Lecture 4: Model-Free Prediction

The environment can be presented as MDP but no one give us MDP. We want the best action.

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Outline

Model-Free Reinforcement Learning

- Last lecture:
  - Planning by dynamic programming
  - Solve a *known* MDP
- This lecture:
  - Model-free prediction
  - Estimate the value function of an *unknown* MDP Evaluate only in this lec4
- Next lecture:
  - Model-free control
  - Optimise the value function of an *unknown* MDP Use this evaluate result to optimize

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is *model-free*: no knowledge of MDP transitions / rewards
  /possibility
- MC learns from *complete* episodes: no bootstrapping
  VS TD
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to *episodic* MDPs
    - All episodes must terminate

Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of experience under policy π

$$S_1, A_1, R_2, ..., S_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}_\pi \left[G_t \mid S_t = s \right]$$

- Monte-Carlo policy evaluation uses *empirical* mean return instead of *expected* return

But how do we do this without reset out status back repeatly each iteration? 2 ways:

First-Visit Monte-Carlo Policy Evaluation

For example from Point A to B, we show 3 episodes here:
1. green, go over state M once.
2. black, go over state M twice.
3. orange, go over state M once.

N(M) = 3. In next slide example, N(M) = 4.

- To evaluate state s
- The first time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- By law of large numbers, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

MC ask for looking ahead complete episode. So at status A, we look forwards all future possible episodes and walk down to the end of each episodes to calculate the Return. Then decide which action to choose (here only evaluation no action choosing.)
So we record current status A,
then walk through episode green from A to B and get the return,
then walk through episode black from A to B and get the return,
then walk through episode orange from A to B and get the return.
We are always at status A, we don't move we don't take action yet.

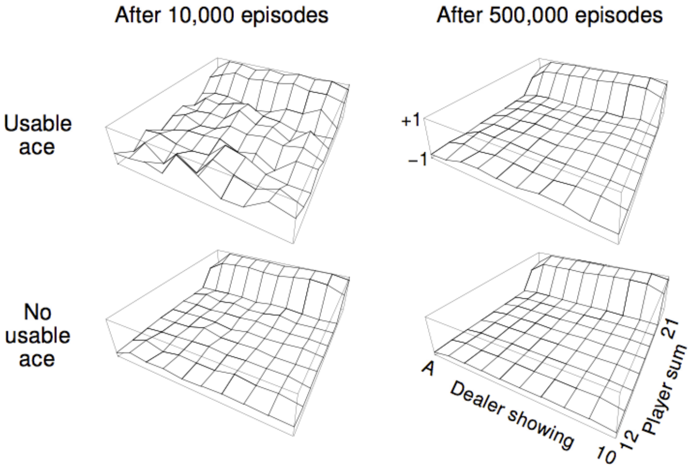
Every-Visit Monte-Carlo Policy Evaluation

- To evaluate state s
- Every time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- Again, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Blackjack Example

- States (200 of them):
  - Current sum (12-21)
  - Dealer's showing card (ace-10)
  - Do I have a "useable" ace? (yes-no)
- Action stick: Stop receiving cards (and terminate)
- Action twist: Take another card (no replacement)
- Reward for stick:
  - $+1$ if sum of cards $>$ sum of dealer cards
  - 0 if sum of cards $=$ sum of dealer cards
  - -1 if sum of cards $<$ sum of dealer cards
- Reward for twist:
  - -1 if sum of cards > 21 (and terminate)
  - 0 otherwise
- Transitions: automatically twist if sum of cards < 12

Blackjack Value Function after Monte-Carlo Learning



Policy: stick if sum of cards ≥ 20, otherwise twist

Incremental Mean

The mean $\mu_1, \mu_2, ...$ of a sequence $x_1, x_2, ...$ can be computed incrementally, Instead of using sum/final_counter

$$
\begin{aligned}
\mu_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\
&= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right) \\
&= \frac{1}{k}\left(x_k + (k-1)\mu_{k-1}\right) \\
&= \mu_{k-1} + \frac{1}{k}\left(x_k - \mu_{k-1}\right)
\end{aligned}
$$

Mean of 1, 2, 3, 4, 5:
Method 1:
(1+2+3+4+5)/5=3

Method 2:
(1+2+3+4)/4=2.5
2.5+(5−2.5)/5=3

Incremental Monte-Carlo Updates

- Update $V(s)$ incrementally after episode $S_1, A_1, R_2, ..., S_T$
- For each state S_t with return G_t

$$N(S_t) \leftarrow N(S_t) + 1$$
$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)}\left(G_t - V(S_t)\right)$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes.

