Topomodels in Haskell

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Abstract

In this project, we provide a library for working with general topological spaces as well as topomodels for modal logic. We also implement a well-known construction for converting topomodels to **S4** Kripke models and back. We also implemented correctness tests and benchmarks which are meant to be extended by users. Thus, this work serves as a solid starting point for investigating hypotheses about general topology, modal logic, and their intersection.

Contents

1 Introduction

In §2, we describe the syntax of basic modal logic. This will define the formal language used throughout the rest of the paper. In §3, we implement several functions that are necessary for working with relations, topologies, and generating arbitrary sets. In §4, we define a library for working with topological spaces. In §5 through §7 , we define the models for our modal language. In §8, we define what it means for our formulas to be true on these models. In §9, we define how to convert between these models. Finally in §10 and §11 we describe how our library was tested and benchmarked.

2 Modal logic

In this section we define the syntax of a basic propositional modal logic.

```
module Syntax where

import Test.QuickCheck
```

2.1 Syntax

Our language will be the formulas of the following shape.

$$\varphi := \top \mid \bot \mid p_n \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Box\varphi$$

```
data Form
    = Top
    | Bot
    | P Int
    | Form `Dis` Form
    | Form `Con` Form
    | Form `Imp` Form
    | Neg Form
    | Dia Form
    | Box Form
    deriving (Eq, Show, Ord)
```

2.2 Arbitrary modal formula generation

When testing, we can generate arbitrary `Form`'s via the following instance of `Arbitrary`.

```
instance Arbitrary Form where
    arbitrary = sized randomForm
      where
        randomForm :: Int -> Gen Form
        randomForm 0 = P <$> elements [1 .. 5] -- Fixed vocabulary
        randomForm n =
```

```haskell
            oneof
                [ P <$> elements [1 .. 5]
                , Dis
                    <$> randomForm (n `div` 2)
                    <*> randomForm (n `div` 2)
                , Con
                    <$> randomForm (n `div` 2)
                    <*> randomForm (n `div` 2)
                , Imp
                    <$> randomForm (n `div` 2)
                    <*> randomForm (n `div` 2)
                , Neg <$> randomForm (n `div` 2)
                , Dia <$> randomForm (n `div` 2)
                , Box <$> randomForm (n `div` 2)
                ]
```

3 Set theory

In this section we define some set-theoretic helpers that will come in handy in the following sections.

```haskell
{-# LANGUAGE ImportQualifiedPost #-}

module SetTheory where

import Data.Set (Set, cartesianProduct, elemAt, intersection, member, union)
import Data.Set qualified as S

import Test.QuickCheck (Arbitrary, Gen, elements, listOf1, oneof, sublistOf, vectorOf)
```

3.1 Unions and intersections

A set of sets S is called *closed under unions* if $T, V \in S$ implies that $T \cup V \in S$. Similarly, S is called *closed under intersections* if $T, V \in S$ implies that $T \cap V \in S$.

The following functions close a passed set under unions and intersections.

```haskell
onceCloseUnderUnion :: (Ord a) => Set (Set a) -> Set (Set a)
onceCloseUnderUnion sets = S.map (uncurry union) (cartesianProduct sets sets)

onceCloseUnderIntersection :: (Ord a) => Set (Set a) -> Set (Set a)
onceCloseUnderIntersection sets = S.map (uncurry intersection) (cartesianProduct sets sets)

closeUnderUnion :: (Ord a) => Set (Set a) -> Set (Set a)
closeUnderUnion sets = do
    let oneUp = onceCloseUnderUnion sets
    if sets == oneUp
        then sets
        else closeUnderUnion oneUp

closeUnderIntersection :: (Ord a) => Set (Set a) -> Set (Set a)
closeUnderIntersection sets = do
    let oneUp = onceCloseUnderIntersection sets
    if sets == oneUp
        then sets
```

```
        else closeUnderIntersection oneUp
```

We also include, for convenience, the following functions which correspond to $\bigcup$ and $\bigcap$ respectively.

```
arbUnion :: (Ord a) => Set (Set a) -> Set a
arbUnion = S.foldr union S.empty

arbIntersection :: (Eq a, Ord a) => Set (Set a) -> Set a
arbIntersection sets
    | sets == S.empty = error "Cannot take the intersection of the empty set."
    | length sets == 1 = firstSet
    | otherwise = firstSet `intersection` arbIntersection restOfSets
  where
    firstSet = elemAt 0 sets
    restOfSets = S.drop 1 sets
```

3.2 Relations

Below are a couple of simple helper functions for working with binary relations.

```
type Relation a = Set (a, a)

field :: (Ord a) => Relation a -> Set a
field relation = domain `union` range
  where
    domain = S.map fst relation
    range = S.map snd relation

imageIn :: (Ord a) => a -> Relation a -> Set a
imageIn element relation = S.map snd $ S.filter (\(x, _) -> x == element) relation
```

Given a set X and a relation $R \subseteq X \times X$, we say that R is *transitive* if it satisfies, for all $x, y, z \in X$,

$$xRy \text{ and } yRz \text{ implies } xRz$$

Below is a function for making a passed relation transitive.

```
onceMakeTransitive :: (Ord a) => Relation a -> Relation a
onceMakeTransitive relation = do
    let relField = field relation
    let fieldCubed = cartesianProduct (cartesianProduct relField relField) relField
    let relTriples = S.filter (\((x, y), z) -> (x, y) `member` relation && (y, z) `member`
        relation) fieldCubed
    let additions = S.map (\((x, _), z) -> (x, z)) relTriples
    relation `union` additions

makeTransitive :: (Ord a) => Relation a -> Relation a
makeTransitive relation = do
    let oneUp = onceMakeTransitive relation
    if relation == oneUp
        then relation
        else makeTransitive oneUp
```

