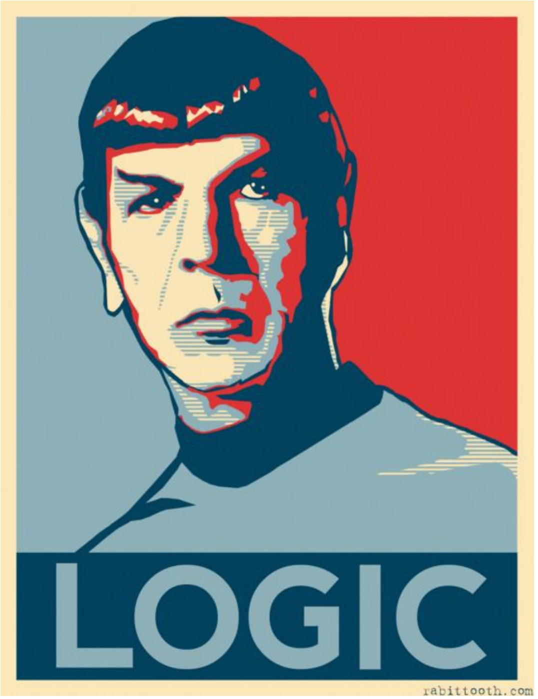
Propositional Logic

Entailment and Inference



Computational Inference

- computers cannot reason informally ("common sense")
  - they don't know the interpretation of the sentences
  - they usually don't have access to the state of the real world to check the correspondence between sentences and facts
- computers can be used to check the validity of sentences
  - "if the sentences in a knowledge base are true, then the sentence under consideration must be true, regardless of its possible interpretations"
  - can be applied to rather complex sentences

Computational Approaches to Inference

- model checking based on truth tables
  - generate all possible models and check them for validity or satisfiability
  - exponential complexity,
    - all combinations of truth values need to be considered
- search
  - use inference rules as successor functions for a search algorithm
  - also exponential, but only worst-case
    - in practice, many problems have shorter proofs
    - only relevant propositions need to be considered

Propositional Logic

- a relatively simple framework for reasoning
- can be extended for more expressiveness at the cost of computational overhead
- important aspects
  - syntax
  - semantics
  - validity and inference
  - models
  - inference rules
  - complexity

Propositional Logic

Propositional logic is the simplest logic—illustrates basic ideas

The proposition symbols P_1, P_2 etc are sentences

If S is a sentence, $\neg S$ is a sentence (negation)

If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)

If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)

If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)

If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Order of Precedence

If and Only If

If S1 then S2

P	Q	$P \land Q$	$P \lor Q$	$\neg P$	$P \to Q$	$P \leftrightarrow Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Propositional Logic: Semantics

Each model specifies true/false for each proposition symbol

E.g.	$P_{1,2}$	$P_{2,2}$	$P_{3,1}$
	$true$	$true$	$false$

If S1 is true, then I am claiming that S2 is true

(With these symbols, 8 possible models, can be enumerated automatically.)

Rules for evaluating truth with respect to a model m:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true **and**	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true **or**	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false **or**	S_2	is true
i.e.,	is false iff	S_1	is true **and**	S_2	is false
$S_1 \Leftrightarrow S_2$	is true iff	$S_1 \Rightarrow S_2$	is true **and**	$S_2 \Rightarrow S_1$	is true

Simple recursive process evaluates an arbitrary sentence, e.g.,
$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = true \wedge (false \vee true) = true \wedge true = true$

Models

- To make things precise, we use the term **model** in place of "possible world."
- Logicians typically think in terms of **models**, which are formally **structured worlds**
  - With respect to which **truth** can be evaluated
- We say $\boldsymbol{m}$ is a model of a sentence $\boldsymbol{\alpha}$ if $\boldsymbol{\alpha}$ is true in $\boldsymbol{m}$
  - or sometimes m satisfies $\boldsymbol{\alpha}$.
- $M(\boldsymbol{\alpha})$: Set of all models of $\boldsymbol{\alpha}$

Entailment

- Entailment: a sentence *follows logically* from another sentence
- Entailment means that one thing follows from another:

$$\alpha \models \beta$$

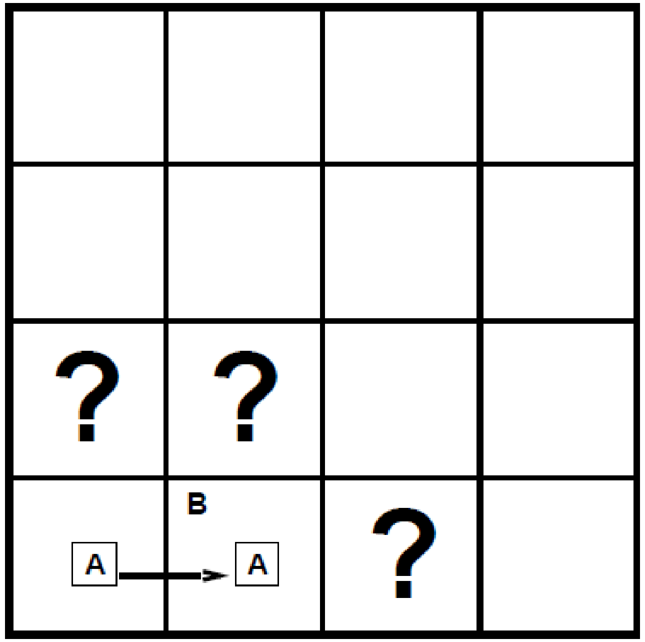
- Knowledge base *KB* entails sentence α if and only if α is true β is also true
  - E.g., x+y = 4 entails 4 = x+y
  - Entailment is a relationship between sentences (i.e., syntax) that is based on **semantics**

