Itic notes

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1 Motivation

Fix $0 \leqslant m \leqslant n \leqslant \infty$. An (n, m) category is a "category-like" structure consisting of a class of objects, notions of 1- morphism, 2-morphism, ... , n-morphism (i.e. k-morphsim $0 < k \leqslant n$) with a "suitable composition law" (satisfying "suitable axioms") and such that $\forall m < k \leqslant n$ the k-morphisms are "invertible".

$$
\begin{aligned}
&(0,0)\text{-cat.} = \text{set}\\
&(1,0)\text{-cat.} = \text{groupoid, i.e. 1-groupoid}\\
&(1,1)\text{-cat.} = \text{category, i.e. 1-category}\\
&(2,0)\text{-cat.} = \text{2-groupoids}\\
&(2,1)\text{-cat.}\\
&(2,2)\text{-cat.} = \text{2-categories}\\
&\vdots\\
&(n,0)\text{-cat.} = \text{n-groupoid}\\
&(n,n)\text{-cat.} = \text{n-cat.}\\
&\vdots\\
&(\infty,0) = \infty\text{-groupoids}\\
&(\infty,1) = \infty\text{-categories (see. Boardman-Vogt)}
\end{aligned}
$$

Reminder 1.1. *A map $f\colon X \to Y$ between topological spaces is a weak homotopy equivalence if $\forall x \in X, \forall n \in \mathbb{N}$ the map*

$$\pi_n(f)\colon \pi_n(X,x) \to \pi_n(Y, f(x))$$

is a bijection.

Theorem 1.2 (Grothendieck's Homotopy Hypothesis)**.** *There is an $(\infty,1)$-category of topological spaces up to weak homotopy equivalence and there is an $(\infty,1)$-category of ∞-groupoids up to eqivalence. There is furthermore an ∞-functor assigning to each topological space X its Poincare ∞-groupoid $\pi_\infty(X)$, this is an equivalence.*

Remark 1.3. Let $\mathcal{C}$ be an $(\infty,1)$-category, then for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an ∞-groupoid/ "space". We have the homtopy category of $\mathcal{C}$ denoted by $\mathrm{Ho}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and for all objects $X, Y \in \mathrm{Ho}(\mathcal{C})$ we have that $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Warning! 1.4. The passage from $\mathcal{C}$ often results in a tremendous loss of information, that is essential for various purposes.

- Computing co-/limits within $\mathcal{C}$.
- Computing co-/limits with $\mathcal{C}$.

- Define invariants associated to $\mathcal{C}$ (f.e. Hochschild cohomology).

Reminder 1.5. *Many important 1-categories arise as homotopy categories of genuine $(\infty, 1)$-categories, that is derived categories Recall for R: ring,* mod_R *its category of right R-modules and* $\mathrm{Ch}(\mathrm{mod}_R)$ *the category of chain complexes in* mod_R*, that a morphism of chain complexes $f^\bullet \colon X^\bullet \to Y^\bullet$ in* $\mathrm{Ch}(\mathrm{mod}_R)$ *is a quasi-isomorphism if $\forall n \in \mathbb{Z}$ we have that $H_n(f^\bullet) = H_n(X^\bullet) \xrightarrow{\sim} H_n(Y^\bullet)$ is an isomorphism. The derived category is defined as follows $D(\mathrm{mod}_R) :=$* $\mathrm{Ch}(\mathrm{mod}_R)[qiso^{-1}]$ *the localisation at the quasi-isomorphisms. Furthermore we have that* $\mathrm{Ho}(\mathcal{D}(\mathrm{mod}_R)) = D(\mathrm{mod}_R)$*. In the first case we obtain it by building it from the ground up so to say and in the second case we obtain by forgetting information from a higher structure.*

Warning! 1.6. The homotopy theory of $(\infty, 1)$-categories has many equivalent (Quillen) implementations:

- Topological categories (Ilias)
- Simplicial categories (Bergner)
- Complete Segal spaces (Rezk)
- Relative categories (Barwick-Kan)
- Pre-derivations
- ∞-categories (Joyal, Lurie)

In the k-linear setting, for k a field we have:

1. Differentially graded k-categories
2. A_∞-categories (Lefèvre-Hasegawa)

The Plan for the lecture is to start of with investigating 2-categories, then give definition and examples of ∞-categories, then do enriched category theory and end on the homotopy theory of ∞-categories.

2 Heuristics: Isomorphism vs. Equivalence

Makkai's Principle of Isomorphism (1998) says:

"All grammatically correct properties about objects in a fixed category are to be invariant under isomorphism."

