MacNeille completions

David Álvarez Lombardi

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Under the supervision of
Nick Bezhanishvili and Rodrigo Nicolau Almeida

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2 Basic definitions

Definition 2.1 (Partial order) - A *partial order* (or *poset*) is a tuple $(P, \leqslant)$ where P is a set and $\leqslant$ is a relation $\leqslant \subseteq P \times P$ that satisfies the following formulas for all $x, y, z \in P$.

- $x \leqslant x$
- $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$
- $x \leqslant y$ and $y \leqslant x$ implies $x = y$

We denote the class of partial orders by **POSet**.

Definition 2.2 (Upset and downset) - Given a partial order $(P, \leqslant)$ and a subset $S \subseteq P$, the *upset* of S, denoted by $\uparrow S$, is defined as follows.

$$\begin{aligned} \uparrow : \wp(P) &\longrightarrow \wp(P) \\ S &\longmapsto \{x \in P \mid (\exists s \in S)(s \leqslant x)\} \end{aligned}$$

The *downset* of S, denoted by $\downarrow S$, is defined symmetrically as follows.

$$\begin{aligned} \downarrow : \wp(P) &\longrightarrow \wp(P) \\ S &\longmapsto \{x \in P \mid (\exists s \in S)(x \leqslant s)\} \end{aligned}$$

Note that when S is a singleton $\{x\}$, it is common to write $\downarrow x$ instead of $\downarrow\{x\}$.

Definition 2.3 (Upper and lower bounds) - Given a partial order $(P, \leqslant)$ and a subset $S \subseteq P$, the *upper bounds* of S, denoted by $\mathbf{u}(S)$, is defined as follows.

$$\begin{aligned} \mathbf{u} : \wp(P) &\longrightarrow \wp(P) \\ S &\longmapsto \{x \in P \mid (\forall s \in S)(s \leqslant x)\} \end{aligned}$$

The *lower bounds* of S, denoted by $\mathbf{l}(S)$, is defined symmetrically as follows.

$$\begin{aligned} \mathbf{l} : \wp(P) &\longrightarrow \wp(P) \\ S &\longmapsto \{x \in P \mid (\forall s \in S)(x \leqslant s)\} \end{aligned}$$

Observe how the definitions of $\mathbf{u}$ and $\mathbf{l}$ are, in some sense, *logically dual* to the definitions of $\uparrow$ and $\downarrow$ and *order dual* to each other.

Fact ? - Given a poset $\mathfrak{P} := (P, \leqslant)$ and a subset $S \subseteq P$,

$$\mathbf{u}(S) = \bigcap {\uparrow}[S] \quad \text{and} \quad \mathbf{l}(S) = \bigcap {\downarrow}[S]$$

Proof -

$$\mathbf{u}(S) = \{x \in P \mid (\forall s \in S)(s \leqslant x)\} = \{x \in P \mid (\forall s \in S)(x \in {\uparrow}s)\} = \bigcap {\uparrow}[S]$$

$$\mathbf{l}(S) = \{x \in P \mid (\forall s \in S)(x \leqslant s)\} = \{x \in P \mid (\forall s \in S)(x \in {\downarrow}s)\} = \bigcap {\downarrow}[S]$$

■

Observation ? - Given a poset $\mathfrak{P}$, the pair of operations $(\mathbf{u}, \mathbf{l})$ form a *Galois connection* on $\wp(\mathfrak{P})$. Because of this fact, the following formulas hold for any subsets $S, T \subseteq \mathfrak{P}$.

- $S \subseteq \mathbf{lu}(S)$ and $S \subseteq \mathbf{ul}(S)$
- $S \subseteq T \Rightarrow \mathbf{u}(S) \supseteq \mathbf{u}(T)$ and $S \subseteq T \Rightarrow \mathbf{l}(S) \supseteq \mathbf{l}(T)$
- $\mathbf{ulu}(S) = \mathbf{u}(S)$ and $\mathbf{lul}(S) = \mathbf{l}(S)$

These facts will not be proven, but will be used extensively. For more information on Galois connections, see **Finish me.**