Entailment

- Entailment is when a sentence follows from another $\alpha \vDash \beta$
  - $\alpha \vDash \beta$ iff in every model where α is true, β is also true.
  - $\alpha \vDash \beta$ iff $M(\alpha) \subseteq M(\beta)$

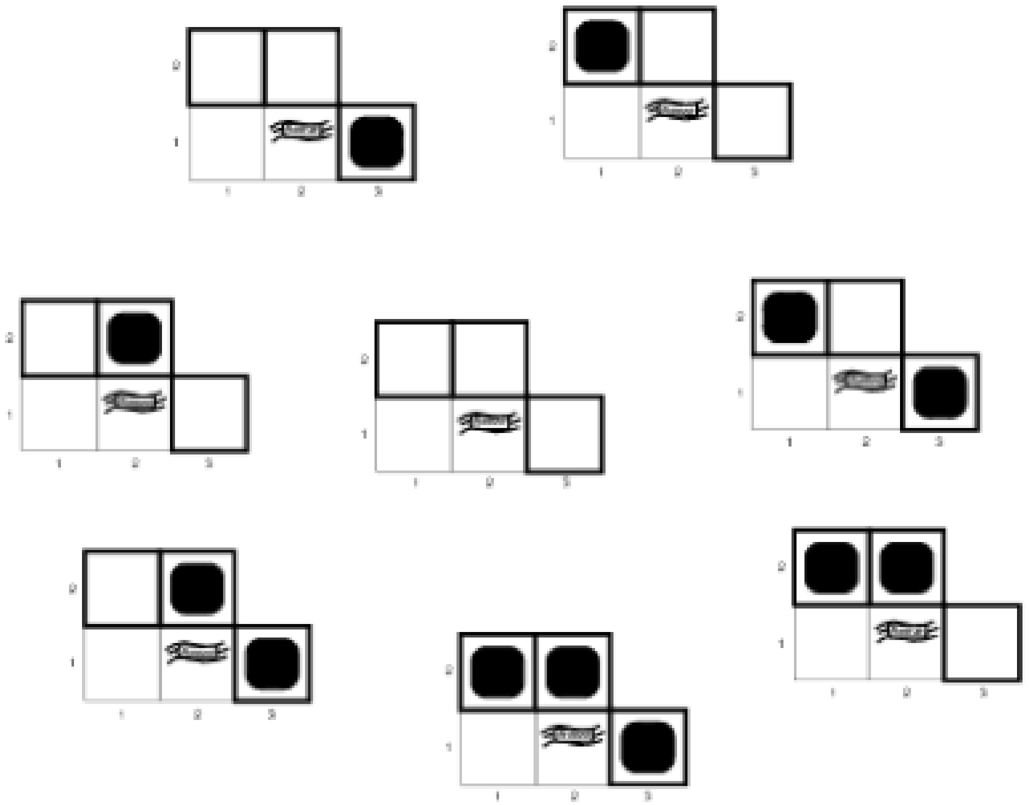
- Examples:
  - $(x = 0) \vDash (xy = 0)$
  - $(p = TRUE) \vDash (p \vee q)$
  - $(p \wedge q) \vDash (p \vee q)$
  - $\big((q \Rightarrow p) \vee r\big) \vDash (q \Rightarrow p)$