Fix a category $\mathcal{C}$ and a small category A and take the functor category $\mathrm{Fun}(A,\mathcal{C})$, that is A-shaped diagrams in $\mathcal{C}$. For $X \in \mathcal{C}$ define:

$$\lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) := \left\{ (p_a\colon X \to D(a))_{a\in A} \;\middle|\; \begin{tikzcd} X \arrow[d, "f_a"'] \arrow[dr, "f_b"] & \\ D(a) \arrow[r] & D(b) \end{tikzcd} \quad \forall f\colon a \to b \text{ in } A \right\}$$

For $\phi\colon Y \to X$ define the function

$$\begin{aligned} \phi^*\colon \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) &\to \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \\ p = (p_a\colon X \to D(a))_{a\in A} &\mapsto \phi^*(p) = (p_a \circ \phi\colon Y \to D(a))_{a\in A} \end{aligned}$$

Thus $\lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf of sets on $\mathcal{C}$.

Definition 2.1. A limit of a diagram $D\colon A \to \mathcal{C}$ is a cone $p \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ that is universal in the sense that $\forall Y \in \mathcal{C}, \forall q \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(Y, D(a)) \exists! \varphi\colon Y \to X$ such that $\varphi^*(p) = q$. We write $\lim_{a\in A} D(a)$ for any limit of D (which may or may not exist).

Reminder 2.2 (Yoneda Lemma)**.** *Let $X \in \mathcal{C}$ and let*

$$\begin{aligned} \nu\colon \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, X), \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a))) &\xrightarrow{\sim} \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a)) \\ \eta = (\eta_Y\colon \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \lim_{a\in \mathcal{C}}(Y, D(a))_{Y\in\mathcal{C}} &\mapsto \eta_X(\mathrm{id}_X) \\ (\eta^p_Y(p) := \varphi^*(p))_{Y\in\mathcal{C}} &\leftarrow\!\shortmid p \end{aligned}$$

If $p \in \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(X, D(a))$ is a limit of $D\colon A \to \mathcal{C}$ then
$\eta^p\colon \mathrm{Hom}_{\mathcal{C}}(-, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, D(a))$ is a natural isomorphism. We furthermore obtain, that for an isomorphism $\psi\colon Y \to X$ in $\mathcal{C}$ we have

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(-, X) \arrow[dr, "\eta^p", "\sim"'] & \\ \mathrm{Hom}_{\mathcal{C}}(-, Y) \arrow[ur, "\psi^*", "\sim"'] \arrow[rr, "\sim"] & & \lim_{a\in A} \mathrm{Hom}_{\mathcal{C}}(-, D(a)) \end{tikzcd}$$

Example 2.3. Let the following be a diagram in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

then the limit of the diagram (if it exists) is called a pullback.

$$\begin{tikzcd}
W \arrow[dr, "\exists ! p"] \arrow[drr, bend left, "\forall f''"] \arrow[ddr, bend right, "\forall g''"'] & & \\
& X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

For example if $\mathcal{C} = \text{Set}$ then $X \times_Z Y = \{(X, Y) \in X \times Y \mid f(x) = g(y)\}$.

The commutativity condition takes place in $\mathrm{Hom}_{\mathcal{C}}(X \times_Z Y, Z) \ni g \circ f' = f \circ g'$

Makkai's Principle of Equivalence All grammatically correct properties of objects in a fixed 2-category are to be invariant under equivalence.

Remark 2.4. We want to Cat be the strict 2-category of (small) categories with functors as 1-morphisms and natural transformations as 2-morphisms. Now natural transformation allow for a notion of equivalence of morphisms, that is in a 1-category we only knew what it meant for two morphisms to be equal, but now we can talk about two functors being naturally isomorphic given us a notion of equivalence of 1-morphisms, via the 2-morphisms.

Definition/Proposition 2.5 (Godement Product)**.** Consider natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"{name=U}] \arrow[r, bend right=50, "F_2"'{name=D}] & \mathcal{D} \arrow[r, bend left=50, "G_1"{name=U2}] \arrow[r, bend right=50, "G_2"'{name=D2}] & \mathcal{E}
\arrow[from=U, to=D, phantom, "\Downarrow \alpha"]
\arrow[from=U2, to=D2, phantom, "\Downarrow \beta"]
\end{tikzcd}$$

Their Godement product is the natural transformation.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "G_1 \circ F_1"{name=U}] \arrow[r, bend right=50, "G_2 \circ F_2"'{name=D}] & \mathcal{E}
\arrow[from=U, to=D, phantom, "\Downarrow \beta * \alpha"]
\end{tikzcd}$$

Let $X \in \mathcal{C}$, we obtain the following diagram

$$\begin{tikzcd}
F_1(X) \arrow[d, "\alpha_x"] & G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta * \alpha)_X"] & G_2(F_1(X)) \arrow[d, "G_2(\alpha_X)"] \\
F_2(X) & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] & G_2(F_2(X))
\end{tikzcd}$$

in $\mathcal{D}$.

