ f_{10}) \arrow[rrrr, Rightarrow, "\alpha"] \arrow[ddddd, Rightarrow, "\mu_{310}"] & & & & (f_{43} \circ f_{31}) \circ f_{10} \arrow[ddddd, Rightarrow, "\mu_{431}"] \\
 & f_{43} \circ ((f_{32} \circ f_{21}) \circ f_{10}) \arrow[ul, Rightarrow, "\mu_{321}"] \arrow[rr, Rightarrow, "\alpha"] & & (f_{43} \circ (f_{32} \circ f_{23})) \circ f_{10} \arrow[ur, Rightarrow] \arrow[d, Rightarrow, "\alpha"] & \\
 & f_{43} \circ (f_{32} \circ (f_{21} \circ f_{10})) \arrow[u, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\mu_{210}"] \arrow[dr, Rightarrow, "\alpha"] & & ((f_{43} \circ f_{32})f_{21})f_{10} \arrow[d, Rightarrow, "\mu_{432}"] & \\
 & f_{43} \circ (f_{32} \circ f_{20}) \arrow[d, Rightarrow, "\alpha"] \arrow[ddl, Rightarrow, "\mu_{320}"'] & (f_{43} \circ f_{32}) \circ (f_{21} \circ f_{10}) \arrow[ur, Rightarrow, "\alpha"] \arrow[dl, Rightarrow, "\mu_{210}"] \arrow[dr, Rightarrow, "\mu_{432}"] & (f_{42} \circ f_{21}) \circ f_{10} \arrow[d, Rightarrow, "\alpha"] \arrow[ddr, Rightarrow, "\mu_{421}"] & \\
 & (f_{43} \circ f_{32}) \circ f_{20} \arrow[r, Rightarrow, "\mu_{432}"'] & f_{42} \circ f_{20} \arrow[d, Rightarrow, "\mu_{420}"] & f_{42} \circ (f_{21} \circ f_{10}) \arrow[l, Rightarrow, "\mu_{210}"] & \\
f_{43} \circ f_{30} \arrow[rr, Rightarrow, "\mu_{430}"] & & f_{40} & & f_{41} \circ f_{10} \arrow[ll, Rightarrow, "\mu_{410}"]
\end{tikzcd}
$$

Note that composition constraint and associativity constraint are isomorphisms, thus if the inner diagrams commute, so does the outer one. The left and right quadrilaterals commute by the compatibility of composition, given by the other boundary 3-simplices of the horn and since composition with a morphism (f_{43} on the left side, f_{42} on the right side) is fully faithful, the upper square commutes by the naturality of α, the pentagon by the Pentagon identity, the triangles below the Pentagon by the naturality of the associativity constraint, the bottom left and right square commute by the composition compatibility. $\square$

Proposition 7.12. *Let $\mathcal{C}$ be a 2-category and*

$$\sigma := \begin{tikzcd} & X_1 \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & X_2 \end{tikzcd} \in N^D(\mathcal{C})_2$$

such that $\forall n \in \{3,4\}$ the following diagram commutes

$$\begin{tikzcd} \Delta^{\{0,1,2\}} \arrow[d, hook] \arrow[dr, bend left, "\sigma"] & \\ \Lambda_1^n \arrow[r, "u"] \arrow[d, hook] & N^D(\mathcal{C}) \\ \Delta^n \arrow[ur, "\exists"'] & \end{tikzcd}$$

then γ is invertible.

Proof. Let the following quadrilaterals be the boundary of a 3-simplex.