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

Temporal-Difference Learning

- TD methods learn directly from episodes of experience
- TD is *model-free*: no knowledge of MDP transitions / rewards
- TD learns from *incomplete* episodes, by *bootstrapping* VS MC
- TD updates a guess towards a guess Use partial episode, estimate how many rewards instead of actual return.

MC and TD

- Goal: learn v_π online from experience under policy π

vs
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ toward *actual* return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

- Simplest temporal-difference learning algorithm: TD(0)
  - Update value $V(S_t)$ toward *estimated* return $R_{t+1} + \gamma V(S_{t+1})$

Immediate reward + discounted value of next step

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$

  - $R_{t+1} + \gamma V(S_{t+1})$ is called the *TD target*

Why this good idea?
  - $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the *TD error*

E.g.: you are driving a car (status 0), you see a car coming to you and you see you are going to crash each other (status 1); then that car turn right at last second and avoid you so you don't crash actually (status 2).
In MC, you don't get the feedback of "almost crashing" because you actually don't crash.
In TD, when you go to status 1 you can go back to status 0 and adjust your value immediately to avoid the "almost crashing".

Driving Home Example

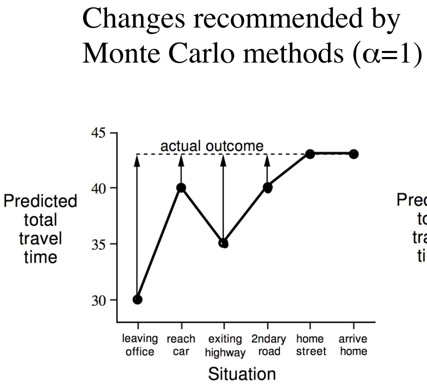
After work, from office to home.

State	Elapsed Time (minutes)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

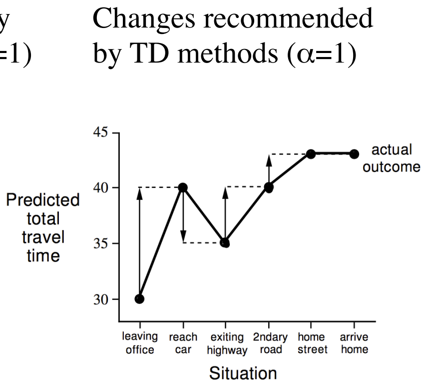
Guess need more time because of raining

Last estimation is 15 and now 10 past so time–to–go should be 5. But guess need more time bc the truck.

Driving Home Example: MC vs. TD



MC update each step estimated travel time all to 43



TD update each step estimated travel time to different value

Advantages and Disadvantages of MC vs. TD

- TD can learn *before* knowing the final outcome
    - TD can learn online after every step
    - MC must wait until end of episode before return is known
- TD can learn *without* the final outcome
    - TD can learn from incomplete sequences
    - MC can only learn from complete sequences
    - TD works in continuing (non-terminating) environments
    - MC only works for episodic (terminating) environments

