1.

$\mathrm{rank}(A)=1$ 故A有 $n-1$ 个零特征值，对应的特征向量可取 $[1,-1,0,\cdots,0]^T$, $[1,0,-1,\cdots,0]^T,\cdots,[1,0,\cdots,-1]^T$

设非零特征值为 λ，对应的特征向量为 x，$Ax=\lambda x$，有

$2(x_1+x_2+\cdots+x_n)=\lambda x_1=\lambda x_2=\cdots=\lambda x_n$

取 $x=[1,1,\cdots,1]^T$

$\Rightarrow \lambda=2n$

综上，A的 n 个特征值为 $2n,0,0,\cdots,0$.

2.

令 $D=P^{-1}AP$，其中 P 为可逆矩阵，$D=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$，$\lambda_i$ 是A的特征值，则 $A=PDP^{-1}$

令 $X=P^{-1}BP$，则 $B=PXP^{-1}$

$AB=PDXP^{-1}$ $\quad BA=PXDP^{-1}$ $\quad AB=BA \Leftrightarrow DX=XD$.

即
$$\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\begin{pmatrix}x_{11} & \cdots & x_{1n}\\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn}\end{pmatrix}=\begin{pmatrix}x_{11} & \cdots & x_{1n}\\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn}\end{pmatrix}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$
且 λ_i 互不相同，

故 X 是对角阵，即 B 可对角化

3.

设 λ 是A的特征值，对应的特征向量为 x，则 $(A^2-5A+6I)x=(\lambda^2-5\lambda+6)x=0$

$x\neq 0 \Rightarrow \lambda^2-5\lambda+6=0 \Rightarrow \lambda=2$ 或 3

而 $A^2-5A+6I=(A-2I)(A-3I)=0 \Rightarrow r(A-2I)+r(A-3I)\leq n$

又 $I=(A-2I)-(A-3I) \Rightarrow r(A-2I)+r(A-3I)\geq n$

故 $r(A-2I)+r(A-3I)=n \Rightarrow [n-r(A-2I)]+[n-r(A-3I)]=n$

即 $(A-2I)x=0$ 和 $(A-3I)x=0$ 的基础解系共含 n 个向量，A有 n 个线性无关的特征向量

故A可对角化.

4.

注意到 $A_{i+1}=\begin{bmatrix} A_i & \alpha \\ \alpha^T & a_{i+1,i+1} \end{bmatrix}$，其中 $\alpha=[a_{1,i+1},a_{2,i+1},\cdots,a_{i,i+1}]^T$，记 x_i 为 i 维向量

有 $$\lambda_k=\min_{S,\dim(S)=n-k+1}\ \max_{x\in S,x\neq 0}\left\{\frac{x^HAx}{x^Hx}\right\}$$

$$\lambda_k(A_{i+1})=\min_{S,\dim(S)=i+1-k+1}\ \max_{x_{i+1}\in S,x_{i+1}\neq 0}\left\{\frac{x_{i+1}^HA_{i+1}x_{i+1}}{x_{i+1}^Hx_{i+1}}\right\}$$

$$\geq \min_{S,\dim(S)=i+1-k+1}\ \max_{x_{i+1}\in S,x_{i+1}\neq 0,\ x_{i+1}=[x_i,0]^T}\left\{\frac{x_{i+1}^HA_{i+1}x_{i+1}}{x_{i+1}^Hx_{i+1}}\right\}$$

$$=\min_{S,\dim(S)=i-k+1}\ \max_{x_i\in S,x_i\neq 0}\left\{\frac{x_i^HA_ix_i}{x_i^Hx_i}\right\}$$

$$=\lambda_k(A_i)$$

即 $\lambda_k(A_i)\leq\lambda_k(A_{i+1})$

同理

$$\lambda_{k+1}(A_{i+1})=\max_{S,\dim(S)=k+1}\ \min_{x_{i+1}\in S,x_{i+1}\neq 0}\left\{\frac{x_{i+1}^HA_{i+1}x_{i+1}}{x_{i+1}^Hx_{i+1}}\right\}$$

$$\leq \max_{S,\dim(S)=k+1}\ \min_{x_{i+1}\in S,x_{i+1}\neq 0,\ x_{i+1}=[x_i,0]^T}\left\{\frac{x_{i+1}^HA_{i+1}x_{i+1}}{x_{i+1}^Hx_{i+1}}\right\}$$

$$=\max_{S,\dim(S)=k}\ \min_{x_i\in S,x_i\neq 0}\left\{\frac{x_i^HA_ix_i}{x_i^Hx_i}\right\}$$

$$=\lambda_k(A_i)$$

即 $\lambda_{k+1}(A_{i+1})\leq\lambda_k(A_i)$．综上，$\lambda_{k+1}(A_{i+1})\leq\lambda_k(A_i)\leq\lambda_k(A_{i+1})$

5.

