1.

令 $A=\begin{bmatrix}0.1 & 0.7\\ 0.3 & 0.6\end{bmatrix}$

$\|A\|_\infty=0.9<1$，故该幂级数收敛，$\sum_{k=0}^{\infty}A^k=(I-A)^{-1}=\begin{bmatrix}\frac{8}{3} & \frac{14}{3}\\ 2 & 6\end{bmatrix}$

2.

(1) $C^2=\begin{bmatrix}A & B\\ 0 & I\end{bmatrix}\begin{bmatrix}A & B\\ 0 & I\end{bmatrix}=\begin{bmatrix}A^2 & (A+I)B\\ 0 & I\end{bmatrix}$

$C^3=\begin{bmatrix}A & B\\ 0 & I\end{bmatrix}\begin{bmatrix}A^2 & (A+I)B\\ 0 & I\end{bmatrix}=\begin{bmatrix}A^3 & (A^2+A+I)B\\ 0 & I\end{bmatrix}$

以此类推，$C^k=\begin{bmatrix}A^k & (A^{k-1}+\cdots+A+I)B\\ 0 & I\end{bmatrix}$

$\Rightarrow \lim_{k\to\infty}C^k=\begin{bmatrix}M & (I-A)^{-1}B\\ 0 & I\end{bmatrix}$

(2) $A=\begin{bmatrix}0.2 & 0\\ 0.3 & 0.7\end{bmatrix}$ $B=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$ $M=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ $(I-A)^{-1}=\begin{bmatrix}\frac{5}{4} & 0\\ \frac{5}{4} & \frac{10}{3}\end{bmatrix}$ $(I-A)^{-1}B=\begin{bmatrix}\frac{5}{4} & \frac{5}{4}\\ \frac{5}{4} & \frac{95}{12}\end{bmatrix}$

$C=\begin{bmatrix}A & B\\ 0 & I\end{bmatrix}$，$\lim_{k\to\infty}C^k=\begin{bmatrix}0 & 0 & \frac{5}{4} & \frac{5}{4}\\ 0 & 0 & \frac{5}{4} & \frac{95}{12}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

3.

(1) $A = PJP^{-1}$，其中 $J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ $P = \begin{bmatrix} 4 & 2 & 1 \\ -8 & 0 & 0 \\ 16 & -8 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0 & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{8} \\ 1 & 1 & \frac{1}{4} \end{bmatrix}$

$$f(A) = e^{tA} = Pf(J)P^{-1} = P\begin{bmatrix} f(-2) & f'(-2) & \frac{1}{2!}f''(-2) \\ 0 & f(-2) & f'(-2) \\ 0 & 0 & f(-2) \end{bmatrix}P^{-1} = e^{-2t}\begin{bmatrix} 2t^2+2t+1 & 2t^2+t & \frac{1}{2}t^2 \\ -4t^2 & -4t^2+2t+1 & -t(t-1) \\ 8t(t-1) & 4t(2t-3) & 2t^2-4t+1 \end{bmatrix}$$

(2)

$A = CDC^{-1}$，其中 $D = \mathrm{diag}(1,5,1)$，$C = \begin{bmatrix} -\frac{379}{419} & \frac{780}{1351} & \frac{285}{6962} \\ \frac{379}{1257} & \frac{780}{1351} & -\frac{245}{529} \\ \frac{379}{1257} & \frac{780}{1351} & \frac{1081}{1221} \end{bmatrix}$

$$A^{1000} = C\,D^{1000}C^{-1}$$

(3) 令 $f(\lambda) = |\lambda I - A| = \lambda^3 - \lambda^2$　$f(A) = A^3 - A^2 = 0 \Rightarrow A^2 = A^3$

而 $A^4 = A^3\cdot A = A^2\cdot A = A^3 = A^2$，$A^5 = A^3\cdot A^2 = A^2\cdot A^2 = A^4 = A^2$，类推可得 $A^k = A^2$，$k \geq 2$

故 $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$

$= I + A + A^2\left(\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}A^k + \cdots\right)$

$= I + A + (e-2)A^2$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + (e-2)\begin{bmatrix} & & \\ & & \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 3-e & e-2 & e \end{bmatrix}$$

$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 + \cdots + (-1)^k\cdot\frac{1}{(2k+1)!}A^{2k+1} + \cdots$

$= A + A^3\left(-\frac{1}{3!} + \frac{1}{5!} + \cdots + (-1)^k\cdot\frac{1}{(2k+1)!} + \cdots\right)$

$$= A + (\sin 1 - 1)A^3 = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + (\sin 1 - 1)\begin{bmatrix} & & \\ & & \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 2-\sin 1 & \sin 1 - 1 & \sin 1 \end{bmatrix}$$

4.

