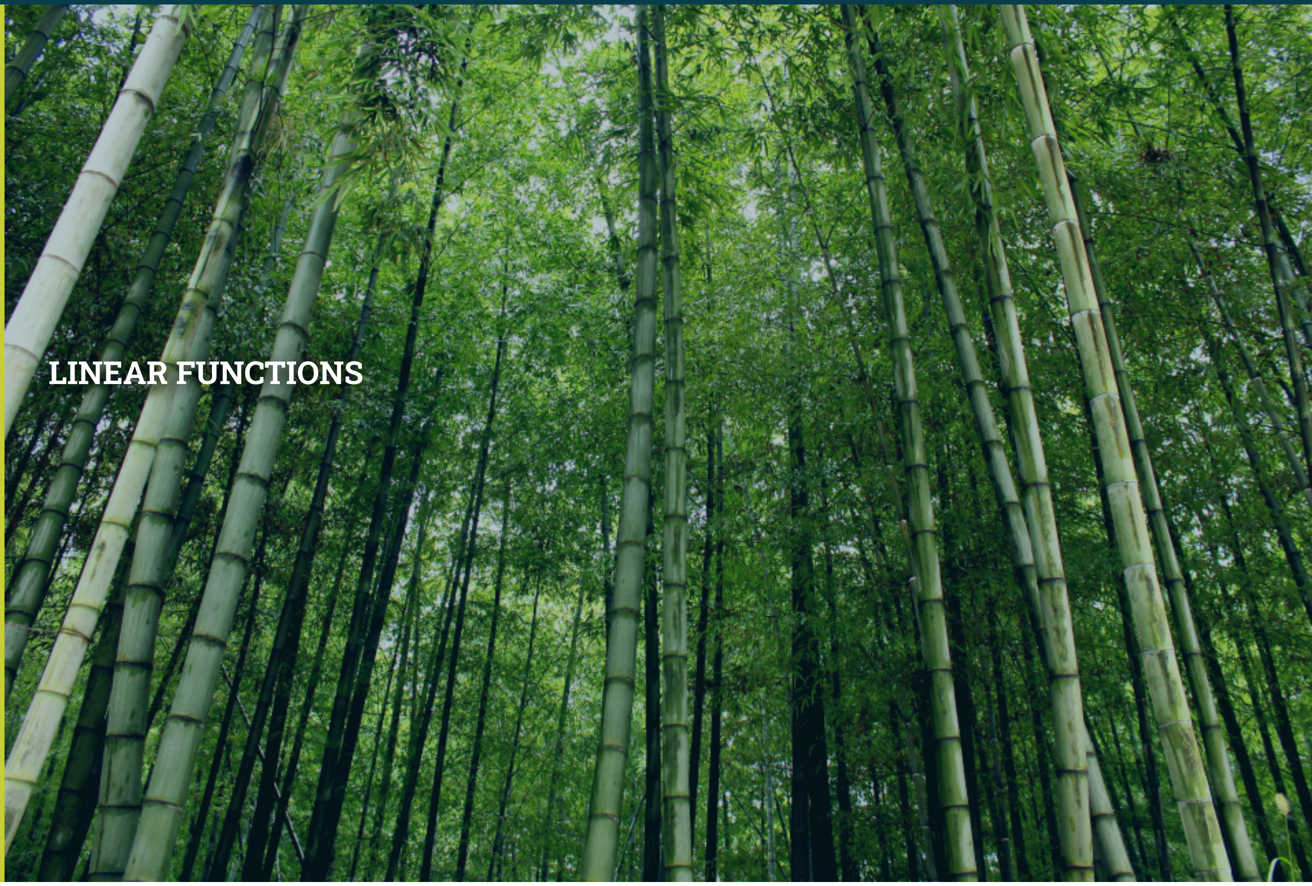
2

LINEAR FUNCTIONS



A bamboo forest in China (credit: “JFXie”/Flickr)

Chapter Outline

Introduction to Linear Functions

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour.[1] In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

1 http://www.guinnessworldrecords.com/records-3000/fastest-growing-plant/

2.1 Linear Functions

Learning Objectives

In this section, you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Calculate and interpret slope.
- Write the point-slope form of an equation.
- Write and interpret a linear function.

Figure 1 Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (Figure 1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.[2]

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train's motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the **slope-intercept form** of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Equation notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The

2 http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm

rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

1 second 1 second 1 second

t	0	1	2	3
$D(t)$	250	333	416	499

83 meters 83 meters 83 meters

Figure 2 Tabular representation of the function D showing selected input and output values

Q&A **Can the input in the previous example be any real number?**

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or ***y*-intercept**, of the line. We can see from the graph in Figure 3 that the *y*-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

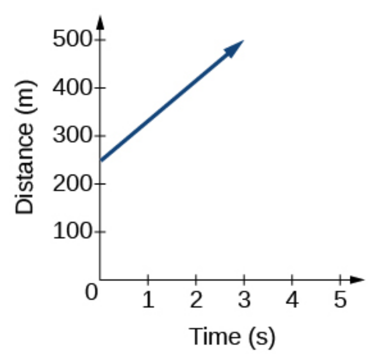


Figure 3 The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x+1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The ***y*-intercept** is at $(0, b)$.

EXAMPLE 1

Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

Figure 4 (credit: Ilse Reijs and Jan-Noud Hutten)

Solution

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in Figure 5**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 5**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 5**(c)**.

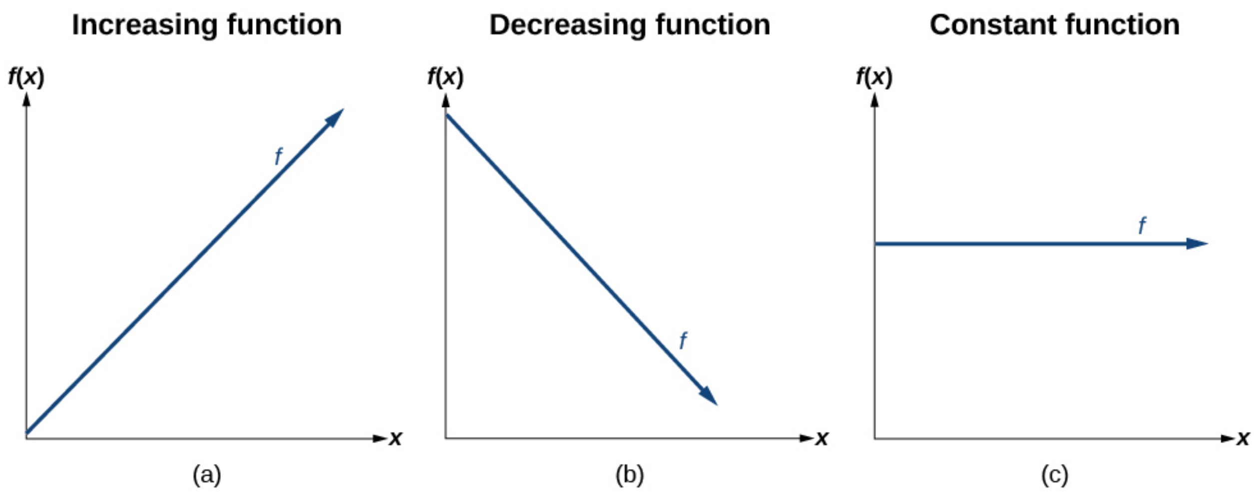


Figure 5

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

EXAMPLE 2

Deciding whether a Function Is Increasing, Decreasing, or Constant

Studies from the early 2010s indicated that teens sent about 60 texts a day, while more recent data indicates much higher messaging rates among all users, particularly considering the various apps with which people can communicate. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

ⓐ The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
ⓑ A person has a limit of 500 texts per month in their data plan. The input is the number of days, and output is the total number of texts remaining for the month.
ⓒ A person has an unlimited number of texts in their data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

ⓐ The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
ⓑ The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
ⓒ The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2 —which can be represented by a set of points, (x_1, y_1) and (x_2, y_2) —we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

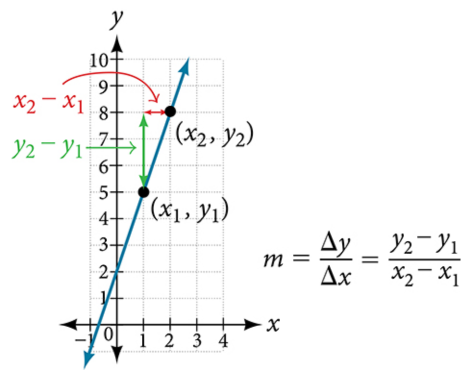


Figure 6 The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Q&A **Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?**

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

HOW TO

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

EXAMPLE 3

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

TRY IT #1 If $f(x)$ is a linear function, and $(2, \ 3)$ and $(0, \ 4)$ are points on the line, find the slope. Is this function increasing or decreasing?

EXAMPLE 4

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

Solution

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4{,}400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4{,}400 \text{ people}}{4 \text{ years}} = 1{,}100 \ \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

TRY IT #2 The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1)$$

The point-slope form is derived from the slope formula.

$$\begin{aligned}
m &= \tfrac{y-y_1}{x-x_1} && \text{assuming } x \neq x_1 \\
m(x - x_1) &= \tfrac{y-y_1}{x-x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1). \\
m(x - x_1) &= y - y_1 && \text{Simplify.} \\
y - y_1 &= m(x - x_1) && \text{Rearrange.}
\end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y - 4 = -\frac{1}{2}(x - 6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y - 4 &= -\tfrac{1}{2}(x - 6) \\ y - 4 &= -\tfrac{1}{2}x + 3 && \text{Distribute the } -\tfrac{1}{2}. \\ y &= -\tfrac{1}{2}x + 7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = -\frac{1}{2}x + 7$.

Point-Slope Form of a Linear Equation

The **point-slope form** of a linear equation takes the form

$$y - y_1 = m(x - x_1)$$

where m is the slope, x_1 and y_1 are the x- and y- coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y - 1 &= 2(x - 4) \\ y - 1 &= 2x - 8 && \text{Distribute the 2.} \\ y &= 2x - 7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y - 1 = 2(x - 4)$ and $y = 2x - 7$, describe the same line. See Figure 7.

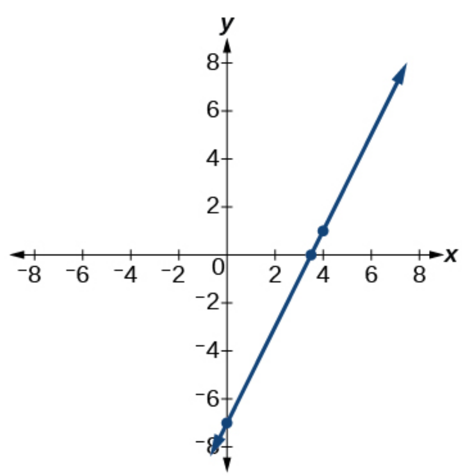


Figure 7

EXAMPLE 5

Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6, -1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = -1$. Now we can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-1) &= 3(x - 6) && \text{Substitute known values.} \\ y + 1 &= 3(x - 6) && \text{Distribute } -1 \text{ to find point-slope form.} \end{aligned}$$

Then we use algebra to find the slope-intercept form.

$$\begin{aligned} y + 1 &= 3(x - 6) \\ y + 1 &= 3x - 18 && \text{Distribute 3.} \\ y &= 3x - 19 && \text{Simplify to slope-intercept form.} \end{aligned}$$

TRY IT #3 Write the point-slope form of an equation of a line with a slope of -2 that passes through the point $(-2,\ 2)$. Then rewrite it in the slope-intercept form.

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0,\ 1)$ and $(3,\ 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-1}{3-0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use (0, 1) for our point.

$$\begin{aligned} y - y_1 &= m\,(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 8.

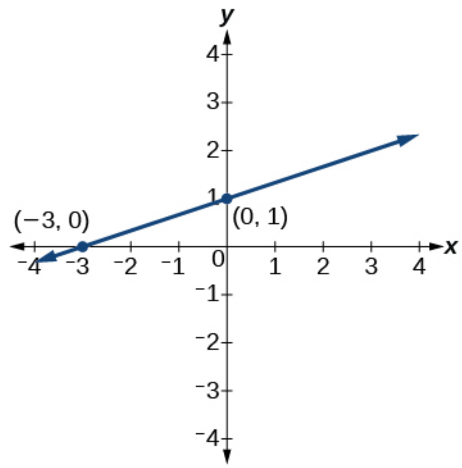


Figure 8

EXAMPLE 6

Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points (5, 1) and (8, 7). Then rewrite it in the slope-intercept form.

Solution

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2 - 1 = 2(x_2 - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

TRY IT #4 Write the point-slope form of an equation of a line that passes through the points $(-1, 3)$ and $(0, 0)$. Then rewrite it in the slope-intercept form.

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 9.

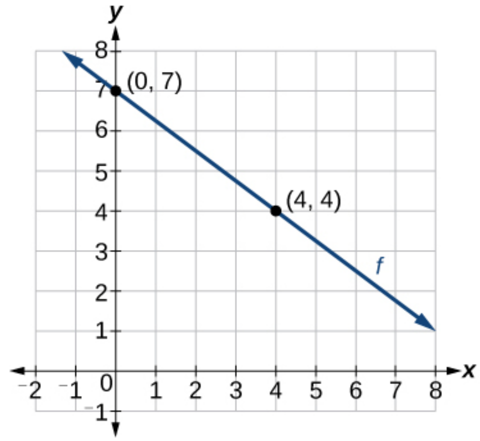


Figure 9

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0,\ 7)$ and $(4,\ 4)$. We can use these points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-7}{4-0} \\ &= -\frac{3}{4} \end{aligned}$$

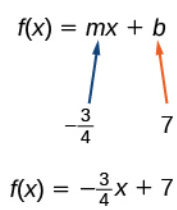
Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

HOW TO

Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

EXAMPLE 7

Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 10.

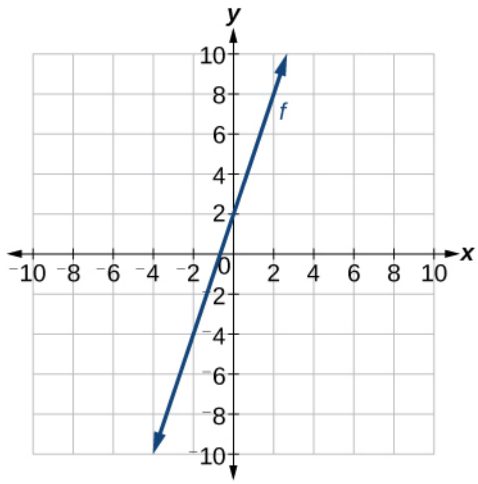


Figure 10

Solution

Identify two points on the line, such as $(0,\ \ 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4-2}{-2-0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m\,(x - x_1) \\ y - (-4) &= 3\,(x - (-2)) \\ y + 4 &= 3\,(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 11 that the line crosses the y-axis at the point $(0,\ \ 2)$, which is the y-intercept, so $b = 2$.

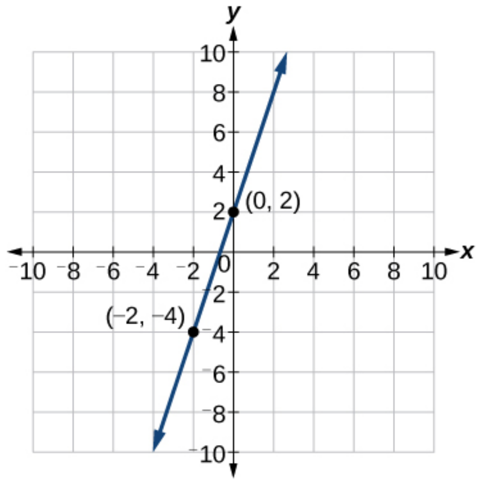


Figure 11

EXAMPLE 8

Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

EXAMPLE 9

Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) = -2 &\to (3, -2) \\ f(8) = 1 &\to (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-2)}{8-3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \tfrac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 2 &= \tfrac{3}{5}(x - 3) \\ y + 2 &= \tfrac{3}{5}x - \tfrac{9}{5} \\ y &= \tfrac{3}{5}x - \tfrac{19}{5} \end{aligned}$$

TRY IT #5 If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

HOW TO

Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

EXAMPLE 10

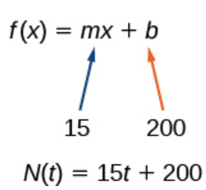
Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.



