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UNIT 3: NUMERICAL INTEGRATION

Mathematics for Modern Computing

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1 Numerical Integration

1.1 Introduction

Given a set of tabulated values of the integrand $f(x)$, to determine the value of integral

$$\int_{x_o}^{x_n} f(x)\,\mathrm{d}x$$

is called Numerical Integration. We divide the given interval of integration into a large number of sub intervals of equal width h and replace the function tabulated at the points of subdivision by any one of the interpolating polynomials and evaluate the integration.
In this topic we are studying following formula for numerical integration.

i) Trapezoidal rule

ii) Simpson's one-third rule

iii) Simpson's three-eighth rule

iv) Weddles Rule

v) Romberg's formula

1.2 Newton Cote's Quadrature Formula

Let

$$I = \int_a^b y\,\mathrm{d}x$$

where y takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$. Let the interval of integration $(a,\ b)$ be divided into n equal sub-intervals, each of width $h = \dfrac{b-a}{n}$. So that $x_0 = a,\ x_1 = x_0 + h, \ldots, x_n = x_0 + nh = b$

$$\mathrm{I} = \int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x$$

Since any x is given by $x = x_0 + rh$ and $\mathrm{d}x = h\,\mathrm{d}r$

$$\begin{aligned}
\mathrm{I} &= h\int_0^n f(x_0+rh)\,\mathrm{d}r \\
&= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots\right]\mathrm{d}r \\
&= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \cdots\right]_0^n \\
\mathrm{I} &= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]
\end{aligned}$$

This is general quadrature formula and is known as **Newton Cote's Quadrature Formula**. A number of important deduction viz. Trapezoidal rule, Simpson's one-third rule, Simpson's three-eighth rule and Wedddles's rule can be immediately deduced by putting $n = 1, 2, 3$ and 6 respectively.

1.3 Trapezoidal Rule(n=1)

Putting $n = 1$ Newton Cote's quadrature formula and taking the curve through $(x_0,\ y_0)$ and $(x_1,\ y_1)$ as a polynomial of degree one so that differences of order higher than one vanishes, we get

$$\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = \frac{h}{2}\left[2y_0 + (y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

Similarly, for the next sub-interval $(x_0 + h,\ x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)\,\mathrm{d}x = \frac{h}{2}(y_1 + y_2)$$

Continue ...

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{I = \int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]}$$

which is known as Trapezoidal rule. By increasing the number of sub intervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

1.4 Simpsons One-Third Rule (n=2)

Putting $n = 2$ Newton-Cote's quadrature formula and taking the curve through $(x_0,\ y_0)$, $(x_1,\ y_1)$ and $(x_2,\ y_2)$ as a polynomial of degree two so that differences of order higher than two vanishes, we get

$$\begin{aligned}
\int_{x_0}^{x_0+2h} f(x)\,\mathrm{d}x &= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] \\
&= \frac{2h}{6}\left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)\right] \\
&= \frac{h}{3}[y_0 + 4y_1 + y_2]
\end{aligned}$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,\mathrm{d}x = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

Continue . . .

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]}$$

which is known as Simpsons One-Third rule rule.
Remark: While using this formula, the given interval of integration must

be divided into an even number of sub-intervals, since we find the area over two sub-interval at a time.

1.5 Simpson Three-Eighth Rule (n=3)

Putting $n = 3$ Newton-Cote's quadrature formula and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) as a polynomial of degree two so that differences of order higher than three vanishes, we get

$$\begin{aligned}
\int_{x_0}^{x_0+3h} f(x)\,\mathrm{d}x &= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right] \\
&= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right] \\
&= \frac{3h}{8}[y_0 + 3y_1 + y_2 + y_3]
\end{aligned}$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,\mathrm{d}x = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

Continue . . .

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\begin{aligned}
\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}&[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) \\
&+ 2(y_3 + y_6 + \cdots + y_{n-3})]
\end{aligned}$$

which is known as Simpsons Three-Eighth rule rule.

Remark: While using this formula, the given interval of integration must be divided into sub-intervals whose number n is multiple of 3.

