Bound on distance for NND

For the 4*4 covariance matrix, the determinant looks like-

$$\begin{aligned}
f(d) &= 1 + e^{-2(\phi_{11}+\phi_{22})d} + 4\sigma_{12}^2(e^{-(\phi_{12}+\phi_{11})d} + e^{-(\phi_{12}+\phi_{22})d}) + \sigma_{12}^4 + \sigma_{12}^4 e^{-4\phi_{12}d} \\
&- e^{-2\phi_{11}d} - e^{-2\phi_{22}d} - 2(e^{-2\phi_{12}d}+1)\sigma_{12}^2 e^{-(\phi_{11}+\phi_{22})d} - 2\sigma_{12}^2 e^{-2\phi_{12}d} - 2\sigma_{12}^2 - 2\sigma_{12}^4 e^{-2\phi_{12}d} \\
&= (1-\sigma_{12}^2)^2 + 4\sigma_{12}^2(e^{-(\phi_{12}+\phi_{11})d} + e^{-(\phi_{12}+\phi_{22})d}) + (e^{-(\phi_{11}+\phi_{22})d} - \sigma_{12}^2 e^{-2\phi_{12}d})^2 \\
&- e^{-2\phi_{11}d} - e^{-2\phi_{22}d} - 2\sigma_{12}^2 e^{-(\phi_{11}+\phi_{22})d} - 2\sigma_{12}^2 e^{-2\phi_{12}d} - 2\sigma_{12}^4 e^{-2\phi_{12}d}
\end{aligned}$$

Clearly, you can see, $\lim_{d\to\infty} f(d) = 1 - 2\sigma_{12}^2 + \sigma_{12}^4 = (1-\sigma_{12}^2)^2 > 0$

Now, as, $0 < \sigma_{12}^2 < 1$, $f(d) > (1-\sigma_{12}^2)^2 - e^{-2\phi_{11}d} - e^{-2\phi_{22}d} - 2e^{-(\phi_{11}+\phi_{22})d} - 2e^{-2\phi_{12}d} - 2e^{-2\phi_{12}d} = g(d)$. \
Clearly, $g(d)$ is strictly increasing function of d and $\lim_{d\to\infty} g(d) = (1-\sigma_{12}^2)^2 > 0$.

For fixed $\phi_{11}, \phi_{12}, \phi_{22}, \sigma_{12}$, we can find d_0 such that $g(d) > 0 \forall d > d_0$. And, as $f(d) > g(d)$ always, we get, $f(d) > 0 \forall d > d_0$.

Now for practical purposes, if we can't estimate σ_{12} before, then we can assume $0 < \sigma_{12}^2 < .975$, say, then we can pick our favourite ϕ_{12}, take $g(d) = .025^2 - e^{-2\phi_{11}d} - e^{-2\phi_{22}d} - 2e^{-(\phi_{11}+\phi_{22})d} - 2e^{-2\phi_{12}d} - 2e^{-2\phi_{12}d}$. Observe that $g(d)$ is now a known function of d as we have estimated ϕ_{11},ϕ_{22} before and we can run numerical methods to find d_0 now. (Ideally, we should pick ϕ_{12} such that d_0 is minimum)