Topomodels

An implementation of topological semantics for modal logic in Haskell

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Presentation Overview

Motivation

...There are no libraries for general topology ?

...There are no libraries for topomodels ?

Let’s implement them to practice topology, modal logic, their marriage, and Haskell!

Normal modal logics

Syntax:

$$\varphi := \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Box\varphi$$

A *normal modal logic* is a set of formulas of the above form containing K and Dual and closed under *modus ponens*, *uniform substitution*, and *necessitation*.

(K) $\Box(p \to q) \to (\Box p \to \Box q)$
(Dual) $\Box p \leftrightarrow \neg\Diamond\neg p$

The smallest such logic is denoted by **K**.

The logic **S4** is defined as $\mathbf{K} \cup \{\Box p \to \Box\Box p, \Box p \to p\}$

Relational semantics for modal logic

A *Kripke frame* is a tuple (X, R) where X is a set and $R \subseteq X \times X$.

A *Kripke model* is a triple (X, R, V) where (X, R) is a Kripke frame and $V : \mathbf{Prop} \to \wp(X)$.

A *pointed Kripke model* is a 4-tuple (X, R, V, x) where (X, R, V) is a Kripke model and $x \in X$.

Key semantic definition:

$$(X, R, V, x) \models \Box\varphi \Leftrightarrow (\forall x' \in X)\Big(xRx' \Rightarrow (X, R, V, x') \models \varphi\Big)$$

The class Pre

A *pre-order* is a tuple (X, R) such that the following hold for all $x, y, z \in X$.

- xRx
- xRy and yRz implies xRz

The class of pre-orders is denoted by **Pre**.

Fact: The logic **S4** is sound and complete with respect to the class of frames **Pre**

$$\begin{aligned} \varphi \in \mathbf{S4} &\Leftrightarrow \mathbf{Pre} \models \varphi \\ &\Leftrightarrow (\forall (X, R) \in \mathbf{Pre})(\forall V \in \wp(X)^{\mathbf{Prop}})(\forall x \in X)\Big((X, R, V, x) \models \varphi\Big) \end{aligned}$$

Basic topology

A *topological space* is a tuple (X, τ) where X is a set and $\tau \subseteq \wp(X)$ where τ satisfies the following.

- $\varnothing, X \in \tau$
- $S \subseteq \tau$ and $|S| < \omega$ implies $\bigcap S \in \tau$
- $S \subseteq \tau$ implies $\bigcup S \in \tau$

A set S is *open* if $S \in \tau$, *closed* if $X - S \in \tau$, and *clopen* if it is both open and closed.

Basic topology cont.

Given a subset $S \subseteq X$, the *interior of* S, denoted by $\mathrm{int}(S)$, is the largest open subset of S, or, equivalently,

$$\bigcup\{U \in \tau \mid U \subseteq S\}$$

The *closure of* S, denoted by $\mathrm{cl}(S)$, is the smallest closed superset of S, or, equivalently,

$$\bigcap\{C \subseteq X \mid X - C \in \tau \text{ and } S \subseteq C\}$$

The class **Alx**

Recall that a topospace (X, τ) satisfies the following.

- $\varnothing, X \in \tau$
- $S \subseteq \tau$ and $|S| < \omega$ implies $\bigcap S \in \tau$
- $S \subseteq \tau$ implies $\bigcup S \in \tau$

A topospace is called *Alexandrov* if it also satisfies the following *strengthening* of the second requirement above.

- $S \subseteq \tau$ $\cancel{\text{and } |S| < \omega}$ implies $\bigcap S \in \tau$

The class of Alexandrov topospaces is denoted by **Alx**.

Topological semantics for modal logic

A *topomodel* is a triple (X, τ, V) where (X, τ) is a topospace and $V : \mathbf{Prop} \to \wp(X)$.

A *pointed topomodel* is a 4-tuple (X, τ, V, x) where (X, τ, V) is a topomodel and $x \in x$.

Key semantic definition:

$$(X, \tau, V, x) \models \Box\varphi \Leftrightarrow (\exists U \in \tau)\Big(x \in U \text{ and } (\forall y \in U)\Big((X, \tau, V, y) \models \varphi\Big)\Big)$$

This implies that

$$[\![\Box\varphi]\!] = \text{int}\Big([\![\varphi]\!]\Big)$$

$$[\![\Diamond\varphi]\!] = \text{cl}\Big([\![\varphi]\!]\Big)$$

The upset topology

Given a pre-order $\mathbf{X} := (X, R)$, an *upset* is a subset $S \subseteq X$ such that

$$(\forall x \in X)(x \in S \text{ and } xRy \text{ implies } y \in S)$$

We denote that set of all upsets on $\mathbf{X}$ by $\mathrm{Up}(\mathbf{X})$.

Observe that $(X, \mathrm{Up}(\mathbf{X}))$ is an Alexandrov topospace, so given an **S4** Kripke model, we can create a topomodel satisfying the same theory.

The specialisation order

Given an topospace $\mathbf{X} := (X, \tau)$, we can define the *specialisation order on* $\mathbf{X}$ as follows:

$$\begin{aligned} xR_{\mathbf{X}}y :&\iff y \in \mathrm{Cl}(\{x\}) \\ &\iff y \in \bigcap\{C \subseteq X \mid X - C \in \tau \text{ and } \{x\} \subseteq C\} \\ &\iff y \in \bigcap\{C \subseteq X \mid X - C \in \tau \text{ and } x \in C\} \\ &\iff (\forall C \subseteq X)(X - C \in \tau \text{ and } x \in C \Rightarrow y \in C) \end{aligned}$$

Observe that $R_{\mathbf{X}}$ is a pre-order, so given an topomodel, we can create an **S4** Kripke model satisfying the same theory.

Benchmarks implemented

We've implemented the following benchmarks:

- Formula satisfaction for S4KripkeModels and TopoModels
- Arbitrary Generation of S4KripkeModels and TopoModels
- Conversion from S4KripkeModels to TopoModels and the other way

Results to be seen in the report...

Conclusion and future work

Good initial start! The basic mathematical structures are implemented, their correctness tested, benchmarks setup for any future work. Good coding style and package structure.

In the future, this library could be used to...

- Perform hypothesis testing for topology/modal logic areas
- Implementing Formal Learning-theoretic notions
- Implementing topo-evidence models, topo-logic
- Further exploration between Kripke models and topomodels (e.g. **S5** logic is sound and complete w.r.t topomodels whose open sets are closed)

The End

Questions? Comments?