1.6 Examples

Example 1 *Use Trapezoidal rule to evaluate* $\int_0^1 x^3 \, dx$ *considering five sub-intervals.*

Solution: Dividing the interval (0, 1) into 5 equal parts, each of width $h = \frac{1-0}{5} = 0.2$, the value of $f(x) = x^3$ are given below:

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	0	0.008	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule

$$\begin{aligned}\int_0^1 x^3 \, dx &= \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)] \\ &= \frac{02}{2}[(0+1) + 2(0.008 + 0.064 + 0.216 + 0.512)] \\ &= 0.26\end{aligned}$$

$$\boxed{\int_0^1 x^3 \, dx = 0.26}$$

Example 2 *evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

i) Simpsons $\frac{1}{3}^{rd}$ *rule taking* $h = \frac{1}{4}$

ii) Simpsons $\frac{3}{8}^{th}$ *rule taking* $h = \frac{1}{6}$

Hence compute an approximate value of π *in each case.*

Solution:

i) the value of $f(x) = \frac{1}{1+x^2}$ at $x = 0, 1/4, 2/4, 3/4, 1$ are given below:

x	0	1/4	1/2	3/4	1
$f(x)$	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's One-Third rule

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2] \\ &= \frac{1}{12}[(1 + 0.5) + 490.9412 + 0.64) + 290.8)] \\ &= 0.7854\end{aligned}$$

Also $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$

Hence

$$\frac{\pi}{4} = 0.7854 \quad \Rightarrow \pi = 3.1416 \text{ approximately}$$

ii) The value of $f(x) = \frac{1}{1+x^2}$ at $x = 0, 1/6, 2/6, 3/6, 4/6, 5/6, 1$ are given below:

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1.0000	0.9730	0.9000	0.8000	0.6923	0.5902	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's three-eighth rule

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right] \\
&= \frac{3}{48}[(1+0.5)+3(0.9730+0.9+0.6923+0.5902)+2(0.8)] \\
&= 0.7854
\end{aligned}$$

Also $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$, so $\frac{\pi}{4} = 0.7854 \Rightarrow \pi = 3.1416$ approximately

Example 3 *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using*

i) *Simpson's one-third rule*

ii) *Simpson's three-eighth rule*

iii) *Trapezoidal rule*

Solution: Divide the interval (0, 6) into six parts of width $h = 1$. The value of $f(x) = \frac{1}{1+x^2}$ are given below:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.0270
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By Simpson's one-third rule

$$\begin{aligned}
\int_0^6 \frac{dx}{1+x^2} &= \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)] \\
&= \frac{1}{3}[(1+0.0270)+4(0.5+0.1+0.0385)+2(0.2+0.0588)] \\
&= 1.3571
\end{aligned}$$

ii) By Simpson's three-eighth rule

$$\begin{aligned}\int_0^6 \frac{dx}{1+x^2} &= \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]\\ &= \frac{3}{8}[(1+0.0270)+3(0.5+0.2+0.0588+0.0385)+2*0.1)]\\ &= 1.3571\end{aligned}$$

iii) By Trapezoidal rule

$$\begin{aligned}\int_0^6 \frac{dx}{1+x^2} &= \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]\\ &= \frac{1}{2}[(1+0.0270)+2(0.5+0.2+0.1+0.0588+0.0385)]\\ &= 1.4108\end{aligned}$$

Example 4 *The speed, v meters per second of a car t seconds after it starts, is shown in the following table:*

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	5.40	4.50	5.40	9.00
	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}

Using Simpson's rule, find the distance traveled by the car in 2 minutes.

Solution: If s meter is the distance covered in t seconds, then

$$\frac{ds}{dt} = v \quad \Rightarrow [s]_0^{120} = \int_0^{120} v\,dt$$

Since the number of sub-intervals is 10 (even). Hence by using Simpson's one-third rule

$$\begin{aligned}\int_0^{120} v\,dt &= \frac{h}{3}[(v_0+v_{10})+4(v_1+v_3+v_5+v_7+v_9)+2(v_2+v_4+v_6+v_8)]\\ &= \frac{12}{3}[(0+9)+4(3.6+18.9+18.547+5.4+5.4)+2(10.08+21.60+10.26+4.50)]\\ &= 1236.9 meters\end{aligned}$$

Hence the distance traveled by car in 2 min. is 1236.9 meters.

