Geometry 9

9.1 Introduction EMCBT

In previous years, you learned about the geometry of points, lines and various polygons, made up from lines. Here we will discuss the geometry of circle in a lot of depth.

See introductory video: VMhmc at www.everythingmaths.co.za

9.2 Circle Geometry EMCBU

Terminology EMCBV

The following is a recap of terms that are regularly used when referring to circles.

arc An arc is a part of the circumference of a circle.

chord A chord is a straight line joining the ends of an arc.

radius A radius, r, is any straight line from the centre of the circle to a point on the circumference.

diameter A diameter, $\varnothing$, is a special chord that passes through the centre of the circle. A diameter is the length of a straight line segment from one point on the circumference to another point on the circumference, that passes through the centre of the circle.

segment A segment is the part of the circle that is cut off by a chord. A chord divides a circle into two segments.

tangent A tangent is a line that makes contact with a circle at one point on the circumference. (AB is a tangent to the circle at point P) in Figure 9.1.

Axioms EMCBW

An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

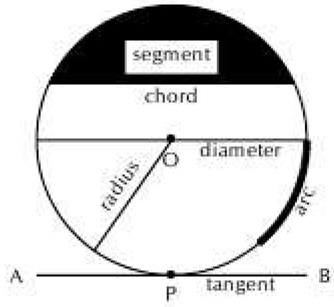


Figure 9.1: Parts of a circle

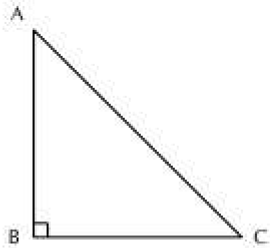


Figure 9.2: A right-angled triangle

1. The Theorem of Pythagoras, which states that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. In $\triangle ABC$, this means that $(AB)^2 + (BC)^2 = (AC)^2$

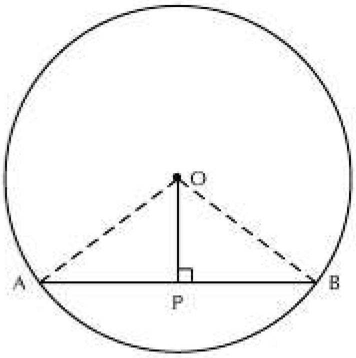
2. A tangent is perpendicular to the radius, drawn at the point of contact with the circle.

Theorems of the Geometry of Circles EMCBX

A theorem is a general proposition that is not self-evident but is proved by reasoning (these proofs need not be learned for examination purposes).

Theorem 1. *The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.*

Proof:



Consider a circle, with centre O. Draw a chord AB and draw a perpendicular line from the centre of the circle to intersect the chord at point P.

The aim is to prove that $AP = BP$

1. $\triangle OAP$ and $\triangle OBP$ are right-angled triangles.
2. $OA = OB$ as both of these are radii and OP is common to both triangles.

Apply the Theorem of Pythagoras to each triangle, to get:

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \\ OB^2 &= OP^2 + BP^2 \end{aligned}$$

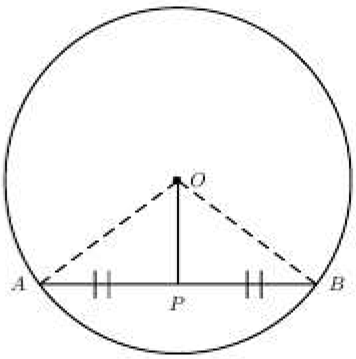
However, $OA = OB$. So,

$$\begin{aligned} OP^2 + AP^2 &= OP^2 + BP^2 \\ \therefore AP^2 &= BP^2 \\ \text{and } AP &= BP \end{aligned}$$

This means that OP bisects AB.

Theorem 2. *The line drawn from the centre of a circle, that bisects a chord, is perpendicular to the chord.*

Proof:



Consider a circle, with centre O. Draw a chord AB and draw a line from the centre of the circle to bisect the chord at point P.

The aim is to prove that $OP \perp AB$

In $\triangle OAP$ and $\triangle OBP$,

1. $AP = PB$ (given)

2. $OA = OB$ (radii)

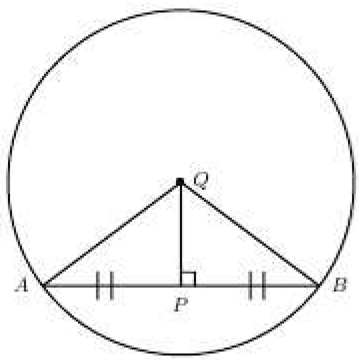
3. OP is common to both triangles.

$\therefore \triangle OAP \equiv \triangle OBP$ (SSS).

$$\begin{aligned} O\hat{P}A &= O\hat{P}B \\ O\hat{P}A + O\hat{P}B &= 180^\circ \quad (APB \text{ is a straight line}) \\ \therefore O\hat{P}A &= O\hat{P}B = 90^\circ \\ \therefore OP &\perp AB \end{aligned}$$

Theorem 3. *The perpendicular bisector of a chord passes through the centre of the circle.*

Proof:



Consider a circle. Draw a chord AB. Draw a line PQ perpendicular to AB such that PQ bisects AB at point P. Draw lines AQ and BQ.

The aim is to prove that Q is the centre of the circle, by showing that $AQ = BQ$.

In $\triangle OAP$ and $\triangle OBP$,

1. $AP = PB$ (given)

2. $\angle QPA = \angle QPB$ ($QP \perp AB$)

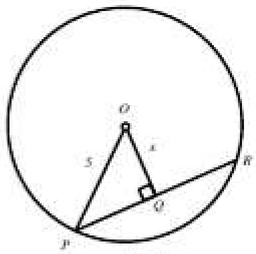
3. QP is common to both triangles.

$\therefore \triangle QAP \equiv \triangle QBP$ (SAS).

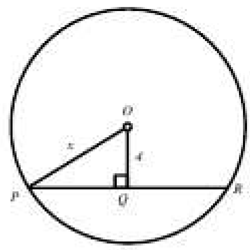
From this, $QA = QB$. Since the centre of a circle is the only point inside a circle that has points on the circumference at an equal distance from it, Q must be the centre of the circle.

Exercise 9 - 1

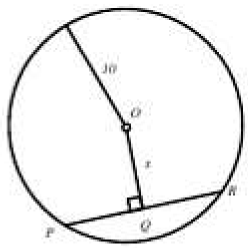
Find the value of x:

1.


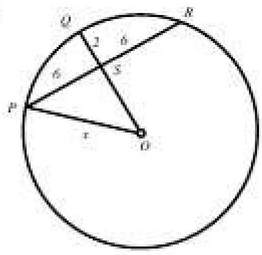
PR=8

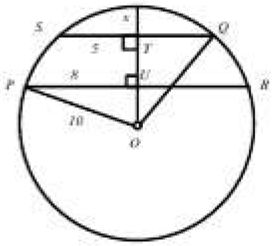
2.


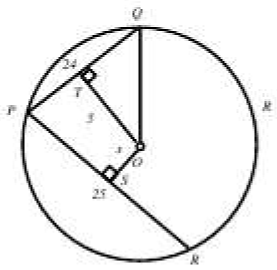
PR=6

3.


PR=8

4.


5.


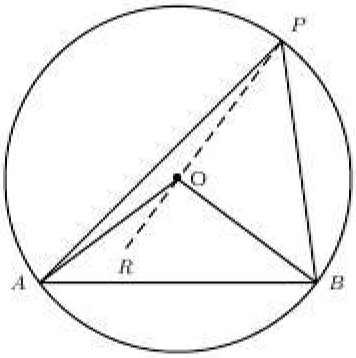
6.