We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

EXAMPLE 11

Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3,760)$ and $(5,920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \quad \text{when } n = 3, \quad I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

EXAMPLE 12

Using Tabular Form to Write an Equation for a Linear Function

Table 1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

***w*, number of weeks**	0	2	4	6
***P(w)*, number of rats**	1000	1080	1160	1240

Table 1

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240-1080}{6-2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Q&A **Is the initial value always provided in a table of values like Table 1?**

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

TRY IT #6 A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Table 2

MEDIA

Access this online resource for additional instruction and practice with linear functions.

Linear Functions (http://openstax.org/l/linearfunctions)

2.1 SECTION EXERCISES

Verbal

1. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

2. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

3. Jessica is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

4. Sonya is currently 10 miles from home and is walking farther away at 2 miles per hour. Write an equation for her distance from home t hours from now.

5. A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

6. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

7. $y = \frac{1}{4}x + 6$

8. $y = 3x - 5$

9. $y = 3x^2 - 2$

10. $3x + 5y = 15$

11. $3x^2 + 5y = 15$

12. $3x + 5y^2 = 15$

13. $-2x^2 + 3y^2 = 6$

14. $-\frac{x-3}{5} = 2y$

For the following exercises, determine whether each function is increasing or decreasing.

15. $f(x) = 4x + 3$

16. $g(x) = 5x + 6$

17. $a(x) = 5 - 2x$

18. $b(x) = 8 - 3x$

19. $h(x) = -2x + 4$

20. $k(x) = -4x + 1$

21. $j(x) = \frac{1}{2}x - 3$

22. $p(x) = \frac{1}{4}x - 5$

23. $n(x) = -\frac{1}{3}x - 2$

24. $m(x) = -\frac{3}{8}x + 3$

For the following exercises, find the slope of the line that passes through the two given points.

25. $(2,\ 4)$ and $(4,\ 10)$

26. $(1,\ 5)$ and $(4,\ 11)$

27. $(-1, 4)$ and $(5, 2)$

28. $(8, -2)$ and $(4, 6)$

29. $(6,\ 11)$ and $(-4,\ 3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

30. $f(-5) = -4$, and $f(5) = 2$

31. $f(-1) = 4$ and $f(5) = 1$

32. $(2, 4)$ and $(4, 10)$

33. Passes through $(1, 5)$ and $(4, 11)$

34. Passes through $(-1,\ 4)$ and $(5,\ 2)$

35. Passes through $(-2,\ 8)$ and $(4,\ 6)$

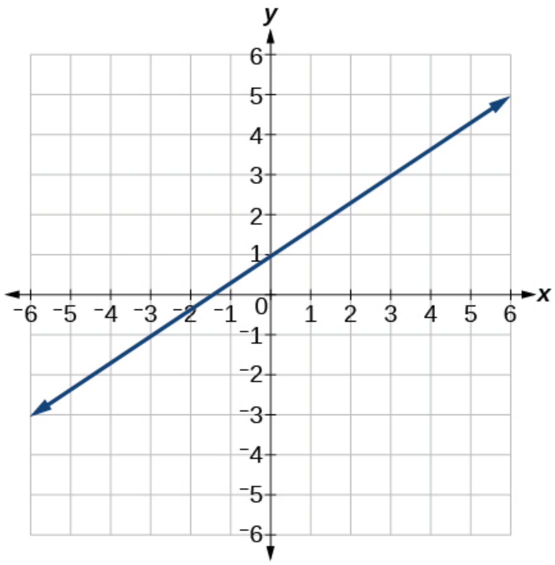
36. x intercept at $(-2,\ 0)$ and y intercept at $(0, -3)$

37. x intercept at $(-5,\ 0)$ and y intercept at $(0,\ 4)$

Graphical

For the following exercises, find the slope of the lines graphed.

38.



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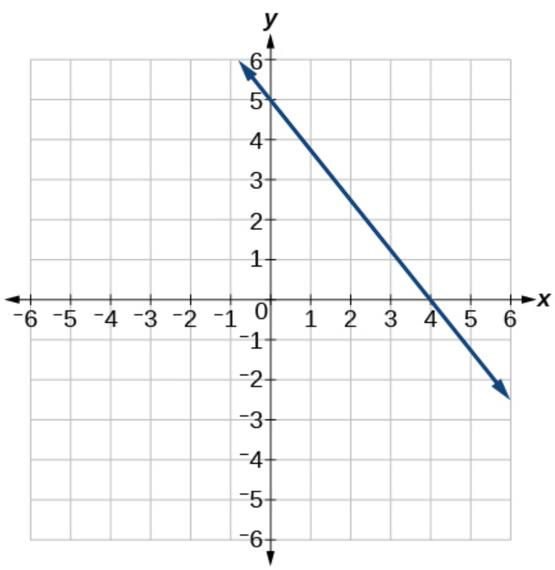

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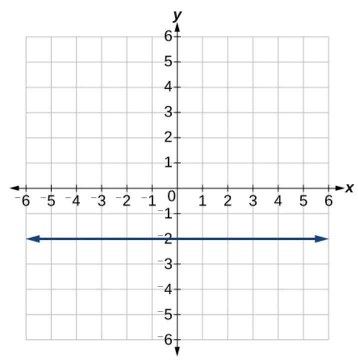
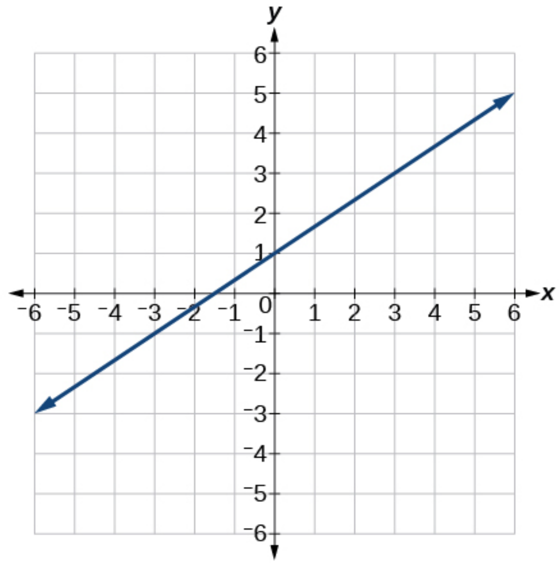


For the following exercises, write an equation for the lines graphed.

41.



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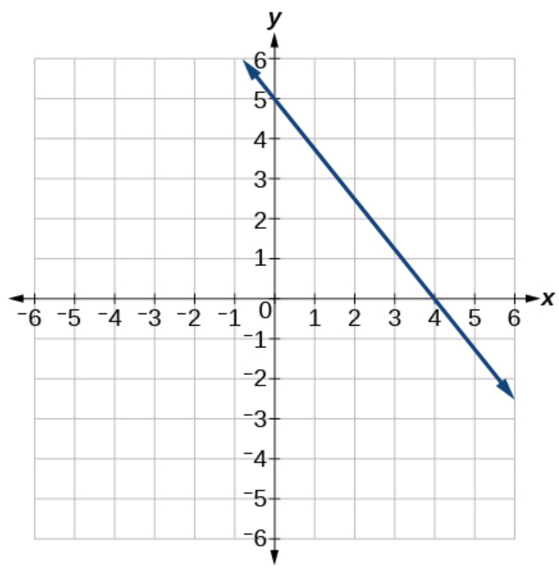

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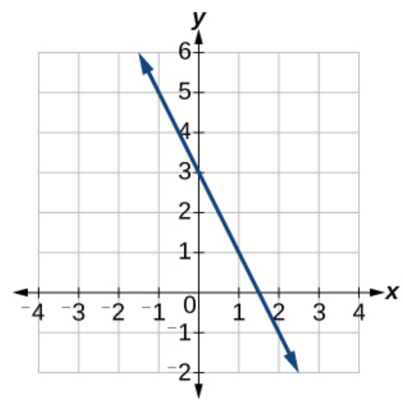

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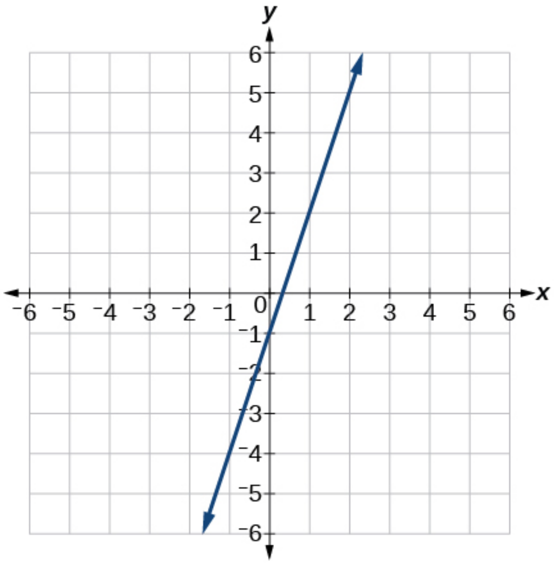

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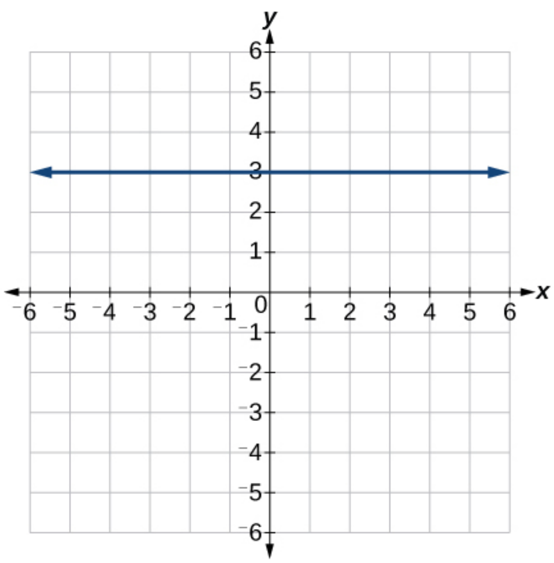

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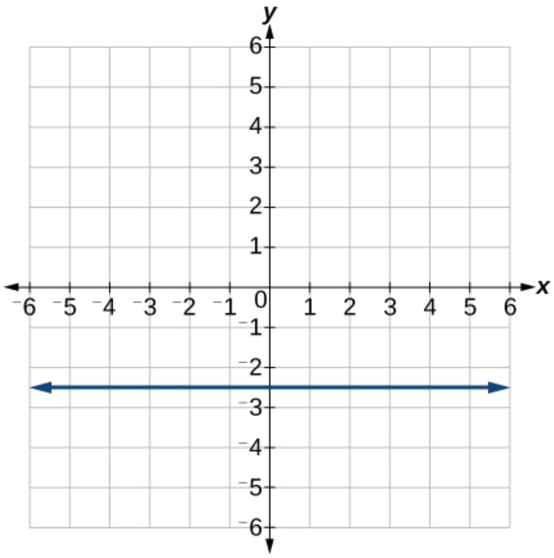


Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

47.

x	0	5	10	15
$g(x)$	5	–10	–25	–40

48.

x	0	5	10	15
$h(x)$	5	30	105	230

49.

x	0	5	10	15
$f(x)$	–5	20	45	70

50.

x	5	10	20	25
$k(x)$	13	28	58	73

51.

x	0	2	4	6
$g(x)$	6	–19	–44	–69

52.

x	2	4	6	8
$f(x)$	–4	16	36	56

53.

x	2	4	6	8
$f(x)$	–4	16	36	56

54.

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

55. If f is a linear function, $f(0.1) = 11.5, \quad \text{and} \quad f(0.4) =\!-5.9,$ find an equation for the function.

56. Graph the function f on a domain of $[-10, \quad 10] : \quad f(x) = 0.02x - 0.01.$ Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be $10.$

57. Graph the function f on a domain of $[-10, \quad 10] : fx) = 2,500x + 4,000$

58. Table 3 shows the input, w, and output, k, for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k, round to 3 decimal places.

w	–10	5.5	67.5	b
k	30	–26	a	–44

Table 3

59. Table 4 shows the input, p, and output, q, for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

p	0.5	0.8	12	b
q	400	700	a	1,000,000

Table 4

60. Graph the linear function f on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and *y*-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

61. Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and *y*-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

62. Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

a. $a = 2; \quad b = 3$
b. $a = 2; \quad b = 4$
c. $a = 2; \quad b =\!-4$
d. $a = 2; \quad b =\!-5$

Extensions

63. Find the value of x if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6), \quad m = 3$

64. Find the value of y if a linear function goes through the following points and has the following slope: $(10, y), (25, 100), \quad m = -5$

65. Find the equation of the line that passes through the following points: $(a, \quad b)$ and $(a, \quad b + 1)$

66. Find the equation of the line that passes through the following points: $(2a, \quad b)$ and $(a, \quad b + 1)$

67. Find the equation of the line that passes through the following points: $(a, \quad 0)$ and $(c, \quad d)$

Real-World Applications

68. At noon, a barista notices that they have \$20 in their tip jar. If the barista makes an average of \$0.50 from each customer, how much will they have in the tip jar if they serve n more customers during the shift?

69. A gym membership with two personal training sessions costs \$125, while gym membership with five personal training sessions costs \$260. What is cost per session?

70. A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$, while 3,000 shirts can be sold at a price of \$22. Find a linear equation in the form $p(n) = mn + b$ that gives the price p they can charge for n shirts.

71. A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

72. A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationship in the form $y = mn + b$ that gives the yield when n stalks are planted.

73. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

74. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

75. Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23,286$, where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

a. As of 1990, average annual income was $23,286.
b. In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
c. Each year in the decade of the 1990s, average annual income increased by $1,054.
d. Average annual income rose to a level of $23,286 by the end of 1999.

76. When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C, the Celsius temperature, $F(C)$.

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
b. Find and interpret $F(28)$.
c. Find and interpret $F(-40)$.

2.2 Graphs of Linear Functions

Learning Objectives

In this section, you will:

- Graph linear functions.
- Write the equation for a linear function from the graph of a line.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.
- Solve a system of linear equations.

Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

Graphing Linear Functions

In Linear Functions, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics.

There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third is by using transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

HOW TO

Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

EXAMPLE 1

Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$\begin{array}{ll} x = 0 & f(0) = -\frac{2}{3}(0) + 5 = 5 \Rightarrow (0, 5) \\ x = 3 & f(3) = -\frac{2}{3}(3) + 5 = 3 \Rightarrow (3, 3) \\ x = 6 & f(6) = -\frac{2}{3}(6) + 5 = 1 \Rightarrow (6, 1) \end{array}$$

Plot the coordinate pairs and draw a line through the points. Figure 1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

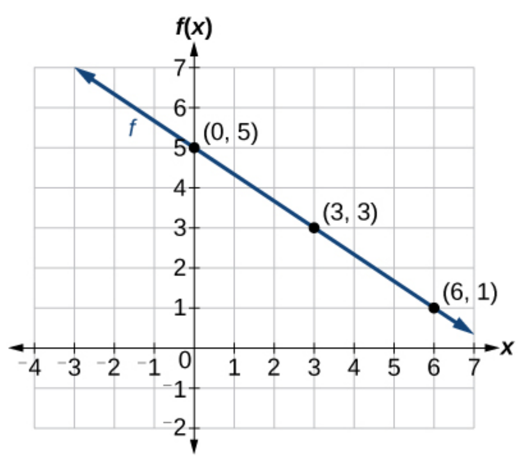


Figure 1 The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

TRY IT #1 Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Graphing a Function Using *y*-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope m, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the *y*-intercept and the slope in Linear Functions.

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x = 0$. The graph crosses the *y*-axis at $(0, 1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0, 1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 2.