Example 5 *Evaluate $\int_0^1 \frac{dx}{1+x}$ dividing the interval of integration into 8 equal parts. Hence find $\log_e 2$ approximately.*

Solution: Since the number of integration is divided into an even number of sub interval, we shall use Simpson's one-third rule. Let $f(x) = \dfrac{1}{1+x}$

x	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
y	1.0000	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's one-third rule

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{1+x} &= \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)] \\ &= \frac{1}{24}[91+0.5)+4(0.8889+0.7273+0.6154+0.5333)+2(0.5+0.6667+0.5714)] \\ &= 0.693154\end{aligned}$$

Since

$$\int_0^1 \frac{\mathrm{d}x}{1+x} = \log(1+x)]_0^1 = \log 2$$

Thus

$$\log_e 2 = 0.693154$$

Example 6 *Find from the following table, the area bounded by the curve and the X-axis from x=7.47 to x=7.52*

x	*7.47*	*7.48*	*7.49*	*7.50*	*7.51*	*7.52*
$f(x)$	*1.93*	*1.95*	*1.98*	*2.01*	*2.03*	*2.06*

Solution: we know that

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,\mathrm{d}x$$

with $h = 0.01$, the Trapezoidal rule gives

$$\text{Area} = \frac{0.01}{2}[(1.93+2.06)+2(1.95+1.98+2.01+2.03)] = 0.09965$$

Example 7 *Use Simpson's rule for evaluating* $\int_{-0.6}^{0.3} f(x0\,\mathrm{d}x$ *from the following table given below*

x	*-0.6*	*-0.5*	*-0.4*	*-0.3*	*-0.2*	*-0.1*	*0*	*0.1*	*0.2*	*0.3*
$f(x)$	*4*	*2*	*5*	*3*	*-2*	*1*	*6*	*4*	*2*	*8*

Solution: Since the number of sub intervals is 9 (a multiple of 3) hence we will use Simpson's three-eighth rule

$$\int_{-0.6}^{0.3} f(x0\,\mathrm{d}x = \frac{3(0.1)}{8}[(4+8)+3(2+5-2+1+4+2)+2(3+6)] = 2.475$$

Example 8 *Evaluate* $\int_1^2 e^{-1/2x}\,\mathrm{d}x$ *using four intervals*

Solution: The table of values is

x	1	1.25	1.5	1.75	2
$y = e^{-1/2x}$	0.60653	0.53526	0.47237	0.41686	0.36788

Since we have four (even) sub intervals here, we will use Simpson's one-third rule

$$\begin{aligned}\int_1^2 e^{-1/2x}\,\mathrm{d}x &= \frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]\\ &= \frac{0.25}{3}[(0.60653+0.36788)+4(0.53526+0.41686)+2(0.47237)]\\ &= 0.4773025\end{aligned}$$

Example 9 *Find* $\int_0^6 \frac{e^x}{1+x}\,\mathrm{d}x$ *approximately suing Simpson's three-eighth rule on integration.*

Solution: Divide the given integral of integration into 6 equal sub intervals, the arguments are 0,,1, 2, 3, 4, 5, 6: $h = 1$. The table is as below

x	0	1	2	3	4	5	6
y	1	1.3591	2.4630	5.0214	10.9196	24.7355	57.6327

Applying Simpson's three-eighth rule, we get

$$\int_0^6 \frac{e^x}{1+x}\,\mathrm{d}x = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3] = 70.1652$$

Note: It is not possible to evaluate $\int_0^6 \frac{e^x}{1+x}\,\mathrm{d}x$ by using usual calculus method. Numerical integration comes to our rescue in such situation.

Example 10 *A train is moving at the speed of 30m/sec. Suddenly brakes are applied. The speed of the train per second after t second is given by*

Time (t)	*0*	*5*	*10*	*15*	*20*	*25*	*30*	*35*	*40*	*45*
Speed (v)	*30*	*24*	*19*	*16*	*13*	*11*	*10*	*8*	*7*	*5*

Apply Simpson 's three-eighth rule to determine the distance moved by the train in 45 seconds.