Entailment in the Wumpus World



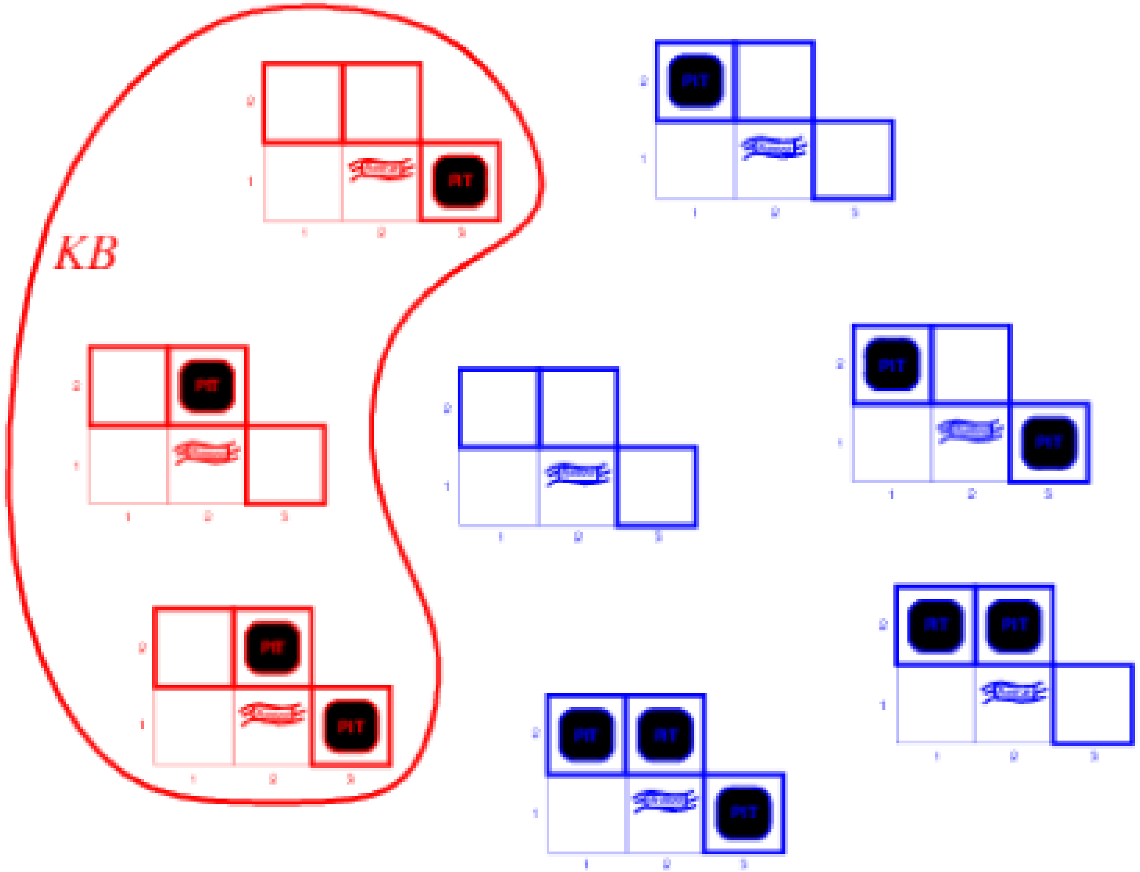
- Situation after detecting nothing in [1,1], **moving right**, breeze in [2,1]
- Consider possible models for ?s, assuming only pits
- 3 Boolean choices, i.e.,
  - $2^3 = 8$ possible models