Fact ? - Given a poset $(P, \leqslant)$, an element $x \in P$, and an arbitrary subset $S \subseteq P$,

$$S \subseteq {\downarrow}x \Leftrightarrow x \in \mathbf{u}(S)$$

Proof -

$$S \subseteq {\downarrow}x \Leftrightarrow (\forall s \in S)(s \in {\downarrow}x) \Leftrightarrow (\forall s \in S)(s \leqslant x) \Leftrightarrow x \in \mathbf{u}(S)$$

■

Lemma ? - Given a poset $(P, \leqslant)$ and arbitrary subsets $S, T \subseteq P$,

$$S \subseteq T \Rightarrow \mathbf{lu}(S) \subseteq \mathbf{lu}(T)$$

Proof - Assuming that $S \subseteq T$, the containment $\mathbf{lu}(S) \subseteq \mathbf{lu}(T)$ will be shown via the contrapositive. Consider an element $x \notin \mathbf{lu}(T)$. This implies there is some element $u \in \mathbf{u}(T)$ such that $x \not\leqslant u$. But since $S \subseteq T$, by *Observation ?*, we know that $\mathbf{u}(T) \subseteq \mathbf{u}(S)$, so $u \in \mathbf{u}(S)$. Recalling that $x \not\leqslant u$, we can conclude that $x \notin \mathbf{lu}(S)$. ■

Example 2.1 - Consider the following poset P and the subset $S := \{x_1, x_2\}$.

$$
\begin{tikzcd}[column sep=small]
 & x_4 \arrow[d, dash] & \\
 & x_3 & \\
x_1 \arrow[ur, dash] & & x_2 \arrow[ul, dash] \\
 & x_0 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
$$

Here we have $\mathbf{u}(S) = \{x_3, x_4\}$, $\mathbf{lu}(S) = \{x_0, x_1, x_2, x_3\}$, $\mathbf{l}(S) = \{x_0\}$, $\mathbf{ul}(S) = P$. Note how, even in this simple case, $\mathbf{l}$ and $\mathbf{u}$ do *not* commute.

Definition 2.4 (Supremum and infemum) - Given a partial order $(P, \leqslant)$ and a subset $S \subseteq P$, the *supremum* (or *least upper bound*) of S, denoted by $\bigvee S$, is the unique element $u \in \mathbf{u}(S)$ such that $u' \in \mathbf{u}(S) \Rightarrow u \leqslant u'$. The *infemum* (or *greatest lower bound*) of S, denoted by $\bigwedge S$, is defined symmetrically as the unique element $l \in \mathbf{l}(S)$ such that $l' \in \mathbf{l}(S) \Rightarrow l' \leqslant l$. Note that the existence of such elements is *not* guaranteed in general, but when they exist it quickly follows that they are, indeed, unique.

Definition 2.5 (Complete partial order) - We call a partial order $(P, \leqslant)$ *complete* when $\bigvee$ and $\bigwedge$ are well-defined functions $\wp(P) \to P$. Another way of stating this is that a poset is *complete* when, given an arbitrary subset $S \subseteq P$, both $\bigvee S$ and $\bigwedge S$ *exist* in P (recall the comment on *uniqueness* above). Note that it is well-known that it is sufficient for *one* of these functions to be well-defined and it follows that the other must be as well.

Definition 2.7 (MacNeille completion) - Given a partial order $\mathfrak{P} := (P, \leqslant)$ the *MacNeille completion* of $\mathfrak{P}$, denoted by $\mathfrak{P}^*$, is defined as follows.

$$
\begin{aligned}
[\,\cdot\,]^* : \mathbf{POSet} &\longrightarrow \mathbf{POSet} \\
\mathfrak{P} &\longmapsto (\{S \subseteq P \mid \mathbf{lu}(S) = S\}, \subseteq)
\end{aligned}
$$

That this is well-defined follows from the fact that $\subseteq$ induces a partial order on any class of sets. Intuitively, the MacNeille completion of a poset is simply the fixpoints of the operation-composition $\mathbf{l} \circ \mathbf{u}$.

Example 2.3 - Returning to the poset $\mathfrak{P}$ and the subset S defined in *Example ?*, we can now observe that $S \notin \mathfrak{P}^*$ because S is *not* a fixpoint of $\mathbf{lu}$. Consider, however, the subset $T := \{x_0, x_1\}$. Then we have $\mathbf{u}(T) = \{x_1, x_3, x_4\}$ and

$\mathbf{lu}(T) = \{x_0, x_1\}$, implying that $T = \mathbf{lu}(T)$ and, therefore, $T \in \mathfrak{P}^*$.

