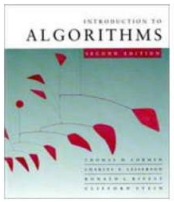

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DAG Shortest Paths

- Bellman-Ford takes O(VE) time.

DAG Shortest Paths

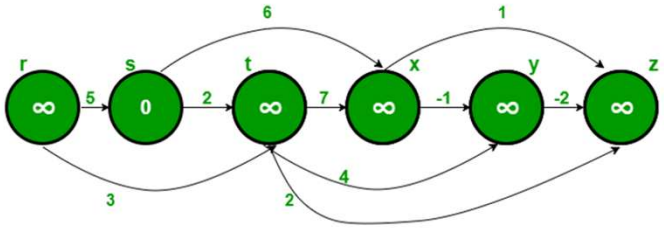
- Bellman-Ford takes O(VE) time.
- Use DAG properties!

```
DAG-Shortest-Paths(G, s)
1.  L = Topological_Sort(G)
2.  for u in V: dist(u) = INF;
3.  dist(s) = 0
4.  for u from s to L.end():
5.      for each vertex v ∈ Adj[u]
6.          if dist(v) > dist(u) + weight(u, v)
7.              dist(v) = dist(u) + weight(u, v)
```

relaxation step

Time: O(V+E)

DAG Shortest Paths



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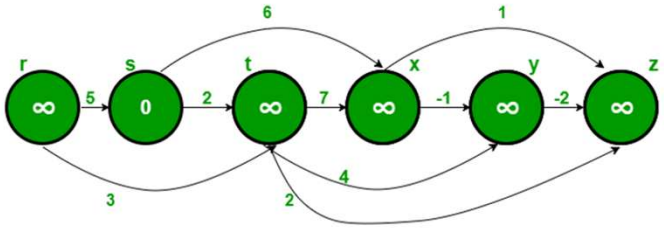

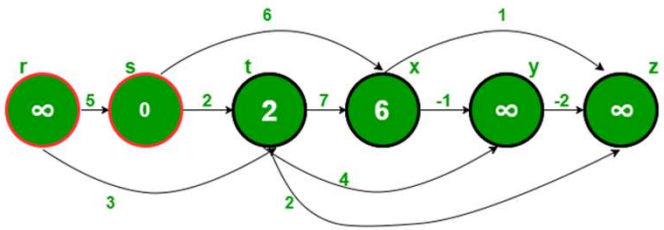

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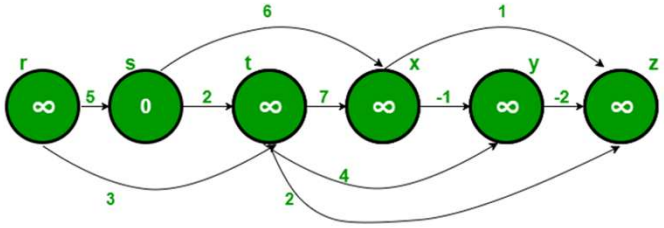

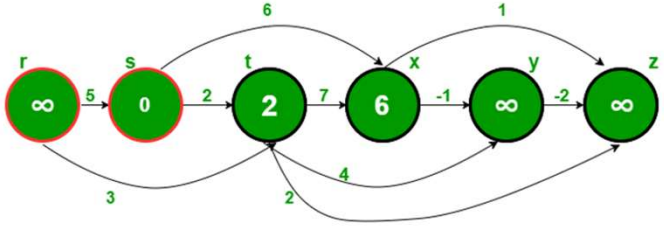

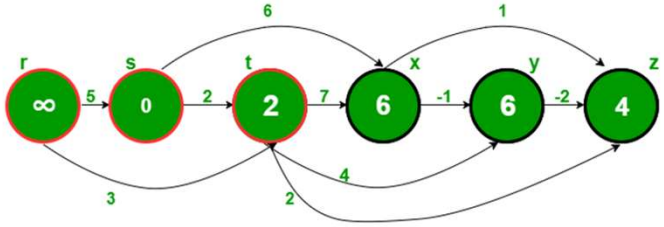

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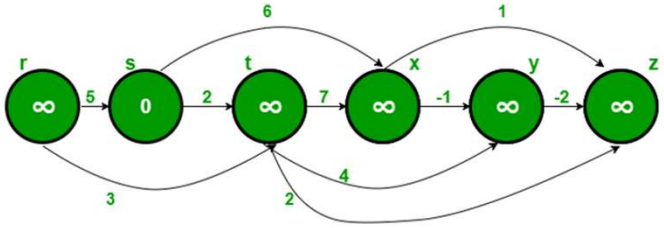

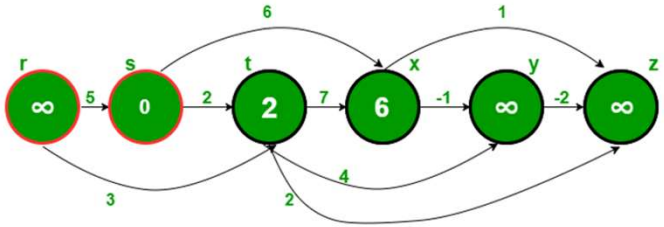

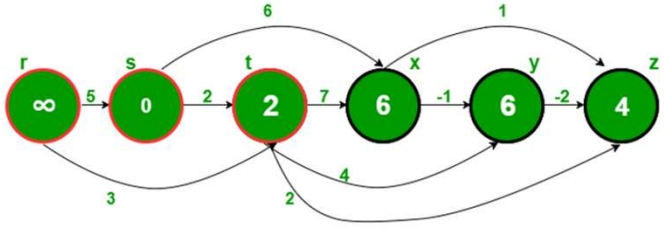

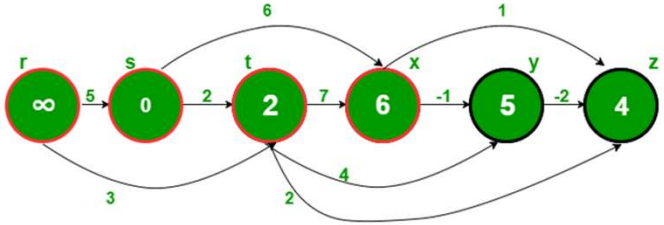

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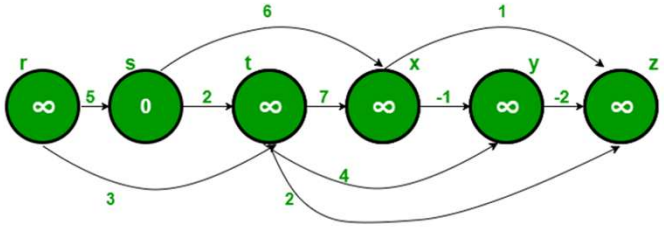

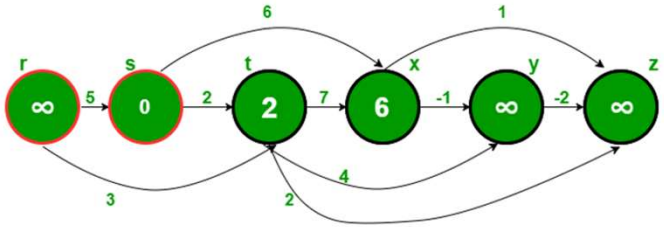

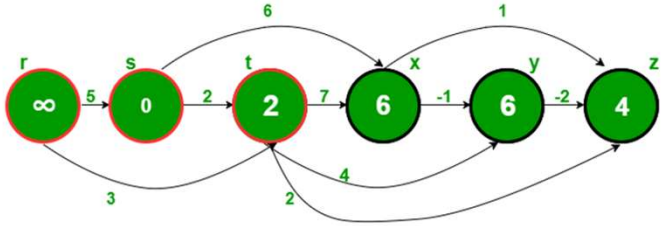

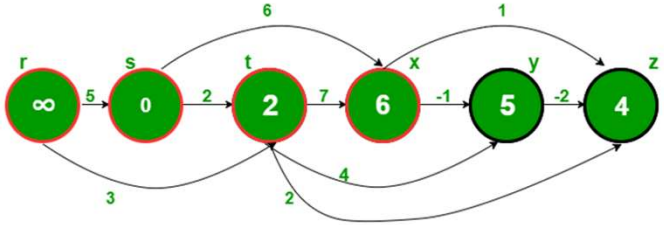

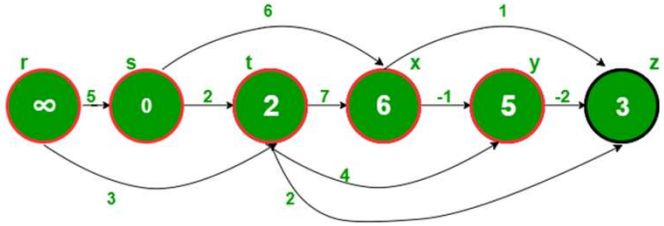

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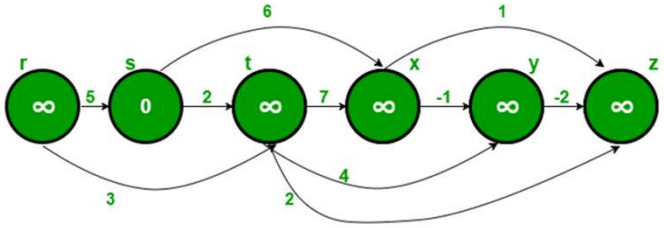
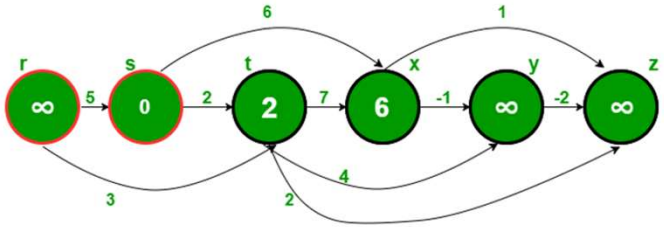
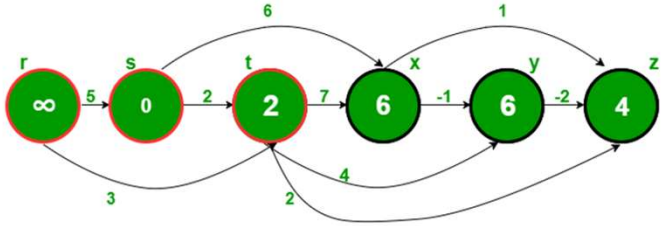
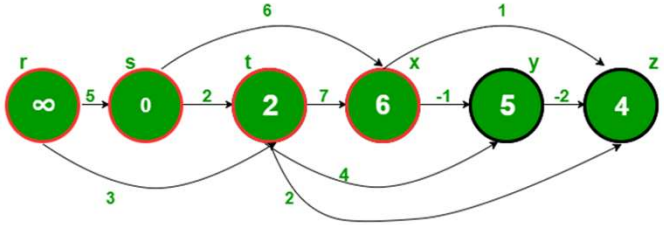
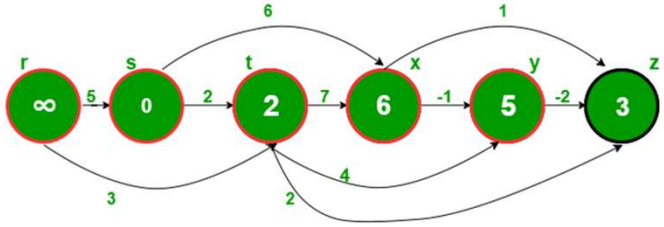
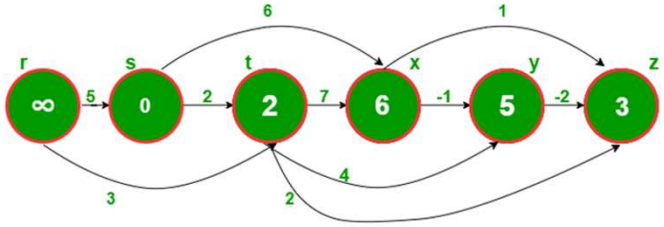


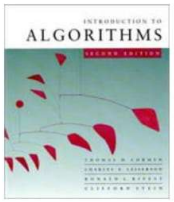








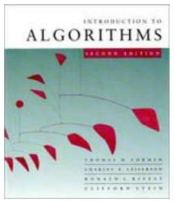




Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(E + V \lg V)$
- General
  - Bellman-Ford: $O(VE)$
- DAG
  - One pass of Bellman-Ford: $O(V + E)$



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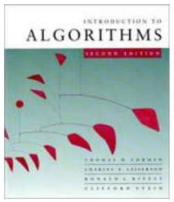

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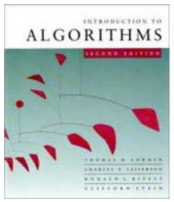

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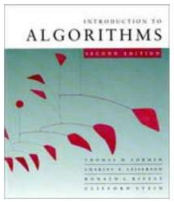

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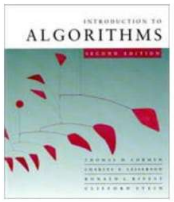
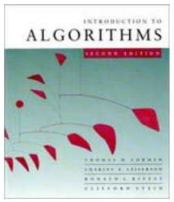


All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, with edge-weight function $w : E \to \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.



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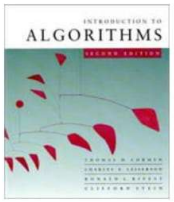

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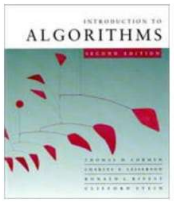
IDEA:

- Run Bellman-Ford once from each vertex.
- Time $= O(V^2E)$.
- Dense graph (n^2 edges) $\Rightarrow \Theta(n^4)$ time in the worst case.

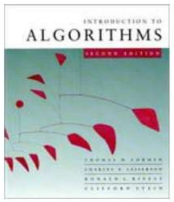
Good first try!



Dynamic programming

Consider the $n \times n$ adjacency matrix $A = (a_{ij})$ of the digraph, and define

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.



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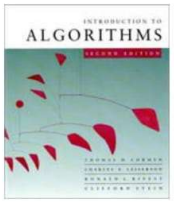
$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j; \end{cases}$$



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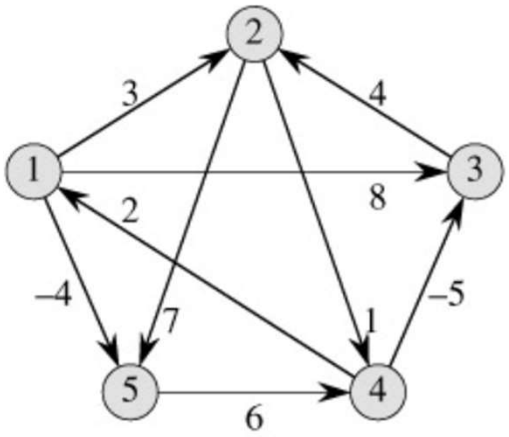
and for $m = 1, 2, \ldots, n-1$,

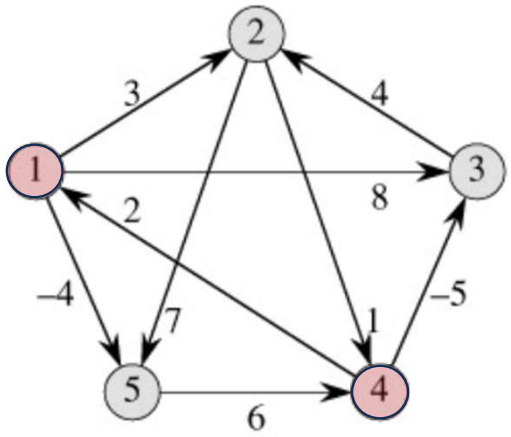
$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}.$$



$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

m=1

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.



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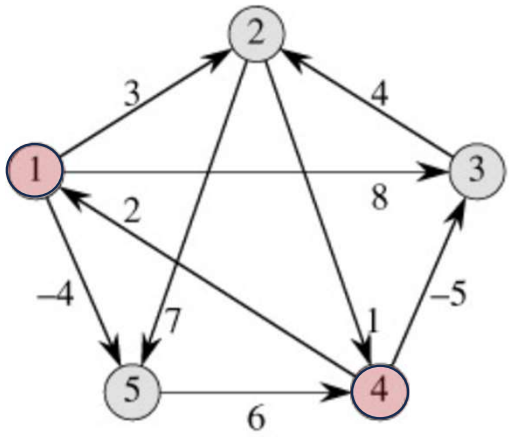
m=2

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.



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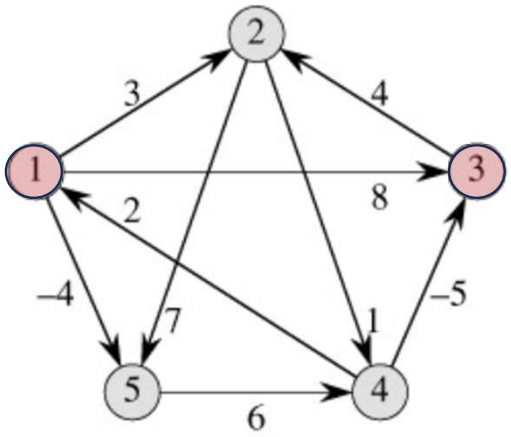

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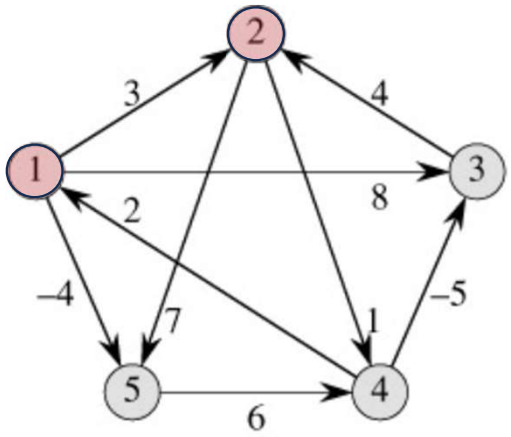
m=3

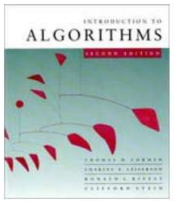
$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.



$$\begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

m=4

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.

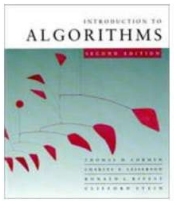


Matrix multiplication

Compute $C = A \cdot B$, where C, A, and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \; .$$

Time = $\Theta(n^3)$ using the standard algorithm.



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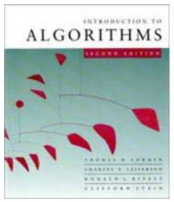

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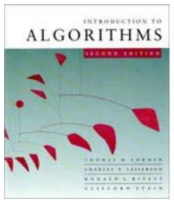
What if we map "+" $\rightarrow$ "min" and "$\cdot$" $\rightarrow$ "+"?

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}.$$

Thus, $D^{(m)} = D^{(m-1)}$ "$\times$" A.

Identity matrix $= \mathrm{I} = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} = D^0 = (d_{ij}^{(0)})$.



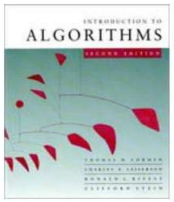
Matrix multiplication (continued)

The (min, +) multiplication is ***associative***, and with the real numbers, it forms an algebraic structure called a ***closed semiring***.

Consequently, we can compute

$$\begin{aligned} D^{(1)} &= D^{(0)} \cdot A &&= A^1 \\ D^{(2)} &= D^{(1)} \cdot A &&= A^2 \\ &\vdots && \vdots \\ D^{(n-1)} &= D^{(n-2)} \cdot A &&= A^{n-1}\,, \end{aligned}$$

yielding $D^{(n-1)} = (\delta(i,j))$.



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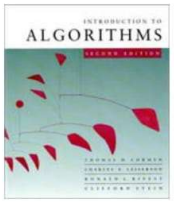
yielding $D^{(n-1)} = (\delta(i,j))$.

Time $= \Theta(n \cdot n^3) = \Theta(n^4)$. No better than $n \times$ B-F.



Improved matrix multiplication algorithm

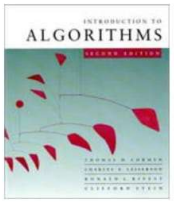
Repeated squaring: $A^{2k} = A^k \times A^k$.

Compute $\underbrace{A^2, A^4, \ldots, A^{2^{\lceil \lg(n-1) \rceil}}}_{O(\lg n) \text{ squarings}}$.

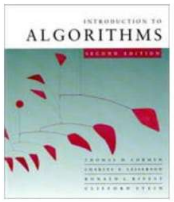
Note: $A^{n-1} = A^n = A^{n+1} = \cdots$.

Time $= \Theta(n^3 \lg n)$.

To detect negative-weight cycles, check the diagonal for negative values in $O(n)$ additional time.