More practice video solutions or help at www.everythingmaths.co.za

(1.) 021i (2.) 021j (3.) 021k (4.) 021m (5.) 021n (6.) 021p

Theorem 4. *The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circumference of the circle.*

Proof:



Consider a circle, with centre O and with A and B on the circumference. Draw a chord AB. Draw radii OA and OB. Select any point P on the circumference of the circle. Draw lines PA and PB. Draw PO and extend to R.

The aim is to prove that $A\hat{O}B = 2\,.\,A\hat{P}B$.

$A\hat{O}R = P\hat{A}O + A\hat{P}O$ (exterior angle = sum of interior opp. angles)

But, $P\hat{A}O = A\hat{P}O$ ($\triangle AOP$ is an isosceles $\triangle$)

$\therefore A\hat{O}R = 2A\hat{P}O$

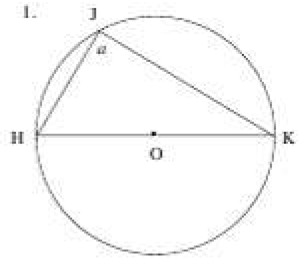
Similarly, $B\hat{O}R = 2B\hat{P}O$.

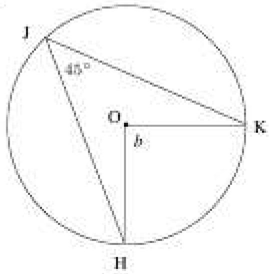
So,

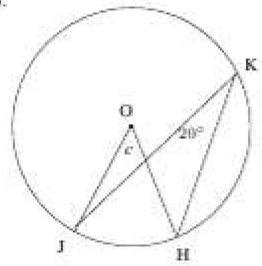
$$\begin{aligned} A\hat{O}B &= A\hat{O}R + B\hat{O}R \\ &= 2A\hat{P}O + 2B\hat{P}O \\ &= 2(A\hat{P}O + B\hat{P}O) \\ &= 2(A\hat{P}B) \end{aligned}$$

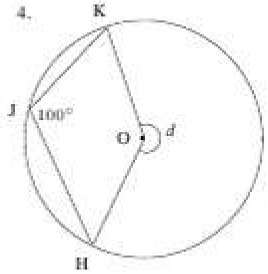
Exercise 9 - 2

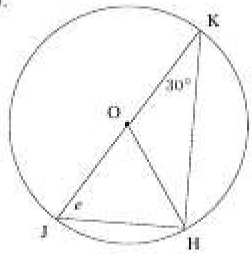
Find the angles (a to f) indicated in each diagram:

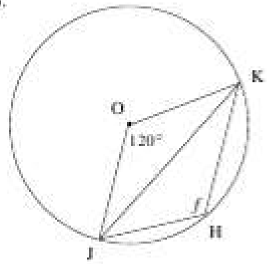
1.


2.


3.


4.


5.


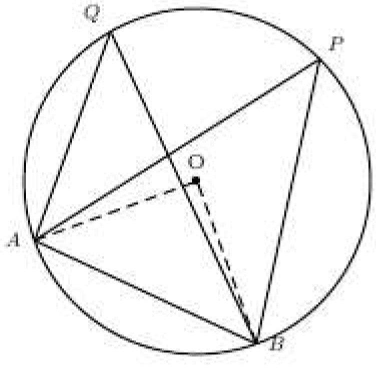
6.


More practice video solutions or help at www.everythingmaths.co.za

(1.) 021q (2.) 021r (3.) 021s (4.) 021t (5.) 021u (6.) 021v

Theorem 5. *The angles subtended by a chord at the circumference of a circle are equal, if the angles are on the same side of the chord.*

Proof:



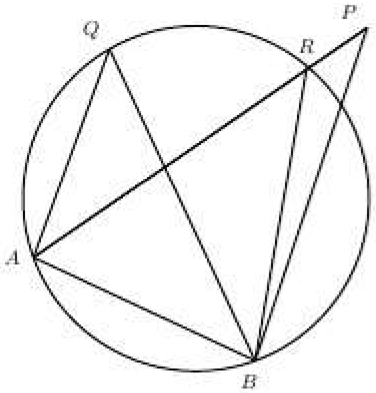
Consider a circle, with centre O. Draw a chord AB. Select any points P and Q on the circumference of the circle, such that both P and Q are on the same side of the chord. Draw lines PA, PB, QA and QB.

The aim is to prove that $A\hat{Q}B = A\hat{P}B$.

$$
\begin{aligned}
A\hat{O}B &= 2A\hat{Q}B \text{ (}\angle\text{ at centre = twice }\angle\text{ at circumference (Theorem 4))} \\
\text{and } A\hat{O}B &= 2A\hat{P}B \text{ (}\angle\text{ at centre = twice }\angle\text{ at circumference (Theorem 4))} \\
\therefore 2A\hat{Q}B &= 2A\hat{P}B \\
\therefore A\hat{Q}B &= A\hat{P}B
\end{aligned}
$$

Theorem 6. *(Converse of Theorem 5) If a line segment subtends equal angles at two other points on the same side of the line, then these four points lie on a circle.*

Proof:



Consider a line segment AB, that subtends equal angles at points P and Q on the same side of AB.

The aim is to prove that points A, B, P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut AP (or AP extended) at point R.

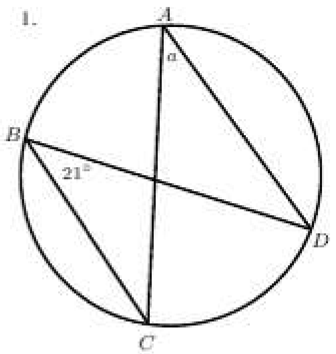
$$
\begin{aligned}
A\hat{Q}B &= A\hat{R}B (\angle\text{s on same side of chord (Theorem 5))} \\
\text{but } A\hat{Q}B &= A\hat{P}B \text{ (given)} \\
\therefore A\hat{R}B &= A\hat{P}B \\
\text{but this cannot be true since } A\hat{R}B &= A\hat{P}B + R\hat{B}P \text{ (exterior }\angle\text{ of }\triangle\text{)}
\end{aligned}
$$

$\therefore$ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle.

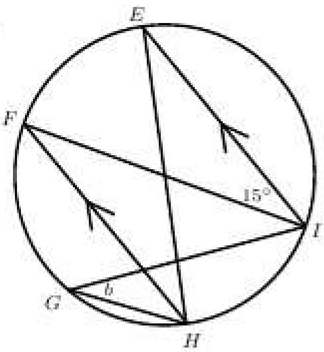
Exercise 9 - 3

Find the values of the unknown letters.

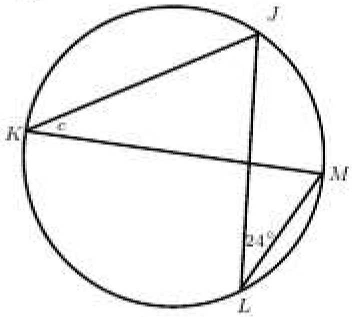
1.



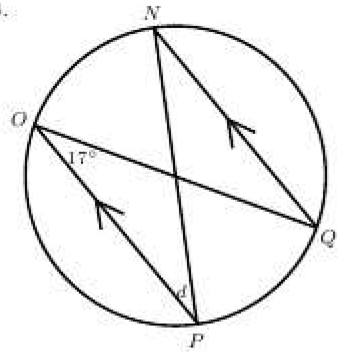
2.



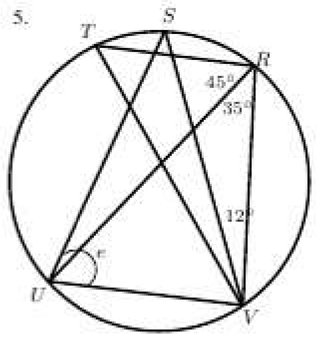
3.