Lemma ? - Given a poset $\mathfrak{P} := (P, \leqslant)$ and a subset $S \subseteq \mathfrak{P}^*$,

$$\mathbf{lu}\left(\bigcup S\right) \in \mathfrak{P}^* \quad \text{and} \quad \bigcap S \in \mathfrak{P}^*$$

Proof - The first result is clear as it is a fact of Galois connections that $\mathbf{lulu}(T) = \mathbf{lu}(T)$ for any T (see *Observation ?*), so it remains only to show that second claim. Recall that to show that $\bigcap S \in \mathfrak{P}^*$, it suffices to show that $\mathbf{lu}(\bigcap S) = \bigcap S$. The inclusion from right-to-left follows from *Observation ?*, so I will show that $\mathbf{lu}(\bigcap S) \subseteq \bigcap S$ via the contrapositive. Consider an element $x \in P$ such that $x \notin \bigcap S$. This implies there exists some $S_i \in S$ such that $x \notin S_i$. Since $S_i \in \mathfrak{P}^*$, we know that $S_i = \mathbf{lu}(S_i)$, implying that $x \notin \mathbf{lu}(S_i)$. So there is some element $u \in \mathbf{u}(S_i)$ such that $x \not\leqslant u$. Now simply observe that $\mathbf{u}(S_i) \subseteq \mathbf{u}(\bigcap S)$ (because $\mathbf{u}$ is antitone on $\wp(P)$ as established in *Observation ?* and $\bigcap S \subseteq S_i$), so $u \in \mathbf{u}(\bigcap S)$. Since $x \not\leqslant u$, it must be the case that $x \notin \mathbf{lu}(\bigcap S)$. ■

Theorem 2.1 - Given a poset $\mathfrak{P} := (P, \leqslant)$, $\mathfrak{P}^*$ is a complete poset with supremum $\mathbf{lu}(\bigcup S)$ and infemum $\bigcap S$ for some $S \subseteq \mathfrak{P}^*$.
Proof - The case of the infemum follows directly from *Lemma ?* along with the fact that there can be no greater lower-bound for S than its intersection (for set-theoretic reasons). So $\bigcap S$ indeed exists and is the infemum of S. Now for the supremum, I have shown in *Lemma ?* that $\mathbf{lu}(\bigcup S) \in \mathfrak{P}^*$, but it remains to show that this is the *least upper bound* of S. Now it is quite clear that $\bigcup S$ is an upper bound for S (set-theoretically), and we know, by *Observation ?*, that $\bigcup S \subseteq \mathbf{lu}(\bigcup S)$, so $\mathbf{lu}(\bigcup S)$ must also be an upper bound for S. Now all that remains is to show that this is the *least* upper bound Consider another upper bound for S denoted by T. Observe that (for set-theoretic reasons) $\bigcup S \subseteq T$. This implies that $\mathbf{lu}(\bigcup S) \subseteq \mathbf{lu}(T)$. Now simply recall that $\mathbf{lu}(T) = T$, implying that $\mathbf{lu}(\bigcup S) \subseteq T$. ■

Theorem 2.2 - There exists a poset-embedding of $\mathfrak{P}$ into $\mathfrak{P}^*$.
Finish me. Rewrite this.
Proof - Consider a map $h : \mathfrak{P} \to \mathfrak{P}^*$ such that $h : x \mapsto {\downarrow}x$. One can quickly check that ${\downarrow}x \in \mathfrak{P}^*$ (because $\mathbf{u}({\downarrow}x) = {\uparrow}x$ so $\mathbf{lu}(\{x\}) = {\downarrow}x$), implying that h is well-defined. Now consider two elements $x, y \in P$ such that $x \leqslant y$. Simply observe that this is the case iff ${\downarrow}x \subseteq {\downarrow}y$, implying that $x \leqslant y \Leftrightarrow h(x) \subseteq h(y)$, further implying that h is a poset-embedding. ■

Remark 2.2 - The implication of *Theorem ?* and *Theorem ?* is that the MacNeille completion provides us a method for *completing* a poset while preserving any and all of the structure present in it. The completion is even more interesting as further investigation will reveal that the MacNeille completion of a poset is a *minimal* completion in that any other completion must *contain* the MacNeille completion. Intuitively we can say the MacNeille completion preserves all structure of the original poset while introducing as little new structure as possible.

