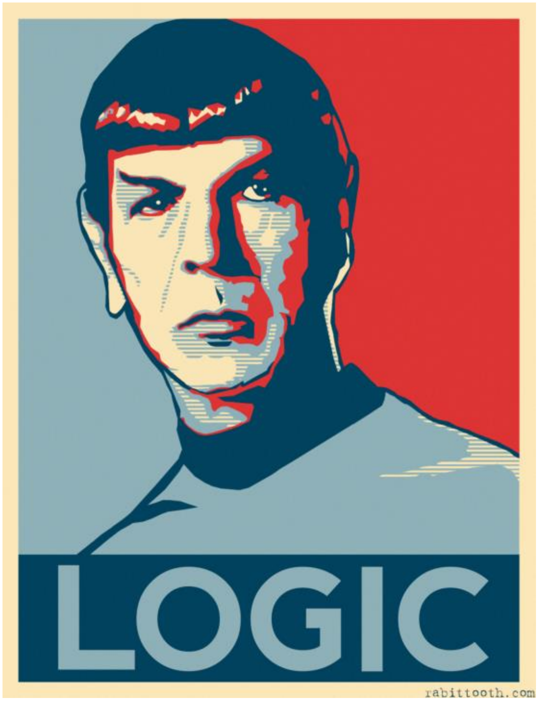
Propositional Logic

Entailment and Inference



Computational Inference

- computers cannot reason informally (“common sense”)
  - they don’t know the interpretation of the sentences
  - they usually don’t have access to the state of the real world to check the correspondence between sentences and facts
- computers can be used to check the validity of sentences
  - “if the sentences in a knowledge base are true, then the sentence under consideration must be true, regardless of its possible interpretations”
  - can be applied to rather complex sentences

Computational Approaches to Inference

- model checking based on truth tables
  - generate all possible models and check them for validity or satisfiability
  - exponential complexity,
    - all combinations of truth values need to be considered
- search
  - use inference rules as successor functions for a search algorithm
  - also exponential, but only worst-case
    - in practice, many problems have shorter proofs
    - only relevant propositions need to be considered

Propositional Logic

- a relatively simple framework for reasoning
- can be extended for more expressiveness at the cost of computational overhead
- important aspects
  - syntax
  - semantics
  - validity and inference
  - models
  - inference rules
  - complexity

Propositional Logic

Propositional logic is the simplest logic—illustrates basic ideas

The proposition symbols P_1, P_2 etc are sentences

If S is a sentence, $\neg S$ is a sentence (negation)

If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)

If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)

If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)

If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Order of Precedence

If and Only If

If S1 then S2

Propositional Logic: Semantics

Each model specifies true/false for each proposition symbol

E.g.	$P_{1,2}$	$P_{2,2}$	$P_{3,1}$
	$true$	$true$	$false$

If S1 is true, then I am claiming that S2 is true

(With these symbols, 8 possible models, can be enumerated automatically.)

Rules for evaluating truth with respect to a model m:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true **and**	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true **or**	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false **or**	S_2	is true
i.e.,	is false iff	S_1	is true **and**	S_2	is false
$S_1 \Leftrightarrow S_2$	is true iff	$S_1 \Rightarrow S_2$	is true **and**	$S_2 \Rightarrow S_1$	is true

Simple recursive process evaluates an arbitrary sentence, e.g.,

$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = true \wedge (false \vee true) = true \wedge true = true$

Models

- To make things precise, we use the term **model** in place of "possible world."
- Logicians typically think in terms of **models**, which are formally **structured worlds**
  - With respect to which **truth** can be evaluated
- We say $\boldsymbol{m}$ is a model of a sentence $\boldsymbol{\alpha}$ if $\boldsymbol{\alpha}$ is true in $\boldsymbol{m}$
  - or sometimes m satisfies $\boldsymbol{\alpha}$.
- $M(\boldsymbol{\alpha})$: Set of all models of $\boldsymbol{\alpha}$