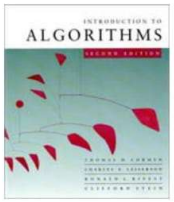
Pseudocode for Floyd-Warshall



Pseudocode for Floyd-Warshall

for $k \leftarrow 1$ **to** n
  do for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
      do if $c_{ij} > c_{ik} + c_{kj}$
        then $c_{ij} \leftarrow c_{ik} + c_{kj}$

} ***relaxation***



Pseudocode for Floyd-Warshall

for $k \leftarrow 1$ **to** n
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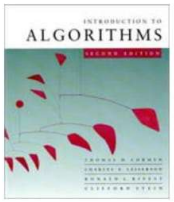
      do if $c_{ij} > c_{ik} + c_{kj}$
        then $c_{ij} \leftarrow c_{ik} + c_{kj}$ } ***relaxation***

Notes:

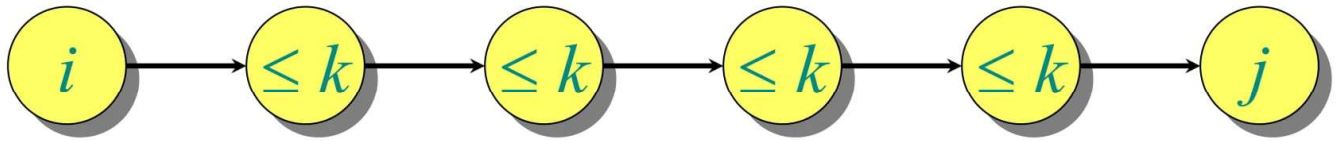
- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code.
- Efficient in practice.



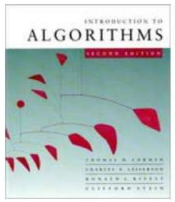
Floyd-Warshall algorithm

Also dynamic programming, but faster!

Define $c_{ij}^{(k)} =$ weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.

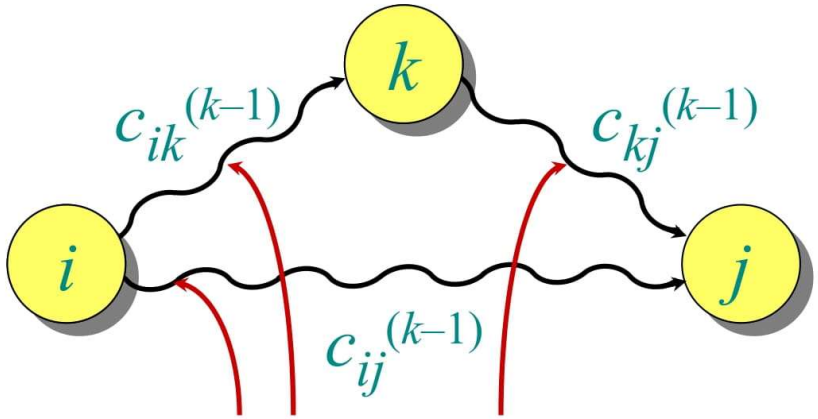


Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

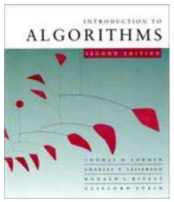


Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min_k \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$



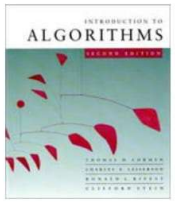
intermediate vertices in $\{1, 2, \ldots, k\}$



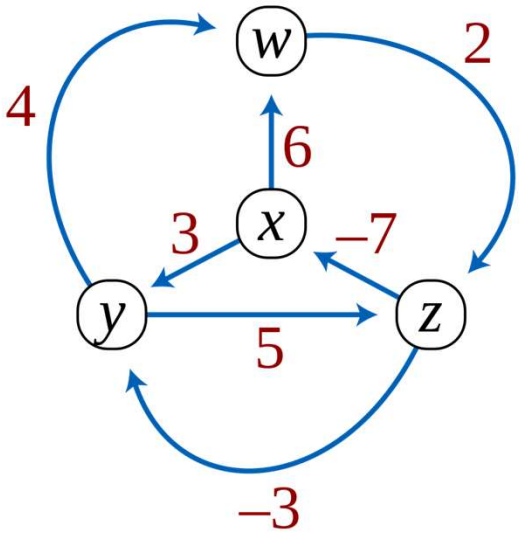
Johnson's algorithm

1. Find a function $h : V \to \mathbb{R}$ such that $w_h(u, v) \geq 0$ for all $(u, v) \in E$ by using Bellman-Ford to solve the difference constraints $h(v) - h(u) \leq w(u, v)$, or determine that a negative-weight cycle exists.
   - Time = $O(VE)$.
2. Run Dijkstra's algorithm using w_h from each vertex $u \in V$ to compute $\delta_h(u, v)$ for all $v \in V$.
   - Time = $O(VE + V^2 \lg V)$.
3. For each $(u, v) \in V \times V$, compute
   $$\delta(u, v) = \delta_h(u, v) - h(u) + h(v) .$$
   - Time = $O(V^2)$.

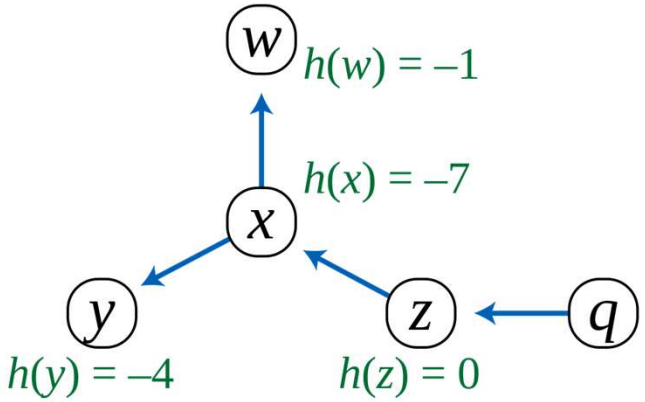
Total time = $O(VE + V^2 \lg V)$.



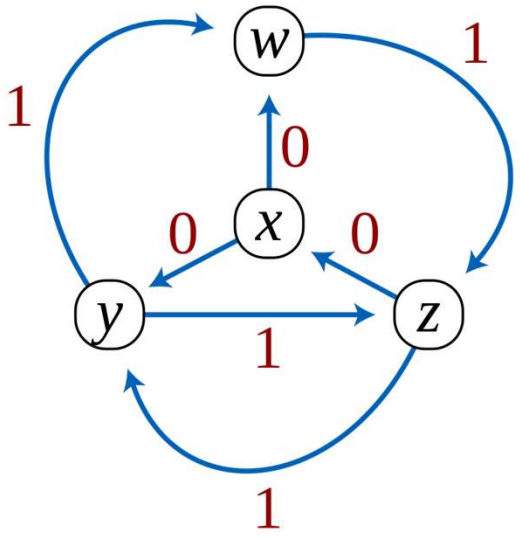
Johnson's algorithm



original graph
with negative edges



shortest path tree
found by Bellman-Ford



reweighted graph with
no negative edges

אלגוריתם קירוב

אלגוריתם קירוב - אינטואיציה

אלגוריתם שמוצא פתרון שאינו בהכרח פתרון אופטימלי לבעיה נתונה, אלא פתרון שקרוב לפתרון אופטימלי.

אלגוריתם קירוב - אינטואיציה

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לבעיה נתונה, אלא פתרון שקרוב לפתרון אופטימלי.

למה להסתפק רק בקירוב ??

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מציאת פתרון אופטימלי היא בלתי מעשית

NP

אלגוריתם קירוב

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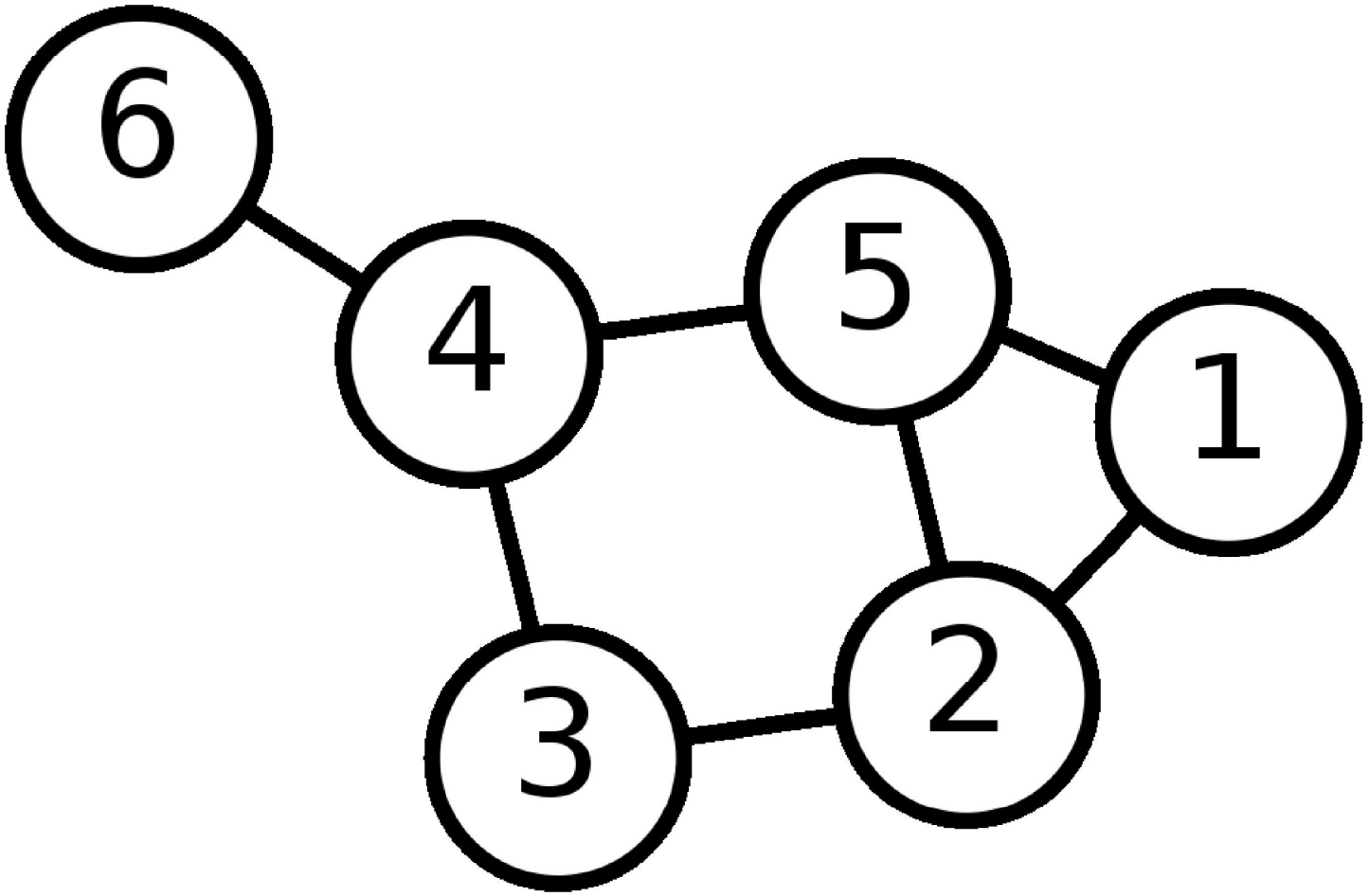
מודדים את הקירוב של אלגוריתם בהתאם ליחס בין הפתרון שנמצא על ידי האלגוריתם לבין הפתרון האופטימלי.

אלגוריתם קירוב – בעיות אופטימיזציה

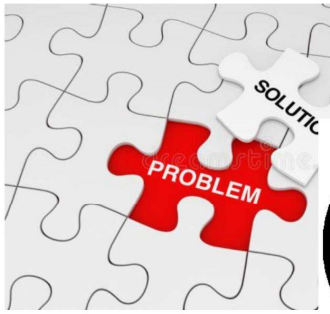
אלגוריתם קירוב **A** הוא בעל יחס קירוב **ρ(n)**
אם עבור כל קלט בגודל **n**,
היחס בין הגודל **C** של הפתרון שמפיק **A**
לגודל הפתרון האופטימלי **C***
אינו עולה על **ρ(n)**

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n)$$

ריחוי הודהודייח



ריחוי הודהודיות



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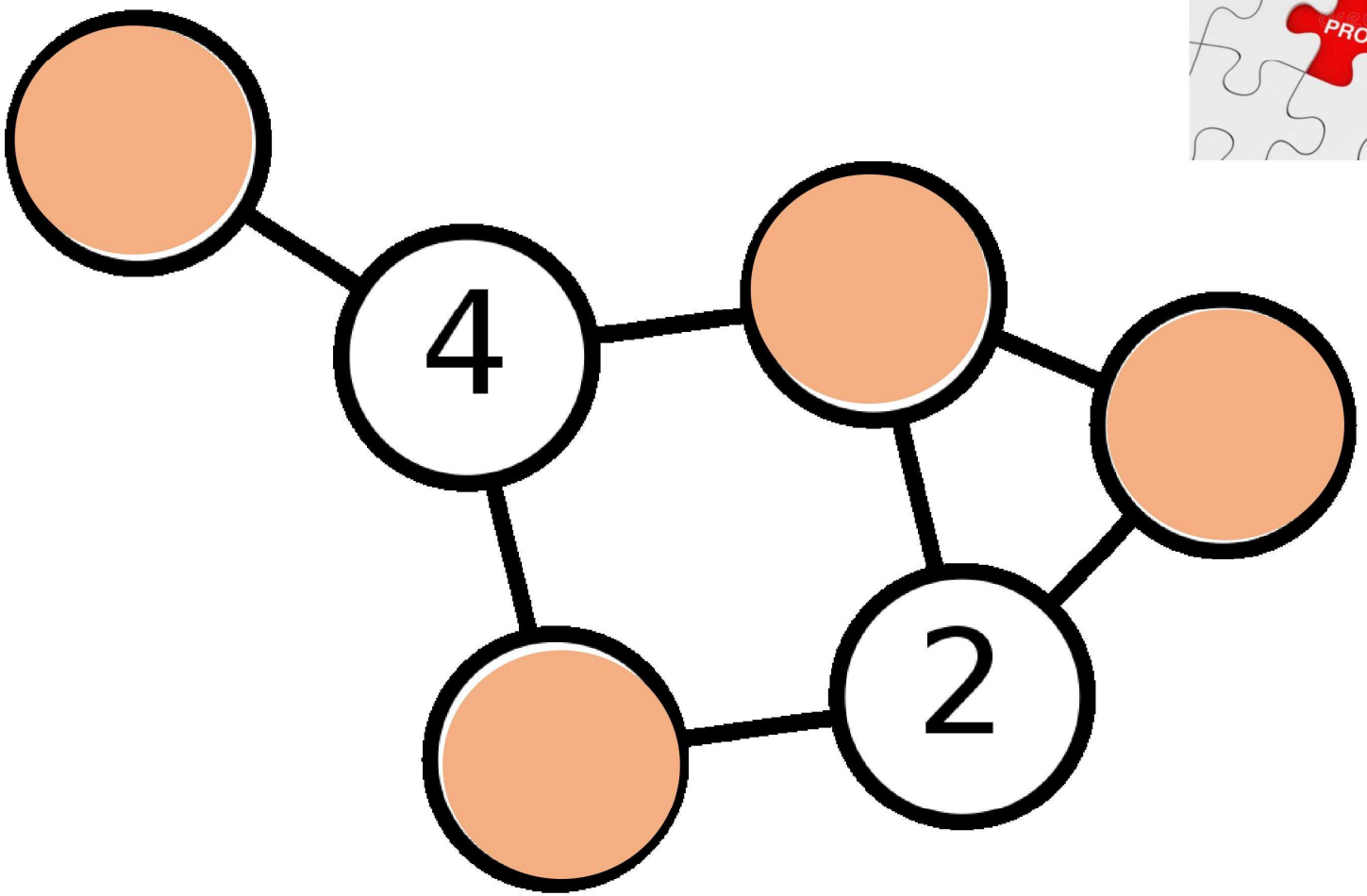

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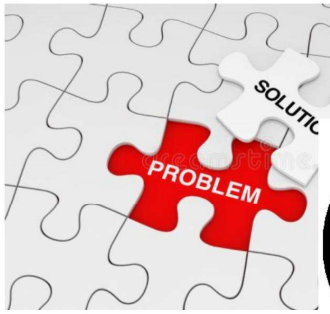

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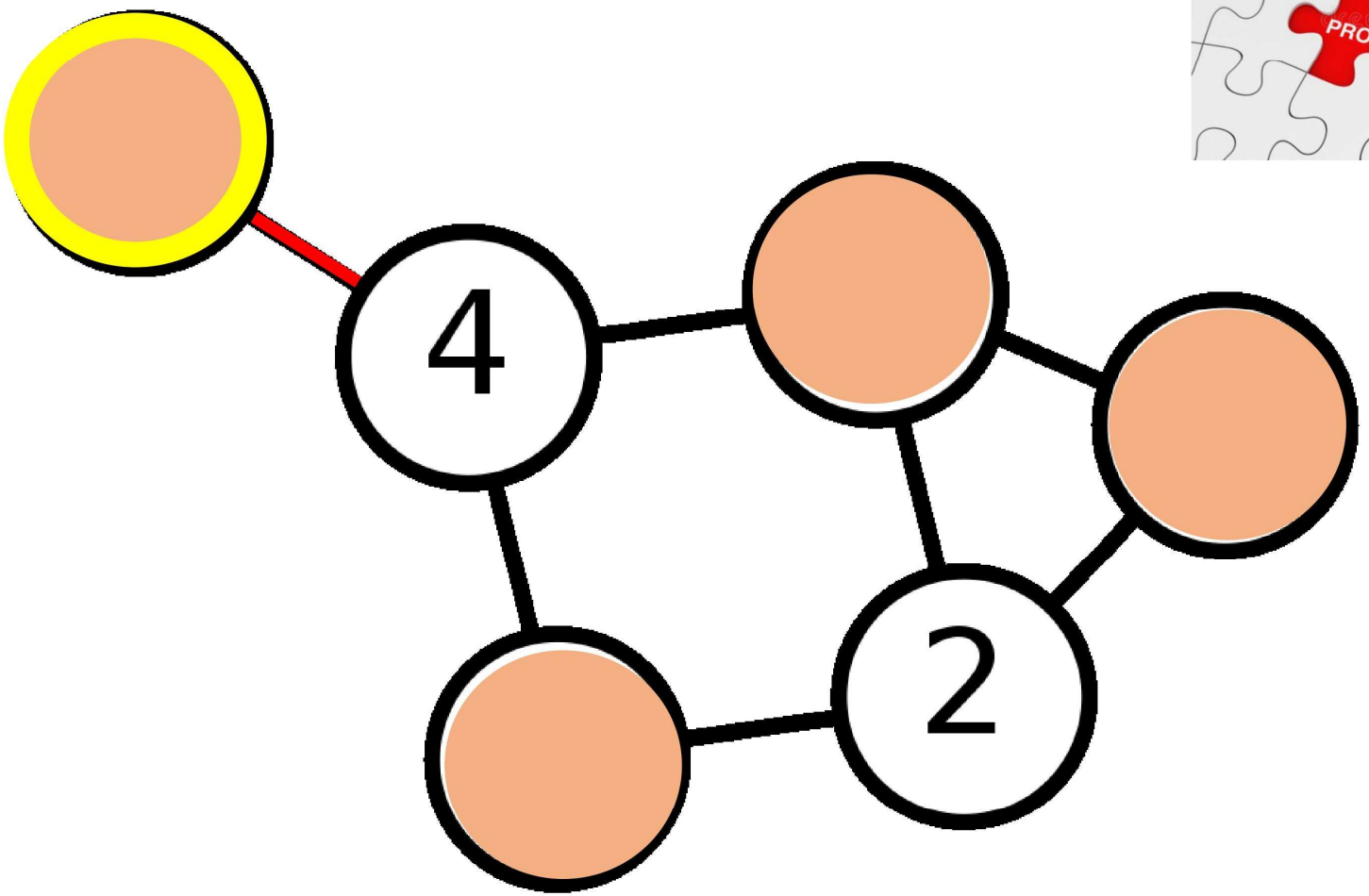
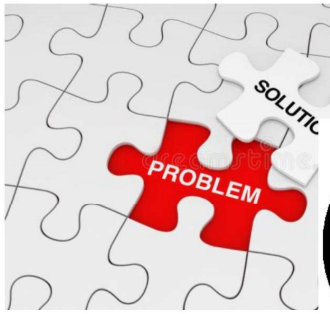