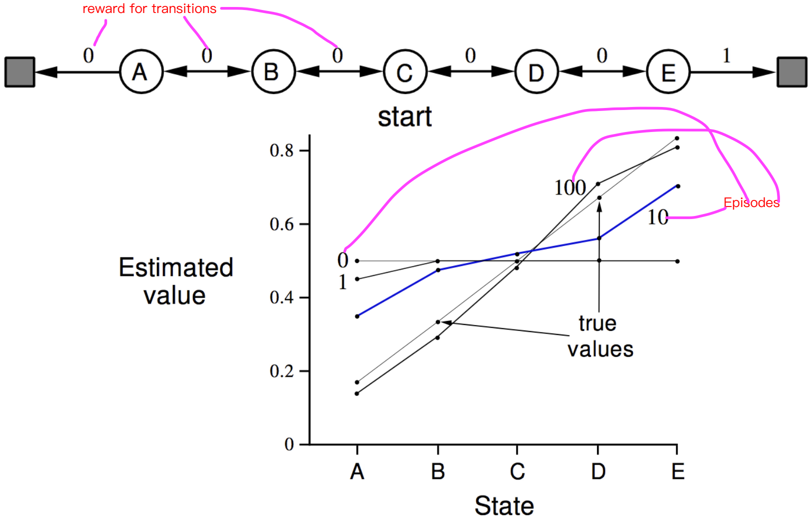
Bias/Variance Trade-Off

- Return $G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T$ is *unbiased* estimate of $v_\pi(S_t)$
Bc it is actual return
- True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ is *unbiased* estimate of $v_\pi(S_t)$
Only care 1 step
- TD target $R_{t+1} + \gamma V(S_{t+1})$ is *biased* estimate of $v_\pi(S_t)$
Bc it's estimated
- TD target is much lower variance than the return:
  - Return depends on *many* random actions, transitions, rewards
  - TD target depends on *one* random action, transition, reward

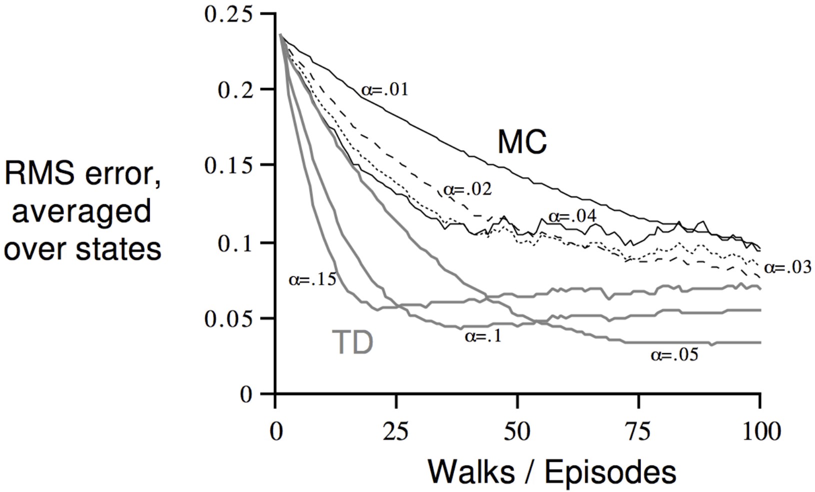
Advantages and Disadvantages of MC vs. TD (2)

- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not very sensitive to initial value Longer or sooner to converge
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to $v_\pi(s)$
  - (but not always with function approximation)
  - More sensitive to initial value

Random Walk Example



Random Walk: MC vs. TD



Batch MC and TD

- MC and TD converge: $V(s) \rightarrow v_\pi(s)$ as experience $\rightarrow \infty$
- But what about batch solution for finite experience?

$$\begin{gathered} s_1^1, a_1^1, r_2^1, \ldots, s_{T_1}^1 \\ \vdots \\ s_1^K, a_1^K, r_2^K, \ldots, s_{T_K}^K \end{gathered}$$

  - e.g. Repeatedly sample episode $k \in [1, K]$
  - Apply MC or TD(0) to episode k

AB Example

Two states A, B; no discounting; 8 episodes of experience

Episode 1: $\mathrm{A}, 0, \mathrm{B}, 0$ Go to status A, get reward 0, go to status B, get reward 0, terminate.

Episode 2: $\mathrm{B}, 1$ Go to status B, get reward 1, terminate.

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 1$

$\mathrm{B}, 0$

What is $V(A), V(B)$?