Entailment

- Entailment: a sentence *follows logically* from another sentence
- Entailment means that one thing follows from another:

$$\alpha \models \beta$$

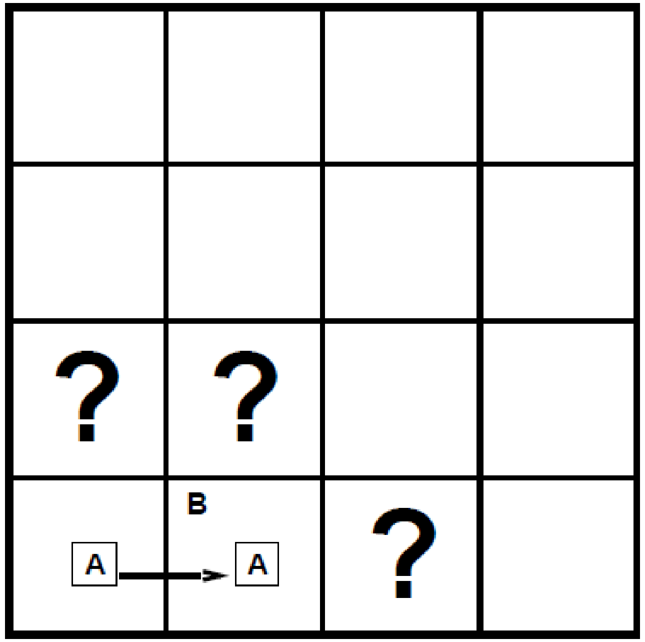
- Knowledge base *KB* entails sentence α if and only if α is true β is also true
  - E.g., x+y = 4 entails 4 = x+y
  - Entailment is a relationship between sentences (i.e., syntax) that is based on **semantics**

Entailment

- Entailment is when a sentence follows from another $\alpha \vDash \beta$
  - $\alpha \vDash \beta$ iff in every model where α is true, β is also true.
  - $\alpha \vDash \beta$ iff $M(\alpha) \subseteq M(\beta)$

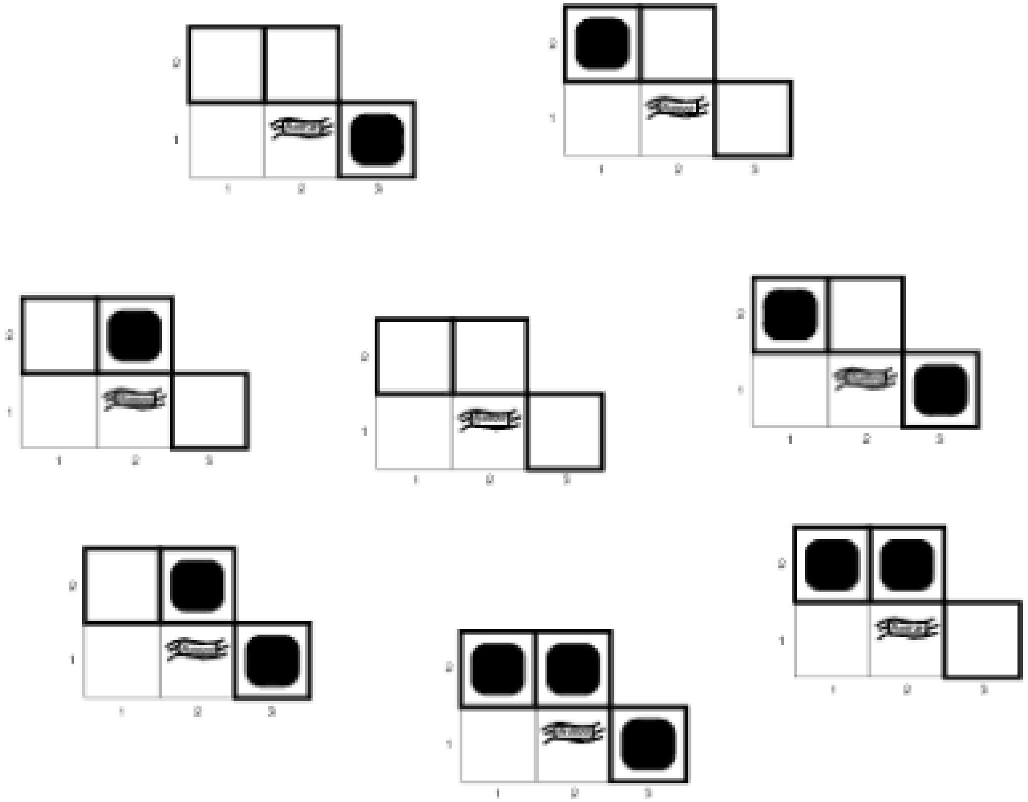
- Examples:
  - $(x = 0) \vDash (xy = 0)$
  - $(p = TRUE) \vDash (p \vee q)$
  - $(p \wedge q) \vDash (p \vee q)$
  - $\big((q \Rightarrow p) \vee r\big) \vDash (q \Rightarrow p)$