3.3 Arbitrary set generation

Here we define functions that are useful in the (constrained) generation of arbitrary sets. These mirror their commonly-used `List`-counterparts, but must be adapted as we work with `Data.Set`. Inspiration for this implementation was taken from here.

```
setOneOf :: Set (Gen a) -> Gen a
setOneOf = oneof . S.toList

subsetOf :: (Arbitrary a, Ord a) =>  Set a -> Gen (Set a)
subsetOf =  fmap S.fromList . sublistOf .  S.toList

setOf1 :: (Arbitrary a, Ord a) => Gen a -> Gen (Set a)
setOf1 = fmap S.fromList . listOf1

setElements :: Set a -> Gen a
setElements = elements . S.toList

isOfSize :: Set a -> Int -> Bool
isOfSize set k = S.size set == k

isOfSizeBetween :: Set a -> Int -> Int -> Bool
isOfSizeBetween set lower upper = lower <= S.size set && S.size set <= upper
```

The following helper functions generate (sub)sets of a specific size. Note that it does NOT guarantee length - it first generates a list and then makes it a set, so if two of the same elements were generated, the resulting set length is smaller than the original list.

```
setSizeOf :: (Ord a) => Gen a -> Int -> Gen (Set a)
setSizeOf g k = fmap S.fromList (vectorOf k g)

subsetSizeOf :: (Ord a) => Set a -> Int -> Gen (Set a)
subsetSizeOf set k = fmap S.fromList (vectorOf k (setElements set))
```

4 Topological preliminaries

In this section we define basic topological concepts that will form the foundation for our subsequent definition of topomodels and toposemantics for modal logic. For a more exhaustive treatment of topological structures, see [2].

```
{-# LANGUAGE ImportQualifiedPost #-}
{-# LANGUAGE ScopedTypeVariables #-}

module Topology where

import Data.Set (Set, isSubsetOf, singleton, union, (\\))
import Data.Set qualified as S
import Test.QuickCheck (Arbitrary (arbitrary), suchThat)

import SetTheory (arbIntersection, arbUnion, closeUnderIntersection, closeUnderUnion,
    isOfSizeBetween, setElements, setOf1)
```

4.1 Topological spaces

A *topological space* (or *topospace*) is a tuple (X, τ) where X is a non-empty set and $\tau \subseteq \wp(X)$ is a family of subsets of X such that

1. $\varnothing, X \in \tau$
2. $S \subseteq \tau$ and $|S| < \omega$ implies $\bigcap S \in \tau$
3. $S \subseteq \tau$ implies $\bigcup S \in \tau$

```
type Topology a = Set (Set a)

data TopoSpace a = TopoSpace (Set a) (Topology a) deriving (Eq, Show)
```

The elements of τ are referred to as *open sets*, so we say a subset $S \subseteq X$ is *open* in τ if $S \in \tau$. Given a point $x \in X$, we call the set of all open sets containing x the *open neighbourhoods of* x.

Additionally, we say that S is *closed* (in τ) if $X - A \in \tau$ (i.e. S is the complement of an open set). The set of closed sets of (X, τ) is denoted by $\overline{\tau}$.

Finally, we say that S is *clopen* if it is both open and closed.

```
isOpenIn :: (Eq a) => Set a -> TopoSpace a -> Bool
isOpenIn set (TopoSpace _ topology) = set `elem` topology

openNbds :: (Eq a) => a -> TopoSpace a -> Set (Set a)
openNbds x (TopoSpace _ topology) = S.filter (x `elem`) topology

isClosedIn :: (Eq a, Ord a) => Set a -> TopoSpace a -> Bool
isClosedIn set (TopoSpace space topology) = space \\ set `elem` topology

closeds :: (Ord a) => TopoSpace a -> Set (Set a)
closeds (TopoSpace space topology) = S.map (space \\) topology

isClopenIn :: (Eq a, Ord a) => Set a -> TopoSpace a -> Bool
isClopenIn set topoSpace = set `isOpenIn` topoSpace && set `isClosedIn` topoSpace
```

Given a topospace (X, τ) and a subset $S \subseteq X$, the *interior* of S, denoted by $\text{int}(S)$, is the union of all open subsets of S, i.e.

$$\text{int}(S) := \bigcup \{U \in \tau \mid U \subseteq S\}$$

The *closure* of S, denoted by $\overline{S}$, is the intersection of all closed supersets of S, i.e.

$$\overline{S} := \bigcap \{C \in \overline{\tau} \mid S \subseteq C\}$$

```
interior :: (Ord a) => Set a -> TopoSpace a -> Set a
interior set topoSpace = arbUnion opensBelowSet
  where
    TopoSpace _ opens = topoSpace
    opensBelowSet = S.filter (`isSubsetOf` set) opens
```

```
closure :: (Ord a) => Set a -> TopoSpace a -> Set a
closure set topoSpace = arbIntersection closedsAboveSet
  where
    closedsAboveSet = S.filter (set `isSubsetOf`) (closeds topoSpace)
```

4.2 Bases and Subbases

Given a topological space $\mathbf{X} := (X, \tau)$, a *basis* for $\mathbf{X}$ is a subset $\beta \subseteq \tau$ such that τ is equal to the closure of β under arbitrary unions.

A *subbasis* for $\mathbf{X}$ is a subset $\sigma \subseteq \tau$ such that the closure of σ under finite intersections forms a basis for $\mathbf{X}$.

```
isBasisFor :: (Ord a) => Set (Set a) -> TopoSpace a -> Bool
isBasisFor sets (TopoSpace _ opens) = closeUnderUnion sets == opens

isSubbasisFor :: (Ord a) => Set (Set a) -> TopoSpace a -> Bool
isSubbasisFor sets topoSpace = closeUnderIntersection sets `isBasisFor` topoSpace
```

4.3 Arbitrary topological spaces

First we include a couple of helper functions.

The first of them checks if the passed TopoSpace is, indeed, a topological space (i.e. respects all of the axioms).