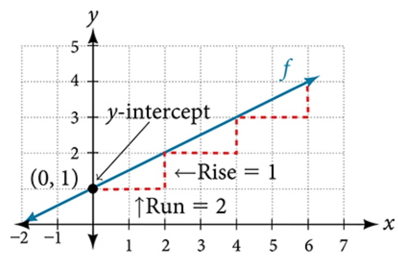


Figure 2

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Q&A **Do all linear functions have *y*-intercepts?**

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: *A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.*)

HOW TO

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

EXAMPLE 2

Graphing by Using the *y*-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

Solution

Evaluate the function at $x = 0$ to find the *y*-intercept. The output value when $x = 0$ is 5, so the graph will cross the *y*-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of -2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the *y*-intercept on the graph in Figure 3. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.

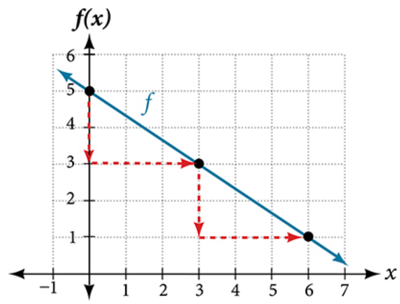


Figure 3

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

TRY IT #2 Find a point on the graph we drew in Example 2 that has a negative *x*-value.

Graphing a Function Using Transformations

Another option for graphing is to use transformations of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x) = mx$, the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure 4 that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of m, the steeper the slope.

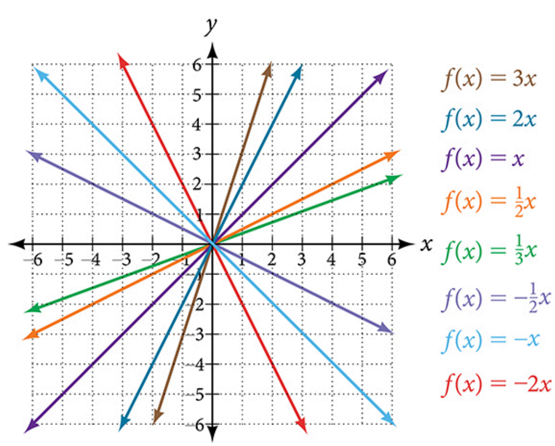


Figure 4 Vertical stretches and compressions and reflections on the function $f(x) = x$.

Vertical Shift

In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line.

Notice in Figure 5 that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

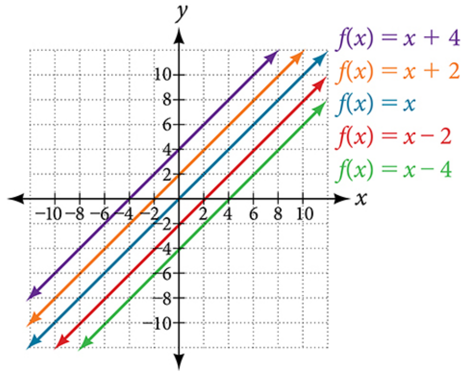


Figure 5 This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

HOW TO

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

EXAMPLE 3

Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Solution

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 6.

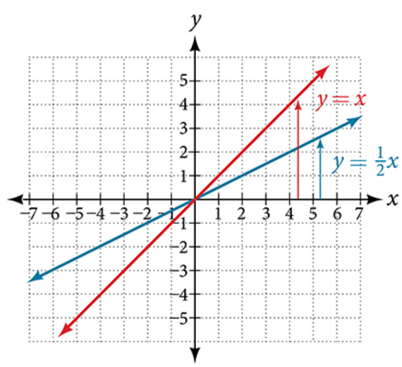


Figure 6 The function, $y = x$, compressed by a factor of $\frac{1}{2}$.

Then show the vertical shift as in Figure 7.

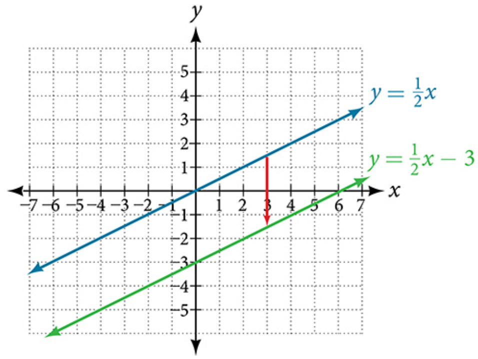


Figure 7 The function $y = \frac{1}{2}x$, shifted down 3 units.

TRY IT #3 Graph $f(x) = 4 + 2x$, using transformations.

Q&A **In Example 3, could we have sketched the graph by reversing the order of the transformations?**

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

$$\begin{aligned} f(2) &= \frac{1}{2}(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Writing the Equation for a Function from the Graph of a Line

Recall that in Linear Functions, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 8. We can see right away that the graph crosses the *y*-axis at the point $(0,\ 4)$ so this is the *y*-intercept.

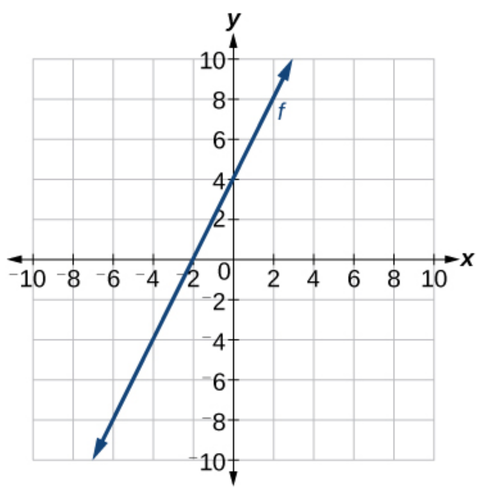


Figure 8

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point $(-2, 0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

$$y = 2x + 4$$

HOW TO

Given a graph of linear function, find the equation to describe the function.

1. Identify the y-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the y-intercept and slope into the slope-intercept form of a line.

EXAMPLE 4

Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 9.

ⓐ $f(x) = 2x + 3$ ⓑ $g(x) = 2x - 3$ ⓒ $h(x) = -2x + 3$ ⓓ $j(x) = \frac{1}{2}x + 3$

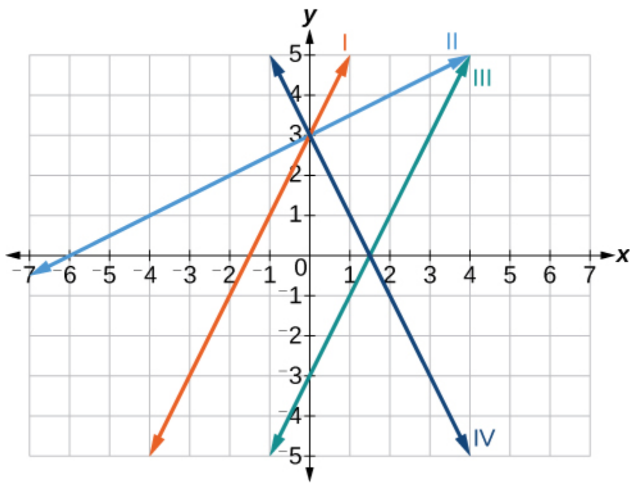


Figure 9

Solution

Analyze the information for each function.

ⓐ This function has a slope of 2 and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0,\ 3)$ so f must be represented by Line I.

ⓑ This function also has a slope of 2, but a y-intercept of -3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by Line III.

ⓒ This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.

ⓓ This function has a slope of $\frac{1}{2}$ and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0,\ 3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 10.

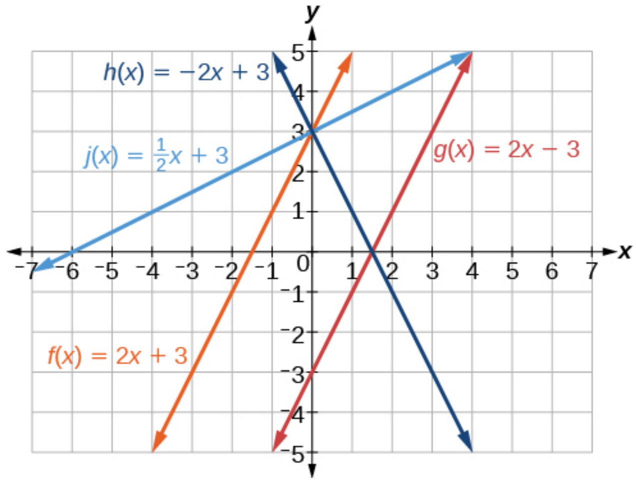


Figure 10

Finding the *x*-intercept of a Line

So far, we have been finding the *y*-intercepts of a function: the point at which the graph of the function crosses the *y*-axis. A function may also have an ***x*-intercept,** which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$$f(x) = 3x - 6$$

Set the function equal to 0 and solve for x.

$$\begin{aligned} 0 &= 3x - 6 \\ 6 &= 3x \\ 2 &= x \\ x &= 2 \end{aligned}$$

The graph of the function crosses the *x*-axis at the point $(2,\ 0)$.

Q&A **Do all linear functions have *x*-intercepts?**

No. However, linear functions of the form $y = c$, where c is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y = 5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in Figure 11.

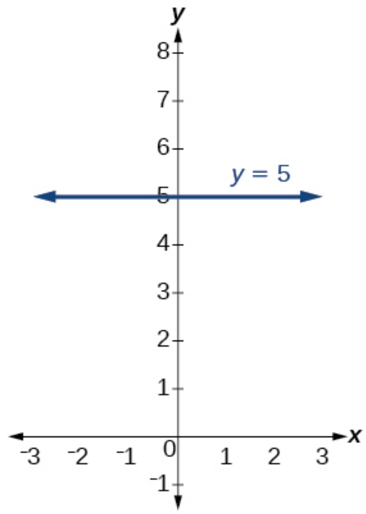


Figure 11

x-intercept

The ***x*-intercept** of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

EXAMPLE 5

Finding an *x*-intercept

Find the x-intercept of $f(x) = \frac{1}{2}x - 3$.

Solution

Set the function equal to zero to solve for x.

$$\begin{aligned} 0 &= \tfrac{1}{2}x - 3 \\ 3 &= \tfrac{1}{2}x \\ 6 &= x \\ x &= 6 \end{aligned}$$

The graph crosses the x-axis at the point $(6,\ 0)$.

Analysis

A graph of the function is shown in Figure 12. We can see that the x-intercept is $(6,\ 0)$ as we expected.

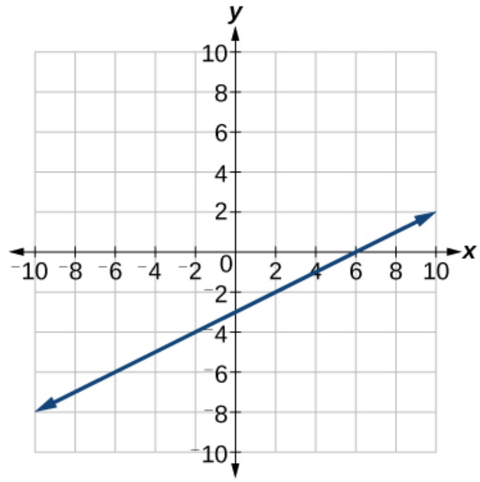
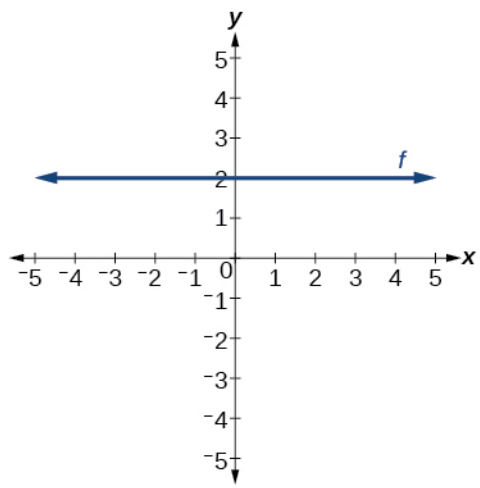


Figure 12 The graph of the linear function $f(x) = \frac{1}{2}x - 3$.

TRY IT #4 Find the x-intercept of $f(x) = \frac{1}{4}x - 4$.

Describing Horizontal and Vertical Lines

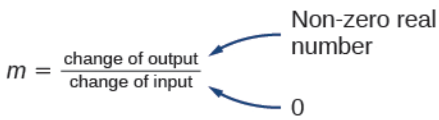
There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or y-value. In Figure 13, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



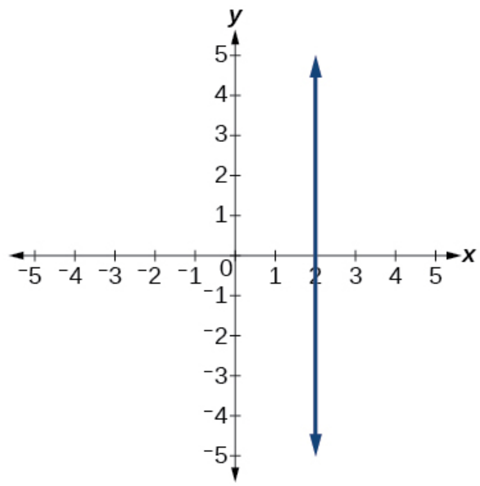
x	−4	−2	0	2	4
y	2	2	2	2	2

Figure 13 A horizontal line representing the function $f(x) = 2$.

A **vertical line** indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.



Notice that a vertical line, such as the one in Figure 14, has an *x*-intercept, but no *y*-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

Figure 14 The vertical line, $x = 2$, which does not represent a function.

Horizontal and Vertical Lines

Lines can be horizontal or vertical.

A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

A **vertical line** is a line defined by an equation in the form $x = a$.

EXAMPLE 6

Writing the Equation of a Horizontal Line
Write the equation of the line graphed in Figure 15.

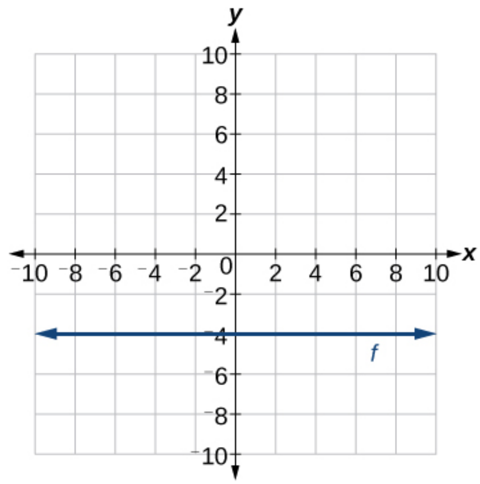


Figure 15

Solution

For any *x*-value, the *y*-value is -4, so the equation is $y = -4$.

EXAMPLE 7

Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 16.

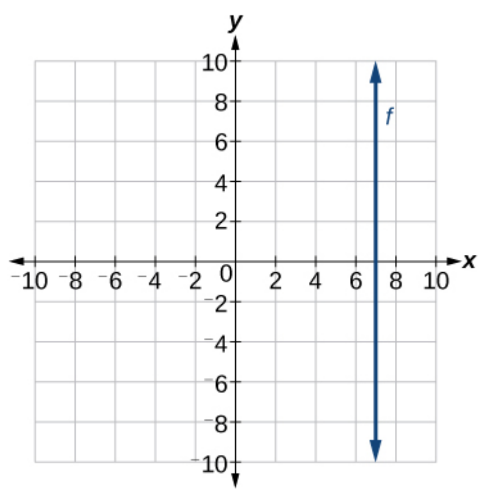


Figure 16

Solution

The constant *x*-value is 7, so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 17 are **parallel lines**: they will never intersect. Notice that they have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the *y*-intercept. If we shifted one line vertically toward the *y*-intercept of the other, they would become the same line.

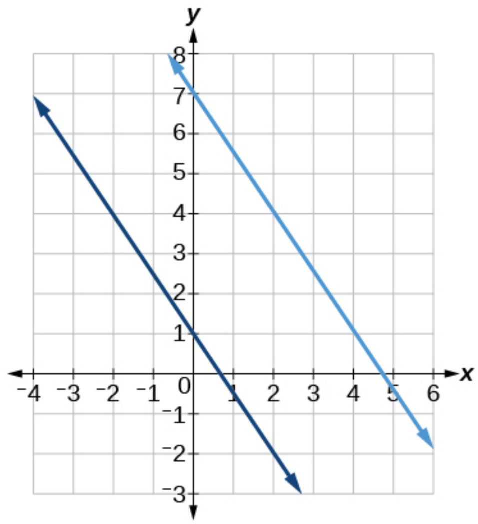


Figure 17 Parallel lines.