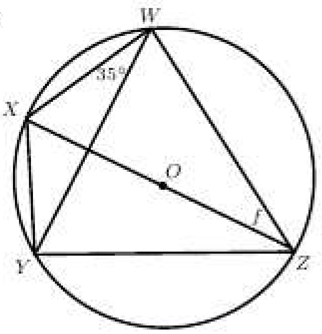
4.



5.



6.



More practice video solutions or help at www.everythingmaths.co.za

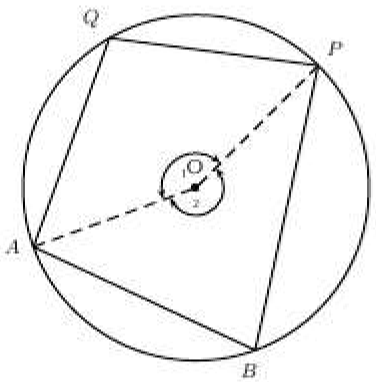
(1.) 021w (2.) 021x (3.) 021y (4.) 021z (5.) 0220 (6.) 0221

Cyclic Quadrilaterals

Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle. The vertices of a cyclic quadrilateral are said to be *concyclic*.

Theorem 7. *The opposite angles of a cyclic quadrilateral are supplementary.*

Proof:



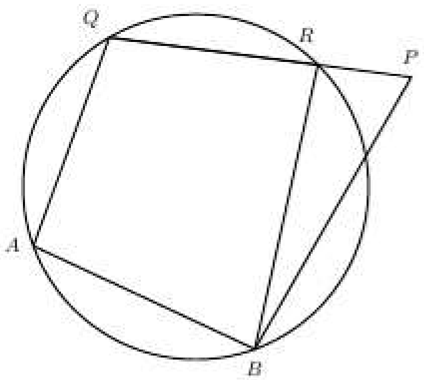
Consider a circle, with centre O. Draw a cyclic quadrilateral $ABPQ$. Draw AO and PO.

The aim is to prove that $A\hat{B}P + A\hat{Q}P = 180^\circ$ and $Q\hat{A}B + Q\hat{P}B = 180^\circ$.

$$
\begin{aligned}
\hat{O}_1 &= 2A\hat{B}P \text{ ($\angle$s at centre (Theorem 4))} \\
\hat{O}_2 &= 2A\hat{Q}P \text{ ($\angle$s at centre (Theorem 4))} \\
\text{But, } \hat{O}_1 + \hat{O}_2 &= 360^\circ \\
\therefore 2A\hat{B}P + 2A\hat{Q}P &= 360^\circ \\
\therefore A\hat{B}P + A\hat{Q}P &= 180^\circ \\
\text{Similarly, } Q\hat{A}B + Q\hat{P}B &= 180^\circ
\end{aligned}
$$

Theorem 8. *(Converse of Theorem 7) If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*

Proof:



Consider a quadrilateral $ABPQ$, such that $A\hat{B}P + A\hat{Q}P = 180^\circ$ and $Q\hat{A}B + Q\hat{P}B = 180^\circ$.

The aim is to prove that points A, B, P and Q lie on the circumference of a circle.

By contradiction. Assume that point P does not lie on a circle drawn through points A, B and Q. Let the circle cut QP (or QP extended) at point R. Draw BR.

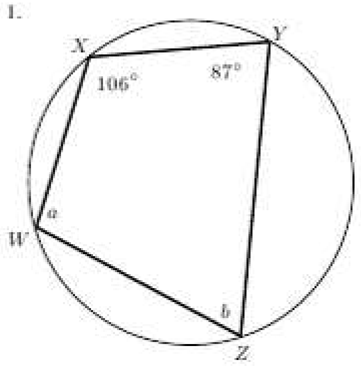
$$
\begin{aligned}
Q\hat{A}B + Q\hat{R}B &= 180^\circ \text{ (opp. $\angle$s of cyclic quad. (Theorem 7))} \\
\text{but } Q\hat{A}B + Q\hat{P}B &= 180^\circ \text{ (given)} \\
\therefore Q\hat{R}B &= Q\hat{P}B \\
\text{but this cannot be true since } Q\hat{R}B &= Q\hat{P}B + R\hat{B}P \text{ (exterior $\angle$ of $\triangle$)}
\end{aligned}
$$

∴ the assumption that the circle does not pass through P, must be false, and A, B, P and Q lie on the circumference of a circle and $ABPQ$ is a cyclic quadrilateral.

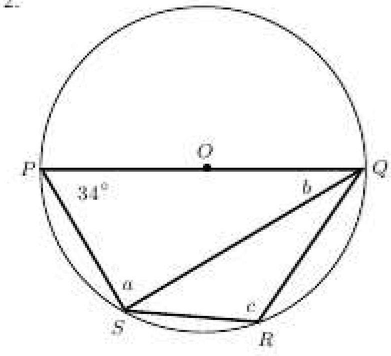
Exercise 9 - 4

Find the values of the unknown letters.

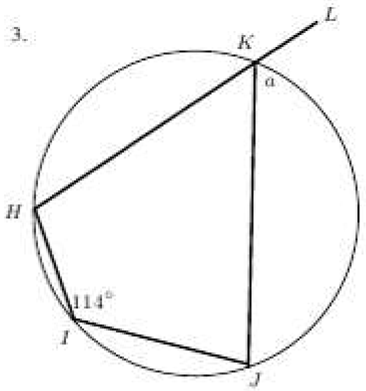
1.



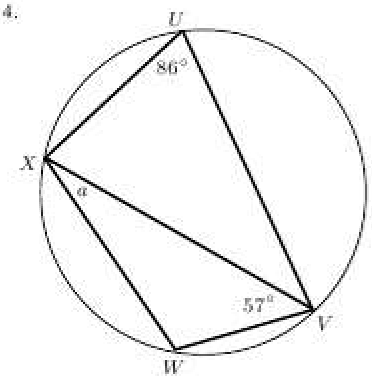
2.



3.



4.

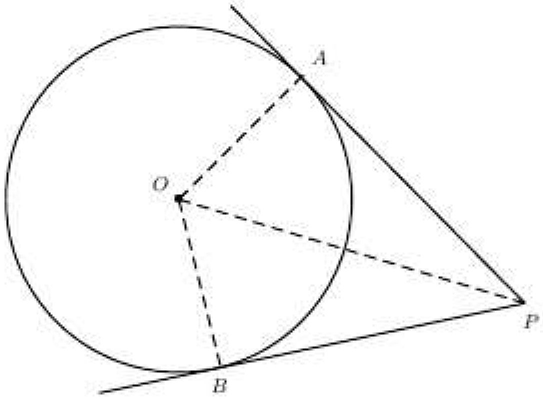


More practice video solutions or help at www.everythingmaths.co.za

(1.) 0222 (2.) 0223 (3.) 0224 (4.) 0225

Theorem 9. *Two tangents drawn to a circle from the same point outside the circle are equal in length.*

Proof:



Consider a circle, with centre O. Choose a point P outside the circle. Draw two tangents to the circle from point P, that meet the circle at A and B. Draw lines OA, OB and OP.

The aim is to prove that $AP = BP$.