The second actually *fixes* a passed TopoSpace in the case that it is not *truly* a topological space (i.e. fails to satisfy one of the axioms). This is necessary for the generation of arbitrary topospaces later on.

```
isTopoSpace :: (Ord a) => TopoSpace a -> Bool
isTopoSpace (TopoSpace space topology)
    -- Passed space is empty
    | space == S.empty = False
    -- Passed topology is not a subset of the power set of passed space
    | not (arbUnion topology `isSubsetOf` space) = False
    -- Passed topology is missing the empty set or the full space
    | S.empty `notElem` topology || space `notElem` topology = False
    -- Passed topology should be closed under intersections and unions
    | otherwise = topology == (closeUnderUnion . closeUnderIntersection) topology

fixTopoSpace :: (Ord a) => TopoSpace a -> TopoSpace a
fixTopoSpace (TopoSpace space topology)
    -- Throw an error since we don't know how the topology should look like
    | not (S.unions topology `isSubsetOf` space) = error "Points in topology are not all
      members of the space"
    | S.empty `notElem` topology = fixTopoSpace (TopoSpace space (topology `union`
      singleton S.empty))
    | space `notElem` topology = fixTopoSpace (TopoSpace space (topology `union` singleton
      space))
    | otherwise = TopoSpace space closedTopology
  where
```

```
    closedTopology = closeUnderUnion . closeUnderIntersection $ topology
```

Now we can define a method for generating arbitrary topospaces.

```
instance (Arbitrary a, Ord a) => Arbitrary (TopoSpace a) where
    arbitrary = do
        (x :: Set a) <- arbitrary `suchThat` (\set -> isOfSizeBetween set 1 10)
        -- Put an artificial bound on the size of the set, otherwise it takes too long to "
            fix" the topology
        subbasis <-
            let basis = setOf1 (setElements x) `suchThat` (\set -> isOfSizeBetween set 0 3)
             in setOf1 basis `suchThat` (\set -> isOfSizeBetween set 0 3)
        let someTopoSpace = TopoSpace x subbasis
        return (fixTopoSpace someTopoSpace)
```

5 Models

In this section we define some concepts that will be used in the subsequent sections on *Kripke models* and *topomodels*.

```
{-# LANGUAGE ImportQualifiedPost #-}

module Models where

import Data.Set (Set, union)
import Data.Set qualified as S
import Test.QuickCheck (Arbitrary, Gen)

import SetTheory (subsetOf)
import Syntax (Form (P))
```

5.1 Arbitrary valuation generation

Given a set X, a *valuation* is a function $V : \mathbf{Prop} \to \wp(X)$ where $\mathbf{Prop}$ is the set of formulas of the shape p_n.

Note: The type of `Valuation` below suggests our valuations should be defined on all `Form` (instead of just on **Prop**), but this is merely an implementation detail and when generating arbitrary `Valuation`'s we only define them on **Prop**.

```
type Valuation a = Set (Form, Set a)

-- Recursively make a valuation function. Go through each propositional variable
-- and assign a subset of points at which it is true
randomVal :: (Arbitrary a, Ord a) => Set a -> [Int] -> Gen (Valuation a)
randomVal _ [] = return S.empty
randomVal points (prop : props)
    | null points = return S.empty
    | otherwise = do
        x <- randomVal points props
```

```
        randSubset <- subsetOf points
        return $ S.singleton (P prop, randSubset) `union` x
```

6 Kripke models

In this section we define relational models of modal logic.

```
{-# LANGUAGE ImportQualifiedPost #-}
{-# LANGUAGE ScopedTypeVariables #-}

module KripkeModels where

import Data.Set (Set, cartesianProduct, union)
import Data.Set qualified as S
import Test.QuickCheck (Arbitrary (arbitrary), chooseInt, suchThat)

import Models (Valuation, randomVal)
import SetTheory (Relation, isOfSizeBetween, makeTransitive, setElements, subsetSizeOf)
```

6.1 Kripke models

An **S4** *Kripke frame* is a tuple (X, R) where X is a set and $R \subseteq X \times X$ and the following are true for all $x, y, z \in X$.

- xRx (Reflexivity)
- xRy and yRz implies xRz (Transitivity)

An **S4** *Kripke model* is a triple (X, R, V) where (X, R) is an **S4** Kripke frame and V is a valuation on X.

A *pointed* **S4** *Kripke model* is a 4-tuple (X, R, V, x) where (X, R, V) is an **S4** Kripke model and $x \in X$.

```
data S4KripkeFrame a = S4KF (Set a) (Relation a)
    deriving (Eq, Show)

data S4KripkeModel a = S4KM (S4KripkeFrame a) (Valuation a)
    deriving (Eq, Show)

data PointedS4KripkeModel a = PS4KM (S4KripkeModel a) a
    deriving (Eq, Show)
```

6.2 Arbitrary Kripke model generation

Below we define a method for generating arbitrary Kripke models. This presented something of an interesting challenge as we cannot simply take *any* relation on *any* carrier set; we must ensure that the generated frame is, indeed, S4 (i.e. reflexive and transitive).

To accomplish this, we generate an arbitrary carrier set and an arbitrary subset of its cartesian product. We then add to this random relation all of the reflexive pairs and close it under transitive triples.

This closure process grows the relation significantly so, in order to avoid ending up with a complete graph, we choose a starting relation that is quite small. Given a carrier set of cardinality n, instead of allowing any subset of the cross product (which could be as large as n^2), we capped the random relation at cardinality $2n$, which ensures that we get interesting frames.

```
instance (Arbitrary a, Ord a) => Arbitrary (S4KripkeFrame a) where
    arbitrary = do
        (carrier :: Set a) <- arbitrary `suchThat` (\set -> isOfSizeBetween set 1 10)
        let carrierSquared = cartesianProduct carrier carrier
        {-
            If no cap is put on this then the resulting frame (after being made
            reflexive and transitive) will almost always be a complete graph,
            which is uninteresting.
        -}
        relationSize <- chooseInt (1, S.size carrier * 3)
        (randomRelation :: Relation a) <- subsetSizeOf carrierSquared relationSize
        let diagonal = S.filter (uncurry (==)) carrierSquared
        let reflexiveRelation = randomRelation `union` diagonal
        let s4Relation = makeTransitive reflexiveRelation
        return (S4KF carrier s4Relation)

instance (Arbitrary a, Ord a) => Arbitrary (S4KripkeModel a) where
    arbitrary = do
        (randomFrame :: S4KripkeFrame a) <- arbitrary
        let (S4KF carrier _) = randomFrame
        (randomValuation :: Valuation a) <- randomVal carrier [1 .. 10]
        return (S4KM randomFrame randomValuation)

instance (Arbitrary a, Ord a) => Arbitrary (PointedS4KripkeModel a) where
    arbitrary = do
        (randomModel :: S4KripkeModel a) <- arbitrary
        let (S4KM frame _) = randomModel
        let (S4KF carrier _) = frame
        point <- setElements carrier
        return (PS4KM randomModel point)
```