Possible Wumpus Models



Valid Wumpus Models

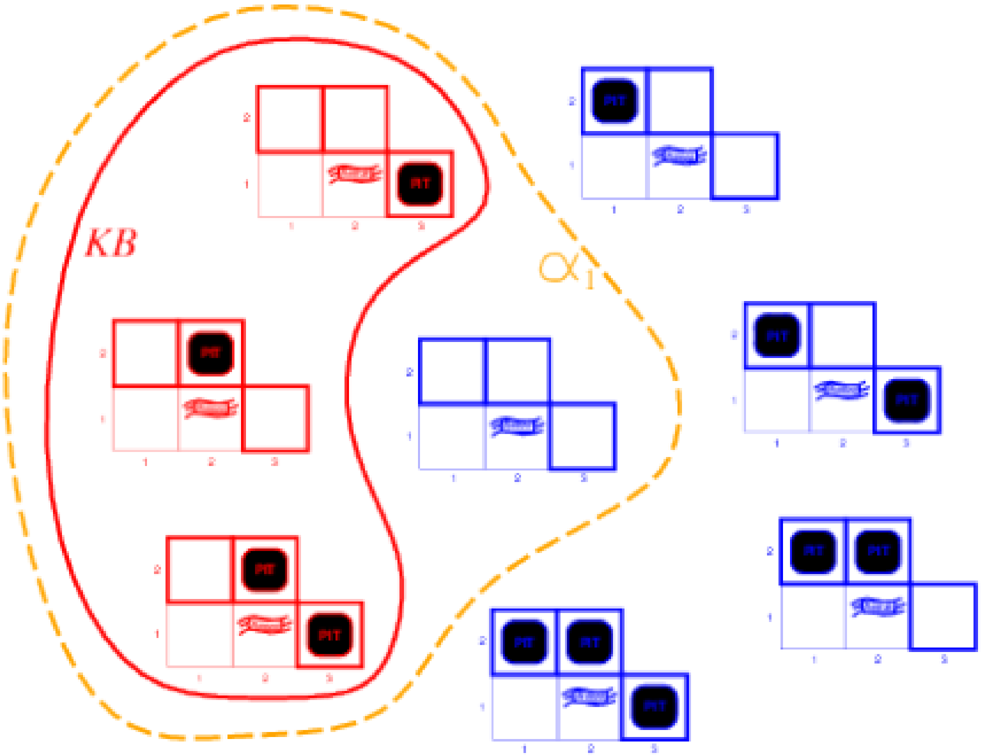
KB = wumpus-world rules + observations



Entailment

KB = wumpus-world rules + observations

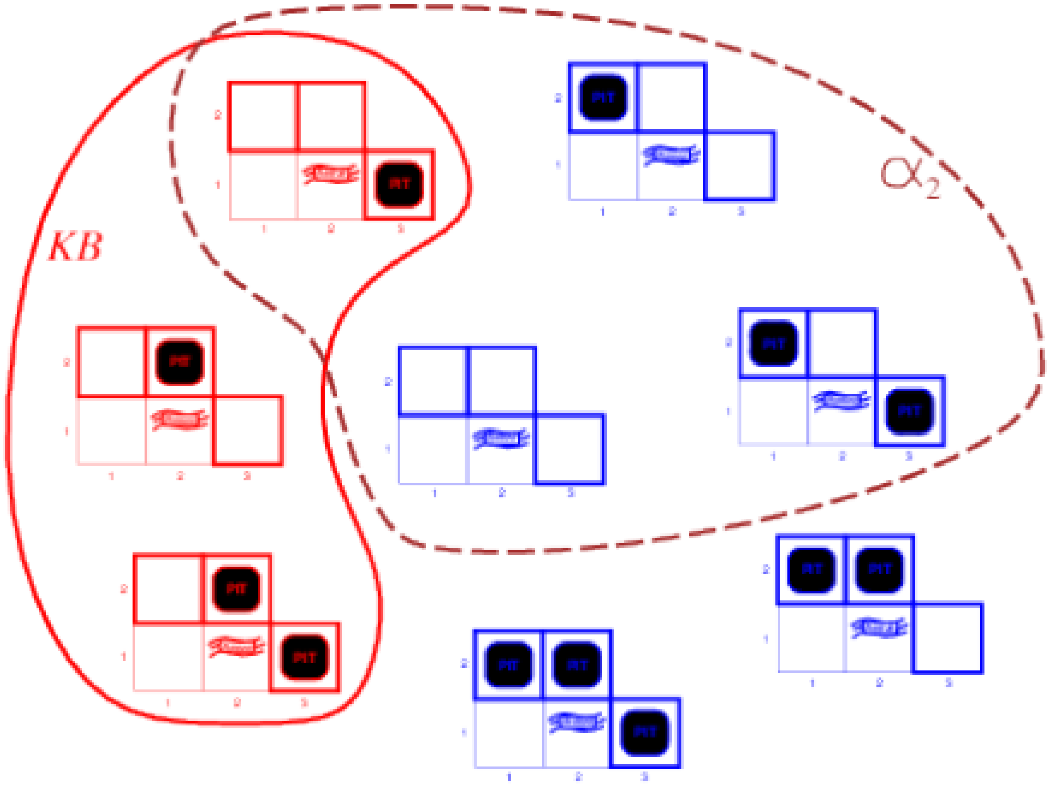
α_1 = "[1,2] is safe", $KB \vdash \alpha_1$, proved by model checking