$$\begin{tikzcd} & X_1 \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\gamma"] & X_2 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[urr, bend left, "h"'] \arrow[rrr, "g\circ f"'] & & & X_3 = X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_1 \arrow[r, "g"] \arrow[drr, bend right, "g"'] \arrow[d, phantom, "\Downarrow\mathrm{id}_{g\circ f}"] & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[d, phantom, "\Downarrow\mathrm{id}_g"] & \\ X_0 \arrow[ur, "f"] \arrow[rrr, "g\circ f"'] & & & X_3 = X_2 \end{tikzcd}$$

by assumption this has a filling and thus there exists a δ such that $\delta\gamma = \mathrm{id}_{g\circ f} \circ \mathrm{id}_g$, we can assume strict unitality of the category (for details on this see [3]) and thus need obtain that $\delta\gamma = \mathrm{id}_{g\circ f}$. Now we need to show it is a right inverse to γ as well. for that we take the 4-horn given by the following boundary data:

$$\begin{tikzcd} & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\ & X_1 \arrow[ur, "g"] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\ X_0 \arrow[ur, "f"] \arrow[uurr, bend left, "h"'] \arrow[urrr, bend right, "g\circ f"'] \arrow[rrrr, "h"'] & & & & X_4 = X_2 \end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[ddrr, bend right, "\mathrm{id}_{X_2}"'] & & \\
 & X_1 \arrow[ur, "g"] \arrow[drrr, bend right=15, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\
 & X_1 \arrow[ur, "g"] \arrow[rr, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[urrr, "g\circ f"'] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & & \\
 & X_1 \arrow[ur, "g"] \arrow[rr, "g"] \arrow[drrr, "g"'] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_2 \arrow[dr, "\mathrm{id}_{X_2}"] \arrow[ddrr, bend right, "\mathrm{id}_{X_2}"'] & & \\
 & X_1 \arrow[ur, "g"] & & X_3 = X_2 \arrow[dr, "\mathrm{id}_{X_2}"] & \\
X_0 \arrow[ur, "f"] \arrow[uurr, bend right, "h"'] \arrow[rrrr, "h"'] & & & & X_4 = X_2
\end{tikzcd}$$

Since the horn given by the simplices above extends, we obtain $\gamma\delta = \mathrm{id}_h$. $\square$

7.1 Exercises

Exercise 1. Given a 2-category $\mathcal{C}$, we defined in the lecture its Duskin Nerve to have as n-simplices the strictly unital lax functors $[n] \to \mathcal{C}$. Hence an n-simplex $X\colon [n] \to \mathcal{C}$ consists of a choice of objects $\{X_i\}_{0\leqslant i\leqslant n}$ and 1-morphisms $f_{j,i}\colon X_i \to X_j$ for $0 \leqslant i \leqslant j \leqslant n$ and the composition constraint, i.e. 2-morphism $\mu_{k,j,i}\colon f_{k,j} \circ f_{j,i} \to f_{k,i}$. We assume the identity constraint to be the identity, we must have $f_{i,i} = \mathrm{id}_{X_i}$ for $0 \leqslant i \leqslant n$ and $\mu_{j,j,i} = \lambda_{f_{j,i}}$ and $\mu_{j,i,i} = \rho_{f_{j,i}}$ for $0 \leqslant i \leqslant j \leqslant n$. Furthermore, the composition constraint must satisfy

$$\mu_{l,k,i} \cdot (\mathrm{id}_{f_{l,k}} * \mu_{k,j,i}) = \mu_{l,j,i} \cdot (\mu_{l,k,j} * \mathrm{id}_{f_{j,i}}) \cdot \alpha_{f_{l,k},f_{k,j},f_{j,i}}$$

for all $0 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n$.

(a) Argue that in the above $\mu_{i,i,i}$ is well defined for $0 \leqslant i \leqslant n$ and if $j = k$, then the composition constraint is automatically fulfilled.

(b) Show that in any 2-category we have that

$$\alpha_{\mathrm{id}_z, g, f} = (\lambda_g^{-1} * \mathrm{id}_f) \cdot \lambda_{g \circ f} \quad \text{and} \quad \alpha_{g,f,\mathrm{id}_x} = \rho_{g \circ f}^{-1} \cdot (\mathrm{id}_g * \rho_f)$$

for any two 1-morphism $f\colon x \to y$ and $g\colon y \to z$ in $\mathcal{C}$.