7 Topomodels

In this section we define topological models of modal logic.

```
{-# LANGUAGE FlexibleContexts #-}
{-# LANGUAGE ScopedTypeVariables #-}

module TopoModels where

import Test.QuickCheck (Arbitrary (arbitrary))
```

```haskell
import Models (Valuation, randomVal)
import SetTheory (setElements)
import Topology (TopoSpace (TopoSpace))
```

7.1 Topomodels

A *topomodel* is a triple (X, τ, V) where (X, τ) is a topospace and V is a valuation on X.

A *pointed topomodel* is a 4-tuple (X, τ, V, x) where (X, τ, V) is an topomodel and $x \in X$.

```haskell
data TopoModel a = TopoModel (TopoSpace a) (Valuation a)
    deriving (Eq, Show)

data PointedTopoModel a = PointedTopoModel (TopoModel a) a
    deriving (Eq, Show)
```

7.2 Arbitrary topomodel generation

Below we define a method for generating arbitrary topomodels.

```haskell
instance (Arbitrary a, Ord a) => Arbitrary (TopoModel a) where
  arbitrary = do
    (TopoSpace space topo) <- arbitrary
    -- Random Valuation depending on the points of the space
    -- Fix the number of propositional variables
    val <- randomVal space [1..10]
    return (TopoModel (TopoSpace space topo) val)

instance (Arbitrary a, Ord a) => Arbitrary (PointedTopoModel a) where
  arbitrary = do
   (TopoModel (TopoSpace space topo) val) <- arbitrary
   (x :: a) <- setElements space
   return (PointedTopoModel (TopoModel (TopoSpace space topo) val) x)
```

8 Semantics

In this section we define the semantics for the formulas defined in `Syntax.lhs` on both `TopoModel`'s and `S4KripkeModel`'s.

```haskell
{-# LANGUAGE ImportQualifiedPost #-}

module Semantics where

import Data.Set qualified as S

import KripkeModels (PointedS4KripkeModel (PS4KM), S4KripkeFrame (S4KF), S4KripkeModel (
    S4KM))
```

```
import SetTheory (imageIn)
import Syntax (Form (..))
import TopoModels (PointedTopoModel (..), TopoModel (..))
import Topology (TopoSpace (TopoSpace), openNbds)
```

Given a formula φ in our language along with a model $\mathfrak{M}$, a *semantics* is a definition of when $\mathfrak{M}$ makes φ *true*. The relation 'makes true' is often written using '$\models$', so we abbreviate the statement '$\mathfrak{M}$ makes φ true' as '$\mathfrak{M} \models \varphi$'.

```
class Semantics m where
    (|=) :: m -> Form -> Bool
```

In both of the below-defined instances of `Semantics`, the Boolean cases are defined in the same, standard way, so we will only comment on the key modal cases. For the Boolean cases, see [1, pp. 17-18].

8.1 Kripke semantics

Given a *pointed* S4 Kripke model (X, R, V, x), we define the following for all formulas φ in our modal language.

$$(X, R, V, x) \models \Box\varphi \;:\Longleftrightarrow\; \forall y \in X(xRy \Rightarrow (X, R, V, y) \models \varphi)$$
$$(X, R, V, x) \models \Diamond\varphi \;:\Longleftrightarrow\; (X, R, V, x) \models \neg\Box\neg\varphi$$

Given an S4 Kripke model (X, R, V) (without a point), we also define the following for all formulas φ in our modal language.

$$(X, R, V) \models \varphi \;:\Longleftrightarrow\; \forall x \in X((X, R, V, x) \models \varphi)$$

```
instance (Eq a, Ord a) => Semantics (PointedS4KripkeModel a) where
    (|=) _ Top = True
    (|=) _ Bot = False
    (|=) pointedModel (P n) = x `elem` worldsWherePnTrue
      where
        PS4KM kripkeModel x = pointedModel
        S4KM _ valuation = kripkeModel
        worldsWherePnTrue = snd . S.elemAt 0 $ S.filter (\(p, _) -> p == P n) valuation
    (|=) pointedModel (phi `Dis` psi) = (pointedModel |= phi) || (pointedModel |= psi)
    (|=) pointedModel (phi `Con` psi) = (pointedModel |= phi) && (pointedModel |= psi)
    (|=) pointedModel (phi `Imp` psi) = pointedModel |= (Neg phi `Dis` psi)
    (|=) pointedModel (Neg phi) = not $ pointedModel |= phi
    (|=) pointedModel (Dia phi) = pointedModel |= Neg (Box (Neg phi))
    (|=) pointedModel (Box phi) = all (\w' -> PS4KM kripkeModel w' |= phi) imageOfWorld
      where
        (PS4KM kripkeModel world) = pointedModel
        S4KM kripkeFrame _ = kripkeModel
        S4KF _ relation = kripkeFrame
        imageOfWorld = world `imageIn` relation

instance (Eq a, Ord a) => Semantics (S4KripkeModel a) where
    kripkeModel |= phi = wholeSetSatisfiesForm carrier phi
      where
        (S4KM frame _) = kripkeModel
        (S4KF carrier _) = frame
        wholeSetSatisfiesForm set psi = all (\x -> PS4KM kripkeModel x |= psi) set
```

8.2 Topo-semantics

Given a *pointed* topomodel (X, τ, V, x), we define the following for all formulas φ in our modal language.

