3 | DERIVATIVES

Figure 3.1 The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

Chapter Outline

Introduction

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat (see **Example 3.8**.)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.1 | Defining the Derivative

Learning Objectives

3.1.1 Recognize the meaning of the tangent to a curve at a point.
3.1.2 Calculate the slope of a tangent line.
3.1.3 Identify the derivative as the limit of a difference quotient.
3.1.4 Calculate the derivative of a given function at a point.
3.1.5 Describe the velocity as a rate of change.
3.1.6 Explain the difference between average velocity and instantaneous velocity.
3.1.7 Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in **Figure 3.2**. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

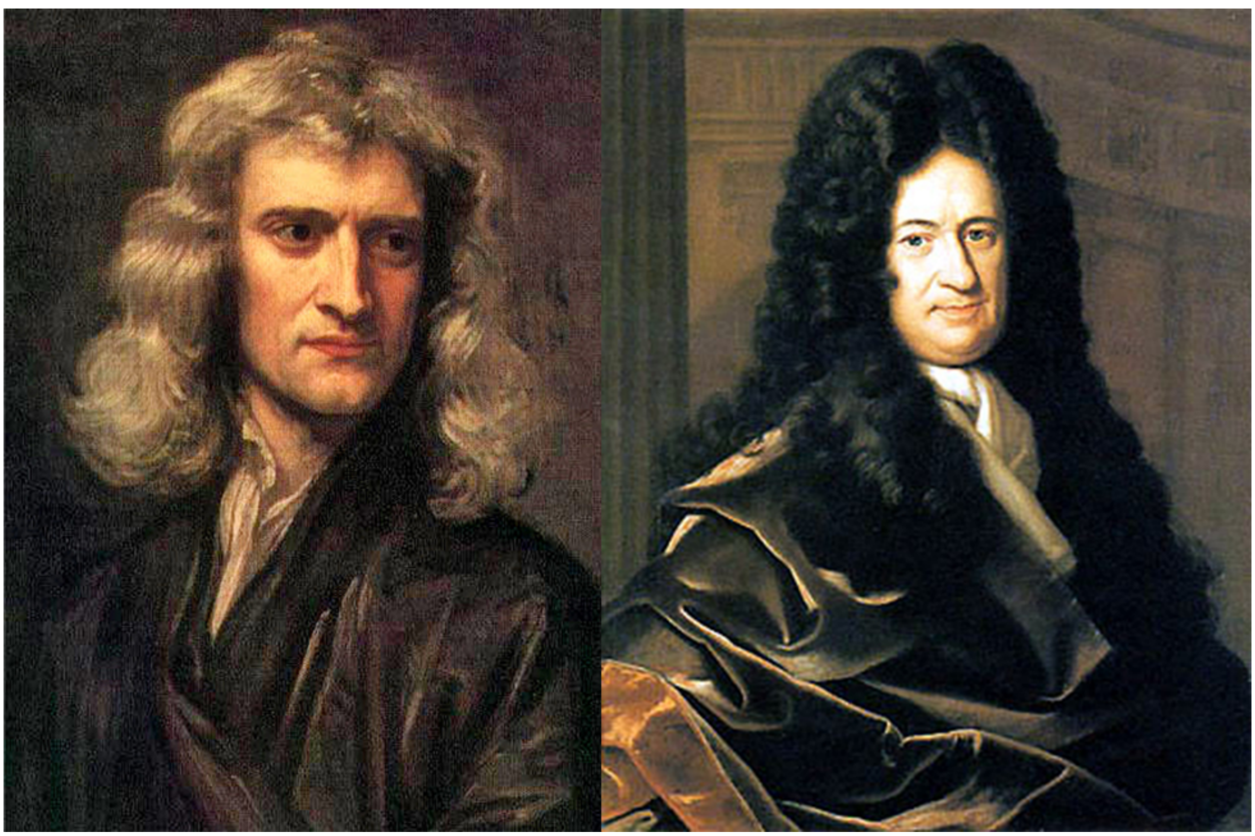


Figure 3.2 Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in **Figure 3.3**. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to 0. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a} \tag{3.1}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a + h) - f(a)}{h} \tag{3.2}$$

is a difference quotient with increment h.

View the development of the **derivative (http://www.openstax.org/l/20_calcapplets)** with this applet.

These two expressions for calculating the slope of a secant line are illustrated in **Figure 3.3**. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

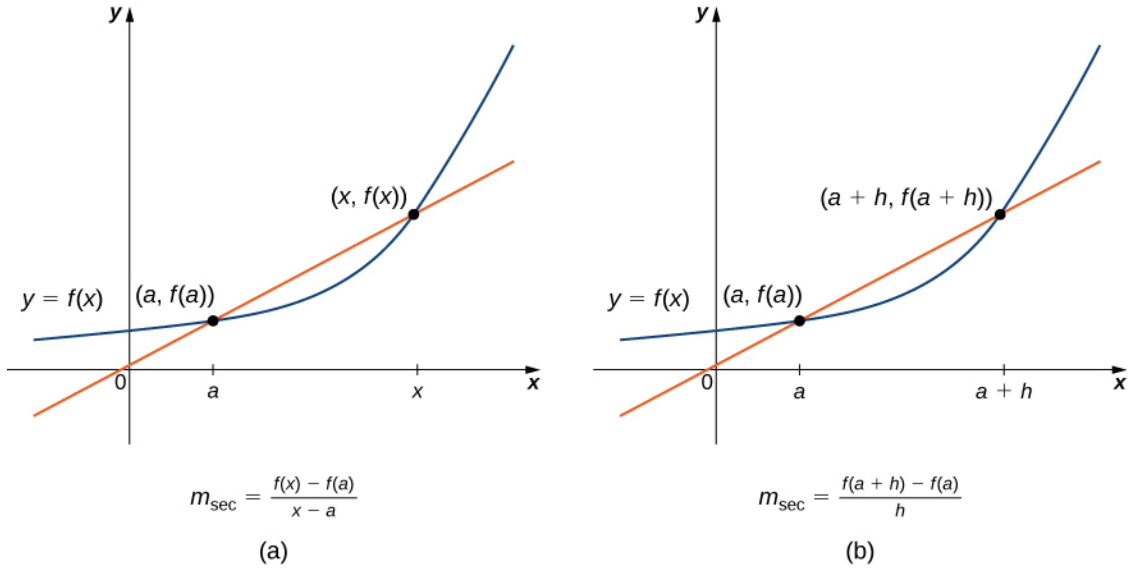


Figure 3.3 We can calculate the slope of a secant line in either of two ways.

In **Figure 3.4**(a) we see that, as the values of x approach $a,$ the slopes of the secant lines provide better estimates of the rate of change of the function at $a.$ Furthermore, the secant lines themselves approach the tangent line to the function at $a,$ which represents the limit of the secant lines. Similarly, **Figure 3.4**(b) shows that as the values of h get closer to $0,$ the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at $a,$ as shown in **Figure 3.4**(c).

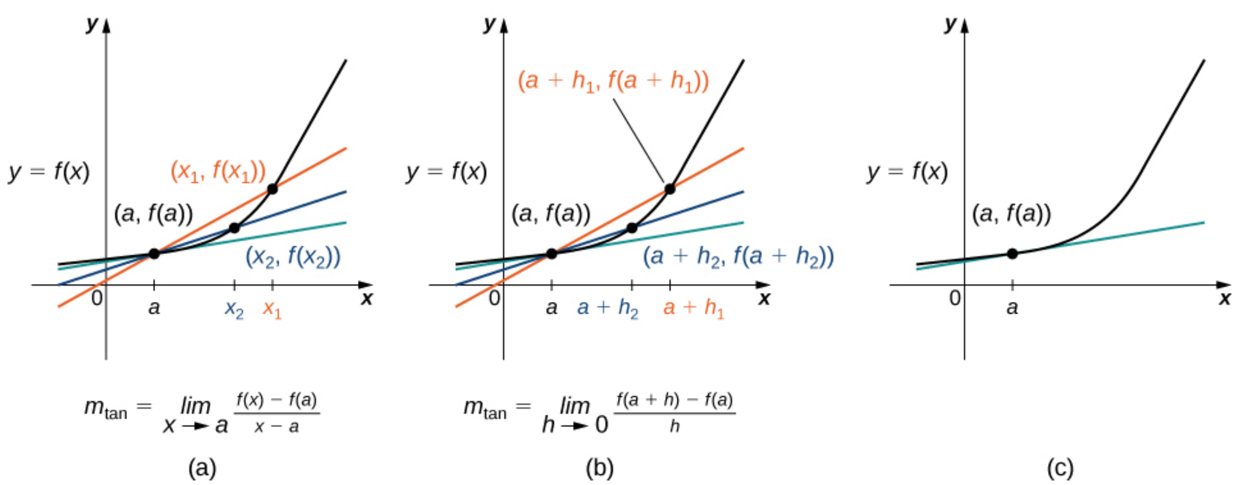


Figure 3.4 The secant lines approach the tangent line (shown in green) as the second point approaches the first.

You can use this **site (http://www.openstax.org/l/20_diffmicros)** to explore graphs to see if they have a tangent line at a point.

In **Figure 3.5** we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1,\ 1)$ in a series of tighter intervals about $x = 1.$ As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to $1.$ In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1.$

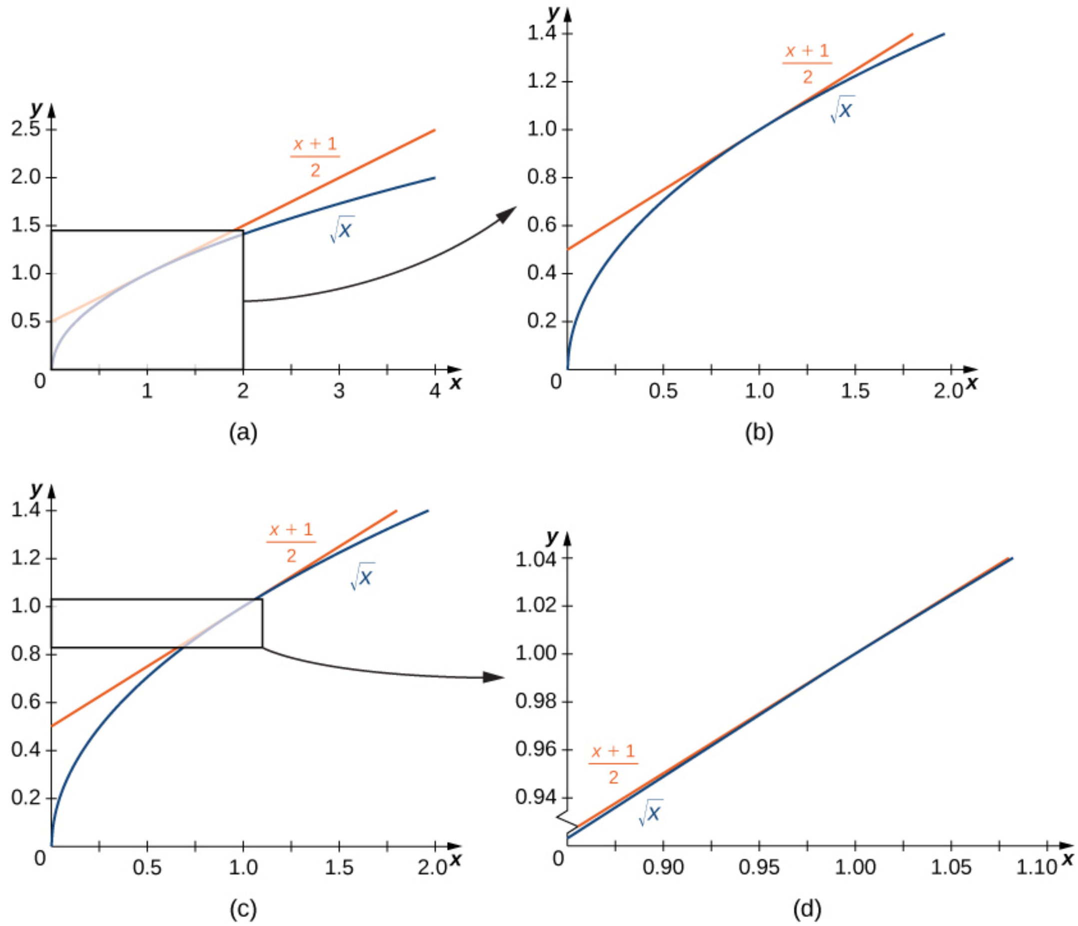


Figure 3.5 For values of x close to $1,$ the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.3}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{3.4}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 3.1

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use **Equation 3.3**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9. \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in **Figure 3.6**.

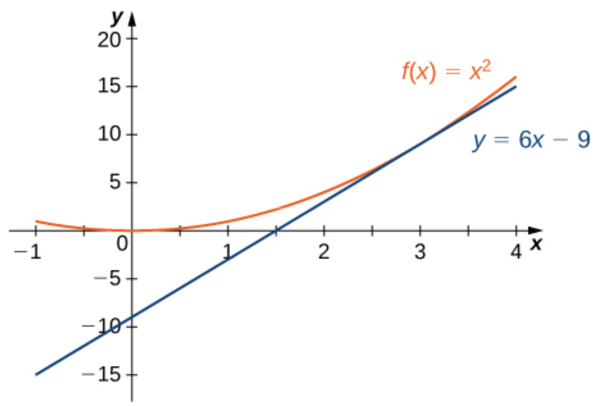


Figure 3.6 The tangent line to $f(x)$ at $x = 3$.

Example 3.2

The Slope of a Tangent Line Revisited

Use **Equation 3.4** to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to **Example 3.1**. See **Equation 3.4** for the definition.

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} && \text{Apply the definition.} \\ &= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} && \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9. \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} && \text{Expand and simplify to evaluate the limit.} \\ &= \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6 \end{aligned}$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 3.3

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use **Equation 3.3**, but as we have seen, the results are the same if we use **Equation 3.4**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} && \text{Substitute } f(x) = \tfrac{1}{x} \text{ and } f(2) = \tfrac{1}{2}. \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \cdot \frac{2x}{2x} && \text{Multiply numerator and denominator by } 2x \text{ to simplify fractions.} \\ &= \lim_{x \to 2} \frac{(2 - x)}{(x - 2)(2x)} && \text{Simplify.} \\ &= \lim_{x \to 2} \frac{-1}{2x} && \text{Simplify using } \tfrac{2 - x}{x - 2} = -1, \text{ for } x \neq 2. \\ &= -\frac{1}{4} && \text{Evaluate the limit.} \end{aligned}$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in **Figure 3.7**.

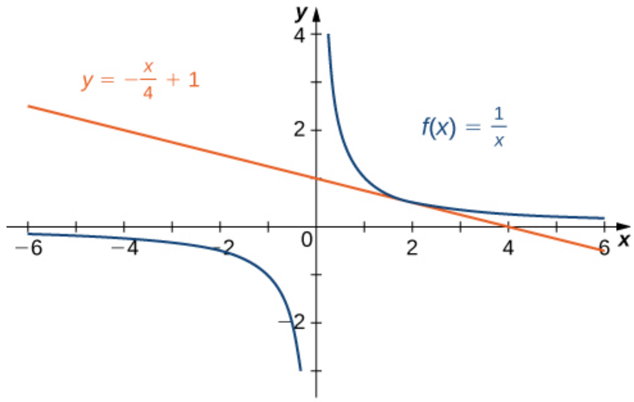


Figure 3.7 The line is tangent to $f(x)$ at $x = 2$.

3.1 Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.5}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{3.6}$$

Example 3.4

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using Equation 3.5.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

3.2 For $f(x) = x^2$, use a table to estimate $f'(3)$ using **Equation 3.6**.

Example 3.5

Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using **Equation 3.5**.

Solution

Substitute the given function and value directly into the equation.

$$\begin{aligned} f'(x) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\ &= \lim_{x \to 2} \frac{\left(3x^2 - 4x + 1\right) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5. \\ &= \lim_{x \to 2} \frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.} \\ &= \lim_{x \to 2} (3x + 2) && \text{Cancel the common factor.} \\ &= 8 && \text{Evaluate the limit.} \end{aligned}$$

Example 3.6

Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1,$ find $f'(2)$ by using **Equation 3.6**.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in **Example 3.5**.

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} && \text{Apply the definition.} \\
&= \lim_{h \to 0} \frac{(3(2+h)^2 - 4(2+h) + 1) - 5}{h} && \text{Substitute } f(2) = 5 \text{ and } f(2+h) = 3(2+h)^2 - 4(2+h) + 1. \\
&= \lim_{h \to 0} \frac{3h^2 + 8h}{h} && \text{Simplify the numerator.} \\
&= \lim_{h \to 0} \frac{h(3h+8)}{h} && \text{Factor the numerator.} \\
&= \lim_{h \to 0} (3h + 8) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}
$$

The results are the same whether we use **Equation 3.5** or **Equation 3.6**.