(c) Deduce, that it suffices to require

$$\mu_{l,k,i} \cdot (\mathrm{id}_{f_{l,k}} * \mu_{k,j,i}) = \mu_{l,j,i} \cdot (\mu_{l,k,j} * \mathrm{id}_{f_{j,i}}) \cdot \alpha_{f_{l,k}, f_{k,j}, f_{j,i}}$$

for all $0 \leqslant i < j < k < l \leqslant n$ in the description of the n-simplices.

(d) Conclude with the help of Exercise 6.3 that the Duskin nerve is 3-coskeletal.

Exercise 2. Let $P = (P, \leqslant)$ be a partially ordered set. The goal of this exercise is to show that strictly unit lax functors $P \to \mathcal{C}$ for a strict 2-category $\mathcal{C}$ are in bijection to strict functors from the path 2-category $\underline{\mathrm{Path}}(P)$. Here $\underline{\mathrm{Path}}(P)$ is the strict 2-category with the same objects as P and for two objects $x, y \in P$ their morphism category $\mathbf{Hom}_{\underline{\mathrm{Path}}(P)}(x, y)$ is given by the opposite category of finite totally ordered subsets $S \subset P$ with $\min(S) = x$ and $\max(S) = y$ ordered by inclusion. The horizontal composition is given by the union with the singletons $\{x\}$ serving as identities.

(a) Compute the homotopy and coarse homotopy category of $\underline{\mathrm{Path}}(P)$ from Exercise 4.4. Which one should be called the path (1-)category of P?

(b) Show that there is a unique strictly unital lax functor $T_P\colon P \to \underline{\mathrm{Path}}(P)$ which is the identity on objects and sends the 1-morphism $x \leqslant y$ to the final element of $\mathrm{Hom}_{\underline{\mathrm{Path}}(P)}(x, y)$.

(c) Show that a strict 2-functor $F\colon \underline{\mathrm{Path}}(P) \to \mathcal{C}$ is completely determined by $f_{y,x} := F(\{x \leqslant y\})$ and $\mu_{z,y,x} := F(\{x \leqslant y \leqslant z\}) \supseteq \{x \leqslant z\})$ and that any such choice of $\{f_{y,x}\}_{x,y \in P}$ and μ determines a unique functor if and only if μ satisfies the associativity constraint of a strictly unital functor $\tilde{F}\colon P \to \mathcal{C}$.

(d) Conclude that precompostion with T_P induces a bijection

$$(T_P)^*\colon \{\text{ strict functors } \underline{\mathrm{Path}}(P) \to \mathcal{C}\} \leftrightarrow \{\text{ strictly unital lax functors } P \to \mathcal{C}\}.$$

(e) Deduce that $\underline{\mathrm{Path}}$ upgrades to a functor from the category of partially ordered sets to the category of strict 2-categories with strict functors. Use this to give an alternative description of the Duskin Nerve of a strict 2-category.

Exercise 3. A functor $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is called a simplicial object in $\mathcal{C}$ where we often identify the functor with objectwise images $F_n := F([n])$ and the images of the face maps $\partial_i^n := F(d_i^n)\colon F_n \to F_{n-1}$ and degeneracy maps $\sigma_i^n := F(s_i^n)\colon F_n \to F_{n+1}$. Recall that we thus have in particular that for the face maps and degeneracy maps obey the following relations

$$\begin{aligned} \partial_i^{n-1}\partial_j^n &= \partial_{j-1}^{n-1}\partial_i^n \quad \text{if } i < j \\ \sigma_i^{n+1}\sigma_j^n &= \sigma_{j+1}^{n+1}\sigma_i^n \quad \text{if } i \leqslant j \end{aligned}$$

$$\partial_i^{n+1}\sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1}\partial_i^n & \text{if } i < j \\ \mathrm{id}_{F_n} & \text{if } i \in \{j, j+1\} \\ \sigma_j^{n-1}\partial_{i-1}^n & \text{if } i > j+1 \end{cases}$$

and that any morphism in the image of F can be written as a composition of face maps followed by degeneracy maps. Let now $A\colon \Delta^{\mathrm{op}} \to \mathrm{Ab}$ be a simplicial abelian group. For $n \in \mathbb{N}_0$ the face maps assemble into a morphism $\partial^n := \sum_{i=0}^n (-1)^i \partial_i^n\colon A_n \to A_{n-1}$.