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the *y*-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{array}{l} f(x) = -2x + 6 \\ f(x) = -2x - 4 \end{array}\right\} \text{parallel} \qquad \left.\begin{array}{l} f(x) = 3x + 2 \\ f(x) = 2x + 2 \end{array}\right\} \text{not parallel}$$

Unlike parallel lines, **perpendicular lines** do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in Figure 18 are perpendicular.

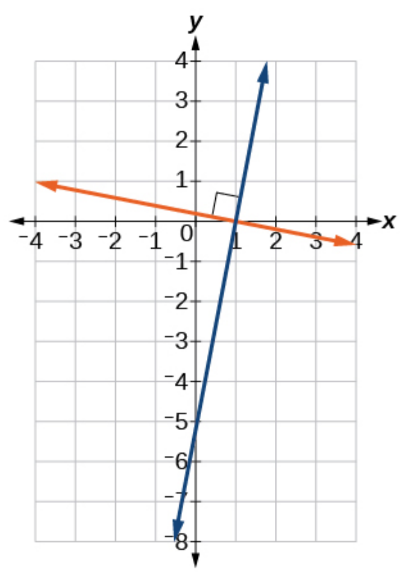


Figure 18 Perpendicular lines.

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield –1.

$$m_1 m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$$f(x) = \frac{1}{4}x + 2 \quad \text{negative reciprocal of} \frac{1}{4} \text{ is } -4$$
$$f(x) = -4x + 3 \quad \text{negative reciprocal of } -4 \text{ is } \frac{1}{4}$$

The product of the slopes is –1.

$$-4\left(\frac{1}{4}\right) = -1$$

Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$$f(x) = m_1 x + b_1 \text{ and } g(x) = m_2 x + b_2 \text{ are parallel if } m_1 = m_2.$$

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect at right angles.

$$f(x) = m_1 x + b_1 \text{ and } g(x) = m_2 x + b_2 \text{ are perpendicular if } m_1 m_2 = -1, \text{ and so } m_2 = -\frac{1}{m_1}.$$

EXAMPLE 8

Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$f(x) = 2x + 3 \qquad h(x) = -2x + 2$$
$$g(x) = \frac{1}{2}x - 4 \qquad j(x) = 2x - 6$$

Solution

Parallel lines have the same slope. Because the functions $f(x) = 2x + 3$ and $j(x) = 2x - 6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because −2 and $\frac{1}{2}$ are negative reciprocals, the equations, $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in Figure 19.

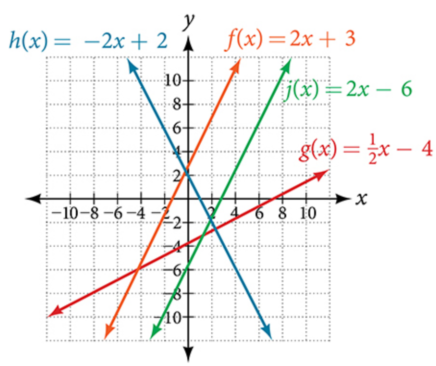


Figure 19

The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x–6$ are parallel, and the lines $g(x) = \frac{1}{2}x–4$ and $h(x) = -2x + 2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$\begin{aligned} g(x) &= 3x + 6 \\ h(x) &= 3x + 1 \\ p(x) &= 3x + \tfrac{2}{3} \end{aligned}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1,\ 7)$. We already know that the slope is 3. We just need to determine which value for b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= 3(x - 1) \\ y - 7 &= 3x - 3 \\ y &= 3x + 4 \end{aligned}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1,\ 7)$.

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

EXAMPLE 9

Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3,\ 0)$.

Solution

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the y-intercept.

$$\begin{aligned} g(x) &= 3x + b \\ 0 &= 3(3) + b \\ b &= -9 \end{aligned}$$

The line parallel to $f(x)$ that passes through $(3,\ 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. Figure 20 shows that the two lines will never intersect.

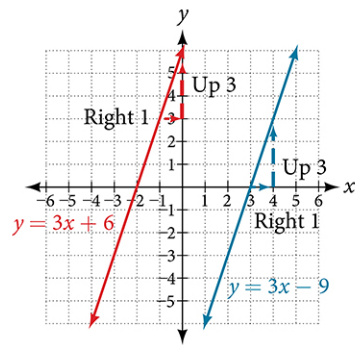


Figure 20

Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$\begin{aligned} g(x) &= -\tfrac{1}{2}x + 4 \\ h(x) &= -\tfrac{1}{2}x + 2 \\ p(x) &= -\tfrac{1}{2}x - \tfrac{1}{2} \end{aligned}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4,\ 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the *y*-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

$$\begin{aligned} g(x) &= mx + b \\ 0 &= -\tfrac{1}{2}(4) + b \\ 0 &= -2 + b \\ 2 &= b \\ b &= 2 \end{aligned}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a *y*-intercept of 2 is

$$g(x) = -\frac{1}{2}x + 2.$$

So $g(x) = -\frac{1}{2}x + 2$ is perpendicular to $f(x) = 2x + 4$ and passes through the point $(4,\ 0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

Q&A **A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?**

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line

perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x) = mx + b$.
4. Solve for b.
5. Write the equation for the line.

EXAMPLE 10

Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x) = 3x + 3$ that passes through the point $(3,\ 0)$.

Solution

The original line has slope $m = 3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation for the line.

$$\begin{aligned} g(x) &= -\tfrac{1}{3}x + b \\ 0 &= -\tfrac{1}{3}(3) + b \\ 1 &= b \\ b &= 1 \end{aligned}$$

The line perpendicular to $f(x)$ that passes through $(3,\ 0)$ is $g(x) = -\frac{1}{3}x + 1$.

Analysis

A graph of the two lines is shown in Figure 21 below.

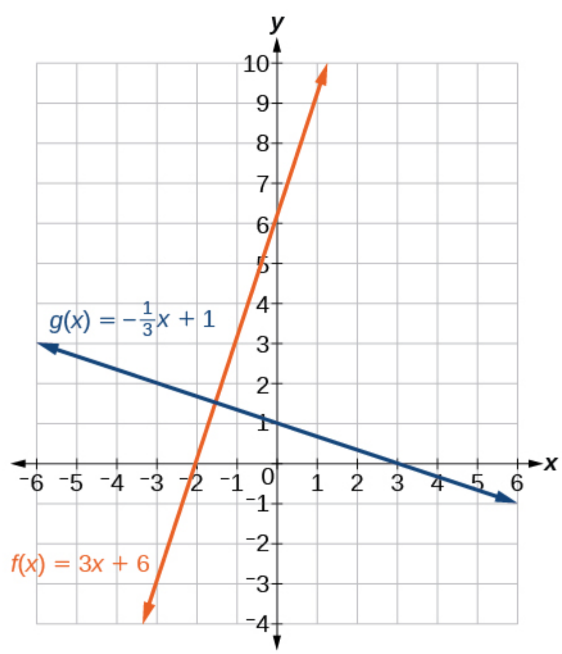


Figure 21

TRY IT #5 Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0, 0)$ that is

ⓐ parallel to $h(x)$ ⓑ perpendicular to $h(x)$

HOW TO

Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

EXAMPLE 11

Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2,\ 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

Solution

From the two points of the given line, we can calculate the slope of that line.

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the y-intercept of the line passing through the point $(4, 5)$.

$$\begin{aligned} g(x) &= 6x + b \\ 5 &= 6(4) + b \\ 5 &= 24 + b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4, 5)$ is

$$y = 6x - 19$$

TRY IT #6 A line passes through the points, $(-2, -15)$ and $(2,-3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.

Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.

EXAMPLE 12

Finding a Point of Intersection Algebraically

Find the point of intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$.

Solution

Set $h(t) = j(t)$.

$$\begin{aligned} 3t - 4 &= 5 - t \\ 4t &= 9 \\ t &= \frac{9}{4} \end{aligned}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input.

$$\begin{aligned} j\left(\frac{9}{4}\right) &= 5 - \frac{9}{4} \\ &= \frac{11}{4} \end{aligned}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$.

Analysis

Looking at Figure 22, this result seems reasonable.

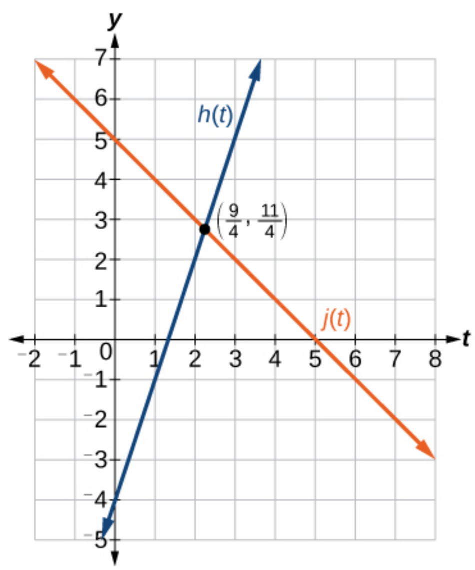


Figure 22

Q&A **If we were asked to find the point of intersection of two distinct parallel lines, should something in the solution process alert us to the fact that there are no solutions?**

Yes. After setting the two equations equal to one another, the result would be the contradiction "0 = non-zero real number".

TRY IT #7 Look at the graph in Figure 22 and identify the following for the function $j(t)$:

ⓐ *y*-intercept ⓑ *x*-intercept(s) ⓒ slope

ⓓ Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)?

ⓔ Is $j(t)$ an increasing or decreasing function (or neither)?

ⓕ Write a transformation description for $j(t)$ from the identity toolkit function $f(x) = x$.

EXAMPLE 13

Finding a Break-Even Point

A company sells sports helmets. The company incurs a one-time fixed cost for $250,000. Each helmet costs $120 to produce, and sells for $140.

ⓐ Find the cost function, C, to produce x helmets, in dollars.

ⓑ Find the revenue function, R, from the sales of x helmets, in dollars.

ⓒ Find the break-even point, the point of intersection of the two graphs C and R.

Solution

ⓐ The cost function in the sum of the fixed cost, $125,000, and the variable cost, $120 per helmet.

$$C(x) = 120x + 250,000$$

ⓑ The revenue function is the total revenue from the sale of x helmets, $R(x) = 140x$.

ⓒ The break-even point is the point of intersection of the graph of the cost and revenue functions. To find the x-coordinate of the coordinate pair of the point of intersection, set the two equations equal, and solve for x.

$$\begin{aligned} C(x) &= R(x) \\ 250,000 + 120x &= 140x \\ 250,000 &= 20x \\ 12,500 &= x \\ x &= 12,500 \end{aligned}$$

To find y, evaluate either the revenue or the cost function at 12,500.

$$\begin{aligned} R(x) &= 140(12,500) \\ &= \$1,750,000 \end{aligned}$$

The break-even point is $(12{,}500, 1{,}750{,}000)$.

Analysis

This means if the company sells 12,500 helmets, they break even; both the sales and cost incurred equaled 1.75 million dollars. See Figure 23

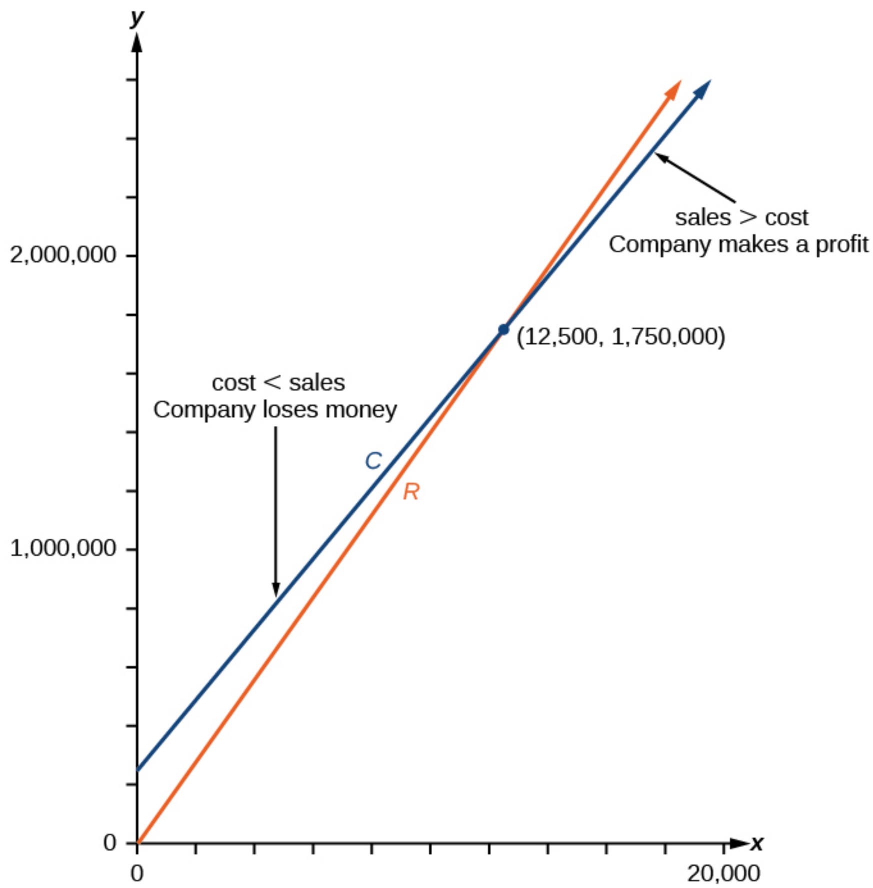


Figure 23

MEDIA

Access these online resources for additional instruction and practice with graphs of linear functions.

[Finding Input of Function from the Output and Graph (http://openstax.org/l/findinginput)](http://openstax.org/l/findinginput)
[Graphing Functions using Tables (http://openstax.org/l/graphwithtable)](http://openstax.org/l/graphwithtable)

2.2 SECTION EXERCISES

Verbal

1. If the graphs of two linear functions are parallel, describe the relationship between the slopes and the y-intercepts.

2. If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the y-intercepts.

3. If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

4. Explain how to find a line parallel to a linear function that passes through a given point.

5. Explain how to find a line perpendicular to a linear function that passes through a given point.

Algebraic

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

6. $4x - 7y = 10$
$7x + 4y = 1$

7. $3y + x = 12$
$-y = 8x + 1$

8. $3y + 4x = 12$
$-6y = 8x + 1$

9. $6x - 9y = 10$
$3x + 2y = 1$

10. $y = \frac{2}{3}x + 1$
$3x + 2y = 1$

11. $y = \frac{3}{4}x + 1$
$-3x + 4y = 1$

For the following exercises, find the x- and y-intercepts of each equation

12. $f(x) = -x + 2$

13. $g(x) = 2x + 4$

14. $h(x) = 3x - 5$

15. $k(x) = -5x + 1$

16. $-2x + 5y = 20$

17. $7x + 2y = 56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

18. Line 1: Passes through $(0, 6)$ and $(3, -24)$
Line 2: Passes through $(-1, 19)$ and $(8, -71)$

19. Line 1: Passes through $(-8, -55)$ and $(10, 89)$
Line 2: Passes through $(9, -44)$ and $(4, -14)$

20. Line 1: Passes through $(2, 3)$ and $(4, -1)$
Line 2: Passes through $(6, 3)$ and $(8, 5)$

21. Line 1: Passes through $(1, 7)$ and $(5, 5)$
Line 2: Passes through $(-1, -3)$ and $(1, 1)$

22. Line 1: Passes through $(0, 5)$ and $(3, 3)$
Line 2: Passes through $(1, -5)$ and $(3, -2)$

23. Line 1: Passes through $(2, 5)$ and $(5, -1)$
Line 2: Passes through $(-3, 7)$ and $(3, -5)$

24. Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point $(2, -12)$.

25. Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

26. Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point (-4, -1).

27. Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

28. Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$.

29. Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$.

30. Use algebra to find the point at which the line $f(x) = -\frac{4}{5}x + \frac{274}{25}$ intersects the line $h(x) = \frac{9}{4}x + \frac{73}{10}$.

31. Use algebra to find the point at which the line $f(x) = \frac{7}{4}x + \frac{457}{60}$ intersects the line $g(x) = \frac{4}{3}x + \frac{31}{5}$.