Solution: If s meters is the distance covered in t seconds, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v \quad \Rightarrow [s]_{t=0}^{t=45} = \int_0^{45} v\,\mathrm{d}t$$

Since the number of sub intervals is 9 (a multiple of 3) hence by using Simpson's three-eighth rule

$$\begin{aligned}\int_0^{45} v\,\mathrm{d}t &= \frac{3h}{8}[(v_0+v_9)+3(v_1+v_2+v_4+v_5+v_7+v_8)+2(v_3+v_6)]\\ &= \frac{15}{8}[(30+5)+3(24+19+13+11+8+7)+2(16+10)]\\ &= 624.375\end{aligned}$$

$\boxed{\text{Distance moved by the train in 45 seconds is 624.375 meters.}}$

Example 11 *A river is 80 m wide. The depth y of the river at a distance x from one bank is given by the following table:*

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the approximate area of cross-section of the river using Simpson's one-third rule.

Solution: The required area of the cross section of the river is given by

$$\text{Area} = \int_0^{80} y\,\mathrm{d}x$$

By Simpson's one-third rule

$$\begin{aligned}\int_0^{80} y\,\mathrm{d}x &= \frac{h}{3}[(y_0+y_8)+49y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]\\ &= \frac{10}{3}[(0+3)+4(4+9+15+8)+297+12+14)]\\ &= 710\end{aligned}$$

$\boxed{\text{Hence the required area of the cross section of the river is 710 sq. m.}}$

Example 12 *A solid of revolution is formed by rotating about X-axis, the line $x=0$ and $x=1$ and a curve through the points with the following coordinates*

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Solution: If v is the volume of the solid formed then we know that

$$v = \pi\int_0^1 y^2\,\mathrm{d}x$$

Hence we need the value of y^2 and these are tabulated below

x	0	0.25	0.5	0.75	1
y^2	1	0.9793	0.9195	0.8261	0.7081

with $h = 0.25$, Simpsons one-third rule gives

$$v = \pi \frac{0.25}{3}[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)] = 2.8192$$

Example 13 *A tank is discharged water through an orifice at a depth of x meter below the surface of the water whose area is Am^2. Following are the vales of x for the corresponding value of A.*

A	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295	2.462	2.650	2.827
x	1.5	1.65	1.8	1.95	2.1	2.25	2.4	2.55	2.7	2.85	3

Using the formula $(0.018)T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}} \, dx$, *calculate T, the time (in seconds) for the level of the water to drop from 3.0 m to 1.5 m above the orifice.*

Solution: here $h = 0.15$. The table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.5	1.65	1.8	1.95	2.1	2.25	2.4	2.55	2.7	2.85	3
$\frac{A}{\sqrt{x}}$	1.025	1.085	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

using Simpson's one-third rule, we get

$$\begin{aligned}
\int_{1.5}^{3} \frac{A}{\sqrt{x}} \, dx &= \frac{0.15}{3}[(1.025 + 1.632) + 4(1081 + 1.182 + 1.308 + 1.438 + 1.571) \\
&+ 2(1.132 + 1.249 + 1.375 + 1.498)] \\
&= 1.9743
\end{aligned}$$

Using the formula

$$(0.018)T = \int_{1.5}^{3} \frac{A}{\sqrt{x}} \, dx \quad \Rightarrow T = 110 \text{ Sec}$$

Example 14 *using the following table of values, approximate by Simpson's rule, the arc length of the graph* $y = \frac{1}{x}$ *between the points (1, 1) and (5,* $\frac{1}{5}$*)*

x	1	2	3	4	5
$\sqrt{\frac{1+x^4}{x^4}}$	1.414	1.031	1.007	1.002	1.001

Solution: The given curve is $y = \dfrac{1}{x} \quad \rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x^2}$

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)} = \sqrt{1 + \frac{1}{x^4}} = \sqrt{\frac{1+x^4}{x^4}}$$

The arc length of the curve between the points (1, 1) and $(5, \frac{1}{5})$ is

$$\text{Arc length} = \int_1^5 \sqrt{\frac{1+x^4}{x^4}} = \frac{h}{3}[(1.414 + 1.001) + 4(1.031 + 1.002) + 2(1.007)] = 4.187$$

Example 15 *A reservoir discharging water through sluices at a depth h below the water surface, has surface area A for various values of h as given below*

h (in meters)	*10*	*11*	*12*	*13*	*14*
A (in sq. meters)	*950*	*1070*	*1200*	*1350*	*1530*

If t denotes time in minutes, the rate of fall of the surface is given by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{48}{A}\sqrt{h}$$

Estimate the time taken for the water level to fall 14 to 10 m above the sluices.