Definition 2.8 (Closed class) - Given a class of structures $\mathbf{K}$ extending **POSet**, we will say, for the purposes of this paper, that $\mathbf{K}$ is a *closed class* if whenever $\mathfrak{X} \in \mathbf{K}$, we also have $\mathfrak{X}^* \in \mathbf{K}$.

Remark 2.3 - We have shown in *Theorem* **?** that **POSet** is, itself, a closed class, but it is far from obvious whether an arbitrary class-extension $\mathbf{K}$ should also be closed. In theory, performing the MacNeille completion on a structure could fail to preserve some crucial property of said structure, carrying it outside of its original class. This observation motivates the remainder of this paper. In the next section, we will survey a variety of extensions of **POSet**, documenting the construction of their MacNeille completions and demonstrating the affirmation or negation of their closed-ness. In the subsequent sections, we will do the same for **Finish me.**

3 Closed classes

Here we will look at several extensions of **POSet** that are well-known to be closed under the MacNeille completion. These include both historically significant classes as well as classes whose merits come from their ubiquity in mathematics, logic, and philosophy.

3.1 Lattices

Definition ? (Lattice) - A *lattice* is a poset $(P, \leqslant)$ where $\bigvee$ and $\bigwedge$ are defined for all $S \in \wp(P)$ such that $|S| = 2$. Such structures are equivalently (and perhaps more intuitively) represented using infix binary operations $\vee, \wedge : P \times P \to P$ where $x \vee y = \bigvee\{x, y\}$ and likewise for $\wedge$. We denote the class of lattices by **Latt**.

Fact ? - Given a lattice $\mathfrak{L} := (L, \leqslant)$ and a subset $S \subseteq L$, if $|S| = n$ for some $n \geqslant 1$, then $\bigvee S$ and $\bigwedge S$ exist in L.
Proof - This will only be shown for $\bigvee$ as the proof for $\bigwedge$ is entirely symmetrical. This will further be shown via induction on the cardinality of S. (Base case) Consider the case where $|S| = 1$, implying $S = \{x\}$ for some $x \in L$. Then clearly $x \in \mathbf{u}(S)$ and is the least such element, so $\bigvee S = x$. (Inductive case) Assume that $\bigvee$ is defined for all subsets of cardinality k and assume that $|S| = k + 1$, so

$S = \{s_i\}_{i=1}^{k+1}$. Consider the element $u := s_{k+1} \vee \bigvee\{s_i\}_{i=1}^{k}$. We know that u exists because, by supposition, $\bigvee$ is defined on subsets of cardinality 2 and k. Now it simply suffices to show that u is the *supremum* of S. We know that $u \in \mathbf{u}(S)$ because it is greater than s_{k+1} and $\bigvee\{s_i\}_{i=1}^{k}$, implying, by transitivity, that it is greater than every s_i. Now consider some u' in $\mathbf{u}(S)$. Then $u' \in \mathbf{u}(\{s_i\}_{i=1}^{k})$, implying that $u' \geqslant \bigvee\{s_i\}_{i=1}^{k}$. But u' in $\mathbf{u}(S)$ also implies that $u' \geqslant s_{k+1}$. Combining these facts, we conclude that $u \leqslant u'$. ■

Theorem ? - The class **Latt** is closed under the MacNeille completion.
Proof - This was implicitly proven in *Theorem ?*, but I will sketch the reasoning again. Given a lattice $\mathfrak{L} \in \mathbf{Latt}$, we know that $\mathfrak{L} \in \mathbf{POSet}$, so by *Theorem ?* we know that $\mathfrak{L}^*$ is a complete poset, implying that $\mathfrak{L}^*$ is a lattice. ■

3.2 Bounded lattices

Definition ? (Top and bottom elements) - Given a partial order $(P, \leqslant)$, the *top element* of P, denoted by $\top$, is the unique element in $\mathbf{u}(P)$. The *bottom element* of P, denoted by $\bot$, is the unique element in $\mathbf{l}(P)$. Note that the existence of such elements is *not* guaranteed in general, but when they exist it quickly follows that they are, indeed, unique.

