Itic notes

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Contents

1 Motivation

Fix $0 \leqslant m \leqslant n \leqslant \infty$. An (n, m) category is a "category-like" structure consisting of a class of objects, notions of 1- morphism, 2-morphism, ... , n-morphism (i.e. k-morphsim $0 < k \leqslant n$) with a "suitable composition law" (satisfying "suitable axioms") and such that $\forall m < k \leqslant n$ the k-morphisms are "invertible".

$$
\begin{aligned}
&(0,0)\text{-cat.} = \text{set}\\
&(1,0)\text{-cat.} = \text{groupoid, i.e. 1-groupoid}\\
&(1,1)\text{-cat.} = \text{category, i.e. 1-category}\\
&(2,0)\text{-cat.} = \text{2-groupoids}\\
&(2,1)\text{-cat.}\\
&(2,2)\text{-cat.} = \text{2-categories}\\
&\vdots\\
&(n,0)\text{-cat.} = \text{n-groupoid}\\
&(n,n)\text{-cat.} = \text{n-cat.}\\
&\vdots\\
&(\infty,0) = \infty\text{-groupoids}\\
&(\infty,1) = \infty\text{-categories (see. Boardman-Vogt)}
\end{aligned}
$$

Reminder 1.1. *A map $f\colon X \to Y$ between topological spaces is a weak homotopy equivalence if $\forall x \in X, \forall n \in \mathbb{N}$ the map*

$$\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection.

Theorem 1.2 (Grothendieck's Homotopy Hypothesis)**.** *There is an $(\infty, 1)$-category of topological spaces up to weak homotopy equivalence and there is an $(\infty, 1)$-category of ∞-groupoids up to eqivalence. There is furthermore an ∞-functor assigning to each topological space X its Poincare ∞-groupoid $\pi_\infty(X)$, this is an equivalence.*

Remark 1.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category, then for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an ∞-groupoid/ "space". We have the homtopy category of $\mathcal{C}$ denoted by $\mathrm{Ho}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and for all objects $X, Y \in \mathrm{Ho}(\mathcal{C})$ we have that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Warning! 1.4. The passage from $\mathcal{C}$ often results in a tremendous loss of information, that is essential for various purposes.

- Computing co-/limits within $\mathcal{C}$.
- Computing co-/limits with $\mathcal{C}$.

- Define invariants associated to $\mathcal{C}$ (f.e. Hochschild cohomology).

Reminder 1.5. *Many important 1-categories arise as homotopy categories of genuine* $(\infty, 1)$*-categories, that is derived categories Recall for* R: *ring,* mod_R *its category of right* R*-modules and* $\mathrm{Ch}(\mathrm{mod}_R)$ *the category of chain complexes in* mod_R*, that a morphism of chain complexes* $f^\bullet \colon X^\bullet \to Y^\bullet$ *in* $\mathrm{Ch}(\mathrm{mod}_R)$ *is a quasi-isomorphism if* $\forall n \in \mathbb{Z}$ *we have that* $H_n(f^\bullet) = H_n(X^\bullet) \xrightarrow{\sim} H_n(Y^\bullet)$ *is an isomorphism. The derived category is defined as follows* $D(\mathrm{mod}_R) := \mathrm{Ch}(\mathrm{mod}_R)[qiso^{-1}]$ *the localisation at the quasi-isomorphisms. Furthermore we have that* $\mathrm{Ho}(\mathcal{D}(\mathrm{mod}_R)) = D(\mathrm{mod}_R)$*. In the first case we obtain it by building it from the ground up so to say and in the second case we obtain by forgetting information from a higher structure.*

Warning! 1.6. The homotopy theory of $(\infty, 1)$-categories has many equivalent (Quillen) implementations:

- Topological categories (Ilias)
- Simplicial categories (Bergner)
- Complete Segal spaces (Rezk)
- Relative categories (Barwick-Kan)
- Pre-derivations
- ∞-categories (Joyal, Lurie)

In the k-linear setting, for k a field we have:

1. Differentially graded k-categories
2. A_∞-categories (Lefèvre-Hasegawa)

The Plan for the lecture is to start of with investigating 2-categories, then give definition and examples of ∞-categories, then do enriched category theory and end on the homotopy theory of ∞-categories.

2 Heuristics: Isomorphism vs. Equivalence

Makkai's Principle of Isomorphism (1998) says:

"All grammatically correct properties about objects in a fixed category are to be invariant under isomorphism."

Fix a category $\mathcal{C}$ and a small category A and take the functor category $\mathrm{Fun}(A, \mathcal{C})$, that is A-shaped diagrams in $\mathcal{C}$. For $X \in \mathcal{C}$ define:

$$\lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) := \{(p_a : X \to D(a))_{a \in A} \mid \begin{tikzcd} X \arrow[d, "f_a"'] \arrow[dr, "f_b"] & \\ D(a) \arrow[r] & D(b) \end{tikzcd} \quad \forall f : a \to b \text{ in } A\}$$

For $\phi : Y \to X$ define the function

$$\begin{aligned} \phi^* : \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) &\to \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \\ p = (p_a : X \to D(a))_{a \in A} &\mapsto \phi^*(p) = (p_a \circ \phi : Y \to D(a))_{a \in A} \end{aligned}$$

Thus $\lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf of sets on $\mathcal{C}$.

Definition 2.1. A limit of a diagram $D : A \to \mathcal{C}$ is a cone $p \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ that is universal in the sense that $\forall Y \in \mathcal{C}, \forall q \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \exists! \varphi : Y \to X$ such that $\varphi^*(p) = q$. We write $\lim_{a \in A} D(a)$ for any limit of D (which may or may not exist).

Reminder 2.2 (Yoneda Lemma)**.** *Let $X \in \mathcal{C}$ and let*

$$\begin{aligned} \nu : \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, X), \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))) &\xrightarrow{\sim} \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) \\ \eta = (\eta_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \lim_{a \in \mathcal{C}} (Y, D(a))_{Y \in \mathcal{C}} &\mapsto \eta_X(\mathrm{id}_X) \\ (\eta^p_Y(p) := \varphi^*(p))_{Y \in \mathcal{C}} &\leftarrow\!\shortmid p \end{aligned}$$

If $p \in \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ is a limit of $D : A \to \mathcal{C}$ then

$\eta^p : \mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, D(a))$ is a natural isomorphism. We furthermore obtain, that for an isomorphism $\psi : Y \to X$ in $\mathcal{C}$ we have

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(-, X) \arrow[dr, "\eta^p", "\sim"'] & \\ \mathrm{Hom}_{\mathcal{C}}(-, Y) \arrow[ur, "\psi^*", "\sim"'] \arrow[rr, "\sim"] & & \lim_{a \in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) \end{tikzcd}$$

Example 2.3. Let the following be a diagram in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

then the limit of the diagram (if it exists) is called a pullback.

$$\begin{tikzcd}
W \arrow[dr, "\exists ! p"] \arrow[drr, bend left, "\forall f''"] \arrow[ddr, bend right, "\forall g''"'] & & \\
& X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

For example if $\mathcal{C} = \text{Set}$ then $X \times_Z Y = \{(X, Y) \in X \times Y \mid f(x) = g(y)\}$.

The commutativity condition takes place in $\text{Hom}_{\mathcal{C}}(X \times_Z Y, Z) \ni g \circ f' = f \circ g'$

Makkai's Principle of Equivalence All grammatically correct properties of objects in a fixed 2-category are to be invariant under equivalence.

