Cavendish Experiment: Measuring Gravitational Constant with Torsional Pendulum.

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Henry Cavendish designed and performed the first experiment to accurately measure gravitational constant G. In this report, the experiment was reproduced and an accurate measurement (within 0.6 %) of G ($G = 6.71 \times 10^{-11}$ m^2/(kg s^2)) was obtained.

INTRODUCTION

Measuring a gravitational constant - an important task that played a major role in advancing physics through observations of objects dominated by gravitational forces. The first accurate measurements of the gravitational constant performed by Henry Cavendish in an experiment similar to the one described here allowed to determine the mass and density of the Earth. Akin to Coulumb's constant in the case of electric charges, the gravitational constant G relates the attractive force between two massive objects to their masses m and M, and the distance b between their centers of mass, thought equation (1):

$$F = G\frac{Mm}{b^2} \tag{1}$$

The experiment employs a delicate torsional balance in a configuration shown in Figure 1. The two small masses m are offset from a gravitational equilibrium with two large masses M, and then oscillate due to the gravitational force between each of the small masses m and the corresponding large masses M, and the torsional force from the suspension thread. Assuming

1. the oscillations are small, i.e. the distance between the canters of the spheres do not change;
2. the mass of the dumbbell torsional balance is fully contained in the masses m;
3. the torsion follows Hook's law precisely

the angle of rotation of the balance follows equation (2) (per the 2$^{\text{nd}}$ Newton's Law):

$$I\ddot{\theta} = -\beta\dot{\theta} - \kappa\theta + 2Fb \tag{2}$$

The coefficients of in the equation above are determined by the physical properties of the system; $I = 2m(d^2 + 2r^2/5)$ - moment of inertial of the balance; β - damping constant, κ - rotational Hook's constant, d - half a length of the dumbbell "handle", and r - radius of the small spheres m.

In case of under-damped oscillations, angular position of the balance is given by equation (3):

$$\theta(t) = \theta_0 + Ae^{-\lambda t}\cos(\omega t + \phi) \tag{3}$$

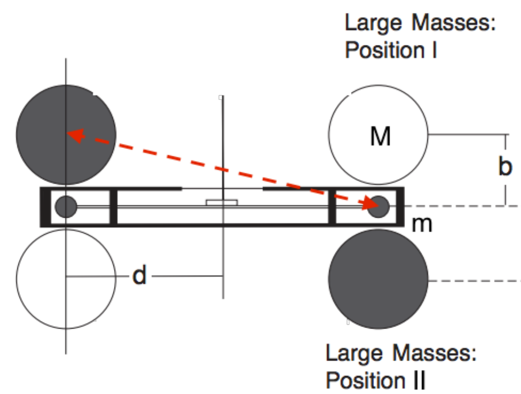

FIG. 1. A schematic drawing of the apparatus, view from above.

where θ_0 is the offset of the equilibrium position under the action of the gravitational force from $\theta = 0$ - the position of zero torsional force; $Ae^{-\lambda t}$ is the damped amplitude, $\lambda = q\omega_0$, $\omega = \omega_0\sqrt{1-q^2}$ - angular frequency, and ϕ - phase offset. Natural frequency of oscillation (i.e. in the absence of damping force) is the given by ω_0 in equation (4):

$$\omega_0 = \sqrt{\kappa/I} = 2\pi/T \tag{4}$$

At the equilibrium position (i.e. at $\theta = \theta_0$), the torsional force is equal in magnitude to that of gravitational attraction between the spheres: $F = \kappa\theta_0$. Thus, the gravitational constant is given by equation (5):

$$G = \frac{\kappa\theta_0 b^2}{2Mmd} \tag{5}$$

METHODOLOGY

To perform the experiment, a commercial gravitational torsion balance made by PASCO® scientific was used.

The schematic drawing of the pendulum is presented in Figure 1. The parameters of the system with uncertainties are presented in Table I. The setup consists of a dumbbell-shaped balance with masses m on each end, suspended from a Beryllium-Copper cord inside a box. A swivel support is fide along the exis of the torsional cord outside of the box, and two masses M can be laced on each end; the arms of the swivel support are of equal lengths.