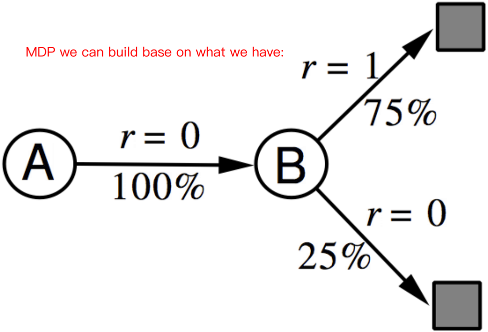
V(B) = 6/8
V(A) = 0

AB Example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0

MDP we can build base on what we have:



What is $V(A), V(B)$?

Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^{K} \sum_{t=1}^{T_k} \left(G_t^k - V(s_t^k) \right)^2$$

  - In the AB example, $V(A) = 0$
- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$ that best fits the data

$$\hat{\mathcal{P}}_{s,s'}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s')$$

$$\hat{\mathcal{R}}_{s}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k = s, a) r_t^k$$

  - In the AB example, $V(A) = 0.75$

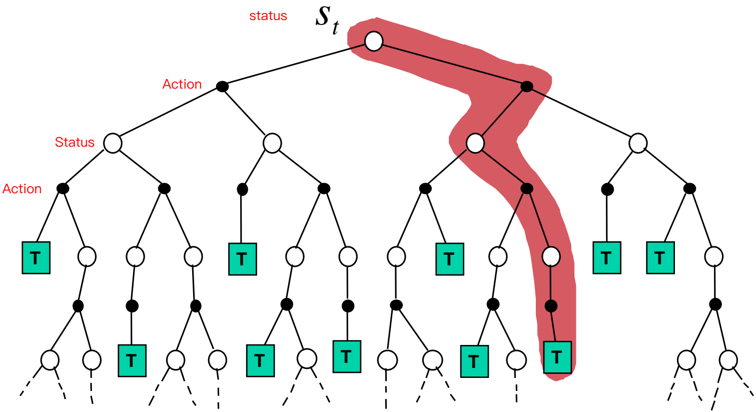
Advantages and Disadvantages of MC vs. TD (3)

- TD exploits Markov property
  - Usually more efficient in Markov environments
- MC does not exploit Markov property
  - Usually more effective in non-Markov environments

Monte-Carlo Backup

We start from status S_t, we have this look ahead tree. How to calculate the value of status S_t?
In MC, we sample one complete episode (until terminate status) labelled red. Run it and get feedback from environment. And use that sample to update S_t.

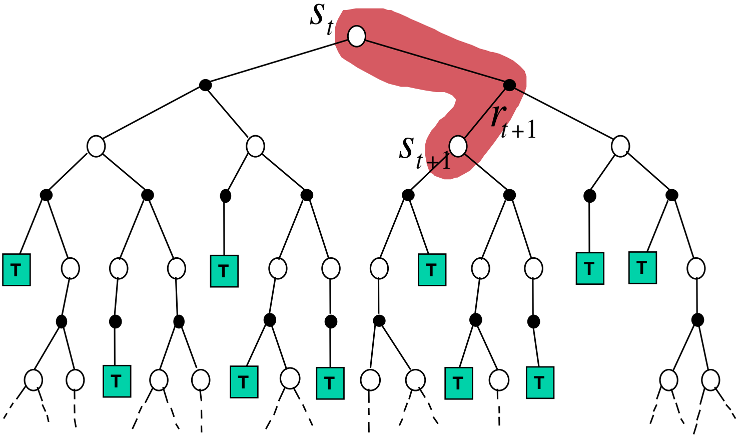
$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$



Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$

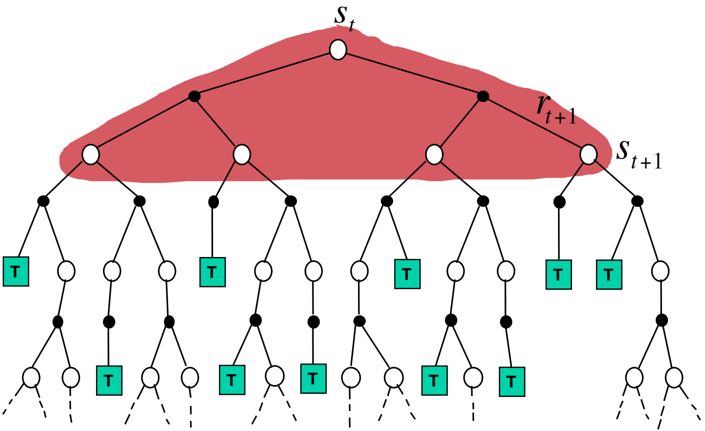
TD only look ahead one step, and sample the result of that step and update S_t.



Dynamic Programming Backup

Also one–step ahead, but we don't sample. We need know those dynamics (value and possibility) and calculate the expectation.

$$V(S_t) \leftarrow \mathbb{E}_\pi \left[R_{t+1} + \gamma V(S_{t+1})\right]$$

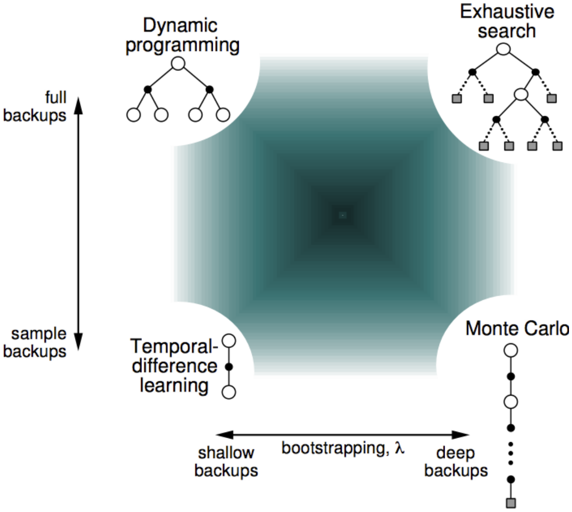


Bootstrapping and Sampling

You don’t use real return, use estimated.

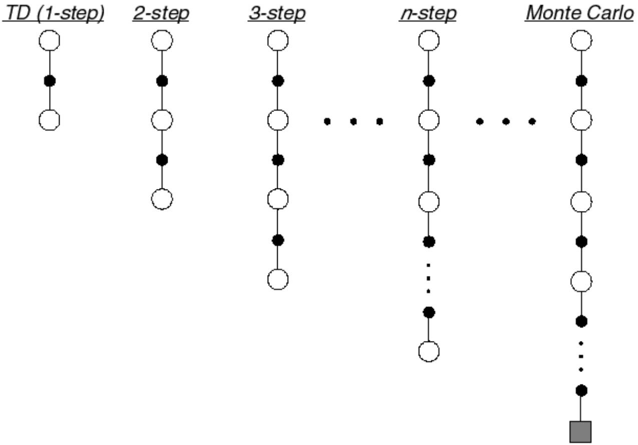
- Bootstrapping: update involves an estimate
  - MC does not bootstrap Use real returns all the way
  - DP bootstraps Use estimated value: = immediate reward you get + value function of next step
  - TD bootstraps
- Sampling: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning



n-Step Prediction

- Let TD target look n steps into the future



n-Step Return

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{array}{lll}
n = 1 & (TD) & G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\
n = 2 & & G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
\vdots & & \vdots \\
n = \infty & (MC) & G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{T-1} R_T
\end{array}
$$

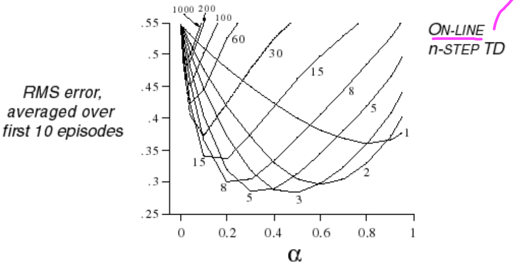
- Define the n-step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