(a) Show that $C_\bullet(A) := (A_n, \partial^n)_{n\in\mathbb{N}_0}$ is an $\mathbb{N}_0$-graded chain complex, i.e. $\partial^n\partial^{n+1} = 0$. Furthermore, show that $D_\bullet(A) := \sum_{0\leqslant i\leqslant n\in\mathbb{N}_0} \mathrm{im}(\sigma_i^n) \subseteq C_\bullet(A)$ is a subcomplex.

(b) Argue that both constructions $C_\bullet$ and $D_\bullet$ are functorial on the category of simplicial groups. Conclude that there is a functor $N_\bullet := C_\bullet/D_\bullet$.

The complex $N_\bullet(A)$ is called the normalized Moore complex of A. Recall that the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$ admits a left adjoint $Fr\colon \mathrm{Set} \to \mathrm{Ab}$ which associates to a set S the free group $\mathbb{Z}S$. The normalised Moore complex of a simplicial set X is then defined to be $N_\bullet(X,\mathbb{Z}) := N_\bullet \circ Fr_*(X) = N_\bullet(Fr \circ X)$.

(c) Argue that for a simplicial set X, its normalised Moore complex $N_\bullet(X)$ is level wise a free abelian group whose basis are the non-degenerate simplices. Describe the differential of $N_\bullet(X)$ in terms of this basis.

8 Nerves of differential graded categories

The reference for this section is [3, ch. 2.5].

Let us begin with a reminder on cochain complexes. Let k be a commutative ring and let mod_k be the category of (right) k-modules and $\mathrm{Ch}(\mathrm{mod}_k)$ the category of cochain complexes of k-modules, so objects are given by

$$(X^\bullet, d^\bullet) = \ldots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots$$

such that $d^2 = 0$. The category $\mathrm{Ch}(\mathrm{mod}_k)$ is a monoidal category with

$$\begin{aligned}(X^\bullet \otimes_k Y^\bullet)^l &:= \coprod_{i+j=l} X^i \otimes_k Y^j \\ d_{X^\bullet \otimes Y^\bullet}(x \otimes y) &:= d_X(x) \otimes y + (-1)^i x \otimes d_Y(y)\end{aligned}$$

where $|x| = i$ is the degree of x. The unit of the monoidal structure is given by k viewed as a chain complex concentrated in degree 0. There is a preferred symmetry constraint

$$\begin{aligned}\tau \colon X^\bullet \otimes Y^\bullet &\xrightarrow{\sim} Y^\bullet \otimes X^\bullet \\ x \otimes y &\mapsto (-1)^{|x||y|} y \otimes x\end{aligned}$$

which is called the Koszul sign rule.

Definition 8.1. A **differential graded category** $\mathcal{A}$ is a category enriched in the monoidal category $\mathrm{Ch}(\mathrm{mod}_k)$. That is:

- A class $\mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$.
- For all $a, b \in \mathrm{Ob}(\mathcal{A})$ a cochain complex $\mathcal{A}(a,b) \in \mathrm{Ch}(\mathrm{mod}_k)$.
- For all $a \in \mathrm{Ob}(\mathcal{A})$ a unit/identity $\mathrm{id}_a \colon k \to \mathcal{A}(a,a)$.
- For all $a, b, c \in \mathrm{Ob}(\mathcal{A})$ a composition law

  $$\mathcal{A}(b,c) \otimes \mathcal{A}(a,b) \xrightarrow{-\circ-} \mathcal{A}(a,b)$$

  given by a morphism in $\mathrm{Ch}(\mathrm{mod}_k)$.

Remark 8.2. This means that if $f \in \mathcal{A}(a,b), |f| = i$ and $g \in \mathcal{A}(b,c), |g| = j$, then

- $|g \circ f| = i + j$ since $|g \otimes f| = i + j$,
- $d_{\mathcal{A}}(g \circ f) = d_{\mathcal{A}}(g) \circ f + (-1)^j g \circ d_{\mathcal{A}}(f), |g| = j$ (Graded Leibniz rule).