In $\triangle OAP$ and $\triangle OBP$,

1. $OA = OB$ (radii)

2. $\angle OAP = \angle OBP = 90°$ ($OA \perp AP$ and $OB \perp BP$)

3. OP is common to both triangles.

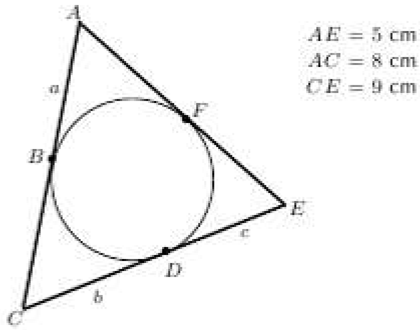
$\triangle OAP \equiv \triangle OBP$ (right angle, hypotenuse, side)

$\therefore AP = BP$

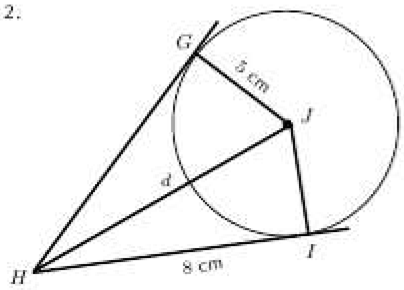
Exercise 9 - 5

Find the value of the unknown lengths.

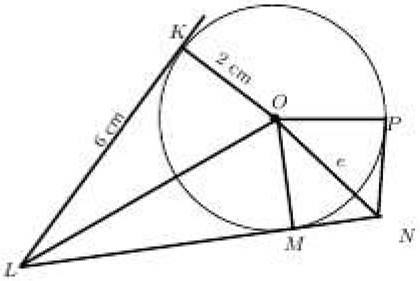
1.



2.

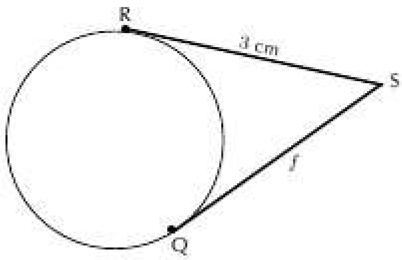


3.



$LN = 7.5$ cm

4.

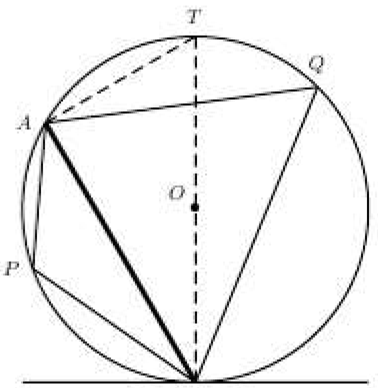


More practice video solutions or help at www.everythingmaths.co.za

(1.) 0226 (2.) 0227 (3.) 0228 (4.) 0229

Theorem 10. *The angle between a tangent and a chord, drawn at the point of contact of the chord, is equal to the angle which the chord subtends in the alternate segment.*

Proof:



Consider a circle, with centre O. Draw a chord AB and a tangent SR to the circle at point B. Chord AB subtends angles at points P and Q on the minor and major arcs, respectively.

Draw a diameter BT and join A to T.

The aim is to prove that $A\hat{P}B = A\hat{B}R$ **and** $A\hat{Q}B = A\hat{B}S$**.**

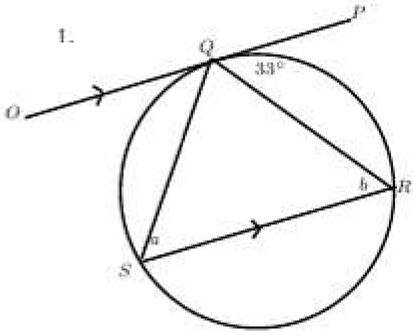
First prove that $A\hat{Q}B = A\hat{B}S$ as this result is needed to prove that $A\hat{P}B = A\hat{B}R$.

$$
\begin{aligned}
A\hat{B}S + A\hat{B}T &= 90^\circ \text{ } (TB \perp SR) \\
B\hat{A}T &= 90^\circ \text{ } (\angle\text{s at centre}) \\
\therefore A\hat{B}T + A\hat{T}B &= 90^\circ \text{ (sum of angles in } \triangle BAT) \\
\therefore A\hat{B}S &= A\hat{T}B \\
\text{However, } A\hat{Q}B &= A\hat{T}B \text{ (angles subtended by same chord } AB \text{ (Theorem 5))} \\
\therefore A\hat{Q}B &= A\hat{B}S \\
A\hat{B}S + A\hat{B}R &= 180^\circ \quad (SBR \text{ is a straight line}) \\
A\hat{P}B + A\hat{Q}B &= 180^\circ \quad (APBQ \text{ is a cyclic quad. (Theorem 7)} \\
\text{From (9.1), } \quad A\hat{Q}B &= A\hat{B}S \\
\therefore 180^\circ - A\hat{Q}B &= 180^\circ - A\hat{B}S \\
\therefore A\hat{P}B &= A\hat{B}R
\end{aligned}
\tag{9.1}
$$

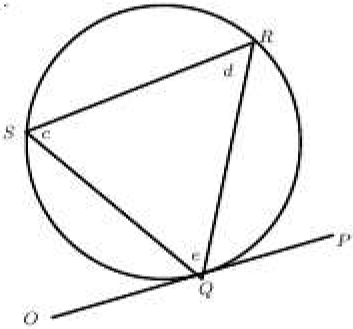
Exercise 9 - 6

Find the values of the unknown letters.

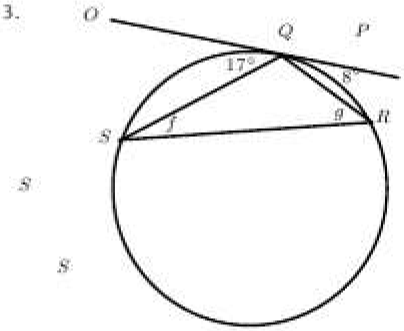
1.


P
Q
33°
O
b
R
a
S

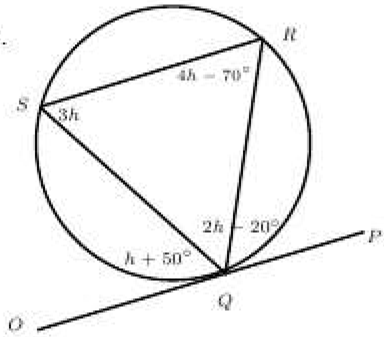
2.


R
d
S
c
e
P
Q
O

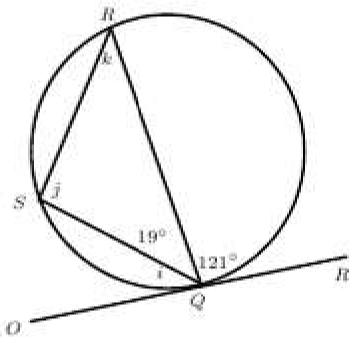
3.


O
Q
P
17°
8°
S
f
g
R
S
S

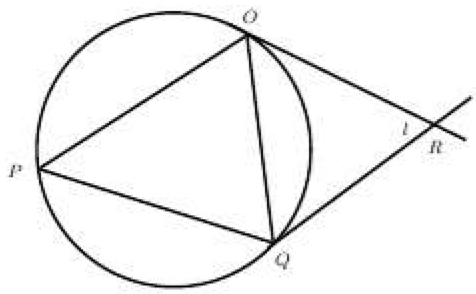
4.


R
S
4h − 70°
3h
2h + 20°
h + 50°
P
Q
O

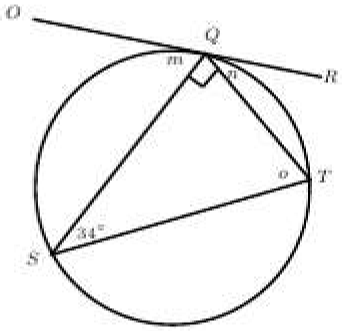
5.


R
k
S
j
19°
121°
i
R
Q
O

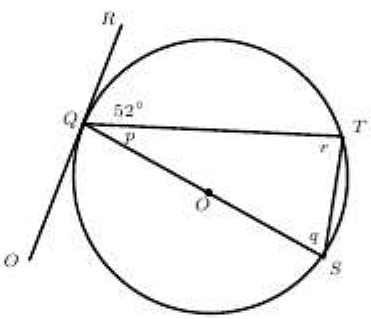
6.