Remark 2.4. We want to Cat be the strict 2-category of (small) categories with functors as 1-morphisms and natural transformations as 2-morphisms. Now natural transformation allow for a notion of equivalence of morphisms, that is in a 1-category we only knew what it meant for two morphisms to be equal, but now we can talk about two functors being naturally isomorphic given us a notion of equivalence of 1-morphisms, via the 2-morphisms.

Definition/Proposition 2.5 (Godement Product)**.** Consider natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"{name=U}] \arrow[r, bend right=50, "F_2"'{name=D}] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E}
\end{tikzcd}$$

Their Godement product is the natural transformation.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "G_1 \circ F_1"] \arrow[r, bend right=50, "G_2 \circ F_2"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & \mathcal{E}
\end{tikzcd}$$

Let $X \in \mathcal{C}$, we obtain the following diagram

$$\begin{tikzcd}
F_1(X) \arrow[d, "\alpha_x"] & G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta * \alpha)_X"] & G_2(F_1(X)) \arrow[d, "G_2(\alpha_X)"] \\
F_2(X) & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] & G_2(F_2(X))
\end{tikzcd}$$

in $\mathcal{D}$.

Proof. We show that $\beta * \alpha \colon G_1 \circ F_1 \Rightarrow g_2 \circ F_2$ is indeed a natural transformation. For that we take the following diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & G_1(F_1(X)) \arrow[r, "G_1(\alpha_X)"] \arrow[d, "G_1(F_1(f))"'] & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(F_2(f))"] & G_2(F_2(X)) \arrow[d, "G_2(F_2(f))"] \\
Y & G_1(F_1(Y)) \arrow[r, "G_1(\alpha_1)"'] & G_1(F_2(Y)) \arrow[r, "\beta_{F_2(Y)}"'] & G_2(F_2(Y))
\end{tikzcd}$$

∎

Proposition 2.6. *Consider natural transformations*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=80, "F_1"] \arrow[r, "F_2"'] \arrow[r, bend right=80, "F_3"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\gamma"] & \mathcal{D} \arrow[r, bend left=80, "G_1"] \arrow[r, "G_2"'] \arrow[r, bend right=80, "G_3"'] \arrow[r, phantom, shift left=4, "\Downarrow\beta"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & \mathcal{E}
\end{tikzcd}$$

Then $(\delta\beta) * (\gamma\alpha) = (\delta * \gamma) \circ (\beta * \alpha)$.

Proof. Let $X \in \mathcal{C}$

$$\begin{tikzcd}
G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta*\alpha)_X"] \arrow[dd, bend right=60, "G_1(\gamma_X\alpha_X)"'] & G_2(F_1(X)) \arrow[r, "\delta_{F_1(X)}"] \arrow[d, "G_2(\alpha_X)"] & G_3(F_1(X)) \arrow[d, "G_3(\alpha_X)"] \arrow[dd, bend left=60, "G_3(\gamma_X\alpha_X)"] \\
G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(\gamma_X)"'] & G_2(F_2(X)) \arrow[r, "\delta_{F_2(X)}"] \arrow[d, "G_2(\gamma_X)"] \arrow[dr, "(\delta*\gamma)_X"] & G_3(F_2(X)) \arrow[d, "G_3(\gamma_X)"] \\
G_1(F_3(X)) \arrow[r, "\beta_{F_3(X)}"] & G_2(F_3(X)) \arrow[r, "\delta_{F_2(X)}"] & G_3(F_3(X))
\end{tikzcd}$$

Now the long diagonal of the diagram corresponds to $(\delta * \gamma) \circ (\beta * \alpha)$ and the outer large square to $(\delta \circ \beta) * (\gamma \circ \alpha)$. ∎

Definition 2.7. The Godement products bellow are called whickerings

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
\mapsto
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=50, "G_1 \circ F"] \arrow[rr, bend right=50, "G_2 \circ F"'] \arrow[rr, phantom, "\Downarrow \beta * \mathrm{id}_F"] & & \mathcal{E}
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E} \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "G \circ F_1"] \arrow[rr, bend right, "G \circ F_2"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_G * \alpha"] & & \mathcal{E} \end{tikzcd}$$

Construction 2.8. Given a cospan of groupoids

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

its 2-pullback is the diagonal of groupoids.

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

The objects are given as $\mathrm{Ob}(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}) = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi \colon F(a) \xrightarrow{\sim} G(b) \text{ in } \mathcal{C})$ and morphisms are given by tuples of morphisms $(u, v) \colon (a, b, \varphi) \to (a', b', \varphi')$, where $u \colon a \to a'$ and $v \colon b \to b'$ are morphisms in the respective groupoids, such that the following square commutes

$$\begin{tikzcd} F(a) \arrow[r, "\varphi", "\sim"'] \arrow[d, "F(a)"'] & G(b) \arrow[d, "G(v)"] \\ F(a') \arrow[r, "\varphi", "\sim"'] & G(b') \end{tikzcd}$$

Lecture 15.04

Let $X, Y, Z \in \mathrm{Set}$ and consider a pullback diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g \circ ?"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where $X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and

Construction 2.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$: groupoids

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} \arrow[r, "\pi_{\mathcal{A}}"] \arrow[d, "\pi_{\mathcal{B}}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

Lecture 15.04

Let $X, Y, Z \in \mathrm{Set}$ and consider a pullback diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Where the fiber product is given by $X \times_Z Y := \{(x, y) \in X \times Y \times f(x) = g(x)\}$ such that the following isomorphism holds.

$$\begin{tikzcd} W \arrow[r, "\pi_Y \circ \varphi"] \arrow[d, "\pi_X \circ \varphi"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

do this properly

Thus the case for pullback of a diagram of objects is clear, but what does the pullback of morphism sets look like?

Construction 2.10. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be groupoids and

$$\begin{tikzcd} \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} \arrow[r, "\pi_{\mathcal{A}}"] \arrow[d, "\pi_{\mathcal{B}}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

be the 2-pullback of $\mathcal{A} \xrightarrow{F} \mathcal{C}$. Its objects are triples $X = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi_X : F(a) \xrightarrow{\sim} G(b))$ and for another triple $X' = (a' \in \mathcal{A}, b' \in \mathcal{B}, \varphi X' : F(a') \xrightarrow{\sim} G(b'))$ the morphisms are given by tuples $(\mathcal{A} \ni u : a \to a', \sqsubseteq \ni v : b \to b')$ such that

$$\begin{tikzcd} F(a) \arrow[r, "\sim"] \arrow[d, "F(u)"] & G(b) \arrow[d, "G(v)"] \\ F(a') \arrow[r, "\sim"] & G(b') \end{tikzcd}$$

that is $\varphi_{X'} \circ F(u) = G(v) \circ \varphi_X$. For a groupoid $\mathcal{D}$ we may consider the induced cospan of groupoids

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D}, \mathcal{B}) \arrow[d, "G \circ ?"] \\ \mathrm{Fun}(\mathcal{D}, \mathcal{A}) \arrow[r, "F \circ ?"] & \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \end{tikzcd}$$

Interlude 2.11. Fix a groupoid G. Then, the construction $\mathcal{D} \in \mathrm{Gpd} \mapsto \mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is suitably functorial, which means

- For all $D \in \mathrm{Gpd}$ it holds that $\mathrm{Fun}(\mathcal{D}, \mathcal{G})$ is a groupoid.