This composition law must be associative and unital in the usual sense.

Example 8.3. Let $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ be a the dg category given as follows:

- The objects are given by complexes of k-modules.
- For $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ a complex $\mathrm{Hom}_k(X^\bullet, Y^\bullet) \in \mathrm{Ch}(\mathrm{mod}_k)$.
- Let $\mathrm{Hom}_k(X^\bullet, Y^\bullet)^j := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k(X^i, Y^{i+j})$ be the degree j maps of graded k-modules endowed with the following differential

$$\begin{aligned}\partial\colon\ \mathrm{Hom}_k(X^\bullet, Y^\bullet)^j &\to \mathrm{Hom}_k(X^\bullet, Y^\bullet)^{j+1}\\ f &\mapsto \partial(f) = (d_Y^{i+j} \circ f^i - (-1)^{|f|} f^{i+1} \circ d_X^i)_{i\in\mathbb{Z}}, |f| = j\end{aligned}$$

Note that $|f| = 0$ and $\partial(f) = 0$ is equivalent to f being a cochain map.

Example 8.4. Let now $\mathrm{Ob}(\mathcal{A}) = \{\star\}$, $A := \mathcal{A}(\star, \star)$, this is a dg algebra.

Construction 8.5. Given a dg category $\mathcal{A}$, its underlying category is denoted by $Z^0(\mathcal{A})$, given by

- $\mathrm{Ob}(Z^0(A)) = \mathrm{Ob}(\mathcal{A})$,
- $\forall a, b \in Z^0(\mathcal{A}), Z^0(a,b) = Z^0(\mathcal{A}(a,b)) = \ker(d_{\mathcal{A}^0}) \subseteq \mathcal{A}(a,b)$.

Definition 8.6. Let $\mathcal{A}$ be a dg category. The homotopy category (0-th cohomology category) of $\mathcal{A}$, denoted $H^0(\mathcal{A})$ has

- $\mathrm{Ob}(H^0(\mathcal{A})) = \mathrm{Ob}(Z^0(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$
- $\forall a, b \in H^0(\mathcal{A}), H^0(\mathcal{A})(a,b) = H^0(\mathcal{A}(a,b))$.

In the case of $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$, we have

$$\begin{aligned}Z^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) &:= \mathrm{Ch}(\mathrm{mod}_k)\\ H^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) &= K(\mathrm{mod}_k) \text{ homotopy category of cochain complexes}\end{aligned}$$

Construction 8.7. Let $\mathcal{A}$ be a dg category, then the dg nerve $N_{\mathrm{dg}}(\mathcal{A})$ is defined as follows:

- $N_{\mathrm{dg}}(\mathcal{A})_0$ are the objects of $\mathcal{A}$,
- $N_{\mathrm{dg}}(\mathcal{A})_1$ are the degree zero cocycles $f \in Z^0(\mathcal{A}(a,b))$, that is the morphisms f such that $|f| = 0$ and $d_{\mathcal{A}}(f) = 0$,
- $N_{\mathrm{dg}}(A)_2$ is given by 2-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "f_{21}"] \arrow[d, phantom, "\Downarrow f_{210}"] & \\ a_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & a_2 \end{tikzcd}$$

where $|f_{ij}| = 0. |f_{210}| = -1, d_{\mathcal{A}}(f_{ij}) = 0$ and $d_{\mathcal{A}}(f_{210}) = f_{20} - f_{21} \circ f_{10}$ as well as $[f_{20}] = [f_{21} \circ f_{10}] \in H^0(\mathcal{A}(a_0, a_2))$,

- $N_{\mathrm{dg}}(\mathcal{A})_3$ is given by 3-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{31}"] & \\ a_0 \arrow[ur, "f_{10}"] \arrow[rr, dashed, "f_{20}"] \arrow[dr, "f_{30}"'] & & a_2 \arrow[dl, "f_{32}"] \\ & a_3 & \end{tikzcd}$$

  with each boundary 2-simplex having its composition given by some morphism $f_{ijk}, i, j, k \in \{0, 1, 2, 3\}$ as well as a morphism f_{3210} such that $d(f_{3210}) = -(f_{321} \circ f_{10} - f_{320}) + (f_{32} \circ f_{210} - f_{310})$.