Graphical

For the following exercises, match the given linear equation with its graph in Figure 24.

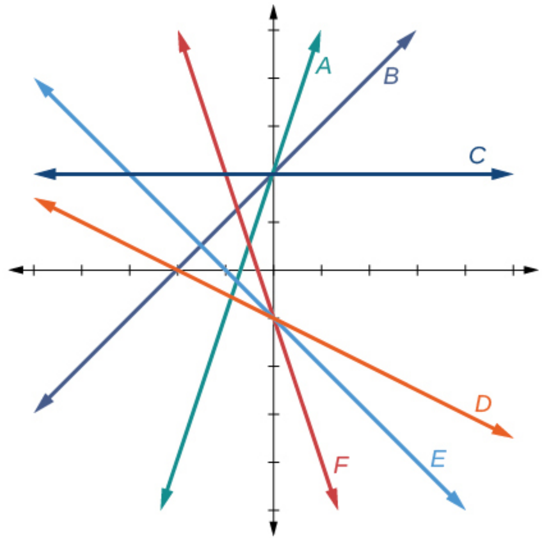


Figure 24

32. $f(x) = -x - 1$

33. $f(x) = -2x - 1$

34. $f(x) = -\frac{1}{2}x - 1$

35. $f(x) = 2$

36. $f(x) = 2 + x$

37. $f(x) = 3x + 2$

For the following exercises, sketch a line with the given features.

38. An x-intercept of $(-4,\ 0)$ and y-intercept of $(0,\ -2)$

39. An x-intercept of $(-2,\ 0)$ and y-intercept of $(0,\ 4)$

40. A y-intercept of $(0,\ 7)$ and slope $-\frac{3}{2}$

41. A y-intercept of $(0,\ 3)$ and slope $\frac{2}{5}$

42. Passing through the points $(-6,\ -2)$ and $(6,\ -6)$

43. Passing through the points $(-3,\ -4)$ and $(3,\ 0)$

For the following exercises, sketch the graph of each equation.

44. $f(x) = -2x - 1$

45. $g(x) = -3x + 2$

46. $h(x) = \frac{1}{3}x + 2$

47. $k(x) = \frac{2}{3}x - 3$

48. $f(t) = 3 + 2t$

49. $p(t) = -2 + 3t$

50. $x = 3$

51. $x = -2$

52. $r(x) = 4$

53. $q(x) = 3$

54. $4x = -9y + 36$

55. $\frac{x}{3} - \frac{y}{4} = 1$

56. $3x - 5y = 15$

57. $3x = 15$

58. $3y = 12$

59. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{3}{4}$, a shift right by 2, and a shift down by 4

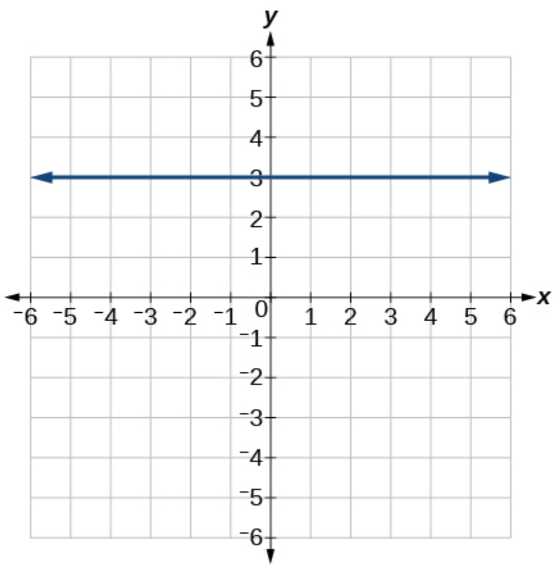
ⓐ Write an equation for $g(x)$.
ⓑ What is the slope of this line?
ⓒ Find the y-intercept of this line.

60. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{1}{3}$, a shift left by 1, and a shift up by 3

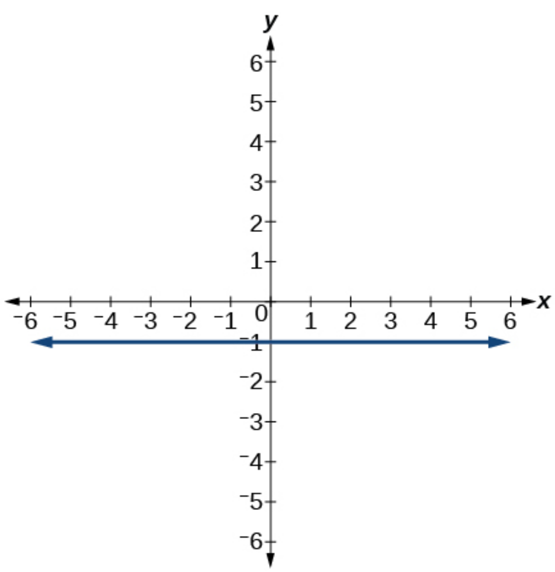
ⓐ Write an equation for $g(x)$.
ⓑ What is the slope of this line?
ⓒ Find the y-intercept of this line.

For the following exercises,, write the equation of the line shown in the graph.

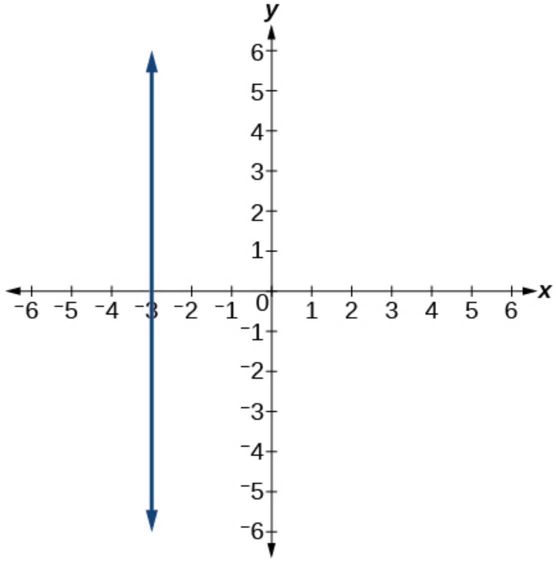
61.



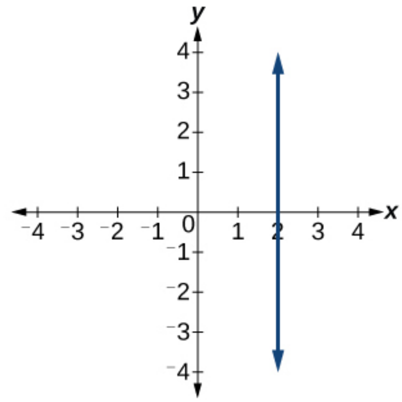
62.



63.



64.



For the following exercises, find the point of intersection of each pair of lines if it exists. If it does not exist, indicate that there is no point of intersection.

65. $$\begin{aligned} y &= \tfrac{3}{4}x + 1 \\ -3x + 4y &= 12 \end{aligned}$$

66. $$\begin{aligned} 2x - 3y &= 12 \\ 5y + x &= 30 \end{aligned}$$

67. $$\begin{aligned} 2x &= y - 3 \\ y + 4x &= 15 \end{aligned}$$

68. $$\begin{aligned} x - 2y + 2 &= 3 \\ x - y &= 3 \end{aligned}$$

69. $$\begin{aligned} 5x + 3y &= -65 \\ x - y &= -5 \end{aligned}$$

Extensions

70. Find the equation of the line parallel to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

71. Find the equation of the line perpendicular to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

For the following exercises, use the functions $f(x) = -0.1x+200$ and $g(x) = 20x + 0.1$.

72. Find the point of intersection of the lines f and g.

73. Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

Real-World Applications

74. A car rental company offers two plans for renting a car.

Plan A: $30 per day and $0.18 per mile
Plan B: $50 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

75. A cell phone company offers two plans for minutes.

Plan A: $20 per month and $1 for every one hundred texts.
Plan B: $50 per month with free unlimited texts.

How many texts would you need to send per month for plan B to save you money?

76. A cell phone company offers two plans for minutes.

Plan A: $15 per month and $2 for every 300 texts.
Plan B: $25 per month and $0.50 for every 100 texts.

How many texts would you need to send per month for plan B to save you money?

2.3 Modeling with Linear Functions

Learning Objectives

In this section, you will:

- Identify steps for modeling and solving.
- Build linear models from verbal descriptions.
- Build systems of linear models.

Figure 1 (credit: EEK Photography/Flickr)

Elan is a college student who plans to spend a summer in Seattle. Elan has saved $3,500 for the trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent the situation? What would be the *x*-intercept, and what can Elan learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

Identifying Steps to Model and Solve Problems

When modeling scenarios with linear functions and solving problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let’s briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let’s take a look at the student in Seattle. In Elan's situation, there are two changing quantities: time and money. The amount of money they have remaining while on vacation depends on how long they stay. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: They saved \$3,500, so \$3,500 is the initial value for M.
- Rate of Change: They anticipate spending \$400 each week, so –\$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because they are spending money each week.

The rate of change is constant, so we can start with the linear model $M\left(t\right) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = \underset{\uparrow \atop -400}{m}t + \underset{\uparrow \atop 3500}{b}$$

$$M(t) = -400t + 3500$$

To find the x- intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \tfrac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x- intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Elan will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn’t make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before Elan saved \$3,500, but the scenario discussed poses the question once they saved \$3,500 because this is when the trip and subsequent spending starts. It is also likely that this model is not valid after the x- intercept, unless Elan will use a credit card and go into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y- intercept, which is the constant or initial value. Once the y- intercept is known, the x- intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is \$1,000. She plans to pay \$250 per month until her balance is \$0. The y- intercept is the initial amount of her debt, or \$1,000. The rate of change, or slope, is -\$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x- intercept.

$$\begin{aligned} 0 &= -250x + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x- intercept is the number of months it takes her to reach a balance of \$0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

HOW TO

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x- value.
6. Use the model to identify an x- value that results in a given y- value.
7. Answer the question posed.

EXAMPLE 1

Using a Linear Model to Investigate a Town’s Population

A town’s population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

ⓐ Predict the population in 2013. ⓑ Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y- intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town’s population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the *y*-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \tfrac{8100-6200}{5-0} \\ &= \tfrac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the *y*-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6,200 \\ &= 9,620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

TRY IT #1 A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

ⓐ Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
ⓑ Find and interpret the *y*-intercept.

TRY IT #2 A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

ⓐ Predict the population in 2014.
ⓑ Identify the year in which the population will reach 54,000.

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

EXAMPLE 2

Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

- Input: t, time in hours.
- Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 2.

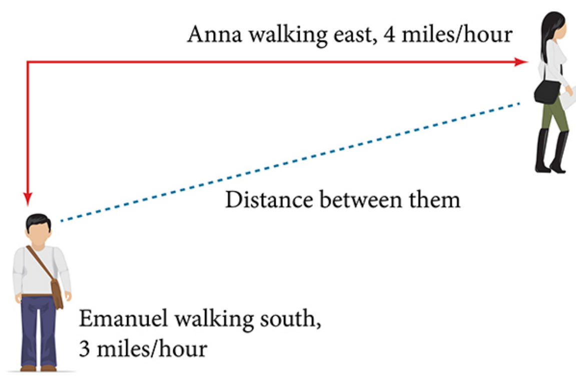


Figure 2

Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.

Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t$$
$$E(t) = 3t$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 3.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

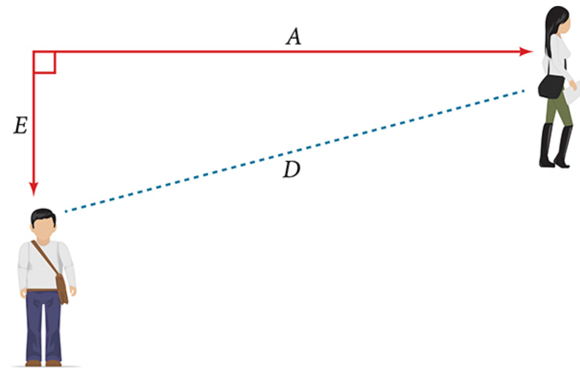


Figure 3

Figure 2 shows us that we can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} D(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} && \text{Solve for } D(t) \text{ using the square root} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \tfrac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Q&A **Should I draw diagrams when given information based on a geometric shape?**

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

EXAMPLE 3

Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 4. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, \ 10)$, and Eastborough at $(20, 0)$.

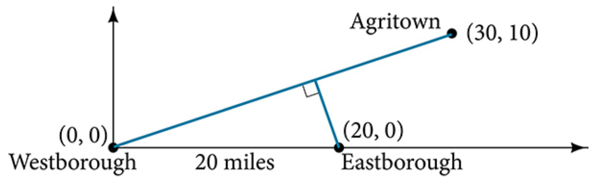


Figure 4

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10 - 0}{30 - 0} = \frac{1}{3}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point (20, 0), we can find the equation:

$$\begin{aligned} E(x) &= -3x + b \\ 0 &= -3(20) + b \quad \text{Substitute in (20, 0)} \\ b &= 60 \\ E(x) &= -3x + 60 \end{aligned}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{aligned} \tfrac{1}{3}x &= -3x + 60 \\ \tfrac{10}{3}x &= 60 \\ 10x &= 180 \\ x &= 18 \qquad \text{Substituting this back into } W(x) \\ y &= W(18) \\ &= \tfrac{1}{3}(18) \\ &= 6 \end{aligned}$$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{aligned} \text{distance} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(18 - 0)^2 + (6 - 0)^2} \\ &\approx 18.974 \text{ miles} \end{aligned}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

TRY IT #3 There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a system of linear equations. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 5.

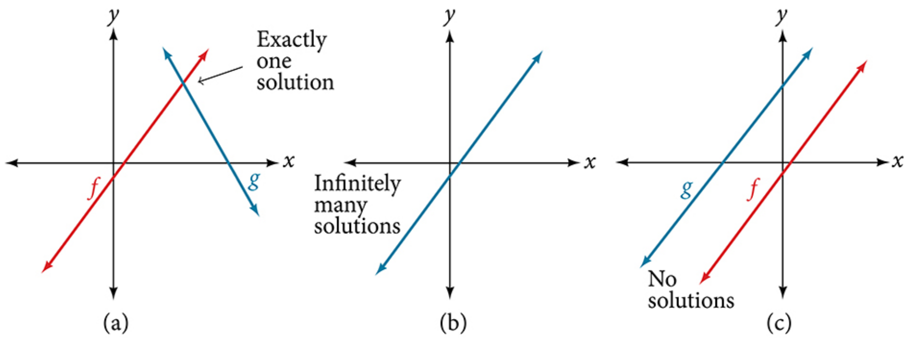


Figure 5

HOW TO

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

1. Identify the input and output of each linear model.
2. Identify the slope and y-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to one another and solving for x, or find the point of intersection on a graph.

EXAMPLE 4

Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile[3]. When will Keep on Trucking, Inc. be the better choice for Jamal?

Solution

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

Input	d, distance driven in miles
Outputs	$K(d)$: cost, in dollars, for renting from Keep on Trucking $M(d)$ cost, in dollars, for renting from Move It Your Way
Initial Value	Up-front fee: $K(0) = 20$ and $M(0) = 16$
Rate of Change	$K(d) = \$0.59$ /mile and $P(d) = \$0.63$ /mile

Table 1

A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

$$K(d) = 0.59d + 20$$
$$M(d) = 0.63d + 16$$

Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

3 Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

These graphs are sketched in Figure 6, with $K(d)$ in blue.