- For all $F\colon \mathcal{D}_{\rightarrow}\mathcal{D}_2$ it holds that $?\circ F\colon\ \mathrm{Fun}(\mathcal{D}_2,\mathcal{G}) \to \mathrm{Fun}(\mathcal{D}_1,\mathcal{G})$ is a functor, which means that

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{G} \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "G_1\circ F"] \arrow[r, bend right, "G_2\circ F"'] \arrow[r, phantom, "\Downarrow\beta_F"] & \mathcal{G} \end{tikzcd}$$

- For a natural transformation between functors between groupoids.

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}_2 \end{tikzcd}$$

there is a natural transformation

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}_2,\mathcal{G}) \arrow[r, bend left, "?\circ F_1"] \arrow[r, bend right, "?\circ F_2"'] \arrow[r, phantom, "\Downarrow ?*\alpha"] & \mathrm{Fun}(\mathcal{D}_1,\mathcal{G}). \end{tikzcd}$$

Let us take the 2- pullback from above $\mathbb{F}\mathcal{D} := \mathrm{Fun}(\mathcal{D},\mathcal{A}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D},\mathcal{B})$ and analyze the map $\mathcal{D} \to \mathbb{F}\mathcal{D}$

- For all groupoids $\mathcal{D}$ the category $\mathbb{F}\mathcal{D}$ is a groupoid.
- For all $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ the attribution $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_1$ is a functor.
- The objects are given as $(p_{\mathcal{A}}\colon \mathcal{D} \to \mathcal{A}, p_{\mathcal{B}}\colon \mathcal{D} \to \mathcal{B}, \phi\colon F\circ p_{\mathcal{A}} \xrightarrow{\sim} G\circ p_{\mathcal{B}})$, that is the 2-pullback, i.e. the datum of a pullback diagram

$$\begin{tikzcd} \mathcal{A}\times_{\mathcal{C}}\mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

- For all functors $H\colon \mathcal{D}_1 \to \mathcal{D}_2$ we get $\mathbb{F}\mathcal{D}_2 \to \mathbb{F}\mathcal{D}_2$, that is for a second pullback square

$$\begin{tikzcd} \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

with a functor between the fiberproducts, is mapped to the following.

$$\begin{tikzcd}
\mathcal{D}_\infty \arrow[dr, "H"] \arrow[drr, bend left, "q_{\mathcal{B}}\circ H"] \arrow[ddr, bend right, "q_{\mathcal{A}}\circ H"'] & & \\
& \mathcal{D}_2 \arrow[r, "q_{\mathcal{B}}"] \arrow[d, "q_{\mathcal{A}}"] \arrow[dr, phantom, "\overset{\psi}{\Longrightarrow}"] & \mathcal{B} \arrow[d, "G"] \\
& \mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

The associated natural transformation is given by $\psi_H \colon F \circ q_{\mathcal{A}} \circ H \implies G \circ q_{\mathcal{B}} \circ H$.

The next question we can naturally ask is what morphisms in $\mathbb{F}(\mathcal{D})$ look like

$$\begin{tikzcd}
(p_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{D}, \arrow[d, Rightarrow, "\alpha"] & p_{\mathcal{B}} \colon \mathcal{D}_2 \longrightarrow \mathcal{B}, \arrow[d, Rightarrow, "\beta"] & F \circ p_{\mathcal{A}} \xRightarrow[\sim]{\phi} \arrow[d, Rightarrow, "F_\alpha"] & G \circ p_{\mathcal{B}}) \arrow[d, Rightarrow, "G_\beta"] \\
(q_{\mathcal{A}} \colon \mathcal{D}_2 \longrightarrow \mathcal{A}, & q_{\mathcal{B}} : \mathcal{D}_2 \longrightarrow \mathcal{B}, & F \circ q_{\mathcal{A}} \xRightarrow[\psi]{\sim} & G \circ q_{\mathcal{B}})
\end{tikzcd}$$

For every $\begin{tikzcd} \mathcal{D}_1 \arrow[r, bend left=50, "H_1"] \arrow[r, bend right=50, "H_2"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{D}_2 \end{tikzcd}$ there is a diagram $\begin{tikzcd} \mathbb{F}\mathcal{D}_1 \arrow[r, bend left=50, "\mathbb{F}H_1"] \arrow[r, bend right=50, "\mathbb{F}H_2"'] \arrow[r, phantom, "\Downarrow\mathbb{F}\gamma"] & \mathbb{F}\mathcal{D}_2 \end{tikzcd}$

Thus for any two pullback diagrams we obtain morphisms

maybe draw this huge diagram

Proposition 2.12. *Let $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ be a cospan of groupoids. Then for all groupoids $\mathcal{D}$, it holds that*

$$\begin{aligned}
\mathbb{X} \colon \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) &\to \mathbb{F}(\mathcal{D} \xleftarrow{\mathbb{F}H} \mathbb{F}(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B})) \\
(\mathcal{D} \xrightarrow{H} \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) &\mapsto H^*(can)
\end{aligned}$$

is an isomorphism.

Example 2.13. Let A, B, C be groups and $\mathbb{B}A, \mathbb{B}B, \mathbb{B}C$ their associated groupoids, then for group homomorphisms $A \xrightarrow{f} C \xleftarrow{g} B$, we get that the objects correspond to triples $(*_A, *_B, *_C \xrightarrow{c \in C} *_C)$

Lecture 17.04

Definition 2.14. A strict 2-category $\mathcal{C}$ consists of:

1. A class $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$
2. For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose objects $f\colon X \to Y$ are called 1-morphisms and whose morphisms $\alpha\colon f \Rightarrow g$ are called 2-morphisms, with a vertical composition of 2-morphisms, that is associative, and unital.
3. For all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a horizontal composition functor

$$- \circ -\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

that is compatible with the vertical composition, in the following way

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left=50, "g \circ f"] \arrow[r, bend right=50, "g' \circ f'"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & Z \end{tikzcd}$$

4. Functoriality of horizontal composition:

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_f"] & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_g"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left=50, "g \circ f"] \arrow[r, bend right=50, "g \circ f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{g\circ f}"] & Z \end{tikzcd}$$

5. There is a composition

$$\begin{tikzcd} X \arrow[r, bend left=70, "f_1"] \arrow[r, "f_2"'] \arrow[r, bend right=70, "f_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\gamma"] & Y \arrow[r, bend left=70, "g_1"] \arrow[r, "g_2"'] \arrow[r, bend right=70, "g_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta"] & Z \arrow[r, phantom, "\mapsto"] & X \arrow[r, bend left=50, "g_1 \circ f_1"] \arrow[r, bend right=50, "g_3 \circ f_3"'] \arrow[r, phantom, "\Downarrow\eta"] & Z \end{tikzcd}$$

where η is the Godement product $(\delta * \gamma) \circ (\beta * \alpha)$.

The above data should satisfy the following axioms:

- (Unitality) For all $X \in \mathrm{Ob}(\mathcal{C}) \exists \, \mathrm{id}_X \in \mathrm{Ob}(\mathrm{Hom}_{\mathcal{C}}(X,X))$ an identity 1-morphism such that $\forall Y \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\begin{gathered}\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(X,X) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)\\ f \mapsto (f, \mathrm{id}_X) \mapsto f \circ \mathrm{id}_X\end{gathered}$$

is equal to the identity functor $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Similarly the functor

$$\begin{gathered}\mathbb{1} \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(X,X) \times \mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)\\ g \mapsto \mathrm{id}_X \circ g = g\end{gathered}$$

is equal to the identity functor$\mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

- (Associativity) For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$: the following square of functors commutes <u>strictly</u>.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(-\circ-)"] \arrow[d, "(-\circ-)\times\mathrm{Hom}_{\mathcal{C}}(W,X)"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[d, "(-\circ-)"] \\
\mathrm{Hom}_{\mathcal{C}}](X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(-\circ-)"] & \mathrm{Hom}_{\mathcal{C}}(W,Z)
\end{tikzcd}$$

Definition 2.15. A strict monoidal category is a strict 2-category with a single object, that is the following data.