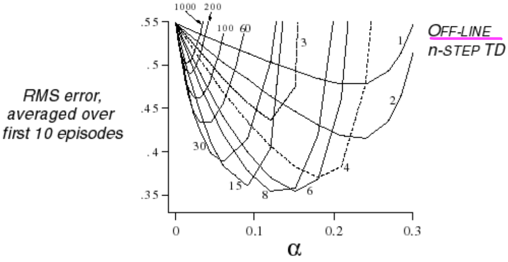
- n-step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha \left(\underbrace{G_t^{(n)} - V(S_t)}_{\text{error}} \right)$$

Large Random Walk Example



immediate update our value function or defer the update until last step.

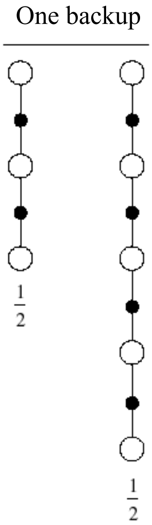


Averaging n-Step Returns

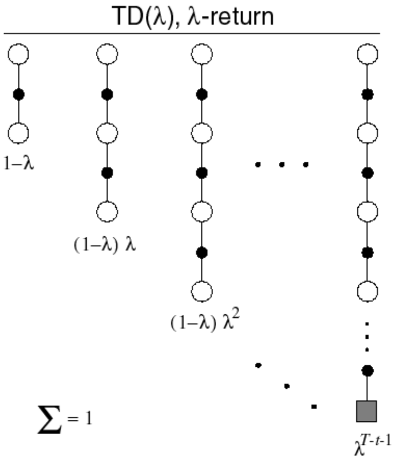
- We can average n-step returns over different n
- e.g. average the 2-step and 4-step returns

$$\frac{1}{2} G^{(2)} + \frac{1}{2} G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps? yes



λ-return



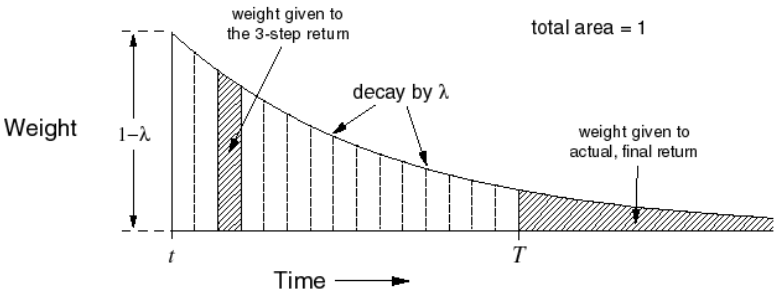
- The *λ-return* G_t^λ combines all *n*-step returns $G_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- Forward-view TD(λ)

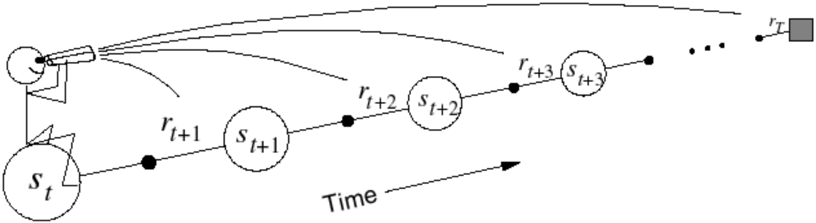
$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