Definition ? (Bounded poset) - We call a poset $(P, \leqslant)$ *bounded* if it has both a bottom and top element.

Note - We denote the class of bounded *lattices* by **BLatt**.

Observation ? - Given a poset $(P, \leqslant)$, $\bigvee$ and $\bigwedge$ are defined on subsets of cardinality 0 iff P is bounded. This is the case because $\mathbf{u}(\varnothing) = P$ (—don't think about this too hard—), so $\mathbf{u}(\varnothing)$ has a least element iff P has a bottom element $\bot$. A symmetrical fact is true for $\bigwedge \varnothing$ and $\top$.

Fact ? - Given a bounded poset $\mathfrak{P} := (P, \leqslant)$ with top and bottom elements $\top$ and $\bot$,

$$\mathbf{lu}(P) = P \quad \text{and} \quad \mathbf{lu}(\{\bot\}) = \{\bot\}$$

Proof - Simply observe that $\mathbf{lu}(P) = \mathbf{l}(\{\top\}) = P$ and $\mathbf{lu}(\{\bot\}) = \mathbf{l}(P) = \{\bot\}$.

Theorem ? - The class **BLatt** is closed under the MacNeille completion.
Proof - Given a bounded lattice $\mathfrak{L} \in \mathbf{BLatt}$ with top and bottom elements $\top$ and $\bot$ respectively, we know that $\mathfrak{L} \in \mathbf{Latt}$, so, by *Theorem ?*, $\mathfrak{L}^* \in \mathbf{Latt}$, so it remains only to show that $\mathfrak{L}^*$ is *bounded*. Recall that we have shown in *Fact*

? that $L, \{\bot\} \in \mathfrak{L}^*$. These will be the top and bottom elements of $\mathfrak{L}^*$ respectively. Clearly L serves as a top element for $\mathfrak{L}^*$, but it remains to show that $(\forall S \in \mathfrak{L}^*)(\{\bot\} \subseteq S)$. But simply observe that for any subset $T \subseteq L$, we have $\bot \in \mathbf{l}(T)$, so, in particular, given an arbitrary $S \in \mathfrak{L}^*$, we know that $\bot \in \mathbf{lu}(S)$, so $\bot \in S$, implying that $\{\bot\} \subseteq S$. ■

Remark ? - Interestingly, the MacNeille completion not only *preserves* bounds, but it actually *generates them*. This is because given a lattice $\mathfrak{L} := (L, \leqslant)$ *without* bounds, we have $L \in \mathfrak{L}^*$ (because $\mathbf{lu}(L) = \mathbf{l}(\varnothing) = L$) and $\varnothing \in \mathfrak{L}^*$ (because $\mathbf{lu}(\varnothing) = \mathbf{l}(L) = \varnothing$) which obviously serve as a top and bottom for $\mathfrak{L}^*$. As a historically significant example, consider the partial order $\mathbb{Q}$ which is a dense linear order *without* bounds. The MacNeille completion of $\mathbb{Q}$ is not $\mathbb{R}$ (as one could expect), but $\mathbb{R} \cup \{\varnothing, \mathbb{Q}\}$ interpreted as $\varnothing = -\infty$ and $\mathbb{Q} = \infty$. To avoid adding endpoints, Dedekind cuts require that both 'sides' of the cut are non-empty. However, this weakens the constructed order which, instead of being truly *complete*, is only *Dedekind complete*, which is completeness only for subsets S such that $\mathbf{u}(S) \neq \varnothing$.

3.3 Interlude

Remark ? - Here we stop to define a *distributive lattice*. We will see that this class does not belong in the section because it is not closed, but the definition is relevant for superclasses which *are* closed.

Definition ? (Distributive lattice) - A lattice $(L, \leqslant)$ is called *distributive* if for all $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{and} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

We denote the class of distributive lattices by **DistLatt**.