3.3 For $f(x) = x^2 + 3x + 2,$ find $f'(1).$

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{3.7}$$

As the values of t approach $a,$ the values of v_{ave} approach the value we call the instantaneous velocity at $a.$ That is, instantaneous velocity at $a,$ denoted $v(a),$ is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{3.8}$$

To better understand the relationship between average velocity and instantaneous velocity, see **Figure 3.8**. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t).$ The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t].$

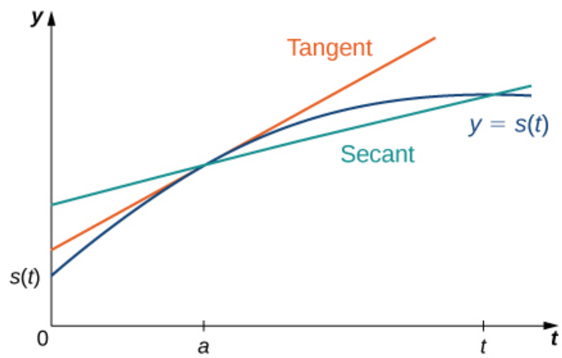


Figure 3.8 The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use **Equation 3.5** to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using **Equation 3.7**.

Example 3.7

Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (**Figure 3.9**). Use a table of values to estimate $v(0)$. Check the estimate by using **Equation 3.5**.

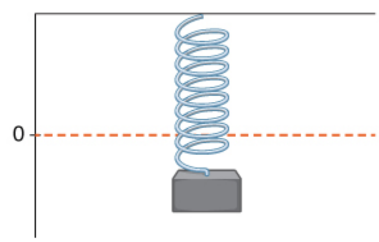


Figure 3.9 A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching $0,$ as shown in **Table 3.1**.

t	$\frac{\sin t - \sin 0}{t - 0} = \frac{\sin t}{t}$
-0.1	0.998334166
-0.01	0.9999833333
-0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

Table 3.1
Average velocities using values of t approaching 0

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is $0.998334166,$ the average velocity over the time interval $[-0.01, 0]$ is $0.9999833333,$ and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1.$

By using **Equation 3.5**, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1.$

3.4 A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2.$ Find its instantaneous velocity 1 second after it is dropped, using **Equation 3.5**.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 3.8

Chapter Opener: Estimating Rate of Change of Velocity

Figure 3.10 (credit: modification of work by Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Solution

First observe that 60 mph = 88 ft/s, 100 mph ≈ 146.67 ft/s, 200 mph ≈ 293.33 ft/s, and 229.9 mph ≈ 337.19 ft/s. We can summarize the information in a table.

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Table 3.2 $v(t)$ at different values of t

Now compute the average acceleration of the car in feet per second per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

t	$\frac{v(t) - v(19.96)}{t - 19.96} = \frac{v(t) - 337.19}{t - 19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

Table 3.3
Average acceleration

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph $(337.19$ ft/s).

Example 3.9

Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°\text{F}$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10,$ where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to **Equation 3.5**.

$$\begin{aligned}
T'(3) &= \lim_{t \to 3} \frac{T(t) - T(3)}{t - 3} && \text{Apply the definition.} \\
&= \lim_{t \to 3} \frac{0.4t^2 - 4t + 70 - 61.6}{t - 3} && \text{Substitute } T(t) = 0.4t^2 - 4t + 70 \text{ and } T(3) = 61.6. \\
&= \lim_{t \to 3} \frac{0.4t^2 - 4t + 8.4}{t - 3} && \text{Simplify.} \\
&= \lim_{t \to 3} \frac{0.4(t-3)(t-7)}{t - 3} && = \lim_{t \to 3} \frac{0.4(t-3)(t-7)}{t-3} \\
&= \lim_{t \to 3} 0.4(t - 7) && \text{Cancel.} \\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6°\text{F}$ per hour.

Example 3.10

Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10{,}000$ dollars. Find the rate of change of profit when $10{,}000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x),$ where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10{,}000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned} P'(10000) &= \lim_{x \to 10000} \frac{P(x) - P(10000)}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 10000 - 1990000}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 2000000}{x - 10000} \\ &= 100. \end{aligned}$$

Since the rate of change of profit $P'(10{,}000) > 0$ and $P(10{,}000) > 0,$ the company should increase production.

3.5 A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10.$ The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25),$ the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

3.1 EXERCISES

For the following exercises, use **Equation 3.1** to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1. $f(x) = 4x + 7;\ x_1 = 2,\ x_2 = 5$

2. $f(x) = 8x - 3;\ x_1 = -1,\ x_2 = 3$

3. $f(x) = x^2 + 2x + 1;\ x_1 = 3,\ x_2 = 3.5$

4. $f(x) = -x^2 + x + 2;\ x_1 = 0.5,\ x_2 = 1.5$

5. $f(x) = \frac{4}{3x - 1};\ x_1 = 1,\ x_2 = 3$

6. $f(x) = \frac{x - 7}{2x + 1};\ x_1 = 0,\ x_2 = 2$

7. $f(x) = \sqrt{x};\ x_1 = 1,\ x_2 = 16$

8. $f(x) = \sqrt{x - 9};\ x_1 = 10,\ x_2 = 13$

9. $f(x) = x^{1/3} + 1;\ x_1 = 0,\ x_2 = 8$

10. $f(x) = 6x^{2/3} + 2x^{1/3};\ x_1 = 1,\ x_2 = 27$

For the following functions,

a. use **Equation 3.4** to find the slope of the tangent line $m_{\tan} = f'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11. $f(x) = 3 - 4x,\ a = 2$

12. $f(x) = \frac{x}{5} + 6,\ a = -1$

13. $f(x) = x^2 + x,\ a = 1$

14. $f(x) = 1 - x - x^2,\ a = 0$

15. $f(x) = \frac{7}{x},\ a = 3$

16. $f(x) = \sqrt{x + 8},\ a = 1$

17. $f(x) = 2 - 3x^2,\ a = -2$

18. $f(x) = \frac{-3}{x - 1},\ a = 4$

19. $f(x) = \frac{2}{x + 3},\ a = -4$

20. $f(x) = \frac{3}{x^2},\ a = 3$

For the following functions $y = f(x)$, find $f'(a)$ using **Equation 3.1**.

21. $f(x) = 5x + 4,\ a = -1$

22. $f(x) = -7x + 1,\ a = 3$

23. $f(x) = x^2 + 9x,\ a = 2$

24. $f(x) = 3x^2 - x + 2,\ a = 1$

25. $f(x) = \sqrt{x},\ a = 4$

26. $f(x) = \sqrt{x - 2},\ a = 6$

27. $f(x) = \frac{1}{x},\ a = 2$

28. $f(x) = \frac{1}{x - 3},\ a = -1$

29. $f(x) = \frac{1}{x^3},\ a = 1$

30. $f(x) = \frac{1}{\sqrt{x}},\ a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x,\ f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31. **[T]** $f(x) = x^2 + 3x + 4,\ P(1,\ 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	*x*	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

32. **[T]** $f(x) = \frac{x+1}{x^2-1},\ P(0,\ -1)$

x	Slope m_{PQ}	*x*	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33. **[T]** $f(x) = 10e^{0.5x},\ P(0,\ 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

34. **[T]** $f(x) = \tan(x),\ P(\pi,\ 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, and (iv) $h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

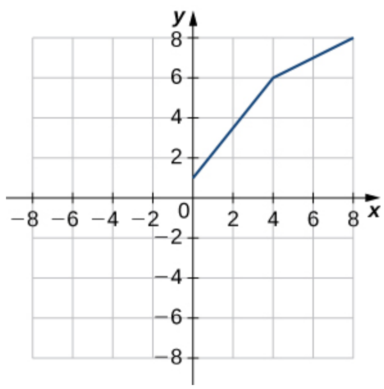
35. $s(t) = \frac{1}{3}t + 5$

36. $s(t) = t^2 - 2t$

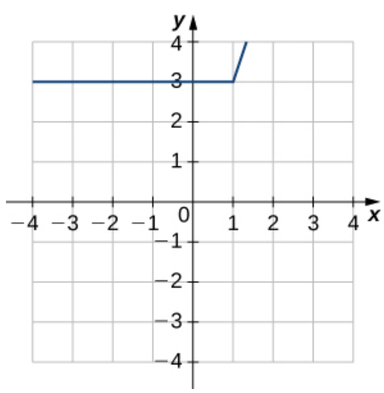
37. $s(t) = 2t^3 + 3$

38. $s(t) = \frac{16}{t^2} - \frac{4}{t}$

39. Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.



40. Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41. $f(x) = x^{1/3},\ x = 0$

42. $f(x) = x^{2/3},\ x = 0$

43. $f(x) = \begin{cases} 1, & x < 1 \\ x, & x \geq 1 \end{cases},\ x = 1$

44. $f(x) = \frac{|x|}{x},\ x = 0$

45. **[T]** The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:
   i. [4, 4.1]
   ii. [4, 4.01]
   iii. [4, 4.001]
   iv. [4, 4.0001]

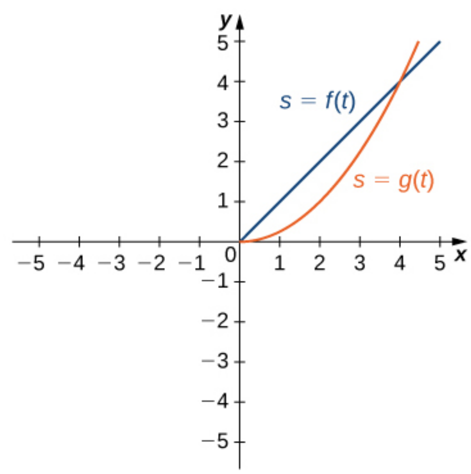
b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

46. **[T]** The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$, where *t* is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:
   i. [5, 5.1]
   ii. [5, 5.01]
   iii. [5, 5.001]
   iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47. Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

48. **[T]** The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300.$

a. Calculate the average cost per jar over the following intervals:
   i. [100, 100.1]
   ii. [100, 100.01]
   iii. [100, 100.001]
   iv. [100, 100.0001]
b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

49. **[T]** For the function $f(x) = x^3 - 2x^2 - 11x + 12,$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{\tan} = f'(a) = 0.$

50. **[T]** For the function $f(x) = \frac{x}{1 + x^2},$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f'(a) = 0.$

51. Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.
b. What is $N(100)$? Explain the physical meaning.
c. What is $N'(100)$? Explain the physical meaning.

52. **[T]** For the function $f(x) = x^4 - 5x^2 + 4,$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2),\ f'(-0.5),\ f'(1.7),$ and $f'(2.718).$

53. **[T]** For the function $f(x) = \frac{x^2}{x^2 + 1},$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function on a graphing calculator to find $f'(-4),\ f'(-2),\ f'(2),$ and $f'(4).$

3.2 | The Derivative as a Function

Learning Objectives

3.2.1 Define the derivative function of a given function.
3.2.2 Graph a derivative function from the graph of a given function.
3.2.3 State the connection between derivatives and continuity.
3.2.4 Describe three conditions for when a function does not have a derivative.
3.2.5 Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.9}$$

A function $f(x)$ is said to be **differentiable at** $\boldsymbol{a}$ if $f'(a)$ exists. More generally, a function is said to be **differentiable on** $\boldsymbol{S}$ if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use **Equation 3.9** to find the derivative of a function.

Example 3.11

Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function. Use **Equation 3.1**.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

Multiply the numerators and simplify.

$$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

Cancel the h.

$$= \frac{1}{2\sqrt{x}}$$

Evaluate the limit.

Example 3.12

Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

$$f'(x) = \lim_{h \to 0} \frac{\left((x+h)^2 - 2(x+h)\right) - \left(x^2 - 2x\right)}{h}$$

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

Expand $(x+h)^2 - 2(x+h)$.

$$= \lim_{h \to 0} \frac{2xh - 2h + h^2}{h}$$

Simplify.

$$= \lim_{h \to 0} \frac{h(2x - 2 + h)}{h}$$

Factor out h from the numerator.

$$= \lim_{h \to 0} (2x - 2 + h)$$

Cancel the common factor of h.

$$= 2x - 2$$

Evaluate the limit.

3.6 Find the derivative of $f(x) = x^2$.

We use a variety of different notations to express the derivative of a function. In **Example 3.12** we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}\left(x^2 - 2x\right) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}(f(x)).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$ Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (**Figure 3.11**). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

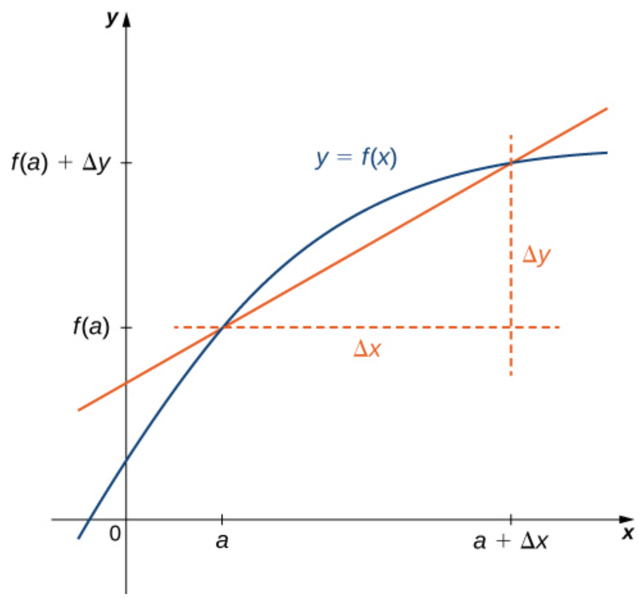


Figure 3.11 The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In **Example 3.11** we found that for $f(x) = \sqrt{x}$, $f'(x) = 1/2\sqrt{x}$. If we graph these functions on the same axes, as in **Figure 3.12**, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

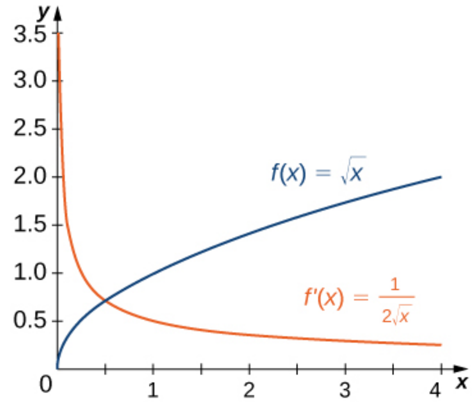


Figure 3.12 The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In **Example 3.12** we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in **Figure 3.13**. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

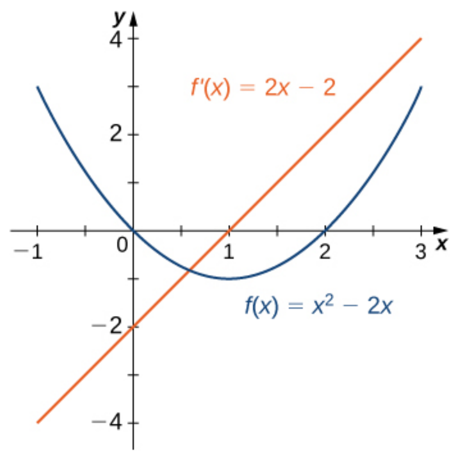
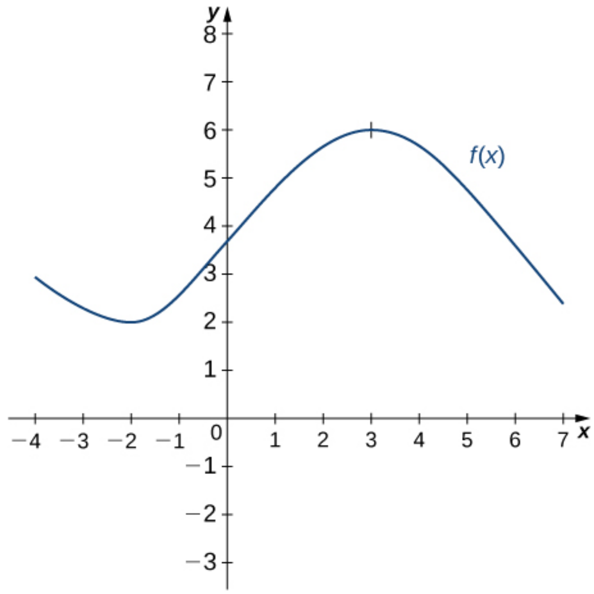


Figure 3.13 The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Example 3.13

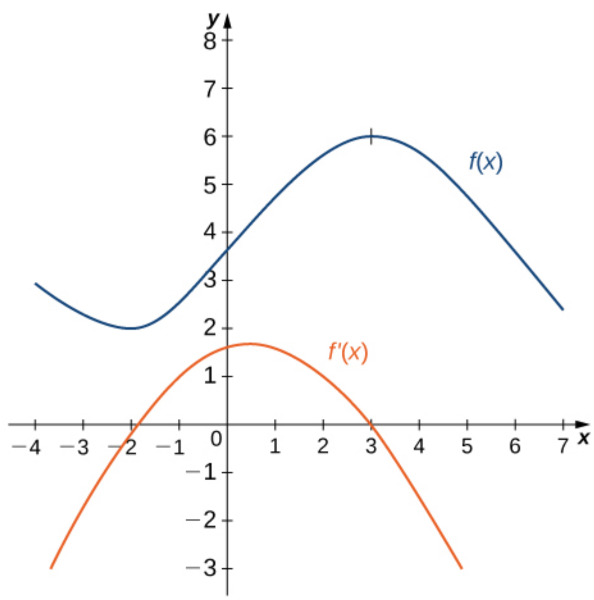
Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



3.7 Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there;

however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Theorem 3.1: Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (f(x) - f(a) + f(a)) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right) && \text{Multiply and divide } f(x) - f(a) \text{ by } x - a. \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left(\lim_{x \to a} (x - a) \right) + \lim_{x \to a} f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a).
\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

See **Figure 3.14**.