1. A strict 2-category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$.
2. A category $\mathcal{M} := \mathrm{Hom}_{B\mathcal{M}}(*,*)$.
3. A monoidal composition $- \otimes -\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, that fullfills the following axioms.
   - (Unitality) There exists $\mathbb{1}_{\mathcal{M}} \in \mathrm{Ob}(\mathcal{M}) = \mathrm{Ob}(Hom_{B\mathcal{M}}(*,*))$ such that the functor

   $$\begin{gathered}\mathcal{M} \to \mathcal{M}\\ M \mapsto M \otimes \mathbb{1}_{\mathcal{M}}\end{gathered}$$

   is the identity, that is gives equalities $M \otimes \mathbb{1}_{\mathcal{M}} = M = \mathbb{1}_{\mathcal{M}} \otimes M$.
   - (Associativity) The following square commutes

   $$\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \times \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{M}} \times (-\circ-)"] \arrow[d, "(-\otimes-)\times\mathrm{id}_{\mathcal{M}}"'] & \mathcal{M} \times \mathcal{M} \arrow[d, "\otimes"] \\
\mathcal{M} \times \mathcal{M} \arrow[r, "-\otimes-"] & \mathcal{M}
\end{tikzcd}$$

   that is for all $M_1, M_2, M_3 \in \mathcal{M}$ it holds that $M_1 \otimes (M_2 \otimes M_3) = (M_1 \otimes M_2) \otimes M_3$.

Definition 2.16. Let $\mathcal{C}, \mathcal{D}$ be strict 2-categories. A strict 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of

1. A map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$ sending an object X to $F(X)$.
2. For all $X, Y \in \operatorname{Ob}(\mathcal{C})$ a functor $F_{XY}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{hom}_{\mathcal{D}}(FX, FY)$ such that the following hold
   - (Unitality) $\forall X \in \operatorname{Ob}(\mathcal{C}) F(\mathrm{id}_X) = \mathrm{id}_{FX} \in \operatorname{Hom}_{\mathcal{D}}(FX, FX)$
   - (Composition) Let $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$

$$\operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y)$$

Let $\mathcal{C}\colon$ 2-category then the opposite category $\mathcal{C}^{\mathrm{op}}$ is also a strict 2-category.

Lecture 22.04

Remark 2.17. Every ordinary 1-category $\mathcal{C}$ can be viewed as a strict 2-category with

- Forall $X, Y \in \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(X,Y)$.
- Horizontal composition = composition in $\mathcal{C}$
- For all $X \in \operatorname{Ob}(\mathcal{C})$ it holds that $\mathrm{id}_X \in \operatorname{Hom}_{\mathcal{C}}(X,X) = \operatorname{Hom}_{\mathcal{C}}(X,X)$ is the identity morphism in the original catgory $\mathcal{C}$.
- Conversely every strict 2-catgory $\mathcal{C}$ has an underlying ordinary category $\mathcal{C}_0$ with $\operatorname{Ob}(\mathcal{C}_0) := \operatorname{Ob}(\mathcal{C})$ and $\forall X, Y \in \operatorname{Ob}(\mathcal{C}) = \operatorname{Ob}(\mathcal{C}), \operatorname{Hom}_{\mathcal{C}_0}(X,Y) := \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,Y))$. The composition law in $\mathcal{C}_0$ is horizontal composition of 1-morphisms in $\mathcal{C}$, this composition is associative since $\mathcal{C}$ is a strict 2-category.

Definition 2.18. A 2-category (bicategory) $\mathcal{C}$ consists of a 2-category $\mathcal{C}$

- A class of objects of $\mathcal{C}$,
- $\forall X, Y \in \operatorname{Ob}\mathcal{C}, \operatorname{Hom}_{\mathcal{C}}(X,Y)$ a category of 1-morphisms,
- for all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ a composition functor

$$(\text{-} \circ \text{-})\colon \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Z),$$

- for all $X \in \operatorname{Ob}(\mathcal{C})$ an object $\mathrm{id}_X \in \operatorname{Ob}(\operatorname{Hom}_{\mathcal{C}}(X,X))$ together with an invertible 2-morphism $v_X = \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$ in $\operatorname{Hom}_{\mathcal{C}}(X,X)$ called unit constraints,

- for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a natural isomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times(\text{-}\circ\text{-})"] \arrow[d, "(\text{-}\circ\text{-})\times \mathrm{Hom}_{\mathcal{C}}(W,X)"'] \arrow[dr, phantom, "\sim\Downarrow\alpha=\alpha_{W,X,Y,Z}"] & \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,Y) \arrow[d, "(\text{-}\circ\text{-})"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "(\text{-}\circ\text{-})"'] & \mathrm{Hom}_{\mathcal{C}}(W,Z)
\end{tikzcd}$$

  where

$$\alpha_{f,g,h} \colon h \circ (g \circ f) \overset{\sim}{\Longrightarrow} (h \circ g) \circ f \in \mathrm{Hom}_{\mathcal{C}}(W,Z)$$

- For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the following functors are fully faithful:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\
f &\mapsto \mathrm{id}_Y \circ f \\
\mathrm{Hom}_{\mathcal{C}}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}}(X,Y) \\
f &\mapsto f \circ \mathrm{id}_X
\end{aligned}$$

- For all $V, W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and all composable 1-morphisms $V \xrightarrow{e} W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ we have the Pentagon-identity

$$\begin{tikzcd}
 & h \circ ((g \circ f) \circ e) \arrow[rr, Rightarrow, "\alpha_{h,g\circ f,e}"] & & (h \circ (g \circ f)) \circ e \arrow[dr, Rightarrow, "\alpha_{h,g,f} * \mathrm{id}_e"] & \\
h \circ (g \circ (f \circ e)) \arrow[ur, Rightarrow, "\mathrm{id}_h * \alpha_{g,f,e}"] \arrow[drr, Rightarrow, "\alpha_{h,g,f\circ e}"] & & & & ((h \circ g) \circ f) \circ e \\
 & & (h \circ g) \circ (f \circ e) \arrow[urr, Rightarrow, "\alpha_{h\circ g,f,e}"] & &
\end{tikzcd}$$

Example 2.19. Every strict 2-category can be viewed as a 2-category with the identity unit and associativity constraints.

Definition 2.20. Monoidal categories are 2-categories with a single object. That is a 2 category $B\mathcal{M}$ with $\mathrm{Ob}(B\mathcal{M}) = \{*\}$ and $\mathcal{M} := \underline{\mathrm{Hom}}_{B\mathcal{M}}(*,*)$. The horizontal composition defines the monoidal composition

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

that is $\alpha_{M_1,M_2,M_3} \colon M_1 \otimes (M_2 \otimes M_3) \xrightarrow{\sim} (M_1 \otimes M_2) \otimes M_3$ in $\mathcal{M}$.