Proof. We show that $\beta * \alpha \colon G_1 \circ F_1 \Rightarrow g_2 \circ F_2$ is indeed a natural transformation. For that we take the following diagram

$$
\begin{tikzcd}
X \arrow[d, "f"] & G_1(F_1(X)) \arrow[r, "G_1(\alpha_X)"] \arrow[d, "G_1(F_1(f))"'] & G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(F_2(f))"] & G_2(F_2(X)) \arrow[d, "G_2(F_2(f))"] \\
Y & G_1(F_1(Y)) \arrow[r, "G_1(\alpha_1)"'] & G_1(F_2(Y)) \arrow[r, "\beta_{F_2(Y)}"'] & G_2(F_2(Y))
\end{tikzcd}
$$

□

Proposition 2.6. *Consider natural transformations*

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2"'] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \gamma"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2"'] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow \beta"] \arrow[r, phantom, shift right=3, "\Downarrow \delta"] & \mathcal{E}
\end{tikzcd}
$$

Then $(\delta\beta) * (\gamma\alpha) = (\delta * \gamma) \circ (\beta * \alpha)$.

Proof. Let $X \in \mathcal{C}$

$$
\begin{tikzcd}
G_1(F_1(X)) \arrow[r, "\beta_{F_1(X)}"] \arrow[d, "G_1(\alpha_X)"'] \arrow[dr, "(\beta * \alpha)_X"] \arrow[dd, bend right=60, "G_1(\gamma_X \alpha_X)"'] & G_2(F_1(X)) \arrow[r, "\delta_{F_1(X)}"] \arrow[d, "G_2(\alpha_X)"] & G_3(F_1(X)) \arrow[d, "G_3(\alpha_X)"] \arrow[dd, bend left=60, "G_3(\gamma_X \alpha_X)"] \\
G_1(F_2(X)) \arrow[r, "\beta_{F_2(X)}"] \arrow[d, "G_1(\gamma_X)"'] & G_2(F_2(X)) \arrow[r, "\delta_{F_2(X)}"] \arrow[d, "G_2(\gamma_X)"] \arrow[dr, "(\delta * \gamma)_X"] & G_3(F_2(X)) \arrow[d, "G_3(\gamma_X)"] \\
G_1(F_3(X)) \arrow[r, "\beta_{F_3(X)}"] & G_2(F_3(X)) \arrow[r, "\delta_{F_2(X)}"] & G_3(F_3(X))
\end{tikzcd}
$$

Now the long diagonal of the diagram corresponds to $(\delta * \gamma) \circ (\beta * \alpha)$ and the outer large square to $(\delta \circ \beta) * (\gamma \circ \alpha)$. □

Definition 2.7. The Godement products bellow are called whickerings

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G_1"] \arrow[r, bend right=50, "G_2"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathcal{E} & \mapsto & \mathcal{C} \arrow[r, bend left=50, "G_1 \circ F"] \arrow[r, bend right=50, "G_2 \circ F"'] \arrow[r, phantom, "\Downarrow \beta * \mathrm{id}_F"] & \mathcal{E}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd}
\mapsto
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "G \circ F_1"] \arrow[rr, bend right, "G \circ F_2"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_G * \alpha"] & & \mathcal{E}
\end{tikzcd}
$$

Construction 2.8. Given a cospan of groupoids

$$
\begin{tikzcd}
 & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
$$

its 2-pullback is the diagonal of groupoids.

$$
\begin{tikzcd}
\mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\pi_{\mathcal{A}}"] & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}
$$

The objects are given as $\mathrm{Ob}(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}) = (a \in \mathcal{A}, b \in \mathcal{B}, \varphi \colon F(a) \xrightarrow{\sim} G(b) \text{ in } \mathcal{C})$ and morphisms are given by tuples of morphisms $(u, v) \colon (a, b, \varphi) \to (a', b', \varphi')$, where $u \colon a \to a'$ and $v \colon b \to b'$ are morphisms in the respective groupoids, such that the following square commutes

$$
\begin{tikzcd}
F(a) \arrow[r, "\varphi", "\sim"'] \arrow[d, "F(a)"'] & G(b) \arrow[d, "G(v)"] \\
F(a') \arrow[r, "\varphi", "\sim"'] & G(b')
\end{tikzcd}
$$