Entailment in the Wumpus World



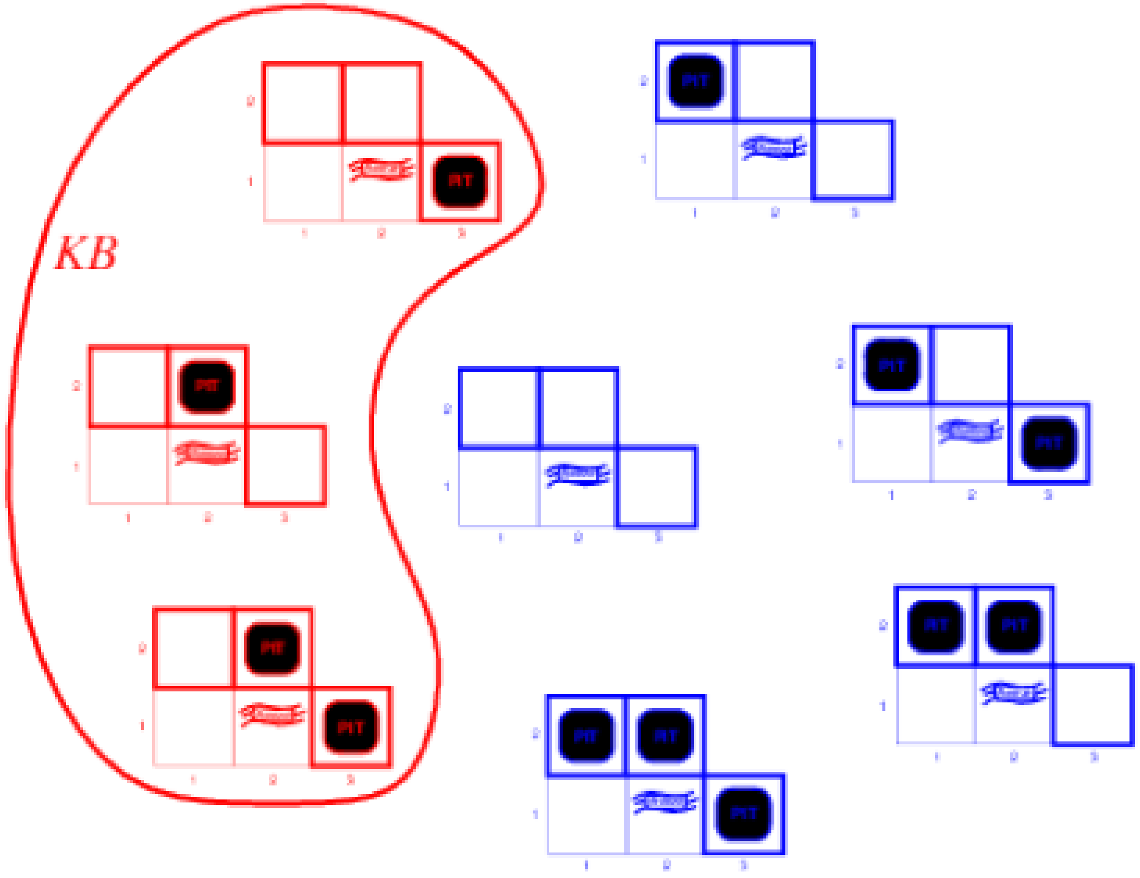
- Situation after detecting nothing in [1,1], **moving right**, breeze in [2,1]
- Consider possible models for ?s, assuming only pits
- 3 Boolean choices, i.e.,
  - $2^3 = 8$ possible models