3.4 Heyting algebras

Definition ? (Heyting algebra) - A *Heyting algebra* $\mathfrak{H}$ is a tuple $(H, \leqslant, \rightarrow)$ where $(H, \leqslant)$ is a bounded distributive lattice and $\rightarrow$ is a binary operation for which the following holds for all $x, y, z \in H$.

$$x \wedge y \leqslant z \Leftrightarrow x \leqslant y \rightarrow z$$

Definition ? (MacNeille completion of a Heyting algebra) - Given a Heyting algebra $\mathfrak{H} := (H, \leqslant, \rightarrow)$, we define the *MacNeille completion of* $\mathfrak{H}$ to be $\mathfrak{H}^* :=$

$(H^*, \subseteq, \overset{*}{\rightarrow})$ where H^* is defined in the normal way and $\overset{*}{\rightarrow}$ is defined as follows.

$$\begin{aligned} \overset{*}{\rightarrow} : H^* \times H^* &\longrightarrow H^* \\ (S, T) &\longmapsto \{x \in H \mid (\forall s \in S)(s \wedge x \in T)\} \end{aligned}$$

That this is map is well defined will follow from *Lemma ?*.

Lemma ? - Given a Heyting algebra $\mathfrak{H} := (H, \leqslant, \rightarrow)$ and sets $S, T \in \mathfrak{H}^*$,

$$\mathbf{lu}(S \overset{*}{\rightarrow} T) = S \overset{*}{\rightarrow} T$$

Proof - **Finish me.**

Lemma ? - Given a Heyting algebra $\mathfrak{H} := (H, \leqslant, \rightarrow)$ and sets $S, T \in \mathfrak{H}^*$,

$$S \wedge^* T \subseteq V \Leftrightarrow S \subseteq T \overset{*}{\rightarrow} V$$

Proof - **Finish me.**

Lemma ? - Given a Heyting algebra $\mathfrak{H} := (H, \leqslant, \rightarrow)$ and sets $S, T, U \in \mathfrak{H}^*$,

$$S \vee (T \wedge V) = (S \vee T) \wedge (S \vee V)$$

Proof - **Finish me.**

Theorem ? - The class **HA** is closed under the MacNeille completion. *Proof* - **Finish me.**

Remark ? - It is a known fact that whenever $(H, \leqslant, \rightarrow)$ is a Heyting algebra, the following is true for any $x, y \in H$.

$$x \rightarrow y = \bigvee \{z \in H \mid z \wedge x \leqslant y\}$$

That this is equal to $\overset{*}{\rightarrow}$ is demonstrated in *Fact ?*.

Fact ? - Given a Heyting algebra $\mathfrak{H} := (H, \vee, \wedge, \rightarrow, \perp, \top)$ and sets $S, T \in \mathfrak{H}^*$,

$$S \overset{*}{\rightarrow} T = \mathbf{lu} \bigcup \{V \in \mathfrak{H}^* \mid V \cap S \subseteq T\}$$

Proof - **Finish me.**

3.5 Boolean algebras

Theorem ? - The class **BA** is closed under the MacNeille completion with the following definitions. Given $\mathfrak{B} := (B, \leqslant) \in \mathbf{BA}$,

$$\begin{aligned} \neg^* : B^* &\longrightarrow B^* \\ S &\longmapsto S \overset{*}{\to} \{\bot\} \end{aligned}$$

Proof - Since we have already shown, in *Theorem ?*, that **HA** is closed, we know that $\neg^*$ is well-defined, so we need only check that $\mathfrak{B}$ satisfy the following equations: (1) $S \cap \{\bot\} = \{\bot\}$, (2) $\mathbf{lu}(S \cup B) = B$, (3) $S \cap \neg^* S = \{\bot\}$, and (4) $\mathbf{lu}(S \cup \neg^* S) = B$. (1) Since $\{\bot\} \subseteq S$, we have $S \cap \{\bot\} = \{\bot\}$. (2) Since $S \subseteq B$, we have $S \cup B = B$, so $\mathbf{lu}(S \cup B) = \mathbf{lu}(B) = B$. (3) **Finish me.**

3.6 Modal algebras

Definition ? (Upper MacNeille completion of a modal algebra) - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$, we define the *upper MacNeille completion of* $\mathfrak{M}$ to be $\mathfrak{M}^* := (\mathfrak{B}^*, \diamondplus)$ where $\diamondplus$ is defined as follows.

$$\begin{aligned} \diamondplus : \mathfrak{B}^* &\longrightarrow \mathfrak{B}^* \\ S &\longmapsto \bigwedge\{\downarrow\Diamond x \mid S \subseteq \downarrow x\} \end{aligned}$$

That this is map is well defined will follow from *Lemma ?*.