TABLE I. Parameters of the experimental setup given by the manufacturer.

name	value	uncertainty	units
b	46.5×10^{-3}	0.5×10^{-3}	m
d	50.0×10^{-3}	-	m
r	9.53×10^{-3}	-	m
M	1.500	0.010	kg
m	0.0383	0.0002	kg

The torsional cord is equipped with a mirror which reflects a laser pointer light onto a wall a distance $L = 2.185 \pm 0.05$ m away from the mirror. At the beginning of the experiment ($t \leq 0$), the large masses were in position 2. Then, at $t = 0$, the large masses were set into position 1, and the balance started oscillating around an equilibrium position θ_1. After several complete periods, the large masses were returned back into position 2, which changed the equilibrium position for the balance to θ_2.

The oscillations of the balance were detected by the motion of the laser dot on the screen - the dot was moving along a horizontally placed meter stick oscillating between $x_l = 48.7 \pm 0.5$ cm (leftmost position during the experiment) and $x_r = 62.8 \pm 0.5$ m (rightmost position). Since the dot was produced by the reflected light, the total angular swing of the equilibrium position of the balance was $\Delta S/(2L)$, where $\Delta S = x_r - x_l$. Hence, experimentally the angular displacement of the equilibrium θ due to the gravitational force between the spheres is:

$$\theta = \frac{\Delta S}{4L} \tag{6}$$

Thus, combining experimental results from (4), (5), and (6), the gravitational constant can be expressed by equation (7):

$$G = \frac{b^2}{2Mmd}\,\frac{\omega_0^2 I\,\Delta S}{4L} = \frac{b^2\omega_0^2\,\Delta S}{4LMd}\left(d^2 + \frac{5}{2}\,r^2\right) \tag{7}$$

Additional correction was introduced into the calculation of G due to the proximity of both heavy spheres M to the balance and the resulting gravitational attraction between non-neighboring spheres. With the correction, the gravitational constant measured with the apparatus

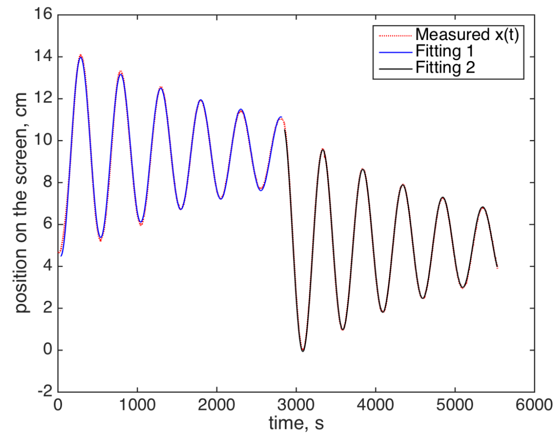

FIG. 2. Position of the dot as a function of time measured in the experiment; two fittings of the form (3) are shown for each equilibration interval.

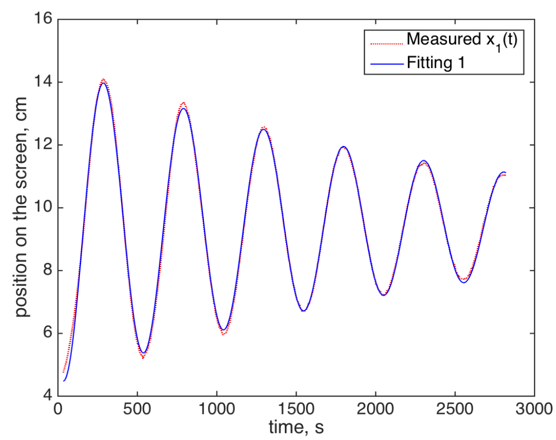

FIG. 3. Position of the dot as a function of time during the first equilibration interval.