Possible Wumpus Models



Valid Wumpus Models

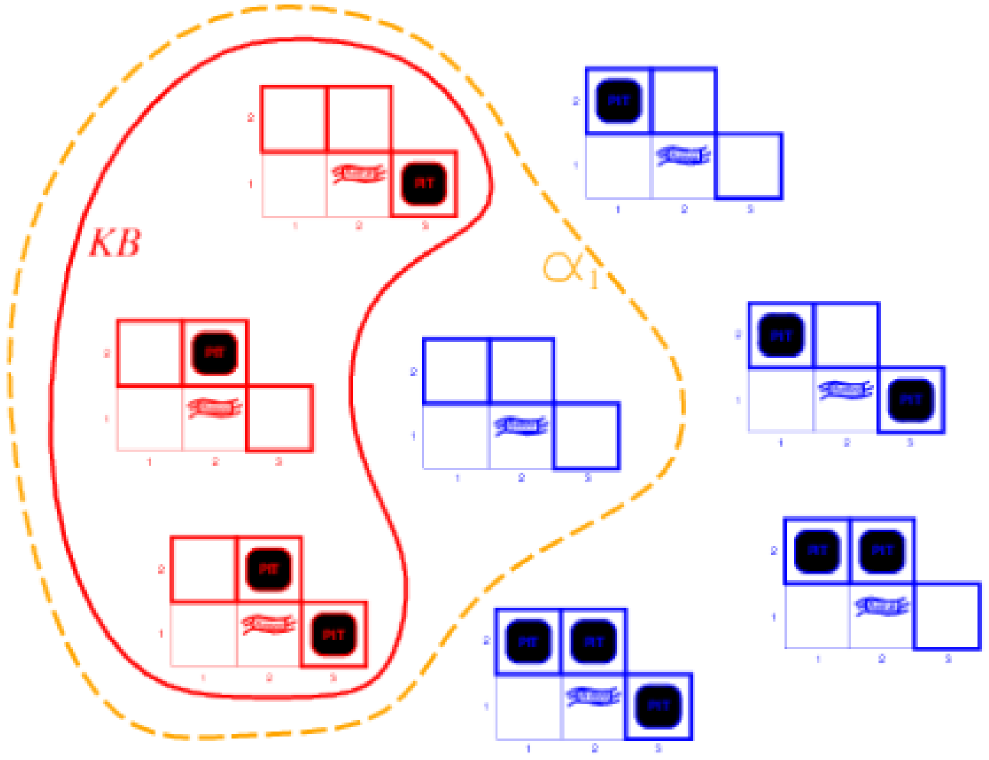
KB = wumpus-world rules + observations



Entailment

KB = wumpus-world rules + observations

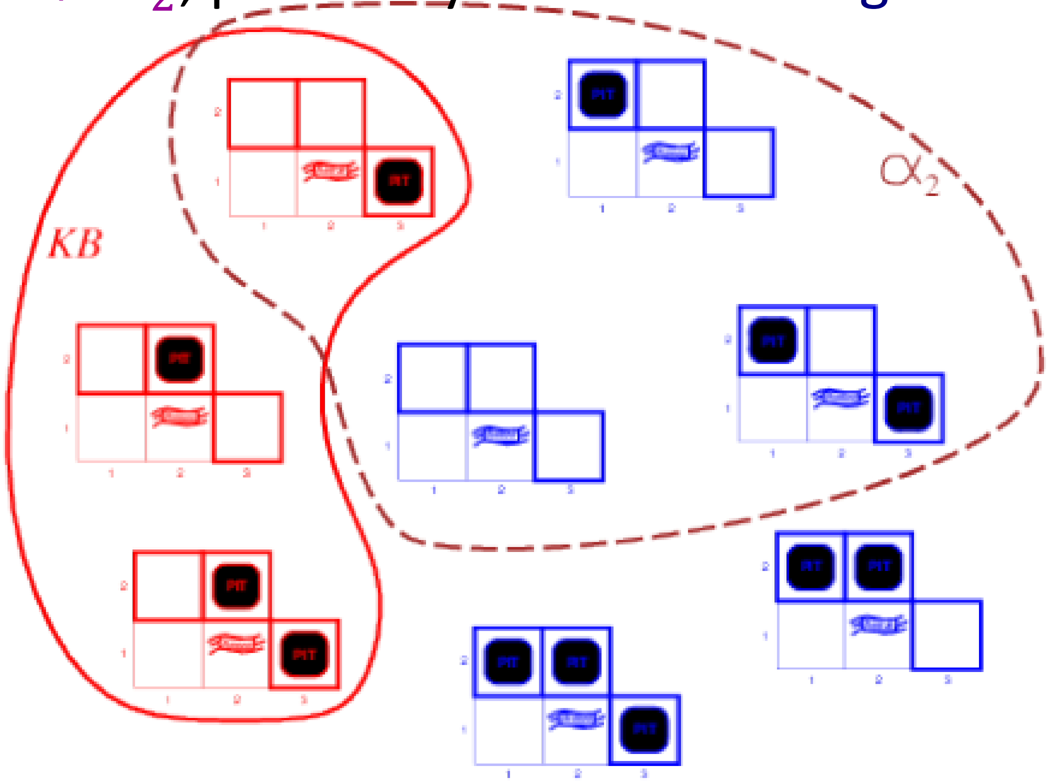
α_1 = “[1,2] is safe”, $KB \vdash \alpha_1$, proved by model checking



Entailment

KB = wumpus-world rules + observations

α_2 = "[2,2] is safe", $KB \not\models \alpha_2$, proved by model checking



Wumpus Models

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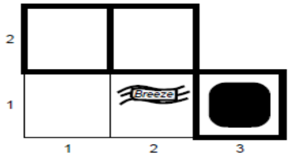

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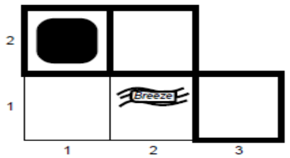

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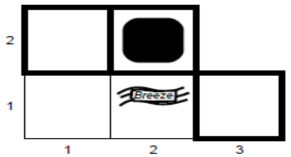

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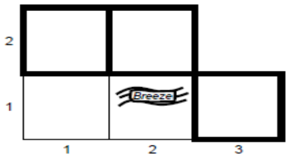

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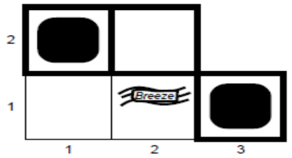
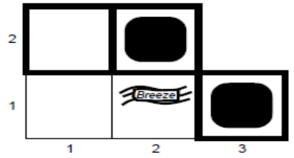


1



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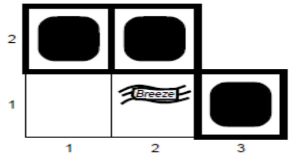