Definition 8.8. Let $\mathcal{A}$ be a dg-category. The dg nerve of $\mathcal{A}$, $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is the simplicial set where for $n \geqslant 0$

$$\begin{aligned} N_{\mathrm{dg}}(\mathcal{A}) &:= \{(a_0, a_1, \ldots, a_n) \in \mathrm{Ob}(\mathcal{A}), (f_I = f_{n \geqslant i_k > \ldots > i_0 \geqslant 0} \in \mathcal{A}(a_{i_0}, a_{i_k}))_{\substack{I \subseteq [n] \\ 2 \leqslant |I|}}\} \\ |f_I| &= -(\#(I \backslash \{i_k, i_0\})) \\ d_{\mathcal{A}}(f_I) &= \sum_{l=1}^{k-1} (-1)^l (f_{i_k > \ldots > i_l} \circ f_{i_l > \ldots > i_0} \circ \ldots \circ f_{i_k > \ldots > i_l > \ldots > i_0}) \end{aligned}$$

Theorem 8.9. *The nerve of a dg-category $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is an ∞-category.*

Exercise 1. Given a partially ordered set P, its differential graded path category $\mathbf{Path}_{\mathrm{dg}}(P)$ has the same objects as P and for every finite totally ordered subset $S \subset P$ with $\min(S) = x \neq y = \max(S)$ a morphism $f_S \colon x \to y$ of degree $|S| - 2$, i.e. the number of elements between x and y in S. Moreover, for every $x \in P$ there is an identity morphism id_x of degree 0 as well as all formal compositions of these morphisms with the identity acting as identity as well as formal sums. We define the differential

$$d(f_S) := \sum_{l=1}^{k} (-1)^l (f_{\{s_l \leqslant s \in S\}} \circ f_{\{s_l \geqslant s \in S\}} - f_{S \backslash \{s_l\}})$$

for $S = \{x < s_1 < \cdots < s_k < y\}$ and extend it obeying the graded Leibniz rule $d(g \circ f) = d(g) \circ f + (-1)^{|g|} g \circ d(f)$ for g homogeneous of degree $|g|$.

(a) Confirm that $d(d(f_S)) = 0$ and that $d(d(g \circ f)) = 0$ if $d \circ d(f) = 0$ and $d \circ d(g) = 0$.

(b) Explain how a morphism of partially ordered sets $f \colon P \to Q$ induces a morphism

$$\mathbf{Path}_{\mathrm{dg}}(f) \colon \mathbf{Path}_{\mathrm{dg}}(P) \to \mathbf{Path}_{\mathrm{dg}}(Q).$$

(Recall that a morphism between dg categories must be degree preserving, compatible with composition and preserve identities.) Conclude that there is a functor $\mathbf{Path}_{\mathrm{dg}} \colon \mathbf{poSet} \to \mathbf{cat}_{\mathrm{dg}}$ from partially ordered sets to differential graded categories.

Restricting, we obtain a functor $p\colon \Delta \to \mathbf{cat}_{\mathrm{dg}}$ and we define the differential graded Nerve as $N_{dg} := p^*\colon\ \mathbf{cat}_{\mathrm{dg}} \to \mathrm{Set}_\Delta$.

(c) Given a differential graded category $\mathcal{D}$, describe the n-smplices of $N_{dg}(\mathcal{D})$.

(d) Show that $N_{dg}(\mathcal{D})$ is an ∞-category.

Exercise 2. The goal of this exercise is to show that the normalised Moore complexes from 7.3 induces an equivalence $N_\bullet\colon \mathrm{Ab}_\Delta \to \mathrm{Ch}_{\mathbb{N}_0}(\mathbb{Z})$ between simplicial abelian groups and non-negative chain complexes of abelian groups. Let $A\colon \Delta^{\mathrm{op}} \to \mathrm{Ab}$ be a simplicial abelian group.