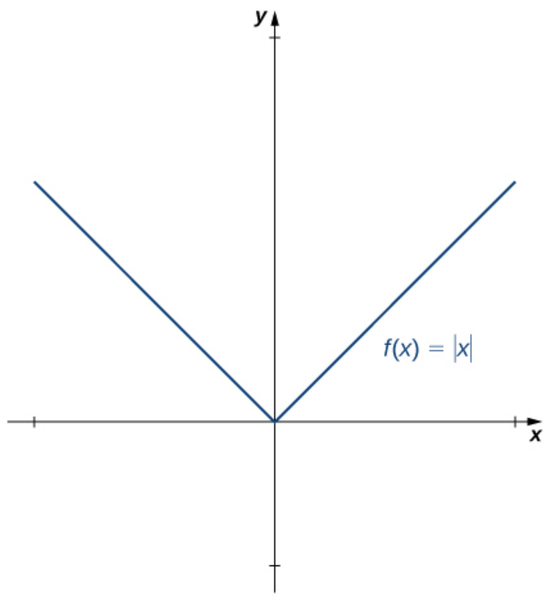


Figure 3.14 The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (**Figure 3.15**).

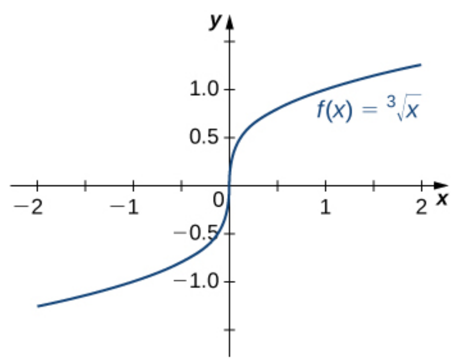


Figure 3.15 The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \ne 0 \\ 0 \text{ if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0. We see that

$$f'(0) = \lim_{x \to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (**Figure 3.16**).

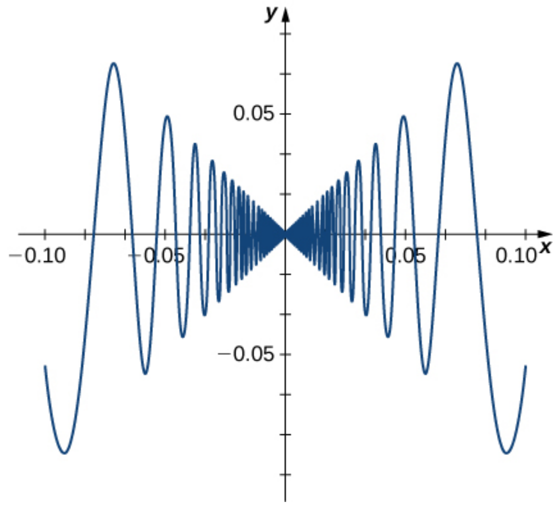


Figure 3.16 The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 3.14

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (**Figure 3.17**). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c \text{ if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2} \text{ if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

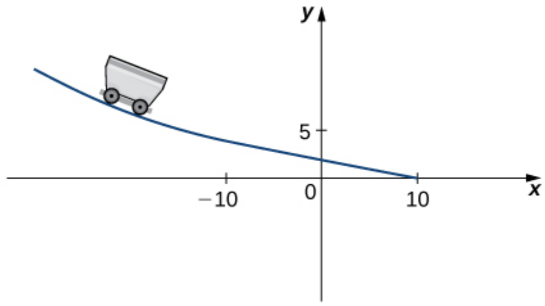


Figure 3.17 For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}
\lim_{x \to -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\
&= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\
&= \lim_{x \to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\
&= \lim_{x \to -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping.} \\
&= b - 2.
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{x \to -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\
&= \lim_{x \to -10^+} \frac{-(x + 10)}{4(x + 10)} \\
&= -\frac{1}{4}.
\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10\left(\frac{7}{4}\right) - 5 = \frac{25}{2}$.

3.8 Find values of a and b that make $f(x) = \begin{cases} ax + b \text{ if } x < 3 \\ x^2 \text{ if } x \ge 3 \end{cases}$ both continuous and differentiable at 3.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ldots,\ f^{(n)}(x)$$
$$y''(x),\ y'''(x),\ y^{(4)}(x),\ldots,\ y^{(n)}(x)$$
$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ldots,\ \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example 3.15

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\left(2(x+h)^2 - 3(x+h) + 1\right) - (2x^2 - 3x + 1)}{h} && \text{Substitute } f(x) = 2x^2 - 3x + 1 \text{ and } f(x+h) = 2(x+h)^2 - 3(x+h) + 1 \text{ into } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} && \text{Simplify the numerator.} \\ &= \lim_{h \to 0} (4x + 2h - 3) && \text{Factor out the } h \text{ in the numerator and cancel with the } h \text{ in the denominator.} \\ &= 4x - 3 && \text{Take the limit.} \end{aligned}$$

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$$\begin{aligned} f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} && \text{Use } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ with } f'(x) \text{ in place of } f(x). \\ &= \lim_{h \to 0} \frac{(4(x+h) - 3) - (4x - 3)}{h} && \text{Substitute } f'(x+h) = 4(x+h) - 3 \text{ and } f'(x) = 4x - 3. \\ &= \lim_{h \to 0} 4 && \text{Simplify.} \\ &= 4 && \text{Take the limit.} \end{aligned}$$

3.9 Find $f''(x)$ for $f(x) = x^2$.

Example 3.16

Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - \left(3t^2 - 4t + 1\right)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

3.10 For $s(t) = t^3$, find $a(t)$.

3.2 EXERCISES

For the following exercises, use the definition of a derivative to find $f'(x)$.

54. $f(x) = 6$

55. $f(x) = 2 - 3x$

56. $f(x) = \frac{2x}{7} + 1$

57. $f(x) = 4x^2$

58. $f(x) = 5x - x^2$

59. $f(x) = \sqrt{2x}$

60. $f(x) = \sqrt{x - 6}$

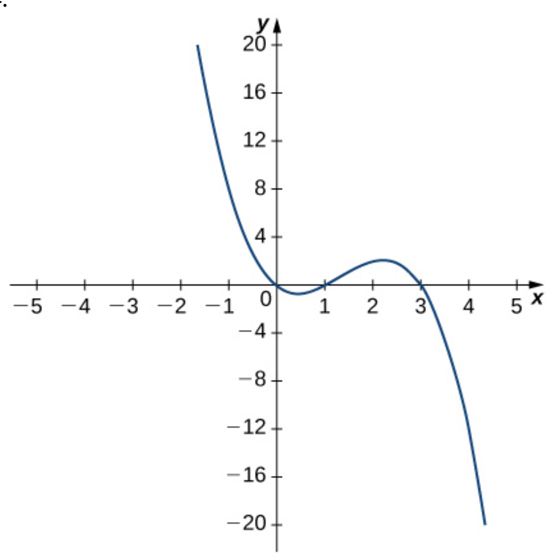
61. $f(x) = \frac{9}{x}$

62. $f(x) = x + \frac{1}{x}$

63. $f(x) = \frac{1}{\sqrt{x}}$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

64.



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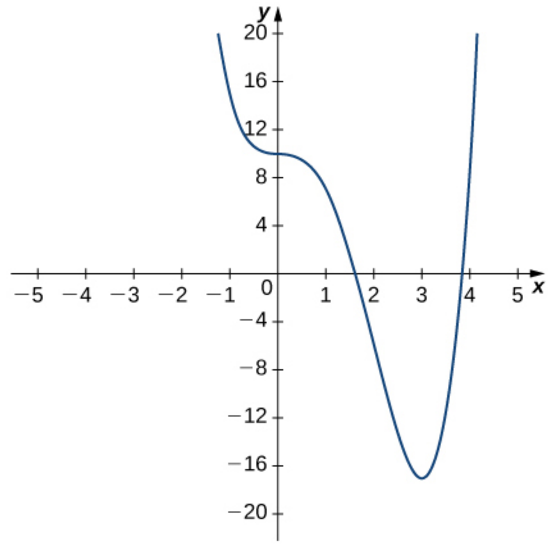

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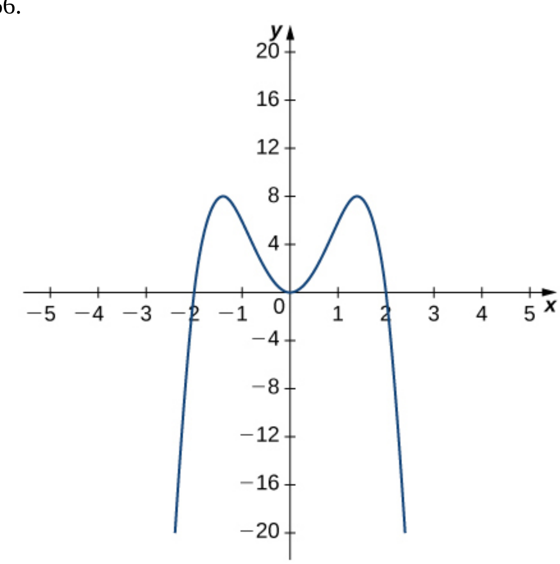

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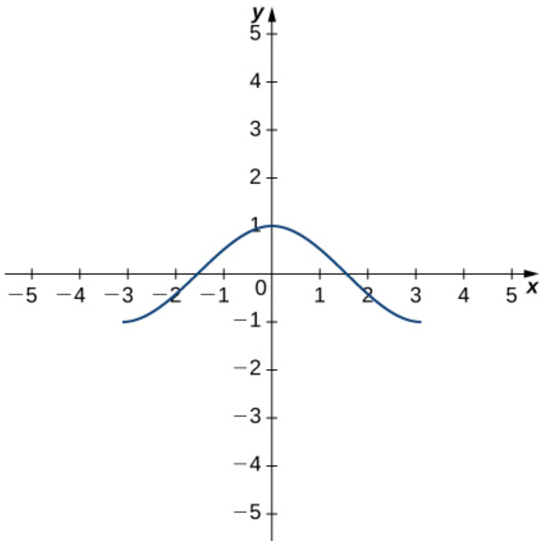


For the following exercises, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

68. $\lim_{h \to 0} \frac{(1+h)^{2/3} - 1}{h}$

69. $\lim_{h \to 0} \frac{\left[3(2+h)^2 + 2\right] - 14}{h}$

70. $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

71. $\lim_{h \to 0} \frac{(2+h)^4 - 16}{h}$

72. $\lim_{h \to 0} \frac{[2(3+h)^2 - (3+h)] - 15}{h}$

73. $\lim_{h \to 0} \frac{e^h - 1}{h}$

For the following functions,

a. sketch the graph and

b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

74. $f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x \le 1 \\ 3x - 1, & x > 1 \end{cases}$

75. $f(x) = \begin{cases} 3, & x < 1 \\ 3x, & x \ge 1 \end{cases}$

76. $f(x) = \begin{cases} -x^2 + 2, & x \le 1 \\ x, & x > 1 \end{cases}$

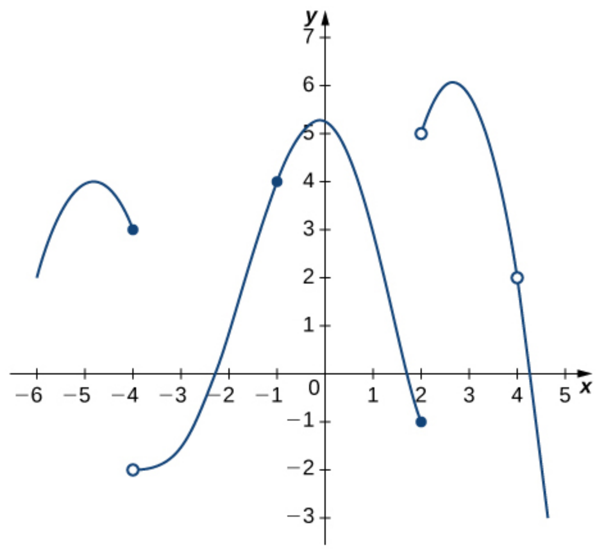
77. $f(x) = \begin{cases} 2x, & x \le 1 \\ \frac{2}{x}, & x > 1 \end{cases}$

For the following graphs,

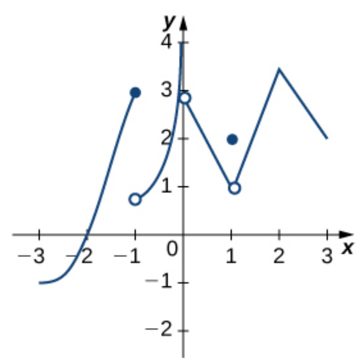
a. determine for which values of $x = a$ the $\lim_{x \to a} f(x)$ exists but f is not continuous at $x = a$, and

b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

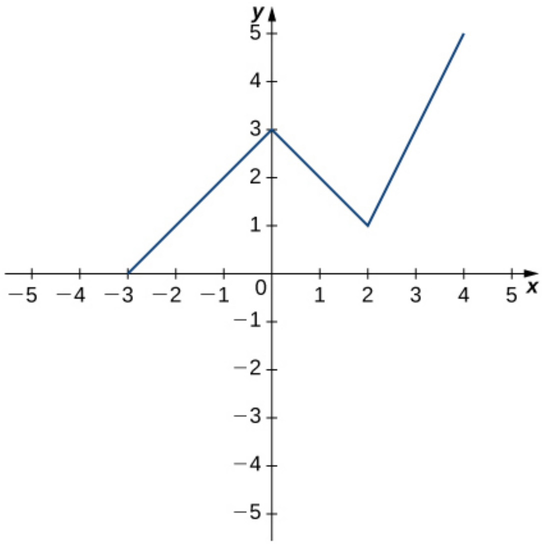
78.



79.



80. Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the following functions, use $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

81. $f(x) = 2 - 3x$

82. $f(x) = 4x^2$

83. $f(x) = x + \frac{1}{x}$

For the following exercises, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

84. **[T]** $f(x) = -\frac{5}{x}$

85. **[T]** $f(x) = 3x^2 + 2x + 4.$

86. **[T]** $f(x) = \sqrt{x} + 3x$

87. **[T]** $f(x) = \frac{1}{\sqrt{2x}}$

88. **[T]** $f(x) = 1 + x + \frac{1}{x}$

89. **[T]** $f(x) = x^3 + 1$

For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\frac{f(x+h) - f(x)}{h}$

b. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

90. $P(x)$ denotes the population of a city at time x in years.

91. $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

92. $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios.

93. $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

94. $B(x)$ denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

95. $p(x)$ denotes atmospheric pressure at an altitude of x feet.

96. Sketch the graph of a function $y = f(x)$ with all of the following properties:

a. $f'(x) > 0$ for $-2 \le x < 1$

b. $f'(2) = 0$

c. $f'(x) > 0$ for $x > 2$

d. $f(2) = 2$ and $f(0) = 1$

e. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = \infty$

f. $f'(1)$ does not exist.

97. Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

a. Give a physical interpretation, with units, of $T'(x)$.

b. If we know that $T'(1000) = -0.1$, explain the physical meaning.

98. Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

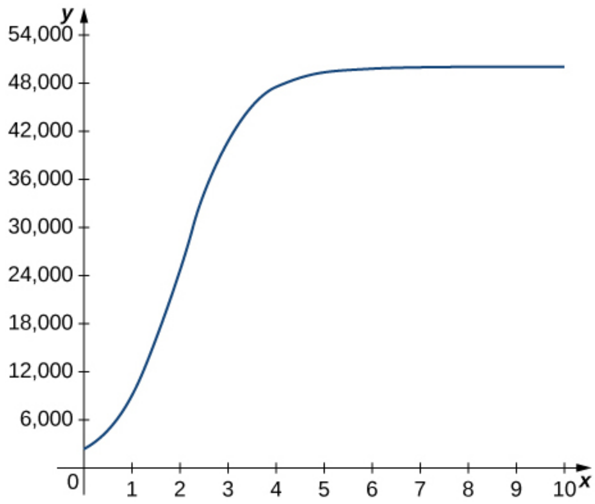
a. What does $\frac{P(b) - P(a)}{b - a}$ for $0 < a < b$ measure, and what are the units?

b. What does $P'(x)$ measure, and what are the units?

c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

99. The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of $50{,}000$ citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.
b. What does the derivative tell us about how this town is affected by the flu outbreak?



