Topomodels

An implementation of topological semantics for modal logic in Haskell

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Motivation

Normal modal logics

Syntax:

$$\varphi := \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Box\varphi$$

A *normal modal logic* is a set of formulas of the above form containing K and Dual and closed under *modus ponens*, *uniform substitution*, and *necessitation*.

(K) $\Box(p \to q) \to (\Box p \to \Box q)$
(Dual) $\Box p \leftrightarrow \neg\Diamond\neg p$

The smallest such logic is denoted by **K**.

The logic **S4** is defined as $\mathbf{K} \cup \{\Box p \to \Box\Box p, \Box p \to p\}$

Relational semantics for modal logic

A *Kripke frame* is a tuple (X, R) where X is a set and $R \subseteq X \times X$.

A *Kripke model* is a triple (X, R, V) where (X, R) is a Kripke frame and $V : \mathbf{Prop} \to \wp(X)$.

A *pointed Kripke model* is a 4-tuple (X, R, V, x) where (X, R, V) is a Kripke model and $x \in X$.

Key semantic definition:

$$(X, R, V, x) \models \Box\varphi \Leftrightarrow (\forall x' \in X)(xRx' \Rightarrow (X, R, V, x') \models \varphi)$$

The class Pre

A *pre-order* is a tuple (X, R) such that the following hold for all $x, y, z \in X$.

- xRx
- xRy and yRz implies xRz

The class of pre-orders is denoted by **Pre**.

Fact: The logic **S4** is sound and complete with respect to the class of frames **Pre**

$$\varphi \in \mathbf{S4} \Leftrightarrow \mathbf{Pre} \models \varphi$$
$$\Leftrightarrow (\forall (X, R) \in \mathbf{Pre})(\forall V \in \wp(X)^{\mathbf{Prop}})(\forall x \in X)\Big((X, R, V, x) \models \varphi\Big)$$

Basic topology

A *topological space* is a tuple (X, τ) where X is a set and $\tau \subseteq \wp(X)$ where τ satisfies the following.

- $\varnothing, X \in \tau$
- $S \subseteq \tau$ and $|S| < \omega$ implies $\bigcap S \in \tau$
- $S \subseteq \tau$ implies $\bigcup S \in \tau$

A set S is *open* if $S \in \tau$, *closed* if $X - S \in \tau$, and *clopen* if it is both open and closed.

Basic topology cont.

Given a subset $S \subseteq X$, the *interior of* S, denoted by $\mathrm{int}(S)$, is the largest open subset of S, or, equivalently,

$$\bigcup\{U \in \tau \mid U \subseteq S\}$$

The *closure of* S, denoted by $\mathrm{cl}(S)$, is the smallest closed superset of S, or, equivalently,

$$\bigcap\{C \subseteq X \mid X - C \in \tau \text{ and } S \subseteq C\}$$

The class **Alx**

Recall that a topospace (X, τ) satisfies the following.

- $\varnothing, X \in \tau$
- $S \subseteq \tau$ and $|S| < \omega$ implies $\bigcap S \in \tau$
- $S \subseteq \tau$ implies $\bigcup S \in \tau$

A topospace is called *Alexandrov* if it also satisfies the following *strengthening* of the second requirement above.

- $S \subseteq \tau$ $\cancel{\text{and } |S| < \omega}$ implies $\bigcap S \in \tau$

The class of Alexandrov topospaces is denoted by **Alx**.

Topological semantics for modal logic

A *topomodel* is a triple (X, τ, V) where (X, τ) is a topospace and V : **Prop** $\rightarrow \wp(X)$.

A *pointed topomodel* is a 4-tuple (X, τ, V, x) where (X, τ, V) is a topomodel and $x \in x$.

Key semantic definition:

$$(X, \tau, V, x) \models \Box\varphi \Leftrightarrow (\exists U \in \tau)\Big(x \in U \text{ and } (\forall y \in U)\big((X, \tau, V, y) \models \varphi\big)\Big)$$

This implies that

$$[\![\Box\varphi]\!] = \text{int}\Big([\![\varphi]\!]\Big)$$

$$[\![\Diamond\varphi]\!] = \text{cl}\Big([\![\varphi]\!]\Big)$$

The upset topology

Given a pre-order $\mathbf{X} := (X, R)$, an *upset* is a subset $S \subseteq X$ such that

$$(\forall x \in X)(x \in S \text{ and } xRy \text{ implies } y \in S)$$

We denote that set of all upsets on $\mathbf{X}$ by $\mathrm{Up}(\mathbf{X})$.

Observe that $(X, \mathrm{Up}(\mathbf{X}))$ is an Alexandrov topospace, so given an **S4** Kripke model, we can create a topomodel satisfying the same theory.

The specialisation order

Given an topospace $\mathbf{X} := (X, \tau)$, we can define the *specialisation order on* $\mathbf{X}$ as follows:

$$\begin{aligned} xR_{\mathbf{X}}y &:\Longleftrightarrow y \in \mathrm{Cl}(\{x\}) \\ &\Longleftrightarrow y \in \bigcap \{C \subseteq X \mid X - C \in \tau \text{ and } \{x\} \subseteq C\} \\ &\Longleftrightarrow y \in \bigcap \{C \subseteq X \mid X - C \in \tau \text{ and } x \in C\} \\ &\Longleftrightarrow (\forall C \subseteq X)(X - C \in \tau \text{ and } x \in C \Rightarrow y \in C) \end{aligned}$$

Observe that $R_{\mathbf{X}}$ is a pre-order, so given an topomodel, we can create an **S4** Kripke model satisfying the same theory.

Future work

The End

Questions? Comments?