Itic notes

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1 Motivation

Fix $0 \le m \le n \le \infty$. An (n,m) category is a "category-like" structure consisting of a class of objects, notions of 1- morphism, 2-morphism, ..., n-morphism (i.e. k-morphsim $0 < k \le n$) with a "suitable composition law" (satisfying "suitable axioms") and such that $\forall m < k \le n$ the k-morphisms are "invertible".

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(0,0)-cat. = set

(1,0)-cat. = groupoid, i.e. 1-groupoid

(1,1)-cat. = category, i.e. 1-category

(2,0)-cat. = 2-groupoids

(2,1)-cat.

(2,2)-cat. = 2-categories

\vdots

(n,0)-cat. = n-groupoid

(n,n)-cat. = n-cat.

\vdots

(\infty,0) = \infty-groupoids

(\infty,1) = \infty-categories (see. Boardman-Vogt)
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Reminder 1.1. A map $f: X \to Y$ between topological spaces is a weak homotopy equivalence if $\forall x \in X, \forall n \in \mathbb{N}$ the map

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

 $is\ a\ bijection.$

Theorem 1.2 (Grothendieck's Homotopy Hypothesis). There is an $(\infty,1)$ -category of topological spaces up to weak homotopy equivalence and there is an $(\infty,1)$ -category of ∞ -groupoids up to eqivalence. There is furthermore an ∞ -functor assigning to each topological space X its Poincare ∞ -groupoid $\pi_{\infty}(X)$, this is an equivalence.

Remark 1.3. Let \mathcal{C} be an $(\infty, 1)$ -category, then for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an ∞ -groupoid/ "space". We have the homtopy category of \mathcal{C} denoted by $\mathrm{Ho}(\mathcal{C})$ whose objects are those of \mathcal{C} and for all objects $X, Y \in \mathrm{Ho}(\mathcal{C})$ we have that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Warning! 1.4. The passage from C often results in a tremendous loss of information, that is essential for various purposes.

- Computing co-/limits within C.
- Computing co-/limits with C.

• Define invariants associated to \mathcal{C} (f.e. Hochschild cohomology).

Reminder 1.5. Many important 1-categories arise as homotopy categories of genuine $(\infty, 1)$ -categories, that is derived categories Recall for R: ring, mod_R its category of right R-modules and $\operatorname{Ch}(\operatorname{mod}_R)$ the category of chain complexes in mod_R , that a morphism of chain complexes $f^{\bullet}\colon X^{\bullet} \to Y^{\bullet}$ in $\operatorname{Ch}(\operatorname{mod}_R)$ is a quasi-isomorphism if $\forall n \in \mathbb{Z}$ we have that $H_n(f^{\bullet}) = H_n(X^{\bullet}) \xrightarrow{\sim} H_n(Y^{\bullet})$ is an isomorphism. The derived category is defined as follows $D(\operatorname{mod}_R) := \operatorname{Ch}(\operatorname{mod}_R)[qiso^{-1}]$ the localisation at the quasi-isomorphisms. Furthermore we have that $\operatorname{Ho}(\mathcal{D}(\operatorname{mod}_R)) = D(\operatorname{mod}_R)$. In the first case we obtain it by building it from the ground up so to say and in the second case we obtain by forgetting information from a higher structure.

Warning! 1.6. The homotopy theory of $(\infty, 1)$ -categories has many equivalent (Quillen) implementations:

- Topological categories (Ilias)
- Simplicial categories (Bergner)
- Complete Segal spaces (Rezk)
- Relative categories (Barwick-Kan)
- Pre-derivations
- ∞-categories (Joyal, Lurie)

In the k-linear setting, for k a field we have:

- 1. Differentially graded k-categories
- 2. A_{∞} -categories (Lefèvre-Hasegawa)

The Plan for the lecture is to start of with investigating 2-categories, then give definition and examples of ∞ -categories, then do enriched category theory and end on the homotopy theory of ∞ -categories.

2 Heuristics: Isomorphism vs. Equivalence

Makkai's Principle of Isomorphism (1998) says:

"All grammatically correct properties about objects in a fixed category are to be invariant under isomorphism."