O
l
R
P
Q

7.


O
Q
m
n
R
o
T
34°
S

8.

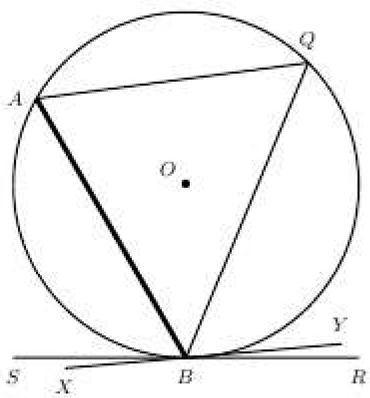

R
Q
52°
T
p
r
O
q
O
S

More practice video solutions or help at www.everythingmaths.co.za

(1.) 022a (2.) 022b (3.) 022c (4.) 022d (5.) 022e (6.) 022f
(7.) 022g (8.) 022h

Theorem 11. *(Converse of 10) If the angle formed between a line, that is drawn through the end point of a chord, and the chord, is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.*

Proof:



Consider a circle, with centre O and chord AB. Let line SR pass through point B. Chord AB subtends an angle at point Q such that $A\hat{B}S = A\hat{Q}B$.

The aim is to prove that SBR is a tangent to the circle.

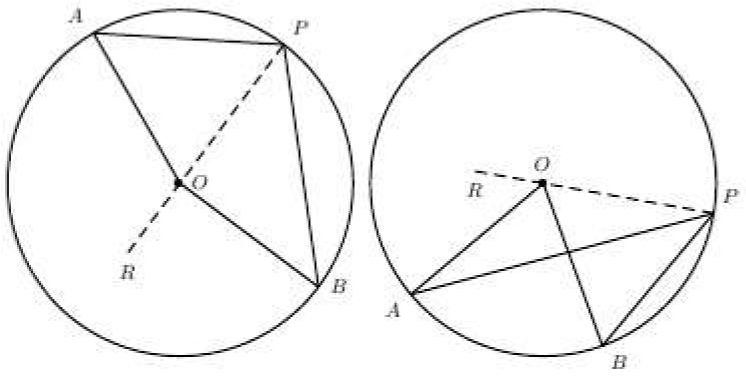
By contradiction. Assume that SBR is not a tangent to the circle and draw XBY such that XBY is a tangent to the circle.

$$\begin{aligned}
A\hat{B}X &= A\hat{Q}B \quad \text{(tan-chord theorem)} \\
\text{However,} \quad A\hat{B}S &= A\hat{Q}B \quad \text{(given)} \\
\therefore A\hat{B}X &= A\hat{B}S \qquad (9.2) \\
\text{But since,} \quad A\hat{B}X &= A\hat{B}S + X\hat{B}S \\
\text{(9.2) can only be true if,} \quad X\hat{B}S &= 0
\end{aligned}$$

If $X\hat{B}S$ is zero, then both XBY and SBR coincide and SBR is a tangent to the circle.

Exercise 9 - 7

1. *Show that Theorem* 4 also applies to the following two cases:

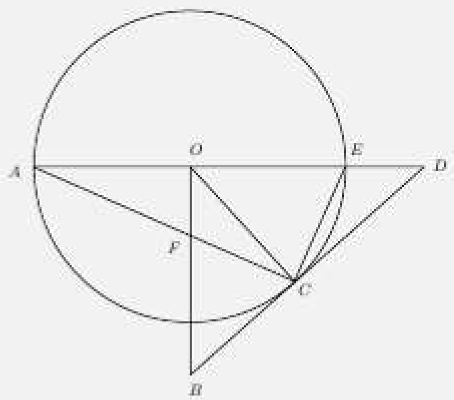


More practice video solutions or help at www.everythingmaths.co.za

(1.) 022i

Example 1: Circle Geometry I

QUESTION



BD is a tangent to the circle with centre O.
$BO \perp AD$.

Prove that:

1. *$CFOE$ is a cyclic quadrilateral*
2. $FB = BC$
3. $\triangle COE///\triangle CBF$
4. $CD^2 = ED \times AD$
5. $\frac{OE}{BC} = \frac{CD}{CO}$

SOLUTION

Step 1 : ***To show a quadrilateral is cyclic, we need a pair of opposite angles to be supplementary, so let's look for that.***

$$\begin{aligned} F\hat{O}E &= 90^\circ\ (BO \perp OD) \\ F\hat{C}E &= 90^\circ\ (\angle \text{ subtended by diameter } AE) \\ &\therefore\ CFOE \text{ is a cyclic quadrilateral (opposite } \angle\text{'s supplementary)} \end{aligned}$$

Step 2 : ***Since these two sides are part of a triangle, we are proving that triangle to be isosceles. The easiest way is to show the angles opposite to those sides to be equal.***

Let $O\hat{E}C = x$.

$\therefore$ $F\hat{C}B = x$ ($\angle$ between tangent BD and chord CE)

$\therefore$ $B\hat{F}C = x$ (exterior $\angle$ to cyclic quadrilateral $CFOE$)

and $BF = BC$ (sides opposite equal $\angle$'s in isosceles $\triangle BFC$)

Step 3 : ***To show these two triangles similar, we will need 3 equal angles. We already have 3 of the 6 needed angles from the previous question. We need only find the missing 3 angles.***

$$\begin{aligned} C\hat{B}F &= 180^\circ - 2x \text{ (sum of } \angle\text{'s in } \triangle BFC) \\ OC &= OE \text{ (radii of circle } O) \\ &\therefore\ E\hat{C}O = x \text{ (isosceles } \triangle COE) \\ &\therefore\ C\hat{O}E = 180^\circ - 2x \text{ (sum of } \angle\text{'s in } \triangle COE) \end{aligned}$$

- $C\hat{O}E = C\hat{B}F$
- $E\hat{C}O = F\hat{C}B$
- $O\hat{E}C = C\hat{F}B$

$\therefore$ $\triangle COE///\triangle CBF$ (3 $\angle$'s equal)

Step 4 : ***This relation reminds us of a proportionality relation between similar triangles. So investigate which triangles contain these sides and prove them similar. In this case 3 equal angles works well. Start with one triangle.***

In $\triangle EDC$

$$\begin{aligned} C\hat{E}D &= 180^\circ - x \text{ ($\angle$'s on a straight line AD)} \\ E\hat{C}D &= 90^\circ - x \text{ (complementary $\angle$'s)} \end{aligned}$$

Step 5 : ***Now look at the angles in the other triangle.***
In $\triangle ADC$

$$\begin{aligned} A\hat{C}D &= 180^\circ - x \text{ (sum of $\angle$'s $A\hat{C}E$ and $E\hat{C}D$)} \\ C\hat{A}D &= 90^\circ - x \text{ (sum of $\angle$'s in $\triangle CAE$)} \end{aligned}$$

Step 6 : ***The third equal angle is an angle both triangles have in common.***
Lastly, $A\hat{D}C = E\hat{D}C$ since they are the same $\angle$.

