Mathematical Induction

a means of proving a theorem by showing that if it is true of any particular case it is true of the next case in a series, and then showing that it is indeed true in one particular case.

What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that it is true for the next highest element

- In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.
- When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know that $P(2)$ is true, because $P(1)$ implies $P(2)$. Further, we know that $P(3)$ is true, because $P(2)$ implies $P(3)$. Continuing along these lines, we see that $P(n)$ is true for every positive integer n.

- Many theorems assert that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function. Mathematical induction is a technique for proving theorems of this kind. In other words, mathematical induction can be used to prove statements of the form $\forall n\, P(n)$, where the domain is the *set of positive integers*. Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form. (Remember, many mathematical assertions include an implicit universal quantifier. The statement "if *n* is a positive integer, then $n^3 - n$ is divisible by 3" is an example of this. Making the implicit universal quantifier explicit yields the statement "for every positive integer *n*, $n^3 - n$ is divisible by 3.)

- Have you heard of the "Domino Effect"?
- Step 1. The **first** domino falls
- Step 2. When **any** domino falls, the **next** domino falls
- So ... **all dominos will fall!**
- That is how Mathematical Induction works.

Induction example

- Show that the sum of the first *n* odd integers is n^2
  - If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show $\forall n\ \mathrm{P}(n)$ where $\mathrm{P}(n) = \sum_{i=1}^{n} 2i - 1 == n^2$
- Basis step: Show that P(1) is true

$$\sum_{i=1}^{1} 2(i) - 1 == 1^2$$

$$1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{\boxed{k}} 2i - 1 == \boxed{k^2}$$

  - Note: we don't yet know if this is true or not!

- Inductive step: show that it is true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i - 1 == \boxed{(k+1)^2}$$

Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i - 1 == k^2$

- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i - 1 == (k+1)^2$$

$$2(k+1) - 1 + \boxed{\sum_{i=1}^{k} 2i - 1} == k^2 + 2k + 1$$

$$2(k+1) - 1 + \boxed{k^2} == k^2 + 2k + 1$$

$$k^2 + 2k + 1 == k^2 + 2k + 1$$

What did we show

- Base case: P(1)
- If P(k) was true, then P(k+1) is true
  - i.e., P(k) → P(k+1)

- We know it's true for P(1)
- Because of P(k) → P(k+1), if it's true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it's true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it's true for P(3), then it's true for P(4)
- Because of P(k) → P(k+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of n

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k\left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Second induction example

- Show the sum of the first *n* positive even integers is $n^2 + n$
- Rephrased: $\forall n \, \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$

- The three parts:
  - Basis step
  - Inductive hypothesis
  - Inductive step

Second induction example, continued

- Basis step: Show P(1):

$$P(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$P(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

Second induction example, continued

- Recall our inductive hypothesis:

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Notes on proofs by induction

- We manipulate the *k*+1 case to make part of it look like the *k* case
- We then replace that part with the other side of the *k* case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Third induction example

- Show $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- Base case: *n* = 1 $$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction example

- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Show: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$$
- Base case: *n* = 1: $$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$ $$1^2 = \frac{7}{6}$$ $$1 \neq \frac{7}{6}$$
- But let's continue anyway…
- Inductive hypothesis: assume $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

- Show that $n! < n^n$ for all $n > 1$

- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$

- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$(k+1)!$	$= (k+1)\boxed{k!}$	$< (k+1)\boxed{k^k}$	$< (k+1)(k+1)^k$	$= (k+1)^{k+1}$

Example: Sum of Odd Integers

- **Proposition:** $1 + 3 + \ldots + (2n-1) = n^2$
  for all integers $n \geq 1$.
- **Proof (by induction):**

  1) Basis step:

  The statement is true for $n=1$: $1=1^2$.

  2) Inductive step:

  Assume the statement is true for some $k \geq 1$
  (inductive hypothesis) ,
  show that it is true for $k+1$.

Example: Sum of Odd Integers

- **Proof (cont.):**

The statement is true for k:

$$1+3+\ldots+(2k-1) = k^2 \qquad \textbf{(1)}$$

We need to show it for k+1:

$$1+3+\ldots+(2(k+1)-1) = (k+1)^2 \qquad \textbf{(2)}$$

Showing (2):

$$1+3+\ldots+(2(k+1)-1) = 1+3+\ldots+(2k+1) =$$

$$1+3+\ldots+(2k-1)+(2k+1) = \qquad \text{by (1)}$$

$$k^2+(2k+1) = (k+1)^2 .$$

We proved the basis and inductive steps,

so we conclude that the given statement true. ■

Proving a divisibility property by mathematical induction

- **Proposition:** For any integer $n \geq 1$,

  $7^n - 2^n$ is divisible by 5. (P(n))

- **Proof** (*by induction*):

  1) Basis step:

  The statement is true for $n=1$: (P(1))

  $7^1 - 2^1 = 7 - 2 = 5$ is divisible by 5.

  2) Inductive step:

  Assume the statement is true for some $k \geq 1$ (P(k))

  (inductive hypothesis) ;

  show that it is true for $k+1$. (P(k+1))

Proving a divisibility property by mathematical induction

- **Proof (cont.):** We are given that

P(k): $7^k - 2^k$ is divisible by 5. **(1)**

Then $7^k - 2^k = 5a$ for some $a \in \mathbf{Z}$. *(by definition)* **(2)**

We need to show:

P(k+1): $7^{k+1} - 2^{k+1}$ is divisible by 5. **(3)**

$$7^{k+1} - 2^{k+1} = 7\cdot 7^k - 2\cdot 2^k = 5\cdot 7^k + 2\cdot 7^k - 2\cdot 2^k$$

$= 5\cdot 7^k + 2\cdot(7^k - 2^k) = 5\cdot 7^k + 2\cdot 5a$ *(by (2))*

$= 5\cdot(7^k + 2a)$ which is divisible by 5. *(by def.)*

Thus, P(n) is true by induction. ■

Strong induction

- Weak mathematical induction assumes P(k) is true, and uses that (and only that!) to show P(k+1) is true

- Strong mathematical induction assumes P(1), P(2), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$\left[\mathrm{P}(1) \wedge \mathrm{P}(2) \wedge \mathrm{P}(3) \wedge \ldots \wedge \mathrm{P}(k)\right] \rightarrow \mathrm{P}(k+1)$$

Strong induction example 1

- Show that any number > 1 can be written as the product of primes

- Base case: P(2)
  - 2 is the product of 2 (remember that 1 is not prime!)
- Inductive hypothesis: P(1), P(2), P(3), …, P(k) are all true
- Inductive step: Show that P(k+1) is true

Strong induction example 1

- Inductive step: Show that P(k+1) is true
- There are two cases:
  - k+1 is prime
    - It can then be written as the product of k+1
  - k+1 is composite
    - It can be written as the product of two composites, a and b, where $2 \le a \le b < k+1$
    - By the inductive hypothesis, both P(a) and P(b) are true