Observation ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and an arbitrary subset $S \subseteq \mathfrak{M}$,

$$\diamondplus S = \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(x \in \mathbf{u}(S) \Rightarrow a \leqslant \Diamond x)\}$$

The set included in *Definition ?* was chosen because it makes more overt use of the embedding $\downarrow$, but the current set is often easier to work with.
Proof -

$$\begin{aligned} \diamondplus S &= \bigwedge\{\downarrow\Diamond x \mid S \subseteq \downarrow x\} \\ &= \bigcap\{\downarrow\Diamond x \mid S \subseteq \downarrow x\} && \text{(by } \textit{Theorem ?}\text{)} \\ &= \bigcap\{\downarrow\Diamond x \mid x \in \mathbf{u}(S)\} && \text{(by } \textit{Fact ?}\text{)} \\ &= \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(x \in \mathbf{u}(S) \Rightarrow a \in \downarrow\Diamond x)\} \\ &= \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(x \in \mathbf{u}(S) \Rightarrow a \leqslant \Diamond x)\} \end{aligned}$$

■

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and an arbitrary subset $S \subseteq \mathfrak{B}$,

$$a \in \mathbf{u}(S) \Rightarrow \Diamond a \in \mathbf{u}(\diamondplus S)$$

Proof - This will be shown via the contrapositive. Assume that $\Diamond a \notin \mathbf{u}(\diamondplus S)$, implying that there exists some $s \in \diamondplus S$ such that $s \not\leqslant \Diamond a$. Note that, since $s \in \diamondplus S$, we know that $(\forall x \in \mathfrak{B})(x \in \mathbf{u}(S) \Rightarrow s \leqslant \Diamond x)$. Now simply observe that, by the contrapositive of the previous formula and that fact that $s \not\leqslant \Diamond a$, we may conclude that $a \notin \mathbf{u}(S)$. ■

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and an arbitrary subset $S \subseteq \mathfrak{B}$,

$$\mathbf{lu}(\diamondplus S) = \diamondplus S$$

Proof - As always, one direction of the containment is immediate due to *Observation ?*, so it will suffice to show that $\mathbf{lu}(\diamondplus S) \subseteq \diamondplus S$ This will be shown via the contrapositive. Consider an element $a \in \mathfrak{B}$ such that $a \notin \diamondplus S$. This implies that there exists some $x \in \mathbf{u}(S)$ (by *Lemma ?*) and $a \not\leqslant \Diamond x$. But simply observe that, by *Lemma ?*, we know that $\Diamond x \in \mathbf{u}(\diamondplus S)$, so since $a \not\leqslant \Diamond x$, it must be the case that $a \notin \mathbf{lu}(\diamondplus S)$. ■

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and arbitrary subsets $S, T \subseteq \mathfrak{B}$,

$$S \subseteq T \Rightarrow \diamondplus S \subseteq \diamondplus T$$

Proof - Let us assume that $S \subseteq T$ and further assume some element $a \in \diamondplus S$, leaving us the task of showing that $a \in \diamondplus T$. Recall that this is the case iff $(\forall x \in \mathfrak{B})(x \in \mathbf{u}(T) \Rightarrow a \leqslant \Diamond x)$. So let us take an arbitrary $x \in \mathbf{u}(T)$ and show that $a \leqslant \Diamond x$. Now since $S \subseteq T$ and $\mathbf{u}$ is *antitone* on $\wp(\mathfrak{B})$, we know that $\mathbf{u}(T) \subseteq \mathbf{u}(S)$, implying that $x \in \mathbf{u}(S)$. Since $a \in \diamondplus S$, we know that $(\forall y \in \mathfrak{B})(y \in \mathbf{u}(S) \Rightarrow a \leqslant \Diamond y)$, so we can conclude that $a \leqslant \Diamond x$. ■

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and an arbitrary subset $S \subseteq \mathfrak{B}$,

$$\diamondplus \mathbf{lu}(S) = \mathbf{lu}(\diamondplus S)$$

Proof - Applying *Lemma ?*, it is clear that it suffices to show that $\diamondplus \mathbf{lu}(S) = \diamondplus S$, but, recalling that $\diamondplus$ is monotone on $\wp(\mathfrak{B})$ (*Lemma ?*), we can get the right-to-left containment. So I will focus on the left-to-right containment. Observe that if $a \in \diamondplus \mathbf{lu}(S)$, then $(\forall x \in \mathfrak{B})(x \in \mathbf{u}(\mathbf{lu}(S)) \Rightarrow a \leqslant \Diamond x)$. But, recalling that $\mathbf{ulu} = \mathbf{u}$ (*Observation ?*), we can conclude that $(\forall x \in \mathfrak{B})(x \in \mathbf{u}(S) \Rightarrow a \leqslant \Diamond x)$, implying that $a \in \diamondplus S$. ■