ריחוי הדדי



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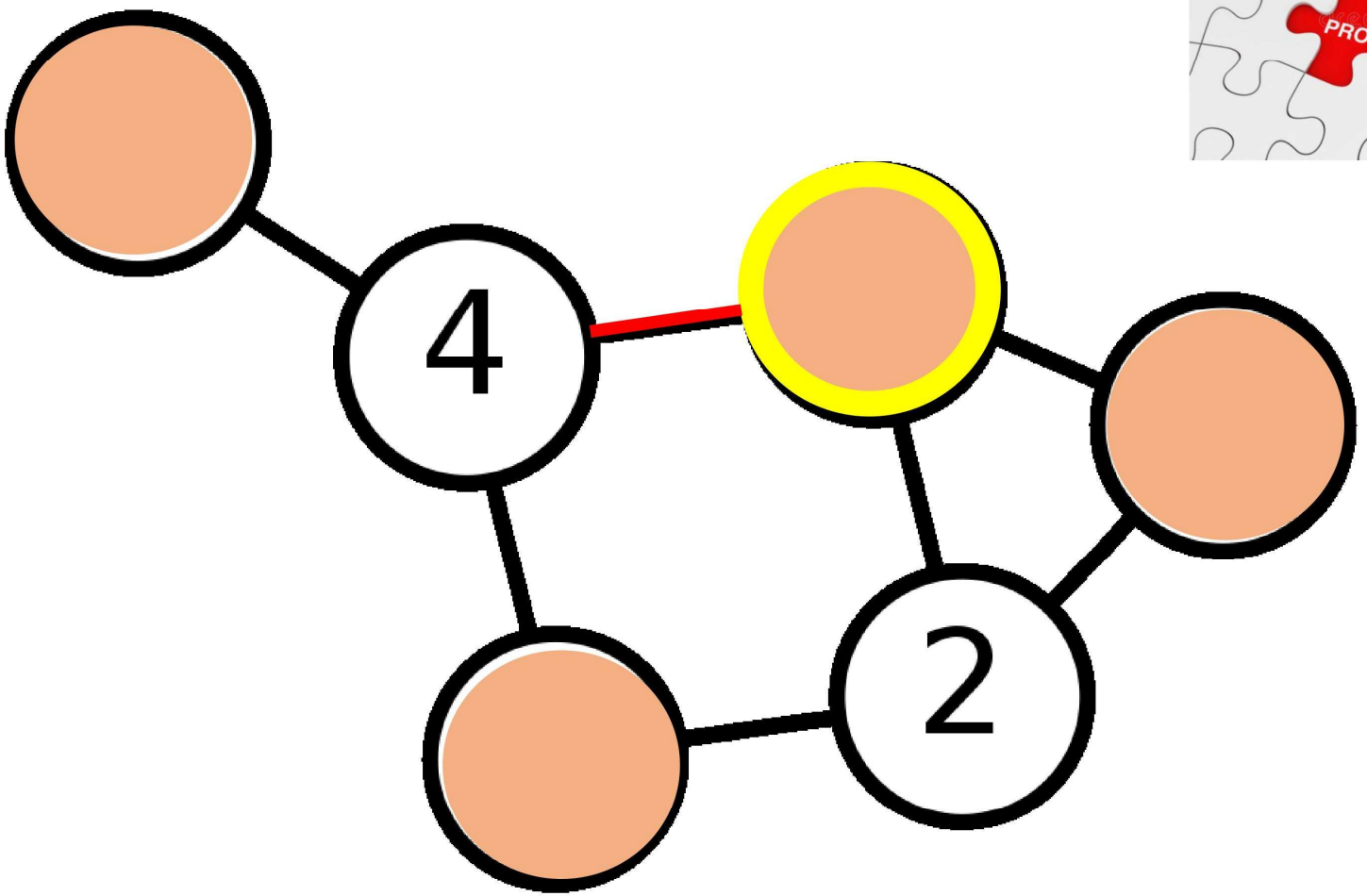

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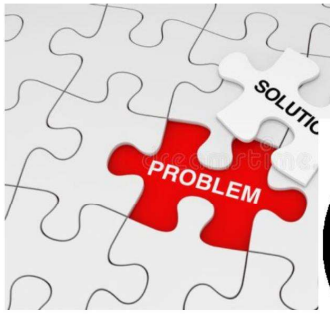

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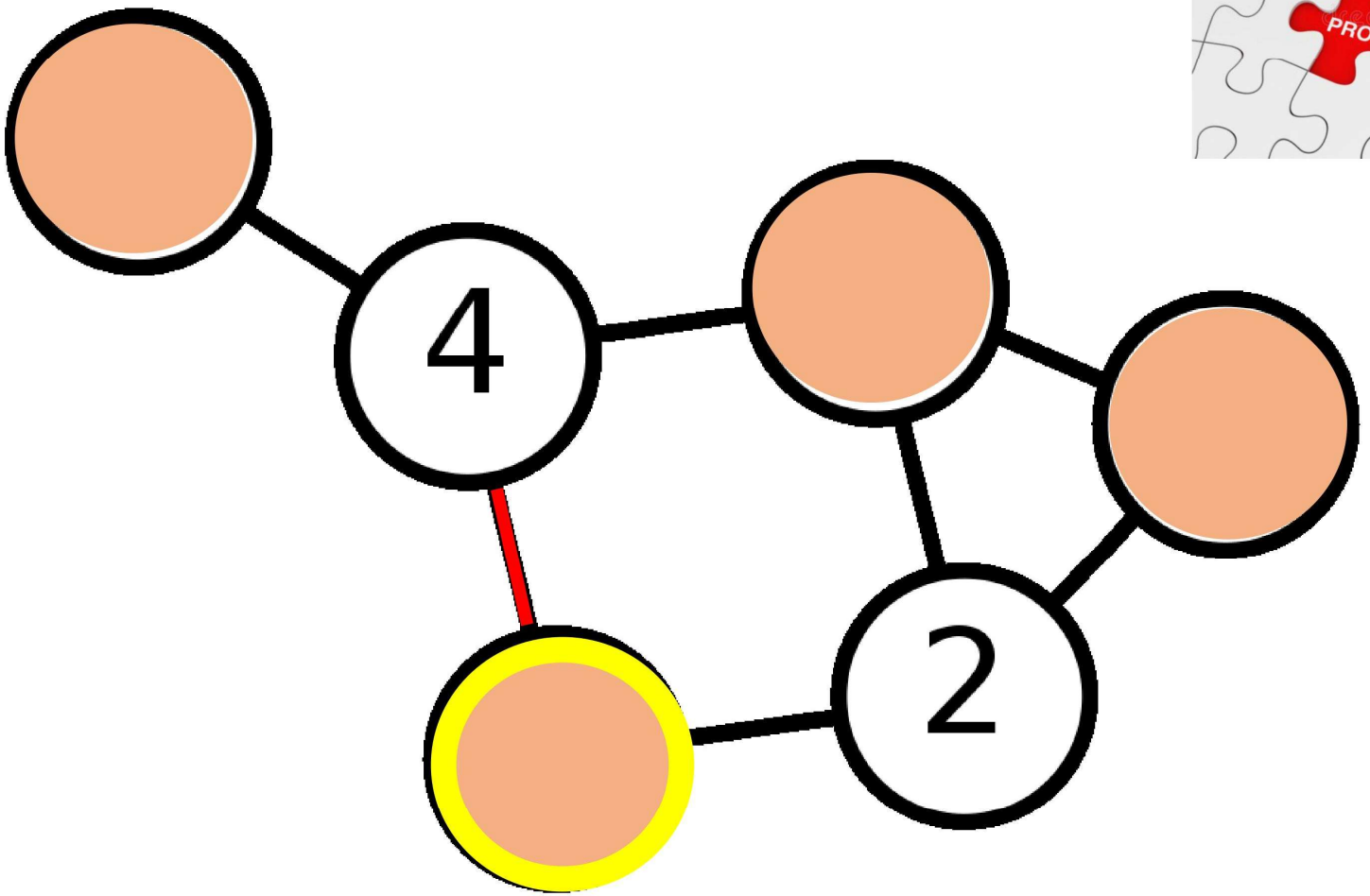

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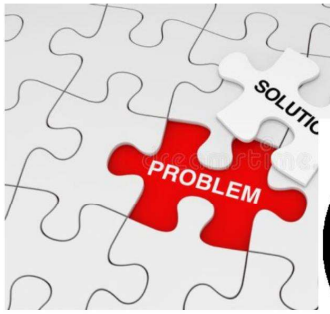

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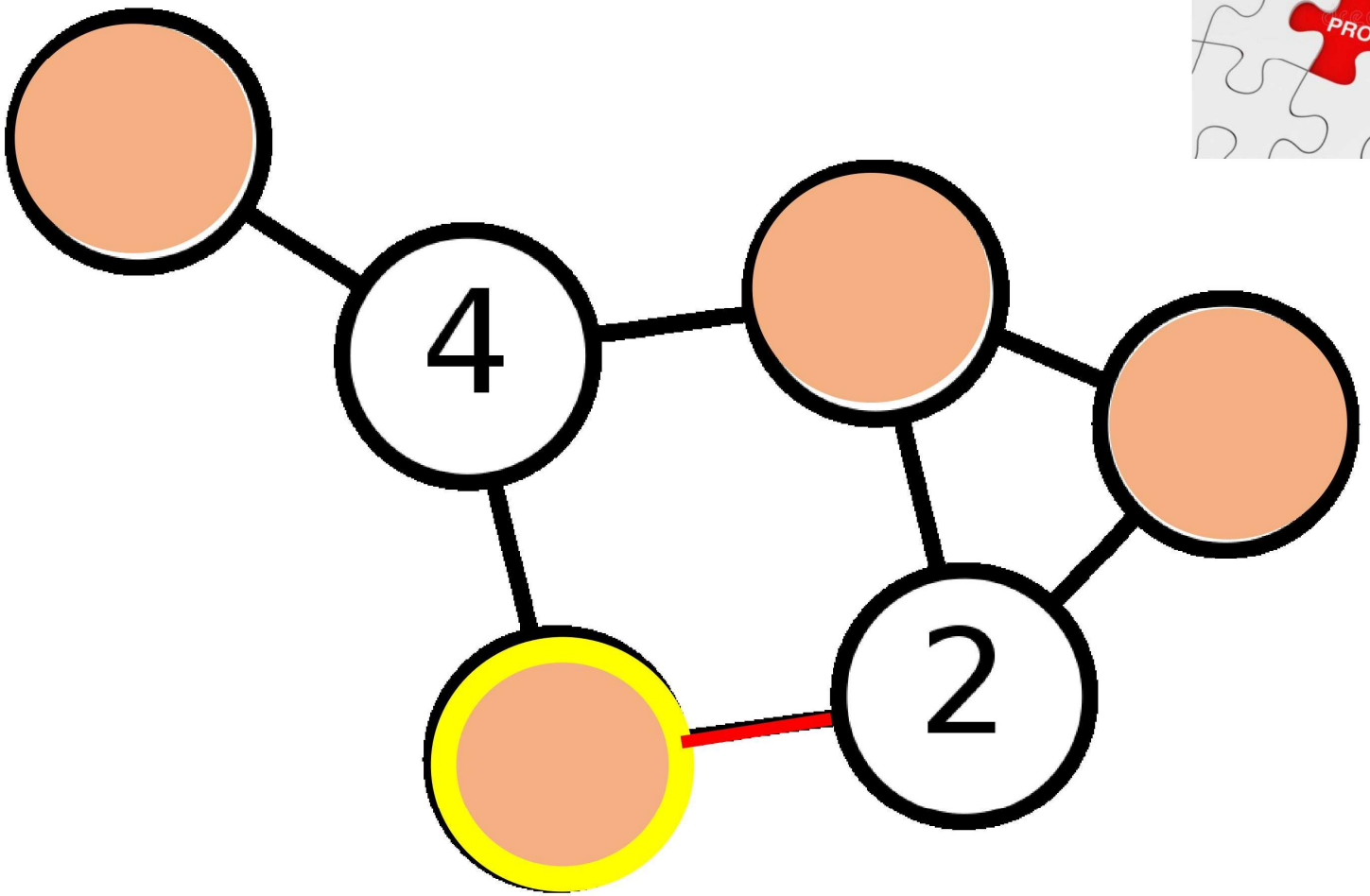

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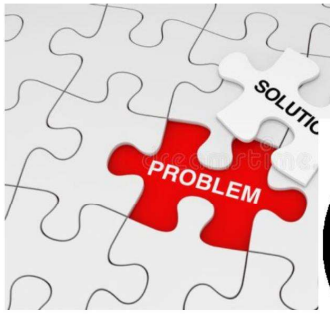

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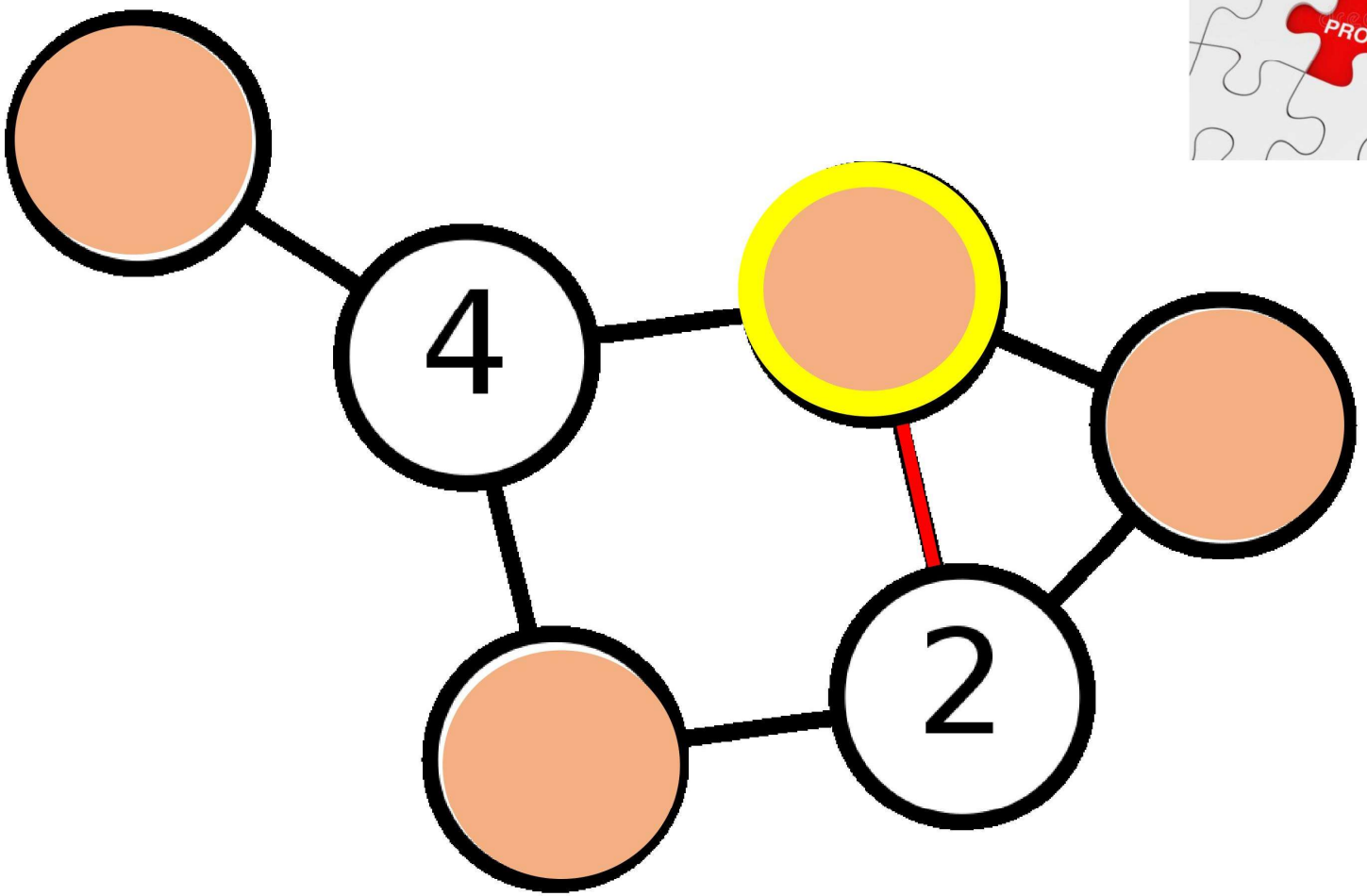
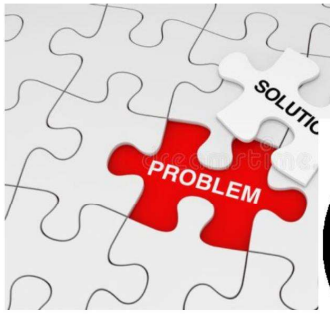


ריחוי הודהודיות



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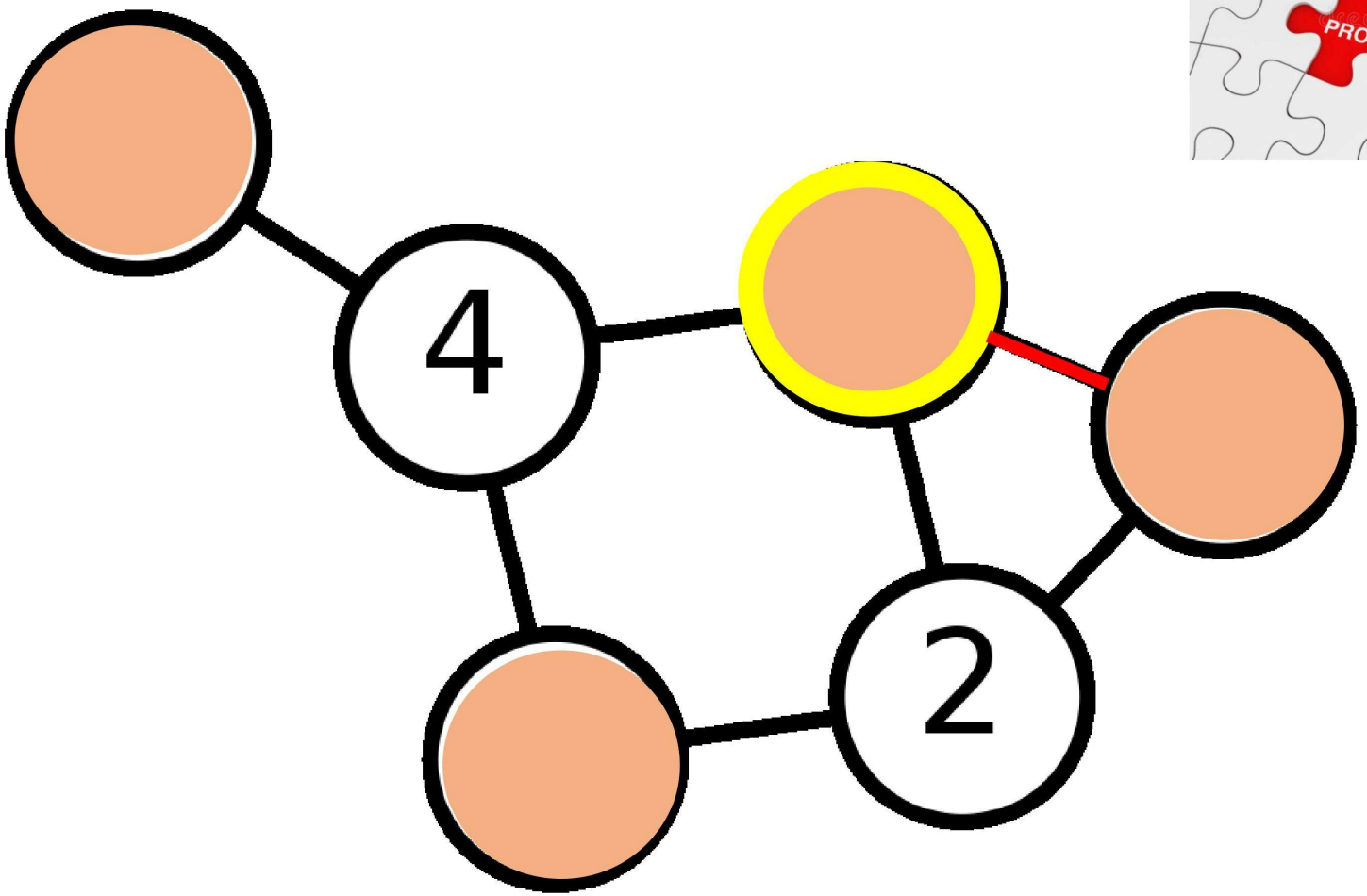
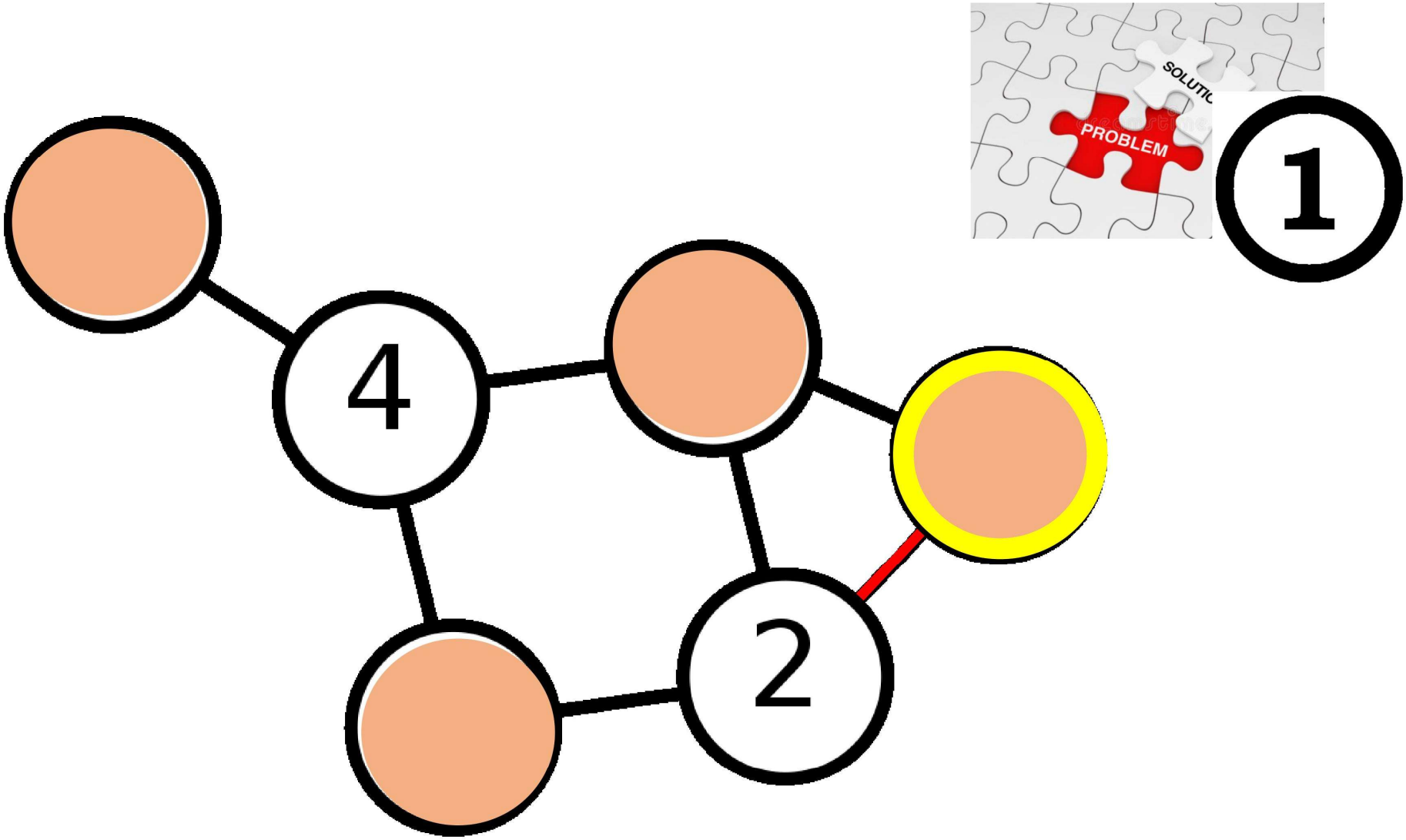



PROBLEM
1
4
2

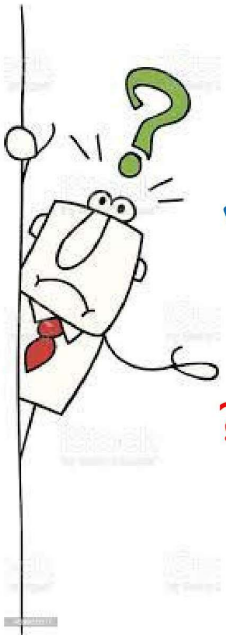
כיסוי קודקודיים

כיסוי על ידי קדקודים **(vertex cover)** של גרף בלתי מכוון $G=(V,E)$ הוא

תת קבוצה $V' \subseteq V$ כך שאם $(u,v) \in E$, אזי $u \in V'$ או $v \in V'$ (או שניהם).