Solution: From

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{48}{\mathrm{A}}\sqrt{h} \quad &\Rightarrow \quad \mathrm{d}t = -\frac{\mathrm{A}}{48}\frac{\mathrm{d}h}{\sqrt{h}} \\ &\Rightarrow \quad t = -\frac{1}{48}\int_{14}^{10} \frac{\mathrm{A}}{\sqrt{h}}\,\mathrm{d}h \\ &\Rightarrow \quad t = \frac{1}{48}\int_{10}^{14} \frac{\mathrm{A}}{\sqrt{h}}\,\mathrm{d}h\end{aligned}$$

Here, $y = \dfrac{\mathrm{A}}{\sqrt{h}}$. table of values is as below

h	10	11	12	13	14
$\dfrac{\mathrm{A}}{\sqrt{h}}$	300.4164	322.6171	346.4102	374.4226	408.9097

Applying Simpson's one-third rule, we get

$$t = \frac{1}{48} * \frac{1}{3}[(300.4164 + 408.9097) + 4(322.6171 + 374.4226) + 2(346.4102)]$$
$$= 29.0993\text{minutes}$$

1.7 Weddle's Rule (n=6)

Let $I = \int_0^b y\,\mathrm{d}x$, where the values $y_0, y_1, \ldots, y_n$ for $x_1, x_2, \ldots, x_n$
By Weddle's rule

$$I = \int_a^b y = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \cdots]$$

Note: Weddle's rule is applicable if n is multiple of 6.

Example 16 *Evaluste* $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$ *by using Weddle's rule by taking* $h = \frac{1}{6}$

Solution: Let $I = \int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$

Here $f(x) = \frac{1}{1+x^2}$
Take $n = 6$

x	0	1/6	2/6	3/6	4/6	5/6	6/6=1
y	1	0.9730	0.9	0.8	0.6923	0.5902	0.5
	y0	y1	y2	y3	y4	y5	y6

$$\begin{aligned} I &= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \\ &= \frac{3 \times 1/6}{10}[1 + 5 \times 0.9730 + 0.9 + 6 \times 0.8 + 0.6923 + 5 \times 0.5902 + 0.5] \\ &= 0.7854 \end{aligned}$$

$$\boxed{I = 0.7854}$$

Example 17 *Evaluate* $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,\mathrm{d}x$ *approximately using Weddles's rule correct to 4 decimal.*

Solution: Let $I = \int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,\mathrm{d}x$
Here $f(x) = \sin x - log_e + e^x$
Let $n = 12$, then $h = \frac{x_n - x_0}{n} = \frac{1.4 - 0.2}{12} = 0.1$

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
y	3.029	2.849	2.797	2.821	2.898	3.015	3.167	3.348	3.551	3.8	4.067	4.37	4.704
	y0	y1	y2	y3	y4	y5	y6	y7	y8	y9	y10	y11	y12

By Weddles rule

$$\begin{aligned} I &= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}] \\ &= 1.8279 \end{aligned}$$

Example 18 *In an experiment, a quantity G was measured as follows:*$G(20) = 95.9$, $G(21) = 96.85$, $G(22)97.77$, $G(23) = 98.68$, $G(24) = 99.56$, $G(25) = 100.41$, $G(26) = 101.24$. *Compute* $\int_{20}^{26} G(x)\,\mathrm{d}x$ *by Weddle's rule.*