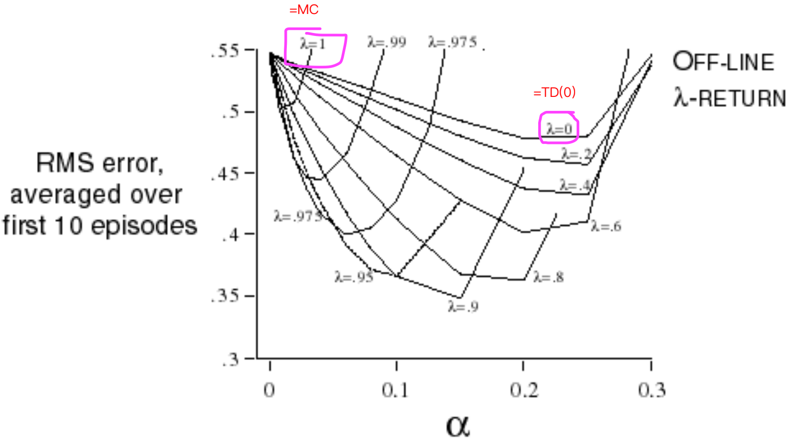
$$G_t^{\lambda} = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

Forward-view TD(λ)



- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes

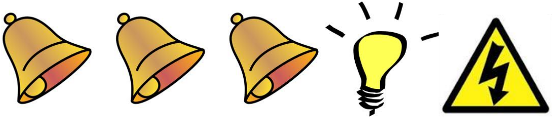
Forward-View TD(λ) on Large Random Walk



Backward View TD(λ)

- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

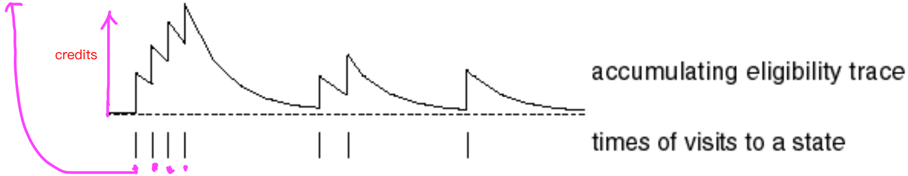
Eligibility Traces



- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- *Eligibility traces* combine both heuristics

$$E_0(s) = 0$$
$$E_t(s) = \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s)$$

Every time we visit a state we increase its credits, while time pass we decrease it.



Backward View TD(λ)

- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

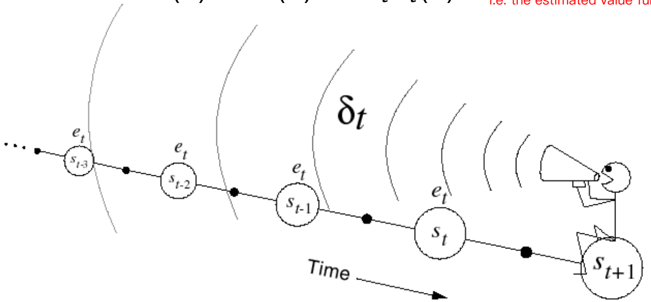
One step TD–error

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

Estimated value function look ahed one step

What we thought the value function going to be. I.e. the estimated value function of next state.

$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$



Look behind

TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

$$E_t(s) = \mathbf{1}(S_t = s)$$
$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha \delta_t$$

TD(λ) and MC

- When $\lambda = 1$, credit is deferred until end of episode
- Consider episodic environments with offline updates
- Over the course of an episode, total update for TD(1) is the same as total update for MC

Theorem

The sum of offline updates is identical for forward-view and backward-view TD(λ)

$$\sum_{t=1}^{T} \alpha \delta_t E_t(s) = \sum_{t=1}^{T} \alpha \left(G_t^{\lambda} - V(S_t) \right) \mathbf{1}(S_t = s)$$

MC and TD(1)

- Consider an episode where s is visited once at time-step k,
- TD(1) eligibility trace discounts time since visit,

$$
\begin{aligned}
E_t(s) &= \gamma E_{t-1}(s) + \mathbf{1}(S_t = s) \\
&= \begin{cases} 0 & \text{if } t < k \\ \gamma^{t-k} & \text{if } t \geq k \end{cases}
\end{aligned}
$$

- TD(1) updates accumulate error *online*

$$
\sum_{t=1}^{T-1} \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^{T-1} \gamma^{t-k} \delta_t = \alpha \left(G_k - V(S_k) \right)
$$

- By end of episode it accumulates total error

$$
\delta_k + \gamma \delta_{k+1} + \gamma^2 \delta_{k+2} + ... + \gamma^{T-1-k} \delta_{T-1}
$$