For the following exercises, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

100. What is the physical meaning of $h'(t)$? What are the units?

101. **[T]** Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (*Hint:* for **interior points**, estimate both the left limit and right limit and average them. An interior point of an interval I is an element of I which is not an endpoint of I.)

102. **[T]** The best linear fit to the data is given by $H(t) = 7.229t - 4.905,$ where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t)$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

103. **[T]** The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429,$ where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

104. **[T]** The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683,$ where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

105. Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

3.3 | Differentiation Rules

Learning Objectives

3.3.1 State the constant, constant multiple, and power rules.
3.3.2 Apply the sum and difference rules to combine derivatives.
3.3.3 Use the product rule for finding the derivative of a product of functions.
3.3.4 Use the quotient rule for finding the derivative of a quotient of functions.
3.3.5 Extend the power rule to functions with negative exponents.
3.3.6 Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Theorem 3.2: The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.17

Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule:

$$f'(x) = 0.$$

3.11 Find the derivative of $g(x) = -3$.

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, note that the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.18

Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution

$$\frac{d}{dx}\left(x^3\right) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2 h + 3xh^2 + h^3 - x^3}{h}$$

Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2 h$. All other terms contain powers of h that are two or greater.

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$$

In this step the x^3 terms have been cancelled, leaving only terms containing h.

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

Factor out the common factor of h.

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

After cancelling the common factor of h, the only term not containing h is $3x^2$.

$$= 3x^2$$

Let h go to 0.

3.12 Find $\frac{d}{dx}\left(x^4\right)$.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

Theorem 3.3: The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$\text{Since } (x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-1} + h^n\right) \\ &= nx^{n-1}. \end{aligned}$$

□

Example 3.19

Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10$, we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

3.13 Find the derivative of $f(x) = x^7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Theorem 3.4: Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) + g(x),\ j'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g:

$$\frac{d}{dx}(f(x)-g(x))=\frac{d}{dx}(f(x))-\frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x)=f(x)-g(x),\ j'(x)=f'(x)-g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } j(x)=kf(x),\ j'(x)=kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $j(x)=f(x)+g(x)$. Using the limit definition of the derivative we have

$$j'(x)=\lim_{h\to 0}\frac{j(x+h)-j(x)}{h}.$$

By substituting $j(x+h)=f(x+h)+g(x+h)$ and $j(x)=f(x)+g(x)$, we obtain

$$j'(x)=\lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}.$$

Rearranging and regrouping the terms, we have

$$j'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$j'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)+\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=f'(x)+g'(x).$$

□

Example 3.20

Applying the Constant Multiple Rule

Find the derivative of $g(x)=3x^2$ and compare it to the derivative of $f(x)=x^2$.

Solution

We use the power rule directly:

$$g'(x)=\frac{d}{dx}(3x^2)=3\frac{d}{dx}(x^2)=3(2x)=6x.$$

Since $f(x)=x^2$ has derivative $f'(x)=2x$, we see that the derivative of $g(x)$ is 3 times the derivative of

$f(x)$. This relationship is illustrated in **Figure 3.18**.

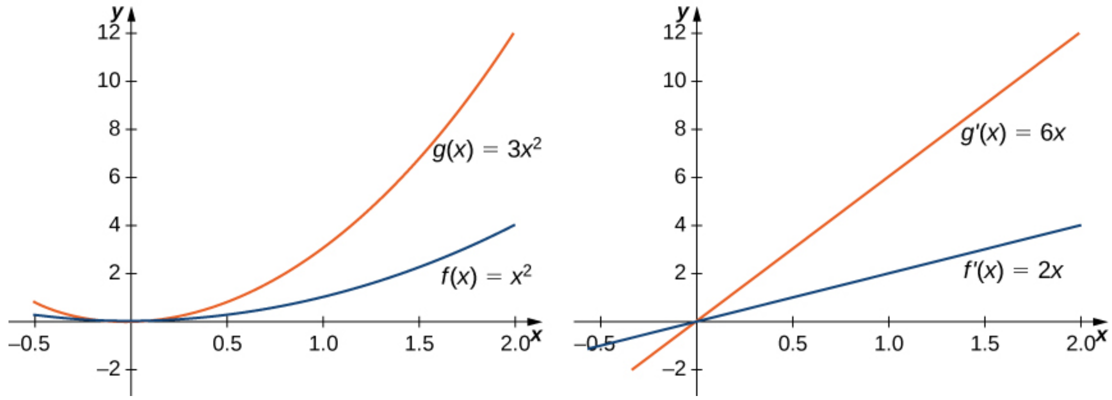


Figure 3.18 The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.21

Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\ &= \frac{d}{dx}\left(2x^5\right) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2\left(5x^4\right) + 0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4. && \text{Simplify.} \end{aligned}$$

3.14 Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Example 3.22

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

3.15 Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$

Theorem 3.5: Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\big(f(x)g(x)\big) = \frac{d}{dx}\big(f(x)\big) \cdot g(x) + \frac{d}{dx}\big(g(x)\big) \cdot f(x).$$

That is,

$$\text{if } j(x) = f(x)g(x), \text{ then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x + h) = g(x).$

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \lim_{h \to 0}\left(\frac{f(x)g(x+h) - f(x)g(x)}{h}\right).$$

Rearranging, we obtain

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right) + \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h} \cdot f(x)\right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 3.23

Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example 3.24

Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)\left(3x^3 - 5x\right) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

$$j'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $j(x) = 3x^5 + x^3 - 10x$ and, consequently, $j'(x) = 15x^4 + 3x^2 - 10$.

3.16 Use the product rule to obtain the derivative of $j(x) = 2x^5\left(4x^2 + x\right)$.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}\left(x^2\right) = 2x, \text{ not } \frac{\frac{d}{dx}\left(x^3\right)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Theorem 3.6: The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.25

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k(x) = \frac{5x^2}{4x + 3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the quotient rule, we have

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x + 3) - 4(5x^2)}{(4x + 3)^2}.$$

Simplifying, we obtain

$$k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}.$$

3.17 Find the derivative of $h(x) = \frac{3x+1}{4x-3}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Theorem 3.7: Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n, $x^{-n} = \frac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{\mathrm{dx}}(x^{-n}) = \frac{0(x^n) - 1\left(nx^{n-1}\right)}{(x^n)^2}.$$

Simplifying, we see that

$$\frac{d}{\mathrm{dx}}(x^{-n}) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

□

Example 3.26

Using the Extended Power Rule

Find $\frac{d}{dx}\left(x^{-4}\right)$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.27

Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find the derivative of $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}\left(6x^{-2}\right) && \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ &= 6\frac{d}{dx}(x^{-2}) && \text{Apply the constant multiple rule.} \\ &= 6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &= -12x^{-3} && \text{Simplify.} \end{aligned}$$

3.18 Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.28

Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution

Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$$\begin{aligned} k'(x) &= \frac{d}{dx}\left(3h(x) + x^2 g(x)\right) = \frac{d}{dx}(3h(x)) + \frac{d}{dx}\left(x^2 g(x)\right) && \text{Apply the sum rule.} \\ &= 3\frac{d}{dx}(h(x)) + \left(\frac{d}{dx}\left(x^2\right)g(x) + \frac{d}{dx}(g(x))x^2\right) && \text{Apply the constant multiple rule to differentiate } 3h(x) \text{ and the product rule to differentiate } x^2 g(x). \\ &= 3h'(x) + 2xg(x) + g'(x)x^2 \end{aligned}$$

Example 3.29

Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$\begin{aligned} k'(x) &= \frac{d}{dx}(f(x)g(x)) \cdot h(x) + \frac{d}{dx}(h(x)) \cdot (f(x)g(x)) && \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x). \\ &= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) && \text{Apply the product rule to } f(x)g(x). \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). && \text{Simplify.} \end{aligned}$$

Example 3.30

Combining the Quotient Rule and the Product Rule

For $h(x) = \frac{2x^3 k(x)}{3x + 2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}\left(2x^3 k(x)\right) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot \left(2x^3 k(x)\right)}{(3x+2)^2} && \text{Apply the quotient rule.} \\ &= \frac{\left(6x^2 k(x) + k'(x) \cdot 2x^3\right)(3x+2) - 3\left(2x^3 k(x)\right)}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}\left(2x^3 k(x)\right). \text{ Use } \frac{d}{dx}(3x+2) = 3. \\ &= \frac{-6x^3 k(x) + 18x^3 k(x) + 12x^2 k(x) + 6x^4 k'(x) + 4x^3 k'(x)}{(3x+2)^2} && \text{Simplify.} \end{aligned}$$

3.19 Find $\frac{d}{dx}(3f(x) - 2g(x))$.

Example 3.31

Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4),$$

we must solve $(3x - 2)(x - 4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$ as shown in the following graph.

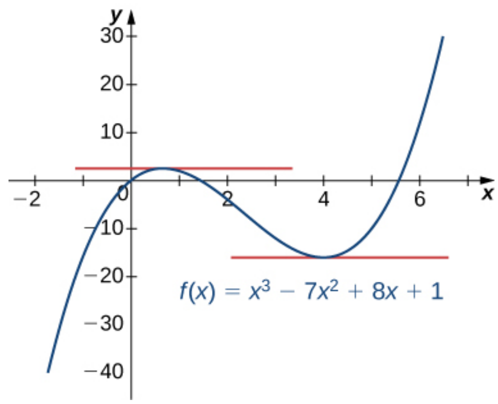


Figure 3.19 This function has horizontal tangent lines at $x = 2/3$ and $x = 4$.

Example 3.32

Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2 + 1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0) = s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t) = \frac{1(t^2 + 1) - 2t(t)}{(t^2 + 1)^2} = \frac{1 - t^2}{(t^2 + 1)^2}.$$

After evaluating, we see that $v(0) = 1$.

3.20 Find the values of x for which the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Student PROJECT

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (**Figure 3.20**).

Figure 3.20 The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (**Figure 3.21**). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

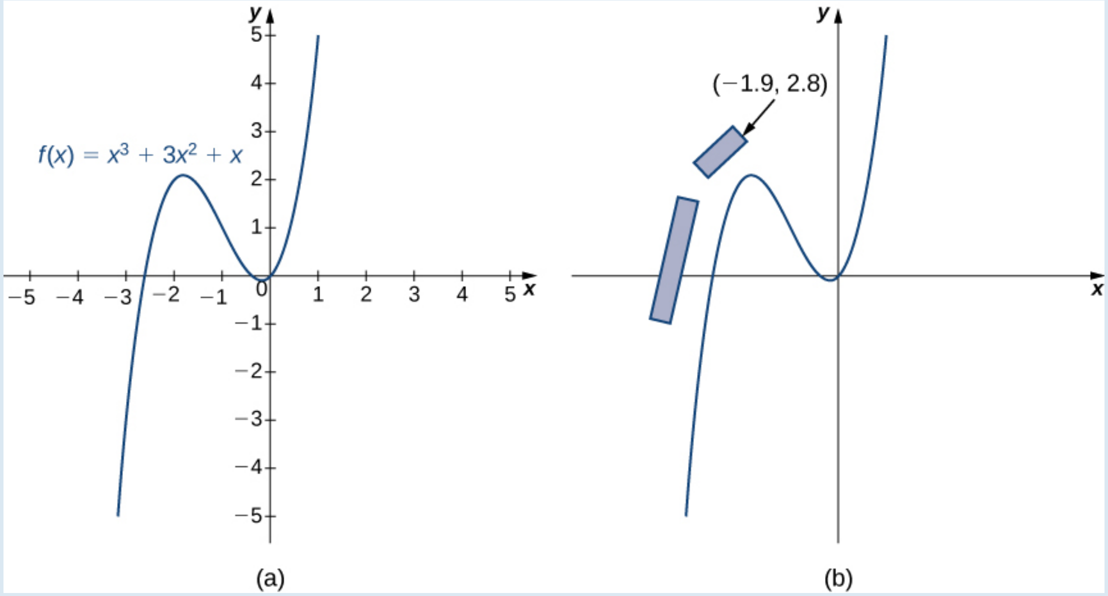


Figure 3.21 (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9,\ 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the $(x,\ y)$ coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the *x*-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5,\ 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

3.3 EXERCISES

For the following exercises, find $f'(x)$ for each function.

106. $f(x) = x^7 + 10$

107. $f(x) = 5x^3 - x + 1$

108. $f(x) = 4x^2 - 7x$

109. $f(x) = 8x^4 + 9x^2 - 1$

110. $f(x) = x^4 + \frac{2}{x}$

111. $f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$

112. $f(x) = (x+2)\left(2x^2 - 3\right)$

113. $f(x) = x^2\left(\frac{2}{x^2} + \frac{5}{x^3}\right)$

114. $f(x) = \frac{x^3 + 2x^2 - 4}{3}$

115. $f(x) = \frac{4x^3 - 2x + 1}{x^2}$

116. $f(x) = \frac{x^2 + 4}{x^2 - 4}$

117. $f(x) = \frac{x + 9}{x^2 - 7x + 1}$

For the following exercises, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

118. **[T]** $y = 3x^2 + 4x + 1$ at $(0, 1)$

119. **[T]** $y = 2\sqrt{x} + 1$ at $(4, 5)$

120. **[T]** $y = \frac{2x}{x - 1}$ at $(-1, 1)$

121. **[T]** $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

122. $h(x) = 4f(x) + \frac{g(x)}{7}$

123. $h(x) = x^3 f(x)$

124. $h(x) = \frac{f(x)g(x)}{2}$

125. $h(x) = \frac{3f(x)}{g(x) + 2}$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f'(x)$	−1	7	8	−3
$g'(x)$	4	1	2	9

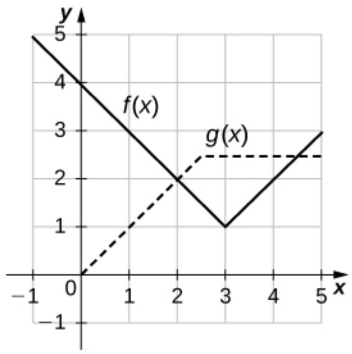
126. Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

127. Find $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$.

128. Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

129. Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

For the following exercises, use the following figure to find the indicated derivatives, if they exist.



130. Let $h(x) = f(x) + g(x)$. Find

a. $h'(1)$,

b. $h'(3)$, and

c. $h'(4)$.

131. Let $h(x) = f(x)g(x)$. Find

a. $h'(1)$,

b. $h'(3)$, and

c. $h'(4)$.

132. Let $h(x) = \frac{f(x)}{g(x)}$. Find

a. $h'(1)$,

b. $h'(3)$, and

c. $h'(4)$.

For the following exercises,

a. evaluate $f'(a)$, and

b. graph the function $f(x)$ and the tangent line at $x = a$.

133. **[T]** $f(x) = 2x^3 + 3x - x^2,\ a = 2$

134. **[T]** $f(x) = \frac{1}{x} - x^2,\ a = 1$

135. **[T]** $f(x) = x^2 - x^{12} + 3x + 2,\ a = 0$

136. **[T]** $f(x) = \frac{1}{x} - x^{2/3},\ a = -1$

137. Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

138. Find the equation of the tangent line to the graph of $f(x) = x^2 + \frac{4}{x} - 10$ at $x = 8$.

139. Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

140. Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x intercept of 6.

141. Find the equation of the line passing through the point $P(3,\ 3)$ and tangent to the graph of $f(x) = \frac{6}{x - 1}$.

142. Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which

a. the tangent line is horizontal

b. the tangent line has a slope of -1.

143. Find a quadratic polynomial such that $f(1) = 5,\ f'(1) = 3$ and $f''(1) = -6$.

144. A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.

b. Determine the acceleration of the car when the velocity is 0.

145. **[T]** A herring swimming along a straight line has traveled $s(t) = \frac{t^2}{t^2 + 2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

146. The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \frac{8t + 3}{0.2t^2 + 1}$, where t is measured in years.

a. Determine the initial flounder population.

b. Determine $P'(10)$ and briefly interpret the result.

147. **[T]** The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \frac{2t^2 + t}{t^3 + 50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

b. Determine the rate of change for $t = 8,\ 12,\ 24$, and 36.

c. Briefly describe what seems to be occurring as the number of hours increases.

148. A book publisher has a cost function given by $C(x) = \frac{x^3 + 2x + 3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

149. **[T]** According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1 m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that $G, m_1,$ and m_2 are constants. Find the rate of change of force F with respect to distance d.
b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/kg^2, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

3.4 | Derivatives as Rates of Change

Learning Objectives

3.4.1 Determine a new value of a quantity from the old value and the amount of change.
3.4.2 Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
3.4.3 Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
3.4.4 Predict the future population from the present value and the population growth rate.
3.4.5 Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of h, $f'(a) \approx \frac{f(a+h) - f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h) \approx f(a) + f'(a)h. \tag{3.10}$$

We can use this formula if we know only $f(a)$ and $f'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in **Figure 3.22**, we are approximating $f(a+h)$ by the y coordinate at $a+h$ on the line tangent to $f(x)$ at $x = a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'(a)$.