כיסוי קודקודיים

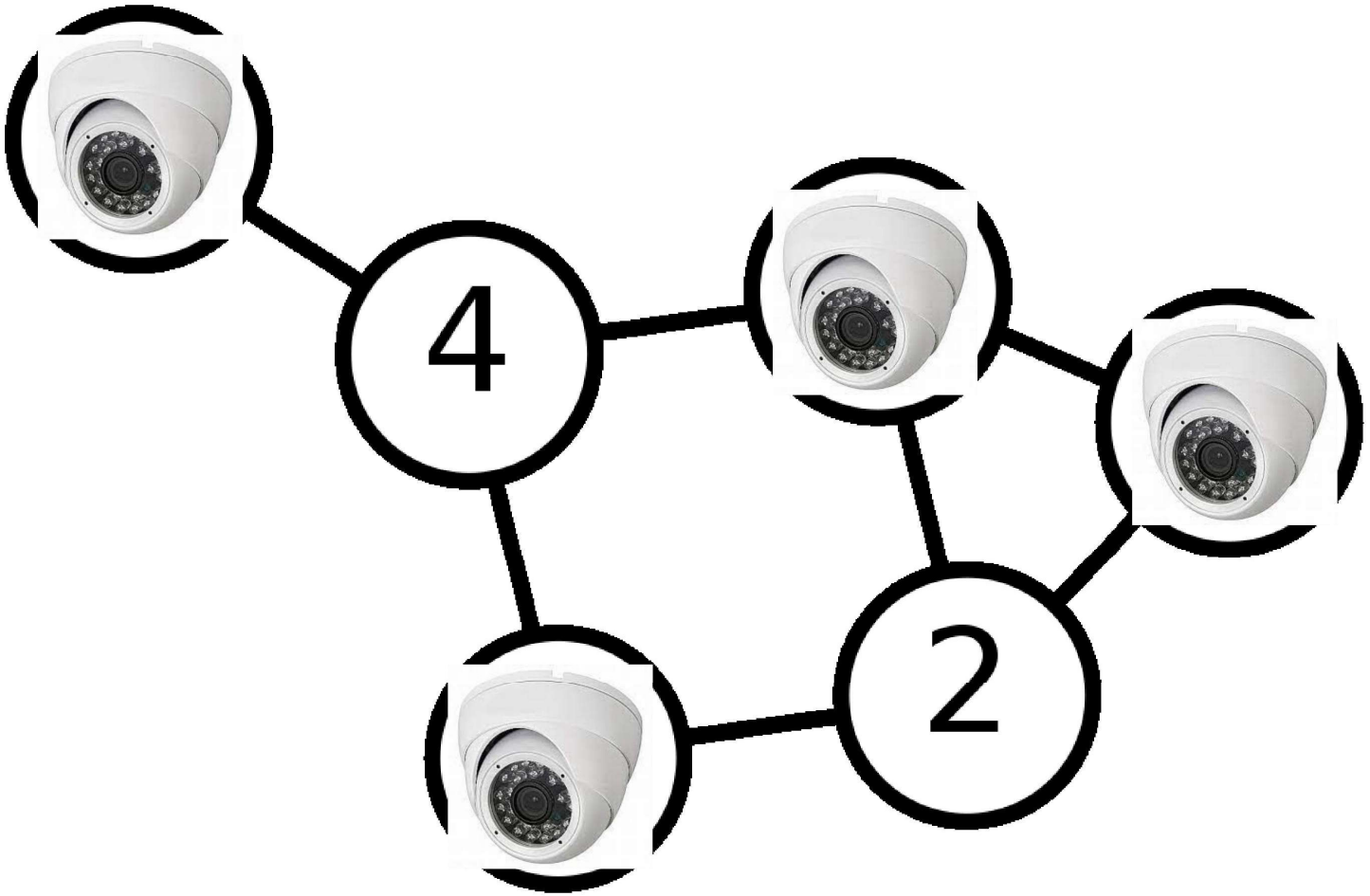
כיסוי על ידי קדקודים (vertex cover) של גרף בלתי מכוון $G=(V,E)$ הוא



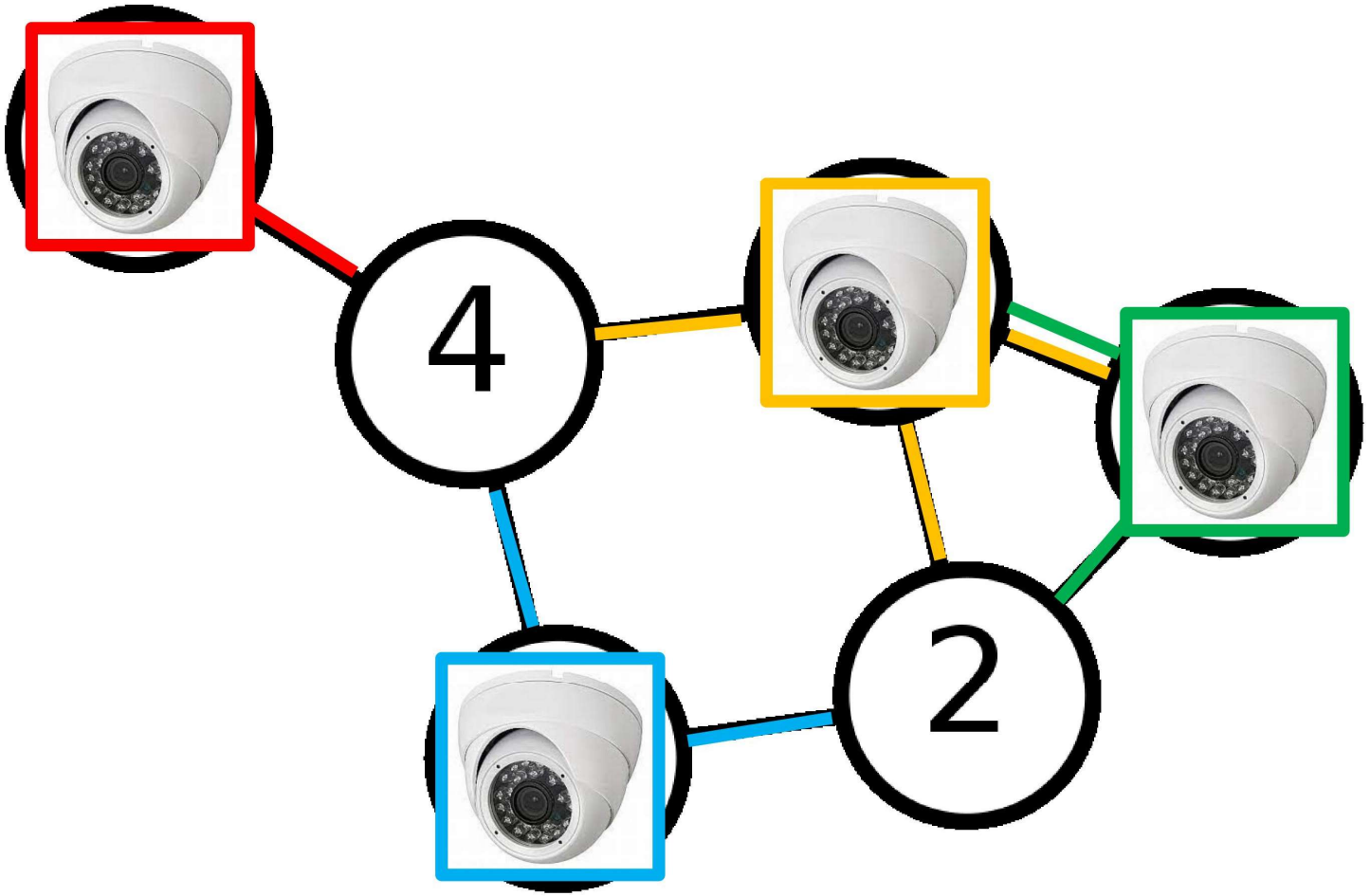
תת קבוצה $V' \subseteq V$ כך שאם $(u,v) \in E$, אזי $u \in V'$ או $v \in V'$ (או שניהם).

למה הבעיה מעניינת?
מה יכולה לייצג בחיים האמתיים?

תיאור חטלחוח ארכוחה רפחחי חחדרוו



תיאור חצלחות ארכיטקטוני חדרון



בעיית כיסוי קודקודיים

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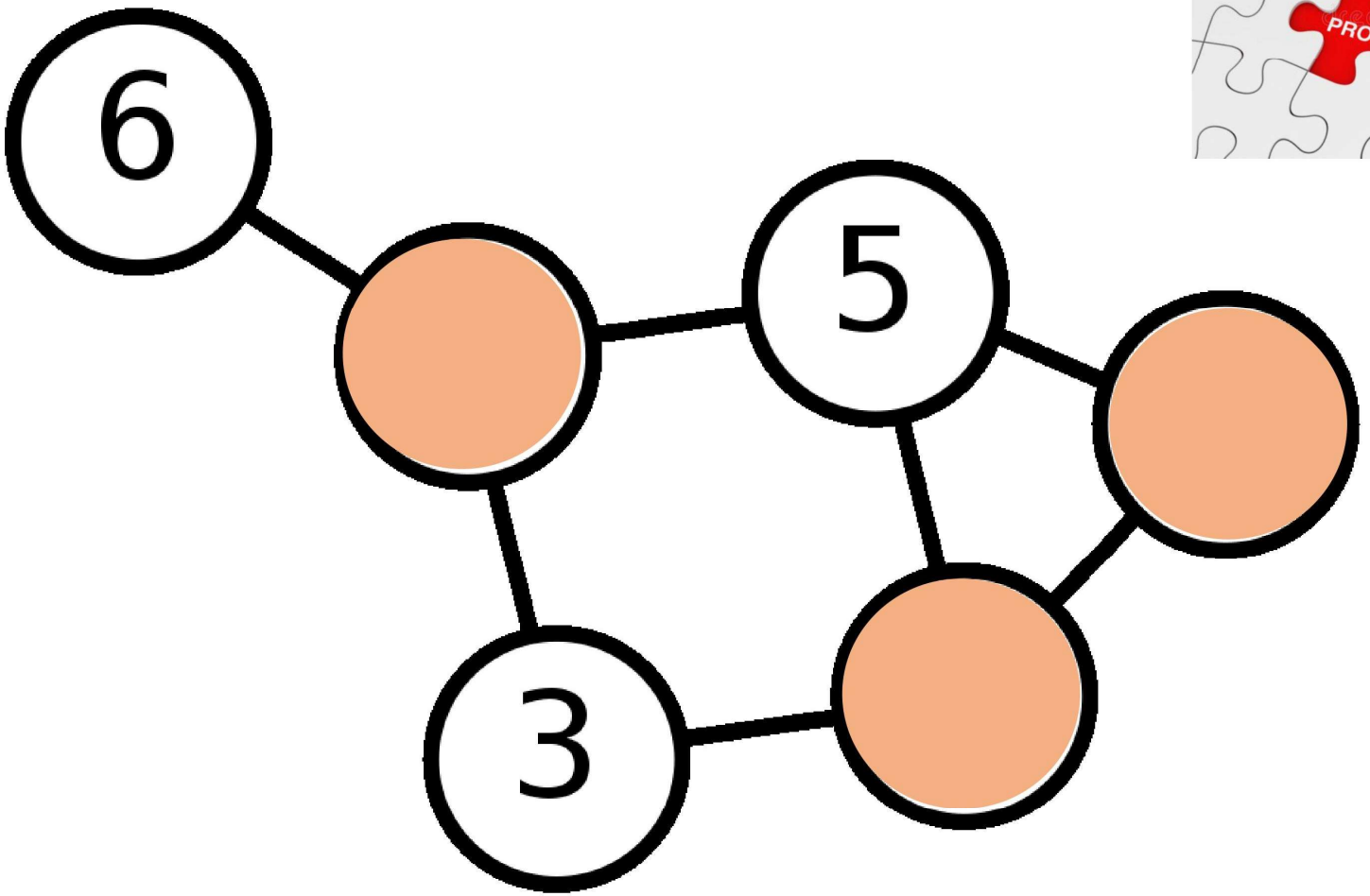
מציאת הכיסוי המינימלי

בעיית ריחוי הדדהודיות

2



רעיון ריחוי הדדהודיוח

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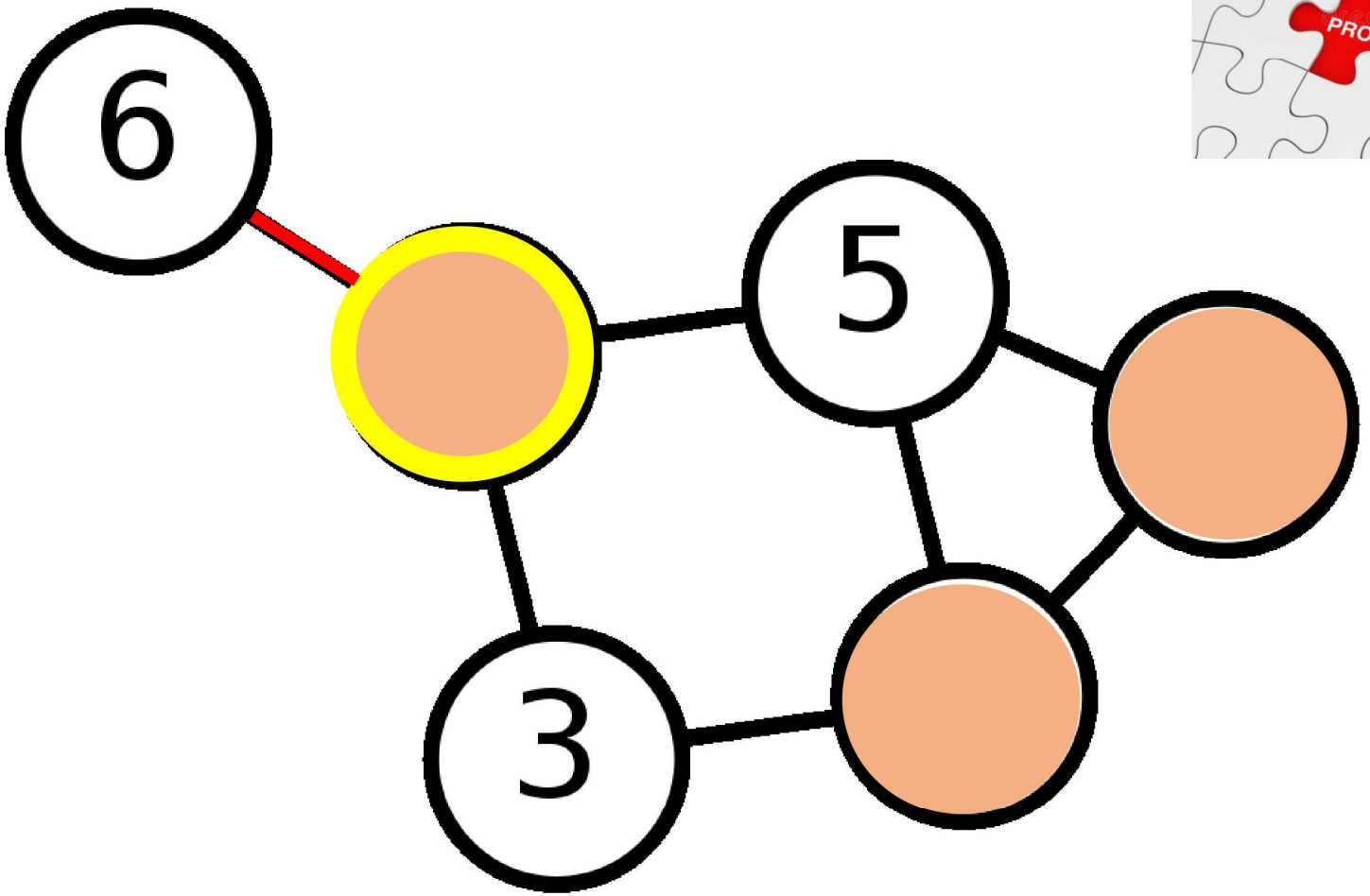
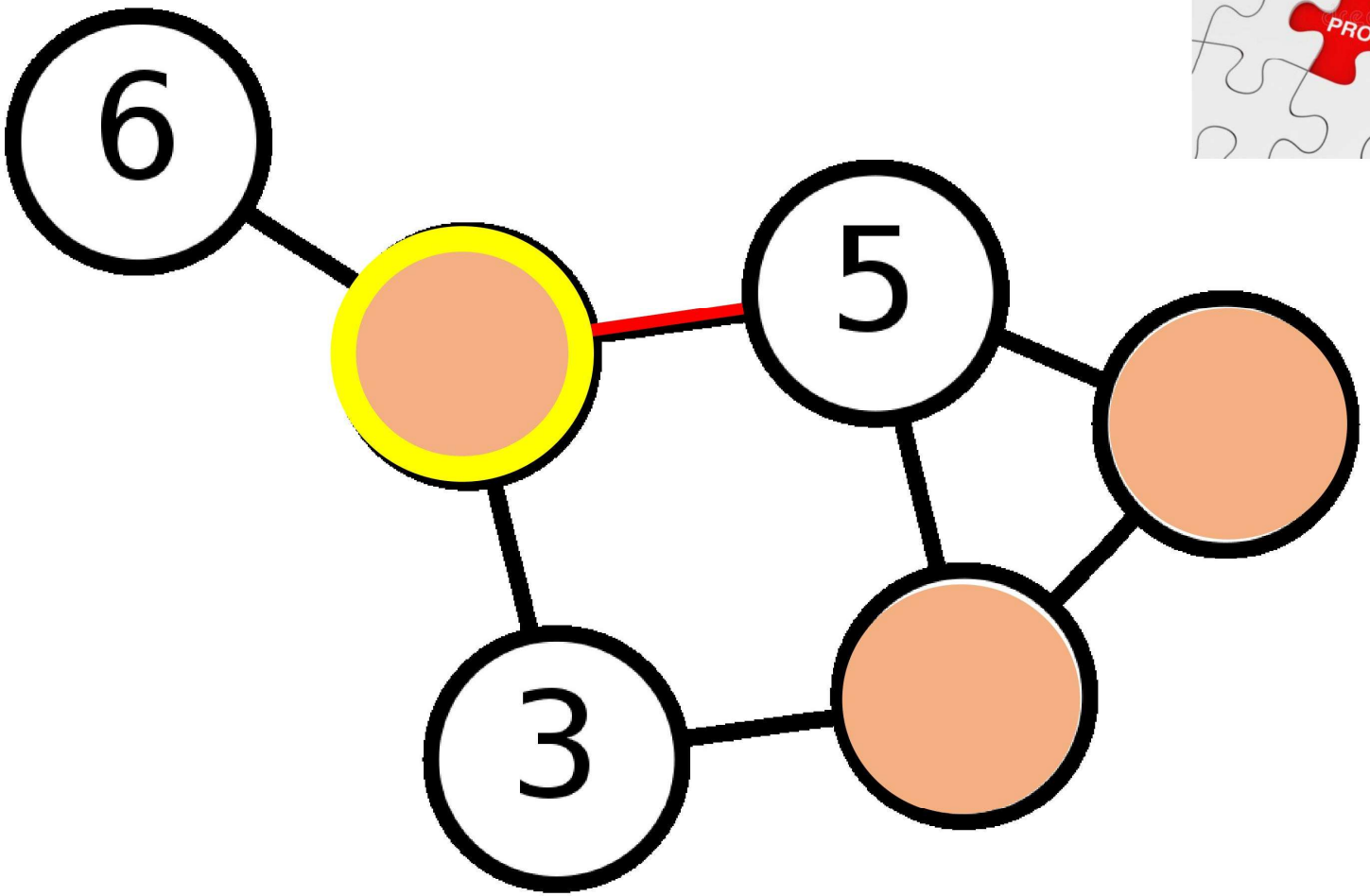