Solution: 591.855

Example 19 *Find by Weddle's rule the value of the integral*

$$\int_{0.4}^{1.6} \frac{x}{\sinh x}\,\mathrm{d}x$$

by taking 12 sub-interval

Ans: 1.0102

2 Romberg's Integration

Let

$$I = \int_a^b f(x)\,\mathrm{d}x$$

Here we calculate

$$\begin{array}{cccc}
I(h) & & & \\
 & I\left(h, \frac{h}{2}\right) & & \\
I\left(\frac{h}{2}\right) & & I\left(h, \frac{h}{2}, \frac{h}{4}\right) & \\
 & I\left(\frac{h}{2}, \frac{h}{4}\right) & & I\left(h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right) \\
I\left(\frac{h}{4}\right) & & I\left(\frac{h}{2}, \frac{h}{4}, \frac{h}{8}\right) & \\
 & I\left(\frac{h}{4}, \frac{h}{8}\right) & & \\
I\left(\frac{h}{8}\right) & & &
\end{array}$$

The computation is continued till successive values are close to each other. This method is called Richardson's deferred approach to the limit and systematic refinement is called Romberg's method.
Where

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$
$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$

and so on.
Also

$$I\left(h,\frac{h}{2},\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2},\frac{h}{4}\right) - I\left(h,\frac{h}{2}\right)\right]$$
$$I\left(\frac{h}{2},\frac{h}{4},\frac{h}{8}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4},\frac{h}{8}\right) - I\left(\frac{h}{2},\frac{h}{4}\right)\right]$$

and so on.

Example 20 *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *correct to three decimal places using Romberg's method.*

Solution: Taking $h = 0.5,\ 0.25$ and 0.125 successively. Let us evaluate the given integral by Trapezoidal rule.

i) **When** $h = 0.5$, the value of $y = \frac{1}{1+x}$ are

x	0	0.5	1
y	1	0.667	0.5
	y_0	y_1	y_2

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}[(y_0+y_2)+2y_1] \\ &= \frac{0.5}{2}[(1+0.5)+2\times 0.667] \\ &= 0.709 \end{aligned}$$

$$\boxed{I(h) = 0.709}$$

ii) **When** $h = 0.25$, the value of $y = \frac{1}{1+x}$ are

x	0	0.25	0.5	0.75	1
y	1	0.8	0.667	0.571	0.5
	y_0	y_1	y_2	y_3	y_4

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)] \\ &= \frac{0.5}{2}[(1+0.5)+2\times(0.8+0.667+0.571)] \\ &= 0.697 \end{aligned}$$

$$\boxed{I\left(\frac{h}{2}\right) = 0.697}$$

iii) **When** $h = 0.125$, the value of $y = \dfrac{1}{1+x}$ are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.889	0.8	0.727	0.667	0.615	0.571	0.5333	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right] \\ &= \frac{0.5}{2}[(1+0.5)+2\times(0.889+0.8+0.727+0.667+0.615+0.571+0.5333)] \\ &= 0.694 \end{aligned}$$

$$\boxed{I\left(\frac{h}{4}\right) = 0.694}$$

Using Romberg,s formula, we obtain

$$I\left(h,\frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = \frac{1}{3}[4\times 0.697 - 0.708] = 0.963$$

$$I\left(\frac{h}{2},\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = \frac{1}{3}[4\times 0.697 - 0.708] = 0.963$$

Hence

$$\boxed{\int_0^1 \frac{dx}{1+x} = 0.693}$$

Example 21 *Use Romberg's method to compute* $\int_0^1 \frac{dx}{1+x^2}$ *correct to four decimal places.*

Solution: Taking $h = 0.5,\ 0.25$ and 0.125 successively. Let us evaluate the given integral by Trapezoidal rule.

i) **When** $h = 0.5$, the value of $y = \dfrac{1}{1+x}$ are

x	0	0.5	1
y	1	0.	0.5
	y_0	y_1	y_2

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}\left[(y_0 + y_2) + 2y_1\right] \\ &= \frac{0.5}{2}\left[(1 + 0.5) + 2 \times 0.8\right] \\ &= 0.0.775 \end{aligned}$$

$$\boxed{I\,(h) = 0.775}$$

ii) **When** $h = 0.25$, the value of $y = \dfrac{1}{1+x}$ are

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right] \\ &= \frac{0.5}{2}\left[(1 + 0.5) + 2 \times (0.9412 + 0.8 + 0.64)\right] \\ &= 0.7828 \end{aligned}$$