$$\begin{aligned}(X, R, V, x) \models \Box\varphi \ &:\Longleftrightarrow\ \exists U \in \tau(x \in U \text{ and } \forall y \in U((X, \tau, V, y) \models \varphi)) \\ (X, \tau, V, x) \models \Diamond\varphi \ &:\Longleftrightarrow\ (X, \tau, V, x) \models \neg\Box\neg\varphi\end{aligned}$$

Given a topomodel (X, τ, V) (without a point), we also define the following for all formulas φ in our modal language.

$$(X, \tau, V) \models \varphi \ :\Longleftrightarrow\ \forall x \in X((X, \tau, V, x) \models \varphi)$$

```
instance (Eq a) => Semantics (PointedTopoModel a) where
    (|=) _ Top = True
    (|=) _ Bot = False
    (|=) pointedModel (P n) = x `elem` worldsWherePnTrue
      where
        PointedTopoModel topoModel x = pointedModel
        TopoModel _ valuation = topoModel
        worldsWherePnTrue = snd . S.elemAt 0 $ S.filter (\(p, _) -> p == P n) valuation
    (|=) pointedModel (phi `Dis` psi) = (pointedModel |= phi) || (pointedModel |= psi)
    (|=) pointedModel (phi `Con` psi) = (pointedModel |= phi) && (pointedModel |= psi)
    (|=) pointedModel (phi `Imp` psi) = pointedModel |= (Neg phi `Dis` psi)
    (|=) pointedModel (Neg phi) = not $ pointedModel |= phi
    (|=) pointedModel (Dia phi) = pointedModel |= Neg (Box (Neg phi))
    (|=) pointedModel (Box phi) = not (null openNbdsSatisfyingFormula)
      where
        PointedTopoModel topoModel point = pointedModel
        TopoModel topoSpace _ = topoModel
        wholeSetSatisfiesForm set psi = all (\x -> PointedTopoModel topoModel x |= psi) set
        openNbdsOfPoint = openNbds point topoSpace
        openNbdsSatisfyingFormula = S.filter (`wholeSetSatisfiesForm` phi) openNbdsOfPoint

instance (Eq a) => Semantics (TopoModel a) where
    topoModel |= phi = wholeSetSatisfiesForm space phi
      where
        (TopoModel topoSpace _) = topoModel
        TopoSpace space _ = topoSpace
        wholeSetSatisfiesForm set psi = all (\x -> PointedTopoModel topoModel x |= psi) set
```

9 Model conversion

In this sections, we implement a method for converting `S4KripkeModel`'s to `TopoModel`'s and vice-versa. We follow the construction described in [3, pp. 22-23].

```
{-# LANGUAGE ImportQualifiedPost #-}

module ModelConversion where

import Data.Set (cartesianProduct, member, singleton)
import Data.Set qualified as S

import KripkeModels (PointedS4KripkeModel (PS4KM), S4KripkeFrame (S4KF), S4KripkeModel (
    S4KM))
import SetTheory (closeUnderUnion, imageIn)
```

```
import TopoModels (PointedTopoModel (PointedTopoModel), TopoModel (TopoModel))
import Topology (TopoSpace (TopoSpace), closure)
```

Given a set-relation pair $\mathbf{X} := (X, R)$, an *upset* is a subset $S \subseteq X$ that satisfies the following for all $x, y \in X$.

$$(x \in S \text{ and } xRy) \text{ implies } y \in S$$

The term 'upset' is used because orders are often depicted using Hasse diagrams where xRy is depicted by the point y being above x, connected by a line. We denote the set of all upsets of $\mathbf{X}$ by $\mathtt{Up}(\mathbf{X})$.

Given an **S4** Kripke frame $\mathbf{X} := (X, R)$, it is a well known fact that $(X, \mathtt{Up}(\mathbf{X}))$ is a topological space. What is more, for all modal formulas φ, all valuations V on X, and all points $x \in X$, we have

$$(X, R, V, x) \models \varphi \Leftrightarrow (X, \mathtt{Up}(\mathbf{X}), V, x) \models \varphi$$

Observe how the '$\models$' on the left-hand-side is a relational semantics while on the right-hand-side it is a topo-semantics.

Below we implement this conversion from `S4KripkeModel`'s to `TopoModel`'s. Since we are working with finite models, we can generate all upsets by closing all of the principle upsets under unions (along with the empty set).

```
toTopoSpace :: (Ord a) => S4KripkeFrame a -> TopoSpace a
toTopoSpace kripkeFrame = TopoSpace carrier opens
  where
    S4KF carrier relation = kripkeFrame
    nonEmptyUpsets = closeUnderUnion $ S.map (`imageIn` relation) carrier
    opens = S.insert S.empty nonEmptyUpsets
```

Now we turn to the other conversion.

Given a topospace $(\mathbf{X} := (X, \tau))$, the *specialisation order* $R_{\mathbf{X}}$ *on* $\mathbf{X}$ is defined as follows for all $x, y \in \mathbf{X}$.

$$xR_{\mathbf{X}}y \;:\Longleftrightarrow\; y \in \overline{\{x\}}$$

It follows quite easily that this relation is reflexive and transitive, implying that $(X, R_{\mathbf{X}})$ is an **S4** Kripke frame.