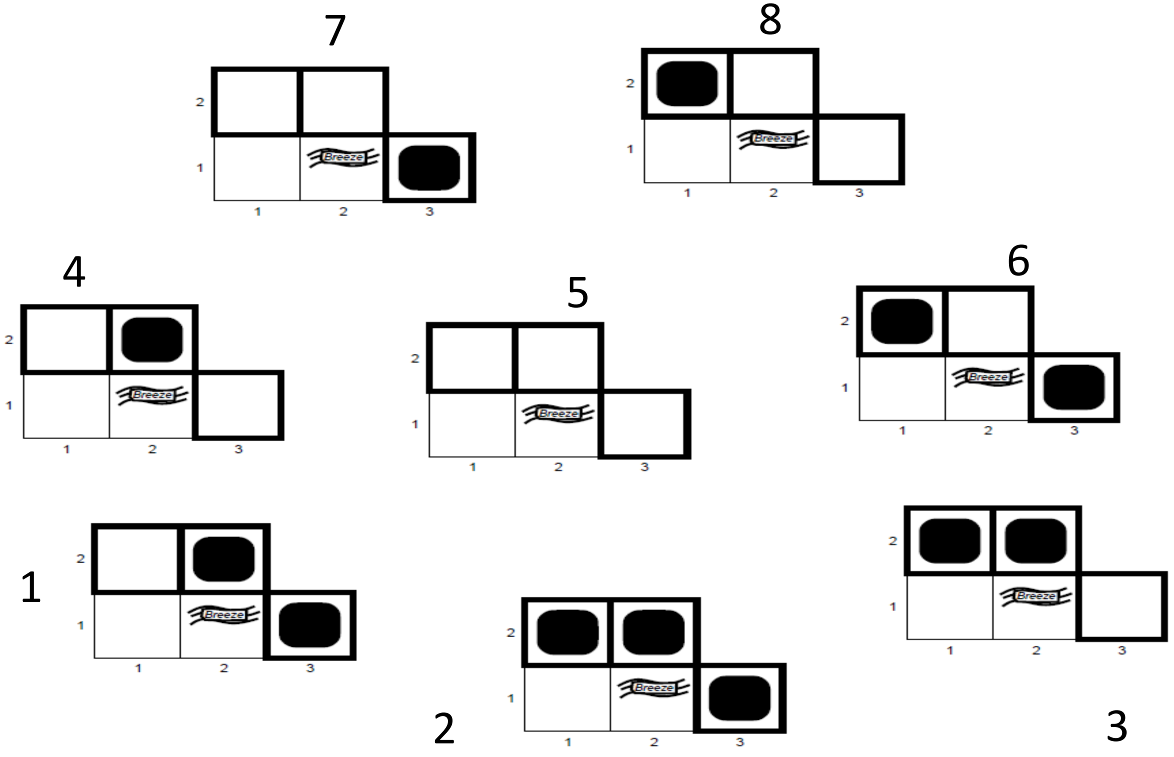
Entailment

KB = wumpus-world rules + observations

α_2 = "[2,2] is safe", $KB \not\models \alpha_2$, proved by model checking



Wumpus Models

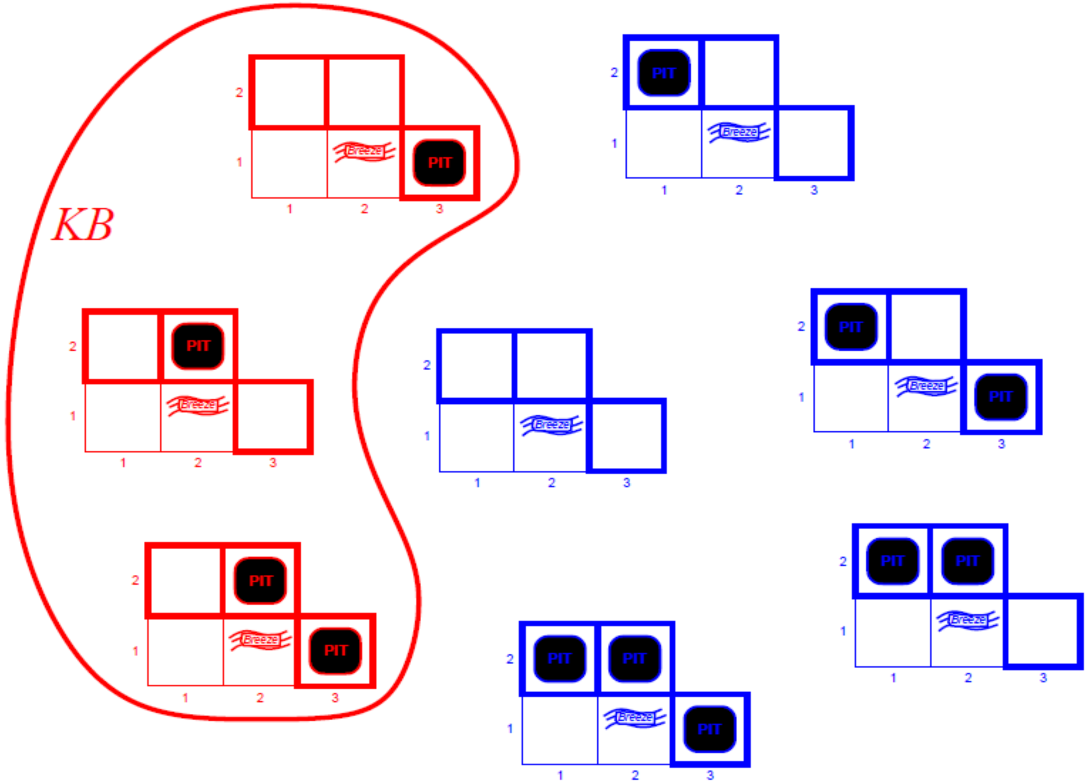


Columns, Rows

	1,1	2,1	3,1	1,2	2,2
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

A simple entailment procedure

KB = Wumpus World Rules
+
Observations

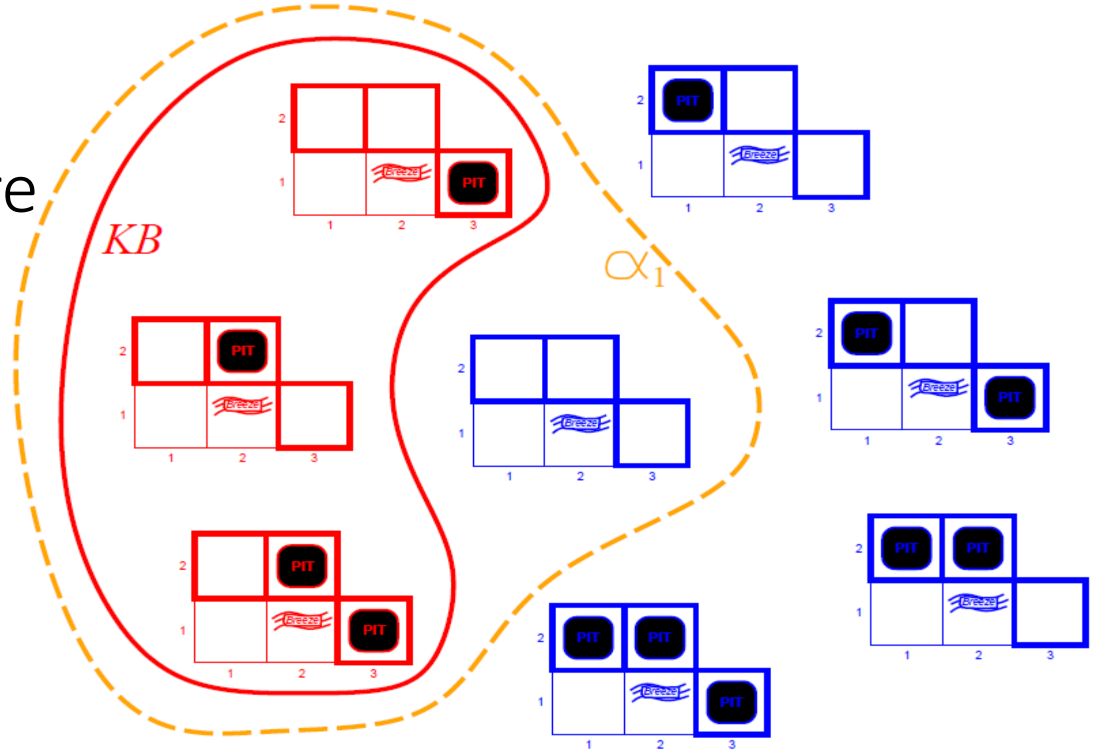


Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

KB

A simple entailment procedure



KB = Wumpus World Rules

+

Observations

α_1 = "No pit in (1,2)"