Example 2.21. Let k be a field and $(\mathrm{Vect}_k, \otimes_k, k)$ a monoidal category, that is we have

$$\begin{aligned}
can \colon V_1 \otimes (V_2 \otimes V_3) &\xrightarrow{\sim} (V_1 \otimes V_2) \otimes V_3 \\
v_1 \otimes (v_2 \otimes v_3) &\mapsto (v_1 \otimes v_2) \otimes v_3
\end{aligned}$$

Example 2.22. Let V be a category with finite products, then $(V, x, *)$ is a monoidal category, with $*$ its terminal object. We need a functor $- \times - \colon V \times V \to V$, such that $V_1 \times (V_2 \times V_3) \xrightarrow{\sim} (V_1 \times V_2) \times V_3$.

Example 2.23. The 2-category Bim of all bimodules has

- Objects Ob(Bim) given by all associative unital rings,
- for $R, S \in \mathrm{Ob}(\mathrm{Bim}), \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,S) := {}_S\mathrm{mod}_R \simeq L\,\mathrm{Fun}({}_R\mathrm{mod}, {}_S\mathrm{mod})$,
- for all $R, S, T \in \mathrm{Ob}(\mathrm{Bim})$ the horizontal composition is given by the functor

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\,\mathrm{Bim}(S,T) \times \underline{\mathrm{Hom}}_{\mathrm{Bim}}(R,S) \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}\,\mathrm{Bim}(R,T) \arrow[d, equal] \\
{}_T\mathrm{mod}_S \times {}_S\mathrm{mod}_R \arrow[r] & {}_T\mathrm{mod}_R
\end{tikzcd}$$

$({}_TM_S, {}_SN_R) \mapsto ({}_TM \otimes_S N_R)$

- For all $R \in \mathrm{Ob}(\mathrm{Bim}), \mathrm{id}_R = {}_RR_R \in {}_R\mathrm{mod}_R = \underline{\mathrm{Hom}}\,\mathrm{Bim}(R,R)$.

$$\begin{gathered}
{}_RR \otimes_R R_R \xrightarrow{\sim} {}_RR_R \\
{}_UL \otimes_T (M \otimes_S N) \xrightarrow{\sim}_{\mathrm{can}} ({}_UL \otimes_T M_S) \otimes_S N_R
\end{gathered}$$

Construction 2.24. Let $\mathcal{C}$ be a 2-category $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, there is a fully faithful functor

$$\begin{gathered}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \\
f \mapsto \mathrm{id}_Y \circ f
\end{gathered}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ and there is a bijection of morphisms called the left unit constraint

$$\mathrm{id}_Y \circ ?\colon\ \underline{\mathrm{Hom}}\,\mathcal{C}(\mathrm{id}_Y \circ f, f) \xrightarrow{\sim} \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)(\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f), \mathrm{id}_Y \circ f)$$

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\exists!\lambda_f"] & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim"] & & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\sim"', "v_Y * \mathrm{id}_f"] \\
f & & \mathrm{id}_Y \circ f &
\end{tikzcd} \quad \mapsto$$

Furthermore there is a fully faithful functor.

$$\begin{gathered}
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \xhookrightarrow{f.f.} \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \\
f \mapsto f \circ \mathrm{id}_X
\end{gathered}$$

as well as a bijection of morphism called the right unit constraint.

$$\begin{tikzcd}
f \circ \mathrm{id}_X \arrow[d, Rightarrow, "\exists!\rho_f"] & f \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_f * v_X"'] & & (f \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "\sim"'] \\
f & & f \circ \mathrm{id}_X &
\end{tikzcd} \quad \mapsto$$

Proposition 2.25. *Let $\mathcal{C}$ be a 2-category. The left and right unit constraints determine natural isomorphisms.*

$$
\begin{tikzcd}[column sep=huge]
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[r, bend left=50, "f \mapsto \mathrm{id}_Y \circ f"] \arrow[r, bend right=50, "\mathbb{1}"'] \arrow[r, phantom, "\Downarrow \lambda"] & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge]
\underline{\mathrm{Hom}}\,\mathcal{C}(X,Y) \arrow[r, bend left=50, "f \mapsto f \circ \mathrm{id}_X"] \arrow[r, bend right=50, "\mathbb{1}"'] \arrow[r, phantom, "\Downarrow \rho"] & \underline{\mathrm{Hom}}\,\mathcal{C}(X,Y)
\end{tikzcd}
$$

Lecture 24.4

Proof. Exercise. Let $\forall\colon X \to Y, \lambda_f$ is an isomorphism. We only prove $\lambda = (\lambda_f\colon \mathrm{id}_Y \circ f \Rightarrow f)_{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)}$ is a natural transformation. Let

$$f \overset{\eta}{\Longrightarrow} g$$

be a morphism in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$
\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"] & f \arrow[d, Rightarrow, "\eta"] \arrow[r, phantom, "\xrightarrow{\mathrm{id}_Y \circ -}"] & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[r, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & \mathrm{id}_Y \circ f \arrow[d, Rightarrow, "\mathrm{id}_Y \circ \eta"] \\
\mathrm{id}_Y \circ g \arrow[r, Rightarrow, "\lambda_g"] & g & \mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[r, Rightarrow] & \mathrm{id}_Y \circ g
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_f"] \arrow[dr, Rightarrow, "\alpha", "\sim"'] \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ (\mathrm{id}_Y \circ \eta)"'] & & \mathrm{id} \circ f \arrow[ddd, Rightarrow, "\mathrm{id}_Y \circ \eta = \mathrm{id}_{\mathrm{id}_Y} * \eta"] \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[ur, Rightarrow, "v_y * \mathrm{id}_f", "\sim"'] \arrow[d, Rightarrow, "(\mathrm{id}_Y \circ \mathrm{id}_Y) \circ \eta"] & \\
& (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ g \arrow[dr, Rightarrow, "v_Y * \mathrm{id}_g", "\sim"'] & \\
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ g) \arrow[ur, Rightarrow, "\sim", "\alpha"'] \arrow[rr, Rightarrow, "\mathrm{id}_Y \circ \lambda_g"] & & \mathrm{id}_Y \circ g
\end{tikzcd}
$$

where the left square commutes by the naturality of the associator constraint, and the top and bottom triangle commute by the left unit constraint. For the right square we use the interchange law for composition and the Godement product to obtain $(\mathrm{id}_{\mathrm{id}_Y} * \eta) \circ (v_Y * \mathrm{id}_f) = (v_\eta * \eta) = (v_Y * \mathrm{id}_g) \circ (\mathrm{id}_Y \circ \eta)$ $\square$

Proposition 2.26. *Let $\mathcal{C}$ be a 2-category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ two composable 1-morphisms in $\mathcal{C}$. Then the following triangle*

$$\begin{tikzcd}
g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_g \times \lambda_f"'] & & (g \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\rho_g \times \mathrm{id}_f"] \\
& g \circ f &
\end{tikzcd}$$

commutes.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& g \circ ((\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "v_Y"] & & (g \circ (\mathrm{id}_Y \circ \mathrm{id}_Y)) \circ f \arrow[d, Rightarrow, "u_Y"] \arrow[ddr, Rightarrow, "\alpha"] & \\
& g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "\alpha", "\sim"'] & & (g \circ id_Y) \circ f \arrow[dll, equal] & \\
g \circ (\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f)) \arrow[uur, Rightarrow, "\alpha"] \arrow[ur, Rightarrow, "\lambda_f"'] \arrow[drr, Rightarrow, "\alpha"'] & (g \circ \mathrm{id}_Y) \circ f \arrow[r, Rightarrow, "\rho_g", "\sim"'] & g \circ f & g \circ (\mathrm{id}_Y \circ f) \arrow[l, Rightarrow, "\lambda_f"', "\sim"] \arrow[u, Rightarrow, "\alpha", "\sim"'] & ((g \circ \mathrm{id}_Y) \circ \mathrm{id}_Y) \circ f \arrow[ul, Rightarrow, "\rho_g"'] \\
& & (g \circ \mathrm{id}_Y) \circ (\mathrm{id}_Y \circ f) \arrow[ul, Rightarrow, "\lambda_f"] \arrow[ur, Rightarrow, "\rho_g"'] \arrow[urr, Rightarrow, "\alpha"'] \arrow[u, phantom, "*)"] & &
\end{tikzcd}$$