Telescoping in TD(1)

When $\lambda = 1$, sum of TD errors telescopes into MC error,

$$
\begin{aligned}
&\delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2} + ... + \gamma^{T-1-t}\delta_{T-1} \\
= \; & R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\
+ \; & \gamma R_{t+2} + \gamma^2 V(S_{t+2}) - \gamma V(S_{t+1}) \\
+ \; & \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) - \gamma^2 V(S_{t+2}) \\
& \quad \vdots \\
+ \; & \gamma^{T-1-t} R_T + \gamma^{T-t} V(S_T) - \gamma^{T-1-t} V(S_{T-1}) \\
= \; & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} ... + \gamma^{T-1-t} R_T - V(S_t) \\
= \; & G_t - V(S_t)
\end{aligned}
$$

TD(λ) and TD(1)

- TD(1) is roughly equivalent to every-visit Monte-Carlo
- Error is accumulated online, step-by-step
- If value function is only updated offline at end of episode
- Then total update is exactly the same as MC

Telescoping in TD(λ)

For general λ, TD errors also telescope to λ-error, $G_t^\lambda - V(S_t)$

$$\begin{aligned}
G_t^\lambda - V(S_t) = -V(S_t) \quad & + \quad (1-\lambda)\lambda^0 \left(R_{t+1} + \gamma V(S_{t+1})\right) \\
& + \quad (1-\lambda)\lambda^1 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2})\right) \\
& + \quad (1-\lambda)\lambda^2 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3})\right) \\
& + \quad \ldots \\
= -V(S_t) \quad & + \quad (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - \gamma\lambda V(S_{t+1})\right) \\
& + \quad (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - \gamma\lambda V(S_{t+2})\right) \\
& + \quad (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - \gamma\lambda V(S_{t+3})\right) \\
& + \quad \ldots \\
= \quad & \quad\quad\; (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right) \\
& + \quad (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - V(S_{t+1})\right) \\
& + \quad (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - V(S_{t+2})\right) \\
& + \quad \ldots \\
= \delta_t + \gamma\lambda\delta_{t+1} + (\gamma\lambda)^2\delta_{t+2} + \ldots &
\end{aligned}$$

Forwards and Backwards TD(λ)

- Consider an episode where s is visited once at time-step k,
- TD(λ) eligibility trace discounts time since visit,

$$\begin{aligned} E_t(s) &= \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s) \\ &= \begin{cases} 0 & \text{if } t < k \\ (\gamma\lambda)^{t-k} & \text{if } t \geq k \end{cases} \end{aligned}$$

- Backward TD(λ) updates accumulate error *online*

$$\sum_{t=1}^{T} \alpha\delta_t E_t(s) = \alpha \sum_{t=k}^{T} (\gamma\lambda)^{t-k} \delta_t = \alpha \left(G_k^\lambda - V(S_k) \right)$$

- By end of episode it accumulates total error for λ-return
- For multiple visits to s, $E_t(s)$ accumulates many errors

Offline Equivalence of Forward and Backward TD

Offline updates

- Updates are accumulated within episode
- but applied in batch at the end of episode

Onine Equivalence of Forward and Backward TD

Online updates

- TD(λ) updates are applied online at each step within episode
- Forward and backward-view TD(λ) are slightly different
- NEW: Exact online TD(λ) achieves perfect equivalence
- By using a slightly different form of eligibility trace
- Sutton and von Seijen, ICML 2014

Summary of Forward and Backward TD(λ)

Offline updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$\parallel$	$\parallel$	$\parallel$
Forward view	TD(0)	Forward TD(λ)	MC
Online updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$\parallel$	$\nparallel$	$\nparallel$
Forward view	TD(0)	Forward TD(λ)	MC
	$\parallel$	$\parallel$	$\parallel$
Exact Online	TD(0)	Exact Online TD(λ)	Exact Online TD(1)

$=$ here indicates equivalence in total update at end of episode.