$\|e^A\| = \|\sum_{k=0}^{\infty} \frac{A^k}{k!}\|$, $e^{\|A\|} = \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$

$\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\| \le \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}$ 是显然的，故$\|e^A\| \le e^{\|A\|}$

5.

$f(x) = \|Ax-b\|_2^2 = (Ax-b)^T(Ax-b) = (x^TA^T - b^T)(Ax-b) = x^TA^TAx - x^TA^Tb - b^TAx + b^Tb$

$\frac{df}{dx} = 2A^TAx - A^Tb - (b^TA)^T = 2(A^TAx - A^Tb)$

6.

$d(AXB) = Ad(X)B \quad d(\mathrm{vec}(AXB)) = (B^T \otimes A)d(\mathrm{vec}X)$

$\Rightarrow D(AXB) = B^T \otimes A$

$d(AX^{-1}B) = Ad(X^{-1})B = -AX^{-1}d(X)X^{-1}B \quad d(\mathrm{vec}(AX^{-1}B)) = -[(X^{-1}B)^T \otimes (AX^{-1})]d(\mathrm{vec}X)$

$\Rightarrow D(AX^{-1}B) = -(X^{-1}B)^T \otimes (AX^{-1})$

7.

$df(x) = d(a^TXX^Ta) = d(\mathrm{tr}(a^TXX^Ta)) = \mathrm{tr}(X^Taa^TdX) + \mathrm{tr}(aa^TXdX^T)$

$\frac{\partial f(x)}{\partial(\mathrm{vec}X)} = \mathrm{vec}(X^Taa^T)^T + \mathrm{vec}(aa^TX) = 2\mathrm{vec}(aa^TX) = 2(I \otimes aa^T)\mathrm{vec}X$

$\Rightarrow H[f(x)] = \frac{\partial}{\partial(\mathrm{vec}X)^T}\left(\frac{\partial f(x)}{\partial(\mathrm{vec}X)}\right) = \frac{\partial}{\partial(\mathrm{vec}X)^T}[2(I \otimes aa^T)\mathrm{vec}X] = 2(I \otimes aa^T)$

8.

$$d(F^+F) = d(F^+FF^+F) = (dF^+F)F^+F + F^+Fd(F^+F) = F^+Fd(F^+F) + [F^+Fd(F^+F)]^T$$

$$dF = dFF^+F = (dF)F^+F + F(dF^+F) \Rightarrow F(dF^+F) = (dF)(I - F^+F)$$，代入上式得到

$$d(F^+F) = F^+(dF)(I - F^+F) + [F^+(dF)(I - F^+F)]^T$$

$$d(FF^+) = d(FF^+FF^+) = d(FF^+)FF^+ + FF^+d(FF^+) = (dFF^+)FF^+ + [(dFF^+)FF^+]^T$$

$$dF = dFF^+F = (dFF^+)F + FF^+dF \Rightarrow (dFF^+)F = (I - FF^+)dF$$

$$d(FF^+) = (I - FF^+)(dF)F^+ + [(I - FF^+)(dF)F^+]^T$$

9.

(1) 设 $X = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}$，则 $X^T = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1m} & \cdots & x_{nm} \end{bmatrix}$

$$f = tr(X^TX) = \sum_{i=1}^{n} x_{i1}^2 + \sum_{i=1}^{n} x_{i2}^2 + \cdots + \sum_{i=1}^{n} x_{im}^2 = \sum_{j=1}^{m}\sum_{i=1}^{n} x_{ij}^2$$

$$\frac{df}{dX} = \left(\frac{\partial f}{\partial x_{ij}}\right)_{n\times m} = 2X$$

(2) 设 $A = (a_{ij})_{m\times n}$，$X = (x_{ij})_{n\times m}$，则 $AX = \left(\sum_{k=1}^{n} a_{ik}x_{kj}\right)_{m\times m}$

$$f = tr(AX) = \sum_{k=1}^{n} a_{1k}x_{k1} + \sum_{k=1}^{n} a_{2k}x_{k2} + \cdots + \sum_{k=1}^{n} a_{mk}x_{km}$$

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial x_{ij}}\right)_{n\times m} = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} = A^T$$