(a) Show that for $x \in A_n$ with $\partial_j^n(x) = 0$ for all $0 < i+1 < j \leqslant n$ we have for

$$x^{\leqslant i} := x - \sigma_i^{n-1}\partial_{i+1}^n(x)$$

that $\partial_j^n(x^{\leqslant i}) = 0$ for all $0 < i < j \leqslant n$. Deduce that any element of $N_n(A)$ can be represented by an $x \in A_n$ such that $\partial_j^n(x) = 0$ for all $0 < j \leqslant n$.

We define the subcomplex $C_\bullet^{\leqslant 0}(A) \subseteq C_\bullet(A)$ by $C_n^{\leqslant 0}(A) := \{a \in A_n \mid \forall 0 < j \leqslant n \quad \partial_j^n(a) = 0\}$.

(b) Show that the canonical morphism

$$\bigoplus_{0\leqslant m\leqslant n} \bigoplus_{\substack{\alpha\colon [n]\twoheadrightarrow[m] \\ \text{surjective}}} A(\alpha)(C_m^{\leqslant 0}(A)) \to C_n(A)$$

is in fact an isomorphism. (Hint: For the injectivity, define a partial ordering

$$\alpha = (k, \alpha\colon [n] \twoheadrightarrow [k]) \leqslant (m, \beta\colon [n] \twoheadrightarrow [m]) = \beta$$

if $k \leqslant m$ and if $k = m$, then $\beta \leqslant \alpha$ pointwise. Then show that for $\sum_\alpha A(\alpha)(x_\alpha) = 0$ we have that $x_\beta = 0$ if $x_\alpha = 0$ for all $\alpha \leqslant \beta$ by applying $A(\gamma)$ for $\gamma\colon [m] \to [n], \gamma(j) := \min \beta^{-1}(j)$.)

(c) Deduce that the canonical map $C_\bullet^{\leqslant 0}(A) \to C_\bullet(A) \to N_\bullet(A)$ is an isomorphism of complexes.

(d) Conclude that the normalised Moore complex is a fully faithful functor by showing that any morphism $f\colon A \to B$ is uniquely determined by a collection of group homomorphisms

$$f_n\leqslant 0\colon C_n^{\leqslant 0}(A) \to C_n^{\leqslant 0}(B)$$

where $n \in \mathbb{N}_0$.

We define the Eilenberg-Maclane functor $K\colon \mathrm{Ch}_{\mathbb{N}_0}(\mathbb{Z}) \to \mathrm{Ab}_\Delta$ as $K := (N_\bullet(-,\mathbb{Z}) \circ \mathcal{Y}_\Delta)^* \circ \mathcal{Y}_{\mathrm{Ch}_{\mathbb{N}_0}(\mathbb{Z})}$.

$$K(C_\bullet)_n := K(C_\bullet)(n) = \mathrm{Hom}_{\mathrm{Ch}_{\mathbb{N}_0}(\mathbb{Z})}(N_\bullet(\Delta^n,\mathbb{Z}), C_\bullet)$$

(e) Make the description of $N_\bullet(\Delta^n, \mathbb{Z})$ from Exercise 7.3(c) explicit. Use this to argue that $f \in C_n^{\leqslant 0}(K(C_\bullet))$ if and only if f annihilates the subcomplex $N_\bullet(\Lambda_0^n, \mathbb{Z})$, i.e. $f\,|_{N_\bullet(\Lambda_0^n,\mathbb{Z})} = 0$.

(f) Deduce that $f \mapsto f(\mathrm{id}_n)$ induces an isomorphism of complexes $C_\bullet^{\leqslant 0}(K(C_\bullet)) \to C_\bullet$.

(g) Conclude that $N_\bullet\colon \mathrm{Ab}_\Delta \to \mathrm{Ch}_{\mathbb{N}_0}(\mathbb{Z})$ is an equivalence of categories.