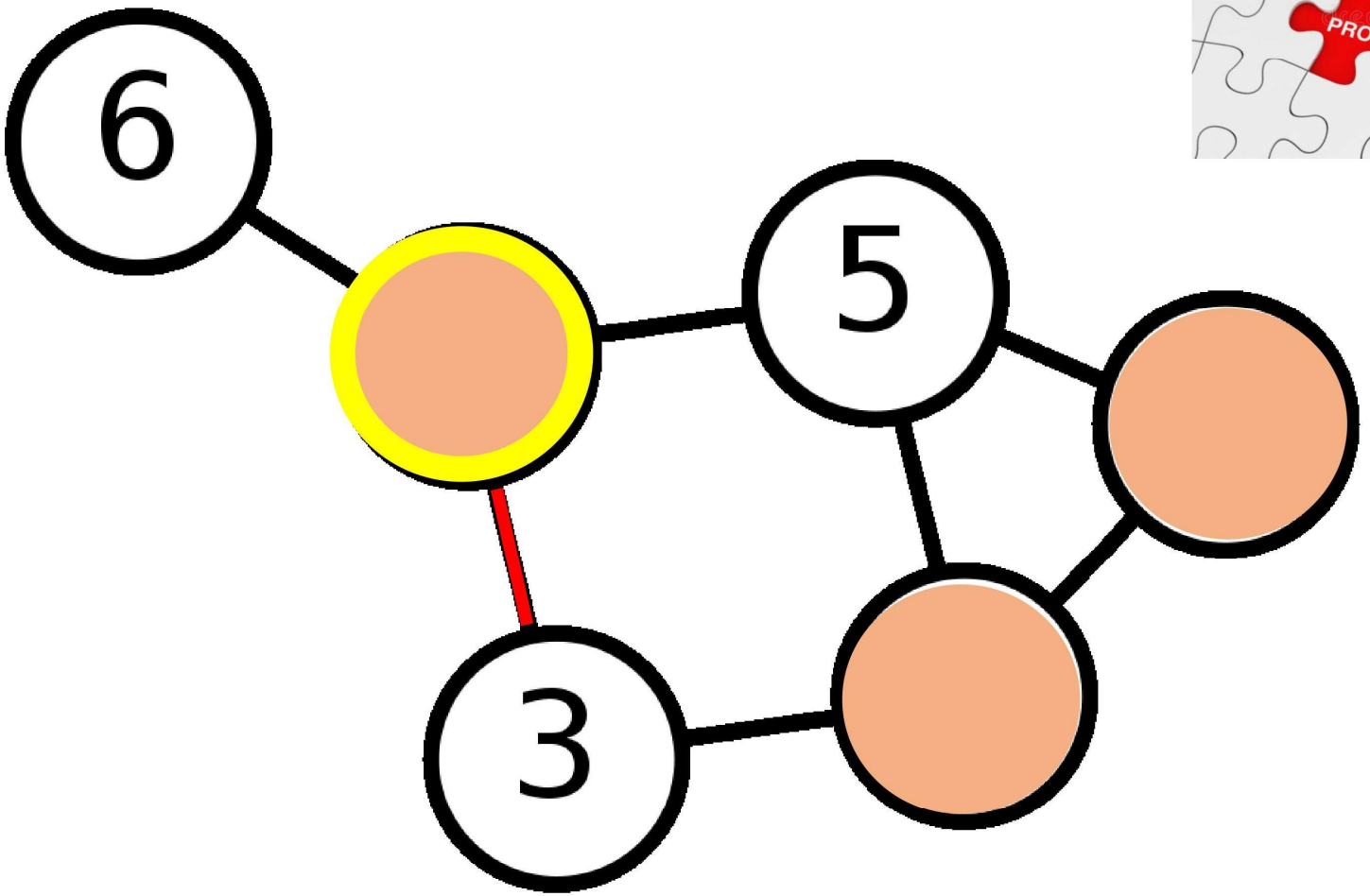
בעיית ריחוי הדדהודיות

2



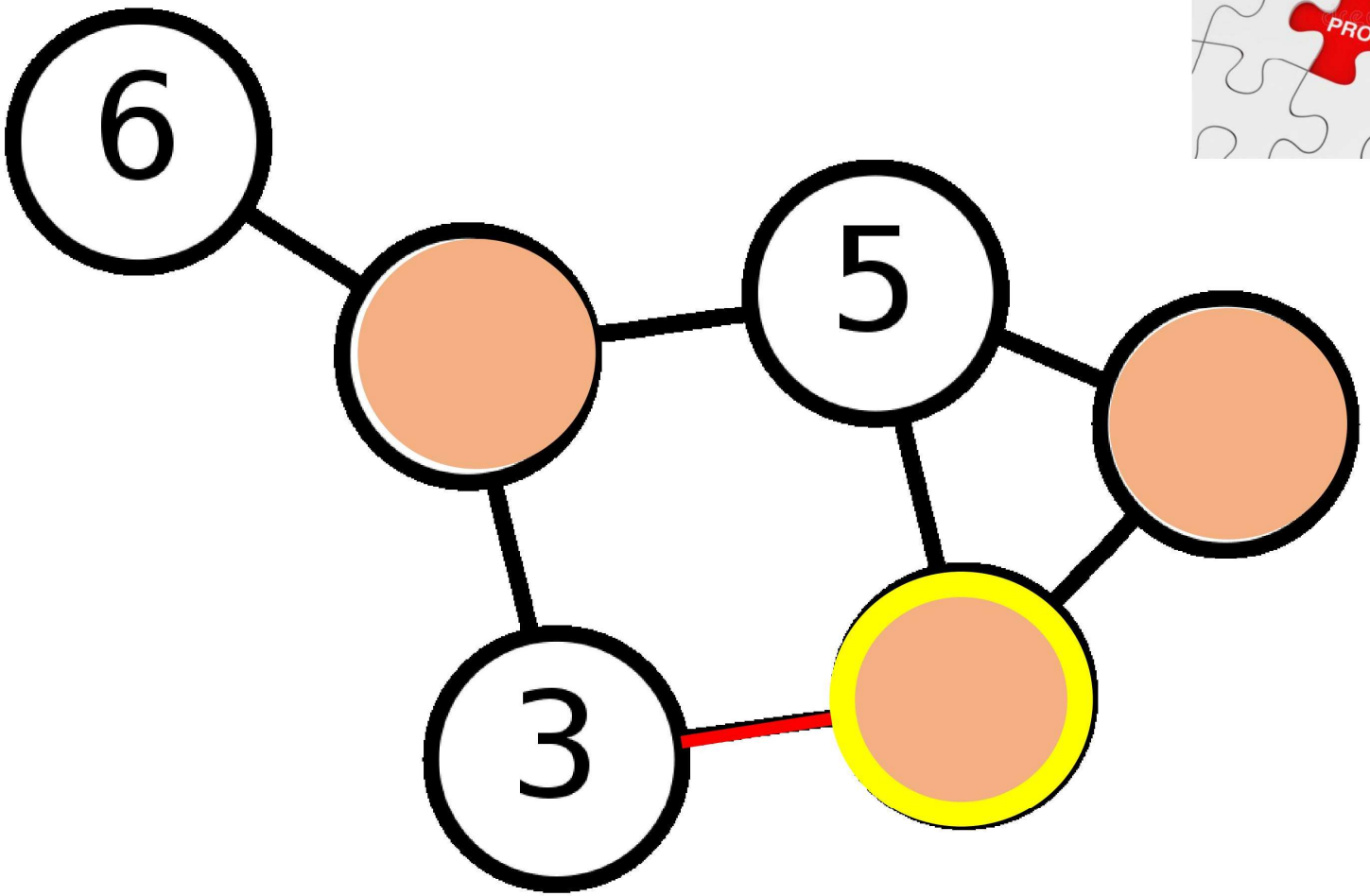
בעיית ריחוי הדדהודיות

2



בעיית ריחוי הדדהודיות

2



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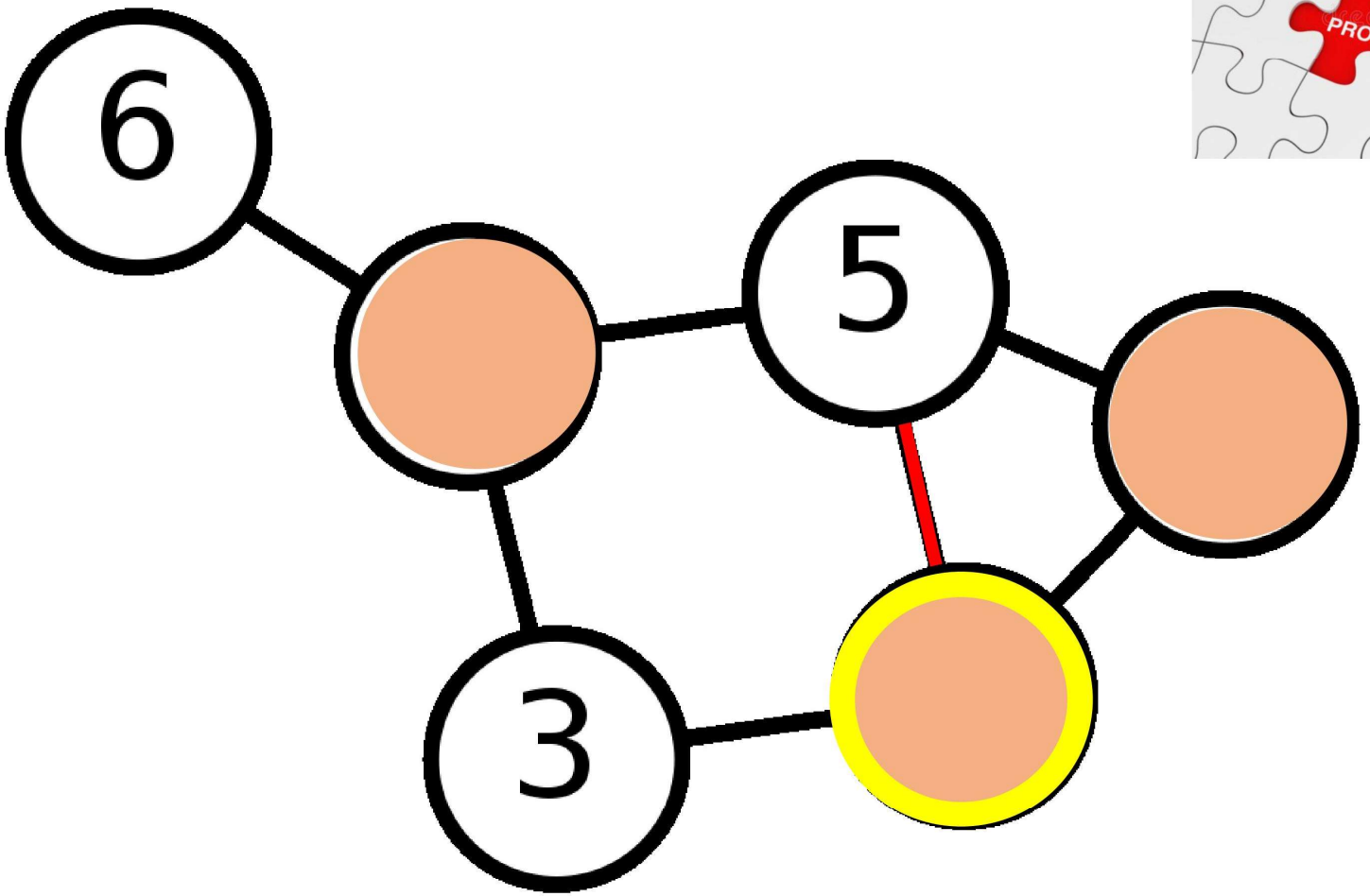
2



רעיון ריחוי הדדהודיוח

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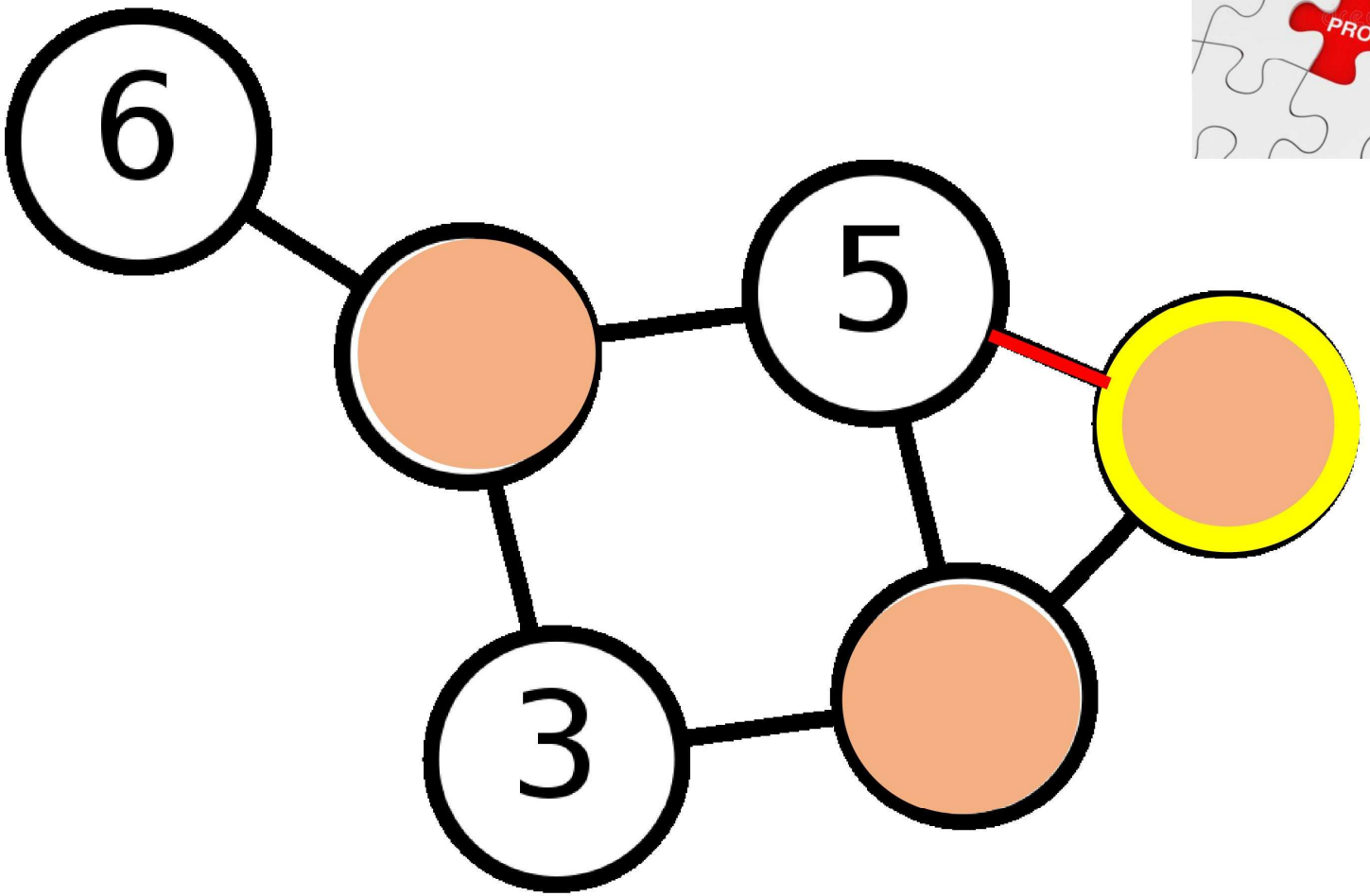
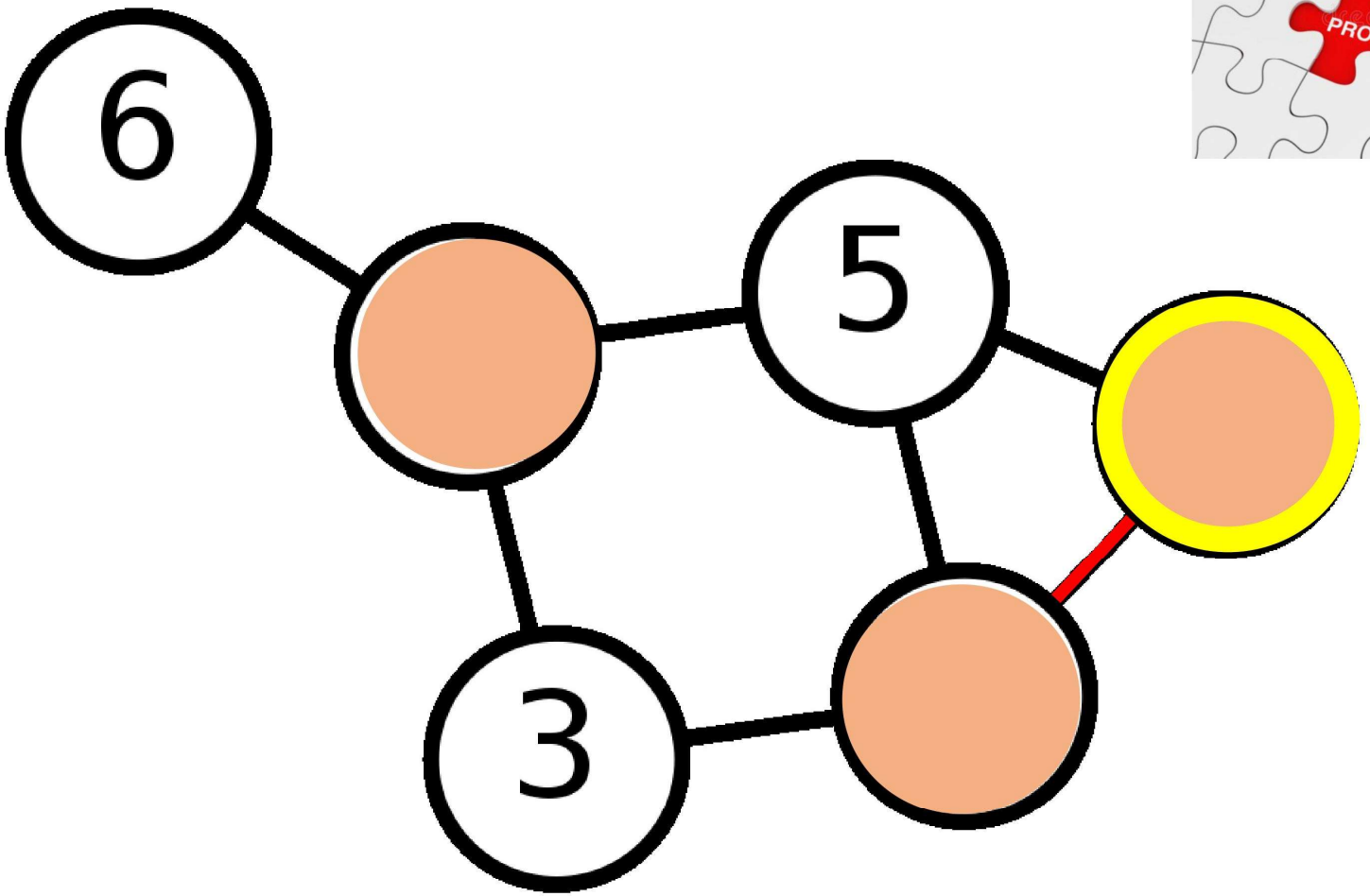