Similarly to the other conversion, we get the following for all modal formulas φ, all valuations V on X, and all points $x \in X$.

$$(X, \tau, V, x) \models \varphi \Leftrightarrow (X, R_{\mathbf{X}}, V, x) \models \varphi$$

```
toS4KripkeFrame :: (Ord a) => TopoSpace a -> S4KripkeFrame a
toS4KripkeFrame topoSpace = S4KF space relation
  where
    (TopoSpace space _) = topoSpace
    relation = S.filter (\(x, y) -> y `member` closure (singleton x) topoSpace) (
        cartesianProduct space space)
```

Since the carrier sets and valuations remain unchanged, we can extend these conversions to (pointed) models.

```
toTopoModel :: (Ord a) => S4KripkeModel a -> TopoModel a
toTopoModel (S4KM frame valuation) = TopoModel (toTopoSpace frame) valuation

toS4KripkeModel :: (Ord a) => TopoModel a -> S4KripkeModel a
toS4KripkeModel (TopoModel topoSpace valuation) = S4KM (toS4KripkeFrame topoSpace)
    valuation

toPointedTopoModel :: (Ord a) => PointedS4KripkeModel a -> PointedTopoModel a
toPointedTopoModel (PS4KM kripkeModel point) = PointedTopoModel (toTopoModel kripkeModel)
    point

toPointedS4KripkeModel :: (Ord a) => PointedTopoModel a -> PointedS4KripkeModel a
toPointedS4KripkeModel (PointedTopoModel topoModel point) = PS4KM (toS4KripkeModel
    topoModel) point
```

10 Testing

```
{-# LANGUAGE ImportQualifiedPost #-}

module Main where

import Control.Exception (evaluate)
import Data.Set (Set, isSubsetOf)
import Data.Set qualified as S
import Test.Hspec (anyException, describe, hspec, it, shouldBe, shouldThrow)
import Test.Hspec.QuickCheck (prop)
import Test.QuickCheck ()

import KripkeModels (PointedS4KripkeModel, S4KripkeModel)
import Semantics (Semantics ((|=)))
import SetTheory (arbUnion, closeUnderIntersection, closeUnderUnion)
import Syntax (Form (Box, Con, Dia, Dis, Imp, Neg, P))
import TestHelpers (
    PointedS4KripkeModelTopoModel (PS4KMTM),
    S4KripkeModelTopoModel (S4KMTM),
    SSubsetTopoSpace (SSTS),
    SubsetTopoSpace (STS),
    badTS,
    fourAxiom,
    goodTS,
    kAxiom,
    s0,
    s1,
    s2,
    s3,
    s4,
    s5,
    s6,
    s7,
    tAxiom,
    topoSpace,
    topology,
 )
import TopoModels (PointedTopoModel, TopoModel)
import Topology (
    TopoSpace (..),
    closeds,
    closure,
    fixTopoSpace,
    interior,
    isClopenIn,
    isClosedIn,
```

```
    isOpenIn,
    isTopoSpace,
    openNbds,
  )
```

The following codeblock contains correctness tests for:

- TopoSpace generation: Check whether Topology axioms are valid.
- Interior and closure operators: Check whether Kuratowski axioms for closure and interior are respected.
- Examples from Topology module: Check whether the result is as expected as we describe in the Topology module.
- TopoModel semantics: Check whether S4KripkeModel's an d TopoModel's validate some propositional and modal tautologies and the following axioms.
  - $\mathbf{K} = \Box(p \to q) \to (\Box p \to \Box q)$
  - $\mathbf{T} = p \to \Diamond p$
  - $\mathbf{4} = \Diamond\Diamond p \to \Diamond p$
- S4KripkeModel and TopoModel correspondence: Similar as the semantics but we first generate TopoModel, then convert to S4KripkeModel, and check if the semantics behave the same.

```
main :: IO ()
main = hspec $ do
  describe "TopoSpace generation" $ do
    prop "Arbitrary TopoSpace satisfies the open set definition of a topo space" $ do
      \ts -> isTopoSpace (ts :: TopoSpace Int)
    prop "The subset in arbitrary SubsetTopoSpace is indeed a subset of the space" $ do
      \(STS setA (TopoSpace space _)) -> (setA :: Set Int) `isSubsetOf` space
    prop "The two subsets in arbitrary SSubsetTopoSpace are indeed subsets of the space" $
        do
      \(SSTS setA setB (TopoSpace space _)) -> (setA :: Set Int) `isSubsetOf` space && (
          setB :: Set Int) `isSubsetOf` space
  describe "Kuratowski Axioms for the closure operator" $ do
    prop "Preserves the empty set" $ do
        \x -> closure S.empty (x :: TopoSpace Int) `shouldBe` S.empty
    prop "Is extensive for all A \\subseteq X" $ do
        \ (STS setA ts) -> (setA :: Set Int) `isSubsetOf` closure setA ts
    prop "Is idempotent for all A \\subseteq X" $ do
        \(STS setA ts) -> closure (setA :: Set Int) ts `shouldBe` closure (closure setA ts)
             ts
    prop "Distributes over binary unions" $ do
        \(SSTS setA setB ts) ->
          closure ((setA :: Set Int) `S.union` setB) ts `shouldBe`
          closure setA ts `S.union` closure setB ts
  describe "Kuratowski Axioms for the interior operator" $ do
    prop "Preserves the whole space" $ do
        \(TopoSpace space topo) -> interior (space :: Set Int) (TopoSpace space topo) `
            shouldBe` space
    prop "Is intensive for all A \\subseteq X" $ do
        \ (STS setA ts) -> interior (setA :: Set Int) ts `isSubsetOf` setA
    prop "Is idempotent for all A \\subseteq X" $ do
         \(STS setA ts) -> interior (setA :: Set Int) ts `shouldBe` interior (interior setA
              ts) ts
    prop "Distributes over binary intersections" $ do
        \(SSTS setA setB ts) ->
          interior ((setA :: Set Int) `S.intersection` setB) ts `shouldBe`
          interior setA ts `S.intersection` interior setB ts
```