Step 7 : ***Now we know that the triangles are similar and can use the proportionality relation accordingly.***

$$\begin{aligned} &\therefore\ \triangle ADC///\triangle CDE \text{ (3 $\angle$'s equal)} \\ &\therefore\ \frac{ED}{CD} = \frac{CD}{AD} \\ &\therefore\ CD^2 = ED \times AD \end{aligned}$$

Step 8 : ***This looks like another proportionality relation with a little twist, since not all sides are contained in 2 triangles. There is a quick observation we can make about the odd side out, OE.***

$$OE = OC \text{ ($\triangle OEC$ is isosceles)}$$

Step 9 : ***With this observation we can limit ourselves to proving triangles BOC and ODC similar. Start in one of the triangles.***
In $\triangle BCO$

$$\begin{aligned} O\hat{C}B &= 90^\circ \text{ (radius OC on tangent BD)} \\ C\hat{B}O &= 180^\circ - 2x \text{ (sum of $\angle$'s in $\triangle BFC$)} \\ B\hat{O}C &= 2x - 90^\circ \text{ (sum of $\angle$'s in $\triangle BCO$)} \end{aligned}$$

Step 10 : ***Then we move on to the other one.***
In $\triangle OCD$

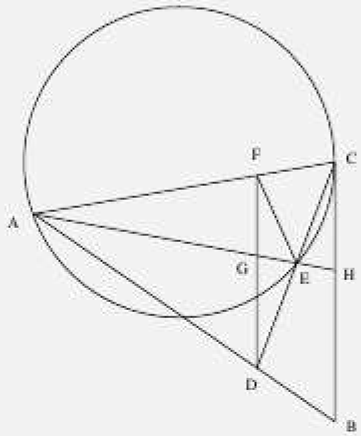
$$\begin{aligned} O\hat{C}D &= 90^\circ \text{ (radius OC on tangent BD)} \\ C\hat{O}D &= 180^\circ - 2x \text{ (sum of $\angle$'s in $\triangle OCE$)} \\ C\hat{D}O &= 2x - 90^\circ \text{ (sum of $\angle$'s in $\triangle OCD$)} \end{aligned}$$

Step 11 : ***Then, once we've shown similarity, we use the proportionality relation, as well as our first observation, appropriately.***

$$\begin{aligned} &\therefore\ \triangle BOC///\triangle ODC \text{ (3 $\angle$'s equal)} \\ &\therefore\ \frac{CO}{BC} = \frac{CD}{CO} \\ &\therefore\ \frac{OE}{BC} = \frac{CD}{CO} \text{ ($OE = CO$ isosceles $\triangle OEC$)} \end{aligned}$$

Example 2: Circle Geometry II

QUESTION



FD is drawn parallel to the tangent CB

Prove that:

1. $FADE$ is cyclic
2. $\triangle AFE///\triangle CBD$
3. $\frac{FC\times AG}{GH} = \frac{DC\times FE}{BD}$

SOLUTION

Step 1 : **In this case, the best way to show $FADE$ is a cyclic quadrilateral is to look for equal angles, subtended by the same chord.**

Let $\angle BCD = x$

$\therefore$ $\angle CAH = x$ ($\angle$ between tangent BC and chord CE)

$\therefore$ $\angle FDC = x$ (alternate $\angle$, $FD \parallel CB$)

$\therefore$ $FADE$ is a cyclic quadrilateral (chord FE subtends equal $\angle$'s)

Step 2 : **To show these 2 triangles similar we will need 3 equal angles. We can use the result from the previous question.**

Let $\angle FEA = y$

$\therefore \quad \angle FDA = y$ ($\angle$'s subtended by same chord AF in cyclic quadrilateral $FADE$)

$\therefore \quad \angle CBD = y$ (corresponding $\angle$'s, $FD \parallel CB$)

$\therefore \quad \angle FEA = \angle CBD$

Step 3 : ***We have already proved 1 pair of angles equal in the previous question.***

$$\angle BCD = \angle FAE \text{ (above)}$$

Step 4 : ***Proving the last set of angles equal is simply a matter of adding up the angles in the triangles. Then we have proved similarity.***

$$\begin{aligned} \angle AFE &= 180^\circ - x - y \text{ ($\angle$'s in $\triangle AFE$)} \\ \angle CDB &= 180^\circ - x - y \text{ ($\angle$'s in $\triangle CBD$)} \\ &\therefore \quad \triangle AFE///\triangle CBD \text{ (3 $\angle$'s equal)} \end{aligned}$$

Step 5 : ***This equation looks like it has to do with proportionality relation of similar triangles. We already showed triangles AFE and CBD similar in the previous question. So lets start there.***

$$\begin{aligned} \frac{DC}{BD} &= \frac{FA}{FE} \\ &\therefore \quad \frac{DC \times FE}{BD} = FA \end{aligned}$$

Step 6 : ***Now we need to look for a hint about side FA. Looking at triangle CAH we see that there is a line FG intersecting it parallel to base CH. This gives us another proportionality relation.***

$$\begin{aligned} \frac{AG}{GH} &= \frac{FA}{FC} \text{ ($FG \parallel CH$ splits up lines AH and AC proportionally)} \\ &\therefore \quad FA = \frac{FC \times AG}{GH} \end{aligned}$$

Step 7 : ***We have 2 expressions for the side FA.***

$$\therefore \quad \frac{FC.AG}{GH} = \frac{DC \times FE}{BD}$$

9.3 Co-ordinate Geometry

EMCBY

Equation of a Circle

EMCBZ

We know that every point on the circumference of a circle is the same distance away from the centre of the circle. Consider a point $(x_1; y_1)$ on the circumference of a circle of radius r with centre at $(x_0; y_0)$.

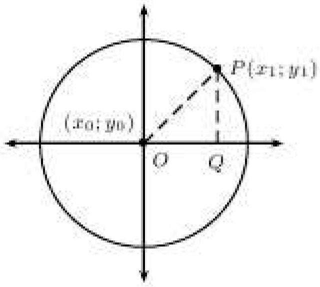


Figure 9.3: Circle with centre $(x_0; y_0)$ and a point P at $(x_1; y_1)$

In Figure 9.3, $\triangle OPQ$ is a right-angled triangle. Therefore, from the Theorem of Pythagoras, we know that:

$$OP^2 = PQ^2 + OQ^2$$

But,

$$\begin{aligned} PQ &= y_1 - y_0 \\ OQ &= x_1 - x_0 \\ OP &= r \\ \therefore \quad r^2 &= (y_1 - y_0)^2 + (x_1 - x_0)^2 \end{aligned}$$

But, this same relation holds for any point P on the circumference. Therefore, we can write:

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{9.3}$$

for a circle with centre at $(x_0; y_0)$ and radius r.

For example, the equation of a circle with centre $(0; 0)$ and radius 4 is:

$$\begin{aligned} (y - y_0)^2 + (x - x_0)^2 &= r^2 \\ (y - 0)^2 + (x - 0)^2 &= 4^2 \\ x^2 + y^2 &= 16 \end{aligned}$$

See video: VMhrm at www.everythingmaths.co.za

Example 3: *Equation of a Circle I*

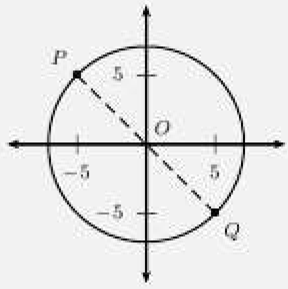
QUESTION

Find the equation of a circle (centre O) with a diameter between two points, P at $(-5;5)$ and Q at $(5;-5)$.

SOLUTION

Step 1 : ***Draw a picture***

Draw a picture of the situation to help you figure out what needs to be done.



Step 2 : ***Find the centre of the circle***

We know that the centre of a circle lies on the midpoint of a diameter. Therefore the co-ordinates of the centre of the circle is found by finding the midpoint of the line between P and Q. Let the co-ordinates of the centre of the circle be $(x_0;y_0)$, let the co-ordinates of P be $(x_1;y_1)$ and let the co-ordinates of Q be $(x_2;y_2)$. Then, the co-ordinates of the midpoint are:

$$\begin{aligned} x_0 &= \frac{x_1+x_2}{2} \\ &= \frac{-5+5}{2} \\ &= 0 \\ y_0 &= \frac{y_1+y_2}{2} \\ &= \frac{5+(-5)}{2} \\ &= 0 \end{aligned}$$

The centre point of line PQ and therefore the centre of the circle is at $(0;0)$.