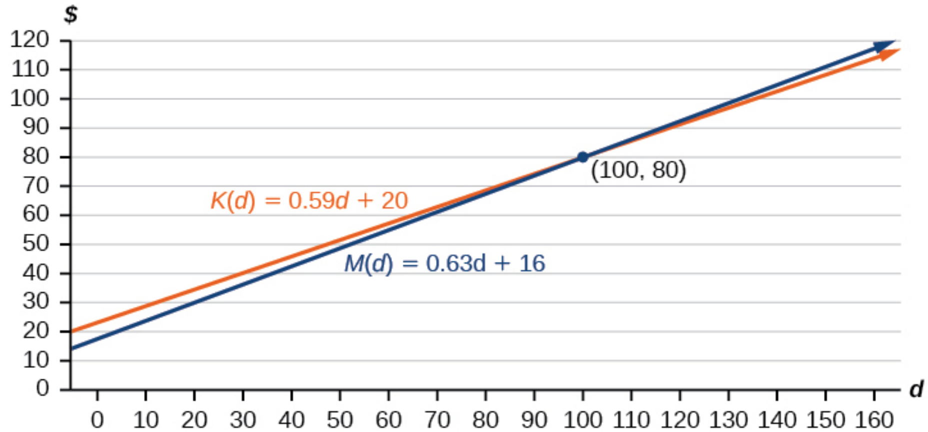


Figure 6

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} K(d) &= M(d) \\ 0.59d + 20 &= 0.63d + 16 \\ 4 &= 0.04d \\ 100 &= d \\ d &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

MEDIA

Access this online resource for additional instruction and practice with linear function models.

Interpreting a Linear Function (http://openstax.org/l/interpretlinear)

2.3 SECTION EXERCISES

Verbal

1. Explain how to find the input variable in a word problem that uses a linear function.

2. Explain how to find the output variable in a word problem that uses a linear function.

3. Explain how to interpret the initial value in a word problem that uses a linear function.

4. Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

5. Find the area of a parallelogram bounded by the *y*-axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through $(2,\ 7)$.

6. Find the area of a triangle bounded by the *x*-axis, the line $f(x) = 12– \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

7. Find the area of a triangle bounded by the *y*-axis, the line $f(x) = 9– \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

8. Find the area of a parallelogram bounded by the *x*-axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through $(6, 1)$.

For the following exercises, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped to 4,700. Assume this trend continues.

9. Predict the population in 2016.

10. Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

11. Predict the population in 2016.

12. Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

13. Find the linear function that models the town's population P as a function of the year, t, where t is the number of years since the model began.

14. Find a reasonable domain and range for the function P.

15. If the function P is graphed, find and interpret the *x*- and *y*-intercepts.

16. If the function P is graphed, find and interpret the slope of the function.

17. When will the output reached 100,000?

18. What is the output in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

19. Find the linear function that models the baby's weight W as a function of the age of the baby, in months, t.

20. Find a reasonable domain and range for the function W.

21. If the function W is graphed, find and interpret the *x*- and *y*-intercepts.

22. If the function W is graphed, find and interpret the slope of the function.

23. When did the baby weight 10.4 pounds?

24. What is the output when the input is 6.2? Interpret your answer.

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were afflicted.

25. Find the linear function that models the number of people inflicted with the common cold C as a function of the year, t.

26. Find a reasonable domain and range for the function C.

27. If the function C is graphed, find and interpret the x- and y-intercepts.

28. If the function C is graphed, find and interpret the slope of the function.

29. When will the output reach 0?

30. In what year will the number of people be 9,700?

Graphical

For the following exercises, use the graph in Figure 7, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

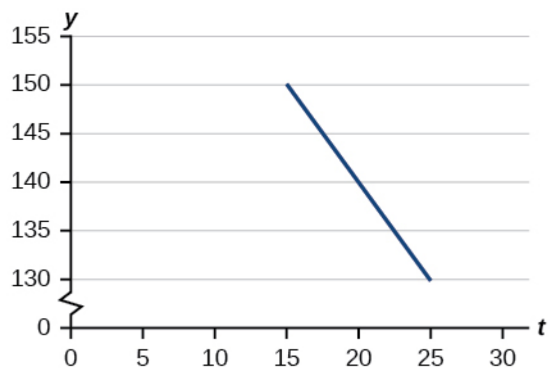


Figure 7

31. Find the linear function y, where y depends on t, the number of years since 1980.

32. Find and interpret the y-intercept.

33. Find and interpret the x-intercept.

34. Find and interpret the slope.

For the following exercises, use the graph in Figure 8, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

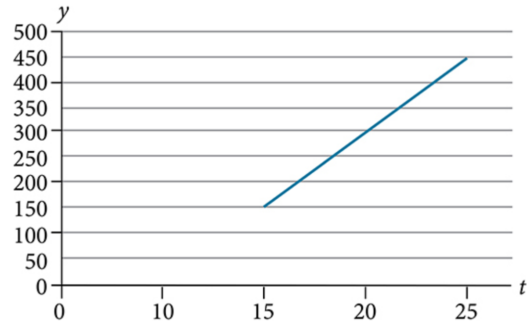


Figure 8

35. Find the linear function y, where y depends on t, the number of years since 1980.

36. Find and interpret the y-intercept.

37. Find and interpret the x-intercept.

38. Find and interpret the slope.

Numeric

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in Table 2. Assume that the house values are changing linearly.

Year	Mississippi	Hawaii
1950	$25,200	$74,400
2000	$71,400	$272,700

Table 2

39. In which state have home values increased at a higher rate?

40. If these trends were to continue, what would be the median home value in Mississippi in 2010?

41. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in Table 3. Assume that the house values are changing linearly.

Year	Indiana	Alabama
1950	$37,700	$27,100

Table 3

Year	Indiana	Alabama
2000	\$94,300	\$85,100

Table 3

42. In which state have home values increased at a higher rate?

43. If these trends were to continue, what would be the median home value in Indiana in 2010?

44. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

Real-World Applications

45. In 2004, a school population was 1,001. By 2008 the population had grown to 1,697. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2004 and 2008?
ⓑ How long did it take the population to grow from 1,001 students to 1,697 students?
ⓒ What is the average population growth per year?
ⓓ What was the population in the year 2000?
ⓔ Find an equation for the population, P, of the school t years after 2000.
ⓕ Using your equation, predict the population of the school in 2011.

46. In 2003, a town's population was 1,431. By 2007 the population had grown to 2,134. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2003 and 2007?
ⓑ How long did it take the population to grow from 1,431 people to 2,134 people?
ⓒ What is the average population growth per year?
ⓓ What was the population in the year 2000?
ⓔ Find an equation for the population, P of the town t years after 2000.
ⓕ Using your equation, predict the population of the town in 2014.

47. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used for voice and video calling. If a customer uses 410 minutes, the monthly cost will be \$71.50. If the customer uses 720 minutes, the monthly cost will be \$118.

ⓐ Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
ⓑ Interpret the slope and y-intercept of the equation.
ⓒ Use your equation to find the total monthly cost if 687 minutes are used.

48. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

ⓐ Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
ⓑ Interpret the slope and *y*-intercept of the equation.
ⓒ Use your equation to find the total monthly cost if 250 MB are used.

49. In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.

ⓐ Find a formula for the moose population, *P* since 1990.
ⓑ What does your model predict the moose population to be in 2003?

50. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 1990.

ⓐ Find a formula for the owl population, P. Let the input be years since 2003.
ⓑ What does your model predict the owl population to be in 2012?

51. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.

ⓐ Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
ⓑ In 2015, what will the helium reserves be?
ⓒ If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

52. Suppose the world's oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

ⓐ Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
ⓑ Seven years from now, what will the oil reserves be?
ⓒ If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

53. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

54. You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

55. When hired at a new job selling jewelry, you are given two pay options:

- Option A: Base salary of $17,000 a year with a commission of 12% of your sales
- Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

56. When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $14,000 a year with a commission of 10% of your sales
- Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

57. When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $20,000 a year with a commission of 12% of your sales
- Option B: Base salary of $26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

58. When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $10,000 a year with a commission of 9% of your sales
- Option B: Base salary of $20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

2.4 Fitting Linear Models to Data

Learning Objectives

In this section, you will:

- Draw and interpret scatter plots.
- Find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Use a linear model to make predictions.

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 1 shows a sample scatter plot.

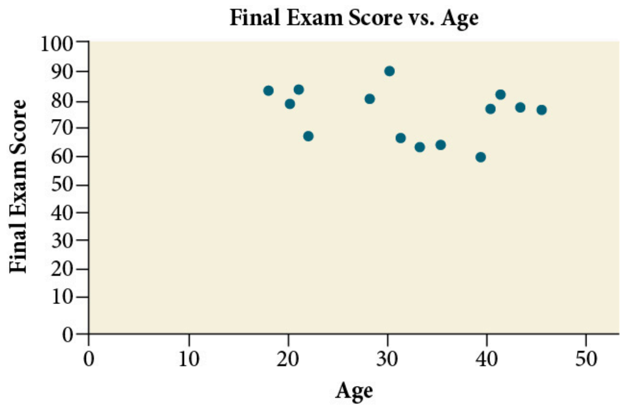


Figure 1 A scatter plot of age and final exam score variables

Notice this scatter plot does *not* indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

EXAMPLE 1

Using a Scatter Plot to Investigate Cricket Chirps

The table below shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[4]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Table 1

Solution

Plotting this data, as depicted in Figure 2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

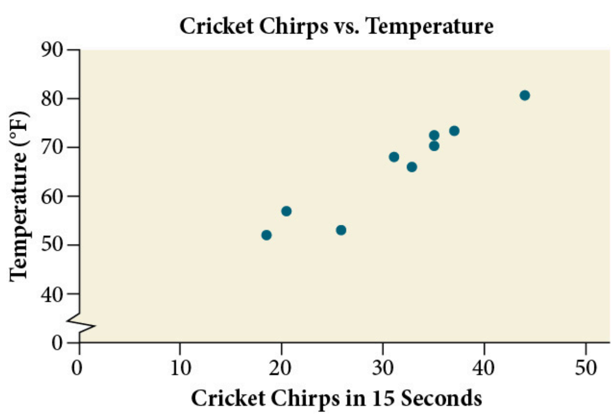


Figure 2

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the *y*-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

4 Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

EXAMPLE 2

Finding a Line of Best Fit

Find a linear function that fits the data in Table 1 by "eyeballing" a line that seems to fit.

Solution

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points (0, 30) and (50, 90), this graph has a slope of

$$m = \frac{60}{50} = 1.2$$

and a y-intercept at 30. This gives an equation of

$$T(c) = 1.2c + 30$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 3.

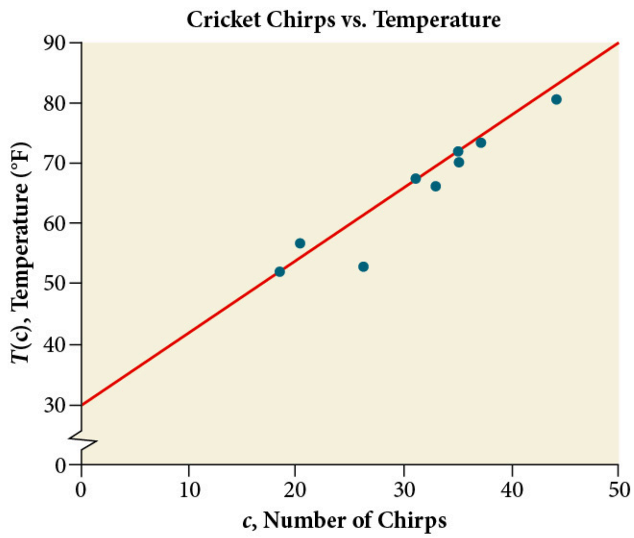


Figure 3

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 4 compares the two processes for the cricket-chirp data addressed in Example 2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

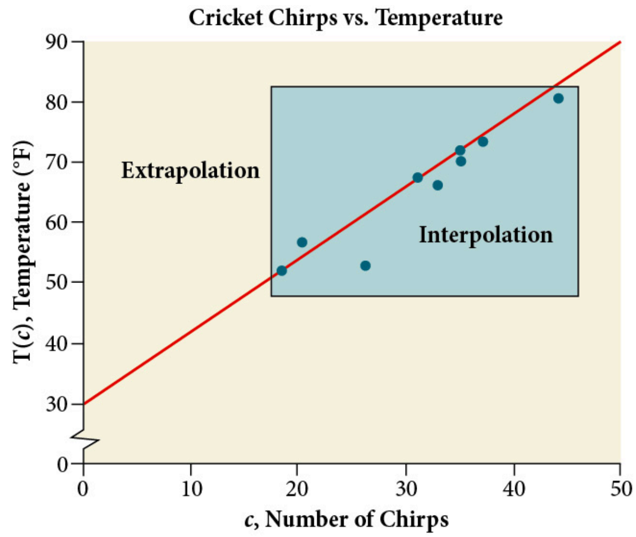


Figure 4 Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called **model breakdown**. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of **interpolation** involves predicting a value inside the domain and/or range of the data.
- The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

EXAMPLE 3

Understanding Interpolation and Extrapolation

Use the cricket data from Table 1 to answer the following questions:

ⓐ Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
ⓑ Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Solution

ⓐ The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$\begin{aligned} T(30) &= 30 + 1.2(30) \\ &= 66 \text{ degrees} \end{aligned}$$

Based on the data we have, this value seems reasonable.

ⓑ The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

We can compare the regions of interpolation and extrapolation using Figure 5.

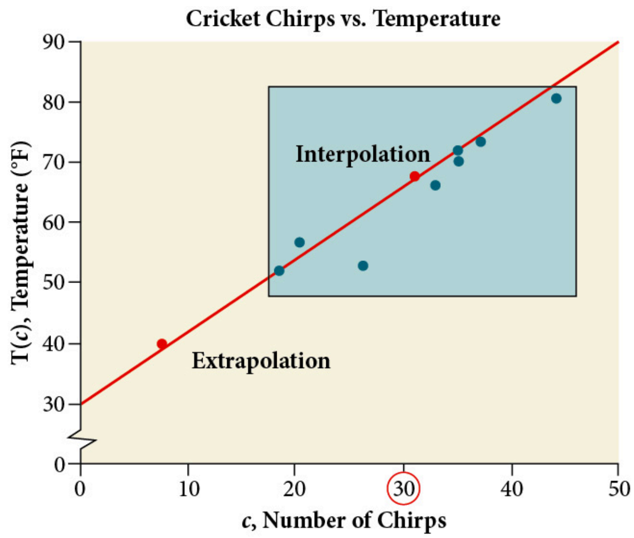


Figure 5

Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

TRY IT #1 According to the data from Table 1, what temperature can we predict it is if we counted 20 chirps in 15 seconds?

Finding the Line of Best Fit Using a Graphing Utility

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[5]. One such technique is called **least squares regression** and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[6]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

HOW TO

Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.

1. Enter the input in List 1 (**L1**).
2. Enter the output in List 2 (**L2**).
3. On a graphing utility, select Linear Regression (**LinReg**).

---

5 Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

6 For example, http://www.shodor.org/unchem/math/lls/leastsq.html

EXAMPLE 4

Finding a Least Squares Regression Line

Find the least squares regression line using the cricket-chirp data in Table 1.

Solution

1. Enter the input (chirps) in List 1 (L1).
2. Enter the output (temperature) in List 2 (L2). See Table 2.

L1	44	35	20.4	33	31	35	18.5	37	26
L2	80.5	70.5	57	66	68	72	52	73.5	53

Table 2

3. On a graphing utility, select Linear Regression (LinReg). Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Analysis

Notice that this line is quite similar to the equation we "eyeballed" but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$\begin{aligned} T(30) &= 30.281 + 1.143(30) \\ &= 64.571 \\ &\approx 64.6 \text{ degrees} \end{aligned}$$

The graph of the scatter plot with the least squares regression line is shown in Figure 6.

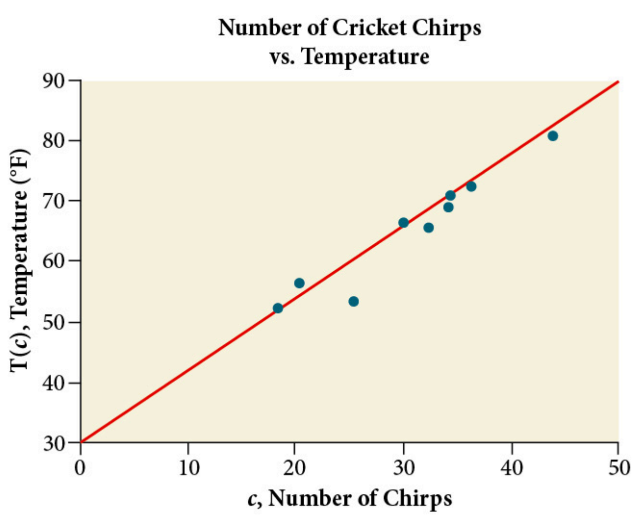


Figure 6

Q&A **Will there ever be a case where two different lines will serve as the best fit for the data?**

No. There is only one best fit line.