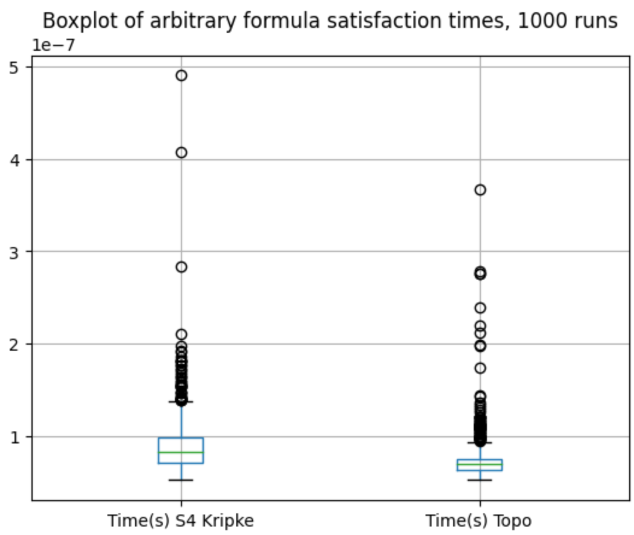
```haskell
describe "Examples from the Topology module" $ do
  it "closeUnderUnion $ Set.fromList [s0, s1, s2]" $ do
    let result = S.fromList [S.fromList [1], S.fromList [1,2], S.fromList [1,2,3,4], S.
        fromList [1,3,4],S.fromList [2],S.fromList [2,3,4],S.fromList [3,4]]
    closeUnderUnion (S.fromList [s0, s1, s2]) `shouldBe` result
  it "closeUnderIntersection $ Set.fromList [s0, s1, s2]" $ do
    let result = S.fromList [S.fromList [], S.fromList [1], S.fromList [2], S.fromList
        [3,4]]
    closeUnderIntersection (S.fromList [s0, s1, s2]) `shouldBe` result
  it "closeUnderUnion $ Set.fromList [s3, s4, s5]" $ do
    let result = S.fromList [S.fromList [1,2,3], S.fromList [1,2,3,4], S.fromList [2,3],
        S.fromList [2,3,4], S.fromList [3,4]]
    closeUnderUnion (S.fromList [s3, s4, s5]) `shouldBe` result
  it "closeUnderIntersection $ Set.fromList [s3, s4, s5]" $ do
    let result = S.fromList [S.fromList [1,2,3], S.fromList [2,3], S.fromList [3], S.
        fromList [3,4]]
    closeUnderIntersection (S.fromList [s3, s4, s5]) `shouldBe` result
  it "(closeUnderUnion . closeUnderIntersection) $ Set.fromList [s5, s6, s7]" $ do
    let result = S.fromList [S.fromList [], S.fromList [1], S.fromList [1,2], S.fromList
        [1,2,3], S.fromList [1,2,3,4], S.fromList [1,3], S.fromList [1,3,4], S.fromList
        [3], S.fromList [3,4]]
    (closeUnderUnion . closeUnderIntersection) (S.fromList [s5, s6, s7]) `shouldBe`
        result
  it "isTopoSpace (TopoSpace (arbUnion Set.fromList [s5, s6, s7]) topology)" $ do
    isTopoSpace (TopoSpace (arbUnion $ S.fromList [s5, s6, s7]) topology)
  it "isTopoSpace badTS" $ do
    not . isTopoSpace $ badTS
  it "isTopoSpace goodTS" $ do
    isTopoSpace goodTS
  it "isTopoSpace (fixTopoSpace goodTS)" $ do
    isTopoSpace (fixTopoSpace goodTS)
  it "closeds topoSpace" $ do
    let result = S.fromList [S.fromList [], S.fromList [1,2], S.fromList [1,2,3,4], S.
        fromList [1,2,4], S.fromList [2], S.fromList [2,3,4], S.fromList [2,4], S.
        fromList [3,4], S.fromList [4]]
    closeds topoSpace `shouldBe` result
  it "openNbds 2 topoSpace" $ do
    let result = S.fromList [S.fromList [1,2], S.fromList [1,2,3], S.fromList [1,2,3,4]]
    openNbds 2 topoSpace `shouldBe` result
  it "(S.fromList [1]) `isOpenIn` topoSpace" $ do
    S.fromList [1] `isOpenIn` topoSpace
  it "(S.fromList [1]) `isClosedIn` topoSpace" $ do
    not (S.fromList [1] `isClosedIn` topoSpace)
  it "(S.fromList []) `isClopenIn` topoSpace" $ do
    S.fromList [] `isClopenIn` topoSpace
  it "interior (Set.fromList [1]) topoSpace" $ do
    let result = S.fromList [1]
    interior (S.fromList [1]) topoSpace `shouldBe` result
  it "closure (Set.fromList [1]) topoSpace" $ do
    let result = S.fromList [1,2]
    closure (S.fromList [1]) topoSpace `shouldBe` result
  it "fixTopoSpace (TopoSpace (S.fromList [1,2,3]) topology)" $ do
     evaluate (fixTopoSpace (TopoSpace (S.fromList [1,2,3]) topology)) `shouldThrow`
         anyException
describe "TopoModel semantics" $ do
  prop "Validates the K axiom" $ do
    \ts -> (ts :: TopoModel Int) |= kAxiom
  prop "Validates tautology: p or not p" $ do
    \ts -> (ts :: TopoModel Int) |= (P 1 `Dis` Neg (P 1))
  prop "Validates tautology: p implies p" $ do
    \ts -> (ts :: TopoModel Int) |= (P 1 `Imp` P 1)
  prop "Validates tautology: p implies (q implies (p and q))" $ do
    \ts -> (ts :: TopoModel Int) |= (P 1 `Imp` (P 2 `Imp` (P 1 `Con` P 2)))
  prop "Validates modal tautology: Dia p or not Dia p"$ do
    \ts -> (ts :: TopoModel Int) |= (Dia (P 1)`Dis` Neg (Dia (P 1)))
  prop "Validates modal tautology: Box p implies Dia p"$ do
    \ts -> (ts :: TopoModel Int) |= (Box (P 1) `Imp` Dia (P 1))
  prop "Cannot satisfy contradiction p and not p" $ do
    \ts -> not ((ts :: PointedTopoModel Int) |= (P 1 `Con` Neg (P 1)))
  prop "Cannot satisfy contradiction ((P or Q) implies R) and not ((P or Q) implies R)" $
      do
    \ts -> not ((ts :: PointedTopoModel Int) |= (((P 1 `Dis` P 2) `Imp` P 3) `Con` Neg ((
        P 1 `Dis` P 2) `Imp` P 3)))
```

```
    prop "Cannot satisfy modal contradiction: Dia p or not Dia p" $ do
      \ts -> not ((ts :: PointedTopoModel Int) |= (Dia (P 1) `Con` Neg (Dia (P 1))))
    prop "Cannot satisfy modal contradiction: Box p and Dia not p" $ do
      \ts -> not ((ts :: PointedTopoModel Int) |= (Box (P 1) `Con` Dia (Neg (P 1))))
  describe "S4 Kripke Model and TopoModel correspondence" $ do
    prop "Both validate the K axiom (distribution of box)" $ do
      \(S4KMTM km tm) -> (km :: S4KripkeModel Int) |= kAxiom && tm |= kAxiom
    prop "Both validate the T axiom (reflexivity)" $ do
      \(S4KMTM km tm) -> (km :: S4KripkeModel Int) |= tAxiom && tm |= tAxiom
    prop "Both validate the 4 axiom (transitivity)" $ do
      \(S4KMTM km tm) -> (km :: S4KripkeModel Int) |= fourAxiom && tm |= fourAxiom
    prop "Corresponding KMs and TMs satisfy the same formulas" $ do
       \(PS4KMTM km tm, f) -> (km :: PointedS4KripkeModel Int) |= (f :: Form) == tm |= f
```