$$\boxed{I\left(\frac{h}{2}\right) = 0.7828}$$

iii) **When** $h = 0.125$, the value of $y = \dfrac{1}{1+x}$ are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal rule

$$\begin{aligned} I &= \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right] \\ &= \frac{0.5}{2}\left[(1 + 0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right] \\ &= 0.7848 \end{aligned}$$

$$\boxed{I\left(\frac{h}{4}\right) = 0.7848}$$

Using Romberg,s formula, we obtain

$$I\left(h,\frac{h}{2}\right)=\frac{1}{3}\left[4I\left(\frac{h}{2}\right)-I(h)\right]=\frac{1}{3}[4\times 0.7828-0.775]=0.7854$$

$$I\left(\frac{h}{2},\frac{h}{4}\right)=\frac{1}{3}\left[4I\left(\frac{h}{4}\right)-I\left(\frac{h}{2}\right)\right]=\frac{1}{3}[4\times 0.7848-0.7828]=0.7828$$

$$I\left(h,\frac{h}{2},\frac{h}{4}\right)=\frac{1}{3}\left[4I\left(\frac{h}{2},\frac{h}{4}\right)-\left(h,\frac{h}{2}\right)\right]=\frac{1}{3}[4\times 0.7855-0.7854]=0.7855$$

Hence

$$\boxed{\int_0^1\frac{\mathrm{d}x}{1+x^2}=0.7855}$$

3 Gaussian Integration Formula

Let

$$I=\int_a^b f(x)\,\mathrm{d}x$$

In the preceding sections, we derived some integration formulae which require values of the function at equally spaced point of the interval. Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

Gauss formula is expressed in the form:

$$\int_{-1}^{1}F(u)\,\mathrm{d}u=W_1F(u_1)+W_2F(u_2)+\cdot+W_nF(u_n)$$
$$=\sum_{i=1}^{n}W_iF(u_i)$$

Where the W_i and u_i are called the weight and abscissa respectively

1) **Gauss formula for** $n=2$

$$\int_{-1}^{1}F(u)\,\mathrm{d}u=W_1F(u_1)+W_2F(u_2)$$

Here $W_1=W_2=1$ and $u_1=-\frac{1}{\sqrt{3}},\ \ u_2=\frac{1}{\sqrt{3}}$

Which is the correct value of the integral of $F(u)$ you in the range $(-1,1)$ for any function up to third order.

2) **Gauss formula for** $n=3$

$$\int_{-1}^{1}F(u)\,\mathrm{d}u=W_1F(u_1)+W_2F(u_2)+W_3F(u_3)$$
$$=\frac{8}{9}F(0)+\frac{5}{9}\left[F\left(-\sqrt{\frac{3}{5}}\right)+F\left(\sqrt{\frac{3}{5}}\right)\right]$$

Note: Gauss formula impose restrictions on the limits of integration to be from -1 to 1. In general the limits of the integral $I = \int_a^b f(x)\,\mathrm{d}x$ are changed -1 to 1 by means of the transformation:

$$\boxed{x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)}$$

Example 22 *Evaluate* $I = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2}$, *using Gauss formula for* $n = 2$ *and* $n = 3$.

solution: Let

$$I = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2}$$

Here $f(x) = \dfrac{1}{1+x^2}$

1) **Gauss formula for** $n = 2$

$$\begin{aligned} I &= F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}} \\ &= \frac{3}{4} + \frac{3}{4} \\ &= 1.5 \end{aligned}$$

2) **Gauss formula for** $n = 3$

$$\begin{aligned} \int_{-1}^{1} F(u)\,\mathrm{d}u &= \frac{8}{9}F(0) + \frac{5}{9}\left[F\left(-\sqrt{\frac{3}{5}}\right) + F\left(\sqrt{\frac{3}{5}}\right)\right] \\ &= \frac{8}{9} + \frac{5}{9}\left[\frac{5}{8} + \frac{5}{8}\right] \\ &= \frac{8}{9} + \frac{50}{72} \\ &= 1.5833 \end{aligned}$$

Example 23 *Evaluate* $I = \int_{-2}^{2} e^{-x/2}\,\mathrm{d}x$, *using Gauss formula for* $n = 2$.