רעיון ריחוי הדדהודיוח

2



בעיית כיסוי קודקודיים

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NP שלמה

מציאת הכיסוי המינימלי

בעיית כיסוי קודקודיים

כיסוי על ידי קדקודים (vertex cover) של גרף בלתי מכוון $G=(V,E)$ הוא

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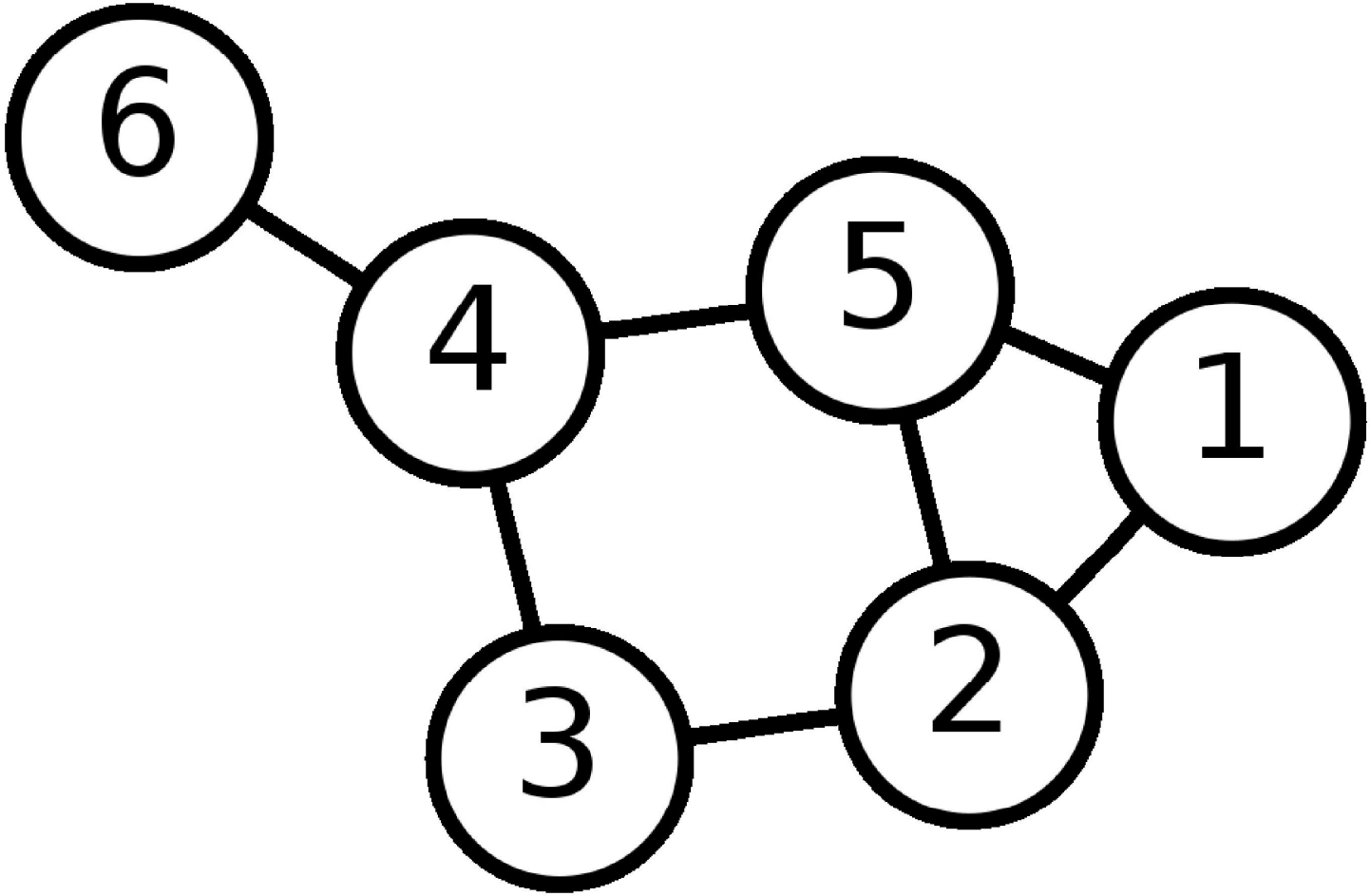
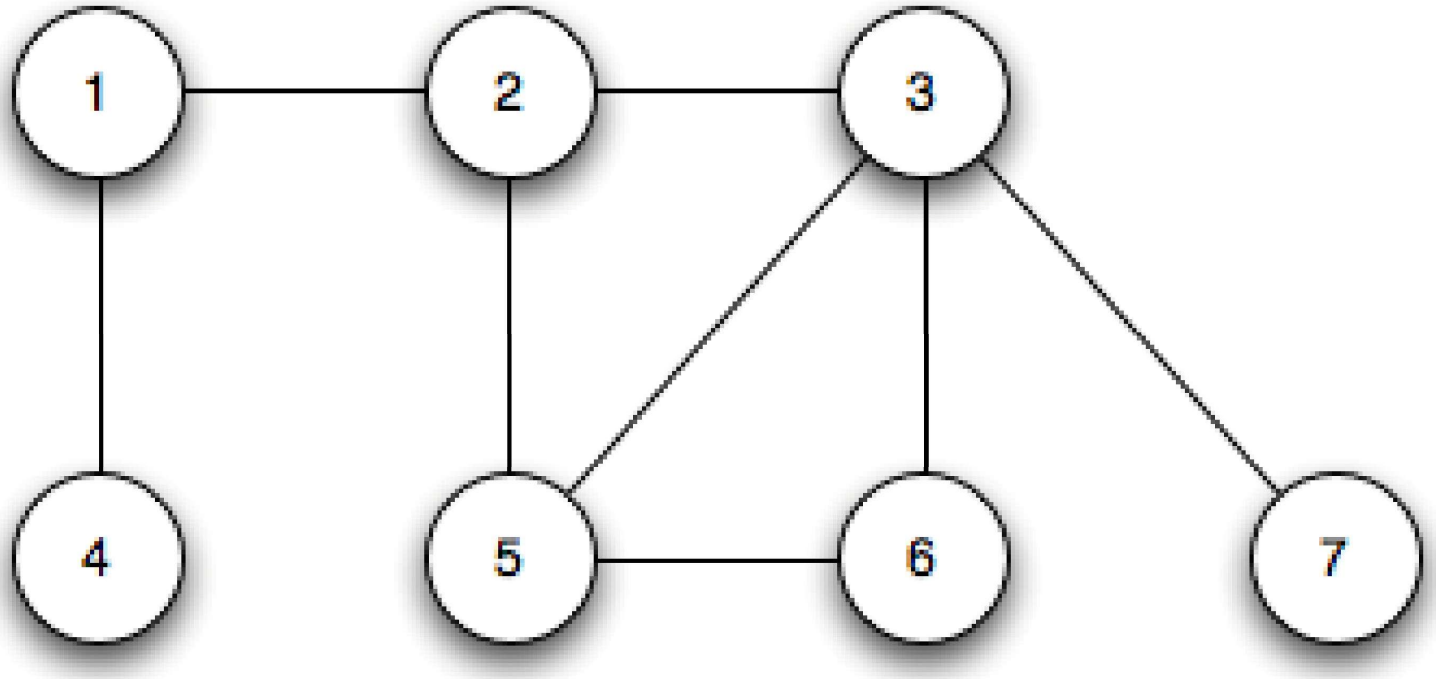
מציאת הכיסוי המינימלי

אלגוריתם הירור לרעיית ריחוי הודהודיים



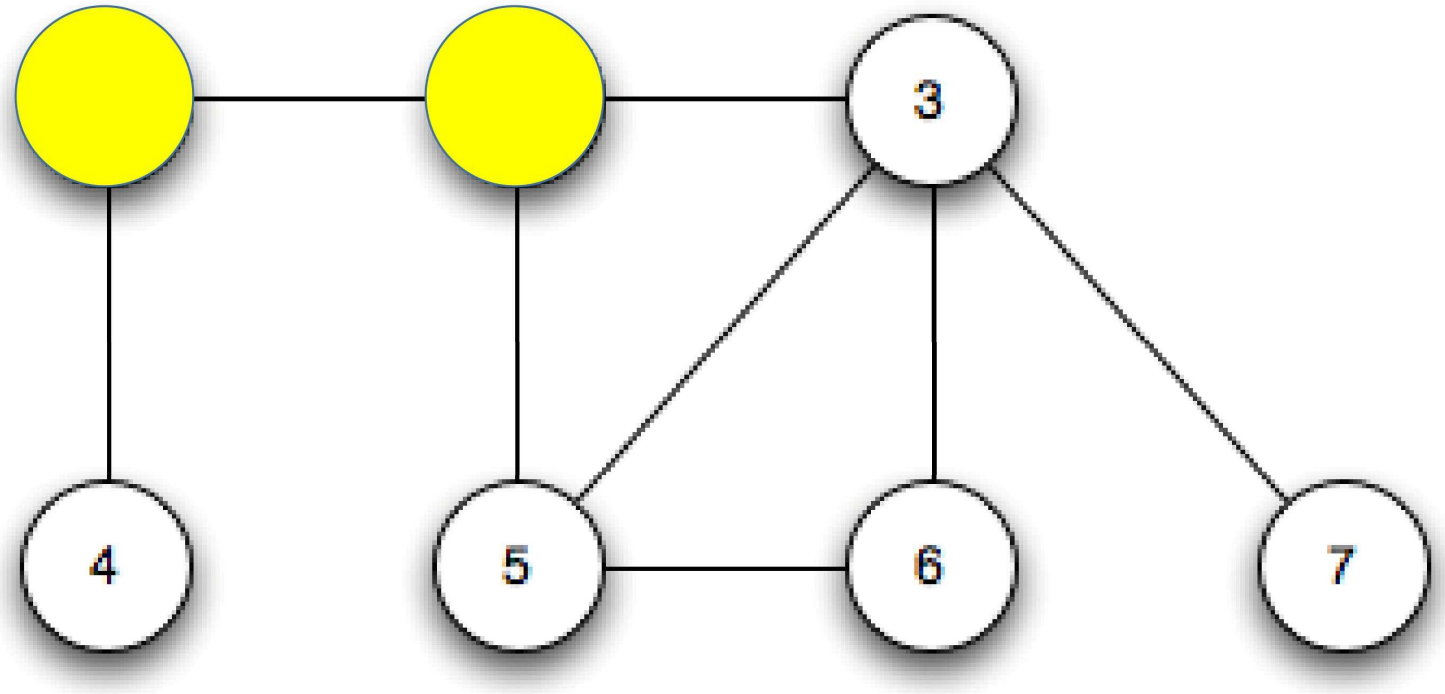
אתחול
אלגוריתם קירוב לבעיית כיסוי קודקודיים



E′	C
(1,4)	
(1,2)	
(2,5)	
(2,3)	
(3,5)	
(3,6)	
(3,7)	
(5,6)	

בחר צלע מהקבוצה E' - הכנס את קדקודיה ל-C

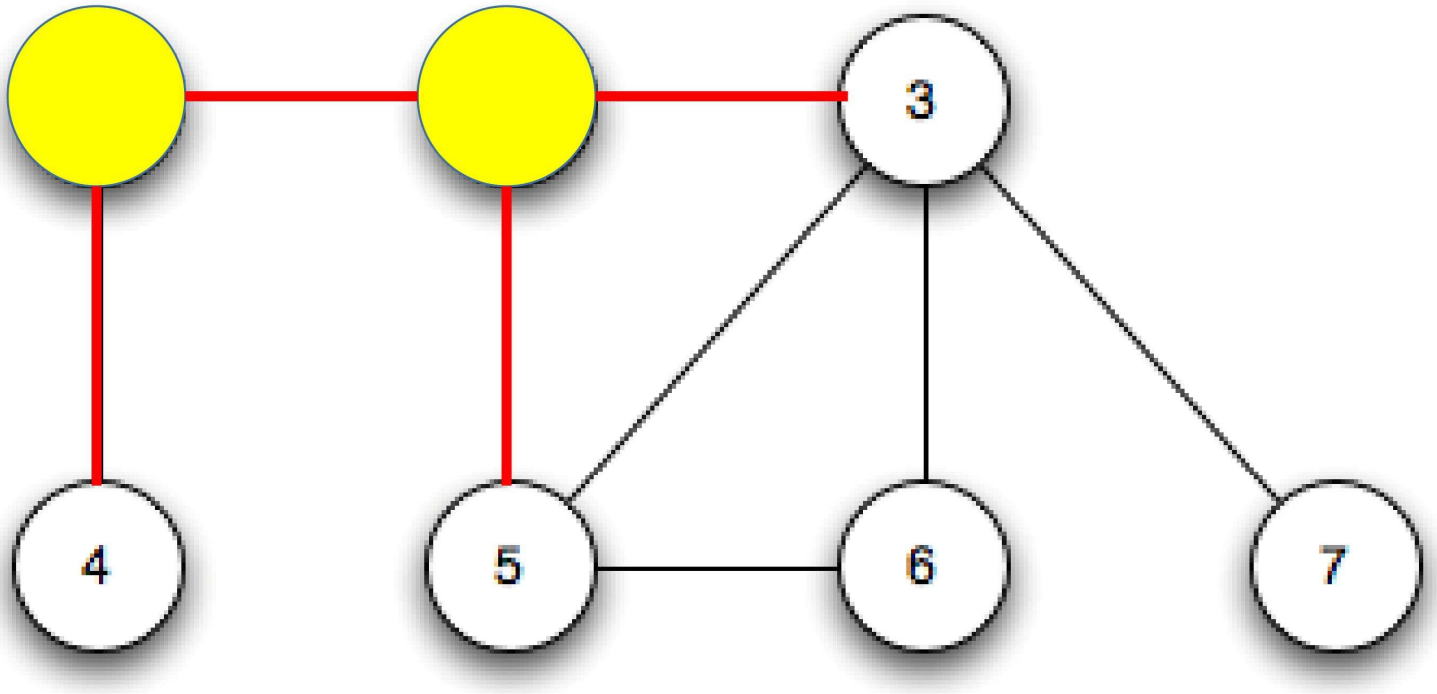
אלגוריתם קירוב לבעיית כיסוי קודקודיים



E'	C
(1,4)	1
(1,2)	2
(2,5)	
(2,3)	
(3,5)	
(3,6)	
(3,7)	
(5,6)	

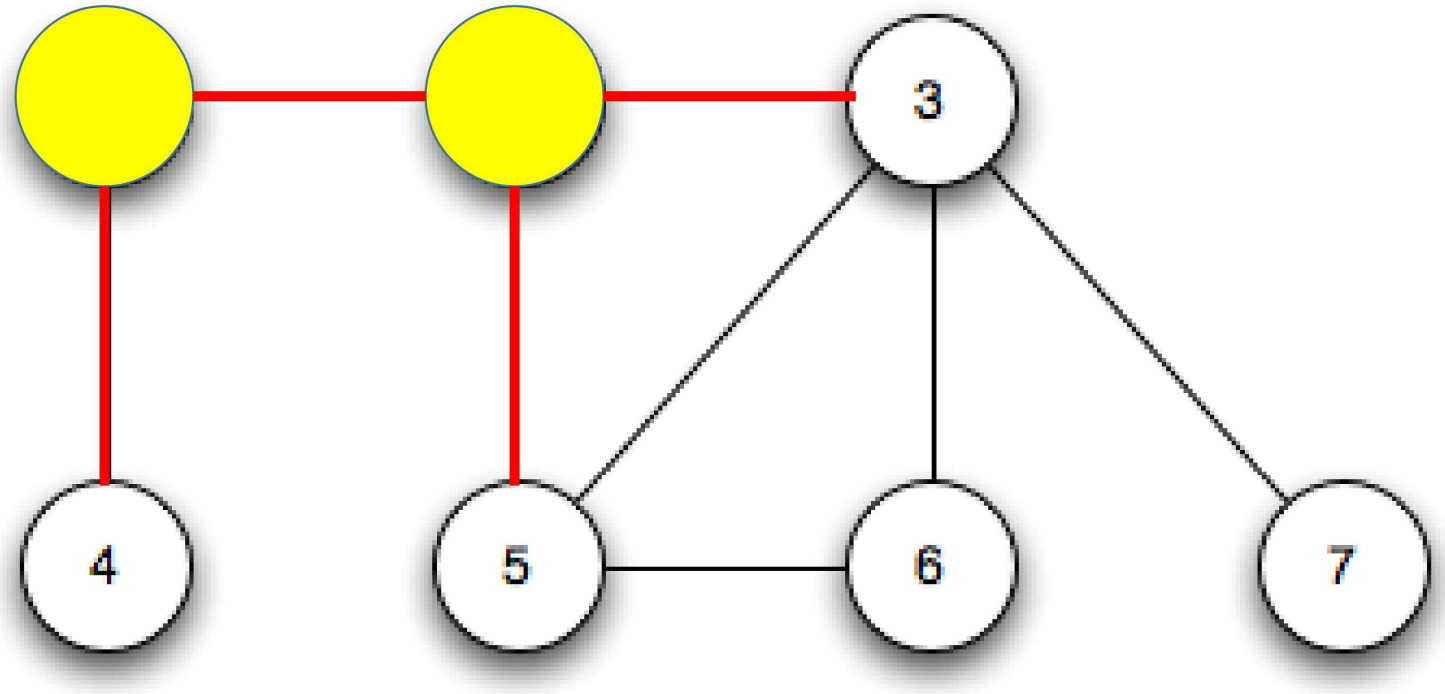
אלגוריתם קירוב לבעיית כיסוי קודקודיים

הסר את הצלעות מ-E′ המכוסות על די הקודקודיים שנוספו ל-C



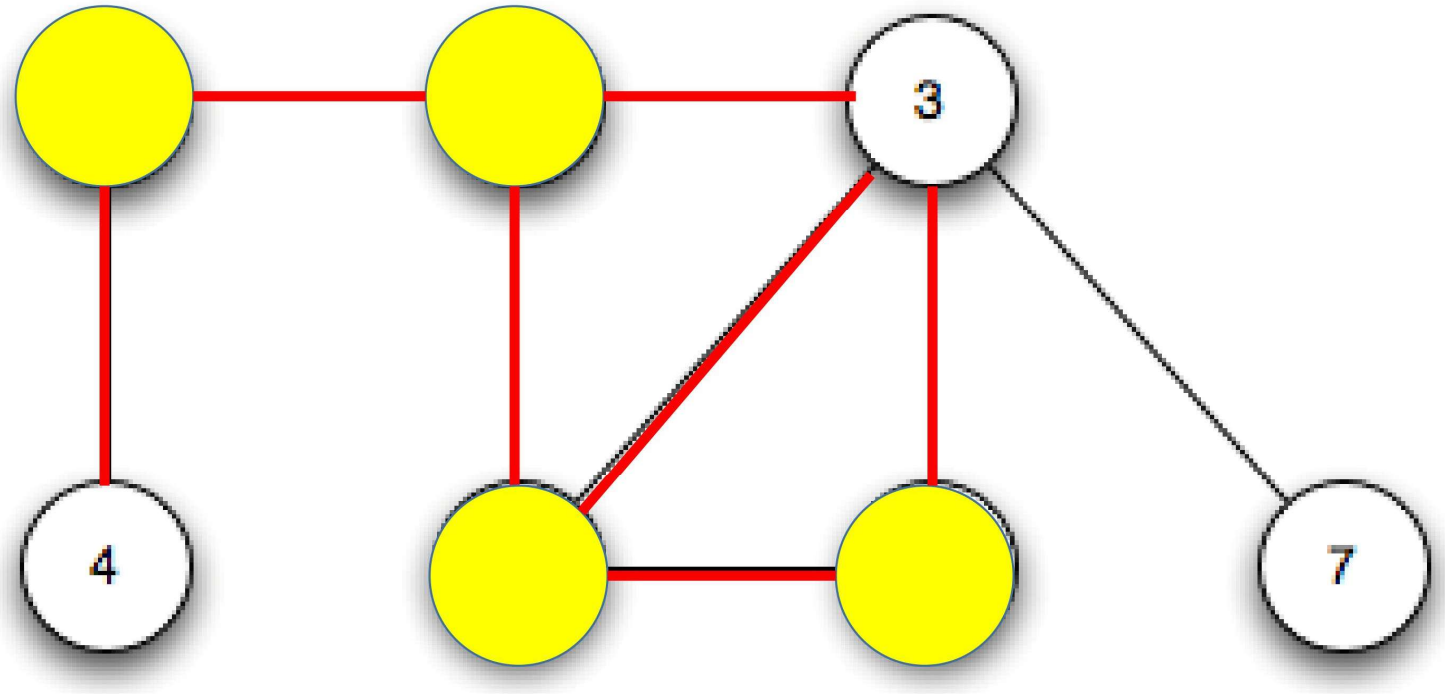
E′	C
(1,4)	1
(1,2)	2
(2,5)	
(2,3)	
(3,5)	
(3,6)	
(3,7)	
(5,6)	