A的一个化零多项式为 $p(x)=|xI-A|=x(x-1)(x+1)$

令 $f(x)=p(x)q(x)+r(x)$，设 $r(x)=ax^2+bx+c$，代入 $x=0,1,-1$ 得到

$$\begin{cases} f(0)=r(0) \\ f(1)=r(1) \\ f(-1)=r(-1) \end{cases} \Rightarrow \begin{cases} c=0 \\ a+b+c=-1 \\ a-b+c=1 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=-1 \\ c=0 \end{cases} \Rightarrow r(x)=-x$$

故 $A^{23}-2A^{13}=-A$.

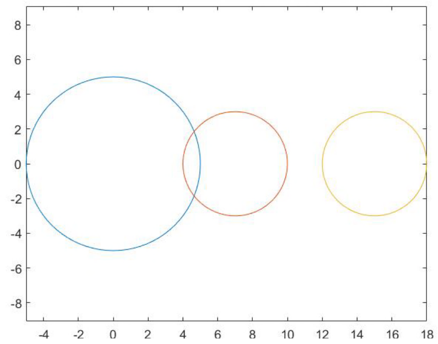
6.

求得A的特征值及盖尔圆估计为

$\lambda_1=-0.8182$ $\qquad G_1=|z|<5$

$\lambda_2=7.3473$ $\qquad G_2=|z-7|<3$

$\lambda_3=15.4709$ $\qquad G_3=|z-15|<3$



$G_1-G_2-G_3$

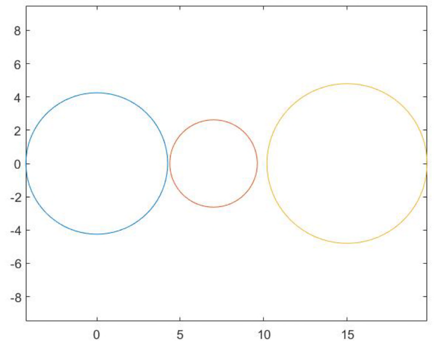
令 $D_1=\mathrm{diag}(\frac{1}{2},\frac{1}{2},\frac{4}{5})$

$$B_1=D_1AD_1^{-1}=\begin{bmatrix} 0 & 3 & \frac{5}{4} \\ 2 & 7 & \frac{5}{8} \\ \frac{8}{5} & \frac{16}{5} & 15 \end{bmatrix}$$

$G_1'=|z|<4.25$

$G_2'=|z-7|<2.625$

$G_3'=|z-15|<4.8$



$G_1'-G_2'-G_3'$

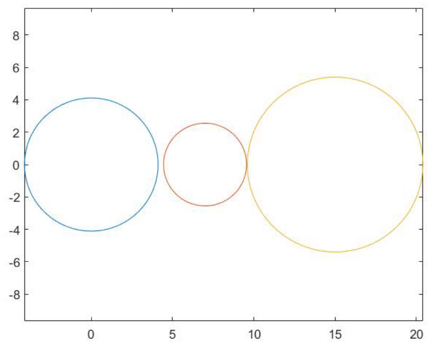
令 $D_2=\mathrm{diag}(\frac{1}{2},\frac{1}{2},\frac{9}{10})$

$$B_2=D_2AD_2^{-1}=\begin{bmatrix} 0 & 3 & \frac{10}{9} \\ 2 & 7 & \frac{5}{9} \\ \frac{9}{5} & \frac{18}{5} & 15 \end{bmatrix}$$