The triangles commute by applying unit constraints **??** 2.24 and the square commute and the squares that include an alpha commute by the associator constraints. The only square that remains is $*$), here we use the interchange law for the Godement product to obtain $(\rho_g * \mathrm{id}_f)(\mathrm{id}_g * \lambda_f) = \rho_g * \lambda_f = (\mathrm{id}_g * \lambda_f) \circ (\rho_g * \mathrm{id}_f)$. $\square$

Corollary 2.27. *Let $\mathcal{C}$ be a 2-category and $X \in \mathcal{C}$, consider $\mathrm{id}_X : X \to X$. Then*

$$\lambda_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

$$\rho_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

are both equal to $v_X : \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$.

Proof. We only do the case $\lambda_{\mathrm{id}_X} = v_X$. By the triangle identity and definition of λ_{id_X} we get that

$$\begin{tikzcd}
\mathrm{id}_X \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[dr, Rightarrow, "\mathrm{id}_X \circ \lambda_{\mathrm{id}_X}"'] & & (\mathrm{id}_X \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "v_X * \mathrm{id}_X"] \arrow[dl, Rightarrow, bend left=60, "\rho_{\mathrm{id}_X} * \mathrm{id}_X"] \\
& \mathrm{id}_X \circ \mathrm{id}_X &
\end{tikzcd}$$

and thus $v_X * \mathrm{id}_X = \rho_{\mathrm{id}_X} * \mathrm{id}_X$ which implies that $v_X = \rho_{\mathrm{id}_X}$ since the composition with the identity is fully faithful. $\square$

Definition 2.28. Let $\mathcal{C}$ be a 2-category. The conjugate of $\mathcal{C}$ is the 2-category $\mathcal{C}^c = \mathcal{C}^{co}$ with $\mathrm{Ob}(\mathcal{C}^c) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^c}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)^{\mathrm{op}}$.

Definition 2.29. A $(2,1)$-category is a 2-category such that $\forall X, Y \in \mathrm{Ob}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a groupoid.

Definition 2.30. Let $\mathcal{C}$ be a 2-category. The coarse homotopy category of $\mathcal{C}$ is the 1-ccategory $h\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(\mathcal{C})$ and with sets of morphisms, $\mathrm{Hom}_{h\mathcal{C}}(X, Y) = \pi_0(L\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) = \pi_0(N\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y))$ with the induced composition law, where L is the localisation functor from Cat to Gpd.

Definition 2.31. Let $\mathcal{C}$ be a 2-category. The pith of $\mathcal{C}$ is the 2-category $\mathrm{Pith}(\mathcal{C})$ with objects $\mathrm{Ob}(\mathrm{Pith}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ with $\underline{\mathrm{Hom}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) := \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\cong}$.

Definition 2.32. The homotopy category of $\mathcal{C}$ is $\mathrm{hPith}(\mathcal{C})$.

3 Functors in 2-category theory

Definition 3.1. Let $\mathcal{C}, \mathcal{D}$ be 2-categories: A <u>lax 2-functor</u> $F \colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map $F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$ where $X \mapsto F(X)$.
- For all $X, Y \in \operatorname{Ob}(\mathcal{C}) F = F_{X,Y} \colon \underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\operatorname{Hom}}_{\mathcal{D}}(FX, FY)$ a functor.
- For all $X, Y, Z \in \operatorname{Ob}(\mathcal{C}), \epsilon_X \colon \operatorname{id}_{FX} \Rightarrow F(\operatorname{id}_X)$ in $\underline{\operatorname{Hom}}_{\mathcal{C}}(X,X) \overset{F}{\Longrightarrow} \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FX)$.
- For all $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ the composition constraint

$$\begin{tikzcd} \underline{\operatorname{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "- \circ -"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"] \arrow[dr, phantom, "\overset{\mu}{\Longrightarrow}"] & \operatorname{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\ \underline{\operatorname{Hom}}_{\mathcal{D}}(FY,FZ) \times \operatorname{Hom}_{\mathcal{D}}(FX,FY) \arrow[r, "- \circ -"] & \underline{\operatorname{Hom}}_{\mathcal{D}}(FX,FZ) \end{tikzcd}$$

  where $\mu_{g,f} \colon F(g) \circ F(f) \Rightarrow F(g \circ f)$. The above data is required to satisfy the following:

  (a) For all $X \to Y$ 1-morphisms in $\mathcal{C}$ and

$$\begin{tikzcd} F(\operatorname{id}_Y) \circ F(f) \arrow[r, Rightarrow, "\mu"] & F(\operatorname{id}_Y \circ f) \arrow[d, Rightarrow, "F(\lambda^e_f)"] \\ \operatorname{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\epsilon_Y * \operatorname{id}_{F(f)}"] \arrow[r, Rightarrow, "\lambda_{F(f)}"] & F(f) \end{tikzcd}$$

  in $\underline{\operatorname{Hom}}(FX, FY)$.

$$\begin{tikzcd} F(f) \circ F(\operatorname{id}_X) \arrow[r, Rightarrow, "\mu"] & F(f \circ \operatorname{id}_X) \arrow[d, Rightarrow, "F(\rho_f)"] \\ \operatorname{id}_{FY} \circ Ff \arrow[u, Rightarrow, "\operatorname{id}_{F(f)} * \epsilon_X"] \arrow[r, Rightarrow, "\rho_{F(f)}"] & F(f) \end{tikzcd}$$

  in $\underline{\operatorname{Hom}}(FX, FY)$.

  (b) For all $W, X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and for all $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ 1-morphisms in $\mathcal{C}$

$$\begin{tikzcd} F(h) \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\alpha^{\mathcal{D}}"] \arrow[d, Rightarrow, "\operatorname{id}_{Fh} * \mu"] & (F(h) \circ F(g)) \circ F(f) \arrow[d, Rightarrow, "\mu * \operatorname{id}_{F(f)}"] \\ F(h) \circ F(g \circ f) \arrow[d, Rightarrow, "\mu"] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\mu"] \\ F(h \circ (g \circ f)) \arrow[r, Rightarrow, "F(\alpha^e)"] & F((h \circ g) \circ f) \end{tikzcd}$$

  in $\underline{\operatorname{Hom}}_{\mathcal{D}}(FW, FZ)$.

Definition 3.2. A 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a lax 2-functor such that for all $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\epsilon_X\colon \mathrm{id}_{FX} \overset{\sim}{\Longrightarrow} F(\mathrm{id}_X)$ is invertible, that is $\forall X, Y \in \mathrm{Ob}(\mathcal{C}), \forall X \xrightarrow{f} Y \xrightarrow{g} Z$ the morphism $\mu_{g,f}\colon F(g) \circ F(f) \overset{\sim}{\Longrightarrow} F(g \circ f)$ is invertible.