Columns, Rows

$$KB \models \alpha_1$$

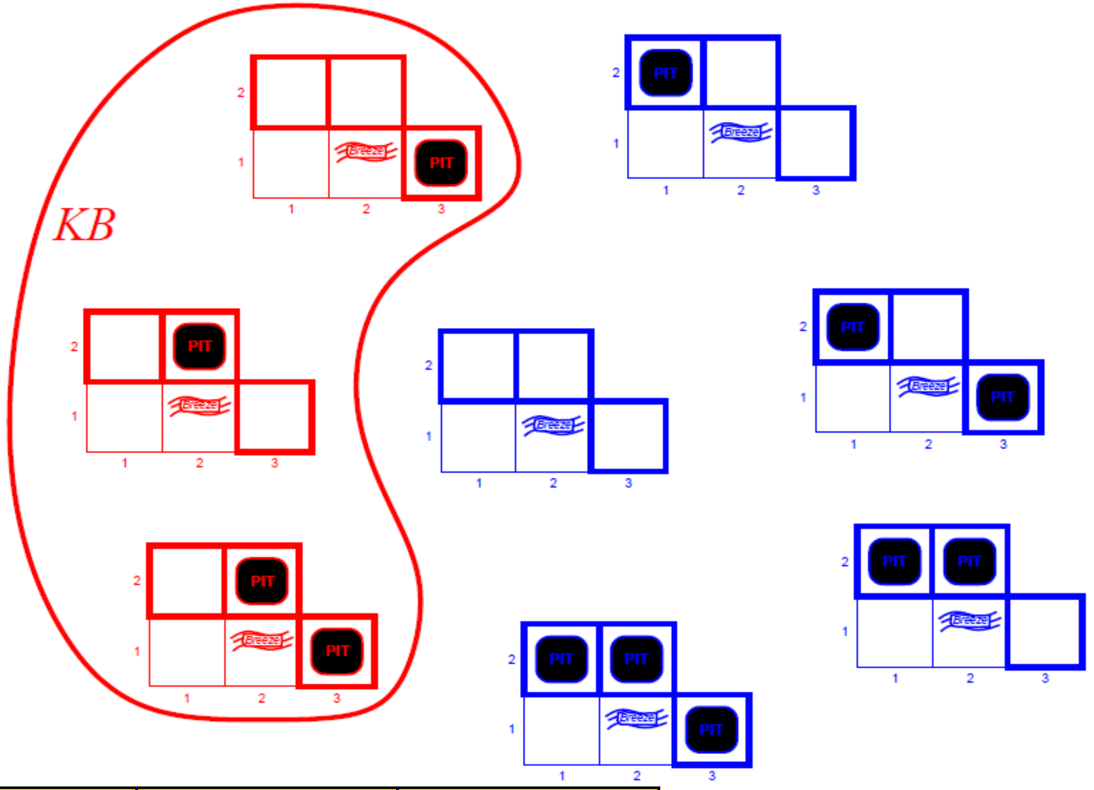
	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

α_1

KB

A simple entailment procedure

KB = Wumpus World Rules
+
Observations



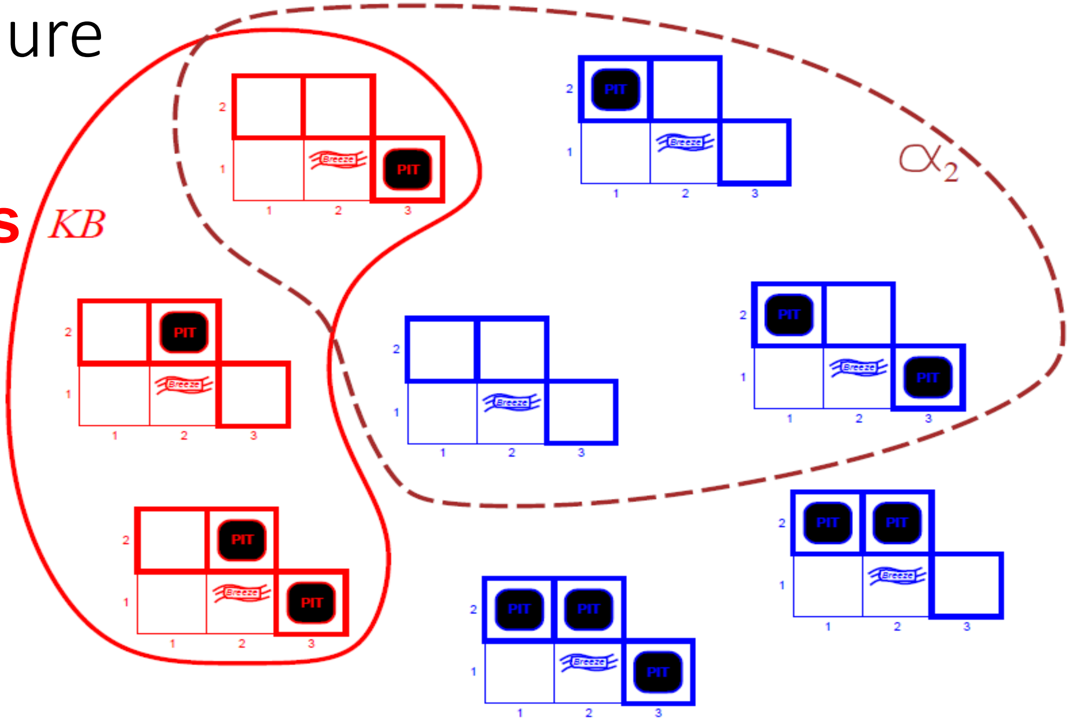
Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