$G_1''=|z|<4.11$

$G_2''=|z-7|<2.56$

$G_3''=|z-15|<5.4$



$G_1''-G_2''-G_3''$

7. 代码和结果请见matlab发布文件

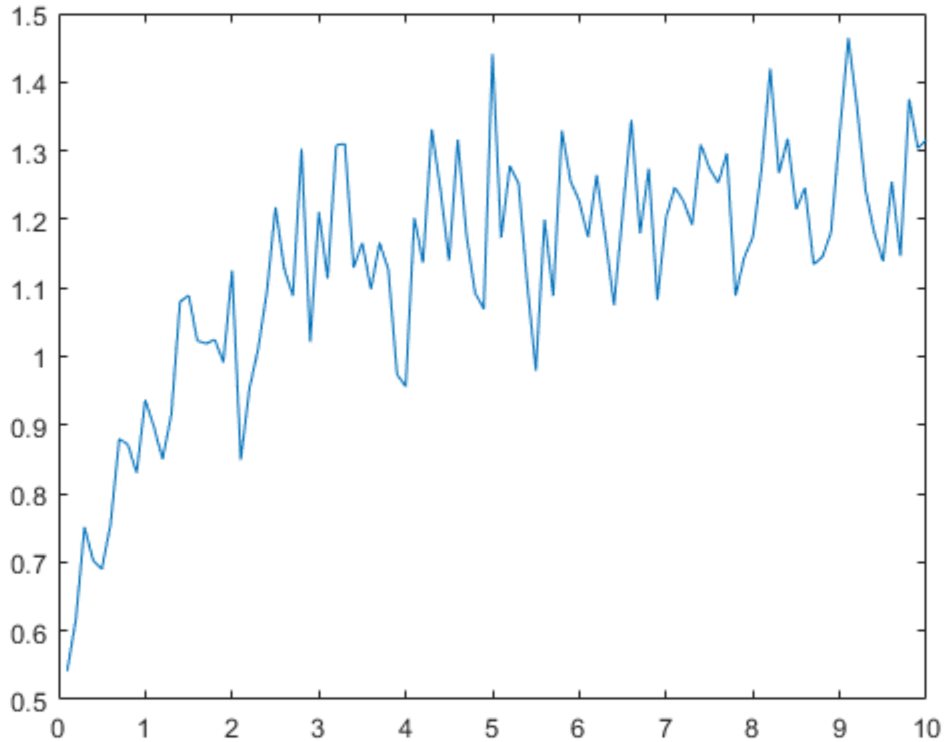
(1)

```
clear all;clc;close all;
N=100;      %signal length
sigma=0;
m = N/2;
L=N-m;
s_amp=[1.31*exp(1i*pi/4),2.07*exp(1i*pi/3),1.88*exp(1i*pi/5)];
s_omega=[0.12*pi,0.37*pi,0.72*pi];
MSE=[];

for t=1:100
    sigma=sigma+0.1;
    tmp=0;
    for K=1:200
        x=zeros(1,N);%initialize
        w = sqrt(sigma)*randn(1,N);
        n = [1:N];
        for slen=1:length(s_omega)
            x = x+s_amp(slen)*exp(1j*s_omega(slen)*n)   ;
        end
        x=x+w;
        for n = 1:L
            X(:,n) = x(n:(n+m-1));
        end
        for n = 1:L
            Y(:,n) = x((n+1):(n+m));
        end
        %Rxx\Rxy
        Rxx = 0;
        for i = 1:L
            Rxx = Rxx+X(:,i)*X(:,i)';
        end
        Rxx = Rxx/L;
        Rxy = 0;
        for i = 1:L
            Rxy = Rxy+X(:,i)*Y(:,i)';
        end
        Rxy = Rxy/L;
        [A,B] = eig(Rxx);
        var = min(diag(B));
        I = eye(m);
        Z = diag(ones(1,m-1),-1);
        Cxx = Rxx - I*var;
        Cxy = Rxy - Z*var;
        [~,B] = eig(Cxx,Cxy);
        f=angle(diag(B));
        [~,fpos]=sort(abs(abs(diag(B))-1));
        f=f(fpos);
        fval=f(f>0);
        omega_est=sort(fval(1:length(s_amp)));
        tmp=tmp+sum((abs(omega_est-s_omega')).^2);
    end
    MSE=[MSE,tmp/K];
```

```
end

xax=1:length(MSE);
xax=xax/10;
plot(xax,MSE)
```