Fix a category \mathcal{C} and a small category A and take the functor category $\operatorname{Fun}(A,\mathcal{C})$, that is A-shaped diagrams in \mathcal{C} . For $X \in \mathcal{C}$ define:

$$\lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(X, D(a)) := \{ (p_a \colon X \to D(a))_{a \in A} \mid \begin{array}{c} X \\ f_a \\ D(a) \end{array} \quad \forall f \colon a \to b \text{ in } A \}$$

For $\phi \colon Y \to X$ define the function

$$\phi^* \colon \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(X, D(a)) \to \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(Y, D(a))$$
$$p = (p_a \colon X \to D(a))_{a \in A} \mapsto \phi^*(p) = (p_a \circ \phi \colon Y \to D(a))_{a \in A}$$

Thus $\lim_{a\in A} \operatorname{Hom}_{\mathcal{C}}(-,D(a)): \mathcal{C}^{\operatorname{op}} \to \operatorname{Set}$ is a presheaf of sets on \mathcal{C} .

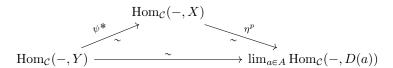
Definition 2.1. A limit of a diagram $D: A \to \mathcal{C}$ is a cone $p \in \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(X, D(a))$ that is universal in the sense that $\forall Y \in \mathcal{C}, \forall q \in \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(Y, D(a)) \exists ! \varphi \colon Y \to X$ such that $\varphi^*(p) = q$. We write $\lim_{a \in A} D(a)$ for any limit of D (which may or may not exist).

Reminder 2.2 (Yoneda Lemma). Let $X \in \mathcal{C}$ and let

$$\nu \colon \operatorname{Nat}(\operatorname{Hom}_{\mathcal{C}}(-,X), \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(-,D(a))) \xrightarrow{\sim} \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(X,D(a))$$
$$\eta = (\eta_Y \colon \operatorname{Hom}_{\mathcal{C}}(Y,X) \to \lim_{a \in \mathcal{C}} (Y,D(a))_{Y \in \mathcal{C}} \mapsto \eta_X(\operatorname{id}_X)$$
$$(\eta_V^p(p) := \varphi^*(p))_{Y \in \mathcal{C}} \longleftrightarrow p$$

If $p \in \lim_{a \in A} \operatorname{Hom}_{\mathcal{C}}(X, D(a))$ is a limit of $D : A \to \mathcal{C}$ then

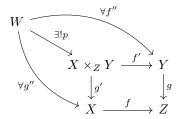
 $\eta^p \colon \operatorname{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(-,D(a))$ is a natural isomorphism. We furthermore obtain, that for an isomorphism $\psi \colon Y \to X$ in \mathcal{C} we have



Example 2.3. Let the following be a diagram in \mathcal{C}

$$\begin{array}{c} Y \\ \downarrow^g \\ X \stackrel{f}{\longrightarrow} Z \end{array}$$

then the limit of the diagram (if it exists) is called a pullback.

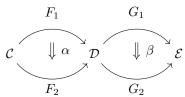


For example if C = Set then $X \times_Z Y = \{(X, Y) \in X \times Y \mid f(x) = g(y)\}.$

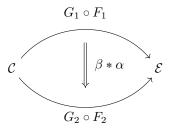
The commutativity condition takes place in $\operatorname{Hom}_{\mathcal{C}}(X \times_Z Y, Z) \ni g \circ f' = f \circ g'$ <u>Makkai's Pronciple of Equivalence</u> All grammatically correct properties of objects in a fixed 2-category are to be invariant under equivalence.

Remark 2.4. We want to Cat be the strict 2-category of (small) categories with functors as 1-morphisms and natural transformations as 2-morphisms. Now natural transformation allow for a notion of equivalence of morphisms, that is in a 1-category we only knew what it meant for two morphisms to be equal, but now we can talk about two functors being naturally isomorphic given us a notion of equivalence of 1-morphisms, via the 2-morphisms.

 $\textbf{Definition/Proposition 2.5} \ (\textbf{Godement Product}). \ \textbf{Consider natural transformations}$



Their Godement product is the natural transformation.