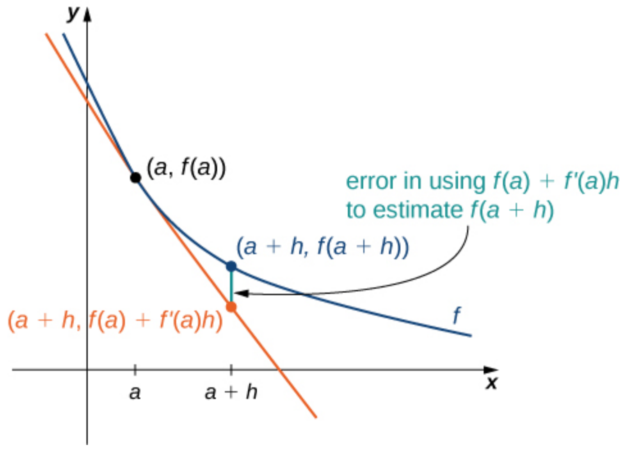


Figure 3.22 The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)$h.

Here is an interesting **demonstration (http://www.openstax.org/l/20_chainrule)** of rate of change.

Example 3.33

Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

3.21 Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

Definition

Let $s(t)$ be a function giving the position of an object at time t.

The velocity of the object at time t is given by $v(t) = s'(t)$.

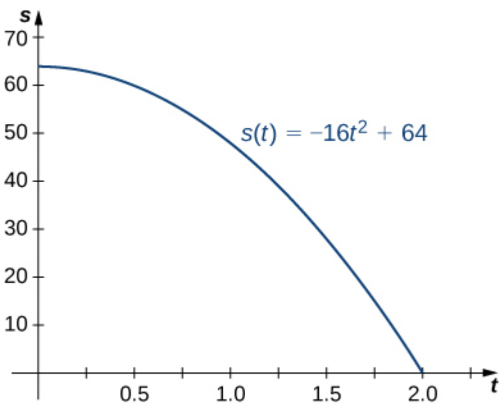
The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.34

Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64.$



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0.$ Solving $-16t^2 + 64 = 0,$ we get $t = 2,$ so it take 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2).$ Since $v(t) = s'(t) = -32t,$ we obtain $v(t) = -64$ ft/s.
b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s.}$$

Example 3.35

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2.$ Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t).$

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t.$

Evaluating these functions at $t = 1,$ we obtain $v(1) = -1$ and $a(1) = 6.$

a. Because $v(1) < 0,$ the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0,$ velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.36

Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4, t \geq 0.$

a. Find $v(t).$

b. At what time(s) is the particle at rest?

c. On what time intervals is the particle moving from left to right? From right to left?

d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0,$ so set $3t^2 - 18t + 24 = 0.$ Factoring the left-hand side of the equation produces $3(t - 2)(t - 4) = 0.$ Solving, we find that the particle is at rest at $t = 2$ and $t = 4.$

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0.$ **Figure 3.23** gives the analysis of the sign of $v(t)$ for $t \geq 0,$ but it does not represent the axis along which the particle is moving.

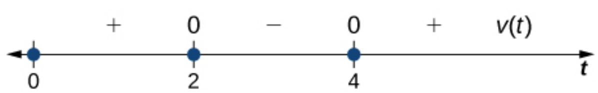


Figure 3.23 The sign of v(t) determines the direction of the particle.

Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (2, +\infty),$ the particle is moving from left to right on these intervals.

Since $3t^2 - 18t + 24 < 0$ on $(2, 4),$ the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4).$ We have $s(0) = 4, s(2) = 24,$ and $s(4) = 20.$ This means that the particle begins on the coordinate axis at 4 and changes direction at 0 and

20 on the coordinate axis. The path of the particle is shown on a coordinate axis in **Figure 3.24**.

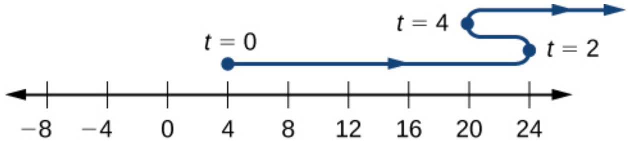


Figure 3.24 The path of the particle can be determined by analyzing v(t).

3.22 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1.$ Is the particle moving from right to left or from left to right at time $t = 3$?

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 3.37

Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30.$ Now estimate $P'(0),$ the current growth rate, using

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4.$$

By applying **Equation 3.10** to $P(t),$ we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

3.23 The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3{,}000.$ If $P'(0) = 100,$ estimate the size of the population in 3 days, where t is measured in days.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.

If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.

If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.38

Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x,\ 0 \le x \le 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \le x \le 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

3.24 Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50.$ Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

3.4 EXERCISES

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

150. $s(t) = 2t^3 - 3t^2 - 12t + 8$

151. $s(t) = 2t^3 - 15t^2 + 36t - 10$

152. $s(t) = \frac{t}{1 + t^2}$

153. A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.
b. Find the acceleration of the rocket 3 seconds after being fired.

154. A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.
b. Determine the velocity of the ball when it hits the ground.

155. The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.
b. Determine the velocity of the car when $s(t) = 14$.

156. The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.
b. Determine the acceleration of the bird at $t = 1$ sec.
c. Determine the acceleration of the bird when the velocity equals 0.

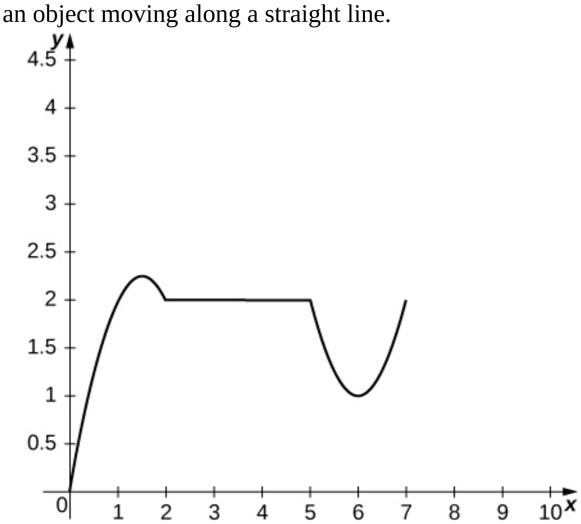
157. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.
b. Find the speed of the potato at 0.5 s and 5.75 s.
c. Determine when the potato reaches its maximum height.
d. Find the acceleration of the potato at 0.5 s and 1.5 s.
e. Determine how long the potato is in the air.
f. Determine the velocity of the potato upon hitting the ground.

158. The position function $s(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t) = 0$.
b. Determine the direction the train is traveling when $a(t) = 0$.
c. Determine the time intervals when the train is slowing down or speeding up.

159. The following graph shows the position $y = s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.
b. Sketch the graph of the velocity function.
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.
d. Determine the time intervals when the object is speeding up or slowing down.

160. The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.
b. Use the marginal cost function to estimate the cost of manufacturing the thirteenth food processor.
c. Find the actual cost of manufacturing the thirteenth food processor.

161. The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$.
b. Find the marginal revenue function.
c. Find the marginal revenue at $x = 2000$ and 5000.

162. **[T]** A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.
b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

163. **[T]** In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$. Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75{,}000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.
b. Find the revenue and marginal revenue functions.
c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.
d. Find the profit and marginal profit functions.
e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

164. A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.
b. Find $P'(1), P'(2), P'(3)$, and $P'(4)$. Interpret what the results mean for the town.
c. Find $P''(1), P''(2), P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

165. **[T]** A culture of bacteria grows in number according to the function $N(t) = 3000\left(1 + \frac{4t}{t^2 + 100}\right)$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.
b. Find $N'(0), N'(10), N'(20)$, and $N'(30)$.
c. Interpret the results in (b).
d. Find $N''(0), N''(10), N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

166. The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.
b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Years since 1800	Population (millions)
1	0.8795
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Table 3.4 Population of London **Source: http://en.wikipedia.org/wiki/Demographics_of_London.**

167. **[T]**
   a. Using a calculator or a computer program, find the best-fit linear function to measure the population.
   b. Find the derivative of the equation in a. and explain its physical meaning.
   c. Find the second derivative of the equation and explain its physical meaning.

168. **[T]**
   a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.
   b. Find the derivative of the equation and explain its physical meaning.
   c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	−1
2	−2
3	−5
4	−7
5	−14

169. **[T]**
   a. Using a calculator or computer program, find the best-fit quadratic curve to the data.
   b. Find the derivative of the position function and explain its physical meaning.
   c. Find the second derivative of the position function and explain its physical meaning.

170. **[T]**
   a. Using a calculator or computer program, find the best-fit cubic curve to the data.
   b. Find the derivative of the position function and explain its physical meaning.
   c. Find the second derivative of the position function and explain its physical meaning.
   d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

171. **[T]** The Holling type I equation is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.
   a. Graph the Holling type I equation, given $a = 0.5$.
   b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.
   c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.
   d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

172. **[T]** The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?
b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.
c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.
d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

173. **[T]** The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2 + x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.
c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).
d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

174. **[T]** The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
55	15
65	55
95	60

Table 3.5 Snowshoe Hare and Lynx Populations **Source: http://www.biotopics.co.uk/newgcse/predatorprey.html.**

a. Graph the data points and determine which Holling-type function fits the data best.
b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.
c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

3.5 | Derivatives of Trigonometric Functions

Learning Objectives

- **3.5.1** Find the derivatives of the sine and cosine function.
- **3.5.2** Find the derivatives of the standard trigonometric functions.
- **3.5.3** Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting $D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (**Figure 3.25**).

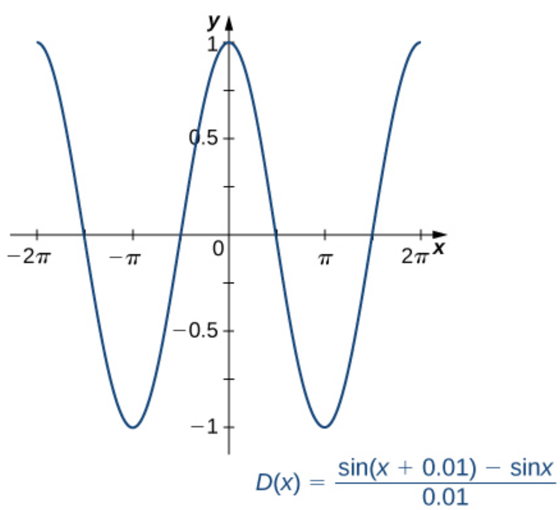


Figure 3.25 The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.8: The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x \tag{3.11}$$

$$\frac{d}{dx}(\cos x) = -\sin x \tag{3.12}$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in **Introduction to Limits**:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in **Figure 3.26**.

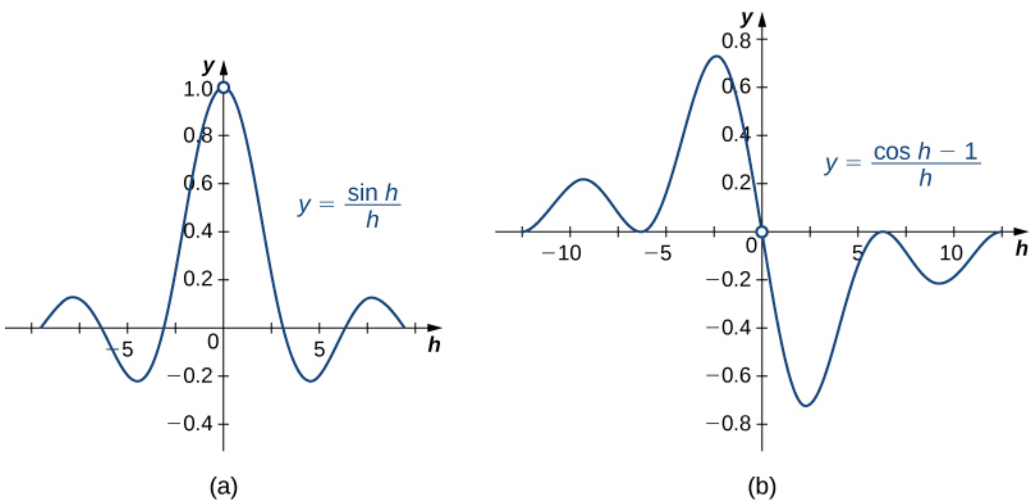


Figure 3.26 These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Apply the definition of the derivative.} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.} \\
&= \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) && \text{Regroup.} \\
&= \lim_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) && \text{Factor out } \sin x \text{ and } \cos x. \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Apply trig limit formulas.} \\
&= \cos x && \text{Simplify.}
\end{aligned}$$

□

Figure 3.27 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.

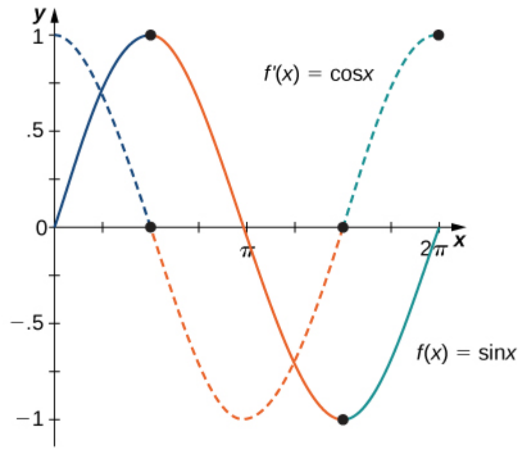


Figure 3.27 Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$ that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.

Example 3.39

Differentiating a Function Containing sin *x*

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution

Using the product rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^3\right) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

3.25 Find the derivative of $f(x) = \sin x \cos x$.

Example 3.40

Finding the Derivative of a Function Containing cos *x*

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{\left(4x^2\right)^2}.$$

Simplifying, we obtain

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x\cos x}{16x^4} \\ &= \frac{-x\sin x - 2\cos x}{4x^3}. \end{aligned}$$

3.26 Find the derivative of $f(x) = \frac{x}{\cos x}$.

Example 3.41

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2\cos t - 1,$$

so we must solve

$$2\cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

3.27 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example 3.42

The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1,$ by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

3.28 Find the derivative of $f(x) = \cot x$.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Theorem 3.9: Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.13}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.14}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.15}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{3.16}$$

Example 3.43

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example 3.44

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x\tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x\tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x\cot x,$ and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x\tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x\cot x + \tan x + x\sec^2 x.$$

3.29 Find the derivative of $f(x) = 2\tan x - 3\cot x$.

3.30 Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example 3.45

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2 y}{dx^2} &= -\sin x \\ \frac{d^3 y}{dx^3} &= -\cos x \\ \frac{d^4 y}{dx^4} &= \sin x. \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of sin x equals sin x, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$
$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

3.31 For $y = \cos x$, find $\frac{d^4 y}{dx^4}$.

Example 3.46

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

3.32 For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Example 3.47

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s'(t)$:

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

3.33 A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

3.5 EXERCISES

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

175. $y = x^2 - \sec x + 1$

176. $y = 3\csc x + \frac{5}{x}$

177. $y = x^2 \cot x$

178. $y = x - x^3 \sin x$

179. $y = \frac{\sec x}{x}$

180. $y = \sin x \tan x$

181. $y = (x + \cos x)(1 - \sin x)$

182. $y = \frac{\tan x}{1 - \sec x}$

183. $y = \frac{1 - \cot x}{1 + \cot x}$

184. $y = \cos x(1 + \csc x)$

For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

185. **[T]** $f(x) = -\sin x,\ x = 0$

186. **[T]** $f(x) = \csc x,\ x = \frac{\pi}{2}$

187. **[T]** $f(x) = 1 + \cos x,\ x = \frac{3\pi}{2}$

188. **[T]** $f(x) = \sec x,\ x = \frac{\pi}{4}$

189. **[T]** $f(x) = x^2 - \tan x,\ \ x = 0$

190. **[T]** $f(x) = 5\cot x,\ \ x = \frac{\pi}{4}$

For the following exercises, find $\frac{d^2y}{dx^2}$ for the given functions.

191. $y = x\sin x - \cos x$

192. $y = \sin x \cos x$

193. $y = x - \frac{1}{2}\sin x$

194. $y = \frac{1}{x} + \tan x$

195. $y = 2\csc x$

196. $y = \sec^2 x$

197. Find all x values on the graph of $f(x) = -3\sin x \cos x$ where the tangent line is horizontal.

198. Find all x values on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

199. Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

200. **[T]** A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6\cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

201. Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is 3 cm/s when $t = 0$, find the values of a and b.

202. After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.
b. Find the velocity function.
c. Sketch one period of the velocity function for $t \geq 0$.
d. Determine the times when the velocity is 0 over one period.
e. Find the acceleration function.
f. Sketch one period of the acceleration function for $t \geq 0$.