Since $N_\bullet$ is an equivalence, it preserves colimits. Hence, we have seen in (a slight generalisation of Exercise 1.3 that $N_\bullet$ admits a right adjoint which is K by construction. Thus K is a quasi inverse of $N_\bullet$.

Lecture 5.6.

9 Simplicial categories/Simplicially enriched categories

The reference for this section is [3, **page** 2.4].

Consider the monoidal category $(\mathrm{Set}_\Delta, \times, \Delta^0)$.

Lecture 17.06

Let $\mathcal{C}_\bullet$ be a simplicial category $\mathcal{C}_\bullet \in \mathrm{Set}_\Delta$-cat.

Definition 9.1. The homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathcal{C}_\bullet) \in \mathrm{Set}_\Delta$ has n-simplices $\mathrm{N}^{\mathrm{hc}}(\mathcal{C}_\bullet)_n = \{$ simplicial functors $\underline{\mathrm{Path}}\,[n]_\bullet \mapsto \mathcal{C}\}$. where more generally, given a poset (Q, S) we let $\underline{\mathrm{Path}}\,[Q]_\bullet$ be the simplicial category associated to the strict 2-category (i.e. the Duskin nerve of the 2-category) $\underline{\mathrm{Path}}_{(2)}\,[Q]$.

Theorem 9.2. *Gordier-Porter Let $\mathcal{C}_\bullet \in \mathrm{Set}_\Delta$-cat be locally Kan then $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})_\bullet \in \mathrm{Set}_\Delta$ is an ∞-category.*

Question: <u>What do we want to achieve?</u> We want to have access to ∞-categorical generalisations of "all statements and constructions in 1-category theory."

1. Let $\mathcal{C} \in \mathrm{Set}_\Delta$ be an infinity category. What is $\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Gpd}_\infty$ (the functorial mapping ∞-groupoid/space) ?

2. What is Gpd_∞ ? Let $X, Y \in \mathrm{Set}_\Delta$ be Kan complexes, then $\mathrm{Fun}(X, Y) := \underline{\mathrm{Hom}}_{\mathrm{Set}_\Delta}(X, Y) \in \mathrm{Set}_\Delta$ is a Kan complex. We want to make sense of the correspondence $\mathrm{Gpd}_\infty \leftrightarrow (\text{Kan complexes})[heq^{-1}/weq^{-1}]_\infty$.

3. What is the ∞-categorical localisation procedure? (e.g. $\mathrm{Ch}(\mathrm{mod}_R)[qiso^{-1}]_\infty) =: \mathcal{D}((\mathrm{mod}_R)_\infty)$.

4. Suppose you have solved 1), then we obtain a functor $\mathcal{Y}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty)$ that is some Yoneda embedding.

5. We need notions of limit and colimit in an ∞-category.

6. We need the notion of a final object and of an ∞-category of cones over a diagram $X \in \mathcal{C}_0$ is trivial if $\mathrm{Map}_{\mathcal{C}}(\forall Y, X) \xrightarrow{\sim} \star = \Delta^0$.

7. What are presheaves of ∞-groupoids? In order to avoid defining Gpd_∞. we need an ∞-categorical variant of the notion of a 1-category fibered in groupoids. For example the right fibrations are given as $\mathrm{RFib}(\mathcal{C})_\infty \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty)$. One can think of this as follows, given a functor $X\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Gpd}_\infty$ there exists the pullback square

$$\begin{tikzcd} X_c \arrow[r] \arrow[d] & \int^{\mathcal{C}} X \arrow[d] \\ \star \arrow[r, "c"] & \mathcal{C} \end{tikzcd}$$

References

[1] Peter T Johnstone. *Sketches of an Elephant A Topos Theory Compendium*. Oxford University Press, **september** 2002. ISBN: 9780198515982. DOI: 10.1093/oso/9780198515982.001.0001. URL: https://doi.org/10.1093/oso/9780198515982.001.0001.

[2] Tom Leinster. *Basic category theory*. **volume** 143. Cambridge University Press, 2014.

[3] Jacob Lurie. *Kerodon*. https://kerodon.net. 2018.