11 Benchmarks

To test the efficiency of our implementation and to investigate whether `S4KripkeModel`'s are more or less efficient than `TopoModel`'s, we implemented a few benchmarks. The benchmark code can be found in the directory `bench`, we will omit them from the report for brevity. We have also structured the benchmarks in a way that we deem them to be extendable easily.

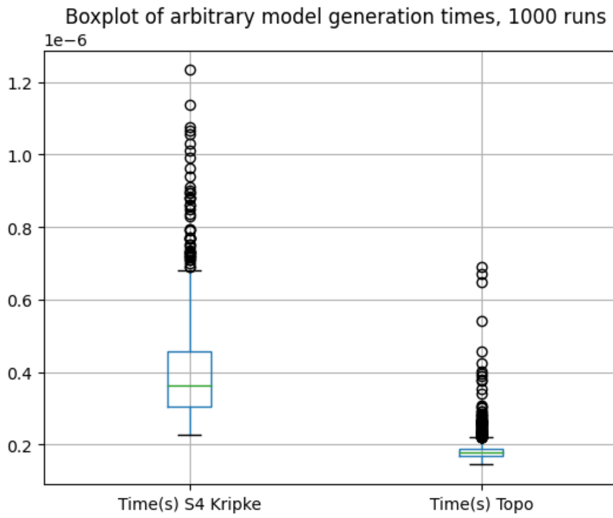
First, we investigated the question "How efficient is arbitrary formula evaluation on `S4KripkeModel`'s as opposed to `TopoModel`'s?". A single run of the benchmark consists of generating a `S4KripkeModel`, then converting to a `TopoModel` and evaluating on an arbitrary formula. This was done 1000 times and the results are pictured below.



Overall, `TopoModel`'s seem to run a bit faster than `S4KripkeModel`'s as can be seen from the boxplot. Indeed, the median times reflect that - 8.3e-08 seconds and 7e-08 second respectively. Is it, however, due to the size of the models? 580 out of 1000 times `S4KripkeModel`'s were larger than Topo Models. And so, the slight difference in formula evaluation speed must be due to Kripke Models being larger.

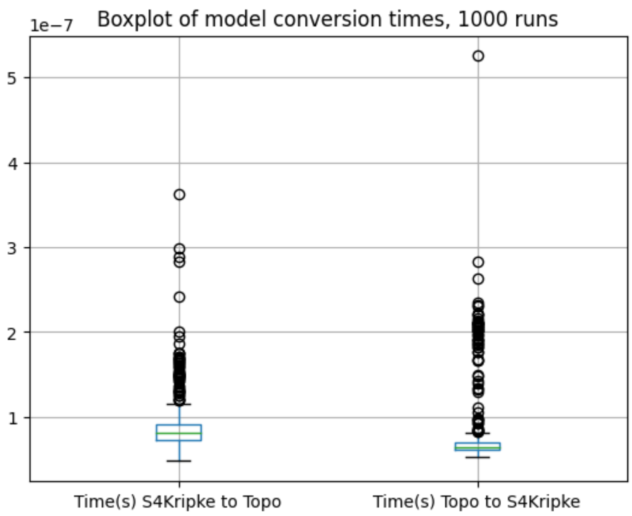
Second, we investigated the question "How efficient is `S4KripkeModel` vs. `TopoModel` generation?". We measured the generation of an arbitrary `S4KripkeModel` and an arbitrary `TopoModel`.

This was done 1000 times and the results are pictured below.



Generating arbitrary `S4KripkeModel`'s (median of 3.65e-07 seconds) takes more time than `TopoModel`'s (median of 1.79e-07 seconds). With the current constraints of generation, we generate larger `S4KripkeModel`'s (823 out of 1000 runs) which also takes more time.

Lastly, we investigated the question "How efficient is the conversion from `S4KripkeModel` to `TopoModel` and the other way around?". We measured the conversions in either direction. This was done 1000 times and the results are pictured below.



Converting from `S4KripkeModel`'s to `TopoModel`'s takes slightly longer (median of 8.1e-08 seconds) compared to converting from `TopoModel`'s to `S4KripkeModel`'s back (median of 6.5e-08 seconds). Similarly as previously, 582 out of 1000 times `S4KripkeModel`'s were larger than Topo Models. Even though the resulting `TopoModel` is smaller than `S4KripkeModel`'s, the process itself involves working with the larger `S4KripkeModel`, which most likely contributes the main runtime overhead.

In conclusion, the differences are somewhat marginal, but `TopoModel`'s seem to represent a semantically equivalent structure that may be sometimes more efficient to work with.

12 Conclusion

We believe this work provides useful and extensible implementations for working with general topology, Kripke models, and topomodels. As demonstrated with the tests and benchmarks, it can be used to test properties of various mathematical structures, implementation details, and their efficiency.

References

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[3] E. Pacuit. *Neighborhood Semantics for Modal Logic*. Springer Cham, 2017.