KB

A simple entailment procedure

KB = Wumpus World Rules
+
Observations



α_2 = "No pit in (2,2)"

KB $\not\models \alpha_2$

Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

α_2

KB

Logical inference problem

- Logical inference problem:
  - Given: – a knowledge base KB (a set of sentences) and
  - a sentence α (called a theorem),
- **Does a KB semantically entail α**? $KB \vDash \alpha$
  - In other words: In all interpretations in which sentences in the KB are true, is also α true?
- **Question**: Is there a procedure (program) that can decide this problem in a finite number of steps?
- **Answer**: Yes. Logical inference problem for the propositional logic is decidable.

Solving logical inference problem

- In the following: How to design the procedure that answers:

$$KB \vDash \alpha$$

- Three approaches:
  - Truth-table approach
  - Inference rules

Modus Ponens:

- Modus ponens is a straightforward inference rule. It states that if you have a conditional statement (an implication) and you know the antecedent (the "if" part) is true, then you can conclude that the consequent (the "then" part) must also be true.
- Symbolically, if $P \rightarrow Q$ is true and P is true, then you can infer that Q is true.
- Example: If it is raining ($P \rightarrow Q$), and it is indeed raining (P), then you can conclude that the ground is wet (Q).

Unit Resolution:

- Unit resolution is a rule used in resolution-based theorem proving. In propositional logic, it involves resolving a clause containing a unit (a single literal) with the negation of another literal. The result is a new clause.
- For example, if you have the clauses ($P \vee Q$) and $\neg Q$, you can resolve them to get the new clause (P).
- Symbolically, if ($A \vee B$) and $\neg B$ are both clauses, then you can resolve them to obtain the new clause (A).

Conjunction Elimination Rule:

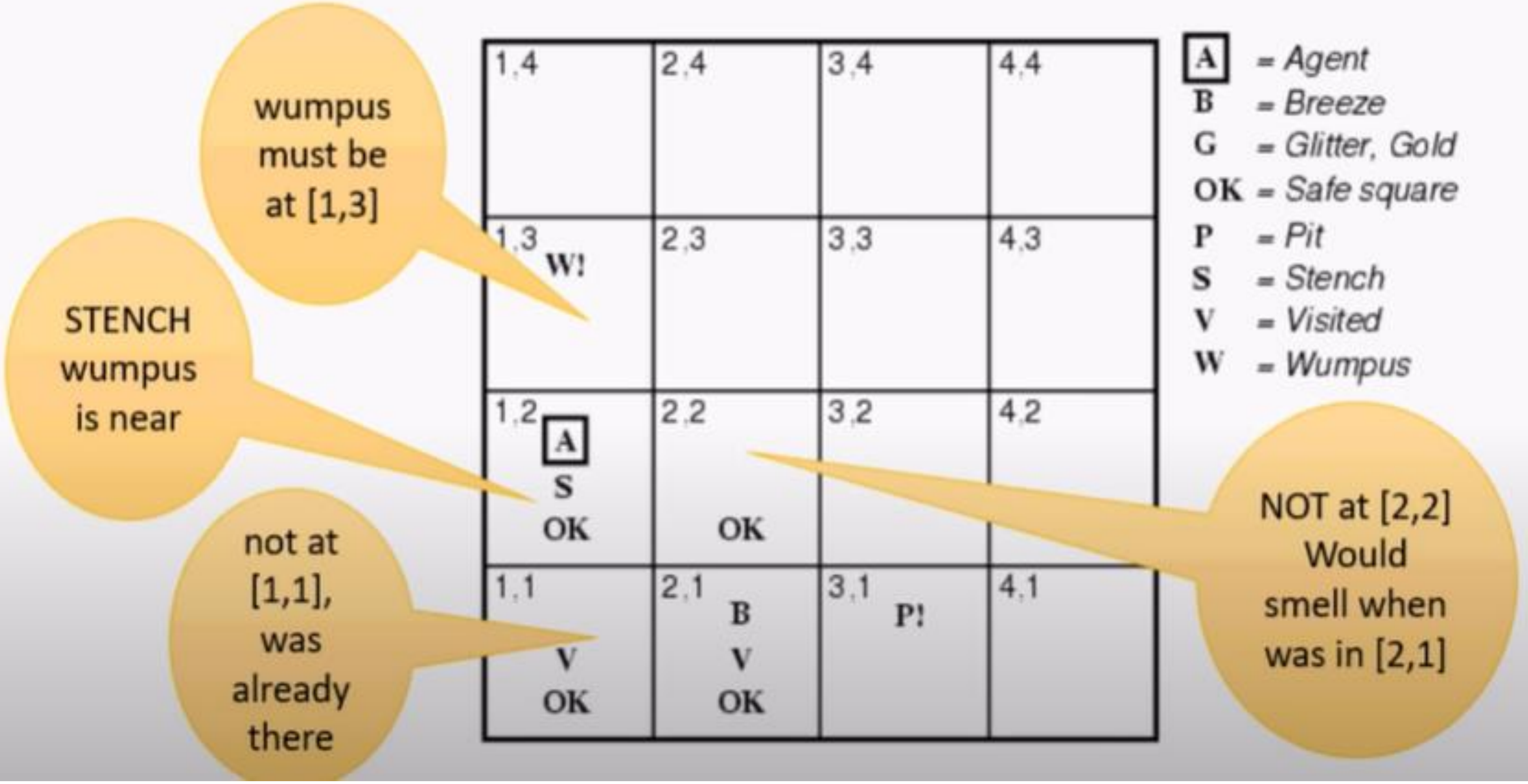
Given a conjunction $P \wedge Q$, you can conclude either P or Q individually.