Let $X \in \mathcal{C}$, we obtain the following diagram

$$F_{1}(X) \qquad G_{1}(F_{1}(X)) \xrightarrow{\beta_{F_{1}(X)}} G_{2}(F_{1}(X))$$

$$\downarrow^{\alpha_{x}} \qquad G_{1}(\alpha_{x}) \downarrow \qquad \downarrow^{(\beta*\alpha)_{X}} \downarrow^{G_{2}(\alpha_{x})}$$

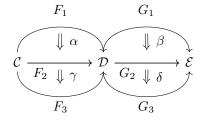
$$F_{2}(X) \qquad G_{1}(F_{2}(X)) \xrightarrow{\beta_{F_{2}(X)}} G_{2}(F_{2}(X))$$

in \mathcal{D} .

Proof. We show that $\beta * \alpha : G_1 \circ F_1 \Rightarrow g_2 \circ F_2$ is indeed a natural transformation. For that we take the following diagram

$$\begin{array}{ccc} X & G_1(F_1(X)) \xrightarrow{G_1(\alpha_X)} G_1(F_2(X)) \xrightarrow{\beta_{F_2(X)}} G_2(F_2(X)) \\ \downarrow^f & G_1(F_1(f)) \downarrow & \downarrow^{G_1(F_2(f))} & \downarrow^{G_2(F_2(f))} \\ Y & G_1(F_1(Y)) \xrightarrow{G_1(\alpha_1)} G_1(F_2(Y)) \xrightarrow{\beta_{F_2(Y)}} G_2(F_2(Y)) \end{array}$$

Proposition 2.6. Consider natural transformations



Then $(\delta\beta) * (\gamma\alpha) = (\delta * \gamma) \circ (\beta * \alpha)$.

Proof. Let $X \in \mathcal{C}$

$$G_{1}(F_{1}(X)) \xrightarrow{\beta_{F_{1}(X)}} G_{2}(F_{1}(X)) \xrightarrow{\delta_{F_{1}(X)}} G_{3}(F_{1}(X))$$

$$G_{1}(\alpha_{X}) \downarrow \qquad \downarrow G_{2}(\alpha_{X}) \qquad \downarrow G_{3}(\alpha_{X})$$

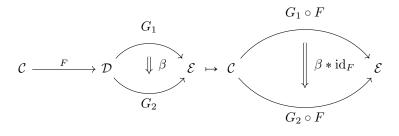
$$G_{1}(F_{2}(X)) \xrightarrow{\beta_{F_{2}(X)}} G_{2}(F_{2}(X)) \xrightarrow{\delta_{F_{2}(X)}} G_{3}(F_{2}(X))$$

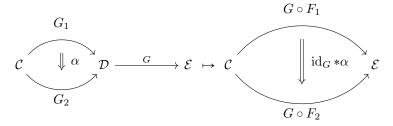
$$G_{1}(\gamma_{X}) \downarrow \qquad \downarrow G_{2}(\gamma_{X}) \xrightarrow{\delta_{F_{2}(X)}} G_{3}(F_{3}(X))$$

$$G_{1}(F_{3}(X)) \xrightarrow{\beta_{F_{3}(X)}} G_{2}(F_{3}(X)) \xrightarrow{\delta_{F_{2}(X)}} G_{3}(F_{3}(X))$$

Now the long diagonal of the diagram corresponds to $(\delta * \gamma) \circ (\beta * \alpha)$ and the outer large square to $(\delta \circ \beta) * (\gamma \circ \alpha)$.

Definition 2.7. The Godement products bellow are called whickerings





Construction 2.8. Given a cospan of groupoids

$$egin{array}{c} \mathcal{B} \ \downarrow_G \ \mathcal{A} \stackrel{F}{\longrightarrow} \mathcal{C} \end{array}$$

its 2-pullback is the diagonal of groupoids.

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{B} \\ \downarrow^{\pi_{\mathcal{A}}} & & \downarrow^{G} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

The objects are given as $\mathrm{Ob}(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}) = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi \colon F(a) \xrightarrow{\sim} G(b) \text{ in } \mathcal{C})$ and morphisms are given by tuples of morphisms $(u,v) \colon (a,b,\varphi) \to (a',b',\varphi')$, where $u \colon a \to a'$ and $v \colon b \to b'$ are morphisms in the respective groupoids, such that the following square commutes

$$F(a) \xrightarrow{\varphi} G(b)$$

$$F(a) \downarrow \qquad \qquad \downarrow G(v)$$

$$F(a') \xrightarrow{\varphi} G(b')$$