Solution: Let

$$I = \int_{-2}^{2} e^{-x/2}\,\mathrm{d}x$$

First we convert the limits $(-2, 2)$ to $(-1, 1)$ by

$$\begin{aligned} x &= \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) \\ &= \frac{1}{2} \times 4u \\ &= 2u \end{aligned}$$

Thus

$$x = 2u, \quad u = \frac{x}{2}$$

Next

$$\begin{aligned} I &= \int_{-2}^{2} e^{-x/2}\,\mathrm{d}x \\ &= \int_{-1}^{1} e^{u} \times 2\,\mathrm{d}u \\ &= 2\int_{-1}^{1} e^{u}\,\mathrm{d}u \\ &= 2 \times \left[F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right] \\ &= 2 \times \left[e^{-\frac{1}{\sqrt{3}}} + e^{\frac{1}{\sqrt{3}}}\right] \\ &= 2 \times 2.3427 \\ &= 4.6854 \end{aligned}$$

Thus

$$\boxed{I = \int_{-2}^{2} e^{-x/2}\,\mathrm{d}x = 4.6854}$$

Example 24 *Using three point Gaussian quadrature formula, evaluate* $I = \int_{0}^{1} \frac{\mathrm{d}x}{1+x}$

Solution: Let

$$I = \int_{0}^{1} \frac{\mathrm{d}x}{1+x}$$

First we convert the limits $(0, 1)$ to $(-1, 1)$ by

$$\begin{aligned}
x &= \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) \\
&= \frac{1}{2}(1-0) + \frac{1}{2}(1) \\
&= \frac{1}{2}(u+1)
\end{aligned}$$

Thus

$$x = \frac{1}{2}(u+1), \quad u = 2x+1$$

Next

$$\begin{aligned}
I &= \int_0^1 \frac{\mathrm{d}x}{1+x} \\
&= \int_{-1}^1 \frac{1}{1+\frac{1}{2}(u+1)}\,\mathrm{d}u \\
&= \int_{-1}^1 \frac{\mathrm{d}u}{u+3} \\
&= \frac{8}{9}F(0) + \frac{5}{9}\left[F\left(-\sqrt{\frac{3}{5}}\right) + F\left(\sqrt{\frac{3}{5}}\right)\right] \\
&= \frac{8}{9} \times \frac{1}{3} + \frac{5}{9}\left[\frac{1}{-\sqrt{\frac{3}{5}}+3} + \frac{1}{-\sqrt{\frac{3}{5}}+3}\right] \\
&= \frac{8}{27} + \frac{25}{63} \\
&= 0.6931
\end{aligned}$$

Thus

$$\boxed{I = \int_0^1 \frac{\mathrm{d}x}{1+x} = 0.6931}$$

Example 25 *Using two point Gaussian quadrature formula, evaluate* $I = \int_2^4 (2x^2 + 1)\,\mathrm{d}x$

Solution: Let

$$I = \int_2^4 (2x^2+1)\,\mathrm{d}x$$

First we convert the limits $(2,4)$ to $(-1,1)$ by

$$\begin{aligned} x &= \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) \\ &= \frac{1}{2}(2u) + \frac{1}{2}(6) \\ &= u+3 \end{aligned}$$

Thus

$$x = u+3, \quad u = x-3$$

Next

$$\begin{aligned} I &= \int_2^4 (2x^2+1)\,\mathrm{d}x \\ &= \int_{-1}^{1} \left[2(u+3)^2+1\right]\,\mathrm{d}u \\ &= \int_{-1}^{1} \frac{\mathrm{d}u}{u+3} \\ &= F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \\ &= 39.3334 \end{aligned}$$

Thus

$$\boxed{\int_2^4 (2x^2+1)\,\mathrm{d}x = 39.3334}$$

Example 26 *Evaluate* $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,\mathrm{d}x$ *by Gaussian 3-point formula.*

Solution:

$$\begin{aligned} x &= u+1 \\ F(u) &= \frac{u^2+4u+4}{(u+2)^4+1} \end{aligned}$$

$$\boxed{\text{I=0.5365}}$$