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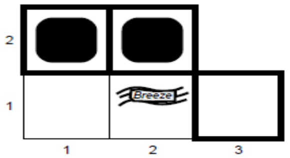
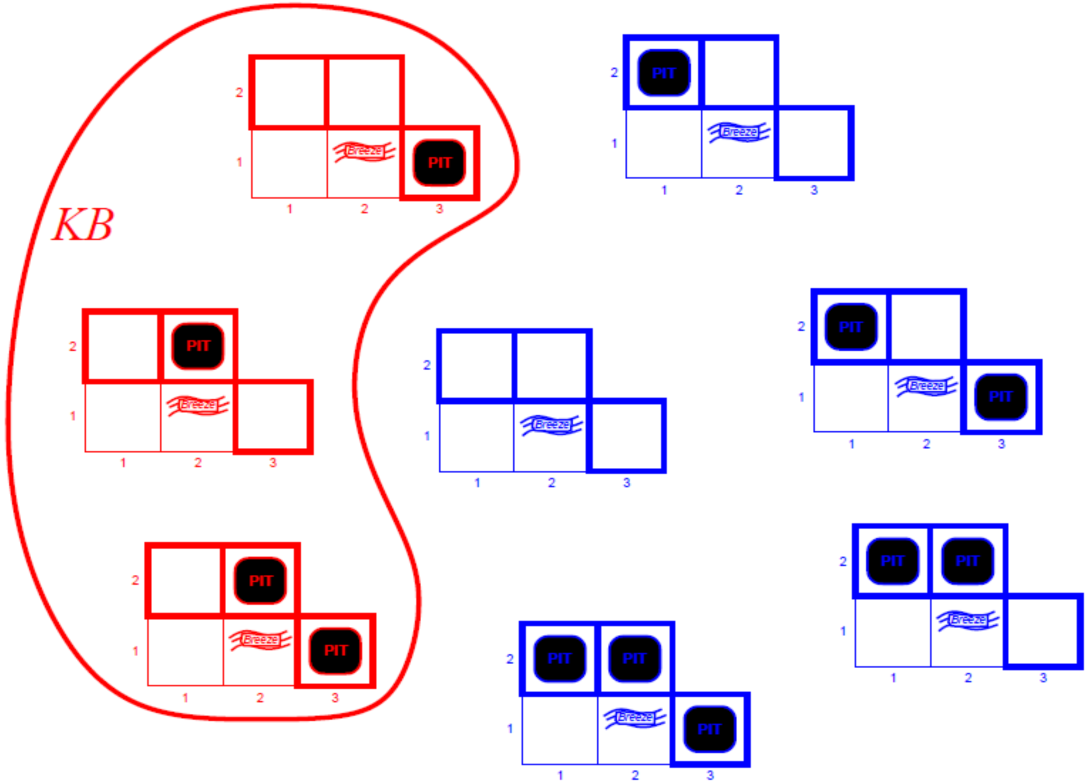


Columns, Rows

	1,1	**2,1**	**3,1**	**1,2**	**2,2**
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

A simple entailment procedure

KB = Wumpus World Rules
+
Observations



Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
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5	Empty	Breeze	Empty	Empty	Empty
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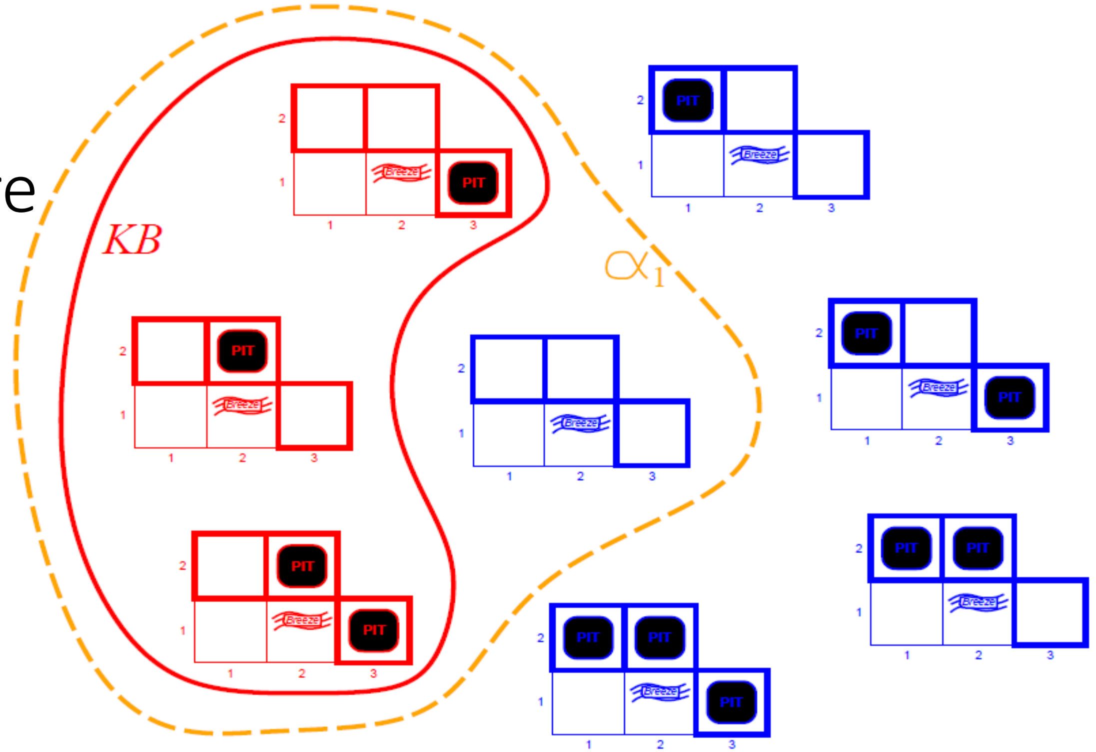
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