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$,

$$\Diamond^* \bot^* = \bot^*$$

Proof - Observe the following string of equalities.

$$\begin{aligned}
\Diamond^* \bot^* &= \Diamond^* \{\bot\} \\
&= \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(x \in \mathbf{u}(\{\bot\}) \Rightarrow a \leqslant \Diamond x)\} \\
&= \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(x \in \mathfrak{B} \Rightarrow a \leqslant \Diamond x)\} \\
&= \{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(a \leqslant \Diamond x)\}
\end{aligned}$$

So it suffices to show that

$$\{a \in \mathfrak{B} \mid (\forall x \in \mathfrak{B})(a \leqslant \Diamond x)\} = \{\bot\}$$

It is clear that $(\forall x \in \mathfrak{B})(\bot \leqslant \Diamond x)$, so the right-to-left containment holds. Now consider an element $a \in$ LHS. Since $(\forall x \in \mathfrak{B})(a \leqslant \Diamond x)$, we know $a \leqslant \Diamond \bot$, so $a \leqslant \bot$ (because $\Diamond \bot = \bot$), implying that $a = \bot$ and $a \in \{\bot\}$, giving us the left-to-right containment. ∎

Lemma ? - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$ and subsets $S, T \in \mathfrak{B}^*$,

$$\Diamond^* S \vee \Diamond^* T = \Diamond^* (S \vee T)$$

Proof - Observe the following string of equalities.

$$\begin{aligned}
\Diamond^* S \vee \Diamond^* T &= \bigwedge \{\downarrow \Diamond x \mid S \subseteq \downarrow x\} \vee \bigwedge \{\downarrow \Diamond y \mid T \subseteq \downarrow y\} \\
&= \bigwedge \{\downarrow \Diamond x \vee \downarrow \Diamond y \mid S \subseteq \downarrow x \text{ and } T \subseteq \downarrow y\} && \text{(by infinite dist. law)} \\
&= \bigwedge \{\downarrow (\Diamond x \vee \Diamond y) \mid S \subseteq \downarrow x \text{ and } T \subseteq \downarrow y\} && \text{(by hom. of } \downarrow) \\
&= \bigwedge \{\downarrow \Diamond (x \vee y) \mid S \subseteq \downarrow x \text{ and } T \subseteq \downarrow y\} && \text{(by dist. of } \Diamond) \\
&= \bigwedge \{\downarrow \Diamond (x \vee y) \mid S \vee T \subseteq \downarrow x \vee \downarrow y\} \\
&= \bigwedge \{\downarrow \Diamond (x \vee y) \mid S \vee T \subseteq \downarrow (x \vee y)\} && \text{(by hom. of } \downarrow) \\
&= \bigwedge \{\downarrow \Diamond z \mid S \vee T \subseteq \downarrow z\} \\
&= \Diamond^* (S \vee T)
\end{aligned}$$

∎

Theorem ? - The class **MA** is closed under the *upper* MacNeille completion.
Proof - Given a modal algebra $\mathfrak{M} := (\mathfrak{B}, \Diamond)$, we know from *Theorem ?* that $\mathfrak{B}^*$ is a Boolean algebra, so it remains only observe that $\Diamond^* \bot^* = \bot^*$ and $\Diamond^* S \vee \Diamond^* T = \Diamond^* (S \vee T)$ (for $S, T \in \mathfrak{B}^*$) which have been shown in *Lemma ?* and *Lemma ?* respectively. These observations imply that $\mathfrak{M}^*$ is itself a modal algebra, so $\mathfrak{M}^* \in \mathbf{MA}$. ∎

4 Non-closed classes

4.1 Distributive lattices

Theorem ? - The class **DistLatt** is *not* closed under the MacNeille completion.
Proof - **Finish me.**

5 TBD variety #1

6 TBD variety #2