Definition 3.3. A <u>strict 2-functor</u> is a 2-functor, such that $\forall X \in \mathrm{Ob}(\mathcal{C})$ the following hold $\epsilon_X = \mathrm{id}\colon \mathrm{id}_{FX} \Rightarrow F(\mathrm{id}_X), \forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $\forall X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\mu_{g,f} = \mathrm{id} = F(g) \circ F(f) \Rightarrow F(g \circ f)$.

Example 3.4. Lax monoidal functors $\mathcal{M} \to \mathcal{N}$ for $\mathcal{M}$ and $\mathcal{N}$ monoidal categories correspond to lax 2-functors $B\mathcal{M} \to B\mathcal{N}$.

Example 3.5. Let S be a set and $\mathcal{E}_S$ be a category with $\mathrm{Ob}(\mathcal{E}_S) = S$, for all $x, y \in S, \underline{\mathrm{Hom}}_{\mathcal{E}_S}(x, y) := \{*\}$.

- Fix $\mathcal{M}$ a monoidal category and let $B\mathcal{M}$ be its delooping and $\underline{C}\colon \mathcal{E}_S \to B\mathcal{M}$ a lax monoidal functor.
- Fix a map $\underline{C}\colon \mathrm{Ob}(\mathcal{E}_S) = S \to \mathrm{Ob}(B\mathcal{M}) = \{*\}$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$ a functor:

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) &\to \mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \\ * &\mapsto \underline{\mathcal{C}}(X, Y) \end{aligned}$$

- For all $X \in S$, $\epsilon_X\colon \mathrm{id}_{\underline{\mathcal{C}}(X)} = \mathrm{id}_*$, such that

$$\mathcal{M} = \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \circlearrowleft_{\mathbb{1}_{\mathcal{M}} = \mathrm{id}_*} \qquad \mathbb{1}_{\mathcal{M}} \otimes M \xrightarrow[\lambda_M]{\sim} M$$

  This is called the identity constraint.

- For all $X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S = S)$

$$\begin{tikzcd} \{*\} \times \{*\} = \underline{\mathrm{Hom}}_{\mathcal{E}_S}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Y) \arrow[r, "\circ", "\sim"'] \arrow[d, "\underline{\mathcal{C}} \times \underline{\mathcal{C}}"] \arrow[dr, phantom, "\overset{\mu}{\Longrightarrow}"] & \underline{\mathrm{Hom}}_{\mathcal{E}_S}(X, Z) = \{*\} \arrow[d, "\underline{\mathcal{C}}"] \\ \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \times \underline{\mathrm{Hom}}_{B\mathcal{M}}(*, *) \arrow[r, "\otimes"] & \mathrm{Hom}_{B\mathcal{M}}(*, *) \end{tikzcd}$$

  where $\mu\colon \underline{\mathcal{C}}(Y, Z) \otimes \underline{\mathcal{C}}(X, Y) \to \mathcal{C}(X, Z)$. The above data should satisfy the following: $\forall X, Y \in \mathrm{Ob}(\mathcal{E}_S) = S$.

$$\begin{tikzcd} \underline{\mathcal{C}}(X, Y) \otimes \underline{\mathcal{C}}(X, Y) \arrow[r, "\mu"] & \mathcal{C}(X, Y) \arrow[d, "\mathrm{id}"] \\ \mathbb{1}_{\mathcal{M}} \otimes \underline{\mathcal{C}}(X, Y) \arrow[u, "\mathcal{E}_Y \otimes \mathrm{id}"] \arrow[r, "\lambda^m_{\underline{\mathcal{C}}(X,Y)}"] & \mathcal{C}(X, Y) \end{tikzcd}$$

- Let $\mathcal{M} = \mathrm{Set}$

$$\begin{tikzcd} (\mathrm{id}_X, f) \arrow[r, mapsto] & \mathrm{id}_X \circ f \arrow[d, equal] \\ (x, f) \arrow[u, mapsto] \arrow[r, mapsto] & f\colon X \to Y \end{tikzcd}$$

  similarly for the right constraint.

- For all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{E}_S)$

$$\begin{tikzcd} \underline{\mathcal{C}}(Y,Z) \otimes (\underline{\mathcal{C}}(X,Z) \otimes \underline{\mathcal{C}}(W,X)) \arrow[r, "\alpha^M"] \arrow[d] & (\underline{\mathcal{C}}(Y,Z) \otimes \underline{\mathcal{C}}(X,Z)) \otimes \underline{\mathcal{C}}(W,X) \arrow[d] \\ \underline{\mathcal{C}}(Y,Z) \otimes \underline{\mathcal{C}}(W,Y) \arrow[d, "\mu"] & \underline{\mathcal{C}}(X,Z) \otimes \underline{\mathcal{C}}(w,Z) \arrow[d] \\ \mathcal{C}(W,Z) \arrow[r, "\underline{\mathcal{C}}(\alpha)=\mathrm{id}"] & \underline{\mathcal{C}}(W,Z) \end{tikzcd}$$

Remark 3.6. Dg-categories are Lax 2-functors

Exercise 1. Let $\mathcal{C}$ be a 2-category and $\mathcal{D}$ a 1-category, then every lax 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ is strict.

Definition 3.7. A $F\colon \mathcal{C} \to \mathcal{D}$ lax functor between 2-categories is

- unitary if $\forall X \in \mathcal{C} \epsilon_X$ is an isomorphism,
- strictly unitary $\forall X \in \mathcal{C}, \epsilon_X = \mathrm{id}_X$,
- copmosition of lax functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

  between 2-categories is the lax functor $G \circ F$ defined as follows:

  - on objects $(G \circ F)(X) = G(F(X))$,
  -

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathcal{E}}(X,Y) \arrow[r, dashed, "GF_{XY}"] \arrow[d, "F"] & \underline{\mathrm{Hom}}_{\mathcal{D}}(GFX, GFY) \arrow[d, equal] \\ \underline{\mathrm{Hom}}_{\mathcal{D}}(FX, FY) \arrow[r, "G_{FX,FY}"] & \underline{\mathrm{Hom}}_{\mathcal{E}}(GFX, GFY) \end{tikzcd}$$

  - identity constraints

$$\begin{tikzcd} & G(\mathrm{id}_{FX}) \arrow[dr, Rightarrow, "G(\epsilon^F_X)"] & & \mathrm{id}_{FX} \arrow[d, Rightarrow, "\epsilon^F_X"] \\ \mathrm{id}_{GFX} \arrow[ur, Rightarrow, "\epsilon^G_{FX}"] \arrow[rr, equal, "\epsilon^{GF}_X"'] & & GF(id_X) & F(\mathrm{id}_X) \end{tikzcd}$$

– $\forall X \xrightarrow{f} Y \xrightarrow{g} Z$ 1-morphisms in $\mathcal{C}$

$$\begin{tikzcd}
& G(F(g) \circ F(f)) \arrow[dr, Rightarrow, "G(\mu^F_{g,f})"] & \\
GF(g) \circ GF(f) \arrow[ur, Rightarrow, "\epsilon^G_{FX}"] \arrow[rr, equal, "\mu^{GF}_{g,f}"'] & & GF(g \circ f)
\end{tikzcd}$$

4 Nerves of differential graded categories

Let us begin with a reminder on cochain complexes. Let k be a commutative ring and let mod_k be the category of (right) k-modules and $\mathrm{Ch}(\mathrm{mod}_k)$ the category of cochain complexes of k-modules, so objects are given by

$$(X^\bullet, d^\bullet) = \ldots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots$$

such that $d^2 = 0$. The category $\mathrm{Ch}(\mathrm{mod}_k)$ is a monoidal category with

$$(X^\bullet \otimes_k Y^\bullet)^l := \coprod_{i+j=l} X^i \otimes_k Y^j$$
$$d_{X^\bullet \otimes Y^\bullet}(x \otimes y) := d_X(x) \otimes y + (-1)^i x \otimes d_Y(y)$$

where $|x| = i$ is the degree of x. The unit of the monoidal structure is given by k viewed as a chain complex concentrated in degree 0. There is a preferred symmetry constraint

$$\tau\colon X^\bullet \otimes Y^\bullet \xrightarrow{\sim} Y^\bullet \otimes X^\bullet$$
$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

which is called the Koszul sign rule.