אלגוריתם קירוב לבעיית כיסוי קודקודיים



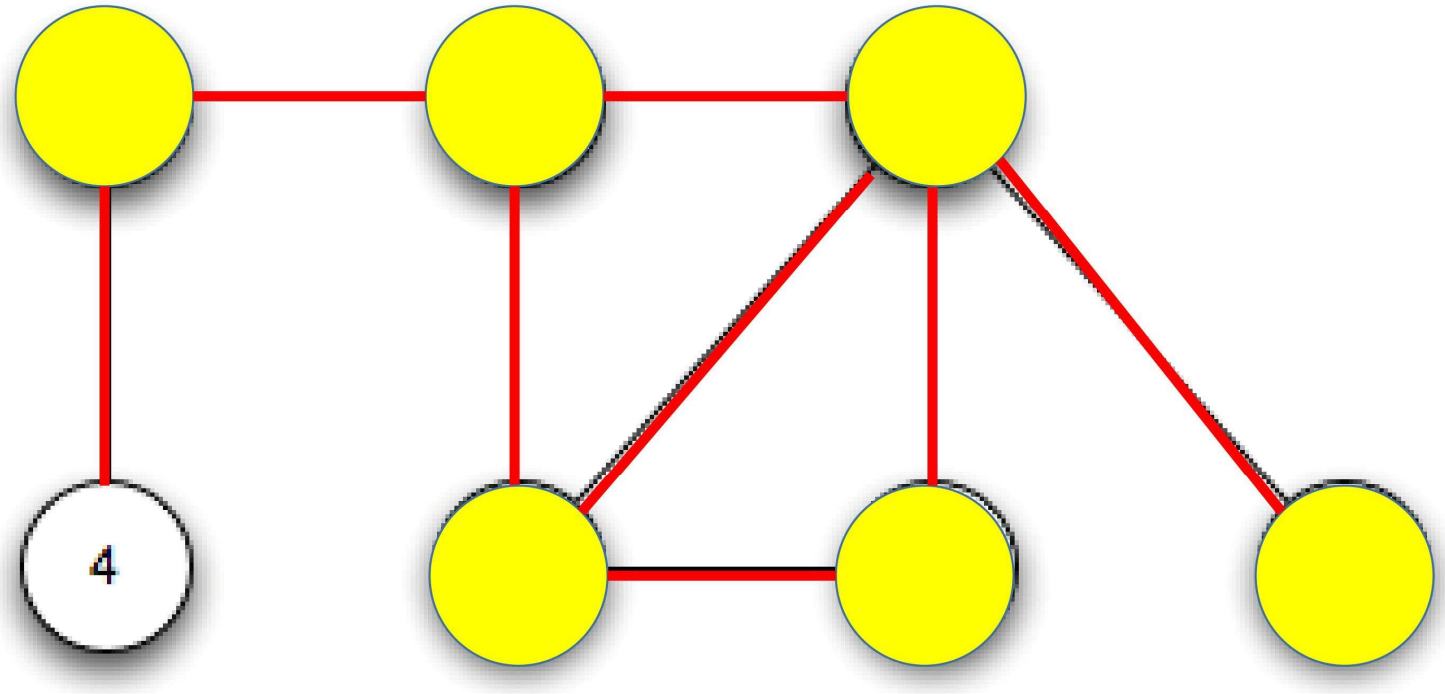
E′	C
	1
(1,2)	2
(3,5)	
(3,6)	
(3,7)	
(5,6)	

אלגוריתם קירוב לבעיית כיסוי קודקודיים



E′	C
	1
(1,2)	2
	5
	6
(3,7)	
(5,6)	

אלגוריתם קירוב לבעיית כיסוי קודקודיים



E'	C
	1
(1,2)	2
	5
	6
	3
	7
(3,7)	
(5,6)	

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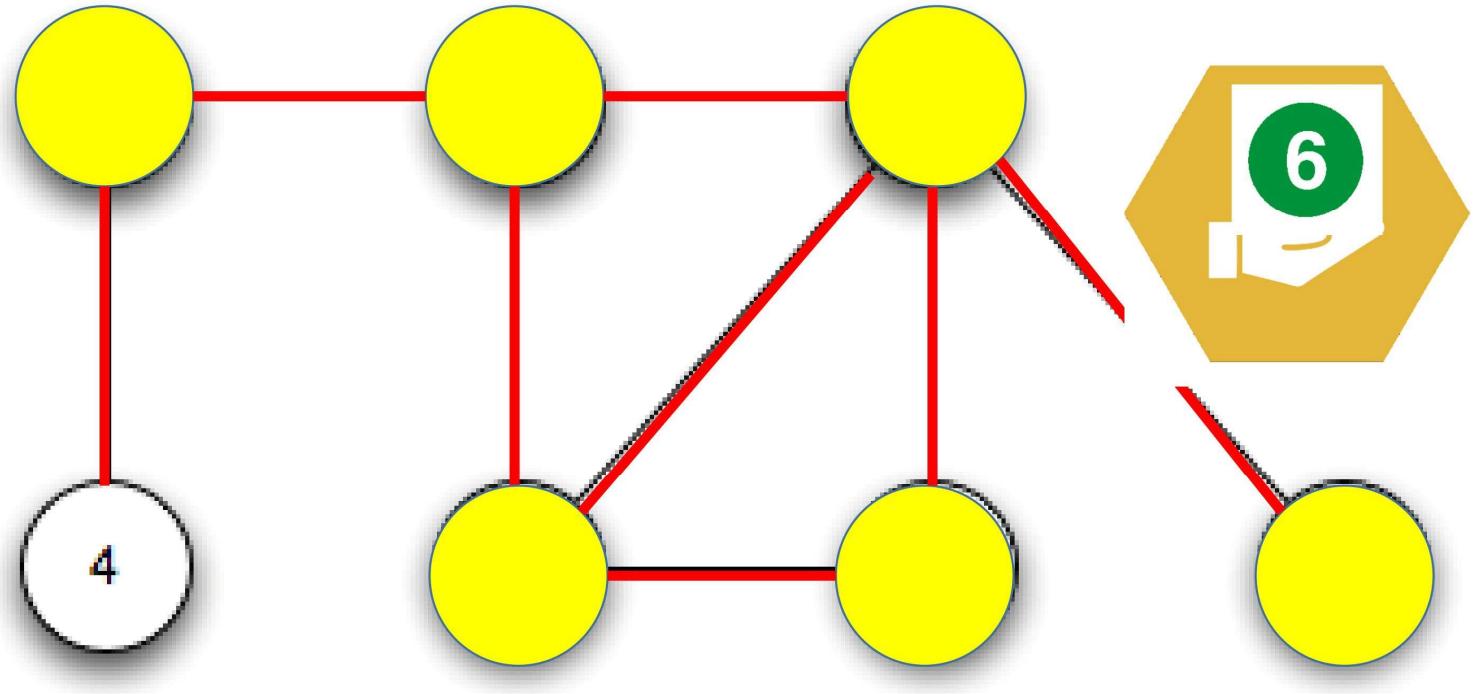
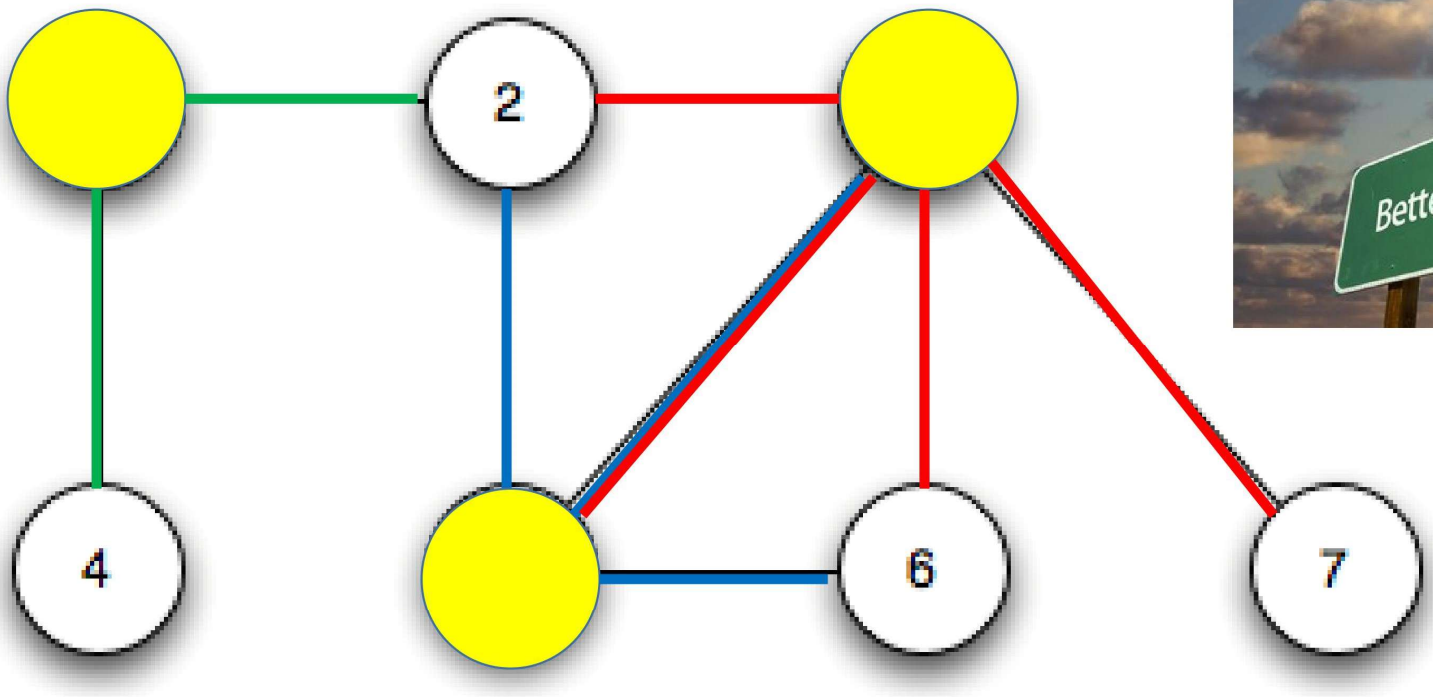


E'	C
	1
(1,2)	2
	5
	6
	3
	7
(3,7)	
(5,6)	

APROX-VERTEX_COVER(G)

1. $C \leftarrow \emptyset$
2. $E' \leftarrow E$
3. **while** $E' \neq \emptyset$
4. **do** let (u,v) be an arbitary edge of E'
5. $C \leftarrow C \cup \{u,v\}$
6. remove from E' every edge incident on either u or v
7. **return** C

אלגוריתם קירוב לבעיית כיסוי קודקודיים



האלגוריתם קירוב לבעיית כיסוי קודקודיים הוא בעל יחס קירוב 2

<u>שלב ראשון:</u> מדוע האלגוריתם מוצא כיסוי ?

האלגוריתם קירוב לבעיית כיסוי קודקודיים הוא בעל יחס קירוב 2

<u>שלב שני:</u> מדוע האלגוריתם מוצא כיסוי ?

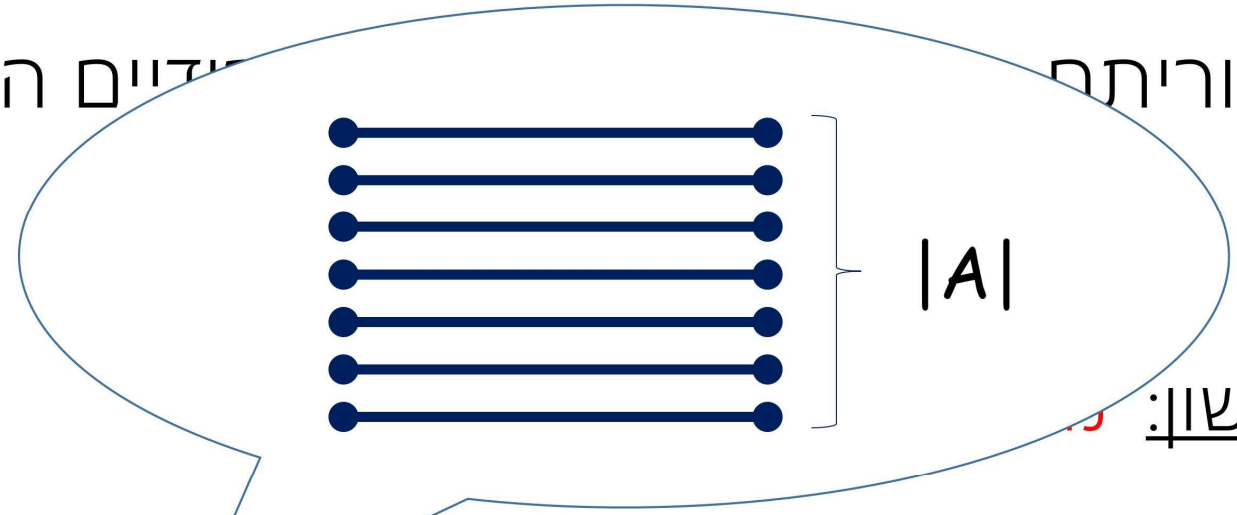
הקבוצה C שמחזיר האלגוריתם היא כיסוי קודקודיים, כיוון שהאלגוריתם חוזר ומבצע את הלולאה עד שכל קשת ב-E מכוסה על ידי איזשהו קדקוד ב-C

האלגוריתם קירוב לבעיית כיסוי קודקודיים הוא בעל יחס קירוב 2

<u>שלב ראשון:</u> מדוע הקירוב הוא 2?

נסמן ב-**A** את קבוצת הקשתות **(u,v)** שנבחרה בשורה 4. קבוצה זו אינה מכילה קשתות שלהן צמתים משותפים, לכן **|C|=2·|A|**.

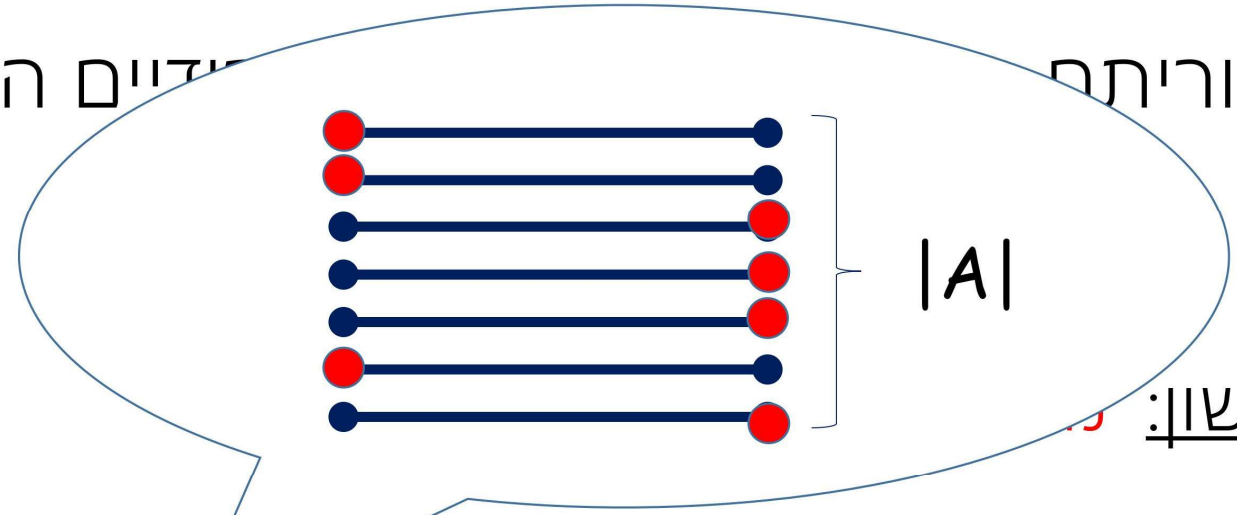
האלגוריתם [illegible] הוא בעל



שלב ראשון: [illegible]

נסמן ב-**A** את קבוצת הקשתות **(u,v)** שנבחרה בשורה 4. קבוצה זו אינה מכילה קשתות שלהן צמתים משותפים, לכן **|C|=2·|A|**.

האלגוריתם [illegible] הוא בעל



שלב ראשון: [illegible]

נסמן ב-A את קבוצת הקשתות (u,v) שנבחרה בשורה 4. קבוצה זו אינה מכילה קשתות שלהן צמתים משותפים, לכן $|C|=2\cdot|A|$.

צמתי הפתרון האופטימלי C^* מכסים את קשתות A ולכן מכילים לפחות צומת אחד לכל קשת ומכאן $|C^*|\geq|A|$.

ולכן נקבל, $|C|\leq 2|C^*|$

Two Glass Balls and a Tower

Amihood Amir

Bar Ilan University

And

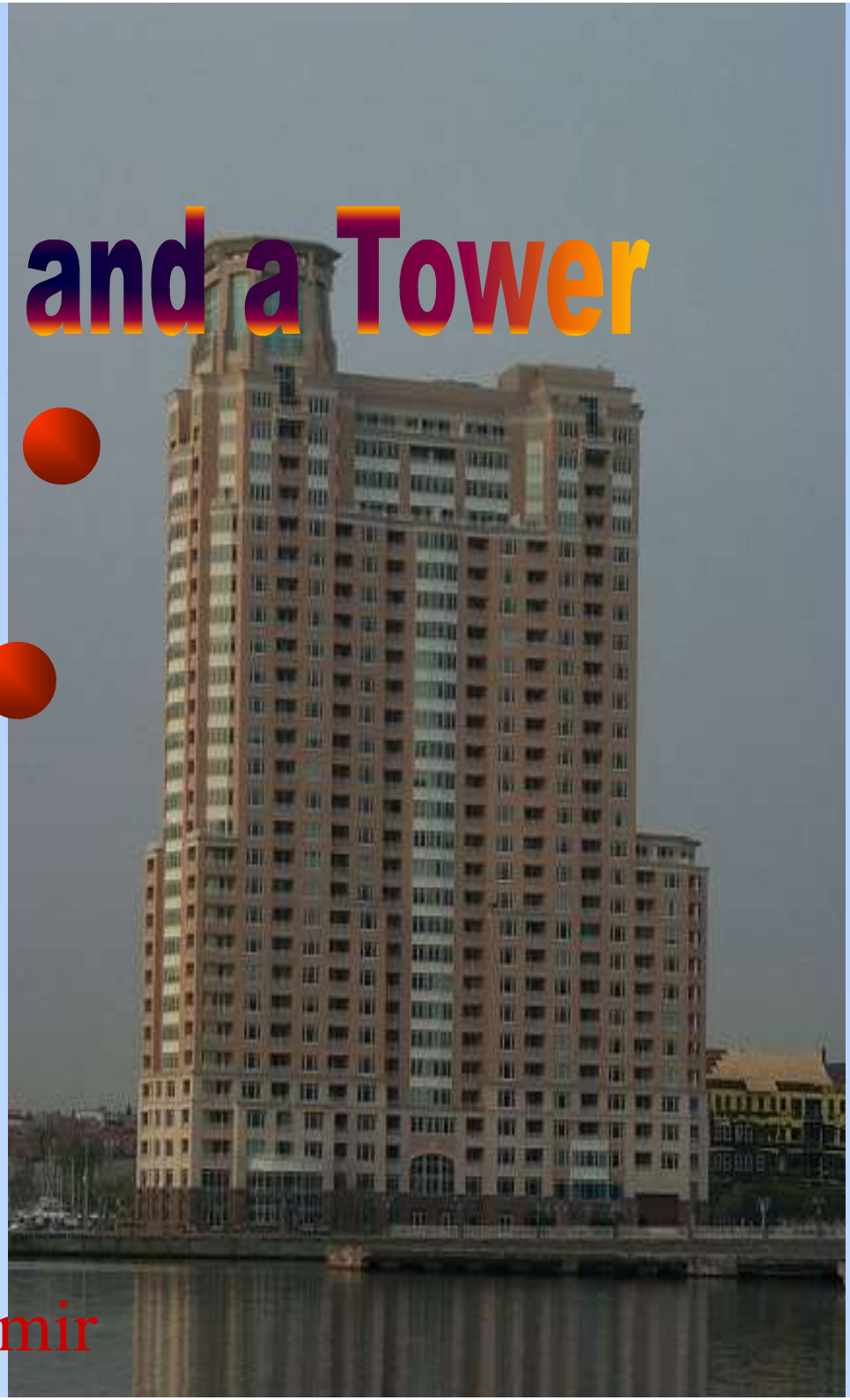
Georgia Tech

Costas Iliopoulos

Oren Kapah

Ely Porat

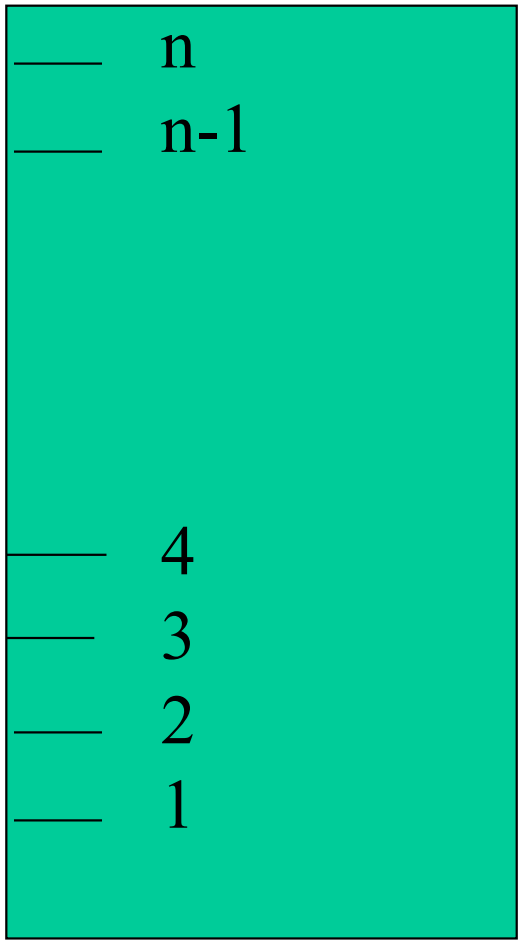
Artistic Consultant: Aviya Amir



Given: A glass ball.
An n storied building.