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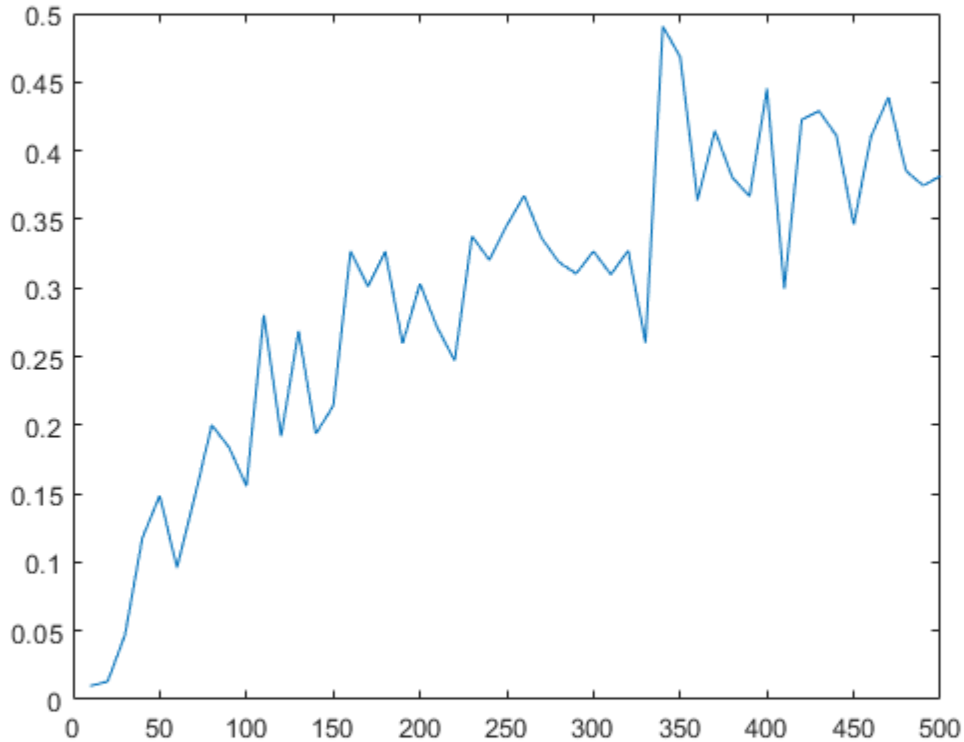
(2)

```
clear all;clc;close all;
N=0;     %signal length
sigma=0.01;
s_amp=[1.31*exp(1i*pi/4),2.07*exp(1i*pi/3),1.88*exp(1i*pi/5)];
s_omega=[0.12*pi,0.37*pi,0.72*pi];
MSE=[];

for t=1:50
    N=N+10;
    m=N/2;
    L=N-m;
    tmp=0;
    for K=1:200
        x=zeros(1,N);%initialize
        w = sqrt(sigma)*randn(1,N);
        for slen=1:length(s_omega)
            x = x+s_amp(slen)*exp(1j*s_omega(slen)*(1:N));
        end
        x=x+w;
        X=[];
        Y=[];
        for n = 1:L
            X(:,n) = x(n:(n+m-1));
        end
        for n = 1:L
            Y(:,n) = x((n+1):(n+m));
        end
        %Rxx\Rxy
        Rxx = 0;
        for i = 1:L
            Rxx = Rxx+X(:,i)*X(:,i)';
        end
        Rxx = Rxx/L;
        Rxy = 0;
        for i = 1:L
            Rxy = Rxy+X(:,i)*Y(:,i)';
        end
        Rxy = Rxy/L;
        [A,B] = eig(Rxx);
        var = min(diag(B));
        I = eye(m);
        Z = diag(ones(1,m-1),-1);
        Cxx = Rxx - I*var;
        Cxy = Rxy - Z*var;
        [~,B] = eig(Cxx,Cxy);
        f=angle(diag(B));
        [~,fpos]=sort(abs(abs(diag(B))-1));
        f=f(fpos);
        fval=f(f>0);
        omega_est=sort(fval(1:length(s_amp)));
        tmp=tmp+sum((abs(omega_est-s_omega')).^2);
    end
```

```
    MSE=[MSE,tmp/K];
end

figure,
plot((1:length(MSE))*10,MSE)
```



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