Distinguishing Between Linear and Non-Linear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the **correlation coefficient**, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as r. The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of r and the graph of the data, Figure 7 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

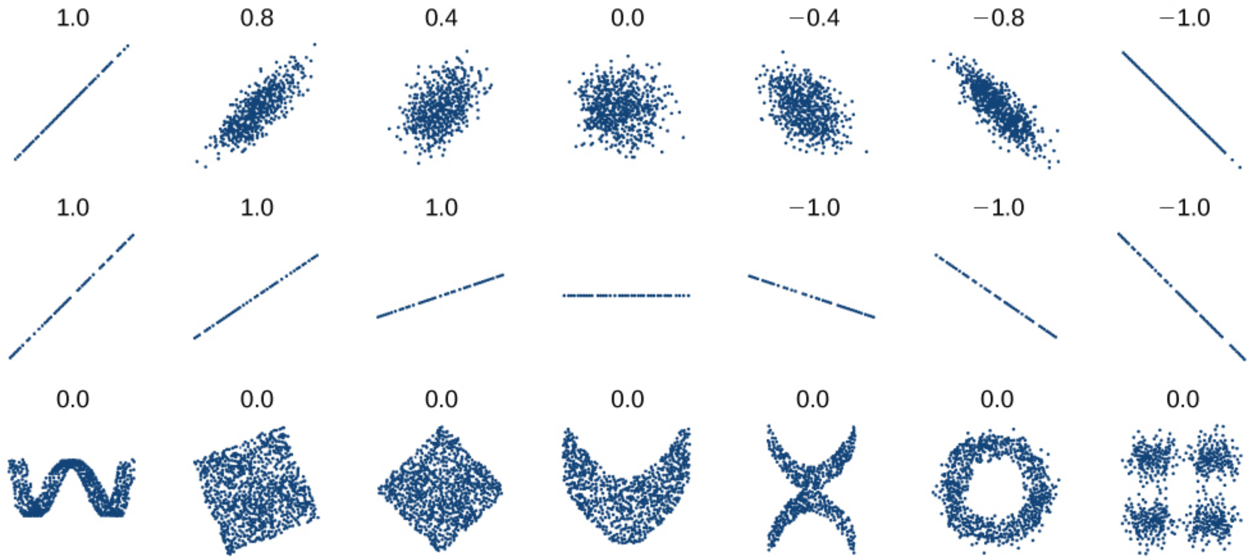


Figure 7 Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

Correlation Coefficient

The **correlation coefficient** is a value, r, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

EXAMPLE 5

Finding a Correlation Coefficient

Calculate the correlation coefficient for cricket-chirp data in Table 1.

Solution

Because the data appear to follow a linear pattern, we can use technology to calculate r. Enter the inputs and corresponding outputs and select the Linear Regression. The calculator will also provide you with the correlation coefficient, $r = 0.9509$. This value is very close to 1, which suggests a strong increasing linear relationship.

Note: For some calculators, the Diagnostics must be turned "on" in order to get the correlation coefficient when linear regression is performed: [2nd]>[0]>[alpha][x –1], then scroll to **DIAGNOSTICSON**.

Predicting with a Regression Line

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

EXAMPLE 6

Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 3[7]. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

Table 3

The scatter plot of the data, including the least squares regression line, is shown in Figure 8.

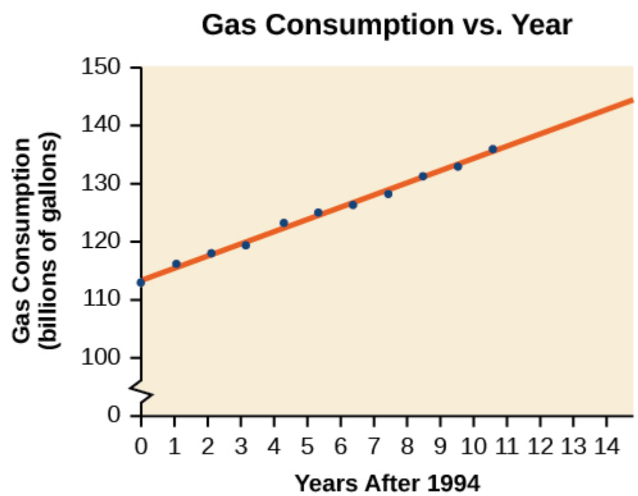


Figure 8

Solution

We can introduce new input variable, t, representing years since 1994.

The least squares regression equation is:

$$C(t) = 113.318 + 2.209t$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 ($t = 14$),

$$\begin{aligned} C(14) &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

TRY IT #2 Use the model we created using technology in Example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Access these online resources for additional instruction and practice with fitting linear models to data.

- Introduction to Regression Analysis (http://openstax.org/l/introregress)
- Linear Regression (http://openstax.org/l/linearregress)

7 http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

2.4 SECTION EXERCISES

Verbal

1. Describe what it means if there is a model breakdown when using a linear model.

2. What is interpolation when using a linear model?

3. What is extrapolation when using a linear model?

4. Explain the difference between a positive and a negative correlation coefficient.

5. Explain how to interpret the absolute value of a correlation coefficient.

Algebraic

6. A regression was run to determine whether there is a relationship between hours of TV watched per day (x) and number of sit-ups a person can do (y). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

$$y = ax + b$$
$$a = -1.341$$
$$b = 32.234$$
$$r = -0.896$$

7. A regression was run to determine whether there is a relationship between the diameter of a tree (x, in inches) and the tree's age (y, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

$$y = ax + b$$
$$a = 6.301$$
$$b = -1.044$$
$$r = -0.970$$

For the following exercises, draw a scatter plot for the data provided. Does the data appear to be linearly related?

8.

0	2	4	6	8	10
–22	–19	–15	–11	–6	–2

9.

1	2	3	4	5	6
46	50	59	75	100	136

10.

100	250	300	450	600	750
12	12.6	13.1	14	14.5	15.2

11.

1	3	5	7	9	11
1	9	28	65	125	216

12. For the following data, draw a scatter plot. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

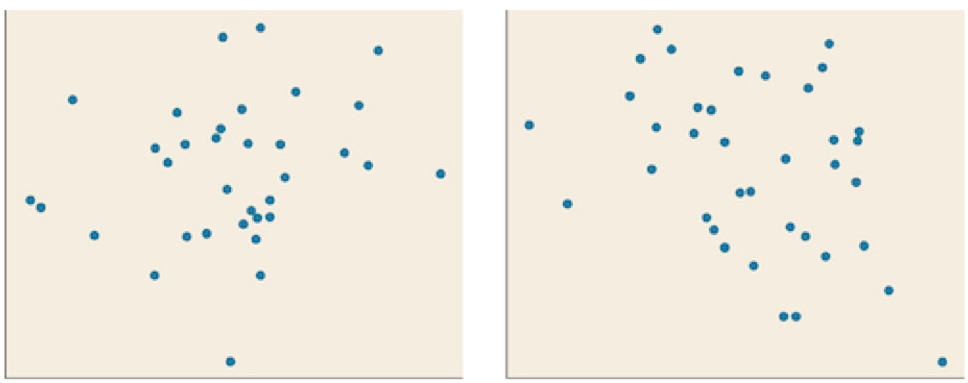
Year	Population
1990	11,500
1995	12,100
2000	12,700
2005	13,000
2010	13,750

13. For the following data, draw a scatter plot. If we wanted to know when the temperature would reach 28 °F, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

Temperature, °F	16	18	20	25	30
Time, seconds	46	50	54	55	62

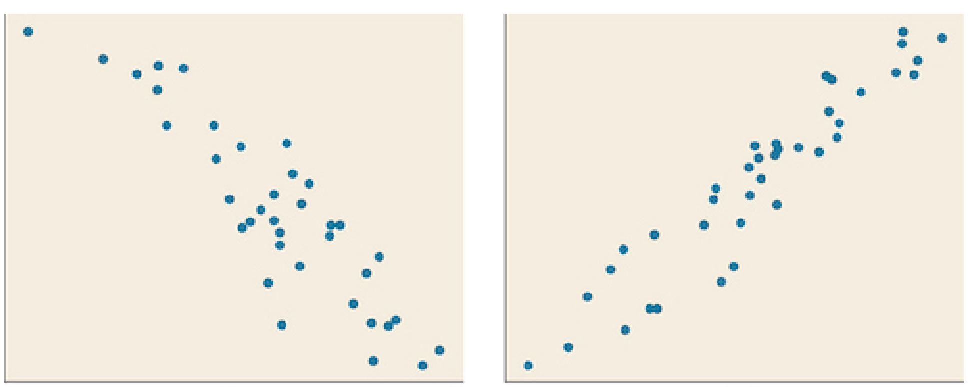
Graphical

For the following exercises, match each scatterplot with one of the four specified correlations in Figure 9 and Figure 10.



(a) (b)

Figure 9



(c) (d)

Figure 10

14. $r = 0.95$

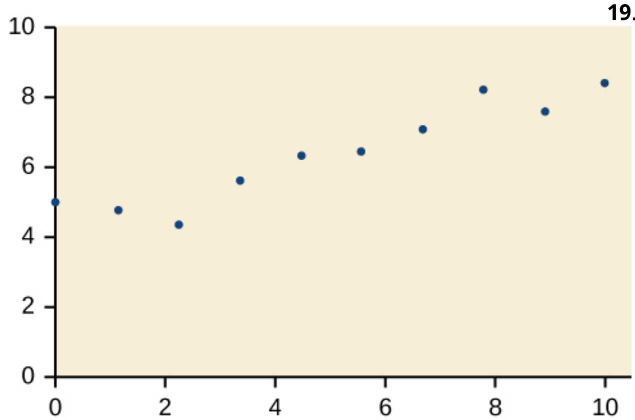
15. $r = -0.89$

16. $r = 0.26$

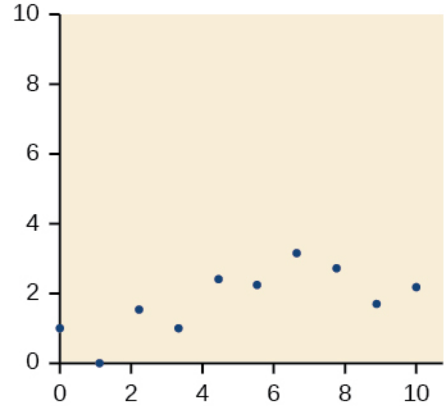
17. $r = -0.39$

For the following exercises, draw a best-fit line for the plotted data.

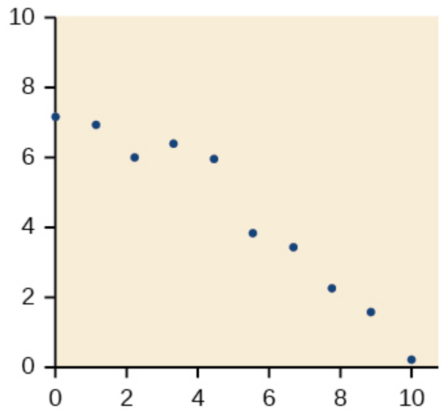
18.



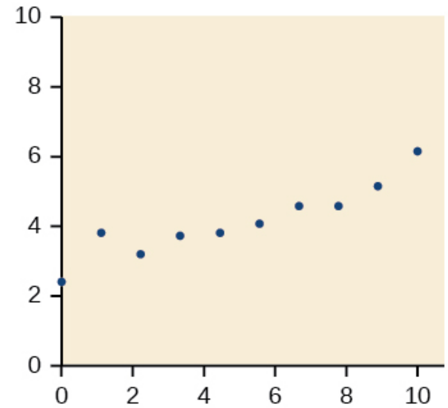
19.



20.



21.



Numeric

22. The U.S. Census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given in Table 4[8]. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed 35%?

Year	Percent Graduates
1990	21.3
1992	21.4
1994	22.2
1996	23.6
1998	24.4
2000	25.6
2002	26.7
2004	27.7
2006	28
2008	29.4

Table 4

23. The U.S. import of wine (in hectoliters) for several years is given in Table 5. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	Imports
1992	2665
1994	2688
1996	3565
1998	4129
2000	4584
2002	5655
2004	6549
2006	7950
2008	8487
2009	9462

Table 5

24. Table 6 shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach 5?

Year	Number Unemployed
1990	750
1992	670
1994	650
1996	605
1998	550
2000	510
2002	460
2004	420
2006	380
2008	320

Table 6

Technology

For the following exercises, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to 3 decimal places of accuracy.

25.

x	8	15	26	31	56
y	23	41	53	72	103

26.

x	5	7	10	12	15
y	4	12	17	22	24

8 http://www.census.gov/hhes/socdemo/education/data/cps/historical/index.html. Accessed 5/1/2014.

27.

x	y	x	y
3	21.9	11	15.76
4	22.22	12	13.68
5	22.74	13	14.1
6	22.26	14	14.02
7	20.78	15	11.94
8	17.6	16	12.76
9	16.52	17	11.28
10	18.54	18	9.1

28.

x	y
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8

29.

x	21	25	30	31	40	50
y	17	11	2	-1	-18	-40

30.

x	100	80	60	55	40	20
y	2000	1798	1589	1580	1390	1202

31.

x	900	988	1000	1010	1200	1205
y	70	80	82	84	105	108

Extensions

32. Graph $f(x) = 0.5x + 10$. Pick a set of 5 ordered pairs using inputs $x = -2,\ 1,\ 5,\ 6,\ 9$ and use linear regression to verify that the function is a good fit for the data.

33. Graph $f(x) = -2x - 10$. Pick a set of 5 ordered pairs using inputs $x = -2,\ 1,\ 5,\ 6,\ 9$ and use linear regression to verify the function.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:

$(46,\ 1,600),\ \ (48,\ 1,550),\ \ (50,\ 1,505),\ \ (52,\ 1,540),\ \ (54,\ 1,495).$

34. Use linear regression to determine a function P where the profit in thousands of dollars depends on the number of units sold in hundreds.

35. Find to the nearest tenth and interpret the *x*-intercept.

36. Find to the nearest tenth and interpret the *y*-intercept.

Real-World Applications

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population and the year over the ten-year span, (population, year) for specific recorded years:

$(2500, 2000), (2650, 2001), (3000, 2003), (3500, 2006), (4200, 2010)$

37. Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

38. Predict when the population will hit 8,000.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:

$(46,\ 250),\ \ (48,\ 305),\ \ (50,\ 350),\ \ (52,\ 390),\ \ (54,\ 410).$

39. Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds .

40. Predict when the profit will exceed one million dollars.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs show dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span (number of units sold, profit) for specific recorded years:

$(46, 250), (48, 225), (50, 205), (52, 180), (54, 165).$

41. Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds .

42. Predict when the profit will dip below the $25,000 threshold.

Chapter Review

Key Terms

correlation coefficient a value, r, between –1 and 1 that indicates the degree of linear correlation of variables, or how closely a regression line fits a data set.

decreasing linear function a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

extrapolation predicting a value outside the domain and range of the data

horizontal line a line defined by $f(x) = b$, where b is a real number. The slope of a horizontal line is 0.

increasing linear function a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

interpolation predicting a value inside the domain and range of the data

least squares regression a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values

linear function a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

model breakdown when a model no longer applies after a certain point

parallel lines two or more lines with the same slope

perpendicular lines two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form the equation for a line that represents a linear function of the form $y - y_1 = m(x - x_1)$

slope the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form the equation for a line that represents a linear function in the form $f(x) = mx + b$

vertical line a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

***x*-intercept** the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

***y*-intercept** the value of a function when the input value is zero; also known as initial value

Key Equations

slope-intercept form of a line	$f(x) = mx + b$
slope	$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
point-slope form of a line	$y - y_1 = m(x - x_1)$

Key Concepts

2.1 Linear Functions

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form. See Example 1.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or *y*-intercept, is the output value when the input of a linear function is zero. It is the *y*-value of the point at which the line crosses the *y*-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant. See Example 2.
- The slope of a linear function can be calculated by dividing the difference between *y*-values by the difference in corresponding *x*-values of any two points on the line. See Example 3 and Example 4.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point. See Example 5.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes. See Example 6.
- The equation for a linear function can be written if the slope m and initial value b are known. See Example 7,

Example 8, and Example 9.
- A linear function can be used to solve real-world problems. See Example 10 and Example 11.
- A linear function can be written from tabular form. See Example 12.