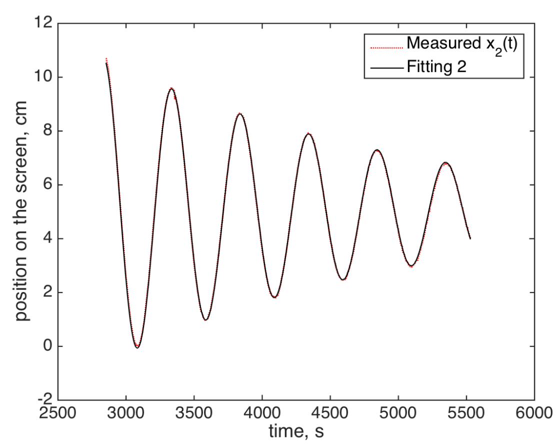

FIG. 4. Position of the dot as a function of time during the second equilibration interval.

used is:

$$G = \frac{b^2\omega_0^2\,\Delta S}{4LMd}\left(d^2 + \frac{5}{2}r^2\right)\frac{1}{1 - \frac{b^3}{(b^2+4d^2)^{3/2}}} \tag{8}$$

From the above equation, ΔS and ω_0, as well as their uncertainties, were obtained by fitting a model given in equation (3) to the horizontal position of the laser dot extracted from the video for two time interval corresponding to two equilibration . The `fit` function in MATLAB yielded estimates for x_1, x_2, ω and q; using the estimates, the quantities ΔS and ω_0 were computed.

TABLE II. Parameters obtained from fitting the position curves to the model in (3)

name	eq. int. 1	eq. int. 2
eq. pos., x	9.46 ± 0.01 cm	5.03 ± 0.01 cm
Amp.	5.05 ± 0.032 cm	20.8 ± 0.02 cm
λ	$(3.93 \pm 0.05) \times 10^{-4}$ t^{-1}	$(4.57 \pm 0.02) \times 10^{-4}$ t^{-1}
ω	0.0125 ± 0.0001 rad/s	0.0125 rad/s
ϕ	2.69 ± 0.01 rad	2.30 ± 0.01 rad
ω_0	0.0125 ± 0.0001 rad/s	0.0125 ± 0.0001 rad/s

RESULTS AND DISCUSSION

Gravitational Constant

The horizontal position $x(t)$ of the laser light during the experiment is presented in Figure 2 (red dots); two fitted models (eq. (3)) are shown on the plot as well: blue line for the first equilibration interval, and black line - for the second. The estimates for the parameters from he model in (3) with the uncertainties are given in Table II

As readily observed from the error in fitting (taken at 95% confidence intervals), the accuracy of the fitting is high. In addition, this is demonstrated in Figures 3 and 4 that closer show the fitted models from Figure 2. The model fit the data closely.

The gravitational constant obtained using the values in Tables I and II is

$$G_e = (6.71 \pm 0.38) \times 10^{-11}\ \mathrm{m^3/(kg\ s^2)} \tag{9}$$

The uncertainty results from the propagated uncertainties in the parameters of experimental hardware, measurements, and fittings done in the experiment. The fractional uncertainty of of the experimental G_e is 0.057, or approximately 5.7%. The accepted value $G = 6.67 \times 10^{-11}$ m^3/(kg s^2) is within the experimental uncertainty, and the error in G_e is under 0.6%.

Error Analysis

The uncertainty was evaluated by the matlab. The uncertainty of hardware dimensions from the PASCO® manual; the uncertainty on the fitted parameter was taken as the 95 % confidence interval obtained from the fitting function. The uncertainty of the measured parameters (L, x_l, and x_r) were incorporated into the calculation of the final uncertainty throught the matlab function as well.

The resulting uncertainty of approximately 5% corresponds to the once stated by the manufacturer serving as an evidence of a proper colculation procedure.

CONCLUSION

The measurement of the gravitational constant G was performed; resulting in an estimate:

$$G_e = (6.71 \pm 0.38) \times 10^{-11}\ \mathrm{m^3/(kg\ s^2)}$$

The obtained result is accurate as the error (0.6 %) is less that the uncertainty (5.7%) of the measurement, and the 'true' G lies within the uncertainty bounds. Hence, an accurate measurement of the gravitational constant G can be accomplished with a non-sophisticated equipment to a significant degree of accuracy, which is highly facilitated by the use of software.