定步长搜索步长推导:

由题有 $\nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}$ 令 $m=1$, $M=100$, 则有 $mI \le \nabla^2 f(x) \le MI$

由不等式 $f(x) + \nabla f(x)^T(y-x) + \frac{m}{2}\|y-x\|^2 \le f(y) \le f(x) + \nabla f(x)^T(y-x) + \frac{M}{2}\|y-x\|^2$

和不等式 $f(y) + \nabla f(y)^T(x-y) + \frac{m}{2}\|x-y\|^2 \le f(x) \le f(y) + \nabla f(y)^T(x-y) + \frac{M}{2}\|x-y\|^2$

推出 $m\|x-y\|^2 \le (\nabla f(x) - \nabla f(y))^T(x-y) \le \|\nabla f(x) - \nabla f(y)\| \, \|x-y\|$

$\Rightarrow m\|x-y\| \le \|\nabla f(x) - \nabla f(y)\|$,同理 $\|\nabla f(x) - \nabla f(y)\| \le M\|x-y\|$

考虑 $Tx = x - \alpha \nabla f(x)$

$Tx - Ty = x - y - \alpha(\nabla f(x) - \nabla f(y))$

根据三角不等式,有

$$\begin{aligned}
\|Tx - Ty\| &\le \big| \|x-y\| - \alpha\|\nabla f(x) - \nabla f(y)\| \big| \\
&= \max(\|x-y\| - \alpha\|\nabla f(x) - \nabla f(y)\|, \alpha\|\nabla f(x) - \nabla f(y)\| - \|x-y\|) \\
&\le \max(1-\alpha m, \alpha M - 1)\|x-y\|
\end{aligned}$$

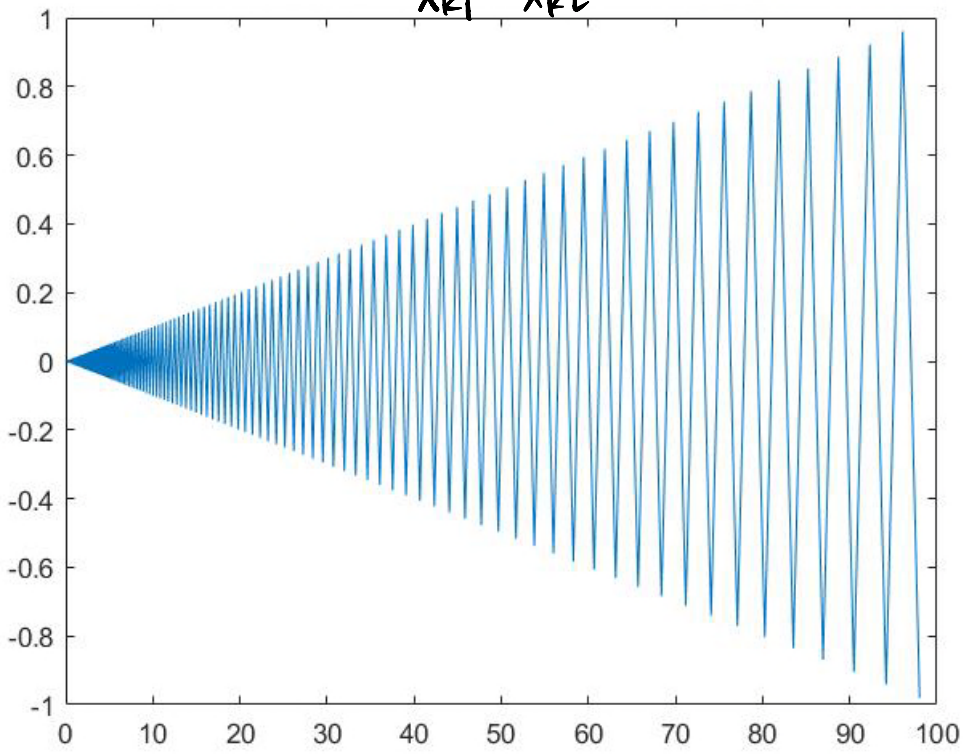
要使 $\max(1-\alpha m, \alpha M-1)$ 最小, 对应的 $\alpha = \frac{2}{m+M}$, 也即定步长搜索中的步长应为 $\frac{2}{101}$

另外两种算法按教材步骤编写即可, 运行结果如下:

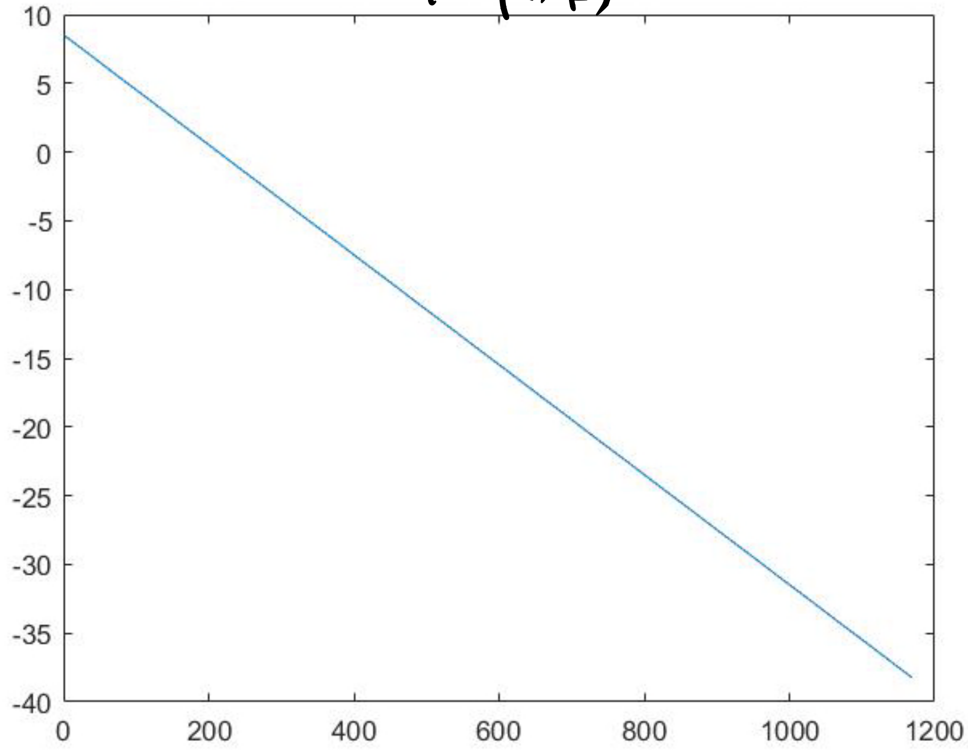
1. 定步长

$K = 1169 \qquad x^* = (7.0122e-9,\ -7.0122e-11)^T$

$x_{k1} - x_{k2}$



$k - f(x_k)$



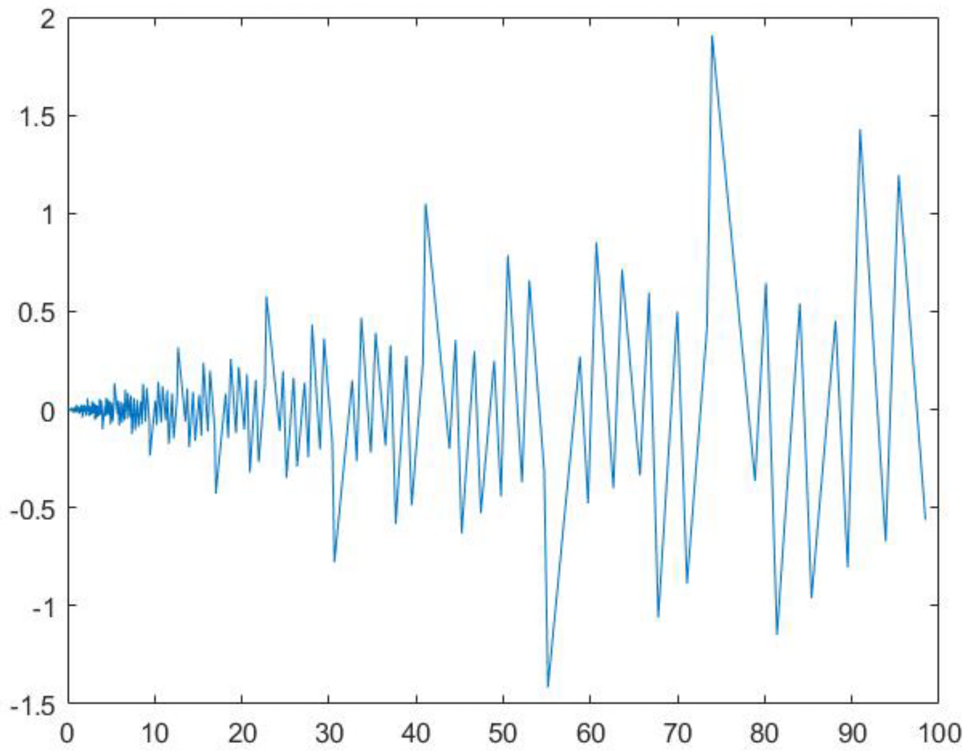
2. 精确

由于精确计算出的步长保持为 $\frac{2}{\text{...}}$，故与上述定步长搜索结果相同.

3、回溯

$k = 973$ $\quad x^* = (8.2212e-9,\ 4.5671e-11)^T$

$x_{k1} - x_{k2}$



$k - f(x_k)$