Step 3 : ***Find the radius of the circle***

If P and Q are two points on a diameter, then the radius is half the distance between them.

The distance between the two points is:

$$
\begin{aligned}
r = \frac{1}{2}PQ &= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \frac{1}{2}\sqrt{(5 - (-5))^2 + (-5 - 5)^2} \\
&= \frac{1}{2}\sqrt{(10)^2 + (-10)^2} \\
&= \frac{1}{2}\sqrt{100 + 100} \\
&= \sqrt{\frac{200}{4}} \\
&= \sqrt{50}
\end{aligned}
$$

Step 4 : ***Write the equation of the circle***

$$x^2 + y^2 = 50$$

Example 4: *Equation of a Circle II*

QUESTION

Find the centre and radius of the circle
$x^2 - 14x + y^2 + 4y = -28$.

SOLUTION

Step 1 : ***Change to standard form***
We need to rewrite the equation in the form $(x - x_0) + (y - y_0) = r^2$
To do this we need to complete the square
i.e. add and subtract $(\frac{1}{2}$ cooefficient of $x)^2$ and $(\frac{1}{2}$ cooefficient of $y)^2$

Step 2 : ***Adding cooefficients***

$$x^2 - 14x + y^2 + 4y = -28$$
$$\therefore \quad x^2 - 14x + (7)^2 - (7)^2 + y^2 + 4y + (2)^2 - (2)^2 = -28$$

Step 3 : ***Complete the squares***

$$\therefore \quad (x - 7)^2 - (7)^2 + (y + 2)^2 - (2)^2 = -28$$

Step 4 : ***Take the constants to the other side***

$$\therefore \quad (x - 7)^2 - 49 + (y + 2)^2 - 4 = -28$$
$$\therefore \quad (x - 7)^2 + (y + 2)^2 = -28 + 49 + 4$$
$$\therefore \quad (x - 7)^2 + (y + 2)^2 = 25$$

Step 5 : ***Read the values from the equation***

$\therefore$ centre is $(7; -2)$ and the radius is 5 units

Equation of a Tangent to a Circle at a Point on the Circle

EMCCA

We are given that a tangent to a circle is drawn through a point P with co-ordinates $(x_1; y_1)$. In this section, we find out how to determine the equation of that tangent.

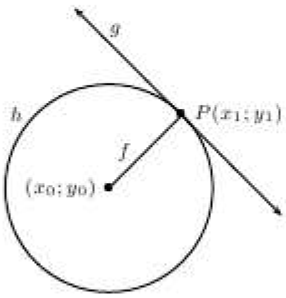


Figure 9.4: Circle h with centre $(x_0; y_0)$ has a tangent, g passing through point P at $(x_1; y_1)$. Line f passes through the centre and point P.

We start by making a list of what we know:

1. We know that the equation of the circle with centre $(x_0; y_0)$ and radius r is $(x-x_0)^2+(y-y_0)^2 = r^2$.
2. We know that a tangent is perpendicular to the radius, drawn at the point of contact with the circle.

As we have seen in earlier grades, there are two steps to determining the equation of a straight line:

Step 1: Calculate the gradient of the line, m.

Step 2: Calculate the y-intercept of the line, c.

The same method is used to determine the equation of the tangent. First we need to find the gradient of the tangent. We do this by finding the gradient of the line that passes through the centre of the circle and point P (line f in Figure 9.4), because this line is a radius and the tangent is perpendicular to it.

$$m_f = \frac{y_1 - y_0}{x_1 - x_0} \tag{9.4}$$

The tangent (line g) is perpendicular to this line. Therefore,

$$m_f \times m_g = -1$$

So,

$$m_g = -\frac{1}{m_f}$$

Now, we know that the tangent passes through $(x_1; y_1)$ so the equation is given by:

$$\begin{aligned}
y - y_1 &= m_g(x - x_1)\\
y - y_1 &= -\frac{1}{m_f}(x - x_1)\\
y - y_1 &= -\frac{1}{\frac{y_1 - y_0}{x_1 - x_0}}(x - x_1)\\
y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1)
\end{aligned}$$

For example, find the equation of the tangent to the circle at point $(1; 1)$. The centre of the circle is at $(0; 0)$. The equation of the circle is $x^2 + y^2 = 2$.

Use

$$y - y_1 = -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1)$$

with $(x_0; y_0) = (0; 0)$ and $(x_1; y_1) = (1; 1)$.

$$\begin{aligned}
y - y_1 &= -\frac{x_1 - x_0}{y_1 - y_0}(x - x_1)\\
y - 1 &= -\frac{1 - 0}{1 - 0}(x - 1)\\
y - 1 &= -\frac{1}{1}(x - 1)\\
y &= -(x - 1) + 1\\
y &= -x + 1 + 1\\
y &= -x + 2
\end{aligned}$$

Exercise 9 - 8

1. Find the equation of the circle:
   (a) with centre $(0; 5)$ and radius 5
   (b) with centre $(2; 0)$ and radius 4
   (c) with centre $(5; 7)$ and radius 18
   (d) with centre $(-2; 0)$ and radius 6
   (e) with centre $(-5; -3)$ and radius $\sqrt{3}$
2. (a) Find the equation of the circle with centre $(2; 1)$ which passes through $(4; 1)$.
   (b) Where does it cut the line $y = x + 1$?
   (c) Draw a sketch to illustrate your answers.
3. (a) Find the equation of the circle with centre $(-3; -2)$ which passes through $(1; -4)$.
   (b) Find the equation of the circle with centre $(3; 1)$ which passes through $(2; 5)$.
   (c) Find the point where these two circles cut each other.
4. Find the centre and radius of the following circles:
   (a) $(x - 9)^2 + (y - 6)^2 = 36$
   (b) $(x - 2)^2 + (y - 9)^2 = 1$
   (c) $(x + 5)^2 + (y + 7)^2 = 12$
   (d) $(x + 4)^2 + (y + 4)^2 = 23$
   (e) $3(x - 2)^2 + 3(y + 3)^2 = 12$

(f) $x^2 - 3x + 9 = y^2 + 5y + 25 = 17$

5. Find the x and y intercepts of the following graphs and draw a sketch to illustrate your answer:

   (a) $(x+7)^2 + (y-2)^2 = 8$

   (b) $x^2 + (y-6)^2 = 100$

   (c) $(x+4)^2 + y^2 = 16$

   (d) $(x-5)^2 + (y+1)^2 = 25$

6. Find the centre and radius of the following circles:

   (a) $x^2 + 6x + y^2 - 12y = -20$

   (b) $x^2 + 4x + y^2 - 8y = 0$

   (c) $x^2 + y^2 + 8y = 7$

   (d) $x^2 - 6x + y^2 = 16$

   (e) $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$

   (f) $x^2 - 6nx + y^2 + 10ny = 9n^2$

7. Find the equation of the tangent to each circle:

   (a) $x^2 + y^2 = 17$ at the point $(1;4)$

   (b) $x^2 + y^2 = 25$ at the point $(3;4)$

   (c) $(x+1)^2 + (y-2)^2 = 25$ at the point $(3;5)$

   (d) $(x-2)^2 + (y-1)^2 = 13$ at the point $(5;3)$

More practice   video solutions   or help at www.everythingmaths.co.za

(1.) 01gt (2.) 01gu (3.) 01gv (4.) 01gw (5.) 01gx (6.) 01gy
(7.) 01gz

9.4 Transformations

EMCCB

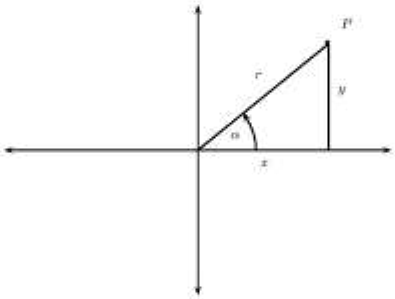
Rotation of a Point About an Angle θ

EMCCC

First we will find a formula for the co-ordinates of P after a rotation of θ.