203. The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5\sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

204. **[T]** The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3\cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

205. $\frac{d}{dx}(\cot x) = -\csc^2 x$

206. $\frac{d}{dx}(\sec x) = \sec x \tan x$

207. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

208. Use the definition of derivative and the identity $\cos(x + h) = \cos x \cos h - \sin x \sin h$ to prove that $\frac{d(\cos x)}{dx} = -\sin x.$

For the following exercises, find the requested higher-order derivative for the given functions.

209. $\frac{d^3 y}{dx^3}$ of $y = 3\cos x$

210. $\frac{d^2 y}{dx^2}$ of $y = 3\sin x + x^2 \cos x$

211. $\frac{d^4 y}{dx^4}$ of $y = 5\cos x$

212. $\frac{d^2 y}{dx^2}$ of $y = \sec x + \cot x$

213. $\frac{d^3 y}{dx^3}$ of $y = x^{10} - \sec x$

3.6 | The Chain Rule

Learning Objectives

3.6.1 State the chain rule for the composition of two functions.
3.6.2 Apply the chain rule together with the power rule.
3.6.3 Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
3.6.4 Recognize the chain rule for a composition of three or more functions.
3.6.5 Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions $(x^n, \sin x, \cos x, \text{etc.})$ as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3)_{x = a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a$, $u \to a^3$:

$$\begin{aligned}\lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin\left(a^3\right)}{u - a^3} \\ &= \frac{d}{du}(\sin u)_{u = a^3} \\ &= \cos\left(a^3\right).\end{aligned}$$

Thus, $h'(a) = \cos\left(a^3\right) \cdot 3a^2$.

In other words, if $h(x) = \sin\left(x^3\right)$, then $h'(x) = \cos\left(x^3\right) \cdot 3x^2$. Thus, if we think of $h(x) = \sin\left(x^3\right)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) =$ sin x and $g(x) = x^3$, then the derivative of $h(x) = \sin\left(x^3\right)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin(g(x))$, it is quite likely that $h'(x) = \cos(g(x))g'(x)$. As we determined above, this is the case for $h(x) = \sin\left(x^3\right)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x))g'(x). \tag{3.17}$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Watch an **animation (http://www.openstax.org/l/20_chainrule2)** of the chain rule.

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n(g(x))^{n-1} g'(x)$$

Rule: Power Rule for Composition of Functions

For all values of x for which the derivative is defined, if

$$h(x) = (g(x))^n.$$

Then

$$h'(x) = n(g(x))^{n-1} g'(x). \tag{3.18}$$

Example 3.48

Using the Chain and Power Rules

Find the derivative of $h(x) = \dfrac{1}{\left(3x^2 + 1\right)^2}$.

Solution

First, rewrite $h(x) = \dfrac{1}{\left(3x^2 + 1\right)^2} = \left(3x^2 + 1\right)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2\left(3x^2 + 1\right)^{-3}(6x).$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}.$$

3.34 Find the derivative of $h(x) = \left(2x^3 + 2x - 1\right)^4$.

Example 3.49

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 3.50

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$. Therefore, the line has equation $y - 1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x + 13$.

3.35 Find the equation of the line tangent to the graph of $f(x) = \left(x^2 - 2\right)^3$ at $x = -2$.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 3.51

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution

Think of $h(x)=\cos(g(x))$ as $f(g(x))$ where $f(x)=\cos x$. Since $f'(x)=-\sin x$. we have $f'(g(x))=-\sin(g(x))$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))g'(x) && \text{Substitute } f'(g(x)) = -\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x)=\cos(g(x))$ is given by $h'(x)=-\sin(g(x))g'(x)$.

In the following example we apply the rule that we have just derived.

Example 3.52

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x)=\cos\left(5x^2\right)$.

Solution

Let $g(x)=5x^2$. Then $g'(x)=10x$. Using the result from the previous example,

$$\begin{aligned} h'(x) &= -\sin\left(5x^2\right)\cdot 10x \\ &= -10x\sin\left(5x^2\right). \end{aligned}$$

Example 3.53

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x)=\sec\left(4x^5+2x\right)$.

Solution

Apply the chain rule to $h(x)=\sec(g(x))$ to obtain

$$h'(x)=\sec(g(x)\tan(g(x))g'(x).$$

In this problem, $g(x)=4x^5+2x$, so we have $g'(x)=20x^4+2$. Therefore, we obtain

$$\begin{aligned} h'(x) &= \sec\left(4x^5+2x\right)\tan\left(4x^5+2x\right)\left(20x^4+2\right) \\ &= (20x^4+2)\sec\left(4x^5+2x\right)\tan\left(4x^5+2x\right). \end{aligned}$$

3.36 Find the derivative of $h(x)=\sin(7x+2)$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in **Example 3.51** and **Example 3.53**. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Theorem 3.10: Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$$\begin{aligned}
\frac{d}{dx}(\sin(g(x)) &= \cos(g(x))g'(x) & \frac{d}{dx}\sin u &= \cos u\frac{du}{dx} \\
\frac{d}{dx}(\cos(g(x)) &= -\sin(g(x))g'(x) & \frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}(\tan(g(x)) &= \sec^2(g(x))g'(x) & \frac{d}{dx}\tan u &= \sec^2 u\frac{du}{dx} \\
\frac{d}{dx}(\cot(g(x)) &= -\csc^2(g(x))g'(x) & \frac{d}{dx}\cot u &= -\csc^2 u\frac{du}{dx} \\
\frac{d}{dx}(\sec(g(x)) &= \sec(g(x)\tan(g(x))g'(x) & \frac{d}{dx}\sec u &= \sec u\tan u\frac{du}{dx} \\
\frac{d}{dx}(\csc(g(x)) &= -\csc(g(x))\cot(g(x))g'(x) & \frac{d}{dx}\csc u &= -\csc u\cot u\frac{du}{dx}.
\end{aligned}$$

Example 3.54

Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left((2x+1)^5\right)\cdot(3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right)\cdot(2x+1)^5 && \text{Apply the product rule.} \\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.} \\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\
&= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6. \\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}$$

3.37 Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h(f(g(x))).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}(h(f(g(x))) = h'(f(g(x))) \cdot \frac{d}{dx} f((g(x))).$$

Applying the chain rule again, we obtain

$$k'(x) = h'(f(g(x))f'(g(x))g'(x)).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 3.55

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \left(\cos\left(7x^2 + 1\right)\right)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(\frac{d}{dx}\cos\left(7x^2+1\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)\left(\frac{d}{dx}\left(7x^2+1\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)(14x) && \text{Apply the chain rule.} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify.} \end{aligned}$$

3.38 Find the derivative of $h(x) = \sin^6\left(x^3\right)$.

Example 3.56

Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Substituting $t = \frac{\pi}{6}$ into $v(t)$, we obtain $v\left(\frac{\pi}{6}\right) = -2.$

3.39 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(4t)$. Find its acceleration at time t.

Proof

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

Example 3.57

Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'(g(1))g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

3.40 Given $h(x) = f(g(x))$. If $g(2) = -3$, $g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx}, \; f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 3.58

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\ &= \frac{10x^4}{(3x+2)^6} && \text{Simplify.} \end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 3.59

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan\left(4x^2 - 3x + 1\right)$.

Solution

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x-3) && \text{Use } \frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x - 3) && \text{Substitute } u = 4x^2 - 3x + 1. \end{aligned}$$

3.41 Use Leibniz's notation to find the derivative of $y = \cos\left(x^3\right)$. Make sure that the final answer is expressed entirely in terms of the variable x.

3.6 EXERCISES

For the following exercises, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

214. $y = 3u - 6,\ u = 2x^2$

215. $y = 6u^3,\ u = 7x - 4$

216. $y = \sin u,\ u = 5x - 1$

217. $y = \cos u,\ u = \frac{-x}{8}$

218. $y = \tan u,\ u = 9x + 2$

219. $y = \sqrt{4u + 3},\ u = x^2 - 6x$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

220. $y = (3x - 2)^6$

221. $y = \left(3x^2 + 1\right)^3$

222. $y = \sin^5(x)$

223. $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

224. $y = \tan(\sec x)$

225. $y = \csc(\pi x + 1)$

226. $y = \cot^2 x$

227. $y = -6\sin^{-3} x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

228. $y = \left(3x^2 + 3x - 1\right)^4$

229. $y = (5 - 2x)^{-2}$

230. $y = \cos^3(\pi x)$

231. $y = \left(2x^3 - x^2 + 6x + 1\right)^3$

232. $y = \frac{1}{\sin^2(x)}$

233. $y = (\tan x + \sin x)^{-3}$

234. $y = x^2 \cos^4 x$

235. $y = \sin(\cos 7x)$

236. $y = \sqrt{6 + \sec \pi x^2}$

237. $y = \cot^3(4x + 1)$

238. Let $y = [f(x)]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

239. Let $y = \left(f(x) + 5x^2\right)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

240. Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

241. **[T]** Find the equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

242. **[T]** Find the equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

243. Find the x-coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

244. **[T]** Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

245. $h(x) = f(g(x));\, a = 0$

246. $h(x) = g(f(x));\, a = 0$

247. $h(x) = \left(x^4 + g(x)\right)^{-2};\, a = 1$

248. $h(x) = \left(\frac{f(x)}{g(x)}\right)^2;\, a = 3$

249. $h(x) = f(x + f(x));\, a = 1$

250. $h(x) = (1 + g(x))^3;\, a = 2$

251. $h(x) = g\left(2 + f\left(x^2\right)\right);\, a = 1$

252. $h(x) = f(g(\sin x));\, a = 0$

253. **[T]** The position function of a freight train is given by $s(t) = 100(t + 1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and
b. acceleration.
c. Using a. and b. is the train speeding up or slowing down?

254. **[T]** A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches: $s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right)$.

a. Determine the position of the spring at $t = 1.5$ s.
b. Find the velocity of the spring at $t = 1.5$ s.

255. **[T]** The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300.$ In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.
b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.
c. Use b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

256. **[T]** The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t + 7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.
b. Use a. to find the rate at which the area is expanding at $t = 4$ s.

257. **[T]** The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t + 1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.
b. Use a. to find the rate at which the volume is changing at $t = 1$ min.

258. **[T]** The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x - 2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

259. **[T]** The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

3.7 | Derivatives of Inverse Functions

Learning Objectives

3.7.1 Calculate the derivative of an inverse function.
3.7.2 Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. **Figure 3.28** shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $\left(a,\ f^{-1}(a)\right)$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $\left(f^{-1}\right)'(a) = \frac{p}{q}$. This point corresponds to a point $\left(f^{-1}(a),\ a\right)$ on the graph of $f(x)$ having a tangent line with a slope of $f'\left(f^{-1}(a)\right) = \frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

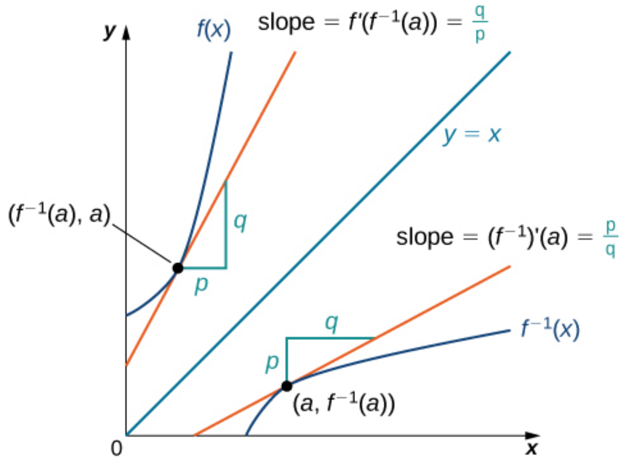


Figure 3.28 The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

$$1 = f'\left(f^{-1}(x)\right)\left(\left(f^{-1}\right)'(x)\right).$$

Solving for $\left(f^{-1}\right)'(x)$, we obtain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}. \tag{3.19}$$

We summarize this result in the following theorem.

Theorem 3.11: Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Example 3.60

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

3.42 Use the inverse function theorem to find the derivative of $g(x) = \frac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Example 3.61

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = 3x^2 \text{ and } f'(g(x)) = 3\left(\sqrt[3]{x}\right)^2 = 3x^{2/3}.$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

3.43 Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\frac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Theorem 3.12: Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}. \tag{3.20}$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}. \tag{3.21}$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = nx^{n-1} \text{ and } f'(g(x)) = n(x^{1/n})^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $\left(x^{1/n}\right)^m$ and apply the chain rule. Thus,

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

□

Example 3.62

Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} \text{ and } \left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

$$y = \frac{1}{3}x + \frac{4}{3}.$$

3.44 Find the derivative of $s(t) = \sqrt{2t+1}$.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example 3.63

Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

Solution

Since for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$, begin by finding $f'(x)$. Since

$$f'(x) = \cos x \text{ and } f'(g(x)) = \cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2},$$

we see that

$$g'(x) = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

Analysis

To see that $\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1 - x^2}$. This triangle is shown in **Figure 3.29**. Using the triangle, we see that $\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}$.

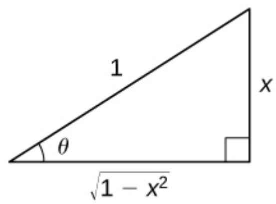


Figure 3.29 Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \cos(-\theta) = \sqrt{1 - x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, $x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1 - x^2} = 0$, we have

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}.$$

Finally, if $\theta = -$, $x = 0$ and $\cos\theta = \sqrt{1} = 1$.

Consequently, in all cases, $\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$.

Example 3.64

Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in **Example 3.61** to find the derivative of $h(x) = \sin^{-1}(g(x))$ and use this result to find the derivative of $h(x) = \sin^{-1}\left(2x^3\right)$.

Solution

Applying the chain rule to $h(x) = \sin^{-1}(g(x))$, we have

$$h'(x) = \frac{1}{\sqrt{1 - (g(x))^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}}. \end{aligned}$$

3.45 Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Theorem 3.13: Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-(x)^2}} \quad (3.22)$$

$$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}} \quad (3.23)$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+(x)^2} \quad (3.24)$$

$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+(x)^2} \quad (3.25)$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{(x)^2-1}} \quad (3.26)$$

$$\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{(x)^2-1}} \quad (3.27)$$

Example 3.65

Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}\left(x^2\right)$.

Solution

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into **Equation 3.24**, we obtain

$$f'(x) = \frac{1}{1+\left(x^2\right)^2} \cdot (2x).$$

Simplifying, we have

$$f'(x) = \frac{2x}{1+x^4}.$$

Example 3.66

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2.$$

3.46 Find the derivative of $h(x) = \cos^{-1}(3x-1)$.

Example 3.67

Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

$$v(t) = s'(t) = \frac{1}{1+\left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2+1}.$$

Thus, $v(1) = -\frac{1}{2}$.

3.47 Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

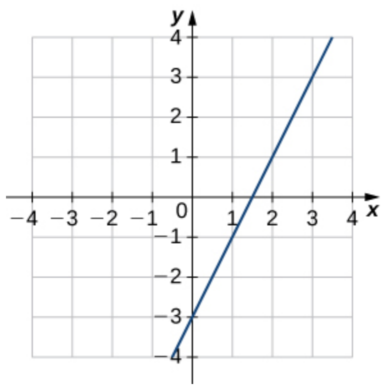
3.7 EXERCISES

For the following exercises, use the graph of $y = f(x)$ to

a. sketch the graph of $y = f^{-1}(x)$, and

b. use part a. to estimate $\left(f^{-1}\right)'(1)$.

260.



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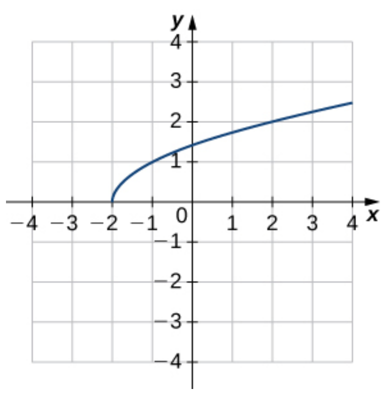

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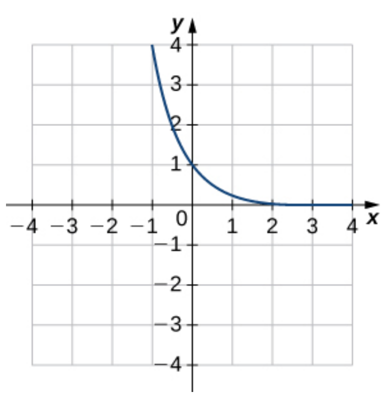

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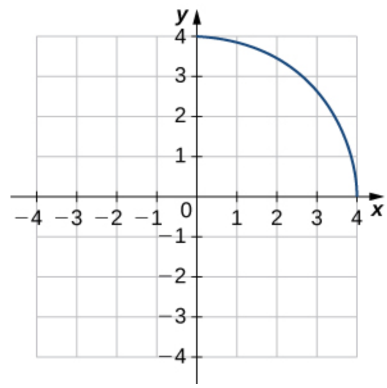


For the following exercises, use the functions $y = f(x)$ to find

a. $\dfrac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\dfrac{df^{-1}}{dy}$ at $y = f(a)$.

