5.27

$L(x,\nu) = \|Ax-b\|_2^2 + \nu^T(Gx-h) = x^TA^TAx + (G^T\nu - 2A^Tb)^T x - \nu^T h$

$g(\nu) = \inf_x L(x,\nu) = -\frac{1}{4}(G^T\nu - 2A^Tb)^T (A^TA)^{-1}(G^T\nu - 2A^Tb) - \nu^T h$, $x = -\frac{1}{2}(A^TA)^{-1}(G^T\nu - 2A^Tb)$

KKT条件：

$2A^T(Ax^* - b) + G^T\nu^* = 0$, $Gx^* = h$

$\Rightarrow x^* = (A^TA)^{-1}(A^Tb - \frac{1}{2}G^T\nu^*)$

$\nu^* = -2(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb)$

5.29

$L(x,\nu) = -3x_1^2 + x_2^2 + 2x_3^2 + 2x_1 + 2x_2 + 2x_3 + \nu(x_1^2 + x_2^2 + x_3^2 - 1)$

KKT条件：

$x_1^2 + x_2^2 + x_3^2 = 1$ $\quad (\nu-3)x_1 + 1 = 0 \quad (\nu+1)x_2 + 1 = 0 \quad (\nu+2)x_3 + 1 = 0$

$\Rightarrow x_1 = \frac{1}{3-\nu} \quad x_2 = -\frac{1}{\nu+1} \quad x_3 = -\frac{1}{\nu+2} \quad (\nu \neq 3, -1, -2)$

$\frac{1}{(3-\nu)^2} + \frac{1}{(\nu+1)^2} + \frac{1}{(\nu+2)^2} = 1$

$\Rightarrow$ ν有4个解，分别为 $\nu_1 = 0.22$, $\nu_2 = 1.89$, $\nu_3 = -3.15$, $\nu_4 = 4.04$

对应的 x 分别为

$(0.36, -0.82, -0.45)$, $(0.90, -0.35, -0.26)$, $(0.16, 0.47, 0.87)$, $(-0.97, -0.20, -0.17)$

对应的 $f_0(x)$ 分别为

$f_0(x)_1 = -1.13 \quad f_0(x)_2 = -1.59 \quad f_0(x)_3 = 4.65 \quad f_0(x)_4 = -5.37$, $f_0(x)_4$ 是最优值

故 $\nu^* = 4.04 \quad x^* = (-0.97, -0.20, -0.17)$

Saddle Point Thm:

考虑 $\sup\limits_{\lambda\succeq 0} L(x,\lambda,\mu) = \sup\limits_{\lambda\succeq 0} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) \right) = \begin{cases} f_0(x), & f_i(x)\le 0,\ h_j(x)=0 \\ \infty, & \text{otherwise} \end{cases}$

那么 $p^* = \inf\limits_{x} \sup\limits_{\lambda\succeq 0} L(x,\lambda,\mu)$. $d^* = \sup\limits_{\lambda\succeq 0} \inf\limits_{x} L(x,\lambda,\mu)$

强对偶性表达为 $p^* = \inf\limits_{x} \sup\limits_{\lambda\succeq 0} L(x,\lambda,\mu) = \sup\limits_{\lambda\succeq 0} \inf\limits_{x} L(x,\lambda,\mu) = d^*$

由鞍点定义可以得到 (a) $\Leftrightarrow$ (b) 且 $L(x^*,\lambda^*,\mu^*) = \inf\limits_{x} \sup\limits_{\lambda\succeq 0} L(x,\lambda,\mu) = \sup\limits_{\lambda\succeq 0} \inf\limits_{x} L(x,\lambda,\mu) = p^* = d^*$