We need to know something about polar co-ordinates and compound angles before we start.

Polar co-ordinates



Notice that $\sin\alpha = \frac{y}{r} \therefore y = r\sin\alpha$
and $\cos\alpha = \frac{x}{r} \therefore x = r\cos\alpha$
so P can be expressed in two ways:

$P(x;y)$ rectangular co-ordinates

or $P(r;\alpha)$ polar co-ordinates.

Compound angles

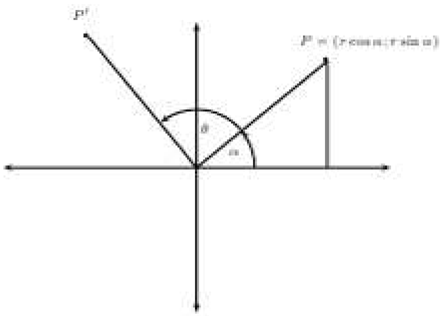
(See derivation of formulae in Chapter 12)

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Now consider P' after a rotation of θ

$$P(x;y) = P(r\cos\alpha; r\sin\alpha)$$
$$P'(r\cos(\alpha+\theta); r\sin(\alpha+\theta))$$

Expand the co-ordinates of P'



$$\begin{aligned}x - \text{co-ordinate} &= r\cos(\alpha+\theta)\\ &= r[\cos\alpha\cos\theta - \sin\alpha\sin\theta]\\ &= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta\\ &= x\cos\theta - y\sin\theta\end{aligned}$$

$$\begin{aligned}y - \text{co-ordinate} &= r\sin(\alpha+\theta)\\ &= r[\sin\alpha\cos\theta + \sin\theta\cos\alpha]\\ &= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta\\ &= y\cos\theta + x\sin\theta\end{aligned}$$

which gives the formula $P' = [(x\cos\theta - y\sin\theta; y\cos\theta + x\sin\theta)]$.

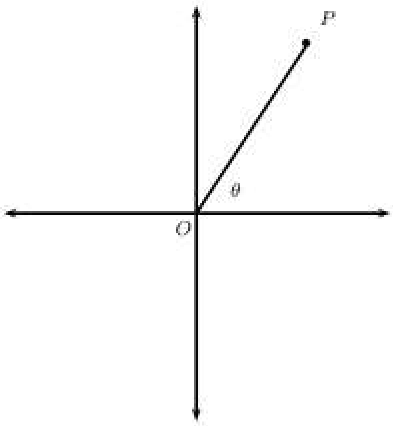
So to find the co-ordinates of $P(1;\sqrt{3})$ after a rotation of 45°, we arrive at:

$$\begin{aligned}P' &= [(x\cos\theta - y\sin\theta);(y\cos\theta + x\sin\theta)]\\ &= \left[(1\cos 45^\circ - \sqrt{3}\sin 45^\circ);(\sqrt{3}\cos 45^\circ + 1\sin 45^\circ)\right]\\ &= \left[\left(\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\right);\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]\\ &= \left(\frac{1-\sqrt{3}}{\sqrt{2}};\frac{\sqrt{3}+1}{\sqrt{2}}\right)\end{aligned}$$

Exercise 9 - 9

Any line OP is drawn (not necessarily in the first quadrant), making an angle of θ degrees with the x-axis. Using the co-ordinates of P and the angle α, calculate the co-ordinates of P', if the line OP is rotated about the origin through α degrees.

	P	α
1.	$(2;6)$	60°
2.	$(4;2)$	30°
3.	$(5;-1)$	45°
4.	$(-3;2)$	120°
5.	$(-4;-1)$	225°
6.	$(2;5)$	-150°



More practice video solutions or help at www.everythingmaths.co.za

(1.) 022j

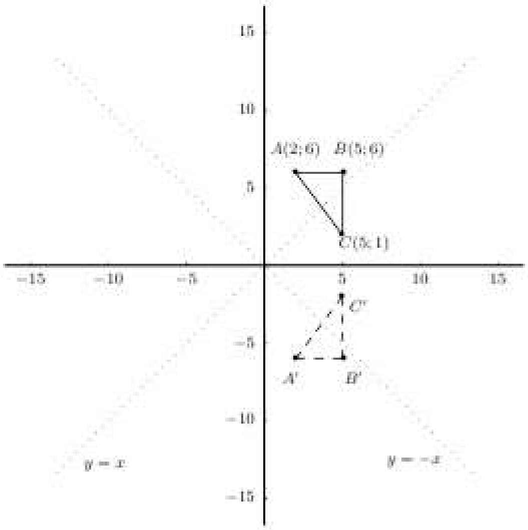
Characteristics of Transformations EMCCD

Rigid transformations like translations, reflections, rotations and glide reflections preserve shape and size, and that enlargement preserves shape but not size.

Activity: Geometric Transformations

Draw a large 15×15 grid and plot $\triangle ABC$ with $A(2;6)$, $B(5;6)$ and $C(5;1)$. Fill in the lines $y = x$ and $y = -x$.

Complete the table below , by drawing the images of $\triangle ABC$ under the given transformations. The first one has been done for you.



Transformation	Description (translation, reflection, rotation, enlargement)	Co-ordinates	Lengths	Angles
$(x;y) \rightarrow (x;-y)$	reflection about the x-axis	$A'(2;-6)$ $B'(5;-6)$ $C'(5;-2)$	$A'B'=3$ $B'C'=4$ $A'C'=5$	$\hat{B}'=90°$ $\tan\hat{A}=4/3$ $\therefore \hat{A}=53°, \hat{C}=37°$
$(x;y) \rightarrow (x+1;y-2)$				
$(x;y) \rightarrow (-x;y)$				
$(x;y) \rightarrow (-y;x)$				
$(x;y) \rightarrow (-x;-y)$				
$(x;y) \rightarrow (2x;2y)$				
$(x;y) \rightarrow (y;x)$				
$(x;y) \rightarrow (y;x+1)$				

A transformation that leaves lengths and angles unchanged is called a rigid transformation. Which of the above transformations are rigid?

Chapter 9 End of Chapter Exercises

1. ΔABC undergoes several transformations forming $\Delta A'B'C'$. Describe the relationship between the angles and sides of ΔABC and $\Delta A'B'C'$ (e.g., they are twice as large, the same, etc.)

Transformation	Sides	Angles	Area
Reflect			
Reduce by a scale factor of 3			
Rotate by $90°$			
Translate 4 units right			
Enlarge by a scale factor of 2			

2. ΔDEF has $\hat{E} = 30°$, $DE = 4$ cm, $EF = 5$ cm. ΔDEF is enlarged by a scale factor of 6 to form $\Delta D'E'F'$.
   (a) Solve ΔDEF
   (b) Hence, solve $\Delta D'E'F'$
3. ΔXYZ has an area of $6\,\text{cm}^2$. Find the area of $\Delta X'Y'Z'$ if the points have been transformed as follows:
   (a) $(x,y) \rightarrow (x+2;y+3)$
   (b) $(x,y) \rightarrow (y;x)$

(c) $(x, y) \rightarrow (4x; y)$
(d) $(x, y) \rightarrow (3x; y+2)$
(e) $(x, y) \rightarrow (-x; -y)$
(f) $(x, y) \rightarrow (x; -y+3)$
(g) $(x, y) \rightarrow (4x; 4y)$
(h) $(x, y) \rightarrow (-3x; 4y)$

More practice video solutions or help at www.everythingmaths.co.za

(1.) 01h0 (2.) 01h1 (3.) 01h2