264. $f(x) = 6x - 1,\ x = -2$

265. $f(x) = 2x^3 - 3,\ x = 1$

266. $f(x) = 9 - x^2,\ 0 \le x \le 3,\ x = 2$

267. $f(x) = \sin x,\ x = 0$

For each of the following functions, find $\left(f^{-1}\right)'(a)$.

268. $f(x) = x^2 + 3x + 2,\ x \ge -\frac{3}{2},\ a = 2$

269. $f(x) = x^3 + 2x + 3,\ a = 0$

270. $f(x) = x + \sqrt{x},\ a = 2$

271. $f(x) = x - \frac{2}{x},\ x < 0,\ a = 1$

272. $f(x) = x + \sin x,\ a = 0$

273. $f(x) = \tan x + 3x^2,\ a = 0$

For each of the given functions $y = f(x)$,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and

b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

274. $f(x) = \frac{4}{1+x^2}$, $P(2,\ 1)$

275. $f(x) = \sqrt{x-4}$, $P(2,\ 8)$

276. $f(x) = \left(x^3+1\right)^4$, $P(16,\ 1)$

277. $f(x) = -x^3 - x + 2$, $P(-8,\ 2)$

278. $f(x) = x^5 + 3x^3 - 4x - 8$, $P(-8,\ 1)$

For the following exercises, find $\frac{dy}{dx}$ for the given function.

279. $y = \sin^{-1}\left(x^2\right)$

280. $y = \cos^{-1}(\sqrt{x})$

281. $y = \sec^{-1}\left(\frac{1}{x}\right)$

282. $y = \sqrt{\csc^{-1} x}$

283. $y = \left(1 + \tan^{-1} x\right)^3$

284. $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

285. $y = \frac{1}{\tan^{-1}(x)}$

286. $y = \sec^{-1}(-x)$

287. $y = \cot^{-1}\sqrt{4-x^2}$

288. $y = x \cdot \csc^{-1} x$

For the following exercises, use the given values to find $\left(f^{-1}\right)'(a)$.

289. $f(\pi) = 0,\ f'(\pi) = -1,\ a = 0$

290. $f(6) = 2,\ f'(6) = \frac{1}{3},\ a = 2$

291. $f\left(\frac{1}{3}\right) = -8,\ f'\left(\frac{1}{3}\right) = 2,\ a = -8$

292. $f(\sqrt{3}) = \frac{1}{2},\ f'(\sqrt{3}) = \frac{2}{3},\ a = \frac{1}{2}$

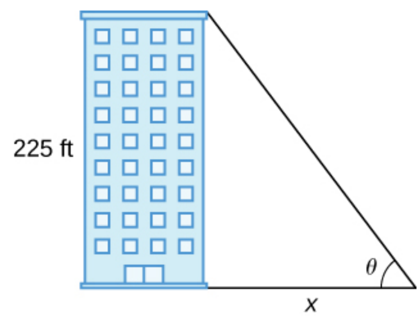
293. $f(1) = -3,\ f'(1) = 10,\ a = -3$

294. $f(1) = 0,\ f'(1) = -2,\ a = 0$

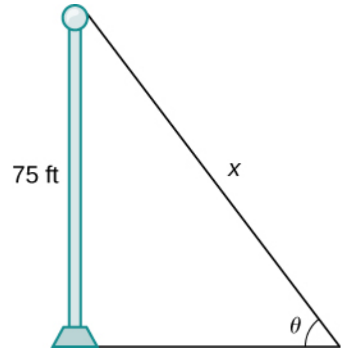
295. **[T]** The position of a moving hockey puck after t seconds is $s(t) = \tan^{-1} t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.
b. Find the acceleration of the puck at any time t.
c. Evaluate a. and b. for $t = 2,\ 4,$ and 6 seconds.
d. What conclusion can be drawn from the results in c.?

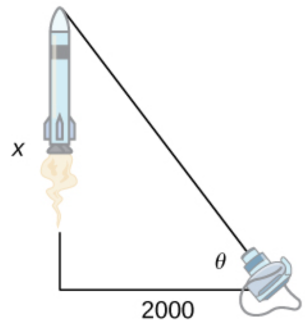
296. **[T]** A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



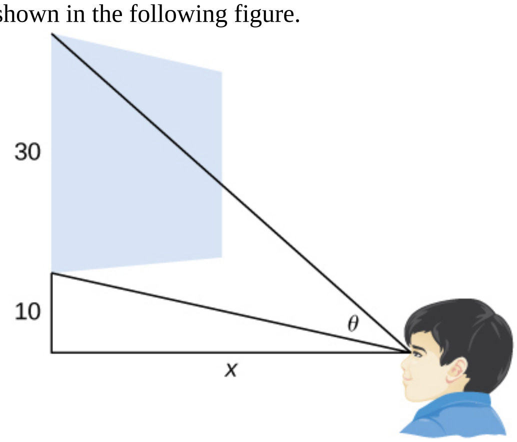
297. **[T]** A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



298. **[T]** A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



299. **[T]** A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$, where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.
b. Evaluate $\frac{d\theta}{dx}$ for $x = 5,\ 10,\ 15,$ and 20.
c. Interpret the results in b..
d. Evaluate $\frac{d\theta}{dx}$ for $x = 25,\ 30,\ 35,$ and 40
e. Interpret the results in d. At what distance x should the person stand to maximize his or her viewing angle?

3.8 | Implicit Differentiation

Learning Objectives

3.8.1 Find the derivative of a complicated function by using implicit differentiation.
3.8.2 Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$y = \sqrt{25 - x^2}$ and $y = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 < x < 0 \\ -\sqrt{25 - x^2} & \text{if } 0 < x < 25 \end{cases}$, which are illustrated in **Figure 3.30**, are just three of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

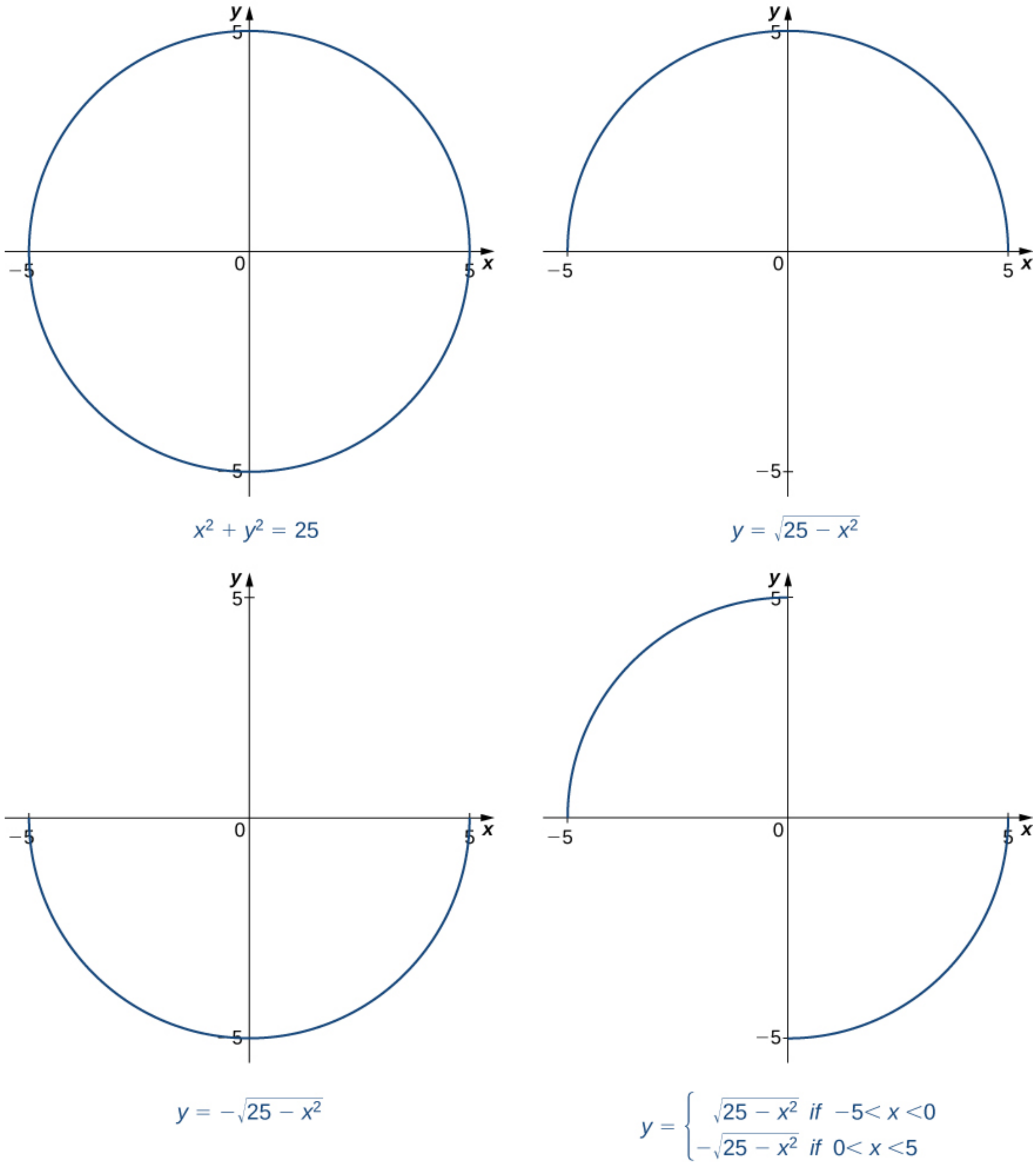


Figure 3.30 The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.68

Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

Step 1. Differentiate both sides of the equation.

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

Step 1.1. Use the sum rule on the left.
On the right $\frac{d}{dx}(25) = 0$.

$$2x + 2y\frac{dy}{dx} = 0$$

Step 1.2. Take the derivatives, so $\frac{d}{dx}\left(x^2\right) = 2x$ and $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x$$

Step 2. Keep the terms with $\frac{dy}{dx}$ on the left.
Move the remaining terms to the right.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 3.69

Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution

$$\frac{d}{dx}\left(x^3 \sin y + y\right) = \frac{d}{dx}(4x+3)$$

Step 1: Differentiate both sides of the equation.

$$\frac{d}{dx}\left(x^3 \sin y\right) + \frac{d}{dx}(y) = 4$$

Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x+3) = 4$.

$$\left(\frac{d}{dx}\left(x^3\right) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx} = 4$$

Step 1.2: Use the product rule to find $\frac{d}{dx}\left(x^3 \sin y\right)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$$3x^2 \sin y + \left(\cos y \frac{dy}{dx}\right) \cdot x^3 + \frac{dy}{dx} = 4$$

Step 1.3: We know $\frac{d}{dx}\left(x^3\right) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.

$$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.

$$\frac{dy}{dx}\left(x^3 \cos y + 1\right) = 4 - 3x^2 \sin y$$

Step 3: Factor out $\frac{dy}{dx}$ on the left.

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

Example 3.70

Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2 y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution

In **Example 3.68**, we showed that $\frac{dy}{dx} = -\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2 y}{dx^2}$.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dy}\left(-\frac{x}{y}\right) && \text{Differentiate both sides of } \frac{dy}{dx} = -\frac{x}{y}. \\
&= -\frac{\left(1 \cdot y - x\frac{dy}{dx}\right)}{y^2} && \text{Use the quotient rule to find } \frac{d}{dy}\left(-\frac{x}{y}\right). \\
&= \frac{-y + x\frac{dy}{dx}}{y^2} && \text{Simplify.} \\
&= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} && \text{Substitute } \frac{dy}{dx} = -\frac{x}{y}. \\
&= \frac{-y^2 - x^2}{y^3} && \text{Simplify.}
\end{aligned}
$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

3.48 Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 3.71

Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In **Example 3.68**, we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ (**Figure 3.31**).

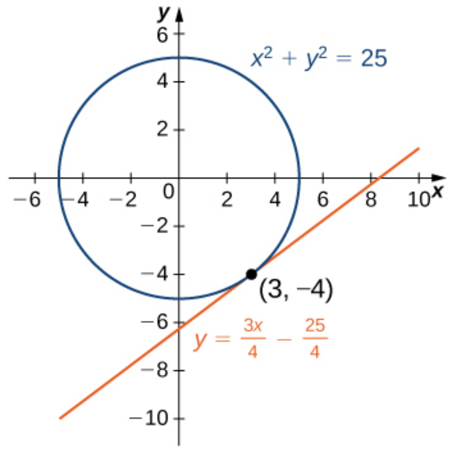


Figure 3.31 The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, −4).

Example 3.72

Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (**Figure 3.32**). This curve is known as the folium (or leaf) of Descartes.

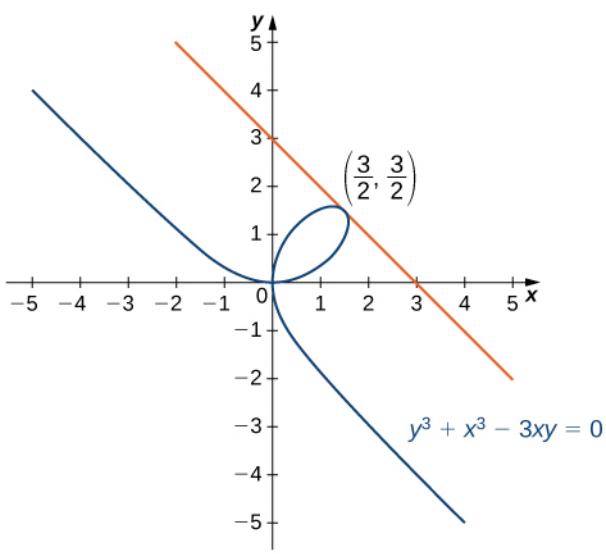


Figure 3.32 Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}\left(y^3 + x^3 - 3xy\right) &= \frac{d}{dx}(0) \\ 3y^2\frac{dy}{dx} + 3x^2 - \left(3y + \frac{dy}{dx}3x\right) &= 0 \\ \frac{dy}{dx} &= \frac{3y - 3x^2}{3y^2 - 3x}. \end{aligned}$$

Next, substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 3.73

Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the x-axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of $4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x + 50y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3, \frac{8}{5}\right)} = -\frac{3}{10}$. The equation of the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$. To determine where the line intersects the x-axis, solve $0 = -\frac{3}{10}x + \frac{5}{2}$. The solution is $x = \frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3}, 0\right)$.

3.49 Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

3.8 EXERCISES

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

300. $x^2 - y^2 = 4$

301. $6x^2 + 3y^2 = 12$

302. $x^2 y = y - 7$

303. $3x^3 + 9xy^2 = 5x^3$

304. $xy - \cos(xy) = 1$

305. $y\sqrt{x+4} = xy + 8$

306. $-xy - 2 = \frac{x}{7}$

307. $y\sin(xy) = y^2 + 2$

308. $(xy)^2 + 3x = y^2$

309. $x^3 y + xy^3 = -8$

For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

310. **[T]** $x^4 y - xy^3 = -2, \ (-1, -1)$

311. **[T]** $x^2 y^2 + 5xy = 14, \ (2, 1)$

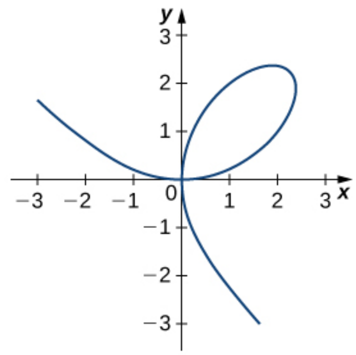
312. **[T]** $\tan(xy) = y, \ \left(\frac{\pi}{4}, 1\right)$

313. **[T]** $xy^2 + \sin(\pi y) - 2x^2 = 10, \ (2, -3)$

314. **[T]** $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, \ (1, 2)$

315. **[T]** $xy + \sin(x) = 1, \ \left(\frac{\pi}{2}, 0\right)$

316. **[T]** The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.
b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

317. For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal to the tangent line at the point $(1, 1)$.
b. At what other point does the normal line in a. intersect the graph of the equation?

318. Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

319. For the equation $x^2 + xy + y^2 = 7$,

a. Find the x-intercept(s).
b. Find the slope of the tangent line(s) at the x-intercept(s).
c. What does the value(s) in b. indicate about the tangent line(s)?

320. Find the equation of the tangent line to the graph of the equation $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

321. Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

322. Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

323. **[T]** The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240.$

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16).$

b. Interpret the result of a.

324. **[T]** The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360.$ (Both x and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8).$

b. Interpret the result of a.

325. The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y.$ Suppose that the volume of the cone is $85\pi\,\text{cm}^3.$ Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16.$

For the following exercises, consider a closed rectangular box with a square base with side x and height $y.$

326. Find an equation for the surface area of the rectangular box, $S(x, y).$

327. If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

For the following exercises, use implicit differentiation to determine $y'.$ Does the answer agree with the formulas we have previously determined?