Find: The floor k such that the ball breaks when dropped from it, but does not break if dropped from floor $k-1$.

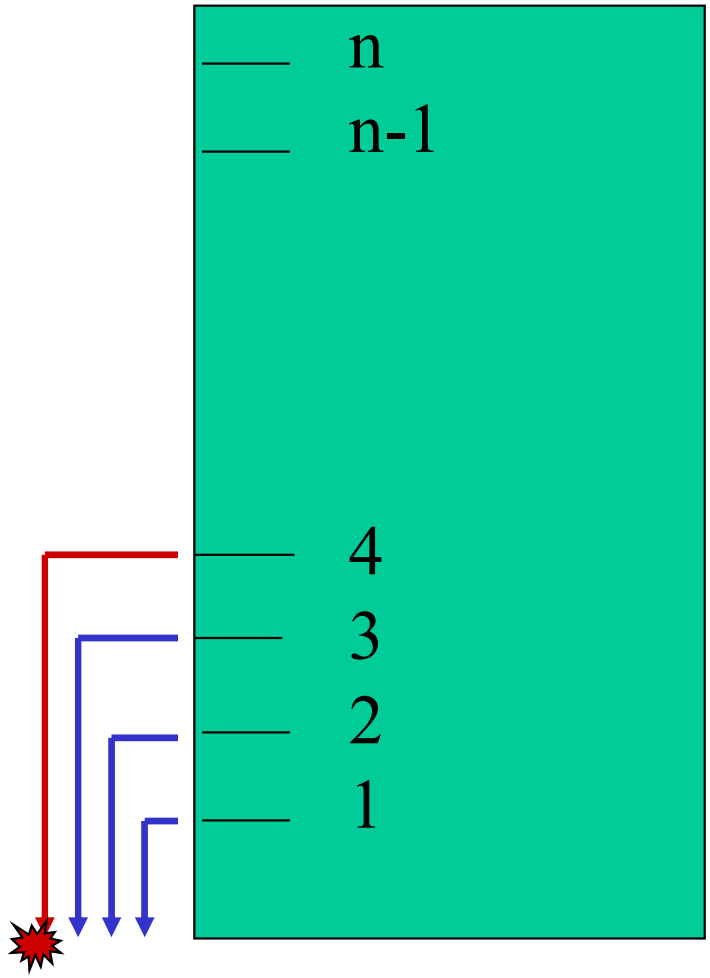
STRATEGY 1: Only *one* ball given to experiment with.



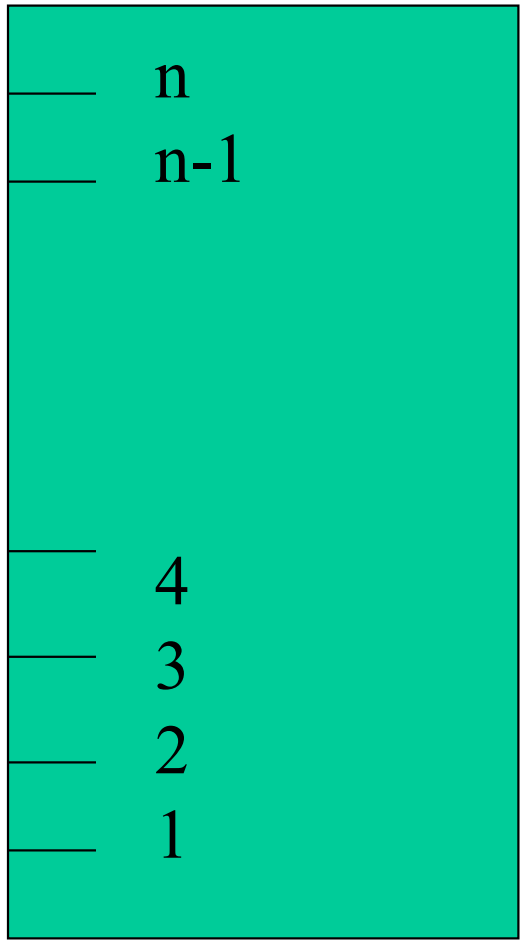
STRATEGY 1: Only *one* ball given to experiment with.

O(n) experiments necessary.

Sequential search.



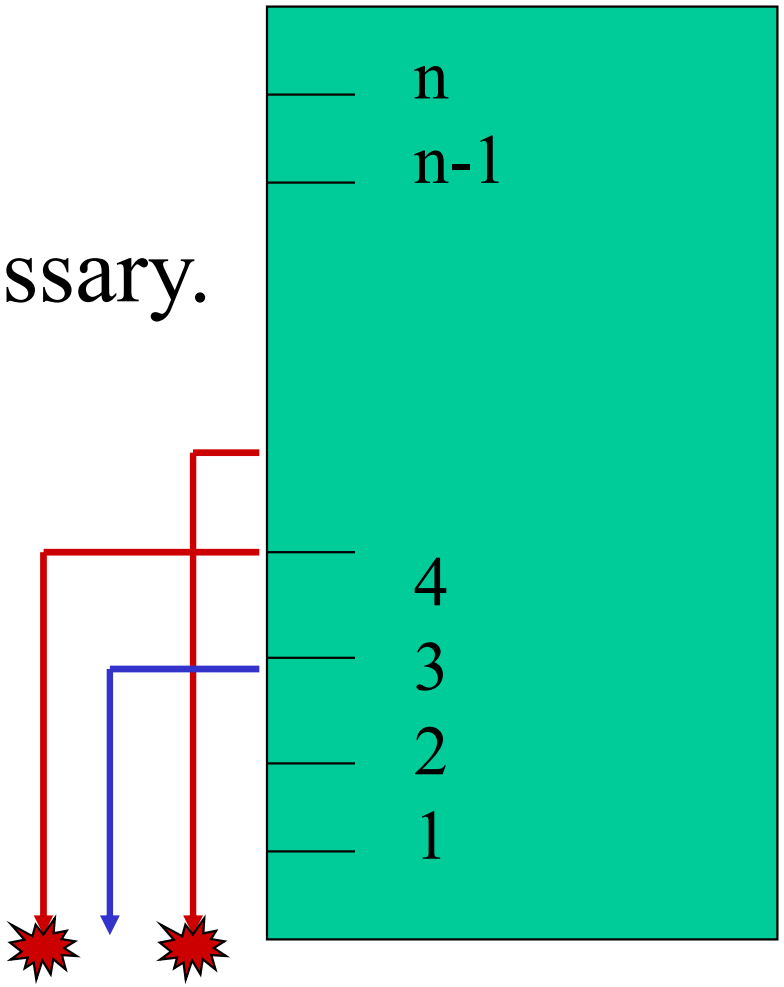
STRATEGY 2: As many balls as necessary are given to experiment with.



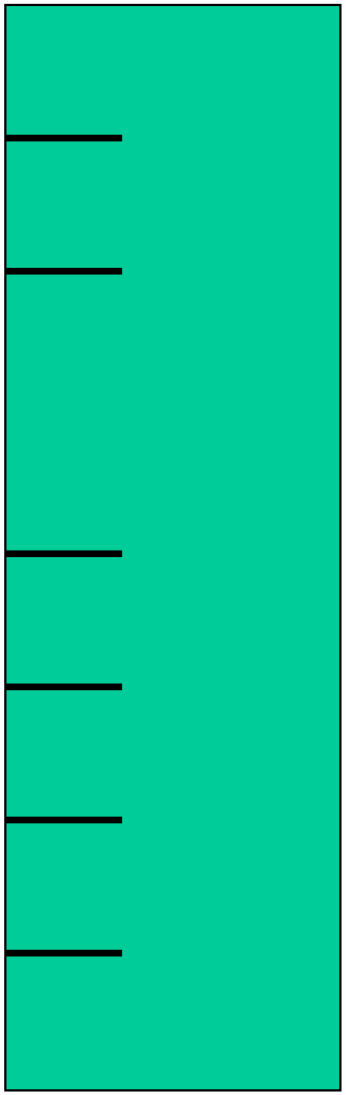
STRATEGY 2: As many balls as necessary are given to experiment with.

$O(\log n)$ experiments necessary.

Binary search.



STRATEGY 3: Only *two* balls given to experiment with.



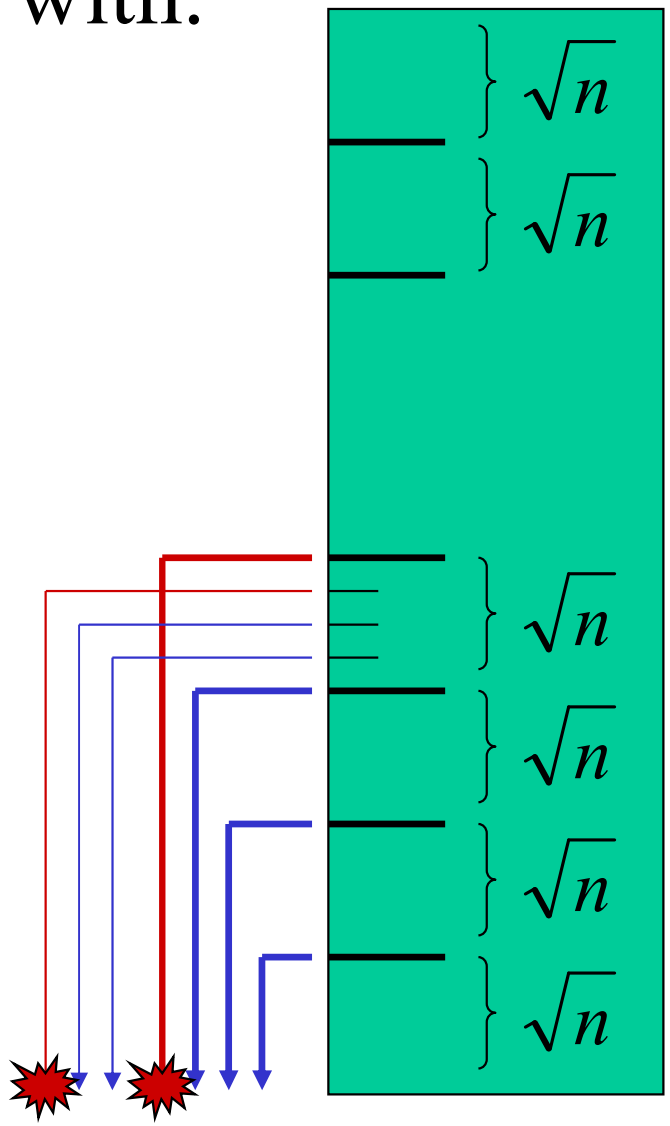
STRATEGY 3: Only *two* balls given to experiment with.

$O(\sqrt{n})$ experiments necessary.

Bounded divide-and-conquer

$O(\sqrt{n})$ Experiments 1st ball

$O(\sqrt{n})$ Experiments 2nd ball



Meaning of two Balls

"Bounded" Divide-and-Conquer.

In reality: Different paradigms.

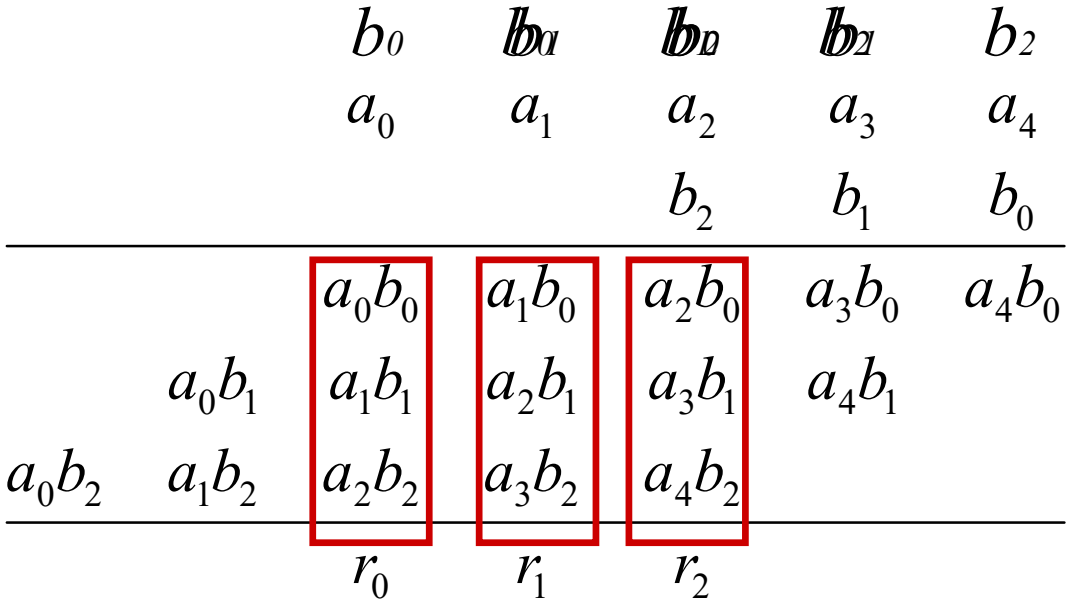
1. Works on large groups.

2. Works within a group.

In Pattern Matching

1. Works on large groups:

Convolutions:



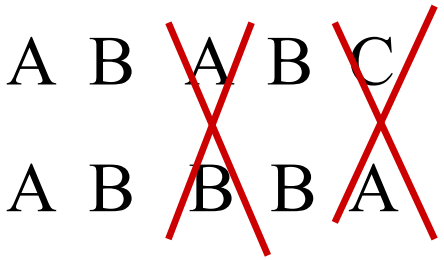
$O(n \log m)$ using FFT

Problem: $O(n \log m)$ only in algebraically closed fields, e.g. $\mathbf{C}$.

Solution: Reduce problem to (Boolean/integer/real) multiplication.

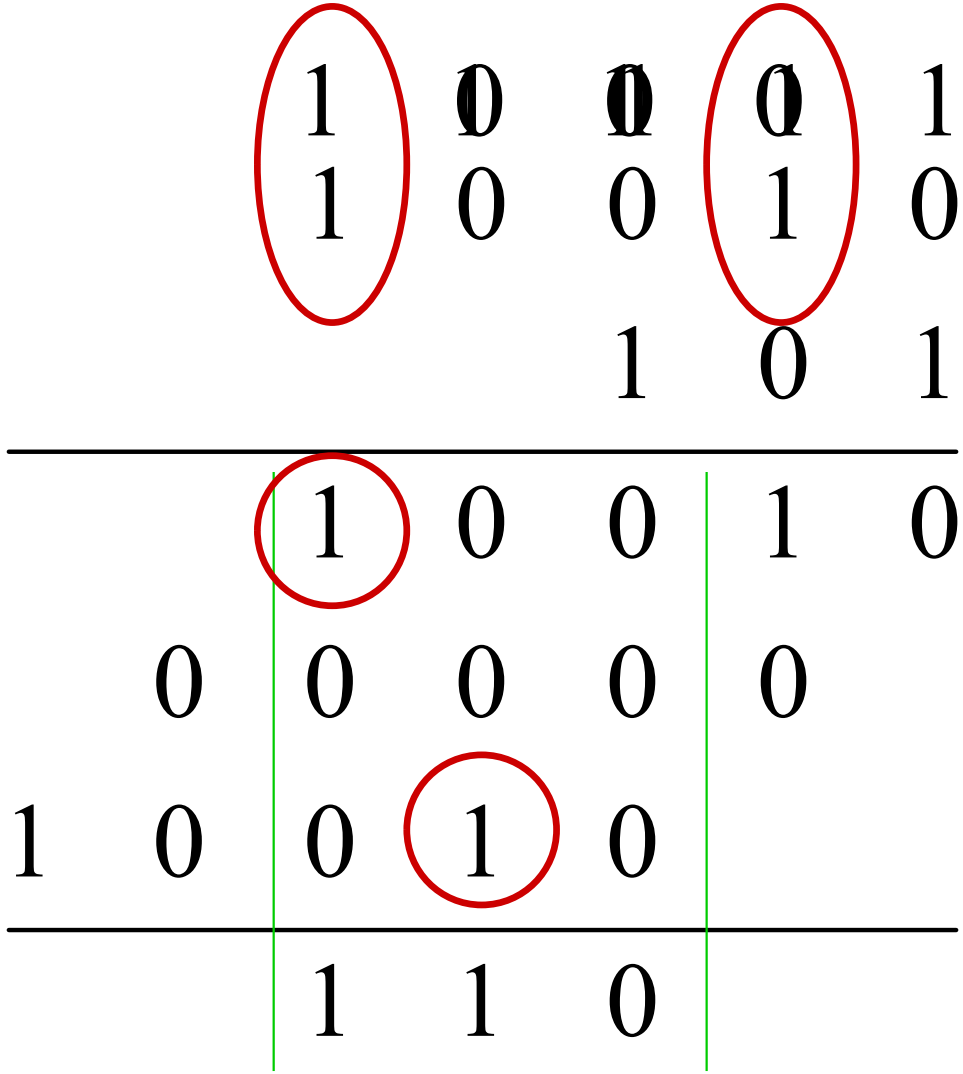
This reduction costs!

Example: Hamming distance.



Counting *mismatches* is equivalent to Counting *matches*

Example:



Count all “hits” of 1 in pattern and 1 in text.

For $a \in \Sigma$

Define:

$$\chi_a(b) = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{o/w} \end{cases}$$

$$\chi_a(S_1 S_2 S_3 \ldots S_n) = \chi_a(S_1)\chi_a(S_2)\chi_a(S_3)\ldots\chi_a(S_n)$$

Example:

$$\chi_a(abbaabb) = 1001100$$

For $\Sigma = \{a, b, c\}$

Do:

$$\begin{aligned} & \chi_a(T) \times \chi_a(P^R) \\ + \; & \chi_b(T) \times \chi_b(P^R) \\ + \; & \chi_c(T) \times \chi_c(P^R) \end{aligned}$$

Result: The number of times a in pattern matches a in text + the number of times b in pattern matches b in text + the number of times c in pattern matches c in text.

So for alphabet with a symbols (a fixed) the time is:

$$O(n\, a \log m) = O(n \log m)$$

Problem: Infinite alphabets.

Without loss of generality:

$$|\Sigma| = m + 1$$

Since every element of T not in P is replaced by some symbol x not in P.

Example:

ABCDEFGH same number of errors as ABXXXXGH

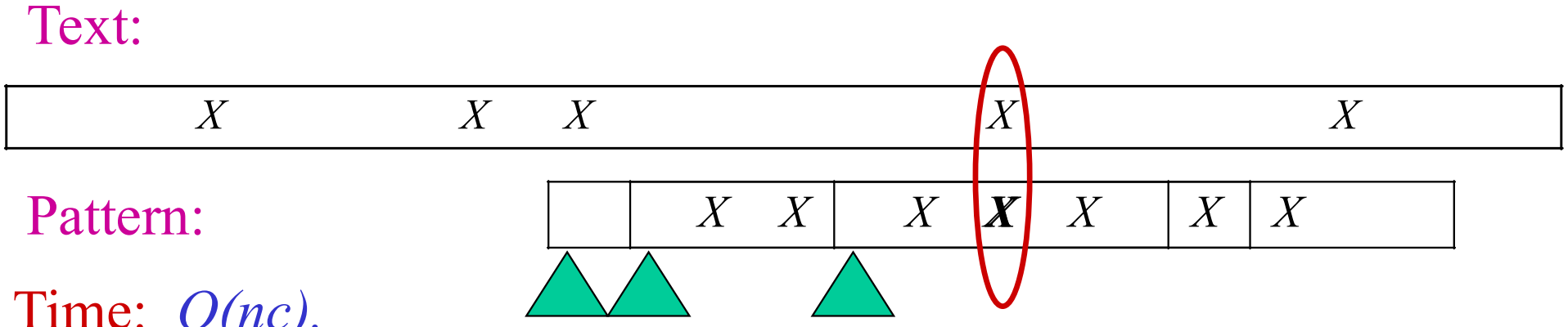
ABBBBBGH ABBBB BGH

Needed: Fast way to compute matches of x to itself.

Such a method exists if x appears in the pattern a very small number of times.

Assume: x appears in pattern c times.

For every occurrence of x in text, update just the appropriate counters of the c occurrences of x in the pattern.



Time: $O(nc)$.

Problem: In general it could be that x occurs in the pattern $O(m)$ times, then total time becomes $O(nm)$. BAD again.

Tradeoff: If x appears in the pattern more than c times, count matches by FFT, in time $O(n \log m)$, per x.

For all x's that appear in the pattern less than c times, count matches (simultaneously) in time $O(nc)$.

How many elements appear at least c times?

$$\frac{m}{c}$$

For these elements , time: *O((m/c) n log m).*

For all other elements, time: *O(nc).*

The optimal case is when they equal, i.e.

$$\frac{mn\log m}{c} = cn$$

$$m\log m = c^2$$

$$\sqrt{m\log m} = c$$

Total Time: $\boxed{O(n\sqrt{m\log m})}$ (A-87,K-87)