Definition 4.1. A **differential graded category** $\mathcal{A}$ is a category enriched in the monoidal category $\mathrm{Ch}(\mathrm{mod}_k)$. That is:

- A class $\mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$.
- For all $a, b \in \mathrm{Ob}(\mathcal{A})$ a cochain complex $\mathcal{A}(a,b) \in \mathrm{Ch}(\mathrm{mod}_k)$.
- For all $a \in \mathrm{Ob}(\mathcal{A})$ a unit/identity $\mathrm{id}_a\colon k \to \mathcal{A}(a,a)$.
- For all $a, b, c \in \mathrm{Ob}(\mathcal{A})$ a composition law

$$\mathcal{A}(b,c) \otimes \mathcal{A}(a,b) \xrightarrow{-\circ-} \mathcal{A}(a,b)$$

  given by a morphism in $\mathrm{Ch}(\mathrm{mod}_k)$.

Remark 4.2. This means that if $f \in \mathcal{A}(a,b), |f| = i$ and $g \in \mathcal{A}(b,c), |g| = j$, then

- $|g \circ f| = i + j$ since $|g \otimes f| = i + j$,
- $d_{\mathcal{A}}(g \circ f) = d_{\mathcal{A}}(g) \circ f + (-1)^j g d_{\mathcal{A}}(f), |g| = j$ (Graded Leibniz rule).

This composition law must be associative and unital in the usual sense.

Example 4.3. Let $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ be a the dg category given as follows:

- The objects are given by complexes of k-modules.

- For $X^\bullet, Y^\bullet \in \mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$ a complex $\mathrm{Hom}_k(X^\bullet, Y^\bullet) \in \mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$.
- Let $\mathrm{Hom}_k(X^\bullet, Y^\bullet)^j := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k(X^i, Y^{i+j})$ be the degree j maps of graded k-modules endowed with the following differential

$$\begin{aligned} \partial\colon \mathrm{Hom}_k(X^\bullet, Y^\bullet)^j &\to \mathrm{Hom}_k(X^\bullet, Y^\bullet)^{j+1} \\ f &\mapsto \partial(f) = (d_Y^{i+j} \circ f^i - (-1)^{|f|} f^{i+1} \circ d_X^i)_{i\in\mathbb{Z}}, |f| = j \end{aligned}$$

Note that $|f| = 0$ and $\partial(f) = 0$ is equivalent to f being a cochain map.

Example 4.4. Let now $\mathrm{Ob}(\mathcal{A}) = \{\star\}$, $A := \mathcal{A}(\star, \star)$, this is a dg algebra.

Construction 4.5. Given a dg category $\mathcal{A}$, its underlying category is denoted by $Z^0(\mathcal{A})$, given by

- $\mathrm{Ob}(Z^0(A)) = \mathrm{Ob}(\mathcal{A})$,
- $\forall a, b \in Z^0(\mathcal{A}), Z^0(a,b) = Z^0(\mathcal{A}(a,b)) = \ker(d_{\mathcal{A}^0}) \subseteq \mathcal{A}(a,b)$.

Definition 4.6. Let $\mathcal{A}$ be a dg category. The homotopy category (0-th cohomology category) of $\mathcal{A}$, denoted $H^0(\mathcal{A})$ has

- $\mathrm{Ob}(H^0(\mathcal{A})) = \mathrm{Ob}(Z^0(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$
- $\forall a, b \in H^0(\mathcal{A}), H^0(\mathcal{A})(a,b) = H^0(\mathcal{A}(a,b))$.

In the case of $\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}$, we have

$$\begin{aligned} Z^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) &:= \mathrm{Ch}(\mathrm{mod}_k) \\ H^0(\mathrm{Ch}(\mathrm{mod}_k)_{\mathrm{dg}}) &= K(\mathrm{mod}_k) \text{ homotopy category of cochain complexes} \end{aligned}$$

Construction 4.7. Let $\mathcal{A}$ be a dg category, then the dg nerve $N_{\mathrm{dg}}(\mathcal{A})$ is defined as follows:

- $N_{\mathrm{dg}}(\mathcal{A})_0$ are the objects of $\mathcal{A}$,
- $N_{\mathrm{dg}}(\mathcal{A})_1$ are the degree zero cocycles $f \in Z^0(\mathcal{A}(a,b))$, that is the morphisms f such that $|f| = 0$ and $d_{\mathcal{A}}(f) = 0$,
- $N_{\mathrm{dg}}(A)_2$ is given by 2-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "f_{21}"] \arrow[d, phantom, "\Downarrow f_{210}"] & \\ a_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & a_2 \end{tikzcd}$$

where $|f_{ij}| = 0. |f_{210}| = -1, d_{\mathcal{A}}(f_{ij}) = 0$ and $d_{\mathcal{A}}(f_{210}) = f_{20} - f_{21} \circ f_{10}$ as well as $[f_{20}] = [f_{21} \circ f_{10}] \in H^0(\mathcal{A}(a_0, a_2))$,

- $N_{\mathrm{dg}}(\mathcal{A})_3$ is given by 3-simplices

$$\begin{tikzcd} & a_1 \arrow[dr, "f_{21}"] \arrow[dd, "f_{31}"] & \\ a_0 \arrow[ur, "f_{10}"] \arrow[rr, dashed, "f_{20}"] \arrow[dr, "f_{30}"'] & & a_2 \arrow[dl, "f_{32}"] \\ & a_3 & \end{tikzcd}$$

with each boundary 2-simplex having its composition given by some morphism $f_{ijk}, i, j, k \in \{0, 1, 2, 3\}$ as well as a morphism f_{3210} such that $d(f_{3210}) = -(f_{321} \circ f_{10} - f_{320}) + (f_{32} \circ f_{210} - f_{310})$.

Definition 4.8. Let $\mathcal{A}$ be a dg-category. The dg nerve of $\mathcal{A}$, $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is the simplicial set where for $n \geqslant 0$

$$\begin{aligned} N_{\mathrm{dg}}(\mathcal{A}) &:= \{(a_0, a_1, \ldots, a_n) \in \mathrm{Ob}(\mathcal{A}), (f_I = f_{n \geqslant i_k > \ldots > i_0 \geqslant 0} \in \mathcal{A}(a_{i_0}, a_{i_k}))_{\substack{I \subseteq [n] \\ 2 \leqslant |I|}}\} \\ |f_I| &= -(\#(I \backslash \{i_k, i_0\})) \\ d_{\mathcal{A}}(f_I) &= \sum_{l=1}^{k-1} (-1)^l (f_{i_k > \ldots > i_l} \circ f_{i_l > \ldots > i_0} \circ \ldots \circ f_{i_k > \ldots > i_l > \ldots > i_0}) \end{aligned}$$

Theorem 4.9. *The nerve of a dg-category $N_{\mathrm{dg}}(\mathcal{A}) \in \mathrm{Set}_\Delta$ is an ∞-category.*