328. $x = \sin y$

329. $x = \cos y$

330. $x = \tan y$

3.9 | Derivatives of Exponential and Logarithmic Functions

Learning Objectives

3.9.1 Find the derivative of exponential functions.
3.9.2 Find the derivative of logarithmic functions.
3.9.3 Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in **Introduction to Functions and Graphs**, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x,\ b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of $b^n,$ where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1,\ b^{-n} = \frac{1}{b^n},$ for a positive integer $n,$ and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the *continuity* of $B(x) = b^x,\ b > 0,$ we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4,\ 4^{3.1} < 4^\pi < 4^{3.2},\ 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142},\ 4^{3.1415} < 4^\pi < 4^{3.1416},\ldots.$$

As we see in the following table, $4^\pi \approx 77.88.$

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Table 3.6 Approximating a Value of 4^{π}

We also assume that for $B(x) = b^x, b > 0,$ the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1.$ We define e to be this unique value, as we did in **Introduction to Functions and Graphs**. **Figure 3.33** provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.

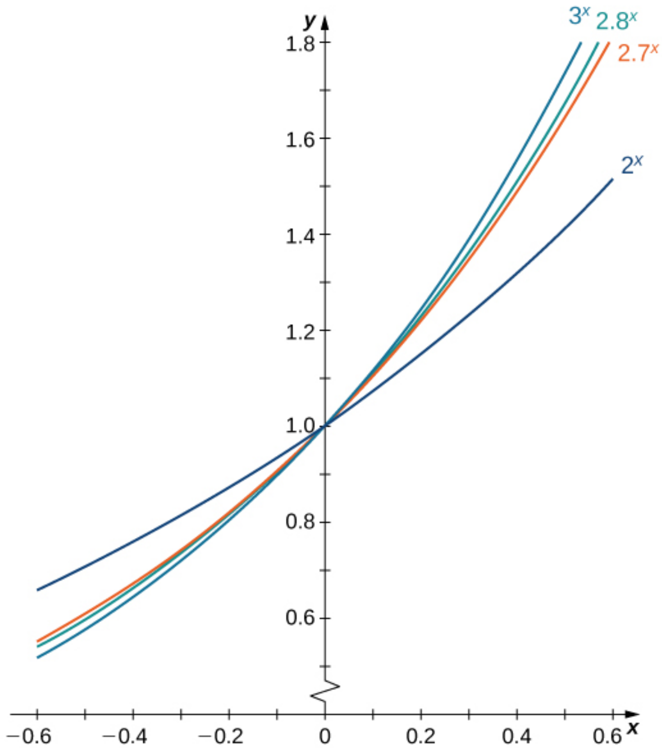


Figure 3.33 The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Table 3.7 Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in **Figure 3.34**. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

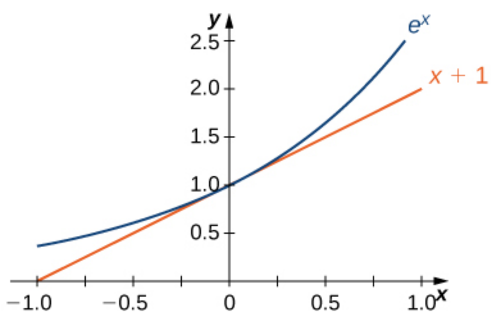


Figure 3.34 The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}. \tag{3.28}$$

Turning to $B'(x)$, we obtain the following.

$$
\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}
$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0). \tag{3.29}$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Theorem 3.14: Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).$$

Example 3.74

Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution

Using the derivative formula and the chain rule,

$$
\begin{aligned}
f'(x) &= e^{\tan(2x)} \frac{d}{dx}(\tan(2x)) \\
&= e^{\tan(2x)} \sec^2(2x) \cdot 2.
\end{aligned}
$$

Example 3.75

Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$\begin{aligned} y' &= \frac{\left(e^{x^2} \cdot 2\right)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}\left(2x^2 - 1\right)}{x^2} && \text{Simplify.} \end{aligned}$$

3.50 Find the derivative of $h(x) = xe^{2x}$.

Example 3.76

Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$A'(t) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

3.51 If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Theorem 3.15: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}. \tag{3.30}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of

$h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} g'(x). \tag{3.31}$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in **Figure 3.35**.

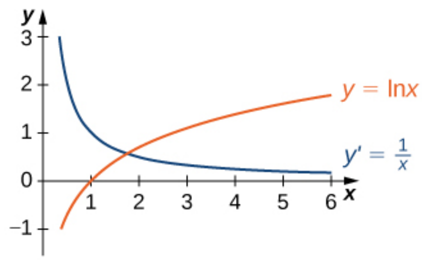


Figure 3.35 The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example 3.77

Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln\left(x^3 + 3x - 4\right)$.

Solution

Use **Equation 3.31** directly.

$$\begin{aligned} f'(x) &= \frac{1}{x^3 + 3x - 4} \cdot \left(3x^2 + 3\right) && \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x). \\ &= \frac{3x^2 + 3}{x^3 + 3x - 4} && \text{Rewrite.} \end{aligned}$$

Example 3.78

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = 2\ln x + \ln(\sin x) - \ln(2x + 1) && \text{Apply properties of logarithms.} \\ f'(x) &= \frac{2}{x} + \cot x - \frac{2}{2x + 1} && \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x). \end{aligned}$$

3.52 Differentiate: $f(x) = \ln(3x + 2)^5$.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = log_b x$ and $y = b^x$ for $b > 0,\ b \neq 1$.

Theorem 3.16: Derivatives of General Exponential and Logarithmic Functions

Let $b > 0,\ b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x,$ then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{3.32}$$

More generally, if $h(x) = \log_b(g(x)),$ then for all values of x for which $g(x) > 0,$

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{3.33}$$

ii. If $y = b^x,$ then

$$\frac{dy}{dx} = b^x \ln b. \tag{3.34}$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b. \tag{3.35}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in **Equation 3.33** now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (**Equation 3.35**) follows from the chain rule.

□

Example 3.79

Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and **Derivatives of General Exponential and Logarithmic Functions**.

$$\begin{aligned} h'(x) &= \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} && \text{Apply the quotient rule.} \\ &= \frac{2 \cdot 3^x \ln 3}{(3^x + 2)^2} && \text{Simplify.} \end{aligned}$$

Example 3.80

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using **Equation 3.33**, we see that

$$\frac{dy}{dx} = \frac{3}{(3x + 1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x = 1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

3.53 Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^{\pi}$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x + 1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$$\begin{aligned}
\ln y &= \ln\left(2x^4+1\right)^{\tan x} && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= \tan x \ln\left(2x^4+1\right) && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x && \text{Step 3. Differentiate both sides. Use the product rule on the right.}\\
\frac{dy}{dx} &= y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 5. Substitute } y=\left(2x^4+1\right)^{\tan x}.
\end{aligned}$$

Example 3.82

Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$$\begin{aligned}
\ln y &= \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x} && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln\sin x && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x} && \text{Step 3. Differentiate both sides.}\\
\frac{dy}{dx} &= y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) && \text{Step 5. Substitute } y=\frac{x\sqrt{2x+1}}{e^x \sin^3 x}.
\end{aligned}$$

Example 3.83

Extending the Power Rule

Find the derivative of $y = x^r$ where r is an arbitrary real number.

Solution

The process is the same as in **Example 3.82**, though with fewer complications.

$$\begin{aligned}
\ln y &= \ln x^r && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= r \ln x && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} && \text{Step 3. Differentiate both sides.}\\
\frac{dy}{dx} &= y\frac{r}{x} && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= x^r\frac{r}{x} && \text{Step 5. Substitute } y = x^r.\\
\frac{dy}{dx} &= rx^{r-1} && \text{Simplify.}
\end{aligned}$$

3.54 Use logarithmic differentiation to find the derivative of $y = x^x$.

3.55 Find the derivative of $y = (\tan x)^{\pi}$.

3.9 EXERCISES

For the following exercises, find $f'(x)$ for each function.

331. $f(x) = x^2 e^x$

332. $f(x) = \frac{e^{-x}}{x}$

333. $f(x) = e^{x^3 \ln x}$

334. $f(x) = \sqrt{e^{2x} + 2x}$

335. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

336. $f(x) = \frac{10^x}{\ln 10}$

337. $f(x) = 2^{4x} + 4x^2$

338. $f(x) = 3^{\sin 3x}$

339. $f(x) = x^\pi \cdot \pi^x$

340. $f(x) = \ln\left(4x^3 + x\right)$

341. $f(x) = \ln\sqrt{5x - 7}$

342. $f(x) = x^2 \ln 9x$

343. $f(x) = \log(\sec x)$

344. $f(x) = \log_7\left(6x^4 + 3\right)^5$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

347. $y = (\sin 2x)^{4x}$

348. $y = (\ln x)^{\ln x}$

349. $y = x^{\log_2 x}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

351. $y = x^{\cot x}$

352. $y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where $x = -1$. Graph both the function and the tangent line.

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

357. Consider the function $y = x^{1/x}$ for $x > 0$.
   a. Determine the points on the graph where the tangent line is horizontal.
   b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

358. The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

359. **[T]** The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.
b. Use a. to determine the rate at which the population is increasing in t years.
c. Use b. to determine the rate at which the population is increasing in 10 years.

360. **[T]** An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.
b. Use a. to determine the rate at which the substance is decaying in t hours.
c. Use b. to determine the rate of decay at $t = 4$ hours.

361. **[T]** The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \le t \le 4),$ where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

362. **[T]** The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where $a, b,$ and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.
b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204, b = 0.0198,$ and $c = 0.15.$
c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

Table 3.8 New York City Population Over Time **Source: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.**

363. **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

364. **[T]** Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

365. **[T]** Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

366. **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:

   a. Will the model be accurate in predicting the future population of New York City? Why or why not?
   b. Estimate the population in 2010. Was the prediction correct from a.?

CHAPTER 3 REVIEW

KEY TERMS

acceleration is the rate of change of the velocity, that is, the derivative of velocity

amount of change the amount of a function $f(x)$ over an interval $[x, x+h]$ is $f(x+h) - f(x)$

average rate of change is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h) - f(a)}{b-a}$

chain rule the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

constant multiple rule the derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative: $\frac{d}{dx}(cf(x)) = cf'(x)$

constant rule the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0,$ where *c* is a constant

derivative the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

difference quotient of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x-a}$$

difference rule the derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*: $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

differentiable at *a* a function for which $f'(a)$ exists is differentiable at a

differentiable function a function for which $f'(x)$ exists is a differentiable function

differentiable on *S* a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiation the process of taking a derivative

higher-order derivative a derivative of a derivative, from the second derivative to the *n*th derivative, is called a higher-order derivative

implicit differentiation is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

instantaneous rate of change the rate of change of a function at any point along the function $a,$ also called $f'(a),$ or the derivative of the function at a

logarithmic differentiation is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

marginal cost is the derivative of the cost function, or the approximate cost of producing one more item

marginal profit is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

marginal revenue is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

population growth rate is the derivative of the population with respect to time

power rule the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}x^n = nx^{n-1}$

product rule the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

quotient rule the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

speed is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

sum rule the derivative of the sum of a function *f* and a function *g* is the same as the sum of the derivative of *f* and the derivative of *g*: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

KEY EQUATIONS

- **Difference quotient**
  $Q = \frac{f(x) - f(a)}{x - a}$
- **Difference quotient with increment** h
  $Q = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$
- **Slope of tangent line**
  $m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
  $m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$
- **Derivative of** $f(x)$ **at** a
  $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
  $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$
- **Average velocity**
  $v_{ave} = \frac{s(t) - s(a)}{t - a}$
- **Instantaneous velocity**
  $v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$
- **The derivative function**
  $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
- **Derivative of sine function**
  $\frac{d}{dx}(\sin x) = \cos x$
- **Derivative of cosine function**
  $\frac{d}{dx}(\cos x) = -\sin x$
- **Derivative of tangent function**

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

- **Derivative of cotangent function**
  $$\frac{d}{dx}(\cot x) = -\csc^2 x$$
- **Derivative of secant function**
  $$\frac{d}{dx}(\sec x) = \sec x \tan x$$
- **Derivative of cosecant function**
  $$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
- **The chain rule**
  $$h'(x) = f'(g(x))g'(x)$$
- **The power rule for functions**
  $$h'(x) = n(g(x))^{n-1} g'(x)$$
- **Inverse function theorem**
  $$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} \text{ whenever } f'\left(f^{-1}(x)\right) \neq 0 \text{ and } f(x) \text{ is differentiable.}$$
- **Power rule with rational exponents**
  $$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$
- **Derivative of inverse sine function**
  $$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-(x)^2}}$$
- **Derivative of inverse cosine function**
  $$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}}$$
- **Derivative of inverse tangent function**
  $$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+(x)^2}$$
- **Derivative of inverse cotangent function**
  $$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+(x)^2}$$
- **Derivative of inverse secant function**
  $$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{(x)^2-1}}$$
- **Derivative of inverse cosecant function**
  $$\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{(x)^2-1}}$$
- **Derivative of the natural exponential function**
  $$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$
- **Derivative of the natural logarithmic function**
  $$\frac{d}{dx}(\ln g(x)) = \frac{1}{g(x)}g'(x)$$
- **Derivative of the general exponential function**

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} g'(x) \ln b$$

- **Derivative of the general logarithmic function**

$$\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

KEY CONCEPTS

3.1 Defining the Derivative

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

  Instantaneous velocity is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

3.2 The Derivative as a Function

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

3.3 Differentiation Rules

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function *f* and a function *g* is the same as the sum of the derivative of *f* and the derivative of *g*.
- The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus

the derivative of the second function times the first function, all divided by the square of the second function.

- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

3.4 Derivatives as Rates of Change

- Using $f(a+h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

3.5 Derivatives of Trigonometric Functions

- We can find the derivatives of sin x and cos x by using the definition of derivative and the limit formulas found earlier. The results are

$$\frac{d}{dx}\sin x = \cos x \frac{d}{dx}\cos x = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

3.6 The Chain Rule

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f(g(x))$,

$$h'(x) = f'(g(x))g'(x).$$

In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = (g(x))^n, \text{ then } h'(x) = n(g(x))^{n-1} g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f(g(k(x)))$, then $h'(x) = f'(g(k(x))g'(k(x))k'(x)$.

3.7 Derivatives of Inverse Functions

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

3.8 Implicit Differentiation

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

3.9 Derivatives of Exponential and Logarithmic Functions

- On the basis of the assumption that the exponential function $y = b^x$, $b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.

- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

CHAPTER 3 REVIEW EXERCISES

True or False? Justify the answer with a proof or a counterexample.

367. Every function has a derivative.

368. A continuous function has a continuous derivative.

369. A continuous function has a derivative.

370. If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

371. $f(x) = \sqrt{x+4}$

372. $f(x) = \frac{3}{x}$

Find the derivatives of the following functions.

373. $f(x) = 3x^3 - \frac{4}{x^2}$

374. $f(x) = \left(4 - x^2\right)^3$

375. $f(x) = e^{\sin x}$

376. $f(x) = \ln(x+2)$

377. $f(x) = x^2 \cos x + x \tan(x)$

378. $f(x) = \sqrt{3x^2 + 2}$

379. $f(x) = \frac{x}{4} \sin^{-1}(x)$

380. $x^2 y = (y+2) + xy \sin(x)$

Find the following derivatives of various orders.

381. First derivative of $y = x \ln(x) \cos x$

382. Third derivative of $y = (3x+2)^2$

383. Second derivative of $y = 4^x + x^2 \sin(x)$

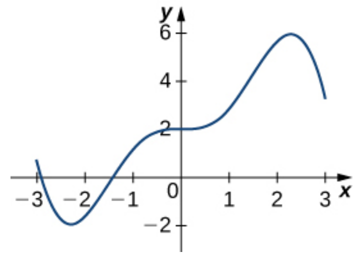
Find the equation of the tangent line to the following equations at the specified point.

384. $y = \cos^{-1}(x) + x$ at $x = 0$

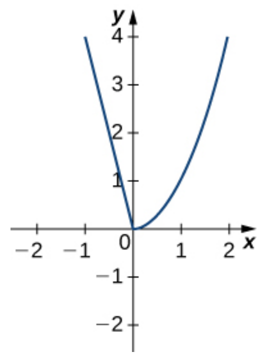
385. $y = x + e^x - \frac{1}{x}$ at $x = 1$

Draw the derivative for the following graphs.

386.



387.



The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos\left(\frac{\pi}{6}t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

388. Find and graph the derivative. What is the physical meaning?

389. Find $w'(3)$. What is the physical meaning of this value?

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Table 3.9 Wind Speeds of Hurricane Katrina **Source: http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.**

390. Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

391. Estimate the derivative of the wind speed at hour 83. What is the physical meaning?