KB

A simple entailment procedure



KB = Wumpus World Rules
+
Observations

α_1 = "No pit in (1,2)"

Columns, Rows **$KB \models \alpha_1$**

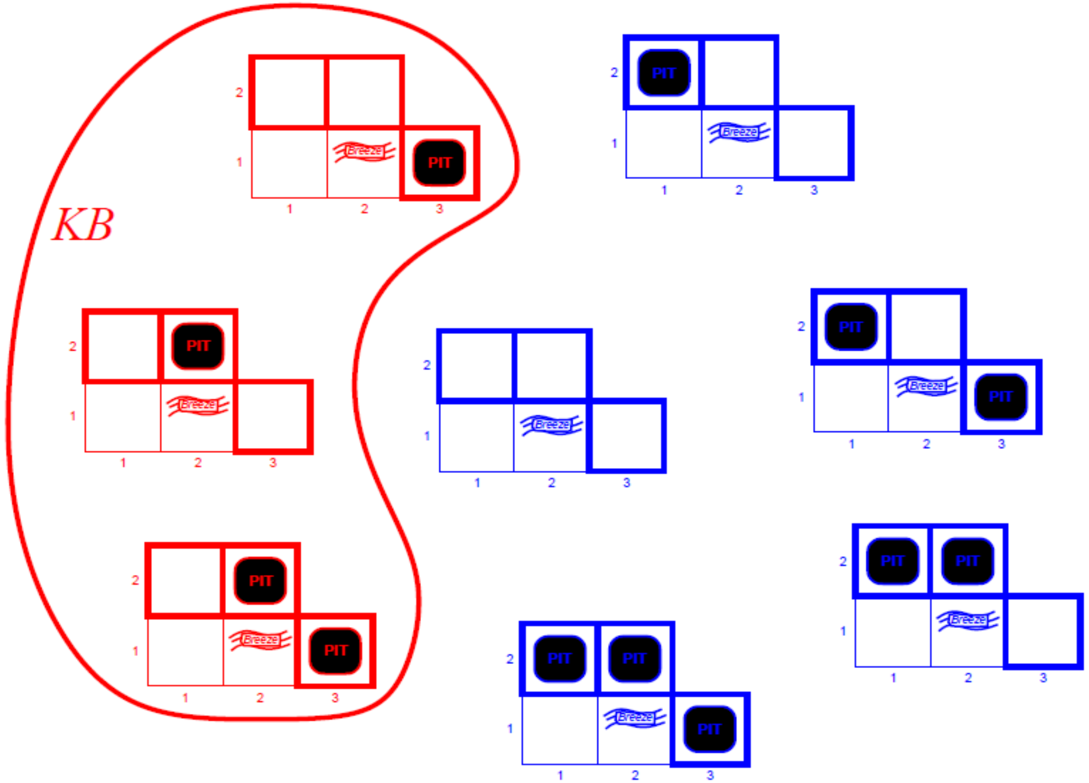
	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