2.2 Graphs of Linear Functions

- Linear functions may be graphed by plotting points or by using the *y*-intercept and slope. See Example 1 and Example 2.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See Example 3.
- The *y*-intercept and slope of a line may be used to write the equation of a line.
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See Example 4 and Example 5.
- Horizontal lines are written in the form, $f(x) = b$. See Example 6.
- Vertical lines are written in the form, $x = b$. See Example 7.
- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See Example 8.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x) = mx + b$, and using the b that results. Similarly, the point-slope form of an equation can also be used. See Example 9**.**
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See Example 10 and Example 11.
- A system of linear equations may be solved setting the two equations equal to one another and solving for x. The *y*-value may be found by evaluating either one of the original equations using this *x*-value.
- A system of linear equations may also be solved by finding the point of intersection on a graph. See Example 12 and Example 13.

2.3 Modeling with Linear Functions

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and *y*-intercept. See Example 1.
- Draw a diagram, where appropriate. See Example 2 and Example 3.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the *y*-intercept.
- The *x*-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the *x*- and *y*-values are the same. See Example 4.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

2.4 Fitting Linear Models to Data

- Scatter plots show the relationship between two sets of data. See Example 1.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated, using a calculator or statistical software. See Example 2.
- Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data. See Example 3.
- The correlation coefficient, r, indicates the degree of linear relationship between data. See Example 5.
- A regression line best fits the data. See Example 6.
- The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables. See Example 4.

Exercises

Review Exercises

Linear Functions

1. Determine whether the algebraic equation is linear. $2x + 3y = 7$

2. Determine whether the algebraic equation is linear. $6x^2 - y = 5$

3. Determine whether the function is increasing or decreasing.

$$f(x) = 7x - 2$$

4. Determine whether the function is increasing or decreasing.

$g(x) = -x + 2$

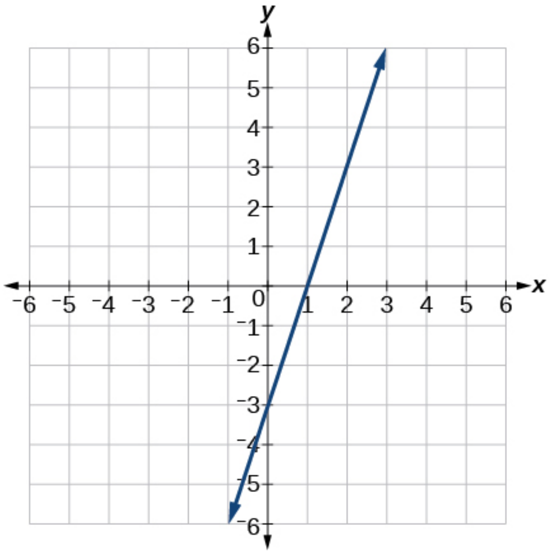
5. Given each set of information, find a linear equation that satisfies the given conditions, if possible.

Passes through $(7, 5)$ and $(3, 17)$

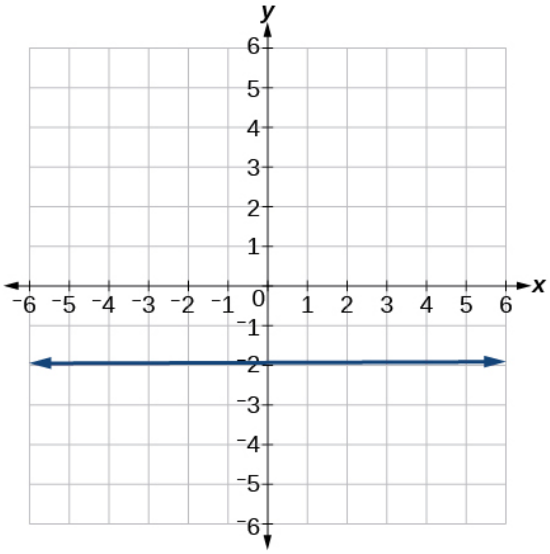
6. Given each set of information, find a linear equation that satisfies the given conditions, if possible.

x-intercept at $(6, 0)$ and y-intercept at $(0, 10)$

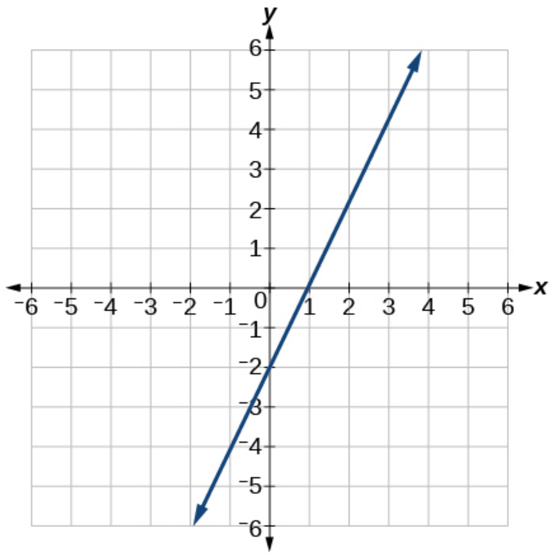
7. Find the slope of the line shown in the line graph.



8. Find the slope of the line graphed.



9. Write an equation in slope-intercept form for the line shown.



10. Does the following table represent a linear function? If so, find the linear equation that models the data.

x	–4	0	2	10
$g(x)$	18	–2	–12	–52

11. Does the following table represent a linear function? If so, find the linear equation that models the data.

x	6	8	12	26
$g(x)$	–8	–12	–18	–46

12. On June 1st, a company has $4,000,000 profit. If the company then loses 150,000 dollars per day thereafter in the month of June, what is the company's profit n^{th}day after June 1st?

Graphs of Linear Functions

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

13. $\begin{aligned} 2x - 6y &= 12 \\ -x + 3y &= 1 \end{aligned}$

14. $y = \frac{1}{3}x - 2$
$3x + y = -9$

For the following exercises, find the x- and y- intercepts of the given equation

15. $7x + 9y = -63$

16. $f(x) = 2x - 1$

For the following exercises, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

17. Line 1: Passes through $(5, 11)$ and $(10, 1)$
Line 2: Passes through $(-1, 3)$ and $(-5, 11)$

18. Line 1: Passes through $(8, -10)$ and $(0, -26)$
Line 2: Passes through $(2, 5)$ and $(4, 4)$

19. Write an equation for a line perpendicular to $f(x) = 5x - 1$ and passing through the point (5, 20).

20. Find the equation of a line with a y- intercept of $(0,\ \ 2)$ and slope $-\frac{1}{2}$.

21. Sketch a graph of the linear function $f(t) = 2t - 5$.

22. Find the point of intersection for the 2 linear functions: $\begin{array}{l} x = y + 6 \\ 2x - y = 13 \end{array}$

23. A car rental company offers two plans for renting a car.

Plan A: 25 dollars per day and 10 cents per mile
Plan B: 50 dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Modeling with Linear Functions

24. Find the area of a triangle bounded by the *y* axis, the line $f(x) = 10 - 2x$, and the line perpendicular to f that passes through the origin.

25. A town's population increases at a constant rate. In 2010 the population was 55,000. By 2012 the population had increased to 76,000. If this trend continues, predict the population in 2016.

26. The number of people afflicted with the common cold in the winter months dropped steadily by 50 each year since 2004 until 2010. In 2004, 875 people were inflicted.

Find the linear function that models the number of people afflicted with the common cold *C* as a function of the year, t. When will no one be afflicted?

For the following exercises, use the graph in Figure 1 showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

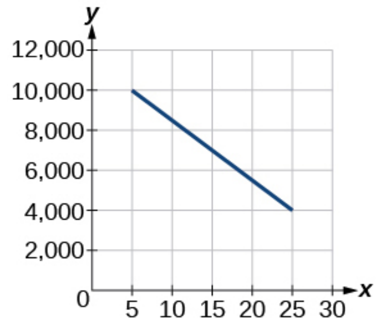


Figure 1

27. Find the linear function *y*, where *y* depends on x, the number of years since 1980.

28. Find and interpret the *y*-intercept.

For the following exercise, consider this scenario: In 2004, a school population was 1,700. By 2012 the population had grown to 2,500.

29. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2004 and 2012?
ⓑ What is the average population growth per year?
ⓒ Find an equation for the population, *P*, of the school *t* years after 2004.

For the following exercises, consider this scenario: In 2000, the moose population in a park was measured to be 6,500. By 2010, the population was measured to be 12,500. Assume the population continues to change linearly.

30. Find a formula for the moose population, P.

31. What does your model predict the moose population to be in 2020?

For the following exercises, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in Table 1. Assume that the house values are changing linearly.

Year	Pima Central	East Valley
1970	32,000	120,250
2010	85,000	150,000

Table 1

32. In which subdivision have home values increased at a higher rate?

33. If these trends were to continue, what would be the median home value in Pima Central in 2015?

Fitting Linear Models to Data

34. Draw a scatter plot for the data in Table 2. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–105	–50	1	55	105	160

Table 2

35. Draw a scatter plot for the data in Table 3. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation?

Year	Population
1990	5,600
1995	5,950
2000	6,300
2005	6,600
2010	6,900

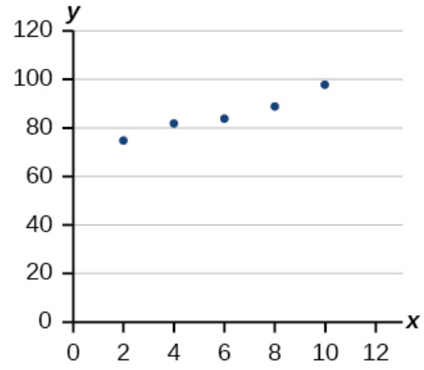
Table 3

36. Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in Table 4. Plot the points, then sketch a line that fits the data.

Predicted	Actual
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

Table 4

37. Draw a best-fit line for the plotted data.



For the following exercises, consider the data in Table 5, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	6.5	7.0	7.4	8.2	9.0

Table 5

38. Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

39. In what year will the percentage exceed 12%?

40. Based on the set of data given in Table 6, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	17	20	23	26	29
y	15	25	31	37	40

Table 6

41. Based on the set of data given in Table 7, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	10	12	15	18	20
y	36	34	30	28	22

Table 7

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:

$(3{,}600,\ 2000);\ (4{,}000,\ 2001);\ (4{,}700,\ 2003);\ (6{,}000,\ 2006)$

42. Use linear regression to determine a function y, where the year depends on the population, to three decimal places of accuracy.

43. Predict when the population will hit 12,000.

44. What is the correlation coefficient for this model to three decimal places of accuracy?

45. According to the model, what is the population in 2014?

Practice Test

1. Determine whether the following algebraic equation can be written as a linear function. $2x + 3y = 7$

2. Determine whether the following function is increasing or decreasing. $f(x) = -2x + 5$

3. Determine whether the following function is increasing or decreasing. $f(x) = 7x + 9$

4. Given the following set of information, find a linear equation satisfying the conditions, if possible.

Passes through (5, 1) and (3, –9)

5. Given the following set of information, find a linear equation satisfying the conditions, if possible.

x intercept at (–4, 0) and y-intercept at (0, –6)

6. Find the slope of the line in Figure 1.

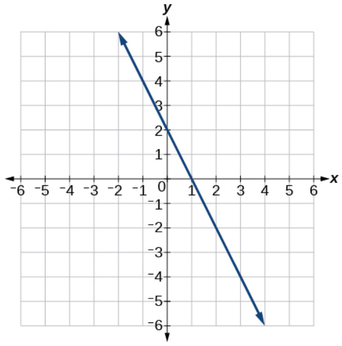


Figure 1

7. Write an equation for line in Figure 2.

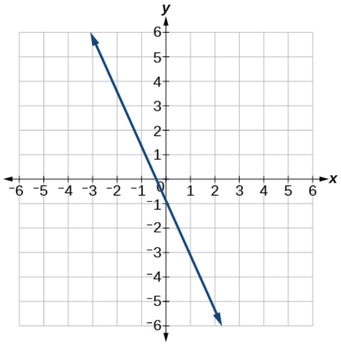


Figure 2

8. Does Table 1 represent a linear function? If so, find a linear equation that models the data.

x	–6	0	2	4
$g(x)$	14	32	38	44

Table 1

9. Does Table 2 represent a linear function? If so, find a linear equation that models the data.

x	1	3	7	11
$g(x)$	4	9	19	12

Table 2

10. At 6 am, an online company has sold 120 items that day. If the company sells an average of 30 items per hour for the remainder of the day, write an expression to represent the number of items that were sold n after 6 am.

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

11. $y = \frac{3}{4}x - 9$
$-4x - 3y = 8$

12. $-2x + y = 3$
$3x + \frac{3}{2}y = 5$

13. Find the x- and y-intercepts of the equation $2x + 7y = -14$.

14. Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(-2, -6)$ and $(3, 14)$

Line 2: Passes through $(2, 6)$ and $(4, 14)$

15. Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

16. Sketch a line with a *y*-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

17. Graph of the linear function $f(x) = -\mathrm{x} + 6$.

18. For the two linear functions, find the point of intersection:

$$x = y + 2$$
$$2x - 3y = -1$$

19. A car rental company offers two plans for renting a car.

Plan A: $25 per day and $0.10 per mile
Plan B: $40 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

20. Find the area of a triangle bounded by the *y* axis, the line $f(x) = 12 - 4x$, and the line perpendicular to f that passes through the origin.

21. A town's population increases at a constant rate. In 2010 the population was 65,000. By 2012 the population had increased to 90,000. Assuming this trend continues, predict the population in 2018.

22. The number of people afflicted with the common cold in the winter months dropped steadily by 25 each year since 2002 until 2012. In 2002, 8,040 people were inflicted. Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will less than 6,000 people be afflicted?

For the following exercises, use the graph in Figure 3, showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

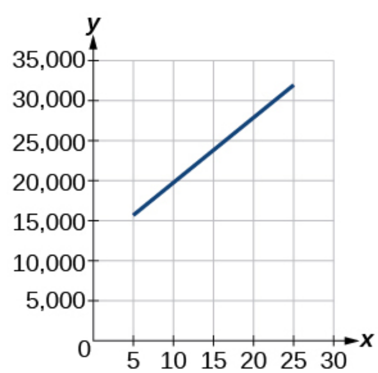


Figure 3

23. Find the linear function y, where y depends on x, the number of years since 1980.

24. Find and interpret the y-intercept.

25. In 2004, a school population was 1250. By 2012 the population had dropped to 875. Assume the population is changing linearly.

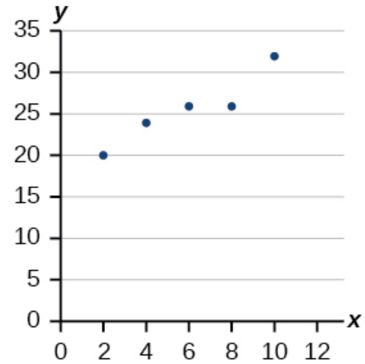
ⓐ How much did the population drop between the year 2004 and 2012?
ⓑ What is the average population decline per year?
ⓒ Find an equation for the population, *P*, of the school *t* years after 2004.

26. Draw a scatter plot for the data provided in Table 3. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–450	–200	10	265	500	755

Table 3

27. Draw a best-fit line for the plotted data.



For the following exercises, use Table 4, which shows the percent of unemployed persons 25 years or older who are college graduates in a particular city, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	8.5	8.0	7.2	6.7	6.4

Table 4

28. Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

29. In what year will the percentage drop below 4%?

30. Based on the set of data given in Table 5, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

x	16	18	20	24	26
y	106	110	115	120	125

Table 5

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population (in hundreds) and the year over the ten-year span, (population, year) for specific recorded years:

$(4{,}500, 2000); (4{,}700, 2001); (5{,}200, 2003); (5{,}800, 2006)$

31. Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

32. Predict when the population will hit 20,000.

33. What is the correlation coefficient for this model?