Symbolically:

$$\frac{P \wedge Q}{P} \quad \text{or} \quad \frac{P \wedge Q}{Q}$$

This rule is based on the idea that if both P and Q are true in a conjunction, you can assert the truth of either P or Q separately.

After the third move



Rules and Atomic Propositions

- Some Atomic Propositions

  S12 = There is a stench in cell (1,2)

  B34 = There is a breeze in cell (3,4)

  W22 = Wumpus is in cell (2,2)

  V11 = We've visited cell (1,1)

  OK11 = Cell (1,1) is safe   etc.

- Some rules

  (R1) $\neg S11 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W21$

  (R2) $\neg S21 \rightarrow \neg W11 \wedge \neg W21 \wedge \neg W22 \wedge \neg W31$

  (R3) $\neg S12 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W22 \wedge \neg W13$

  (R4) $S12 \rightarrow W13 \vee W12 \vee W22 \vee W11$ etc.

- Note that the lack of variables requires us to give similar rules for each cell.

Prove that the Wumpus is in (1,3)

Apply MP with $\neg S11$ and R1:

$\neg W11 \land \neg W12 \land \neg W21$

(R1) $\neg S11 \rightarrow \neg W11 \land \neg W12 \land \neg W21$

$\neg S11$

$\neg W11 \land \neg W12 \land \neg W21$

Apply And-Elimination to this we get 3 sentences:

$\neg W11, \neg W12, \neg W21$

Prove that the Wumpus is in (1,3)

Apply MP to $\neg$ S21 and R2, then apply And-elimination:

$\neg W22, \neg W21, \neg W31$

(R2) $\neg S21 \rightarrow \neg W11 \land \neg W21 \land \neg W22 \land \neg W31$

$\neg S21$

$\neg W22 , \neg W21 , \neg W31$

Prove that the Wumpus is in (1,3)

Apply MP to S12 and R4 to obtain:

W13 ∨ W12 ∨ W22 ∨ W11

(R4) S12 → W13 V W12 V W22 V W11

S12

(W13 V W12 V W22 V W11)

Prove that the Wumpus is in (1,3)

Apply Unit resolution on $(W13 \lor W12 \lor W22 \lor W11)$ and $\neg W11$:

$W13 \lor W12 \lor W22$

$$\begin{tikzcd}
\boxed{W13 \lor W12 \lor W22 \lor W11} \arrow[dr] & & \boxed{\neg W11} \arrow[dl] \\
& \boxed{(W13 \lor W12 \lor W22)} &
\end{tikzcd}$$

Prove that the Wumpus is in (1,3)

Apply Unit Resolution with (W13 ∨ W12 ∨ W22) and ¬W22:

W13 ∨ W12

$$
\begin{tikzcd}
\boxed{W13 \vee W12 \vee W22} \arrow[dr] & & \boxed{\neg W22} \arrow[dl] \\
& \boxed{(W13 \vee W12)} &
\end{tikzcd}
$$

Prove that the Wumpus is in (1,3)

Apply Unit Resolution with ($W13 \lor W12$) and $\neg W12$:

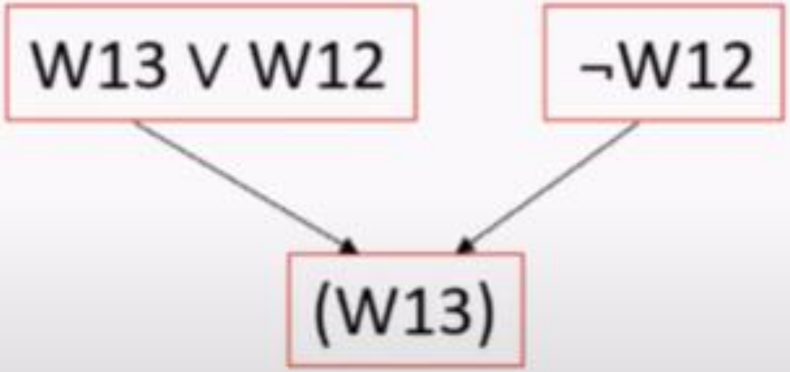
W13

$$
\begin{tikzcd}
\boxed{W13 \lor W12} \arrow[dr] & & \boxed{\neg W12} \arrow[dl] \\
& \boxed{(W13)} &
\end{tikzcd}
$$

Prove that the Wumpus is in (1,3)

Apply Unit Resolution with (W13 ∨ W12) and ¬W12:

W13



Proved