α_1

KB

A simple entailment procedure

KB = Wumpus World Rules
+
Observations

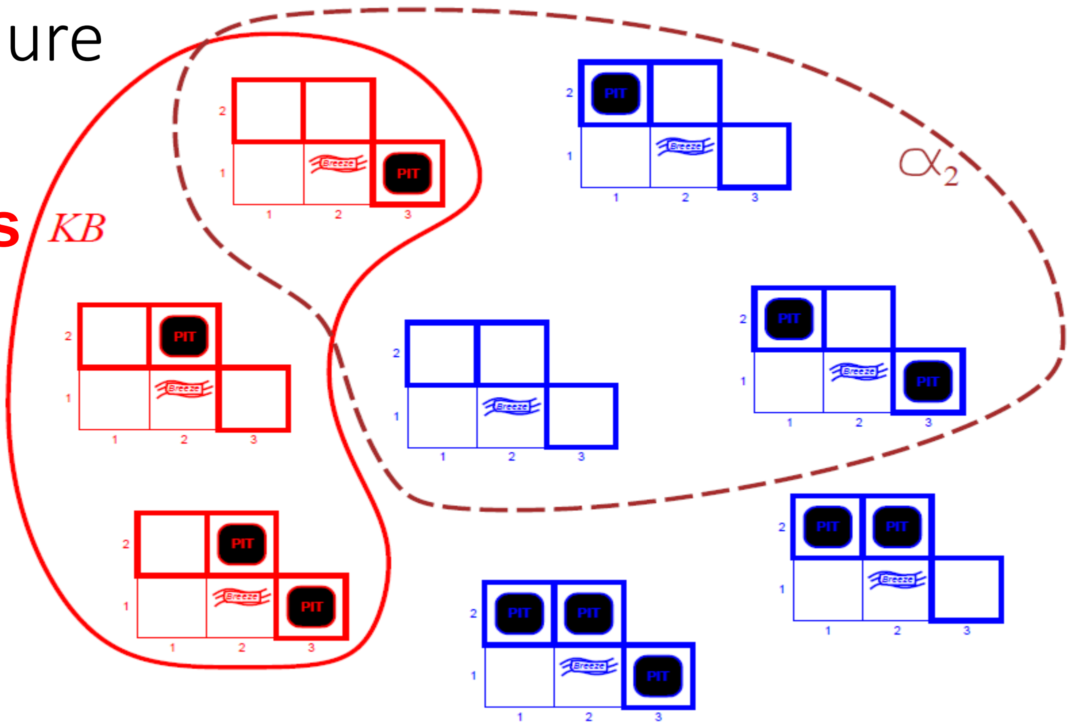


Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

KB

A simple entailment procedure



KB = Wumpus World Rules

+

Observations

α_2 = "No pit in (2,2)"

KB $\not\models$ α_2

Columns, Rows

	[1,1]	[2,1]	[3,1]	[1,2]	[2,2]
1	Empty	Breeze	Pit	Empty	Pit
2	Empty	Breeze	Pit	Pit	Pit
3	Empty	Breeze	Empty	Pit	Pit
4	Empty	Breeze	Empty	Empty	Pit
5	Empty	Breeze	Empty	Empty	Empty
6	Empty	Breeze	Pit	Pit	Empty
7	Empty	Breeze	Pit	Empty	Empty
8	Empty	Breeze	Empty	Pit	Empty

α_2

KB

Logical inference problem

- Logical inference problem:
  - Given: – a knowledge base KB (a set of sentences) and
  - a sentence α (called a theorem),
- **Does a KB semantically entail α**? $KB \vDash \alpha$
  - In other words: In all interpretations in which sentences in the KB are true, is also α true?
- **Question**: Is there a procedure (program) that can decide this problem in a finite number of steps?
- **Answer**: Yes. Logical inference problem for the propositional logic is decidable.

Solving logical inference problem

- In the following: How to design the procedure that answers:
$$KB \vDash \alpha$$
- Three approaches:
  - Truth-table approach
  - Inference rules